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ScienceDirect
Advances in Mathematics 221 (2009) 1047–1068
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Hurwitz–Hodge integrals, the E_6 and D_4 root systems, and the Crepant Resolution Conjecture

Jim Bryan *, Amin Gholampour

Dept. of Math., University of British Columbia, Vancouver, BC, Canada

Received 15 October 2007; accepted 3 February 2009

Available online 24 February 2009

Communicated by Ravi Vakil

Abstract

Let G be the group A_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$. We compute the integral of λ_g on the Hurwitz locus $\overline{H}_G \subset \overline{M}_g$ of curves admitting a degree 4 cover of $\mathbb{P}^1$ having monodromy group G. We compute the generating functions for these integrals and write them as a trigonometric expression summed over the positive roots of the E_6 and D_4 root systems, respectively. As an application, we prove the Crepant Resolution Conjecture for the orbifolds $[\mathbb{C}^3/A_4]$ and $[\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)]$.

Keywords: Gromov–Witten; Root systems; Hodge integrals; Crepant resolution conjecture; Orbifolds

1. Introduction

In his seminal 1983 paper [13], Mumford developed an enumerative geometry for the moduli space of curves analogous to Schubert calculus. On the Grassmannian, one can integrate the Chern classes of the tautological bundle over Schubert cycles, namely cycles given by the loci of linear spaces satisfying various incidence conditions. On the moduli space of stable curves, one can integrate Chern classes of the Hodge bundle over Hurwitz cycles, namely cycles defined by the loci of curves satisfying some Hurwitz conditions. Such integrals (and their variants) are called Hurwitz–Hodge integrals and they arise in various contexts, notably in orbifold Gromov–Witten theory, e.g. [5–8].

* Corresponding author.
E-mail addresses: jbryan@math.ubc.ca (J. Bryan), amin@math.ubc.ca (A. Gholampour).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.02.004

In this paper, we consider Hurwitz–Hodge integrals over a natural class of g-dimensional cycles on $\overline{M}_g$ which are defined as follows.

Let G be either A_4, the alternating group on 4 letters, or $\mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein four group. Let

$$H_G \subset M_g$$

be the locus of genus g curves C admitting a degree 4 map

$$f : C \to \mathbb{P}^1$$

whose monodromy group is contained in G.

The branch points $p_1, \ldots, p_n \in \mathbb{P}^1$ of f are then such that $f^{-1}(p_i)$ consists of exactly 2 points. By the Riemann–Hurwitz formula,

$$g = n - 3$$

and consequently H_G has dimension g.

In modern terms, H_G can be described as $M_{0,n}(BG)$, the moduli stack of twisted maps to the classifying stack BG. As such, there is a natural compactification $\overline{H}_G \subset \overline{M}_g$ given by twisted stable maps $\overline{M}_{0,n}(BG)$.

The G-Hurwitz space

$$\overline{H}_G = \overline{M}_{0,n}(BG)$$

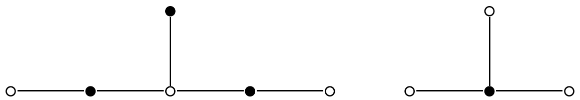
has components indexed by the monodromy around the n points. These are given by n-tuples of non-trivial conjugacy classes in G. Since each component has dimension g, we can evaluate the Hodge class $(-1)^g \lambda_g$ on each component to obtain a rational number. There are three non-trivial conjugacy classes in G, so the natural generating functions for these G-Hurwitz–Hodge integrals are formal power series $F_G(x_1, x_2, x_3)$ in three variables (defined in detail in Section 2). Our main result is an explicit formula for F_G written in terms of the E_6 and D_4 root systems for G equal to A_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$, respectively.

To write our expression for F_G, we will need to introduce some concepts which will relate conjugacy classes of G to the E_6 and D_4 root systems. Both A_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are naturally subgroups of $SO(3)$ (they are the symmetry groups of the tetrahedron and the prism over the 2-gon). Let $\widehat{G}$ be the binary version of G, that is the preimage of G in $SU(2)$ (namely the binary tetrahedral group $\widehat{A}_4$ and the quaternion 8 group Q).

$$\begin{tikzcd} \widehat{G} \arrow[r, hook] \arrow[d] & SU(2) \arrow[d] \\ G \arrow[r, hook] & SO(3) \end{tikzcd}$$

By the classical McKay correspondence [11,12,14], finite subgroups of $SU(2)$ admit an ADE classification where the non-trivial irreducible representations of a group naturally correspond to the nodes of the associated Dynkin diagram. In this classification, the binary tetrahedral group $\widehat{A}_4$ and the quaternion 8 group Q correspond to the E_6 and D_4 Dynkin diagrams respectively. The non-trivial irreducible representations of $\widehat{A}_4$ and Q that pullback from representations of A_4

and $\mathbb{Z}_2 \times \mathbb{Z}_2$ correspond to the white nodes in the Dynkin diagrams as follows:



ADE Dynkin diagrams also correspond to simply laced root systems where the nodes of the diagram correspond to simple roots of the root system. Let R be the E_6 or D_4 root system, let ρ be a non-trivial irreducible representation of G, and let e_ρ be the simple root which corresponds to the same node in the Dynkin diagram as ρ. For any positive root $\alpha \in R^+$, let α^ρ denote the coefficient of e_ρ in α.

We index non-trivial conjugacy classes of G by $i \in \{1, 2, 3\}$ and we let χ_ρ^i be the value of the character of a representation ρ on the ith conjugacy class. Let z_i be the order of the centralizer of the ith conjugacy class, and let V be the 3-dimensional representation of G arising from the embedding $G \subset SO(3)$. We define the following matrix which is a modification of the character table of G:

$$L_\rho^i = \frac{1}{z_i}\sqrt{3 - \chi_V^i}\,\chi_\rho^i.$$

Our main result is the following theorem.

Theorem 1.1. *Let G be A_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and let R be the E_6 or D_4 root system, respectively. The generating function for the G-Hurwitz–Hodge integrals is given by*

$$F_G(x_1, x_2, x_3) = \frac{1}{2}\sum_{\alpha \in R^+} \mathbf{h}\left(\pi + \sum_\rho \alpha^\rho \left(\frac{2\pi \dim \rho}{|G|} + \sum_i L_\rho^i x_i\right)\right)$$

where R^+ is the set of positive roots of R, the sum over ρ is over non-trivial irreducible representations of G, and $\mathbf{h}(u)$ is the series defined by

$$\mathbf{h}'''(u) = \frac{1}{2}\tan\left(\frac{-u}{2}\right).$$

The above formula is expanded out explicitly for $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Proposition 2.1 and for A_4 in Proposition 2.2. Note that since the constant, linear, and quadratic terms of the series $\mathbf{h}(u)$ are undefined, the same is true for $F_G(x_1, x_2, x_3)$. This corresponds to the fact that $\overline{M}_{0,n}(BG)$ is not defined for $n < 3$.

We prove in Proposition 7.1 that $F_G(x_1, x_2, x_3)$ is equal to the (non-classical part of the) genus zero Gromov–Witten potential for the orbifold $[\mathbb{C}^3/G]$. In [4] it is conjectured that for a crepant resolution $Y \to X$ of an orbifold $\mathcal{X}$ satisfying the Hard Lefschetz condition, the Gromov–Witten potentials of Y and $\mathcal{X}$ are related by a linear change of variables and a specialization of quantum parameters of Y to roots of unity.

The singular space $X = \mathbb{C}^3/G$ underlying the orbifold $\mathcal{X} = [\mathbb{C}^3/G]$ admits a preferred Calabi–Yau resolution

$$\pi : \text{G-Hilb}(\mathbb{C}^3) \to X$$

given by Nakamura's Hilbert scheme of G clusters [2]. In [3], we completely compute the Gromov–Witten theory of G-Hilb$(\mathbb{C}^3)$ for all finite subgroups $G \subset SO(3)$ and we use the Crepant Resolution Conjecture to obtain a predicition for the genus zero Gromov–Witten potential of the orbifold $\mathcal{X} = [\mathbb{C}^3/G]$. The change of variables matrix for the conjecture in this example is given by $\sqrt{-1}\,L^i_\rho$ and the roots of unity are given by

$$q_\rho = \exp\left(\frac{2\pi i \dim \rho}{|G|}\right).$$

Note that the matrix L^i_ρ, the roots of unity q_ρ, and the formula in Theorem 1.1 make sense for any finite subgroup $G \subset SO(3)$ (although the number of variables differs from three in general). Indeed, the conjectural formula for $F_{\mathcal{X}}$, the (non-classical part of the) Gromov–Witten potential of $\mathcal{X} = [\mathbb{C}^3/G]$ given in [3] is exactly the same as the formula for F_G in Theorem 1.1.

An immediate consequence of Proposition 7.1 and Theorem 1.1 is the proof of the conjecture in [3]. In particular, we have proven the following.

Theorem 1.2. *The genus zero Crepant Resolution Conjecture is true for the orbifold* $[\mathbb{C}^3/G]$ *with its crepant resolution given by* G-Hilb$(\mathbb{C}^3)$ *when* G *is* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *or* A_4.

The Gromov–Witten invariants for $\mathcal{X} = [\mathbb{C}^3/G]$ in general can also be described as integrals over G-Hurwitz loci in $\overline{M}_g$ but for other $G \subset SO(3)$, the integral of λ_g is replaced with slightly more exotic Hodge classes obtained from Chern classes of eigen-subbundles of the Hodge bundle.

2. Notation and results

Let G be a finite group, and $\overline{M}_{0,n}(BG)$ be the moduli space of genus 0, n-marked twisted stable maps to BG. The evaluation maps, denoted by ev_i for $i \in \{1, \dots, n\}$, take values in the inertia stack IBG. The coarse space of this stack is a finite collection of points, one for each conjugacy class in G.

Let $S = (c_1, \dots, c_n)$ be an n-tuple of conjugacy classes in G. We define the following open and closed substack of $\overline{M}_{0,n}(BG)$:

$$\overline{M}_S(BG) = \bigcap_{i=1}^{n} \mathrm{ev}_i^{-1}(c_i).$$

Concretely, $\overline{M}_S(BG)$ parametrizes G covers of an n marked genus zero curve with monodromy c_i around the ith marked point.

In this paper we will deal with the cases that the group G is either $\mathbb{Z}_2 \times \mathbb{Z}_2$, A_4, or S_4. We fix a notation for the conjugacy classes in these groups that we will use throughout the paper:

- Let $1, \tau, \sigma, \rho, \zeta$, denote the conjugacy classes in S_4 corresponding to the elements (1), $(1\,2)$, $(1\,2\,3)$, $(1\,2\,3\,4)$, $(1\,2)(3\,4)$, respectively.
- Let $1, \sigma_1, \sigma_2, \zeta$, denote the conjugacy classes in A_4 corresponding to the elements (1), $(1\,2\,3)$, $(1\,3\,2)$, $(1\,2)(3\,4)$, respectively.
- Let $1, \zeta_1, \zeta_2, \zeta_3$, denote the conjugacy classes in $\mathbb{Z}_2 \times \mathbb{Z}_2$ corresponding to its four elements.

All the above groups have a natural action on the set of four elements. Thus to any element

$$[f : C \to BG] \in \overline{M}_S(BG)$$

we can associate a degree 4 cover

$$\overline{C} \to C$$

with monodromy type c_i over the ith marked point. Let

$$\overline{\pi} : \overline{\mathcal{C}} \to \overline{M}_S(BG)$$

be the universal family of the four fold covers and let

$$\mathbb{E}^\vee = R^1\overline{\pi}_*(\mathcal{O}_{\overline{\mathcal{C}}})$$

be the dual Hodge bundle of this family.

We now define the *G-Hurwitz–Hodge integrals* as follows:

$$\langle c_1 \cdots c_n \rangle^G = \int_{[\overline{M}_S(BG)]} c\big(\mathbb{E}^\vee\big).$$

When G is A_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\overline{C}$ is connected, it is genus g and the integrand in the above definition is $(-1)^g \lambda_g$. The conjugacy classes c_i in $\langle c_1 \cdots c_n \rangle$ are called *insertions* and the total number of insertions will be called the *length*. We will drop the superscript G, when it is understood from context.

We define F_G, the generating functions for these integrals, by

$$F_{A_4}(x_1, x_2, x_3) = \sum_{n_1+n_2+n_3 \geqslant 3} \big\langle \sigma_1^{n_1} \sigma_2^{n_2} \zeta^{n_3} \big\rangle^{A_4} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \frac{x_3^{n_3}}{n_3!},$$

$$F_{\mathbb{Z}_2 \times \mathbb{Z}_2}(x_1, x_2, x_3) = \sum_{n_1+n_2+n_3 \geqslant 3} \big\langle \zeta_1^{n_1} \zeta_2^{n_2} \zeta_3^{n_3} \big\rangle^{\mathbb{Z}_2 \times \mathbb{Z}_2} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \frac{x_3^{n_3}}{n_3!},$$

$$F_{S_4}(x_1, x_2, x_3, x_4) = \sum_{n_1+n_2+n_3+n_4 \geqslant 3} \big\langle \tau^{n_1} \sigma^{n_2} \rho^{n_3} \zeta^{n_4} \big\rangle^{S_4} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \frac{x_3^{n_3}}{n_3!} \frac{x_4^{n_4}}{n_4!}.$$

Our use of F_{S_4} is auxiliary to our computations of F_{A_4} and $F_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ and we do not determine it completely.

For concreteness, we write out the formula in Theorem 1.1 explicitly for the two cases of A_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$. As before, we define the series $\mathbf{h}(u)$ by

$$\mathbf{h}'''(u) = \frac{1}{2} \tan\left(-\frac{u}{2}\right).$$

By a theorem of Faber and Pandharipande [10], $\mathbf{h}(u)$ is the generating series[1] for $\mathbb{Z}_2$-Hurwitz–Hodge integrals, namely the integral of $-\lambda_g\lambda_{g-1}$ over the hyperelliptic locus $\overline{H}_{\mathbb{Z}_2} \subset \overline{M}_g$.

Proposition 2.1. *The generating function for $\mathbb{Z}_2 \times \mathbb{Z}_2$-Hurwitz–Hodge integrals is given by the formula*

$$\begin{aligned} F_{\mathbb{Z}_2\times\mathbb{Z}_2} = {} & \mathbf{h}\left(\frac{1}{2}(x_1+x_2+x_3)-\frac{\pi}{2}\right)+\mathbf{h}\left(\frac{1}{2}(-x_1+x_2-x_3)-\frac{\pi}{2}\right) \\ & + \mathbf{h}\left(\frac{1}{2}(x_1-x_2-x_3)-\frac{\pi}{2}\right)+\mathbf{h}\left(\frac{1}{2}(-x_1-x_2+x_3)-\frac{\pi}{2}\right) \\ & + \frac{1}{2}\mathbf{h}(x_1)+\frac{1}{2}\mathbf{h}(x_2)+\frac{1}{2}\mathbf{h}(x_3). \end{aligned}$$

Proposition 2.2. *Let $\omega = e^{2\pi i/3}$. The generating function for A_4-Hurwitz–Hodge integrals is given by the formula*

$$\begin{aligned} F_{A_4} = {} & \mathbf{h}\left(\frac{1}{\sqrt{3}}(x_1+x_2)+\frac{x_3}{2}-\frac{5\pi}{6}\right)+ 2\mathbf{h}\left(\frac{1}{\sqrt{3}}(x_1+x_2)-\frac{\pi}{3}\right) \\ & + \mathbf{h}\left(\frac{1}{\sqrt{3}}(\omega x_1+\overline{\omega}x_2)+\frac{x_3}{2}-\frac{5\pi}{6}\right)+2\mathbf{h}\left(\frac{1}{\sqrt{3}}(\omega x_1+\overline{\omega}x_2)-\frac{\pi}{3}\right) \\ & + \mathbf{h}\left(\frac{1}{\sqrt{3}}(\overline{\omega}x_1+\omega x_2)+\frac{x_3}{2}-\frac{5\pi}{6}\right)+2\mathbf{h}\left(\frac{1}{\sqrt{3}}(\overline{\omega}x_1+\omega x_2)-\frac{\pi}{3}\right) \\ & + \mathbf{h}\left(\frac{1}{\sqrt{3}}(x_1+x_2)-\frac{x_3}{2}+\frac{\pi}{6}\right) \\ & + \mathbf{h}\left(\frac{1}{\sqrt{3}}(\omega x_1+\overline{\omega}x_2)-\frac{x_3}{2}+\frac{\pi}{6}\right) \\ & + \mathbf{h}\left(\frac{1}{\sqrt{3}}(\overline{\omega}x_1+\omega x_2)-\frac{x_3}{2}+\frac{\pi}{6}\right)+ 4\mathbf{h}\left(\frac{x_3}{2}+\frac{\pi}{2}\right)+\frac{1}{2}\mathbf{h}(x_3). \end{aligned}$$

2.1. Outline of the proof

We prove Theorem 1.1 by first computing $F_{\mathbb{Z}_2\times\mathbb{Z}_2}$ to prove Proposition 2.1 and then by computing F_{A_4} to prove Proposition 2.2. For each of $F_{\mathbb{Z}_2\times\mathbb{Z}_2}$ and F_{A_4} we first prove that F is uniquely determined by the WDVV equations along with certain specializations. We then prove that our formulas for F satisfy the WDVV equations and specialize correctly. The required specialization for $F_{\mathbb{Z}_2\times\mathbb{Z}_2}$ uses the Faber–Pandharipande computation of $F_{\mathbb{Z}_2}$. The required specializations for F_{A_4} uses the previously proven formula for $F_{\mathbb{Z}_2\times\mathbb{Z}_2}$ as well as certain generating series for S_4-Hurwitz–Hodge integrals. These are in turn determined by a WDVV argument in Section 6 using the $\mathbb{Z}_3$-Hurwitz–Hodge integrals computed in [5].

[1] Note that our series $\mathbf{h}(u)$ is equal to $-u^2H(u)$ where $H(u)$ is the series defined in Faber–Pandharipande [10].

3. The WDVV equations

The G-Hurwitz–Hodge integrals $\langle c_1 \cdots c_n \rangle^G$ satisfy the following version of the WDVV equations. These equations are the primary tools of this paper.

Theorem 3.1. *For any n-tuple $(c_1, \dots, c_n)$ of conjugacy classes of G and any subset $I \subset \{1, \dots, n\}$ of cardinality $|I|$, let c_I denote the corresponding $|I|$-tuple of conjugacy classes and let I^c be the complement of I. For $g \in G$ let (g) denote the corresponding conjugacy class and let $z(g)$ be the order of the centralizer of g. Then*

$$\big\langle c_1 \cdots c_n (a_1 a_2 | a_3 a_4)\big\rangle^G = \big\langle c_1 \cdots c_n (a_1 a_3 | a_2 a_4)\big\rangle^G$$

where $\langle c_1 \cdots c_n (a_i a_j | a_k a_l)\rangle^G$ is given by:

$$\sum_{(g) \subset G} \sum_{I \subset \{1,\dots,n\}} z(g) \big\langle c_I a_i a_j (g)\big\rangle^G \big\langle \big(g^{-1}\big) a_k a_l c_{I^c}\big\rangle^G.$$

From this theorem, one easily derives the PDE version of the WDVV equations:

Corollary 3.2. *Let F_G be the generating function for the G-Hurwitz–Hodge integrals. Let*

$$F_{ijk} = \frac{\partial^3 F_G}{\partial x_i \partial x_j \partial x_k}$$

and let

$$g_{ij} = \frac{1}{z_i} \delta_{i\bar{j}}, \qquad g^{ij} = z_i \delta_{i\bar{j}}$$

where z_i is the order of the centralizer of the ith conjugacy class and if (g) is the jth conjugacy class then (g^{-1}) is the $\bar{j}$th conjugacy class. Then the following expression is symmetric in $\{i, j, n, m\}$:

$$g_{ij} g_{nm} |G| + \sum_{k,l} F_{ijk} g^{kl} F_{lnm}.$$

The constant term in the above expression corresponds to terms containing an insertion of the trivial conjugacy class. These terms occur separately from the derivative terms since our variables only correspond to non-trivial conjugacy classes.

The proof of Theorem 3.1 is substantially no different than the proof of the WDVV equations in orbifold Gromov–Witten theory given in [1, Section 6.2]. The only difference is that we are integrating the total Chern class of the dual Hodge bundle and so we need to check that the Hodge bundle behaves well on the boundary. Indeed, the Hodge bundle restricted to the boundary component where two domain curves are glued along a marked point is equal in K-theory (up to a trivial factor) to the sum of the Hodge bundles of each factor.

Remark 3.3. *We may assume that the series F_G given in Theorem* 1.1 *satisfies the WDVV equations, before we actually prove that it is the generating function for the G-Hurwitz–Hodge integrals.* The formula in Theorem 1.1 for F_G was obtained from F_Y, the Gromov–Witten potential for the Calabi–Yau threefold $Y = \text{G-Hilb}(\mathbb{C}^3)$ by a linear change of variables and a specialization of the quantum parameters (see [3]). Since the change of variables transforms the Poincaré pairing on Y to the pairing g_{ij} defined above, it transforms the WDVV equations for F_Y into the WDVV equations for F_G. Thus the predicted formula for F_G automatically satisfies the WDVV equations. This is a feature common to all predictions obtained via the Crepant Resolution Conjecture.

4. Computing $F_{\mathbb{Z}_2\times\mathbb{Z}_2}$

In this section, we fix $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. For the integral $\langle c_1 \cdots c_n\rangle$ to be non-zero, we must have the monodromy condition satisfied: the product of the insertions must be trivial. This is equivalent to

$$\left\langle \zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}\right\rangle = 0 \quad \text{unless } n_1 \equiv n_2 \equiv n_3 \mod 2. \tag{1}$$

Consequently, the only non-trivial integrals of length three are

$$\langle \zeta_1\zeta_2\zeta_3\rangle = \langle 1\zeta_1\zeta_1\rangle = \langle 1\zeta_2\zeta_2\rangle = \langle 1\zeta_3\zeta_3\rangle = \frac{1}{4}.$$

We also note the integrals $\langle \zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}\rangle$ are symmetric under permutations of $(\zeta_1, \zeta_2, \zeta_3)$.

Lemma 4.1. *The integrals $\langle \zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}\rangle$ are uniquely determined by the length three integrals, the integrals $\langle \zeta_1^{n}\rangle$, and the WDVV equations.*

Proof. We proceed by induction on length (total number of insertions). The length three integrals start the induction and so we fix $n \geqslant 4$ and we assume that all integrals of length less than n are known. We introduce the notation

$$\mathsf{Length}(<n)$$

to stand for any combination of integrals of length less than n.

Fix (k_1, k_2, k_3) with

$$k_1 + k_2 + k_3 = n - 3$$

and consider the WDVV relation

$$\left\langle \zeta_1^{k_1}\zeta_2^{k_2}\zeta_3^{k_3}(\zeta_1\zeta_1|\zeta_2\zeta_2)\right\rangle = \left\langle \zeta_1^{k_1}\zeta_2^{k_2}\zeta_3^{k_3}(\zeta_1\zeta_2|\zeta_1\zeta_2)\right\rangle.$$

Expanding out each side into a sum of products of integrals and applying the monodromy condition (1), we find that there is only one non-zero term of length n. This yields:

$$\left\langle \zeta_1^{k_1+1}\zeta_2^{k_2+1}\zeta_3^{k_3+1}\right\rangle = \mathsf{Length}(<n).$$

Since the integrals $\langle \zeta_i^n \rangle$ are known by hypothesis, the only unknowns of length n are of the form

$$\langle \zeta_i^a \zeta_j^b \rangle \quad \text{where } i \neq j \text{ and } a+b=n.$$

Now consider the following WDVV relation with $k_1+k_2=n-3$

$$\langle \zeta_1^{k_1} \zeta_2^{k_2} (\zeta_1\zeta_1 | \zeta_2\zeta_3) \rangle = \langle \zeta_1^{k_1} \zeta_2^{k_2} (\zeta_1\zeta_2 | \zeta_1\zeta_3) \rangle.$$

Expanding out we obtain

$$\langle \zeta_1^{k_1+3} \zeta_2^{k_2} \rangle = \langle \zeta_1^{k_1+1} \zeta_2^{k_2+2} \rangle + \langle \zeta_1^{k_1+1} \zeta_2^{k_2} \zeta_3^2 \rangle + \mathsf{Length}(<n).$$

Solving for $\langle \zeta_1^{k_1+1} \zeta_2^{k_2+2} \rangle$, using the previous equation to write $\langle \zeta_1^{k_1+1} \zeta_2^{k_2} \zeta_3^2 \rangle$ in terms of $\mathsf{Length}(<n)$, and setting $k_1=a-1$ and $k_2=b-2$ we get

$$\langle \zeta_1^a \zeta_2^b \rangle = \langle \zeta_1^{a+2} \zeta_2^{b-2} \rangle + \mathsf{Length}(<n)$$

for any $a \geqslant 1$ and $b \geqslant 2$ with $a+b=n$. By the monodromy condition (1), we have that a and b must both be even, so we can use the above equation to inductively solve for all $\langle \zeta_1^a \zeta_2^b \rangle$ in terms of $\langle \zeta_1^n \rangle$ and integrals of length less than n and the lemma is proved. □

To prove Proposition 2.1, we now show that the series in Proposition 2.1 is the unique solution to the WDVV relations which has the correct cubic terms and the correct specialization $F_{\mathbb{Z}_2\times\mathbb{Z}_2}(x,0,0)$.

Up to symmetry, there are two distinct WDVV relations for the $\mathbb{Z}_2 \times \mathbb{Z}_2$-Hurwitz–Hodge integrals. In generating function form (Corollary 3.2), the relations are

$$F_{121}^2 + F_{122}^2 + F_{123}^2 = F_{111}F_{122} + F_{112}F_{222} + F_{113}F_{322} + \frac{1}{16}$$

and

$$F_{121}F_{133} + F_{122}F_{233} + F_{123}F_{333} = F_{131}F_{123} + F_{132}F_{223} + F_{133}F_{323}.$$

It is a straightforward but tedious exercise in trigonometry to prove that the series given in Proposition 2.1 satisfies the above WDVV equations (see Remark 3.3 for a conceptual proof).

To finish the proof of Proposition 2.1, it remains to check that the formula for F given in Proposition 2.1 has the correct specializations, namely that

$$F_{ijk}(0,0,0) = \langle \zeta_i \zeta_j \zeta_k \rangle$$

and

$$F(u,0,0) = \sum_{n\geqslant 3} \langle \zeta_1^n \rangle \frac{u^n}{n!}.$$

The first is easy to check and the second is equivalent to the following:

Lemma 4.2. *Let $F(x_1, x_2, x_3)$ be the series given in Proposition* 2.1. *Then*

$$F_{111}(u,0,0)=\sum_{n=3}^{\infty}\langle\zeta_1^n\rangle\frac{u^{n-3}}{(n-3)!}.$$

Proof. We compute $F_{111}(u,0,0)$ to get

$$\frac{1}{8}\left(\tan\left(-\frac{u}{4}+\frac{\pi}{4}\right)+\tan\left(-\frac{u}{4}-\frac{\pi}{4}\right)\right)+\frac{1}{4}\tan\left(-\frac{u}{2}\right)=\frac{1}{2}\tan\left(-\frac{u}{2}\right).$$

Since for $S=(\zeta_1^n)$ all the monodromies are equal, the universal cover $\overline{\mathcal{C}}\to\overline{M}_S(BG)$ is disconnected and is the union of two copies of the universal double cover over the hyperelliptic locus $\overline{H}_g$ where $2g+2=n$. Consequently, the dual Hodge bundle of $\overline{\mathcal{C}}$ is *two* copies of the dual Hodge bundle on the hyperelliptic locus. Denoting both by $\mathbb{E}^\vee$ we can then write:

$$\begin{aligned}
\sum_{n=3}^{\infty}\langle\zeta_1^n\rangle\frac{u^{n-3}}{(n-3)!}&=\sum_{n=3}^{\infty}\frac{u^{n-3}}{(n-3)!}\int_{[\overline{M}_S(B\mathbb{Z}_2\times\mathbb{Z}_2)]}c(\mathbb{E}^\vee)\\
&=\sum_{g=1}^{\infty}\frac{u^{2g-1}}{(2g-1)!}\frac{1}{2}\int_{[\overline{H}_g]}c(\mathbb{E}^\vee)c(\mathbb{E}^\vee)\\
&=\sum_{g=1}^{\infty}\frac{u^{2g-1}}{(2g-1)!}\int_{[\overline{H}_g]}-\lambda_g\lambda_{g-1}\\
&=\frac{1}{2}\tan\left(-\frac{u}{2}\right).
\end{aligned}$$

The factor of $1/2$ on the second line accounts for the automorphism exchanging the two copies of $\overline{H}_g$. The last equality follows from the computation of Faber–Pandharipande [10] and the lemma is proved. □

This completes the proof of Proposition 2.1.

5. Computing F_{A_4}

In this section, we compute the A_4-Hurwitz–Hodge integrals

$$\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^b\rangle^{A_4}$$

using the WDVV equations along with certain S_4-Hurwitz–Hodge integrals that will be computed in Section 6. The main technical result of this section is the following.

Proposition 5.1. *The generating function $F_{A_4}(x_1,x_2,x_3)$ for the A_4-Hurwitz–Hodge integrals is uniquely determined by the WDVV equations, the cubic coefficients of F_{A_4}, and the specializations $F_{A_4}(x,x,0)$ and $F_{A_4}(0,0,x)$.*

We will prove this proposition in Section 5.1. The specializations that appear in the above proposition can be expressed in terms of S_4-Hurwitz–Hodge integrals and $\mathbb{Z}_2 \times \mathbb{Z}_2$-Hurwitz–Hodge integrals by the following lemma.

Lemma 5.2. *The following equalities hold*:

$$3F_{A_4}(0,0,x) = F_{\mathbb{Z}_2\times\mathbb{Z}_2}(x,x,x),$$
$$2F_{S_4}(0,x,0,0) = F_{A_4}(x,x,0).$$

Proof. The integrals that appear as coefficients of the specialization $F_{A_4}(0,0,x)$ are $\langle\zeta^n\rangle^{A_4}$. These correspond to A_4 covers whose monodromy around every branched point is in ζ, the conjugacy class of disjoint pairs of two cycles. The structure group of such a cover reduces to $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset A_4$ and so the integral is given as a sum of $\mathbb{Z}_2 \times \mathbb{Z}_2$ integrals as follows:

$$3\langle\zeta^n\rangle^{A_4} = \sum_{n_1+n_2+n_3=n} \binom{n}{n_1, n_2, n_3} \langle\zeta_1^{n_1}\zeta_2^{n_2}\zeta_3^{n_3}\rangle^{\mathbb{Z}_2\times\mathbb{Z}_2}.$$

The multinomial coefficient takes into account all the possible choices of the distribution of the monodromy among the n marked points. The factor of 3 occurs because the degree of the map

$$\overline{M}_{0,n}(B\mathbb{Z}_2 \times \mathbb{Z}_2) \to \overline{M}_{0,n}(BA_4)$$

is 3. By a similar argument, we derive

$$2\langle\sigma^n\rangle^{S_4} = \sum_{n_1+n_2=n} \binom{n}{n_1, n_2} \langle\sigma_1^{n_1}\sigma_2^{n_2}\rangle^{A_4}.$$

The lemma follows easily. $\square$

Corollary 5.3. *The validity of Proposition* 2.2*, which gives our formula for F_{A_4}, follows from Propositions* 5.1*,* 2.1*, and* 6.1.

Proof. By Proposition 5.1, we only need to show that the explicit formula for $F_{A_4}(x_1,x_2,x_3)$ given in Proposition 2.2:

(1) satisfies the WDVV equations,
(2) has the correct cubic terms, and
(3) has the correct specializations $F_{A_4}(x,x,0)$ and $F_{A_4}(0,0,x)$.

The fact that the predicted formula satisfies the WDVV equations is once again a tedious but straightforward exercise in trigonometry, or for a more conceptual proof see Remark 3.3.

The cubic terms correspond to the three point A_4-Hurwitz–Hodge integrals. These are simply counts of A_4 covers which can be evaluated using group theory and TQFT methods [9, Section 4]. The non-zero values are given by,

$$\langle\sigma_1\sigma_2\zeta\rangle = 1, \qquad \langle\sigma_1^3\rangle = \langle\sigma_2^3\rangle = \frac{4}{3}, \qquad \langle\zeta^3\rangle = \frac{1}{2}. \tag{2}$$

It is easy to check that cubic terms of the predicted formula for F_{A_4} agrees with the above values.

Finally, in light of Lemma 5.2, we must check that when we specialize the predicted formula for $F_{A_4}(x_1, x_2, x_3)$ to $F_{A_4}(0,0,x)$ and $F_{A_4}(x,x,0)$ we get $\frac{1}{3}F_{\mathbb{Z}_2\times\mathbb{Z}_2}(x,x,x)$ and $\frac{1}{2}F_{S_4}(0,x,0,0)$ which are determined by Proposition 2.1 and Proposition 6.1, respectively. With the use of the following trigonometric identities:

$$\frac{1}{9}\mathbf{h}(3u) = \mathbf{h}(u) + \mathbf{h}\left(u + \frac{2\pi}{3}\right) + \mathbf{h}\left(u - \frac{2\pi}{3}\right),$$
$$\frac{1}{4}\mathbf{h}(2u) = \mathbf{h}\left(u + \frac{\pi}{2}\right) + \mathbf{h}\left(u - \frac{\pi}{2}\right), \tag{3}$$

this is a straightforward check. □

5.1. The WDVV relations for A_4-Hurwitz–Hodge integrals

In this subsection we give the proof of Proposition 5.1. As before, we use the notation

$$\mathsf{Length}(< n)$$

to denote any combination of integrals of length less than n.

We will use induction on the length to prove that the integrals are determined by the WDVV relations from the length three integrals (which start the induction) and the integrals (or combinations of integrals) which occur as the coefficients of the specializations $F_{A_4}(0,0,x)$ and $F_{A_4}(x,x,0)$.

The WDVV relations we need are given in the following:

Lemma 5.4. *Let $n = a_1 + a_2 + b + 3$. We have the following relations among the A_4-Hurwitz–Hodge integrals*:

i) $4\langle\sigma_1^{a_1+2}\sigma_2^{a_2}\zeta^{b+1}\rangle = 4\langle\sigma_1^{a_1}\sigma_2^{a_2+1}\zeta^{b+2}\rangle + \mathsf{Length}(<n)$,
ii) $4\langle\sigma_1^{a_1}\sigma_2^{a_2+2}\zeta^{b+1}\rangle = 4\langle\sigma_1^{a_1+1}\sigma_2^{a_2}\zeta^{b+2}\rangle + \mathsf{Length}(<n)$,
iii) $4\langle\sigma_1^{a_1+1}\sigma_2^{a_2+1}\zeta^{b+1}\rangle = 4\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b+3}\rangle + \mathsf{Length}(<n)$,
iv) $4\langle\sigma_1^{a_1+3}\sigma_2^{a_2}\zeta^{b}\rangle + 4\langle\sigma_1^{a_1}\sigma_2^{a_2+3}\zeta^{b}\rangle = 8\langle\sigma_1^{a_1+1}\sigma_2^{a_2+1}\zeta^{b+1}\rangle + \mathsf{Length}(<n)$.

Proof. We prove the above relations using the following WDVV relations which are expressed using the notation of Theorem 3.1.

i) $\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_1\zeta|\sigma_1\zeta)\rangle = \langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_1\sigma_1|\zeta\zeta)\rangle$,
ii) $\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_2\zeta|\sigma_2\zeta)\rangle = \langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_2\sigma_2|\zeta\zeta)\rangle$,
iii) $\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_1\zeta|\sigma_2\zeta)\rangle = \langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_1\sigma_2|\zeta\zeta)\rangle$,
iv) $\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_1\sigma_1|\sigma_2\sigma_2)\rangle = \langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^{b}(\sigma_1\sigma_2|\sigma_1\sigma_2)\rangle$.

After expanding i) for $a_1 + a_2 + b > 0$, the resulting equation is

$$\sum\binom{a_1}{a_1'}\binom{a_2}{a_2'}\binom{b}{b'}\bigl\{4\bigl\langle\sigma_1^{a_1'+1}\sigma_2^{a_2'}\zeta^{b'+2}\bigr\rangle\bigl\langle\sigma_1^{a_1''+1}\sigma_2^{a_2''}\zeta^{b''+2}\bigr\rangle$$

$$
\begin{aligned}
&+3\langle\sigma_1^{a_1'+2}\sigma_2^{a_2'}\zeta^{b'+1}\rangle\langle\sigma_1^{a_1''+1}\sigma_2^{a_2''+1}\zeta^{b''+1}\rangle\\
&+3\langle\sigma_1^{a_1'+1}\sigma_2^{a_2'+1}\zeta^{b'+1}\rangle\langle\sigma_1^{a_1''+2}\sigma_2^{a_2''}\zeta^{b''+1}\rangle\}\\
=&\sum\binom{a_1}{a_1'}\binom{a_2}{a_2'}\binom{b}{b'}\{4\langle\sigma_1^{a_1'+2}\sigma_2^{a_2'}\zeta^{b'+1}\rangle\langle\sigma_1^{a_1''}\sigma_2^{a_2''}\zeta^{b''+3}\rangle\\
&+3\langle\sigma_1^{a_1'+3}\sigma_2^{a_2'}\zeta^{b'}\rangle\langle\sigma_1^{a_1''}\sigma_2^{a_2''+1}\zeta^{b''+2}\rangle\\
&+3\langle\sigma_1^{a_1'+2}\sigma_2^{a_2'+1}\zeta^{b'}\rangle\langle\sigma_1^{a_1''+1}\sigma_2^{a_2''}\zeta^{b''+2}\rangle\}
\end{aligned}
$$

where the sums are over $a_1'+a_1''=a_1$, $a_2'+a_2''=a_2$, and $b'+b''=b$.

All the integrals of length n in the above expression are multiplied by a integrals of length 3 which are given in Eq. (2). All other terms contain only terms of length less than n. Substituting the values of the length three integrals, we obtain the relation i). The proof of relations ii), iii), and iv) is similar. $\square$

Remark 5.5. One can see easily by monodromy considerations that

$$\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^b\rangle$$

is non-zero only when $a_1\equiv a_2 \pmod 3$. Note also that the above integral is symmetric in a_1 and a_2 due to the fact that A_4 has a non-trivial outer automorphism which exchanges σ_1 and σ_2.

We will now use the relations i)–iv) in Lemma 5.4 to show that all the A_4 integrals can be inductively recovered from the length three integrals, the integrals $\langle\zeta^n\rangle^{A_4}$ (which are the coefficients of $F_{A_4}(0,0,x)$), and the integrals $\langle\sigma^n\rangle^{S_4}$ (which by virtue of Lemma 5.2 are the coefficients of $F_{A_4}(x,x,0)$).

The following relations are direct consequences of relations i) and iii) in Lemma 5.4:

$$
\begin{aligned}
\langle\sigma_1^k\zeta^b\rangle&=\langle\sigma_1^{k-2}\sigma_2\zeta^{b+1}\rangle+\mathsf{Length}(<k+b),\quad b>0,\ k>2,\\
\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^b\rangle&=\langle\sigma_1^{a_1-1}\sigma_2^{a_2-1}\zeta^{b+2}\rangle+\mathsf{Length}(<a_1+a_2+b),\quad a_1,a_2,b>0.
\end{aligned}
$$

The following lemma is proven readily by a direct repeated application of two relations above:

Lemma 5.6. *Assume that $b>0$. We have the following relation*:

$$\langle\sigma_1^{a_1}\sigma_2^{a_2}\zeta^b\rangle=\langle\zeta^{a_1+a_2+b}\rangle+\mathsf{Length}(<a_1+a_2+b).$$

In the next lemma we deal with the case $b=0$:

Lemma 5.7. *Let $n=a_1+a_2$. All the integrals of the form $\langle\sigma_1^{a_1}\sigma_2^{a_2}\rangle$ can be written in terms of $\langle\sigma^n\rangle^{S_4}$, length n integrals having at least one ζ insertion, and* $\mathsf{Length}(<n)$.

Proof. Let $0\leqslant k\leqslant 2$ be so that $n\equiv -k \pmod 3$. One can see easily that (see Remark 5.5):

$$\{\langle\sigma_1^{a_1}\sigma_2^{a_2}\rangle \mid a_1+a_2=n\}=\{\langle\sigma_1^{n-k}\sigma_2^k\rangle,\langle\sigma_1^{n-k-3}\sigma_2^{k+3}\rangle,\dots,\langle\sigma_1^k\sigma_2^{n-k}\rangle\}.$$

This is a set of $l+1$ elements, where

$$l=(n-2k)/3.$$

For simplicity we write the elements of this set in the same order form left to right by $x_0, x_2, \dots, x_l$. Note that by symmetry of the integrals (see Remark 5.5) we have

$$x_i = x_{l-i}. \tag{4}$$

Applying relation iv) in Lemma 5.4 l times, we get the following set of relations:

$$\begin{aligned} x_0+x_1&=2\big\langle\sigma_1^{n-k-2}\sigma_2^{k+1}\zeta\big\rangle+\mathsf{Length}(<n),\\ x_1+x_2&=2\big\langle\sigma_1^{n-k-5}\sigma_2^{k+4}\zeta\big\rangle+\mathsf{Length}(<n),\\ &\vdots\\ x_{l-1}+x_l&=2\big\langle\sigma_1^{k+1}\sigma_2^{n-k-2}\zeta\big\rangle+\mathsf{Length}(<n). \end{aligned} \tag{5}$$

Now we consider two cases:

i) ***n is odd.*** We see that l (the number of relations in (5)) is odd as well, and then (4) and (5) boil down into the following set of independent equations among x_i's:

$$\begin{aligned} x_0+x_1&=2\big\langle\sigma_1^{n-k-2}\sigma_2^{k+1}\zeta\big\rangle+\mathsf{Length}(<n),\\ x_1+x_2&=2\big\langle\sigma_1^{n-k-5}\sigma_2^{k+4}\zeta\big\rangle+\mathsf{Length}(<n),\\ &\vdots\\ x_p+x_p&=2\big\langle\sigma_1^{n-k-2-3p}\sigma_2^{k+1+3p}\zeta\big\rangle+\mathsf{Length}(<n), \end{aligned}$$

where $p=(l+1)/2$. From these equations we get

$$\begin{aligned} x_{l-i}=x_i&=-(-1)^{p-i}\big\langle\sigma_1^{n-k-2-3p}\sigma_2^{k+1+3p}\zeta\big\rangle\\ &\quad+2\sum_{j=i}^{p}(-1)^{j-i}\big\langle\sigma_1^{n-k-2-3j}\sigma_2^{k+1+3j}\zeta\big\rangle+\mathsf{Length}(<n), \end{aligned}$$

and the lemma is proven in this case.

ii) ***n is even.*** In this case l is even. One can see that one of the relations in (4) is redundant. However, since the coefficients of $F_{A_4}(x,x,0)=2F_{S_4}(0,x,0,0)$ are known by hypothesis, we get the following extra relation:

$$\binom{n}{n-k}x_0+\binom{n}{n-k-3}x_1+\cdots+\binom{n}{k}x_l=2\big\langle\sigma^n\big\rangle^{S_4}. \tag{6}$$

Adding (6) to (5), we get a system of $l+1$ equations and $l+1$ unknowns (we call x_i's unknowns). One can see that the determinant of the matrix of coefficients is given by (note that n is even)

$$\binom{n}{n-k} - \binom{n}{n-k-3} + \cdots + \binom{n}{k},$$

which is non-zero by [5, Lemma A.6]. This means that we can express x_i's in terms of the right-hand side of the system of $l+1$ equations, and the lemma is proven in this case. $\square$

We may now prove Proposition 5.1 in the following equivalent form:

Proposition 5.8. *The A_4-Hurwitz–Hodge integrals are uniquely determined by the WDVV equations, the length three integrals, and the integrals $\langle \zeta^m \rangle^{A_4}$ and $\langle \sigma^m \rangle^{S_4}$.*

Proof. We use induction on the length n of the integrals. The length three integrals are known by hypothesis.

Let $\langle \sigma_1^{a_1} \sigma_2^{a_2} \zeta^b \rangle^{A_4}$ be an arbitrary integral of length $n > 3$. By Lemmas 5.6 and 5.7, we can write this integral in terms of $\langle \zeta^n \rangle^{A_4}$, $\langle \sigma^n \rangle^{S_4}$, and $\mathsf{Length}(< n)$. Both $\langle \zeta^n \rangle^{A_4}$ and $\langle \sigma^n \rangle^{S_4}$ are known by the assumption, and $\mathsf{Length}(< n)$ is also known by the induction hypothesis. Therefore $\langle \sigma_1^{a_1} \sigma_2^{a_2} \zeta^b \rangle^{A_4}$ is determined, and the proposition is proven. $\square$

6. Computing S_4-Hurwitz–Hodge integrals

In this section we prove the following proposition, which is needed to complete the proof of Proposition 2.2.

Proposition 6.1. *Let $F_{S_4}(x_1, x_2, x_3, x_4)$ be the generating function for the S_4-Hurwitz–Hodge integrals. Then*

$$F_{S_4}(0, x, 0, 0) = \frac{1}{8}\mathbf{h}\left(\frac{2x}{\sqrt{3}} - \frac{2\pi}{3}\right) + 2\mathbf{h}\left(\frac{x}{\sqrt{3}} - \frac{\pi}{3}\right).$$

This follows immediately from the identity in Eqs. (3) and the following:

Theorem 6.2.

$$F_{S_4}(0, u, 0, v) = \frac{1}{2}K(2u, v) + K(-u, v)$$

where

$$\begin{aligned} K(u, v) = {} & \mathbf{h}\left(\frac{u}{\sqrt{3}} + \frac{v}{2} - \frac{5\pi}{6}\right) + 2\mathbf{h}\left(\frac{u}{\sqrt{3}} - \frac{\pi}{3}\right) \\ & + \mathbf{h}\left(\frac{u}{\sqrt{3}} - \frac{v}{2} + \frac{\pi}{6}\right) + 2\mathbf{h}\left(\frac{u}{\sqrt{2}} + \frac{\pi}{2}\right) \\ & + \frac{2}{3}\mathbf{h}\left(\frac{v}{2} + \frac{3\pi}{2}\right). \end{aligned}$$

To prove Theorem 6.2, the basic strategy is once again to use WDVV to inductively determine the integrals. We will not determine all the S_4-Hurwitz–Hodge integrals but we will have to determine a certain set of integrals that have a small number of τ and ρ insertions.

We use the following generating functions. For convenience, we define unstable integrals (those have fewer than three insertions) to be zero.

- $T(u,v) = F_{S_4}(0,u,0,v) = \sum_{a,b} = 0^{\infty} \langle \sigma^a \zeta^b \rangle \frac{u^a}{a!} \frac{v^b}{b!}$,
- $X_a(u) = \sum_{n=0}^{\infty} \langle \sigma^a \zeta^n \rangle \frac{u^n}{n!}$,
- $Y_b(u) = \sum_{n=0}^{\infty} \langle \sigma^n \zeta^b \rangle \frac{u^{n+b-3}}{(n+b-3)!}$,
- $B(u) = \sum_{n=0}^{\infty} \langle \tau^2 \sigma^n \rangle \frac{u^{n-1}}{(n-1)!}$,
- $C(u) = \sum_{n=0}^{\infty} \langle \tau \sigma^n \rho \zeta \rangle \frac{u^n}{n!}$,
- $D(u) = \sum_{n=0}^{\infty} \langle \tau \sigma^n \rho \rangle \frac{u^{n-1}}{(n-1)!}$.

The length three integrals can be evaluated using group theory and TQFT methods [9, Section 4]. The non-zero values are given by:

$$\langle \sigma^2 \zeta \rangle = \langle \tau^2 \sigma \rangle = \langle \tau \rho \sigma \rangle = \langle \rho^2 \sigma \rangle = 1,$$
$$\langle \zeta^3 \rangle = \langle \rho^2 \zeta \rangle = \langle \tau^2 \zeta \rangle = \langle \tau^2 1 \rangle = \langle \rho^2 1 \rangle = \frac{1}{4},$$
$$\langle \sigma^3 \rangle = \frac{4}{3}, \qquad \langle \tau \rho \zeta \rangle = \frac{1}{2}, \qquad \langle \sigma^2 1 \rangle = \frac{1}{3}, \qquad \langle \zeta^2 1 \rangle = \frac{1}{8}.$$

We now determine the series X_0, X_1, and B.

Lemma 6.3. *The series $X_0(u)$, $X_1(u)$, and $B(u)$ are given by*

$$\begin{aligned} X_0(u) &= \frac{1}{6}\mathbf{h}\left(\frac{3u}{2} - \frac{\pi}{2}\right) + \frac{1}{2}\mathbf{h}\left(\frac{u}{2} + \frac{\pi}{2}\right) + \frac{1}{4}\mathbf{h}(u), \\ X_1(u) &= 0, \\ B(u) &= \frac{1}{\sqrt{3}} \tan\left(\frac{-u}{\sqrt{12}} + \frac{\pi}{3}\right). \end{aligned}$$

Proof. The proof of the first formula follows immediately from Proposition 2.1 and the formula

$$X_0(u) = \frac{1}{6} F_{\mathbb{Z}_2 \times \mathbb{Z}_2}(u,u,u).$$

The proof of the above formula is almost identical to the proof of the first formula in Lemma 5.2; the only difference being that the 3 (the index of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in A_4) is replaced with 6 (the index of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in S_4).

The lemma's second formula is a consequence of

$$\langle \sigma \zeta^n \rangle = 0$$

which follows from monodromy considerations (see Remark 5.5).

To prove the lemma's third formula, we need to show that the generating function $B(u)$ has the same coefficients, up to an alternating sign, as the generating function with the same name computed in [5, Proposition A.3]. The coefficients of the generating function in [5] are the λ_g integrals over the Hurwitz locus of curves admitting a *degree three* cover of $\mathbb{P}^1$ with 2 ordinary ramification points and $g+1$ double ramifications. The identification of this integral with the integral $(-1)^g \langle \tau^2 \sigma^{g+1} \rangle^{S_4}$ is because the only chance for a degree four cover to contribute to $\langle \tau^2 \sigma^{g+1} \rangle^{S_4}$ is if the cover is *disconnected*. Indeed, if it is not disconnected, then the genus is $g-1$ and hence the gth Chern class of $\mathbb{E}^\vee$ is zero. The sign is because we work with the dual Hodge bundle instead of the Hodge bundle. □

To determine $T(u,v)$, it clearly suffices to determine $X_a(u)$ for all a. We will do this using an induction on a. The following lemma provides the basic relations that are needed in the induction.

Lemma 6.4. *The series $T(u,v)$, $X_a(u)$, $Y_0(u)$, $Y_1(u)$, $Y_2(u)$, and $Y_3(u)$ satisfy the following relations.*

(i) $3T_{uuv}^2 + 8T_{uvv}^2 - 3T_{uuu}T_{uvv} - 8T_{uuv}T_{vvv} + 1 = 0$,
(ii) $3(X_2')^2 - 8X_2'X_0''' - 1 = 0$,
(iii) $(6Y_2B + 4Y_3 + 1)(3Y_0B + 2Y_1 - 4B^2 + 2) = 2(3Y_1B + 2Y_2 - B)^2$,
(iv) $(6X_2' - 8X_0''')X_{a+2}' - 3aX_2''X_{a+2} = G(X_0, X_1, \ldots, X_{a+1})$,

where in item (iv) $a > 0$, *and $G(X_0, X_1, \ldots, X_{a+1})$ is a function of the series $X_k(u)$ for $0 \leqslant k \leqslant a+1$.*

Proof. These relations are all consequences of the WDVV equations. For fixed a and b, consider the WDVV relation

$$\langle \sigma^a \zeta^b (\sigma\zeta | \sigma\zeta) \rangle = \langle \sigma^a \zeta^b (\sigma\sigma | \zeta\zeta) \rangle.$$

It is given by

$$\begin{aligned} 0 = \delta_{0,a}\delta_{0,b} + \sum_{\substack{a_1+a_2=a\\ b_1+b_2=b}} \binom{a}{a_1}\binom{b}{b_1} \\ \times \big\{ 3\langle \sigma^{a_1+2}\zeta^{b_1+1}\rangle\langle \sigma^{a_2+2}\zeta^{b_2+1}\rangle + 8\langle \sigma^{a_1+1}\zeta^{b_1+2}\rangle\langle \alpha^{a_2+2}\zeta^{b_2+2}\rangle \\ - 3\langle \sigma^{a_1+3}\zeta^{b_1}\rangle\langle \alpha^{a_2+1}\zeta^{b_2+2}\rangle - 8\langle \sigma^{a_1+2}\zeta^{b_1+1}\rangle\langle \sigma^{a_2}\zeta^{b_2+3}\rangle \big\}. \end{aligned}$$

Multiplying the above equation by $\frac{u^a}{a!}\frac{u^b}{b!}$ and then summing over all of a and b yields the PDE in (i). Fixing $a=0$, multiplying by $\frac{u^b}{b!}$, and summing over b yields (ii). To prove (iv), we fix $a>0$, we multiply the above equation by $\frac{u^b}{b!}$, we move to the right-hand side of the equation all the terms containing only integrals with fewer than $a+2$ insertions of σ, and then we sum over b.

The relation (iii) follows easily from the equations:

$$\begin{aligned}6Y_2B+4Y_3+1&=8C^2,\\ 3Y_1B+2Y_2&=4CD+B,\\ 3Y_0B+2Y_1+2&=4B^2+4D^2\end{aligned}$$

which are derived in a fashion similar to the above from the WDVV relations:

$$\begin{aligned}\langle\sigma^a(\tau\tau|\zeta\zeta)\rangle&=\langle\sigma^a(\tau\zeta|\tau\zeta)\rangle,\\ \langle\sigma^a(\tau\tau|\sigma\zeta)\rangle&=\langle\sigma^a(\tau\sigma|\tau\zeta)\rangle,\\ \langle\sigma^a(\tau\tau|\sigma\sigma)\rangle&=\langle\sigma^a(\tau\sigma|\tau\sigma)\rangle.\end{aligned}$$

In the derivation, we use the following crucial fact:

$$\langle\tau^2\zeta\sigma^a\rangle=\begin{cases}0 & \text{if } a>0,\\ \frac{1}{4} & \text{if } a=0.\end{cases}$$

This follows from the fact that a four fold cover of $\mathbb{P}^1$ with 3 branched points of monodromies τ, τ, ζ, and a branched points of monodromy σ is connected and of genus $a-1$. Consequently $c_a(\mathbb{E}^\vee)=0$, namely the ath Chern class of the dual Hodge bundle vanishes, and so the corresponding Hurwitz–Hodge integral is zero. □

Proposition 6.5. *The functions $X_a(u)$ are uniquely determined for all a by the series $B(u)$, $X_0(u)$, $X_1(u)$, the length three integrals, and the relations* (ii), (iii), *and* (iv) *in Lemma* 6.4.

Proof. The relation (ii) of Lemma 6.4 is a quadratic equation for $X_2'(u)$ whose solution is fixed by the condition

$$X_2'(0)=\langle\sigma^2\zeta\rangle=1.$$

Since the constant term of $X_2(u)$ is an unstable integral (and hence zero by convention), $X_2(u)$ is then uniquely determined.

We now proceed to determine $X_a(u)$ by induction on a. We assume that $X_k(u)$ is known for all $k<a+2$ and we need to show that we can determine $X_{a+2}(u)$. Since X_0, X_1, and X_2 are known, we may assume that $a>0$. Then relation (iv) in Lemma 6.4 is a first order ODE for $X_{a+2}(u)$. Since the coefficient of X'_{a+2} in the ODE is an invertible series, the ODE has a solution which is uniquely determined by specifying $X_{a+2}(0)$.

Now $X_{a+2}(0)$ is equal to the coefficient of u^{a-1} in the series $Y_0(u)$, and so we need to determine this coefficient. Since the series $X_0,\dots,X_{a+1}$ are known by the induction hypothesis, we know the coefficients of $Y_b(u)$ up to the u^{a+b-2} term. By examining the u^{a-1} term in the relation (iii) from Lemma 6.4, we find that the only unknown is the coefficient of u^{a-1} of Y_0 which appears exactly once with a non-zero coefficient. Hence we can uniquely solve for this coefficient which provides the initial condition which uniquely determines $X_{a+2}(u)$. $\square$

Thus to complete the proof of Theorem 6.2, we must show that the formula for $T(u,v)$ in the theorem, yields series $X_a(u)$

(1) which predict the correct length three integrals,
(2) which agree with the formulas for X_0 and X_1 given in Lemma 6.3,
(3) and which are solutions to the relations (ii), (iii), and (iv) of Lemma 6.4.

The first two are straightforward checks. The compatibility of the solution with relation (iii) is also a straightforward check. The compatibility with relations (ii) and (iv) is equivalent to the formula for T satisfying the PDE (i). This compatibility can be checked directly (with Maple, for example), but there is a more conceptual proof, along the lines of Remark 3.3 which we outline below.

The formula for $T(u,v)$ can be derived from the formula for $F_{A_4}(x_1,x_2,x_3)$ via the relation

$$T(u,v)=F_{S_4}(0,u,0,v)=\frac{1}{2}F_{A_4}(u,u,v)$$

which is proved by the same method as the proof of Lemma 5.2. The fact that $T(u,v)$ satisfies (i) can be seen to be a consequence of the fact that F_{A_4} satisfies the A_4 WDVV equations, a fact which we proved in Section 5. Recall that our formula for F_{A_4} was derived via the Crepant Resolution Conjecture from the Gromov–Witten potential of the crepant resolution $Y\to\mathbb{C}^3/A_4$ given by the A_4 Hilbert scheme. The derived formula for F_{A_4} thus automatically satisfies the WDVV equations since the Gromov–Witten potential of Y satisfies the WDVV equations and the Crepant Resolution Conjecture is compatible with the WDVV equations. Thus the fact that T satisfies relations (i) in Lemma 6.4 is ultimately a consequence of the fact that the Gromov–Witten potential of Y (which was computed in [3]) satisfies the WDVV equations. $\square$

This completes the proof of Theorem 6.2 and consequently it completes the proof of Proposition 2.2 and hence it completes the proof of our main result, Theorem 1.1. $\square$

7. The relationship with orbifold Gromov–Witten theory

Let G (which is A_4 or $\mathbb{Z}_2\times\mathbb{Z}_2$) act on $\mathbb{C}^3$ by the representation obtained from the embedding $G\subset SO(3)\subset SU(3)$. Let $\mathcal{X}$ be the orbifold given by the quotient:

$$\mathcal{X}=[\mathbb{C}^3/G].$$

The orbifold cohomology $H^*_{\mathrm{orb}}(\mathcal{X})$ has a canonical basis labelled by conjugacy classes of G. Consequently, the insertions for the orbifold Gromov–Witten invariants of $\mathcal{X}$ are conjugacy classes, and the genus zero Gromov–Witten invariants of $\mathcal{X}$ take the form $\langle c_1\cdots c_n\rangle^{\mathcal{X}}$.

In this section we prove:

Proposition 7.1. *The genus zero orbifold Gromov–Witten invariants*[2] *of $\mathcal{X}$ are given by the G-Hurwitz–Hodge integrals, namely*

$$\langle c_1 \cdots c_n \rangle^{\mathcal{X}} = \langle c_1 \cdots c_n \rangle^{G}$$

for any n-tuple of non-trivial conjugacy classes. Consequently, the (*non-classical part of the*) *genus zero Gromov–Witten potential of $\mathcal{X}$ is equal to the generating function of the G-Hurwitz–Hodge integrals*:

$$F_{\mathcal{X}}(x_1, x_2, x_3) = F_G(x_1, x_2, x_3).$$

Proof. By definition, the orbifold invariants are given by

$$\langle c_1 \cdots c_k \rangle^{\mathcal{X}} = \int\limits_{[\overline{M}_{0,k}(\mathcal{X})]^{\mathrm{vir}}} \mathrm{ev}_1^*(c_1) \cup \cdots \cup \mathrm{ev}_k^*(c_k).$$

Using virtual localization with respect to the $\mathbb{C}^*$ action on $\mathcal{X}$, we can express the above integral in terms of an integral over the $\mathbb{C}^*$ fixed locus of $\overline{M}_{0,k}(\mathcal{X})$ which is $\overline{M}_{0,k}(BG)$:

$$\begin{aligned}\langle c_1 \cdots c_k \rangle^{\mathcal{X}} &= \int\limits_{[\overline{M}_{0,k}(BG)]} \mathrm{ev}_1^*(c_1) \cup \cdots \cup \mathrm{ev}_k^*(c_k) \cup e\big(N^{\mathrm{vir}}\big)^{-1} \\ &= \int\limits_{[\overline{M}_S(BG)]} e\big(-N^{\mathrm{vir}}\big)\end{aligned}$$

where $S = (c_1 \cdots c_k)$ and N^{vir} is the virtual normal bundle of $\overline{M}_S(BG)$ in $\overline{M}_{0,k}(\mathcal{X})$ regarded as an element in K-theory. So to prove the proposition, we need to show that

$$\int\limits_{[\overline{M}_S(BG)]} e\big(-N^{\mathrm{vir}}\big) = \int\limits_{[\overline{M}_S(BG)]} c\big(R^1\overline{\pi}_*\mathcal{O}_{\overline{C}}\big). \tag{7}$$

Let

$$V \to BG$$

be the bundle given by the 3-dimensional representation of G induced from the embedding $G \subset SO(3)$. By the standard argument in Gromov–Witten theory, the virtual normal bundle is given by

$$N^{\mathrm{vir}} = -R^\bullet\pi_* f^* V$$

[2] Strictly speaking, some of the orbifold Gromov–Witten invariants of $\mathcal{X}$ are not well defined because the corresponding moduli space of twisted stable maps is non-compact. In these cases, we *define* the invariants by localization with respect to the $\mathbb{C}^*$ action on $\mathcal{X}$.

where $\pi : \mathcal{C} \to \overline{M}_{0,k}(BG)$ is the universal curve and $f : \mathcal{C} \to BG$ is the universal map.

Let $H \subset G$ be the subgroup $\mathbb{Z}_3 \subset A_4$ or $\{0\} \subset \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. Then the action of G on the coset space G/H is the usual permutation action of G on the set of four elements. Consequently, we can construct the universal degree 4 cover $p : \overline{\mathcal{C}} \to \mathcal{C}$ by pulling back the map

$$i : BH \to BG$$

via f. That is, we have the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{C}} \arrow[r, "\overline{f}"] \arrow[d, "p"] \arrow[dd, bend right, "\overline{\pi}"'] & BH \arrow[d, "i"] \\
\mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & BG \\
\overline{M}_{0,k}(BG) &
\end{tikzcd}$$

We now compute in K-theory:

$$\begin{aligned}
R^\bullet \overline{\pi}_* \mathcal{O}_{\overline{\mathcal{C}}} &= R^\bullet \pi_* \big(p_* \overline{f}^* \mathcal{O}_{BH}\big) \\
&= R^\bullet \pi_* \big(f^* i_* \mathcal{O}_{BH}\big) \\
&= R^\bullet \pi_* f^* (V \oplus \mathcal{O}) \\
&= -R^1 \pi_* f^* V + \pi_* \mathcal{O}_{\mathcal{C}}.
\end{aligned}$$

The equality on the top line uses the fact that p is finite. The equality on the second line uses the fact that the commutative square in the diagram is Cartesian. The equality on the third line uses the fact that the G representation induced by the trivial representation is V plus the trivial representation. The equality on the fourth line uses the fact that $\pi : \mathcal{C} \to \overline{M}_{0,k}(BG)$ is genus 0.

Finally, we apply the total Chern class to both sides of the above equality and integrate over $\overline{M}_S(BG)$. The $\mathbb{C}^*$ equivariant Euler class is the same as the total Chern class with the appropriate power of the equivariant parameter appearing in each degree. Since the virtual dimension of $\overline{M}_{0,k}(\mathcal{X})$ is equal to the degree of the integrand, the integral is degree 0 in t the equivariant parameter and is hence independent of t. Eq. (7) is proved. □

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