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Operator on the space of rapidly decreasing functions with all non-zero vectors hypercyclic

Michał Goliński

Adam Mickiewicz University in Poznań, Faculty of Mathematics and Computer Science, Umultowska 87, 61-614 Poznań, Poland

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Abstract

We construct a continuous linear operator T on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions such that each non-zero orbit of T is dense. The construction is inspired by the work of C. Read on similar operators on the space ℓ_1. The construction, due to the structure of a Fréchet space, can be made significantly simpler than the original construction of Read.

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1. Introduction

For a linear operator on a linear topological space $T: X \to X$ a vector $x \in X$ is called *hypercyclic* if $\{x, Tx, T^2x, \ldots\}$ is dense in X; x is called *cyclic* if $\operatorname{span}\{x, Tx, T^2x, \ldots\}$ is dense in X. An operator T which has a hypercyclic vector is called hypercyclic.

Studying hypercyclicity started as a means for better understanding the invariant subspaces of linear operators, but recently has gained momentum of its own, as evidenced by the recent

E-mail address: golinski@amu.edu.pl.

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publication of two books on the subject: [2,6]. It is now known that any separable infinite-dimensional Fréchet space supports a hypercyclic operator, a result due to Ansari [1] and Bonet and Peris [3].

By a result of Herrero [7] and Bourdon [4] any hypercyclic operator on a Fréchet space has a dense subspace of hypercyclic vectors. An interesting problem is to find operators with as many hypercyclic vectors as possible. In [10] Read, building on his construction of an operator without invariant subspaces, could construct an operator with every non-zero vector hypercyclic on any Banach space of the form $\ell_1 \oplus W$, where W is a separable Banach space. This construction seems very complicated, even though the idea behind it is quite simple.

In this paper we will construct a continuous linear operator on the space s of rapidly decreasing sequences such that every non-zero vector is hypercyclic. In [5] the author has constructed an operator on s (and other nuclear Fréchet spaces) for which every vector is cyclic (i.e. the operator has no non-trivial invariant subspaces). That construction was modelled after the exposition of Read's result from [9] given in [2]. The present construction is built upon the previous one and an idea underlying [10] with notation compatible with [5]. The construction, while still quite technical, is much simpler than that in [10]. Unfortunately it cannot be transferred back to a Banach space setting.

We will denote the set $\{0, 1, \ldots\}$ by $\mathbb{N}$ and by $\mathbb{N}_+$ the set $\{1, 2, \ldots\}$. By $\mathbb{K}$ we will denote either the field of real or complex numbers. We define a sequence space:

$$s = \left\{ x = (x_j)_{j=0}^{\infty} \in \mathbb{K}^{\mathbb{N}} : |x|_N := \sum_{j=0}^{\infty} |x_j|(j+1)^N < \infty,\ N = 1, 2, \ldots \right\}$$

endowed with its natural locally convex topology. It is called the *space of rapidly decreasing sequences*. One can show that s is in fact a nuclear Fréchet space (see [8, 29.4.1]).

It is known that many spaces in analysis are in fact isomorphic to s as Fréchet spaces, including

- the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$, where
$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^{\infty}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} \left(1 + |x|^2\right)^N |f(x)| < \infty, \quad N = 1, 2, \ldots \right\},$$
see [8, 31.14];
- the space $C_{2\pi}^{\infty}(\mathbb{R})$ of periodic smooth functions (see [8, 29.5.(1)]);
- the space $C^{\infty}[0, 1]$; see [8, 29.5.(4)] for a direct proof;
- the space $C^{\infty}(K)$ for each compact C^{∞}-manifold K (see [12]);
- the space $\mathcal{D}(K)$ of smooth functions with their support contained in a compact set $K \subset \mathbb{R}^n$, when K has a nonempty interior (see [12]);
- the space of all entire Dirichlet series, i.e. the space of sequences (a_n) such that the series
$$\sum_{n=1}^{\infty} a_n n^{z_1 + \cdots + z_d}$$
is convergent for any $(z_1, \ldots, z_d) \in \mathbb{C}^d$ (see [11, 8.4.1]).

Our main result is:

Theorem 1. *There exists a continuous linear operator $T: s \to s$ such that every non-zero vector is hypercyclic with respect to T, in other words, there are no non-trivial (closed) invariant subsets of T.*

2. Preliminaries

Throughout we will denote by c_{00} the space of all finite sequences — a linear subspace of $\mathbb{K}^{\mathbb{N}}$. The canonical basis of c_{00} will be denoted by $(e_0, e_1, e_2, \ldots)$ and $E_n = \operatorname{span}\{e_0, e_1, \ldots, e_n\}$. For $M \subseteq \mathbb{N}$ we will write π_M for the canonical projection onto $\operatorname{span}\{e_j : j \in M\}$. For brevity we write $\pi_m := \pi_{[0,m]}$.

For a linear basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ of E_n and

$$E_n \ni x = \sum_{i=k}^{n} x_i \gamma_i, \quad x_k \neq 0$$

we write $\operatorname{val}_{\widetilde{\gamma}}(x) = k$. If $x = 0$, then we put $\operatorname{val}_{\widetilde{\gamma}}(x) = +\infty$. For a set $K \subseteq E_n$ we define $\operatorname{val}_{\widetilde{\gamma}}(K) = \sup_{y \in K} \operatorname{val}_{\widetilde{\gamma}}(y)$.

Remark 2. Observe that if

$$\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$$

and

$$\widetilde{\mu} = (\gamma_0, \gamma_1, \ldots, \gamma_n, \mu_{n+1}, \ldots, \mu_m)$$

are bases of E_n and E_m respectively, then for $y \in E_n$

$$\operatorname{val}_{\widetilde{\gamma}}(y) = \operatorname{val}_{\widetilde{\mu}}(y).$$

We will call a linear basis $\gamma = (\gamma_0, \gamma_1, \ldots)$ of c_{00} a *perturbed canonical basis* if $\operatorname{span}\{\gamma_0, \gamma_1, \ldots, \gamma_n\} = E_n$ for every n. Analogously, a linear operator $T: c_{00} \to c_{00}$ will be called a *perturbed weighted forward shift* if $\left(e_0, Te_0, T^2 e_0, \ldots\right)$ is a perturbed canonical basis.

Further on, for a polynomial $P \in \mathbb{K}[t]$ we will write $|P|$ for the sum of the modules of the coefficients of P, obviously $|\cdot|$ is a norm on the space of all polynomials.

Observe that for each $N \in \mathbb{N}_+$ the sequence $\left(\left(\frac{j+1}{j}\right)^N\right)_j$ is decreasing and converges to 1, therefore there exists an increasing sequence $(k_N)_{N \in \mathbb{N}_+}$, $k_1 = 3$, with the property that

$$\sup_{j \in \mathbb{N}} \frac{(j + k_N + 1)^N}{(j + k_N)^N} = \frac{(k_N + 1)^N}{k_N^N} \leq \frac{3}{2}, \quad N = 1, 2, \ldots. \tag{1}$$

Clearly,

$$\frac{(j + k_N)^N}{(j + k_{N+1})^{N+1}} \leq \frac{1}{j + k_{N+1}} < \frac{1}{2} \tag{2}$$

for all $j \in \mathbb{N}$, $N \in \mathbb{N}_+$.

Further on, we fix

$$A_{N,j} := (j + k_N)^N ,$$
$$\|(x_j)\|_N := \sum_{j=0}^{\infty} |x_j| A_{N,j}.$$

For further reference let us restate condition (1):

$$\frac{A_{N,j+1}}{A_{N,j}} \le \frac{3}{2}, \quad N = 1, 2, \ldots, \ j = 0, 1, \ldots. \tag{3}$$

One can easily check that each seminorm $\|\cdot\|_N$ is equivalent to $|\cdot|_N$, therefore $(s, |\cdot|_N)$ is isomorphic to $(s, \|\cdot\|_N)$. Moreover, the unit balls for seminorms $\|\cdot\|_N$ form a base of neighbourhoods of zero in s by (2).

3. The lemma

The following lemma will allow us to "extend" partially defined finite dimensional operators to more dimensions and is a basis for an inductive procedure carried out in Section 5. It is an almost verbatim repetition of a result one finds in the last chapter of [2], only slightly modified to admit also weighted ℓ_1 norms. In [5] a somewhat more complicated version was used, but it turns out that this simplified one will be sufficient for us. We give a full proof for the convenience of the reader.

Lemma 3. *Assume that for some integers $a, \Delta > 0$ there is given a perturbed canonical basis $\widetilde{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_{a+\Delta-1})$ of $E_{a+\Delta-1}$, with $\gamma_a = \varepsilon e_a + e_0$ and $\gamma_0 = e_0$, where $\varepsilon > 0$. Let $\|\cdot\|$ be any weighted ℓ_1-norm on c_{00} and K be any compact set with respect to the topology induced by $\|\cdot\|$ satisfying*

$$K \subseteq \{y \in E_{a+\Delta-1} : \mathrm{val}_{\widetilde{\gamma}}(y) \le a\}.$$

Then there is a finite family of polynomials $\mathcal{P} = \{P_l\}_{l=1}^{L}$ satisfying $\deg P_l < a + \Delta$ *and a number C such that for any $y \in K$ there is a polynomial $P \in \mathcal{P}$ such that for each perturbed weighted forward shift $T : c_{00} \to c_{00}$ with*

$$T^j e_0 = \gamma_j, \quad \text{if } j = 1, 2, \ldots, a + \Delta - 1 \tag{4}$$

we have that

$$\|P(T)y - e_0\| \le 2\varepsilon \|e_a\| + C \max_{a+\Delta \le j \le 2(a+\Delta-1)} \left\| T^j e_0 \right\|.$$

Proof. Let a linear map $T' : E_{a+\Delta-1} \to E_{a+\Delta-1}$ be given by:

$$T'(\gamma_j) = \begin{cases} \gamma_{j+1}, & j < a + \Delta - 1; \\ 0, & j = a + \Delta - 1. \end{cases}$$

Take $z \in K$. It is easy to see that

$$\gamma_a = \varepsilon e_a + e_0 \in \operatorname{span}\{z, T'z, \left(T'\right)^2 z, \ldots, \left(T'\right)^{a+\Delta-1} z\},$$

particularly for every $z \in K$ there is a polynomial P_z of degree smaller that $a + \Delta$ such that

$$\left\| P_z(T')z - e_0 \right\| = \varepsilon \left\| e_a \right\|.$$

Now, because of the continuity of T', the compactness of K assures us that there is a finite family of polynomials $\mathcal{P}$ such that for every $y \in K$ there exists $P \in \mathcal{P}$ such that

$$\left\| P(T')y - e_0 \right\| \le 2\varepsilon \|e_a\|. \tag{5}$$

Let $y = \sum_{k=0}^{a+\Delta-1} \lambda_k \gamma_k \in K$ and $P(t) = \sum_{i=0}^{a+\Delta-1} p_i t^i$ be chosen so that (5) holds. Then we have by (4) that

$$P(T)y = \sum_{i=0}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} p_i \lambda_k T^{i+k} e_0 = \sum_{j=0}^{2(a+\Delta-1)} \sum_{i=0}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} \delta_{j,i+k} p_i \lambda_k T^j e_0$$

$$P(T')y = \sum_{j=0}^{a+\Delta-1} \sum_{i=0}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} \delta_{j,i+k} p_i \lambda_k T^j e_0,$$

where $\delta_{i,j}$ is the Kronecker delta. Therefore

$$\begin{aligned}\|P(T)y - e_0\| &\le \|P(T')y - e_0\| + \left\| \sum_{j=a+\Delta}^{2(a+\Delta-1)} \sum_{i=0}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} \delta_{j,i+k} p_i \lambda_k T^j e_0 \right\| \\ &\le 2\varepsilon \|e_a\| + \max_{a+\Delta \le j \le 2(a+\Delta-1)} \left\| T^j e_0 \right\| \sum_{i=0}^{a+\Delta-1} \sum_{k=0}^{a+\Delta-1} |p_i \lambda_k| \\ &\le 2\varepsilon \|e_a\| + C \max_{a+\Delta \le j \le 2(a+\Delta-1)} \left\| T^j e_0 \right\|,\end{aligned}$$

for some C that depends only on K and $\widetilde{\gamma}$. □

4. The operator

Assume that a strictly increasing sequence $(\Delta_1, a_1, c_1, \Delta_2, a_2, c_2, \ldots)$ of positive integers increases sufficiently rapidly, and let (μ_n) be a sequence of positive integers so that the intervals

$$\begin{aligned}&[a_1, a_1 + \Delta_1), [c_1, c_1 + a_1 + \Delta_1), [c_1^2, c_1^2 + a_1 + \Delta_1), \ldots, [c_1^{\mu_1}, c_1^{\mu_1} + a_1 + \Delta_1), \\ &[a_2, a_2 + \Delta_2), [c_1, c_2 + a_2 + \Delta_2), [c_2^2, c_2^2 + a_1 + \Delta_1), \ldots, [c_2^{\mu_2}, c_2^{\mu_2} + a_2 + \Delta_2), \\ &\ldots\end{aligned}$$

are pairwise disjoint—where $c_n^{\mu_n} < a_{n+1}$. Assume that for every n and $1 \le k \le \mu_n$ some polynomials $P_l^{(n)}, S_w^{(n)}$ are given together with some function ρ_n fixing the mapping between (l, w) and $\{1, 2, \ldots, \mu_n\}$. Assume that $\deg P_l^{(n)} + \deg S_w^{(n)} < c_n$. Moreover, in (6) below let α_j be non-zero real numbers and let C_n be positive real numbers. Let $(N_n) = (1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots)$ be a sequence containing each positive integer infinitely many times.

We define a sequence of finite vectors inductively by

$$T^j e_0 = \begin{cases} \frac{1}{n2^n A_{N_n,a_n}} e_j + T^{j-a_n} e_0, & j \in [a_n, a_n + \Delta_n), \\ \frac{1}{C_n A_{N_n,c_n^k}} e_j + P_l^{(n)}(T) S_w^{(n)}(T) T^{j-c_n^k} e_0, & j \in [c_n^k, c_n^k + a_n + \Delta_n), \\ \alpha_j e_j, & \text{otherwise}, \end{cases} \tag{6}$$

where in the second case $k = \rho_n(l, w) \in \{1, 2, \dots \mu_n\}$.

One can check that our assumptions on all the parameters imply that Eq. (6) uniquely defines a linear operator T on the space of finite sequences. In fact after fixing values for all the parameters in the next section, while proving Proposition 6 we will calculate how T acts on the basic vectors, but it is not important at this point.

Regardless of the precise values of the parameters we have that $\left(e_0, Te_0, T^2 e_0, \dots\right)$ is a perturbed canonical basis of c_{00}, as for any $j \in \mathbb{N}, T^j e_0 \in E_j$ and the coefficient of e_j in $T^j e_0$ is always non-zero.

5. The parameters

We will describe in this section an inductive process by which we will find suitable parameters to plug into (6). As the first step is very similar to the consecutive ones, we only indicate the necessary adjustments/changes.

Assume that we have already defined all the parameters $\Delta_{n-1}, a_{n-1}, c_{n-1}, \mu_{n-1}, C_{n-1}$ together with corresponding numbers α_j in (6). We will show how to choose suitable $\Delta_n, a_n, c_n, \mu_n, C_n$ and the corresponding α_j.

First, we put $\Delta_n = c_{n-1}^{\mu_{n-1}} + a_{n-1} + \Delta_{n-1}$ (we take $\Delta_1 = 1$).

Let $\eta_n = 2^{1/c_{n-1}^{\mu_{n-1}}}$ $\left(\text{we take } \eta_1 = \frac{4}{3}\right)$.

Let d_{n-1} be a number such that

$$\frac{A_{N_{n-1},j}}{A_{N_{n-1}+1,j}} \le 1/2^{c_{n-1}^{\mu_{n-1}}} \quad \text{for } j \ge d_{n-1}. \tag{7}$$

We can assume that $d_{n-1} \ge \Delta_n$. We just skip this step if $n = 1$.

Take a_n to be any number such that the following inequalities are satisfied (skip (8) and (10) for $n = 1$):

$$a_n \ge d_{n-1} + c_{n-1}^{\mu_{n-1}}; \tag{8}$$

$$a_n \ge n 2^{n+\Delta_n} A_{1,0}; \tag{9}$$

$$A_{N_n,a_n} \ge C_{n-1}^2 2^{a_{n-1}+\Delta_{n-1}} \left(A_{N_{n-1}, 2c_{n-1}^{\mu_{n-1}}}\right)^{\mu_{n-1}+1}; \tag{10}$$

$$\frac{\eta_n^{a_n - \Delta_n - 1}}{n 2^n A_{N_n,a_n}} \ge 1. \tag{11}$$

We put

$$\alpha_j = \frac{\eta_n^{j-\Delta_n}}{n 2^n A_{N_n,a_n}} \quad \text{for } \Delta_n \le j < a_n. \tag{12}$$

Using the already defined parameters and definition (6), we can define a basis

$$\widetilde{\gamma}_n = \left(e_0, Te_0, T^2e_0, \ldots, T^{a_n+\Delta_n-1}e_0\right)$$

of $E_{a_n+\Delta_n-1}$. In terms of this basis we define a projection $\tau_n\colon E_{a_n+\Delta_n-1} \to E_{a_n+\Delta_n-1}$ by

$$\sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^j e_0 \overset{\tau_n}{\longmapsto} \sum_{j=0}^{a_n} \lambda_j T^j e_0 \tag{13}$$

and a compact set

$$K_n = \left\{x \in E_{a_n+\Delta_n-1} : \|x\|_1 \le 1 \text{ and } \|\tau_n x\|_1 \ge \frac{1}{2}\right\}. \tag{14}$$

By Lemma 3, applied with $\widetilde{\gamma} = \widetilde{\gamma}_n, K = K_n, \|\cdot\| = \|\cdot\|_{N_n}, a = a_n, \Delta = \Delta_n$, we get a finite family of polynomials $\mathcal{P}^{(n)} = \left\{P_l^{(n)}\right\}_{l=1}^{L_n}$ and a number C_n with the stated properties. Let $\mathcal{S}^{(n)} = \left\{S_w^{(n)}\right\}_{w=1}^{W_n}$ be a $\frac{1}{2^n A_{N_n,0}}$-net with respect to the norm $|\cdot|$ in the set of all polynomials with degree at most n and sum of the absolute values of coefficients also at most n. We put $\mu_n = L_n W_n$ and fix a bijection ρ_n between $\{1, \ldots, L_n\} \times \{1, \ldots, W_n\}$ and $\{1, \ldots, \mu_n\}$.

We can assume that $\left|P_l^{(n)}\right| \le C_n$ for $P_l^{(n)} \in \mathcal{P}^{(n)}$ and if

$$y = \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^j e_0 \in K_n,$$

then $\sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \le C_n$. We additionally require $C_n \ge 1$.

Let us denote $\theta_{n,1} = 2^{1/a_n}$. We fix c_n to be any number greater than $2(a_n + \Delta_n + n)$ such that:

$$\frac{\theta_{n,1}^{c_n-a_n-\Delta_n-1}}{nC_n^2 2^{a_n+\Delta_n+n} A_{N_n,2c_n^{\mu_n}}} \ge nC_n 2^{2(a_n+\Delta_n)+n}; \tag{15}$$

$$\frac{2^{c_n-2}}{nC_n^2 2^{a_n+\Delta_n+n}\left(A_{N_n,2c_n^{\mu_n}}\right)^{\mu_n}} \ge nC_n 2^{2(a_n+\Delta_n)+n}; \tag{16}$$

$$\frac{A_{N_n,j}}{A_{N_n+1,j}} \le \frac{1}{n^2 C_n^3 2^{3(a_n+\Delta_n)+2n} A_{N_n,0}} \quad \text{for } j \ge c_n; \tag{17}$$

$$\frac{A_{N_n,2j}}{A_{N_n+1,j}} \le \frac{1}{2} \quad \text{for } j \ge c_n; \tag{18}$$

$$\frac{A_{N_n,j+a_n+\Delta_n}}{A_{N_n,j}} \le 2 \quad \text{for } j \ge c_n. \tag{19}$$

Let us denote $\theta_{n,k} = 2^{1/c_n^{k-1}}$ for $k = 2, 3, \ldots, \mu_n$. We put

$$\alpha_j = \frac{\theta_{n,1}^{j-a_n-\Delta_n}}{nC_n^2 2^{a_n+\Delta_n+n} A_{N_n,2c_n^{\mu_n}}} \quad \text{for } j \in [a_n+\Delta_n, c_n); \tag{20}$$

$$\alpha_j = \frac{\theta_{n,k}^{j-c_n^{k-1}-a_n-\Delta_n}}{nC_n^2 2^{a_n+\Delta_n+n} \left(A_{N_n,2c_n^{\mu_n}}\right)^k} \quad \text{for } j \in [c_n^{k-1}+a_n+\Delta_n, c_n^k), 2 \le k \le \mu_n. \tag{21}$$

These parameters imply the following properties of the numbers α_j. The guiding principle is that at the beginning of each contiguous interval, the numbers α_j are very small and are large towards the end of each such interval.

Corollary 4. *If all the parameters are chosen as described in this section, we have that*

$$\alpha_{\Delta_n} \overset{(12)}{=} \frac{1}{n2^n A_{N_n,a_n}}; \tag{22}$$

$$\alpha_{a_n-1} \overset{(11)}{\ge} 1; \tag{23}$$

$$\alpha_{a_n+\Delta_n} \overset{(20)}{=} \frac{1}{nC_n^2 2^{a_n+\Delta_n+n} A_{N_n,2c_n^{\mu_n}}}; \tag{24}$$

$$\alpha_{c_n^k+a_n+\Delta_n} \overset{(21)}{=} \frac{1}{nC_n^2 2^{a_n+\Delta_n+n} \left(A_{N_n,2c_n^{\mu_n}}\right)^{k+1}} \quad \textit{for } k = 1, \ldots, \mu_n - 1; \tag{25}$$

$$\alpha_{c_n^{\mu_n}+a_n+\Delta_n} = \alpha_{\Delta_{n+1}} \overset{(12),(10)}{\le} \frac{1}{nC_n^2 2^{a_n+\Delta_n+n} \left(A_{N_n,2c_n^{\mu_n}}\right)^{\mu_n+1}}; \tag{26}$$

$$\alpha_{c_n-1} \overset{(20),(15)}{\ge} nC_n 2^{2(a_n+\Delta_n)+n}; \tag{27}$$

$$\alpha_{c_n^k-1} \overset{(21)}{\ge} \frac{2^{\frac{c_n^k-c_n^{k-1}-a_n-\Delta_n-1}{c_n^{k-1}}}}{nC_n^2 2^{a_n+\Delta_n+n} \left(A_{N_n,2c_n^{\mu_n}}\right)^{\mu_n}} \overset{(16)}{\ge} nC_n 2^{2(a_n+\Delta_n)+n} \quad \textit{for } k = 2, \ldots, \mu_n. \tag{28}$$

In the procedure above we have used Lemma 3, in order to emphasise it and for further reference let us write it as a corollary.

Corollary 5. *If the parameters are chosen in the way specified in this section, then for each n and each $y \in K_n$ there exists a polynomial $P \in \mathcal{P}^{(n)}$ with* $\deg P < a_n + \Delta_n$ *such that*

$$\|P(T)y - e_0\|_{N_n} \le \frac{3}{n2^n}.$$

Proof. By Lemma 3, we have that:

$$\begin{aligned}
\|P(T)y - e_0\|_{N_n} &\le \frac{2}{n2^n} + C_n \cdot \max_{a_n+\Delta_n \le j \le 2(a_n+\Delta_n-1)} \left\|T^j e_0\right\|_{N_n} \\
&\overset{(6)}{=} \frac{2}{n2^n} + C_n \alpha_{2(a_n+\Delta_n-1)} A_{N_n,2(a_n+\Delta_n-1)} \\
&\overset{(20)}{\le} \frac{2}{n2^n} + 4C_n \alpha_{a_n+\Delta_n} A_{N_n,2(a_n+\Delta_n-1)} \overset{(24)}{\le} \frac{3}{n2^n}. \quad \square
\end{aligned}$$

6. Continuity

Proposition 6. *When the parameters are chosen as described in Section 5, then the operator* $T : c_{00} \to c_{00}$ *given by (6) satisfies for all* N

$$\|Tx\|_N \leq 2\|x\|_N.$$

Proof. Because the norms are weighted ℓ_1 norms, we need to show this only for all the basic vectors e_j. The proof consists in checking the possible cases for j and uses induction on j in some of the cases.

- If $j = 0$, then by (3)
$$\frac{\|Te_0\|_N}{\|e_0\|_N} = \frac{\|\alpha_1 e_1\|_N}{\|e_0\|_N} = \frac{1}{2A_{N_1,a_1}} \frac{A_{N,1}}{A_{N,0}} \leq \frac{3}{4}.$$
- If $j \in [\Delta_n, a_n - 1) \cup [a_n + \Delta_n, c_n - 1) \cup \bigcup_{k=2}^{\mu_n} [c_n^{k-1} + a_n + \Delta_n, c_n^k - 1)$ for some n, then
$$\frac{\|Te_j\|_N}{\|e_j\|_N} \overset{(6)}{=} \frac{1}{\alpha_j} \frac{\left\|T^{j+1}e_0\right\|_N}{\|e_j\|_N} \overset{(6)}{=} \frac{\alpha_{j+1}}{\alpha_j} \frac{A_{N,j+1}}{A_{N,j}} \overset{(12),(20),(21)}{\leq} \frac{4}{3} \cdot \frac{3}{2} = 2.$$
- If $j \in [a_n, a_n + \Delta_n - 1) \cup \bigcup_{k=1}^{\mu_n} [c_n^k, c_n^k + a_n + \Delta_n - 1)$ for some n, then one can check that (6) implies $Te_j = e_{j+1}$, hence
$$\frac{\|Te_j\|_N}{\|e_j\|_N} = \frac{A_{N,j+1}}{A_{N,j}} \leq \frac{3}{2} < 2.$$
- If $j = a_n - 1$ for some n, then
$$\begin{aligned}\frac{\|Te_{a_n-1}\|_N}{\|e_{a_n-1}\|_N} &= \frac{\left\|\frac{1}{\alpha_{a_n-1}}\left(\frac{1}{n2^n A_{N_n,a_n}} e_{a_n} + e_0\right)\right\|_N}{\|e_{a_n-1}\|_N} \\ &= \frac{1}{\alpha_{a_n-1}}\left(\frac{1}{n2^n A_{N_n,a_n}} \frac{A_{N,a_n}}{A_{N,a_n-1}} + \frac{A_{N,0}}{A_{N,a_n-1}}\right) \\ &\overset{(23)}{\leq} \frac{1}{2} \cdot \frac{3}{2} + 1 < 2.\end{aligned}$$
- If $j = a_n + \Delta_n - 1$ for some n, then
$$\begin{aligned}\frac{\|Te_{a_n+\Delta_n-1}\|_N}{\|e_{a_n+\Delta_n-1}\|_N} &= \frac{\left\|n2^n A_{N_n,a_n}\left(\alpha_{a_n+\Delta_n} e_{a_n+\Delta_n} - \alpha_{\Delta_n} e_{\Delta_n}\right)\right\|_N}{\|e_{a_n+\Delta_n-1}\|_N} \\ &\overset{(24),(22)}{=} \frac{A_{N_n,a_n}}{C_n^2 2^{a_n+\Delta_n} A_{N_n,2c_n^{\mu_n}}} \frac{A_{N,a_n+\Delta_n}}{A_{N,a_n+\Delta_n-1}} + \frac{A_{N,\Delta_n}}{A_{N,a_n+\Delta_n-1}} \leq 2.\end{aligned}$$
- If $j = c_n^k - 1$ for some $k \in \{1, 2, \ldots, \mu_n\}$ with $k = \rho_n(l, w)$, then (6) implies that
$$Te_{c_n^k-1} = \frac{1}{\alpha_{c_n^k-1}}\left(\frac{1}{C_n A_{N_n,c_n^k}} e_{c_n^k} + P_l^{(n)}(T) S_w^{(n)}(T) e_0\right).$$

  By construction we have that $\deg P_l^{(n)} < a_n + \Delta_n$, $\left|P_l^{(n)}\right| \leq C_n$, $\deg S_w^{(n)} \leq n$ and $\left|S_w^{(n)}\right| \leq n$. Therefore, because $a_n + \Delta_n + n < c_n - 1 \leq j$, we have by induction that
$$\left\|P_l^{(n)}(T) S_w^{(n)}(T) e_0\right\|_N \leq nC_n 2^{a_n+\Delta_n+n} A_{N,0}. \tag{29}$$

Hence

$$\frac{\|Te_{c_n^k-1}\|_N}{\|e_{c_n^k-1}\|_N} \le \frac{1}{\alpha_{c_n^k-1}}\left(\frac{1}{C_n A_{N_n,c_n^k}}\frac{A_{N,c_n^k}}{A_{N,c_n^k-1}} + nC_n 2^{a_n+\Delta_n+n}\frac{A_{N,0}}{A_{N,c_n^k-1}}\right) \overset{(27),(28)}{\le} 2.$$

- If $j = c_n^k + a_n + \Delta_n - 1$ for some $k \in \{1, 2, \dots, \mu_n\}$ with $k = \rho_n(l, w)$, then with a similar reasoning to (29) we get that

$$\begin{aligned}
&\frac{\|Te_{c_n^k+a_n+\Delta_n-1}\|_N}{\|e_{c_n^k+a_n+\Delta_n-1}\|_N} \\
&\overset{(6)}{=} \frac{\left\|C_n A_{N_n,c_n^k}\left(T^{c_n^k+a_n+\Delta_n}e_0 - P_l^{(n)}(T)S_w^{(n)}(T)T^{a_n+\Delta_n}e_0\right)\right\|_N}{\|e_{c_n^k+a_n+\Delta_n-1}\|_N} \\
&\overset{(6)}{\le} C_n A_{N_n,c_n^k}\Bigg(\alpha_{c_n^k+a_n+\Delta_n}\frac{A_{N,c_n^k+a_n+\Delta_n}}{A_{N,c_n^k+a_n+\Delta_n-1}} \\
&\quad + nC_n 2^{a_n+\Delta_n+n}\alpha_{a_n+\Delta_n}\frac{A_{N,a_n+\Delta_n}}{A_{N,c_n^k+a_n+\Delta_n-1}}\Bigg) \\
&\le nC_n^2 2^{a_n+\Delta_n+n}A_{N_n,c_n^{\mu_n}}\left(\frac{3}{2}\alpha_{c_n^k+a_n+\Delta_n} + \alpha_{a_n+\Delta_n}\right) \\
&\overset{(24),(25),(26)}{\le} nC_n^2 2^{a_n+\Delta_n+n}A_{N_n,c_n^{\mu_n}}\alpha_{a_n+\Delta_n}\left(\frac{3}{2A_{N_n,2c_n^{\mu_n}}} + 1\right) \overset{(24)}{\le} 2. \quad \square
\end{aligned}$$

Corollary 7. *When all the parameters in definition (6) are chosen as indicated in Section 5, the operator T can be uniquely extended to a continuous operator acting on the space s. We will still call this extension T. Moreover e_0 is a cyclic vector for T.*

7. Tails

Lemma 8. *If all the parameters are chosen as described in Section 5, then for the linear operator T defined by (6), for each $n \in \mathbb{N}_+$ and any $x \in s$ we have for any $1 \le k \le \mu_n$*

$$\left\|T^{c_n^k}\pi_{[a_n+\Delta_n,+\infty)}x\right\|_{N_n} \le 3\|x\|_{N_n+1}.$$

Proof. Because our norms are weighted ℓ_1 norms, we need only to show that

$$\frac{\left\|T^{c_n^k}\pi_{[a_n+\Delta_n,+\infty)}e_j\right\|_{N_n}}{\|e_j\|_{N_n+1}} \le 3$$

for all the basic vectors e_j. It is trivially true for $j < a_n + \Delta_n$. We will now check all the remaining cases for j, using Proposition 6 where necessary. Let $k = \rho_n(l, w)$, we have that:

- If $j \in [c_n^p, c_n^p + a_n + \Delta_n)$, where $1 \le p \le \mu_n$ and $p = \rho_n(l', w')$, then $c_n^k + j$ falls into the "otherwise" case in (6), because it is at least $c_n^{\max(k,p)} + a_n + \Delta_n$, but smaller than $c_n^{\max(k,p)+1}$.

Hence

$$\begin{aligned}T^{c_n^k}e_j &\overset{(6)}{=} T^{c_n^k}C_nA_{N_n,c_n^p}\left(T^je_0 - P_{l'}^{(n)}(T)S_{w'}^{(n)}(T)T^{j-c_n^p}e_0\right)\\ &= C_nA_{N_n,c_n^p}T^{j+c_n^k}e_0 - C_nA_{N_n,c_n^p}P_{l'}^{(n)}(T)S_{w'}^{(n)}(T)T^{j-c_n^p+c_n^k}e_0\\ &\overset{(6)}{=} C_nA_{N_n,c_n^p}\alpha_{j+c_n^k}e_{j+c_n^k} - \frac{C_nA_{N_n,c_n^p}}{C_nA_{N_n,c_n^k}}P_{l'}^{(n)}(T)S_{w'}^{(n)}(T)e_{j-c_n^p+c_n^k}\\ &\quad - C_nA_{N_n,c_n^p}P_{l'}^{(n)}(T)S_{w'}^{(n)}(T)P_l^{(n)}(T)S_w^{(n)}(T)T^{j-c_n^p}e_0.\end{aligned}$$

Therefore using Proposition 6 and an estimate similar to (29), we get that

$$\begin{aligned}\frac{\left\|T^{c_n^k}e_j\right\|_{N_n}}{\left\|e_j\right\|_{N_n+1}} &\le \frac{C_nA_{N_n,c_n^p}\alpha_{j+c_n^k}A_{N_n,j+c_n^k}}{A_{N_n+1,c_n^p}} + \frac{A_{N_n,c_n^p}}{A_{N_n,c_n^k}}nC_n2^{a_n+\Delta_n+n}\frac{A_{N_n,j+c_n^k-c_n^p}}{A_{N_n+1,c_n^p}}\\ &\quad + C_nA_{N_n,c_n^p}\left(nC_n2^{a_n+\Delta_n+n}\right)^2 2^{a_n+\Delta_n}\frac{A_{N_n,0}}{A_{N_n+1,c_n^p}}.\end{aligned} \tag{30}$$

Now we estimate the three terms. For the first one we have that

$$\begin{aligned}\frac{C_nA_{N_n,c_n^p}\alpha_{j+c_n^k}A_{N_n,j+c_n^k}}{A_{N_n+1,c_n^p}} &\overset{(19),(21)}{\le} 4C_n\alpha_{\max(c_n^p,c_n^k)+a_n+\Delta_n}A_{N_n,c_n^p+c_n^k}\\ &\overset{(25),(26)}{\le} \frac{A_{N_n,c_n^p+c_n^k}}{A_{N_n,2c_n^{\mu_n}}} \le 1.\end{aligned}$$

As for the second term in (30), we have that

$$\frac{A_{N_n,j+c_n^k-c_n^p}}{A_{N_n,c_n^k}}nC_n2^{a_n+\Delta_n+n}\frac{A_{N_n,c_n^p}}{A_{N_n+1,c_n^p}} \overset{(19)}{\le} nC_n2^{a_n+\Delta_n+n+1}\frac{A_{N_n,c_n^p}}{A_{N_n+1,c_n^p}} \overset{(17)}{\le} 1.$$

And the third term:

$$C_n\left(nC_n2^{a_n+\Delta_n+n}\right)^2 2^{a_n+\Delta_n}A_{N_n,0}\frac{A_{N_n,c_n^p}}{A_{N_n+1,c_n^p}} \overset{(17)}{\le} 1.$$

- If $j \in [a_n+\Delta_n, c_n) \cup [c_n+a_n+\Delta_n, c_n^2) \cup \cdots \cup [c_n^{k-1}+a_n+\Delta_n, c_n^k)$, then

$$\frac{\left\|T^{c_n^k}e_j\right\|_{N_n}}{\left\|e_j\right\|_{N_n+1}} \overset{(6)}{=} \frac{1}{\alpha_j}\frac{\left\|T^{c_n^k+j}e_0\right\|_{N_n}}{\left\|e_j\right\|_{N_n+1}} \overset{(6)}{=} \frac{\alpha_{c_n^k+j}}{\alpha_j}\frac{A_{N_n,c_n^k+j}}{A_{N_n+1,j}}.$$

From (20), (21) and (12) it follows that $\alpha_{c_n^k+j} \le \alpha_{2c_n^k}$ and $\alpha_j \ge \alpha_{c_n^{k-1}+a_n+\Delta_n}$, so

$$\frac{\left\|T^{c_n^k}e_j\right\|_{N_n}}{\left\|e_j\right\|_{N_n+1}} \overset{(21)}{\le} \frac{2\alpha_{c_n^k+a_n+\Delta_n}}{\alpha_{c_n^{k-1}+a_n+\Delta_n}}A_{N_n,2c_n^k} \overset{(25),(26)}{\le} \frac{2A_{N_n,2c_n^k}}{A_{N_n,2c_n^{\mu_n}}} \le 2.$$

- If $j \in [c_n^{p-1}+a_n+\Delta_n, c_n^p - c_n^k)$ for some $k < p \le \mu_n$, then

$$\frac{\left\|T^{c_n^k}e_j\right\|_{N_n}}{\left\|e_j\right\|_{N_n+1}} \overset{(6)}{=} \frac{\alpha_{c_n^k+j}}{\alpha_j}\frac{A_{N_n,c_n^k+j}}{A_{N_n+1,j}} \overset{(21)}{\le} 2\frac{A_{N_n,2j}}{A_{N_n+1,j}} \overset{(18)}{\le} 1.$$

- If $j \in [c_n^p - c_n^k, c_n^p - c_n^k + a_n + \Delta_n)$ for some $k < p \le \mu_n$ with $p = \rho_n(l', w')$, then

$$\begin{aligned} \frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} &\overset{(6)}{=} \frac{1}{\alpha_j} \frac{\left\| T^{j+c_n^k} e_0 \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} \\ &\overset{(6)}{\le} \frac{1}{\alpha_j} \left(\frac{1}{C_n A_{N_n, c_n^p}} \frac{A_{N_n, j+c_n^k}}{A_{N_n+1,j}} + \frac{\left\| P_{l'}^{(n)}(T) S_{w'}^{(n)}(T) T^{j+c_n^k - c_n^p} e_0 \right\|_{N_n}}{A_{N_n+1,j}} \right) \\ &\overset{(21)}{\le} \frac{2}{\alpha_{c_n^p - 1}} \left(\frac{1}{C_n A_{N_n, c_n^p}} \frac{A_{N_n, 2j}}{A_{N_n+1,j}} + \frac{n C_n 2^{a_n + \Delta_n + n} 2^{a_n + \Delta_n} A_{N_n, 0}}{A_{N_n+1,j}} \right) \\ &\overset{(28),(18)}{\le} 3. \end{aligned}$$

- If $j \in [c_n^p - c_n^k + a_n + \Delta_n, c_n^p)$ for some $k < p \le \mu_n$, then

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} \overset{(6)}{=} \frac{\alpha_{j+c_n^k}}{\alpha_j} \frac{A_{N_n, j+c_n^k}}{A_{N_n+1,j}} \overset{(12),(21)}{\le} \frac{4\alpha_{c_n^p + a_n + \Delta_n}}{\alpha_{c_n^p - 1}} \frac{A_{N_n, 2j}}{A_{N_n+1,j}} \overset{(28),(25),(18)}{\le} 2.$$

- If $j \in [c_n^{\mu_n} + a_n + \Delta_n, d_n) = [\Delta_{n+1}, d_n)$, then by (8) we are sure that $j + c_n^k < a_{n+1}$, so we have that

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} \overset{(6)}{=} \frac{\alpha_{j+c_n^k}}{\alpha_j} \frac{A_{N_n, j+c_n^k}}{A_{N_n+1,j}} \overset{(12)}{\le} 2 \frac{A_{N_n, 2j}}{A_{N_n+1,j}} \overset{(18)}{\le} 1.$$

- If $j \in [d_n, \infty)$, then by Proposition 6:

$$\frac{\left\| T^{c_n^k} e_j \right\|_{N_n}}{\left\| e_j \right\|_{N_n+1}} \le 2^{c_n^{\mu_n}} \frac{A_{N_n, j}}{A_{N_n+1,j}} \overset{(7)}{\le} 1. \quad \square$$

8. Hypercyclicity

In this section we will show that for $T : s \to s$ defined by (6) every non-zero vector is hypercyclic. To do this we will need a fact about the projections $\tau_n : E_{a_n + \Delta_n - 1} \to E_{a_n + \Delta_n - 1}$ and the sets K_n defined in (13) and (14) respectively.

From Eq. (6) we get that if $j \le a_n$, then for suitable coefficients $e_j = \sum_{i=0}^{j} \lambda_i T^i e_0$, and if $j \in (a_n, a_n + \Delta_n)$, then

$$e_j = n 2^n A_{N_n, a_n} \left(T^j e_0 - T^{j - a_n} e_0 \right).$$

Therefore (13) implies that

$$\tau_n e_j = \begin{cases} e_j, & 0 \le j \le a_n; \\ -n 2^n A_{N_n, a_n} T^{j - a_n} e_0, & a_n < j \le a_n + \Delta_n - 1. \end{cases} \tag{31}$$

Proposition 9. *For any n the projection $\tau_n : E_{a_n + \Delta_n - 1} \to E_{a_n + \Delta_n - 1}$ satisfies*

$$\|\tau_n x\|_1 \le \|x\|_{N_n+1}. \tag{32}$$

Proof. We need to check (32) only for the basic vectors as the norms are weighted ℓ_1 norms. Because of the monotonicity of norms, in view of (31) we get the required inequality automatically for $j \leq a_n$ and we only need to check the other case. But for $j > a_n$ we get by Proposition 6:

$$\frac{\left\|\tau_n e_j\right\|_1}{\left\|e_j\right\|_{N_n+1}} = \frac{n2^n A_{N_n,a_n}\left\|T^{j-a_n}e_0\right\|_1}{A_{N_n+1,a_n}} \leq n2^{n+\Delta_n}A_{1,0}\frac{A_{N_n,a_n}}{A_{N_n+1,a_n}}$$
$$= n2^{n+\Delta_n}A_{1,0}\frac{\left(a_n+k_{N_n}\right)^{N_n}}{\left(a_n+k_{N_n+1}\right)^{N_n+1}} \leq \frac{n2^{n+\Delta_n}A_{1,0}}{a_n} \overset{(9)}{\leq} 1. \quad \square$$

Proposition 10. *Let $N \in \mathbb{N}_+$ and take a sequence (n_k) such that $N_{n_k} = N$. Take $x \in s$ such that $\|x\|_1 = 1$. Then for all but finitely many k*

$$\pi_{a_{n_k}+\Delta_{n_k}-1}x \in K_{n_k}.$$

Proof. In view of (14) we need only to show that $\left\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}x\right\|_1 \geq \frac{1}{2}$ holds for all but finitely many k, but with the help of (31) and (32) we have that:

$$\left\|\tau_{n_k}\pi_{a_{n_k}+\Delta_{n_k}-1}x\right\|_1 \geq \left\|\tau_{n_k}\pi_{a_{n_k}}x\right\|_1 - \left\|\tau_{n_k}\pi_{(a_{n_k},a_{n_k}+\Delta_{n_k}-1]}x\right\|_1$$
$$\geq \left\|\pi_{a_{n_k}}x\right\|_1 - \left\|\pi_{(a_{n_k},a_{n_k}+\Delta_{n_k}-1]}x\right\|_{N+1} \xrightarrow[k\to\infty]{} 1. \quad \square$$

Theorem 11. *Let $x \in s$ satisfy $\|x\|_1 = 1$. Then for any N and $z \in s$ there exist numbers n and k such that*

$$\left\|T^{c_n^k}x - z\right\|_N \leq 10.$$

Proof. By Corollary 7, e_0 is cyclic for T, so we can find a polynomial S such that

$$\|S(T)e_0 - z\|_N \leq 1.$$

Let n be any number such that

- $n \geq \deg S$;
- $n \geq |S|$;
- $N_n = N$;
- $y := \pi_{a_n+\Delta_n-1}x \in K_n$;
- $\|x-y\|_{N+1} = \|\pi_{[a_n+\Delta_n,\infty)}x\|_{N+1} \leq 1$.

This is possible by Proposition 10.

By definition of $\mathcal{S}^{(n)}$, there is a polynomial $S_w^{(n)} \in \mathcal{S}^{(n)}$ with $\deg S_w^{(n)} \leq n$ and $\left|S_w^{(n)}\right| \leq n$ such that

$$\left|S - S_w^{(n)}\right| \leq \frac{1}{n2^n A_{N,0}},$$

in particular $\|S(T)e_0 - S_w^{(n)}(T)e_0\|_N \leq 1$ by an argument as for (29). Now, by Corollary 5, there is a polynomial $P_l^{(n)} \in \mathcal{P}^{(n)}$ such that:

$$\left\|P_l^{(n)}(T)y - e_0\right\|_N \leq \frac{3}{n2^n}.$$

For suitable coefficients λ_j we have $y = \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^j e_0$. As $y \in K_n$, we have that $\sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \le C_n$.

Now, for $k = \rho_n(l, w)$ we get, using Lemma 8:

$$\begin{aligned}
\left\| T^{c_n^k} x - z \right\|_N &\le \left\| T^{c_n^k}(x-y) \right\|_N + \left\| T^{c_n^k} y - P_l^{(n)}(T) S_w^{(n)}(T) y \right\|_N \\
&\quad + \left\| S_w^{(n)}(T)(P_l^{(n)}(T)y - e_0) \right\|_N + \left\| S_w^{(n)}(T)e_0 - S(T)e_0 \right\|_N \\
&\quad + \| S(T)e_0 - z \|_N \\
&\le 3\|x-y\|_{N+1} \\
&\quad + \left\| \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j T^{c_n^k+j} e_0 - \sum_{j=0}^{a_n+\Delta_n-1} \lambda_j P_l^{(n)}(T) S_w^{(n)}(T) T^j e_0 \right\|_N \\
&\quad + n2^n \left\| P_l^{(n)}(T) y - e_0 \right\|_N + 2 \\
&\overset{(6)}{\le} \sum_{j=0}^{a_n+\Delta_n-1} |\lambda_j| \left\| \frac{1}{C_n A_{N_n, c_n^k}} e_{c_n^k + j} \right\|_N + 8 \\
&\le \frac{A_{N, c_n^k + a_n + \Delta_n}}{A_{N, c_n^k}} + 8 \overset{(19)}{\le} 10. \quad \square
\end{aligned}$$

Corollary 12. *Every non-zero vector $x \in s$ is hypercyclic for T.*

Proof. Obviously we can assume that $\|x\|_1 = 1$, so we get the conclusion from Theorem 11 because the norms were chosen so that their unit balls constitute a basis of neighbourhoods of zero in s. □

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