



On weak product recurrence and synchronization of return times

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Abstract

The paper is devoted to a study of product recurrence. First, we prove that notions of \mathcal{F}_{ps} -PR and \mathcal{F}_{pubd} -PR are exactly the same as product recurrence, completing that way results of [P. Dong, S. Shao, X. Ye, Product recurrent properties, disjointness and weak disjointness, Israel J. Math. 188 (1) (2012) 463–507], and consequently, extending the characterization of return times of distal points which originated from the works of Furstenberg. We also study the structure of the set of return times of weakly mixing sets. As a consequence, we obtain new sufficient conditions for \mathcal{F}_s -PR and also find a short proof that weakly mixing systems are disjoint with all minimal distal systems (in particular, our proof does not involve Furstenberg's structure theorem of minimal distal systems).

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1. Introduction

There are two important characterizations related to distality, both observed by Furstenberg more than 30 years ago. If a point is distal, then it is recurrent in pairs with any recurrent point

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in any dynamical system [17] and if a minimal dynamical system is distal then it is disjoint with any weakly mixing system [16]. It is worth emphasizing that later, a full characterization of flows disjoint with all distal flows was provided by Petersen in [31]. While proofs of both the above mentioned characterizations are not that long, their proofs highly rely on other, even more important and highly nontrivial results. Namely, the first of them uses Hindman's theorem on finite partitions of IP-sets, while the second can be obtained as a consequence of an algebraic characterization of distal flows, proved by Furstenberg in [15].

The above characterizations on synchronization of return times of a point with return times of distal points gave motivation to two directions of research. The first of them asks about synchronization of return times between different types of recurrent points and the second asks about disjointness between specified classes of systems. Both these questions lead to partial classifications and hard open problems [11]. It is also worth emphasizing that studies on the above topics were very influential and important for topics which at first sight do not seem to be related very much. For example, Blanchard characterized in [8] systems disjoint with flows with zero entropy giving in that way a good motivation for introducing entropy pairs and systems with uniform positive entropy (u.p.e.) which are two fundamental notions in local entropy theory which were used later for deep and insightful investigations on topological entropy (cf. [19] for more on local entropy theory). In general, results on synchronization of trajectories can be of wide use (e.g. they can help to simplify some arguments in proofs etc.). For example, analysis of return or transfer times of points lead to a simple proof that a distal point is always minimal or that point minimal or recurrent for a dynamical system (X, f) is, respectively, minimal and recurrent for (X, f^n) for every n , etc.

If in place of recurrence in pairs with any recurrent point we demand recurrence in pairs with points in a smaller class of dynamical systems, it can lead to a wider class of points than the class of all distal points. For example, Auslander and Furstenberg in [5] asked about points which are recurrent in pairs with any minimal point. While there is no known full characterization of points with this property, it was proved in [20] that the class of such points is much larger than distal points, in particular it contains many points which are not minimal. Other sufficient conditions for this kind of product recurrence were provided in [11,28]. Moreover, [11] defines product recurrence in terms of Furstenberg families (i.e. upward hereditary sets of subsets of \mathbb{N}), which is a nice tool for a better classification of product recurrence. It is worth emphasizing that the concepts of [11] are not artificial, since it is possible almost immediately to relate these new types of product recurrence with some older results on disjointness.

The present paper completes some previous studies on recurrence and product recurrence from [11,28]. First, we prove that notions of \mathcal{F}_{ps} -PR and \mathcal{F}_{pubd} -PR are exactly the same as product recurrence, that is, if return times of a point can be synchronized with points returning with a piecewise syndetic set of times, then it can be synchronized with any recurrent point. This provides another condition to the list of conditions equivalent to distality as first provided by Furstenberg in [17, Theorem 9.11], and next extended by many authors (e.g. see [11]). Next, we analyze synchronizing properties of points in weakly mixing sets, which allow us to show that any weakly mixing set with dense distal points contains a residual subset of \mathcal{F}_s -PR points which are not distal (this is a question left open in [28]) and also to prove Furstenberg's result on disjointness between weakly mixing systems and minimal distal systems without referring to Furstenberg's structure theorem of distal flows. This is especially nice, since now both results of Furstenberg mentioned in the first paragraph of this introduction can be obtained using only Hindman's theorem plus some topological arguments. This even more bonds these two results together.

2. Preliminaries

In this section, we will provide some basic definitions used later in this paper. The reader is encouraged to refer to the books [1,2,21] for more details.

Denote by \mathbb{N} ($\mathbb{N}_0, \mathbb{Z}, \mathbb{R}$, respectively) the set of all positive integers (non-negative integers, integers, real numbers, respectively). A set $A \subset \mathbb{N}$ is an *IP-set* if there exists a sequence $\{p_i\}_{i=1}^\infty \subset \mathbb{N}$ such that A consists exactly of numbers p_i together with all finite sums

$$p_{n_1} + p_{n_2} + \dots + p_{n_k} \quad \text{with } n_1 < n_2 < \dots < n_k, k \in \mathbb{N}.$$

2.1. Basic notions in topological dynamics

By a (topological) dynamical system, or TDS for short, we mean a pair consisting of a compact metric space (X, d) and a continuous map $f: X \rightarrow X$. We denote the diagonal in $X \times X$ by $\Delta_2(X) = \{(x, x) : x \in X\}$.

The (positive) orbit of $x \in X$ under f is the set $\text{Orb}^+(x, f) = \{f^n(x) : n \in \mathbb{N}\}$. We denote $N_f(x, A) = \{n \in \mathbb{N} : f^n(x) \in A\}$ and similarly $N_f(A, B) = \{n \in \mathbb{N} : f^n(A) \cap B \neq \emptyset\}$.

We say that x is a *periodic point* of (X, f) if $f^n(x) = x$ for some $n \in \mathbb{N}$; a *recurrent point* of (X, f) if $N_f(x, U) \neq \emptyset$ for any open set $U \ni x$; a *transitive point* of (X, f) if $\overline{\text{Orb}^+(x, f)} = X$. Denote by $\text{Per}(X, f)$ ($\text{Rec}(X, f)$ and $\text{Tran}(X, f)$, respectively) the set of all periodic points (recurrent points and transitive points, respectively) of (X, f) .

Recall that (X, f) is *transitive* if $N_f(U, V) \neq \emptyset$ for any non-empty open sets U and V and *minimal* if $\text{Tran}(X, f) = X$. A point $x \in X$ is *minimal* or *uniformly recurrent* if $(\overline{\text{Orb}^+(x, f)}, f)$ is minimal. A dynamical system (X, f) is an *M-system* if it is transitive and the set of all minimal points is dense in X .

A pair of points $x, y \in X$ is *proximal* if there exists an increasing sequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $\lim_{k \rightarrow \infty} d(f^{n_k}(x), f^{n_k}(y)) = 0$. A point x is *distal* if it is not proximal to any point in its orbit closure other than itself. It was first observed by Auslander and Ellis that any point is proximal to a minimal point from its orbit closure and so any distal point is minimal [4,13,17]. The set of all proximal pairs of (X, f) is denoted by $\text{Prox}(f)$ and a proximal cell of $x \in X$ is denoted by $\text{Prox}(f)(x) = \{y \in X : (x, y) \in \text{Prox}(f)\}$. We say that (X, f) is *distal* if every point $x \in X$ is a distal point. Equivalently, it is to say that there is no proper proximal pair in X (i.e. $\text{Prox}(f) = \Delta_2(X)$).

Let f, g be two continuous surjective maps acting on compact metric spaces X and Y respectively. We say that a non-empty closed set $J \subset X \times Y$ is a *joining* of (X, f) and (Y, g) if it is invariant (for the product map $f \times g$) and its projections onto the first and second coordinates are X and Y respectively. If each joining is equal to $X \times Y$ then we say that (X, f) and (Y, g) are *disjoint* and denote this fact by $(X, f) \perp (Y, g)$ or simply by $f \perp g$.

2.2. Families and product recurrence

A (Furstenberg) family \mathcal{F} is a collection of subsets of \mathbb{N} which is *upwards hereditary*, that is:

$$F_1 \in \mathcal{F} \quad \text{and} \quad F_1 \subset F_2 \implies F_2 \in \mathcal{F}.$$

A family \mathcal{F} is *proper* if $\mathbb{N} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$ and is a *filter* if it is a proper family closed under finite intersections (equivalently $A \cap B \in \mathcal{F}$ for every $A, B \in \mathcal{F}$). Among filters there is an important class which is maximal with respect to set inclusion. Any such filter is called an

ultrafilter. Note that by Zorn's Lemma every filter is contained in some ultrafilter. We denote by $\beta\mathbb{N}$ the set of all ultrafilters of \mathbb{N} .

For any $n \in \mathbb{N}$ we can define its *principal ultrafilter* $e(n) = \{A \subset \mathbb{N} : n \in A\}$, therefore we can write $\mathbb{N} \subset \beta\mathbb{N}$ by the natural identification. Given $A \subset \mathbb{N}$ we set $\hat{A} = \{p \in \beta\mathbb{N} : A \in p\}$, and then we can define a topology on $\beta\mathbb{N}$ which has $\{\hat{A} : A \subset \mathbb{N}\}$ as its basis. It can be proved that $\beta\mathbb{N}$, equipped with the above introduced topology, coincides with Stone–Čech compactification of the discrete space \mathbb{N} , in particular, $\beta\mathbb{N}$ is a compact Hausdorff space and the set $\{e(n) : n \in \mathbb{N}\}$ is a dense subset of $\beta\mathbb{N}$ whose points are precisely the isolated points of $\beta\mathbb{N}$ (e.g. see [21, Theorems 3.18 and 3.28]). For any $p, q \in \beta\mathbb{N}$ we define

$$p + q = \{A \subset \mathbb{N} : \{n \in \mathbb{N} : -n + A \in q\} \in p\},$$

and we call p an *idempotent* if $p + p = p$. It can be proved (e.g. see [21, Chapter 4]) that $(\beta\mathbb{N}, +)$ is a right topological semigroup with \mathbb{N} contained in its topological center, in the sense that $e(n) + e(m) = e(n + m)$ for all $n, m \in \mathbb{N}$ and

- (i) for each $q \in \beta\mathbb{N}$, the map $\rho_q : \beta\mathbb{N} \ni p \mapsto p + q \in \beta\mathbb{N}$ is continuous,
- (ii) for each $n \in \mathbb{N}$, the map $\lambda_n : \beta\mathbb{N} \ni p \mapsto e(n) + p \in \beta\mathbb{N}$ is continuous.

For a more detailed exposition on Stone–Čech compactifications, in particular the case of $\beta\mathbb{N}$, the reader is referred to the book [21] by Hindman and Strauss.

Recall that a set $A \subset \mathbb{N}$ is *thick* if for every $n > 0$ there is an $i \in \mathbb{N}$ such that $\{i, i + 1, \dots, i + n\} \subset A$; is *syndetic* if A has a bounded gap, that is, there exists $N \in \mathbb{N}$ such that $[i, i + N] \cap A \neq \emptyset$ for each $i \in \mathbb{N}$. We denote by \mathcal{F}_{inf} , \mathcal{F}_t and \mathcal{F}_s the family of all infinite subsets, thick subsets and syndetic subsets of \mathbb{N} , respectively. It is direct to see that each thick subset intersects all syndetic subsets. We denote by \mathcal{F}_{ps} the family of all *piecewise syndetic* sets, that is sets that can be obtained as the intersection of a thick set and a syndetic set. We denote by \mathcal{F}_{pubd} the family of sets with *positive upper Banach density*, that is sets $F \subset \mathbb{N}$ such that

$$\limsup_{n-m \rightarrow \infty} \frac{\#(F \cap \{m, m + 1, \dots, n\})}{n - m + 1} > 0,$$

where as usual $\#A$ denotes the cardinality of a set A .

Families may be used to state definitions of recurrent points with prescribed types of the set of return times. Namely, for a family \mathcal{F} and $x \in X$, we say that x is \mathcal{F} -*recurrent* if $N_f(x, U) \in \mathcal{F}$ for any open neighborhood U of x . Note that a point is recurrent exactly when it is \mathcal{F}_{inf} -recurrent and is minimal when it is \mathcal{F}_s -recurrent. For an interesting exposition on recurrence properties expressed in terms of families the reader is referred to the book [2] by Akin.

A recurrent point x in a dynamical system (X, f) is *product recurrent* if given any recurrent point y in any dynamical system (Y, g) the pair (x, y) is recurrent for the product system $(X \times Y, f \times g)$. If we demand the above condition only for y which is uniformly recurrent then we say that x is *weakly product recurrent*. Again, let \mathcal{F} be a family. A recurrent point x in a dynamical system (X, f) is \mathcal{F} -*product recurrent* (\mathcal{F} -PR for short) if for any \mathcal{F} -recurrent point y in any dynamical system (Y, g) , the pair (x, y) is recurrent for $(X \times Y, f \times g)$. Thus, a recurrent point is product recurrent if and only if it is \mathcal{F}_{inf} -PR, and is weakly product recurrent if and only if it is \mathcal{F}_s -PR.

It is well known that a point is product recurrent if and only if it is distal [17] and it was recently proved in [20] that there are weakly product recurrent points which are not distal (in fact, they form a much wider class of points). Properties of product recurrence were studied

independently in [11,28], where some further necessary conditions for product recurrence were obtained. It is obvious that

$$\mathcal{F}_{inf}\text{-PR} \implies \mathcal{F}_{pubd}\text{-PR} \implies \mathcal{F}_{ps}\text{-PR}.$$

The question whether any of the above implications can be reverted was left open in [11]. We will answer this question later in this paper.

2.3. Weakly mixing sets

A closed set $A \subset X$ containing at least two points is a *weakly mixing set of order n* if for any choice of open subsets $V_1, U_1, \dots, V_n, U_n$ of X with $A \cap U_i \neq \emptyset, A \cap V_i \neq \emptyset, i = 1, \dots, n$, there exists $k > 0$ such that $f^k(V_i \cap A) \cap U_i \neq \emptyset$ for each $1 \leq i \leq n$. If A is weakly mixing of order n for all $n \geq 2$, then we say that A is *weakly mixing of all orders*, or simply *weakly mixing*.

The idea of weakly mixing sets comes from [10] where it was studied firstly in considerable detail. In particular, it was shown that every system with positive entropy contains weakly mixing sets [10], while we are looking at the *a priori* weaker property of weak mixing of order n and especially 2 in [29,30]. It was proved in [30] that for each $n \geq 2$ there exists a minimal system containing weakly mixing sets of order n but without weakly mixing sets of order $n + 1$ (in [29] such an example was constructed only for the particular case of $n = 2$).

Remark 1. The definition of weakly mixing set (of order n) introduced here is a little more restrictive (i.e. more conditions are put on A) than that in [29], where it was introduced first.

It can be proved easily that each weakly mixing set of order 2 (as introduced here) is perfect [29]. If the whole X is a weakly mixing set then we say that (X, f) is weakly mixing, which equivalently means that $(X \times X, f \times f)$ is transitive. Furstenberg proved that for an invariant subset, weak mixing of order 2 implies weak mixing of all orders [16, Proposition II.3] (i.e. on invariant subsets, these notions coincide).

3. Piecewise syndetic product recurrence

As we mentioned earlier, the question whether any of the implications below can be reverted was left open in [11, Section 5.3]:

$$\mathcal{F}_{inf}\text{-PR} \implies \mathcal{F}_{pubd}\text{-PR} \implies \mathcal{F}_{ps}\text{-PR}.$$

In this section we will prove that all the above properties are equivalent, and then as a corollary of this equivalence we can extend an important characterization of distal points from [17, Theorem 9.11].

To settle the question we need the following notions. A set $S \subset \mathbb{N}$ is a *dynamical syndetic set* if there exists a minimal dynamical system (X, f) with a minimal point $x \in X$ and an open neighborhood U of x such that $S = N_f(x, U)$. A set $J \subset \mathbb{N}$ is an *md-set* if there exists an M -system (Y, g) with a transitive point $y \in Y$ and a neighborhood V of y such that $J = N_g(y, V)$.

It was proved in [11, Proposition 3.3] that every thick set contains an md-set. The following lemma shows a similar result for a finer structure.

Lemma 2. *Let S_1 and S_2 be a dynamical syndetic set and a thick set, respectively. Then $S_1 \cap S_2$ contains an md-set $N_g(z, W)$ defined by an \mathcal{F}_{ps} -recurrent and a transitive point z in an M -system (Y, g) and an open neighborhood W of z .*

Proof. From the definition, there exists a minimal dynamical system (X, f) with a minimal point $x \in X$ and an open neighborhood U of x such that $S_1 = N_f(x, U)$. Moreover, as S_2 is a thick set, there is an increasing sequence n_j such that $n_{j+1} - n_j > j^2$ and $B_j \subset S_2$ with $B_j = [n_j, n_j + j] \cap \mathbb{N}$. Note that sets B_j are pairwise disjoint. Let us renumerate sequence $\{B_j\}_{j=1}^\infty$ creating a double indexed family $\{B_j^{(i)}\}_{i,j=1}^\infty$ with the property that $k < s$ whenever $k \in B_1^{(i)}, s \in B_1^{(i+1)}$ or $k \in B_j^{(i)}, s \in B_{j+1}^{(i)}$ for some indices $i, j = 1, 2, \dots$.

Now we will perform an inductive construction of points $z_x^{(i)} \in \Sigma_2$ and sets $A^{(i)} \subset \mathbb{N}$, where by Σ_2 we denote the two-sided full shift over the alphabet $\{0, 1\}$ (together with the left shift transformation σ). If $u, v \in \{0, 1\}^k$ then we write $u \preceq v$ if $u[i] \leq v[i]$ for every $0 \leq i < k$. We can extend easily this relation onto infinite sequences and next to bi-infinite sequences $y \in \Sigma_2$ such that $y[i] = 0$ for $i < 0$. Define a point $z_x \in \Sigma_2$ by putting $z_x[i] = 0$ if $i < 0, z_x[i] = 0$ if $f^i(x) \notin U$ for some $i \geq 0$ and $z_x[i] = 1$ in all other situations.

Now, let us put $z_x^{(1)}[i] = 1$ when $i = 0$ or when $f^i(x) \in U$ and $i \in \bigcup_{j=1}^\infty B_j$. For all other values of i we put $z_x^{(1)}[i] = 0$. Then $z_x^{(1)} \preceq z_x$. Define

$$A^{(1)} = \left\{ i \in \mathbb{N} : f^i(x) \in U, i \in \bigcup_{j=1}^\infty B_j^{(1)} \right\}.$$

The point x is minimal, so clearly $A^{(1)}$ is piecewise syndetic and $z_x^{(1)}[k] = 1$ for every $k \in A^{(1)}$.

Assume that for some $m \geq 1$ we have constructed points $z_x^{(m)} \preceq z_x^{(m-1)} \preceq \dots \preceq z_x^{(1)}$ and piecewise syndetic sets $A^{(1)}, \dots, A^{(m)}$ with the following additional properties:

- (2.1) $z_x^{(s)}[k] = z_x^{(s+1)}[k]$ provided that $k \notin \bigcup_{j=1}^\infty B_j^{(s+1)}$, where $s < m$;
- (2.2) $(k - s, k + s) \cap \mathbb{Z} \subset \bigcup_{j=1}^\infty B_j^{(s)}$ for every $k \in A^{(s)}$, where $s = 1, \dots, m$; and
- (2.3) $(z_x^{(s)})_{(-s,s)} = (z_x^{(s)})_{(k-s,k+s)}$ for every $k \in A^{(s)}$, where $s = 1, \dots, m$.

Now we are going to construct $z_x^{(m+1)}$ and $A^{(m+1)}$.

First we put $z_x^{(m+1)}[k] = z_x^{(m)}[k]$ for every $k \notin \bigcup_{j=1}^\infty B_j^{(m+1)}$. On other positions we will copy only a part of symbols 1 from $z_x^{(m)}$, changing to 0 some of them. Note that by the construction we have $z_x^{(m)}[k] = z_x^{(1)}[k]$ for every $k \in \bigcup_{s>m} \bigcup_{j=1}^\infty B_j^{(s)}$.

There is an integer t such that if $a \in B_j^{(m+1)}, b \in B_{j+1}^{(m+1)}$ for some $j \geq t$ then $b - a > 3m$. In particular, if $i - m, i + m \in \bigcup_{j=t}^\infty B_j^{(m+1)}$ for some $i \in \mathbb{N}$ then there is $j \geq t$ such that $[i - m, i + m] \cap \mathbb{N} \subset B_j^{(m+1)}$. We choose an open set $V \ni x$ such that, for $0 \leq j \leq m, f^j(x) \in U$ implies $f^j(V) \subset U$. Obviously the following set is piecewise syndetic:

$$A = \left\{ k \in \mathbb{N} : f^k(x) \in V \text{ and } k - m \in \bigcup_{j=t}^\infty B_j^{(m+1)}, k + m \in \bigcup_{j=t}^\infty B_j^{(m+1)} \right\}.$$

We can remove elements in A if necessary, keeping it piecewise syndetic and at the same time ensuring that $|a - b| > 3m$ whenever $a, b \in A$ are distinct. Denote by $A^{(m+1)}$ the set A after this modification. This modification ensures that if we put $(z_x^{(m+1)})_{[k,k+m]} = (z_x^{(m)})_{[0,m]}$ for all $k \in A^{(m+1)}$ and put $z_x^{(m+1)}[i] = 0$ for all $i \in \bigcup_{j=1}^\infty B_j^{(m+1)} \setminus \bigcup_{i \in A^{(m+1)}} [i, i + m]$ then $z_x^{(m+1)}$ is well defined (simply because $|a - b| > 3m$ and so $[a - m, a + m] \cap [b - m, b + m] = \emptyset$ for distinct $a, b \in A^{(m+1)}$).

Now it remains to check that $z_x^{(m+1)}$ and $A^{(m+1)}$ satisfy all desired properties. Directly from the construction we have that $A^{(m+1)}$ is piecewise syndetic and conditions (2.1), (2.2) are satisfied. Note that $\min \bigcup_{j=1}^{\infty} B_j^{(m+1)} > m$ therefore $(z_x^{(m+1)})_{[0,m]} = (z_x^{(m)})_{[0,m]}$ which also gives (2.3) with the help of the construction. The only condition which remains is $z_x^{(m+1)} \preceq z_x^{(m)}$. Fix any $k \in A^{(m+1)}$. Then by the construction we have that $(z_x^{(m+1)})_{[k,k+m]} = (z_x^{(m+1)})_{[0,m]}$. Observe that $f^k(x) \in V$ and therefore by the definition of V , if $f^j(x) \in U$ for some $0 \leq j \leq m$ then $f^{k+j}(x) \in U$. In other words $(z_x)_{[0,m]} \preceq (z_x^{(1)})_{[k,k+m]}$. Additionally by the construction and (2.1) we have that $(z_x^{(m)})_{[k,k+m]} = (z_x^{(1)})_{[k,k+m]}$. Combining all these facts together we obtain the following:

$$\begin{aligned} (z_x^{(m+1)})_{[k,k+m]} &= (z_x^{(m+1)})_{[0,m]} = (z_x^{(m)})_{[0,m]} \\ &\preceq (z_x)_{[0,m]} \preceq (z_x^{(1)})_{[k,k+m]} = (z_x^{(m)})_{[k,k+m]}. \end{aligned}$$

But for all $i \in \mathbb{Z} \setminus \bigcup_{j \in A^{(m+1)}} [j, j+m]$ we also have $z_x^{(m+1)}[i] \leq z_x^{(m)}[i]$ from the construction, and so $z_x^{(m+1)} \preceq z_x^{(m)}$ which completes the induction.

By the definition obviously $\left\{ z_x^{(m)} \right\}_{m=1}^{\infty}$ is a Cauchy sequence in Σ_2 , so the limit $z = \lim_{m \rightarrow \infty} z_x^{(m)}$ is well defined. Furthermore, the only modifications of $z_x^{(s)}$ in the further steps of induction are done on the set $Q_s = \bigcup_{i=s+1}^{\infty} \bigcup_{j=1}^{\infty} B_j^{(i)}$ and so $z[k] = z_x^{(s)}[k]$ for every $k \notin Q_s$. In particular, for every $i \in A^{(s)}$ we have

$$z_{(-s,s)} = (z_x^{(s)})_{(-s,s)} = (z_x^{(s)})_{(i-s,i+s)} = z_{(i-s,i+s)}$$

which equivalently means that $N_{\sigma}(z, W) \supset A^{(s)}$, where σ is the shift transformation over Σ_2 and W is the cylinder set $\{q \in \Sigma_2 : q_{(-s,s)} = z_{(-s,s)}\}$. This proves that z is \mathcal{F}_{ps} -recurrent. Denote by Z the closure of the (positive) orbit $\text{Orb}^+(z, \sigma)$ of z under the shift transformation. It can be proved (e.g. see [23, Lemma 2.1]) that (Y, g) is an M-system if and only if there is a transitive point $y \in Y$ such that $N_g(y, W) \in \mathcal{F}_{ps}$ for any neighborhood W of y . Thus (Z, σ) is an M-system and so from the construction $S_1 \cap S_2$ contains an md-set $N_{\sigma}(z, W_1)$, where W_1 is the cylinder set $\{q \in \Sigma_2 : q[0] = 1\}$. \square

By characterization from [17], a point x is distal if and only if x is \mathcal{F}_{inf} -PR, so to answer our question it is enough to prove the following theorem.

Theorem 3. *If x is \mathcal{F}_{ps} -PR then it is distal.*

Proof. First note that x is minimal by [11, Theorem 3.4]. Let X be the closure of the (positive) orbit of x under the associated map f . Assume on the contrary that x is not distal, which means that there is a minimal point $y \in X \setminus \{x\}$ such that the pair (x, y) is proximal. Let $\varepsilon > 0$ and U, V be two open sets such that $y \in U, x \notin \bar{V}$ and $B(\bar{U}, \varepsilon) \subset V$.

The pair (x, y) is proximal, and so there is an increasing sequence n_j such that $n_{j+1} - n_j > j^2$ and $d(f^i(x), f^i(y)) < \frac{\varepsilon}{j}$ for every $i \in [n_j, n_j + j]$. We denote $S_1 = N_f(y, U)$ and let $S_2 = \bigcup_{j=1}^{\infty} [n_j, n_j + j] \cap \mathbb{N}$. Then S_1 is a dynamical syndetic set and S_2 is a thick set.

By Lemma 2 the intersection $S_1 \cap S_2$ contains a set $N_g(z, W)$, where z is a transitive and \mathcal{F}_{ps} -recurrent point from some M-system (Z, g) , and W is a neighborhood of z . But $(x, z) \in (X \setminus \bar{V}) \times W$ and for any $k > 0$ if $g^k(z) \in W$ then $f^k(y) \in U$ and $d(f^k(x), f^k(y)) < \varepsilon$, which implies that $f^k(x) \in V$. This shows that (x, z) is not recurrent, a contradiction to the assumption that x is \mathcal{F}_{ps} -PR, which ends the proof. \square

So far, we have obtained by **Theorem 3** the equivalence of the properties \mathcal{F}_{inf} -PR, \mathcal{F}_{pubd} -PR and \mathcal{F}_{ps} -PR. Combining this equivalence with recent results from [27], we obtain the following characterization, which is an extension of the classical characterization of distality in [17, Theorem 9.11]. While it is not visible here, the proof strongly relies on the structure of the set of idempotents in $\beta\mathbb{N}$. For a more detailed exposition on this important topic see for example [7,14,18,21,27].

Fix a family \mathcal{F} and any sequence $\{x_n\}_{n=1}^\infty \subset X$. We say that $z \in X$ is an \mathcal{F} -limit of $\{x_n\}_{n=1}^\infty$ if for every open neighborhood U of z the set $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$. Observe that if \mathcal{F} is a filter then there exists at most one such point z . It can be proved that every ultrafilter $p \in \beta\mathbb{N}$ has *Ramsey Property*, that is, if $A \cup B \in p$ then either $A \in p$ or $B \in p$. Therefore, it is not hard to show that for any sequence $\{x_n\}_{n=1}^\infty \subset X$ and any ultrafilter $p \in \beta\mathbb{N}$ there exists a unique point $z \in X$ such that z is a p -limit of $\{x_n\}_{n=1}^\infty$. In what follows, for any $x \in X$ and any $p \in \beta\mathbb{N}$ we will denote by px the p -limit of the sequence $\{f^n(x)\}_{n=1}^\infty$. By the definition of the Stone–Čech compactification, any function $\mathbb{N} \ni n \mapsto x_n \in X$ can be extended to be a continuous function $\beta\mathbb{N} \ni p \mapsto x_p \in X$. But it is also not hard to prove that for every $p \in \beta\mathbb{N}$ the point $x_p \in X$ is just the p -limit of the sequence $\{x_n\}_{n=1}^\infty$ (e.g. see [21, Corollary 3.49.1]).

A family \mathcal{F} is a *filterdual* if its dual family $k\mathcal{F}$ is a filter, where $k\mathcal{F} = \{F \subset \mathbb{N} : \mathbb{N} \setminus F \notin \mathcal{F}\}$. It can be proved that both \mathcal{F}_{pubd} and \mathcal{F}_{ps} are filterduals (e.g. see [2]) and that both sets $h(\mathcal{F}_{pubd})$ and $h(\mathcal{F}_{ps})$ are closed subsemigroups of $(\beta\mathbb{N}, +)$ where $h(\mathcal{F}) = \{p \in \beta\mathbb{N} : p \subset \mathcal{F}\}$ (e.g. see [27]). But then with respect to these families we can apply the following equivalent characterization (it is a shortened version of a list of equivalent conditions in [27, Theorem 4.4]):

Theorem 4. *Let \mathcal{F} be a filterdual and suppose that $h(\mathcal{F})$ is a subsemigroup of $(\beta\mathbb{N}, +)$. Then x is \mathcal{F} -recurrent if and only if there exists an idempotent $p \in h(\mathcal{F})$ such that $px = x$.*

Now we are ready to prove the above announced extension of Furstenberg’s theorem.

Theorem 5. *The following statements are equivalent:*

- (1) x is distal,
- (2) (x, y) is recurrent for any recurrent point y of a system (Y, g) ,
- (3) (x, y) is \mathcal{F}_{pubd} -recurrent for any \mathcal{F}_{pubd} -recurrent point y of a system (Y, g) ,
- (4) (x, y) is \mathcal{F}_{ps} -recurrent for any \mathcal{F}_{ps} -recurrent point y of a system (Y, g) ,
- (5) (x, y) is minimal for any minimal point y of a system (Y, g) .

Proof. The equivalence of (1), (2) and (5) comes from [17, Theorem 9.11].

Implications (3) \implies (1) and (4) \implies (1) follow from the equivalence of the properties \mathcal{F}_{inf} -PR, \mathcal{F}_{pubd} -PR and \mathcal{F}_{ps} -PR.

Now we prove the implication of (1) \implies (3). Fix any \mathcal{F}_{pubd} -recurrent point y of a system (Y, g) . Then by **Theorem 4** there exists an idempotent $p \in \beta\mathbb{N}$ such that $p \subset \mathcal{F}_{pubd}$ and $py = y$. But it is not hard to check that if p is an idempotent in $\beta\mathbb{N}$ and x is distal then $px = x$ (e.g. see [6, Proposition 3.17]). Since p is a filter, for any neighborhood U of x and V of y we have that $\{n \in \mathbb{N} : f^n(x) \in U\} \cap \{m \in \mathbb{N} : g^m(y) \in V\} \in p \subset \mathcal{F}_{pubd}$, which implies that (x, y) is \mathcal{F}_{pubd} -recurrent.

The proof of implication (1) \implies (4) is identical to that of (1) \implies (3), so we leave it to the reader. \square

Denote by \mathcal{F} -PR₀ the restriction in the definition of \mathcal{F} -PR considering recurrence in pairs only with points y from dynamical systems (Y, g) with zero topological entropy. Generally speaking, \mathcal{F} -PR₀ denotes product recurrence with respect to systems of zero entropy.

Observe that in the proof of **Lemma 2** the system generated by z can have positive topological entropy. For example, if x is from a minimal subshift X over $\{0, 1\}$ then it may happen that in the first step of the induction we will incorporate in z arbitrarily long subwords of x , and therefore entropy of the subshift generated by z will be at least as that of X . In fact, this point is not surprising, since it was shown in [11] that there exists minimal systems with positive entropy (therefore not distal) such that every point in this system is \mathcal{F}_{ps} -PR₀ (an important aid when dealing with the \mathcal{F} -PR₀ property, especially for construction of counterexamples, is provided by results of [22] on disjointness with zero entropy systems). In particular, another question from [11] about implication

$$\mathcal{F}_{ps}\text{-PR}_0 \implies \mathcal{F}_{pubd}\text{-PR}_0$$

remains still open.

4. Weak mixing and synchronization

In this section we investigate properties of the set of transfer times of points in weakly mixing sets. One of the possible applications of our analysis is another proof of an important result by Furstenberg [16] stating that every weakly mixing system is disjoint from all minimal distal systems. As another application we show that a weakly mixing set with dense distal points contains a residual subset of \mathcal{F}_s -PR points which are not distal, which completes the results of [28].

For this purpose, given sets $U, V \subset X$ we denote $C_f(U, V) = \{n \in \mathbb{N} : f^n(U) \subset V\}$. Additionally, for any sequence of positive integers p_1, p_2, \dots and $n \geq 1$, denote by $S(p_1, \dots, p_n)$ the set of all sums of subsequences of p_1, \dots, p_n , that is,

$$S(p_1, \dots, p_n) = \{p_{i_1} + \dots + p_{i_k} : 1 \leq k \leq n, i_1 < \dots < i_k\}.$$

Just by the definition, the set $S(p_1, p_2, \dots) = \bigcup_{n=1}^{\infty} S(p_1, \dots, p_n)$ is an IP-set.

The following fact has a straightforward proof.

Lemma 6. *If for some open sets U, V and a point $x \in X$ we have $J \subset C_f(U, V)$ and $l \in N_f(x, U)$ then $l + s \in N_f(x, V)$ for every $s \in J$.*

Before proceeding, we also need the following technical lemma.

Lemma 7. *Assume that A is a weakly mixing set of order 2 for (X, f) . Let U, V be open sets intersecting A and $x \in U \cap A, y \in V \cap A$. If there are $p_1, \dots, p_n \in \mathbb{N}$ satisfying*

$$S(p_1, \dots, p_n) \subset N_f(x, V) \cap N_f(y, V)$$

then there is an integer $p_{n+1} > \sum_{j=1}^n p_j$ and open sets U', V' intersecting A such that

$$\overline{U'} \subset U, \overline{V'} \subset V \quad \text{and} \quad S(p_1, \dots, p_n, p_{n+1}) \subset C_f(U', V) \cap C_f(V', V).$$

Proof. By the assumptions, there are neighborhoods $W \subset V, W' \subset U$ of y and x respectively, such that $S(p_1, \dots, p_n) \subset C_f(W', V) \cap C_f(W, V)$. Since A is weakly mixing of order 2, there are $l > \sum_{j=1}^n p_j$ and $x' \in A \cap W', y' \in A \cap W$, such that

$$l \in N_f(x', W) \cap N_f(y', W).$$

But then, by Lemma 6, $l + s \in N_f(x', V) \cap N_f(y', V)$ for all $s \in S(p_1, \dots, p_n)$. Putting $p_{n+1} = l$ we obtain that

$$S(p_1, \dots, p_{n+1}) \subset N_f(x', V) \cap N_f(y', V).$$

Now, it is enough to choose sufficiently small neighborhoods U', V' of x', y' respectively and the proof is completed. \square

The following fact is the main result of this section. It provides a characterization of transfer times of points in weakly mixing sets.

Theorem 8. *Let A be a weakly mixing set of order 2 for (X, f) and U an open set intersecting A . Then there is $x \in U \cap A$ such that for every open set V intersecting A the set $N_f(x, V)$ contains an IP-set. In particular, for every open set V intersecting A the set $N_f(U \cap A, V)$ contains an IP-set.*

Proof. We choose a set $\{x_i\}_{i=1}^\infty \subset A$ dense in A . It suffices to prove that there is $x \in U \cap A$ such that for every i and every $n > 0$ the set $N_f\left(x, B\left(x_i, \frac{1}{n}\right)\right)$ contains an IP-set.

Arrange sets $\left\{B\left(x_i, \frac{1}{n}\right)\right\}_{i,n=1}^\infty$ into a sequence, let us say V_1, V_2, \dots and order pairs (i, j) by taking consecutive diagonals $\{(i, j) : i + j = n\}$ in $\mathbb{N} \times \mathbb{N}$, say

$$(1, 1) < (1, 2) < (2, 1) < (1, 3) < (2, 2) < (3, 1) < (1, 4) < \dots$$

For technical reasons, we assume that $(0, 0) < (1, 1)$ and put as $U^{(0,0)}$ an open set intersecting A such that $\overline{U^{(0,0)}} \subset U$ and the diameter of $U^{(0,0)}$ is at most 1. By the same reason we put $p_0^{(0)} = 0$ and $V_i^{(0)} = V_i$ for each $i \in \mathbb{N}$. We will perform an inductive construction with respect to the relation $<$ to construct:

- (i) a sequence of sets $U^{(i,j)}$ intersecting A such that the diameter of $U^{(i,j)}$ is at most $\frac{1}{i+j+1}$ and $\overline{U^{(i,j)}} \subset U^{(a,b)}$ if $(a, b) < (i, j)$;
- (ii) for every $j = 1, 2, \dots$ a sequence of sets $V_i^{(j)}$ intersecting A such that $\overline{V_i^{(j)}} \subset V_i^{(j-1)}$ (recall that $V_i^{(0)} = V_i$); and
- (iii) a sequence of integers $p_i^{(j)}$ such that $S(p_i^{(1)}, \dots, p_i^{(j)})$ is a subset of $C_f(U^{(i,j)}, V_i) \cap C_f(V_i^{(j)}, V_i)$ and $p_i^{(j)} > p_i^{(j-1)}$ (recall that $p_0^{(0)} = 0$).

Consider a pair (i, j) and assume that for the pair (i', j') directly preceding (i, j) with respect to the relation $<$ we have already constructed a set $U^{(i',j')}$ such that $U^{(i',j')} \cap A \neq \emptyset$ and $\overline{U^{(i',j')}} \subset U^{(a,b)}$ for every $(a, b) < (i', j')$. We also assume that numbers $p_{j_1}^{(i_1)}$ have already been constructed for all $(i_1, j_1) < (i, j)$.

If $j = 1$, then since A is weakly mixing of order 2, we can find $x \in U^{(i',j')} \cap A$ and $y \in V_i \cap A$ such that there is $l \in N_f(x, V_i) \cap N_f(y, V_i)$. Then, there are open sets $U^{(i,j)} \ni x$ and $\overline{V_i^{(1)}} \ni y$ such that the diameter of $U^{(i,j)}$ is at most $\frac{1}{i+j+1}$ and $l \in C_f(U^{(i,j)}, V_i) \cap C_f(V_i^{(1)}, V_i)$, $\overline{V_i^{(1)}} \subset V_i$, $\overline{U^{(i,j)}} \subset U^{(i',j')}$. We put $p_1^{(i)} = l$.

If $j > 1$ then we have already constructed positive integers $p_i^{(1)} < p_i^{(2)} < \dots < p_i^{(j-1)}$ and sets $V_i^{(1)}, \dots, V_i^{(j-1)}$ intersecting A such that $S(p_i^{(1)}, \dots, p_i^{(j-1)}) \subset C_f(U^{(i,j-1)}, V_i) \cap C_f(V_i^{(j-1)}, V_i)$ and $\overline{V_i^{(k)}} \subset V_i^{(k-1)}$ for all $k = 1, \dots, j - 1$. Fix any $x \in U^{(i',j')} \cap A$ and

$y \in V_i^{(j-1)} \cap A$. Applying Lemma 7 we obtain an integer $p_i^{(j)} > \sum_{k=1}^{j-1} p_i^{(k)}$ and open sets $\overline{U^{(i,j)}} \subset U^{(i',j')}$, $\overline{V_i^{(j)}} \subset V_i^{(j-1)}$, both intersecting A , such that the diameter of $U^{(i,j)}$ is at most $\frac{1}{i+j+1}$ and

$$S(p_i^{(1)}, \dots, p_i^{(j)}) \subset C_f(U^{(i,j)}, V_i) \cap C_f(V_i^{(j)}, V_i).$$

By the above inductive construction, we obtain a nested sequence of sets $U^{(i,j)}$ with respect to the relation \prec . Now fix any

$$z \in \bigcap_{i,j} \overline{U^{(i,j)}} \subset U.$$

Note that each $U^{(i,j)}$ intersects the closed set A and the diameter of $U^{(i,j)}$ is at most $\frac{1}{i+j+1}$, so directly from the construction we have that $z \in U \cap A$ and

$$S(p_i^{(1)}, \dots, p_i^{(j)}) \subset N_f(z, V_i)$$

for any pair (i, j) . Therefore $N_f(z, V_i)$ contains the IP-set generated by the sequence $\left\{ p_i^{(j)} \right\}_{j=1}^\infty$. But $i \in \mathbb{N}$ is arbitrary, and so the proof is completed. \square

Now we are going to present two possible applications of the somewhat technical results in Theorem 8.

The following theorem was first proved by Furstenberg [16, Theorem II.3]. The main argument in his proof is an algebraic description of the structure of minimal distal systems [16, Proposition II.9] which is an advanced result with a complicated proof (see also [15]) and the fact that the Cartesian product of a weakly mixing system with any minimal system is transitive [16, Proposition II.11]. Here we offer another proof of Furstenberg’s theorem using the tool provided by Theorem 8. In particular our proof relies on dynamical properties of distal points rather than algebraic properties of distal minimal systems. One of the important tools in our proof (which is somehow hidden behind the results of [17, Theorem 9.11]) is Hindman’s theorem. While we definitely do not refer to algebraic classification of distal minimal systems, it is not completely honest to say that arguments in our proof are elementary.

Corollary 9. *If (X, f) is weakly mixing then it is disjoint from any minimal distal system.*

Proof. If X is a singleton then the theorem holds. Therefore we can assume that X has at least two points, and so X is perfect. Fix any minimal distal system (Y, g) and let $J \subset X \times Y$ be a joining of (X, f) and (Y, g) . Let $x \in X$ be a point obtained by applying Theorem 8 with $A = U = X$. Since J is a joining, there exists $y \in Y$ such that $(x, y) \in J$. It can be proved that a point y is distal if and only if for each neighborhood W of y , the set $N_g(y, W)$ has a non-empty intersection with any IP-set (mainly because that each IP-set contains a set of return times of some recurrent point to its sufficiently small neighborhood), e.g. see [17, Theorem 9.11]. It immediately implies that $N_f(x, V) \cap N_g(y, W) \neq \emptyset$ for any non-empty open set $V \subset X$ and any open neighborhood W of y . Since V is arbitrary, taking a nested sequence of neighborhoods of y and using the fact that J is a closed set containing a positive limit set of the pair (x, y) under $f \times g$, we obtain the inclusion $X \times \{y\} \subset J$. But y has a dense orbit in Y , hence we get $X \times Y \subset J$ which ends the proof. \square

In [9] the authors introduced scattering systems using topological complexity of open covers and proved that scattering systems are disjoint from all minimal distal systems

[9, Proposition 4.2]. The proof of [9, Proposition 4.2] relies again on the algebraic description from [16, Proposition II.9]. Each weakly mixing system is scattering [9, Proposition 3.4] and for a minimal system scattering and weak mixing are equivalent properties [9, Proposition 3.8]. Unfortunately, we do not know if our proof can be adopted to prove [9, Proposition 4.2], that is, we do not know how to extend our result to work also for non-minimal scattering systems.

The following property was introduced in [28]. A non-empty set $A \subset X$ has the *property (P)* if for any open set $U \subset X$ with $U \cap A \neq \emptyset$ there exists a point $x \in U \cap A$ and an integer $K > 0$ such that $f^{nK}(x) \in U$ for all $n \in \mathbb{N}$. In other words, if we denote by $\mathcal{F}_{s,+}$ the family generated by $\{k\mathbb{N} : k \in \mathbb{N}\}$, then once the subset $\emptyset \neq A \subset X$ has the property (P) then $N_f(U \cap A, U) \in \mathcal{F}_{s,+}$ for any open set $U \subset X$ with $U \cap A \neq \emptyset$.

In [11,28] it was proved that each weakly mixing system with dense distal points is disjoint from all minimal systems and all points with a dense orbit in such a system are weakly product recurrent. Another result of [28] shows that each weakly mixing set with the property (P) contains a residual subset of weakly product recurrent points. It is also possible to prove that (see [23]) every weakly mixing system with the property (P) is disjoint from all minimal systems. Then, a natural claim is that every weakly mixing set (of order 2) with dense distal points should contain weakly product recurrent points. However, it is much harder to synchronize with distal points than with uniformly recurrent points, and because of this difficulty the above claim was left open in [28]. Now, Theorem 8 provides a tool which can be used to finally prove the above mentioned fact, completing the results of [28].

Corollary 10. *Let $A \subset X$ be a weakly mixing set of order 2 for (X, f) . If additionally distal points are dense in A then A contains a residual subset of points which are weakly product recurrent but not product recurrent.*

Proof. Fix any sequence $\{x_i\}_{i=1}^\infty \subset A$ dense in A which consists of distal points in A . We consider the subset $D \subset A$ defined as follows: a point $x \in A$ belongs also to D if for any point x_i and any open set $U \ni x_i$ we have $N_f(x, U) \cap N_f(x_i, U) \neq \emptyset$.

First, let us prove that D is residual in A . Obviously D is a G_δ subset, as $D = \bigcap_{i,j=1}^\infty D_{i,j}$ where $D_{i,j}$ is the open set consisting of all points $x \in A$ such that

$$N_f\left(x, B\left(x_i, \frac{1}{j}\right)\right) \cap N_f\left(x_i, B\left(x_i, \frac{1}{j}\right)\right) \neq \emptyset.$$

It remains to show that each $D_{i,j}$ is a dense subset of A . Fix any $i, j > 0$ and any open set W intersecting A . Let $x \in W \cap A$ be a point obtained from Theorem 8. By the choice of x there is an IP-set $P \subset N_f\left(x, B\left(x_i, \frac{1}{j}\right)\right)$, and so $P \cap N_f\left(x_i, B\left(x_i, \frac{1}{j}\right)\right) \neq \emptyset$ as x_i is a distal point (we apply again the characterization of product recurrence [17, Theorem 9.11]). Thus $D_{i,j}$ is dense in A , and then D is residual in A .

By the construction, $\text{Orb}^+(x, f) \supset A$ for any $x \in D$. It remains to show that each point in D is weakly product recurrent, since it is easy to verify that it cannot be product recurrent. Namely, if x is product recurrent which is equivalent to saying that $x \in D$ is distal, then a contradiction to the fact of $A \subset \text{Orb}^+(x, f)$, since it is easy to verify that in that case $\text{Orb}^+(x, f)$ contains a point (other than x itself) proximal to x . In fact, by the construction of D each point x_i is proximal to x .

To finish the proof, fix any $x \in D$ and any uniformly recurrent point y in a dynamical system (Y, g) , and next fix any open neighborhood $U \subset X$ of x and any open neighborhood $V \subset Y$ of y . There is $i \in \mathbb{N}$ such that $x_i \in U$, and so by [17, Theorem 9.11] the pair (x_i, y) is uniformly recurrent. We choose $\varepsilon > 0$ with $B(x_i, 2\varepsilon) \subset U$. Since (x_i, y) is uniformly recurrent, the set

$N_f(x_i, B(x_i, \varepsilon)) \cap N_g(y, V)$ is syndetic, and so there is $M \in \mathbb{N}$ such that it intersects any block of consecutive M integers in \mathbb{N} . Now we choose an open set $W \ni x_i$ such that the diameter of each $f^j(W)$ is smaller than ε for $j = 0, 1, \dots, M$. But $x \in D$, so by the definition of D there is $t \in N_f(x, W) \cap N_f(x_i, W)$. Then there is $j \in \{0, 1, \dots, M\}$ such that $t + j \in N_f(x_i, B(x_i, \varepsilon)) \cap N_g(y, V)$ and additionally $d(f^{t+j}(x), f^{t+j}(x_i)) \leq \varepsilon$, because the diameter of $f^j(W)$ is smaller than ε and both $f^t(x)$ and $f^t(x_i)$ are contained in W . This gives $f^{t+j}(x) \in B(x_i, 2\varepsilon) \subset U$ and therefore $N_f(x, U) \cap N_f(y, V) \neq \emptyset$, which completes the proof. \square

As another application of **Theorem 8**, we have the following result, which was suggested to us by the anonymous referee of the paper.

Corollary 11. *Let $A \subset X$ be a weakly mixing set of order 2 for (X, f) . If additionally distal points are dense in A then A is weakly mixing of all orders.*

Proof. Fix any $n > 2$ and any open sets $U_1, V_1, \dots, U_n, V_n$ intersecting A . It suffices to prove that $\bigcap_{i=1}^n N_f(U_i \cap A, V_i) \neq \emptyset$.

First, by assumptions we may take distal points $x_i \in V_i \cap A$ for each $i = 2, \dots, n$. By the definitions, clearly (x_2, \dots, x_n) is distal for $f \times \dots \times f$, and hence the following set (which is a subset of $\bigcap_{i=2}^n N_f(V_i \cap A, V_i)$):

$$N_{f \times \dots \times f}((x_2, \dots, x_n), V_2 \times \dots \times V_n)$$

has a non-empty intersection with any IP-set (the same arguments as in **Corollary 9** apply). As a consequence of **Theorem 8** we obtain that $N_f(U_1 \cap A, V_1)$ contains an IP-set, thus

$$N_f(U_1 \cap A, V_1) \cap \bigcap_{i=2}^n N_f(V_i \cap A, V_i) \neq \emptyset.$$

Now, let us assume that, for some $1 \leq j < n$ we have

$$\bigcap_{i=1}^j N_f(U_i \cap A, V_i) \cap \bigcap_{l=j+1}^n N_f(V_l \cap A, V_l) \neq \emptyset.$$

Then there is $k > 0$ and open sets U'_1, \dots, U'_n intersecting A such that $f^k(U'_i) \subset V_i$ for each $i = 1, \dots, n$, and $U'_i \subset U_i$ for $1 \leq i \leq j$ and $U'_l \subset V_l$ for $l = j + 1, \dots, n$. Repeating previous arguments we obtain that the following set of integers is non-empty:

$$M \doteq N_f(U_{j+1} \cap A, U'_{j+1}) \cap \bigcap_{i \in \{1, \dots, j, j+2, \dots, n\}} N_f(U'_i \cap A, U'_i) \neq \emptyset.$$

If we take any $m \in M$ then it is easy to check that

$$m + k \in N_f(U_{j+1} \cap A, f^k(U'_{j+1})) \cap \bigcap_{i \in \{1, \dots, j, j+2, \dots, n\}} N_f(U'_i \cap A, f^k(U'_i)),$$

and so

$$\bigcap_{i=1}^{j+1} N_f(U_i \cap A, V_i) \cap \bigcap_{l=j+2}^n N_f(V_l \cap A, V_l) \neq \emptyset.$$

Hence, by induction we eventually obtain that $\bigcap_{i=1}^n N_f(U_i \cap A, V_i) \neq \emptyset$, completing the proof. \square

The following example shows that there are systems fulfilling assumptions of Corollary 10 which cannot be covered by the results of [28].

Example 12. Let R be an irrational rotation of the unit circle \mathbb{S}^1 and let T be the standard tent map on the unit interval. Consider $F = R \times T$ acting on $X = \mathbb{S}^1 \times [0, 1]$. Note that the set $A = \{x\} \times [0, 1]$ is a weakly mixing set for any $x \in \mathbb{S}^1$. To see this, fix any open sets $W_1, \dots, W_n \subset X$ intersecting A . There exist open intervals $V_1, \dots, V_n \subset [0, 1]$ and $U \subset \mathbb{S}^1$ such that $U \times V_i \subset W_i$ and $U \times V_i \cap A \neq \emptyset$ (i.e. $x \in U$) for all $i = 1, \dots, n$. But T is the tent map, so there exists $N > 0$ such that $T^k(V_i) = [0, 1]$ for every $k > N$ and $i = 1, \dots, n$. There also exists $K > N$ such that $R^K(x) \in U$. This shows that $F^K(W_i \cap A) \supset \{R^K(x)\} \times [0, 1]$ and so $F^K(W_i \cap A) \cap W_j \neq \emptyset$ for any $i, j = 1, \dots, n$. Indeed A is a weakly mixing set.

Additionally A contains a dense set of distal points, since any periodic point of T contained in the fiber A will generate a distal point for F . This shows, by Corollary 10, that there are many weakly product recurrent but not product recurrent points for F . But there is not a set in X which can fulfil the regular condition of return times required by the property (P). Simply, if we fix any $z \in X$ then for every open set $U \times V \subset X$ and every $K > 0$ there is $n \in \mathbb{N}$ such that $\pi_1(F^{nK}(z)) = R^{nK}(\pi_1(z)) \notin U$, where π_1 is the projection onto the first coordinate, and so $F^{nK}(z) \notin U \times V$.

The next example (constructed in Theorem 13) shows that assumptions of Corollary 10 cannot be weakened too much. Strictly speaking, it is not enough if a weakly mixing set is contained in the closure of distal or even regular uniformly recurrent points.

First, let us recall some basic facts on Toeplitz flows (a more detailed exposition on properties of Toeplitz flows can be found in [12]). Suppose that X_ω is a Toeplitz flow, that is $X_\omega = \text{Orb}^+(\omega, \sigma)$ for some Toeplitz sequence ω . For $x \in X_\omega$ and an integer $l > 0$, we denote

$$\text{Per}_l(x) = \{n \in \mathbb{N} : x(n) = x(n + kl) \text{ for every } k = 1, 2, \dots\}.$$

As ω is a Toeplitz sequence then $\mathbb{N} = \bigcup_m \text{Per}_m(\omega)$. By an *essential period* of ω we mean any s such that $\text{Per}_s(\omega) \neq \emptyset$ and does not coincide with $\text{Per}_k(\omega)$ for any $k < s$. A period structure of ω is any sequence $\mathbf{s} = \{s_m\}_{m=1}^\infty$ of essential periods such that each s_m divides s_{m+1} and $\mathbb{N} = \bigcup_m \text{Per}_{s_m}(\omega)$. It is known that a periodic structure always exists, and X_ω is an almost 1–1 extension of the odometer (inverse limit with addition (mod s_m) on each coordinate):

$$G_s = \varprojlim_m \{0, 1, \dots, s_m - 1\}$$

which is well defined as the subset consisting of all “paths”

$$(j_1, j_2, \dots) \in \prod_m \{0, 1, \dots, s_m - 1\}$$

such that $j_{m+1} = j_m \pmod{s_m}$ for each $m \in \mathbb{N}$. So we will always assume that a periodic structure is fixed together with the factor map $\pi_\omega: X_\omega \rightarrow G_s$. If $x \in X_\omega$ is not a Toeplitz sequence itself, then it may happen that periodic parts do not cover the whole \mathbb{N} , that is the *aperiodic part* of x

$$\text{Aper}(x) := \mathbb{N} \setminus \bigcup_{m=1}^\infty \text{Per}_{s_m}(x)$$

is non-empty. However it can also be proved that if we fix any $\mathbf{j} \in G_s$ then $\text{Aper}(x) = \text{Aper}(y)$ for every $x, y \in \pi_\omega^{-1}(\mathbf{j})$ (for further properties of the set $\text{Aper}(x)$ we refer again to [12]), in

particular we can write $\text{Aper}(\mathbf{j}) := \text{Aper}(x)$. Assume that $\text{Aper}(x)$ is infinite and enumerate its elements, say $\text{Aper}(x) = \{0 \leq n_1 < n_2 < \dots\}$. Then by the *aperiodic readout* of x we mean the sequence

$$y = x|_{\text{Aper}(x)} = x[n_1]x[n_2]\dots$$

In other words, we read and write down symbols along $\text{Aper}(x)$. For every $\mathbf{j} \in G_s$ with $\#\text{Aper}(\mathbf{j}) = \infty$, let $Y_{\mathbf{j}}$ be the set of all possible aperiodic readouts of elements in $\pi_{\omega}^{-1}(\mathbf{j})$.

Theorem 13. *There is a minimal dynamical system with dense distal points, such that it contains weakly mixing sets and any of its weakly mixing sets does not contain weakly product recurrent points.*

Proof. Let X_{ω} be a Toeplitz flow which is an extension of an odometer G_s via the factor map $\pi_{\omega}: X_{\omega} \rightarrow G_s$, such that $Y_{\mathbf{j}} = \Sigma_2$ for every $\mathbf{j} \in G_s$ with $\#\text{Aper}(\mathbf{j}) = \infty$, where Σ_2 is the two-sided infinite sequence over the alphabet $\{0, 1\}$ (one of the possible constructions of such a flow is explicitly described in [25]). Since X_{ω} is an almost 1–1 extension of an odometer, the set of all distal points is dense in X_{ω} . It can be proved that X_{ω} has positive topological entropy, and so by results of [10,29] it has at least one weakly mixing subset.

Observe that each odometer is equicontinuous and so contains no weakly mixing subsets. Now, if A is a weakly mixing set of X_{ω} then it must be contained in a fiber $\pi_{\omega}^{-1}(\mathbf{j})$ with $\#Y_{\mathbf{j}} = \infty$, which equivalently means that $\#\text{Aper}(\mathbf{j}) = \infty$. But then the aperiodic part of elements of the fiber $\pi_{\omega}^{-1}(\mathbf{j})$ equals Σ_2 . The aperiodic part occurs at the same positions in each element of A and furthermore any two points in the same fiber have the same symbols on the periodic part.

For any $x \in A$ let $y \in \pi_{\omega}^{-1}(\mathbf{j})$ be such that $y \neq x$ and $(x|_{\text{Aper}(\mathbf{j})})_{[1,\infty)} = (y|_{\text{Aper}(\mathbf{j})})_{[1,\infty)}$. In other words, x and y are defined by points in Σ_2 which differ on the first position. It means that $x_{[i,i+N]} = y_{[i,i+N]}$ for every $N \in \mathbb{N}$ and all $i > K$ where $K \in \mathbb{N}$ depends only on $\text{Aper}(\mathbf{j})$ (in particular, it is independent of $N \in \mathbb{N}$). If we fix sufficiently small neighborhoods U, V of x, y , respectively, then x, y cannot return to U and V respectively synchronously (in fact, it is enough that U, V are defined by cylinders of words longer than K). This proves that (x, y) is not recurrent, and so x is not weakly product recurrent, since y is uniformly recurrent as a member of a minimal system. \square

5. Weakly mixing sets and proximality

As shown by results of [29,30] the limit behavior of points in a weakly mixing set can be quite complex. In this section we shall present some further discussions along this line.

We begin this section with the following easy observation.

Lemma 14. *Let A be a weakly mixing set and $x_1, \dots, x_n \in A$. For every sequence U_1, \dots, U_n of open neighborhoods of points x_1, \dots, x_n , respectively, there is a weakly mixing set $B \subset \bigcup_{i=1}^n U_i \cap A$ such that $x_i \in B$ for all $i = 1, \dots, n$.*

Proof. The proof will be finished by a simple construction using the definition. We leave details to the reader. \square

Recall that for $x \in X$, its (positive) limit set $\omega_f(x)$ is defined as the set of all points $y \in X$ such that $\lim_{k \rightarrow \infty} f^{n_k}x = y$ for some increasing sequence $\{n_1 < n_2 < \dots\}$ in \mathbb{N} . The following example shows that points in weakly mixing sets do not have to have good recurrence properties.

Example 15. There is a weakly mixing set A such that $\omega_f(x) \cap A = \emptyset$ for some $x \in A$.

Proof. Take any non-minimal weakly mixing system (X, f) containing a non-recurrent point x with a minimal limit set. A trivial example is the two-sided full shift over the alphabet $\{0, 1\}$. Next, let $U = X \setminus \overline{B(\omega_f(x), \varepsilon)}$ where $0 < \varepsilon < \frac{1}{2} \text{dist}(\omega_f(x), x)$, where as usual $\text{dist}(\omega_f(x), x)$ is the distance between the set $\omega_f(x)$ and the point x . Then $x \in U$ and so it suffices to apply Lemma 14 with $x_1 = x$ and $U_1 = U$ obtaining a weakly mixing set $B \subset U$ such that $x \in B$. But then $\text{dist}(B, \omega_f(x)) > \varepsilon$ and the result follows. \square

By the above example we cannot guarantee that a member of a weakly mixing set A will return to a neighborhood of A . In particular, points obtained by application of Theorem 8 are very special. Despite this difficulty, we still can guarantee some degree of synchronization of trajectories of points in A . In the case of $X = A$ the following lemma follows directly from the definition. In the local case $A \subsetneq X$ we have to perform a more careful approximation of desired points.

Lemma 16. *Let A be a weakly mixing set and $x \in A$. Then for every open set U intersecting A and each $\varepsilon > 0$ there are $n \in \mathbb{N}$ and $y \in U \cap A$ with $d(f^n(x), f^n(y)) \leq \varepsilon$.*

Proof. Fix any $x \in A$ and any $\varepsilon > 0$. Let U be an open set intersecting A and let $\{V_1, \dots, V_k\}$ be a cover of X consisting of open sets with diameters less than ε . We will perform a construction of y and n in a finite number of steps (at most k).

By weak mixing of A there are $i_1 \in \{1, \dots, k\}$, and integer $n_1 > 0$ and an open set $U_1^{(1)} \subset U$ intersecting A such that $f^{n_1}(U_1^{(1)}) \subset V_{i_1}$, where $x \in V_{i_1}$.

Next, assume that for some $m \geq 1$ we have constructed open sets $U_1^{(m)}, \dots, U_m^{(m)} \subset U$ intersecting A , an integer $n_m > 0$ and pairwise distinct integers $i_1, \dots, i_m \in \{1, \dots, k\}$ such that $f^{n_m}(U_j^{(m)}) \subset V_{i_j}$ for each $j = 1, \dots, m$. If $f^{n_m}(x) \notin \bigcup_{j=1}^m V_{i_j}$, then we can choose $i_{m+1} \in \{1, \dots, k\} \setminus \{i_1, \dots, i_m\}$ and an open set $U_{m+1}^{(m)}$ containing x (and so intersecting A) such that $f^{n_m}(U_{m+1}^{(m)}) \subset V_{i_{m+1}}$. By weak mixing of A there are open sets $U_j^{(m+1)} \subset U$ intersecting A and $s > 0$ such that $f^s(U_j^{(m+1)}) \subset U_j^{(m)}$ for each $j = 1, \dots, m + 1$. Now, if we put $n_{m+1} = n_m + s$ then, for each $j = 1, \dots, m + 1$ we obtain that:

$$f^{n_{m+1}}(U_j^{(m+1)}) = f^{n_m}(f^s(U_j^{(m+1)})) \subset f^{n_m}(U_j^{(m)}) \subset V_{i_j}.$$

Obviously at some step $m \leq k$ we cannot extend the sequence i_1, \dots, i_m any further by the above procedure. Hence, we have that $f^{n_m}(x) \in \bigcup_{j=1}^m V_{i_j}$, in particular $f^{n_m}(x) \in V_{i_l}$ for some $l \in \{1, \dots, k\}$. But then by construction $f^{n_m}(U_l^{(m)}) \subset V_{i_l}$, and so if we fix any $y \in U_l^{(m)} \cap A \subset U \cap A$ then $f^{n_m}(y), f^{n_m}(x) \in V_{i_l}$. We have just proved that $d(f^{n_m}(y), f^{n_m}(x)) < \varepsilon$, as the diameter of V_{i_l} is less than ε , which ends the proof. \square

First, it was proved in [24] that for weakly mixing systems the set of points x at which $\text{Prox}(f)(x)$ is residual in X is itself residual in X , that is, for $\text{Prox}(f)$ there is a residual set of parameters where sections are also residual. Later it was proved by Furstenberg in [17] that $\text{Prox}(f)(x)$ is residual for any x , provided that (X, f) is minimal and weakly mixing. Finally, Akin and Kolyada proved in [3] that in a weakly mixing system $\text{Prox}(f)(x)$ is dense for every point $x \in X$ (hence residual, because it is always a G_δ set).

Now, we have enough tools at hand to provide yet another extension of these classical results.

Theorem 17. For every weakly mixing set A and every $x \in A$ the set $\text{Prox}(f)(x) \cap A$ is residual in A .

Proof. Fix any $x \in A$ and $\varepsilon > 0$. By Lemma 16, if we consider the set A_ε consisting of all points $y \in A$ such that $d(f^n(x), f^n(y)) < \varepsilon$ for some $n > 0$, then A_ε is a dense subset of A . But it is also easy to verify that A_ε is an open subset of A . This, by the inclusion

$$\text{Prox}(f)(x) \cap A \supset \bigcap_{n=1}^{\infty} A_{\frac{1}{n}},$$

proves that $\text{Prox}(f)(x)$ is residual in A which completes the proof. \square

Remark 18. Theorem 17 provides another tool that can be used to prove Corollary 10 similarly in the special case of weakly mixing sets. Unfortunately, so far we do not have any evidence that these cases are different.

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