Available online at www.sciencedirect.com
SciVerse ScienceDirect
Advances in Mathematics 244 (2013) 69–90
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

W*-superrigidity of mixing Gaussian actions of rigid groups ☆

Rémi Boutonnet

ENS Lyon, UMPA UMR 5669, 69364 Lyon cedex 7, France

Received 31 October 2012; accepted 13 May 2013
Available online 7 June 2013

Communicated by Dan Voiculescu

Abstract

We generalize W*-superrigidity results about Bernoulli actions of rigid groups to general mixing Gaussian actions. We thus obtain the following: If Γ is any ICC group which is w-rigid (i.e. it contains an infinite normal subgroup with the relative property (T)) then any mixing Gaussian action $\Gamma \curvearrowright X$ is W*-superrigid. More precisely, if $\Lambda \curvearrowright Y$ is another free ergodic action such that the crossed-product von Neumann algebras are isomorphic $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$, then the actions are conjugate. We prove a similar statement whenever Γ is a non-amenable ICC product of two infinite groups.

Keywords: II_1 factors; Deformation/rigidity theory; W*-superrigidity; Gaussian actions

1. Introduction

Most known examples of finite von Neumann algebras are constructed from discrete groups or equivalence relations. Thus the question of understanding which data of the initial group or equivalence relation is remembered in the construction of the associated von Neumann algebra is fundamental if one wants to classify finite von Neumann algebras. This problem is usually very hard, but a dramatic progress has been made possible in the last decade thanks to Sorin Popa's deformation/rigidity theory (see [18,6,22] for surveys).

☆ Research supported by ANR grant NEUMANN.
E-mail address: remi.boutonnet@ens-lyon.fr.

0001-8708/$ - see front matter
http://dx.doi.org/10.1016/j.aim.2013.05.012

The first rigidity result in the framework of group-measure space constructions is Popa's beautiful strong rigidity theorem [13,14]. Assume that $\Gamma \curvearrowright X = X_0^\Gamma$ is a Bernoulli action and that $\Lambda \curvearrowright Y$ is a probability measure preserving (pmp) free ergodic action of an ICC w-rigid group (i.e. which contains an infinite normal subgroup with the relative property (T)). Popa shows in [14] that if the crossed-product von Neumann algebras of these actions are isomorphic, then the actions are conjugate. This is the first result that deduces conjugacy of two actions out of an isomorphism of their crossed product von Neumann algebra.

Later on, Ioana managed to prove a very general W^*-superrigidity result about Bernoulli shifts, which is a natural continuation to Popa's strong rigidity result. For more historical information and results on W^*-superrigidity, see for instance [11,20,7–9] and the introductions therein.

Definition 1.1. A measure preserving free ergodic action σ of a discrete countable group Γ on a standard probability space (X, μ) is said to be *W^*-superrigid* if the associated crossed-product von Neumann algebra M remembers the action, in the following sense. If $\Lambda \curvearrowright^\rho (Y, \nu)$ is a measure preserving action for which the crossed-product von Neumann algebra is isomorphic to M, then Λ is isomorphic to Γ and the actions σ and ρ are conjugate.

Theorem (*Ioana, [8]*). *Let Γ be an ICC w-rigid group. Then the Bernoulli action $\Gamma \curvearrowright^\sigma [0, 1]^\Gamma$ is W^*-superrigid.*

The rigidity of the von Neumann algebra in the theorem above comes from a tension between the property (T) of the group Γ and the deformability of Bernoulli actions. This tension is exploited via Popa's deformation/rigidity strategy. Using a similar strategy of proof, Ioana, Popa and Vaes later proved the W^*-superrigidity of Bernoulli actions for other groups, relying this time on the spectral gap type rigidity discovered by Popa in [19].

Theorem (*Ioana–Popa–Vaes, [9]*). *Let Γ be a non-amenable ICC group which is the product of two infinite groups $\Gamma = \Gamma_1 \times \Gamma_2$. Then the Bernoulli action $\Gamma \curvearrowright^\sigma [0, 1]^\Gamma$ of Γ is W^*-superrigid.*

The proofs of both Ioana's theorem and Ioana–Popa–Vaes' theorem seemed to deeply rely on the very particular structure of Bernoulli actions. We will show that this is not the case, and generalize these results to *Gaussian actions*.

Let Γ be a countable group and $\pi : \Gamma \to \mathcal{O}(H)$ an orthogonal representation of Γ on a real Hilbert space H. Recall that there exist (see [12] for instance) a canonical standard probability space (X, μ) and a pmp action of Γ on X, such that $H \subset L^2(X)$, as representations of Γ. This action is called the *Gaussian action* induced by the representation π.

Ioana's result is then generalized as follows.

Theorem A. *Let Γ be an ICC w-rigid group, and $\pi : \Gamma \to \mathcal{O}(H)$ be any mixing orthogonal representation of Γ. Then the Gaussian action σ_π associated with π is W^*-superrigid.*

However, in order to apply Popa's spectral gap argument, one has to make an extra assumption on the initial representation π. Ioana–Popa–Vaes' theorem then becomes a particular case of the following result.

Theorem B. *Let Γ be a non-amenable ICC group which is the product of two infinite groups, and consider a mixing orthogonal representation $\pi : \Gamma \to \mathcal{O}(H)$ of Γ. Assume that some tensor power of π is weakly contained in the regular representation. Then the Gaussian action σ_π associated with π is W^*-superrigid.*

In [3], we showed that many mixing Gaussian actions are not conjugate to generalized Bernoulli actions. More precisely, we proved in [3, Proposition 2.8] that any Gaussian action arising from a mixing representation which is not weakly contained in the regular representation is not conjugate to a Bernoulli action.

To prove Theorems A and B we will adapt the proof used by Ioana, and Ioana–Popa–Vaes to the context of Gaussian actions. Let us recall the general strategy of their proof.

Steps of the proof in the Bernoulli case. Let Γ be a group as in Theorem A or B and $\Gamma \curvearrowright X = [0,1]^\Gamma$ the corresponding Bernoulli action. Assume that $\Lambda \curvearrowright (Y, \nu)$ is another pmp, free ergodic action such that

$$L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda.$$

Put $A = L^\infty(X)$, $B = L^\infty(Y)$ and $M = A \rtimes \Gamma$.

Thanks to Popa's orbit equivalence superrigidity theorems [17, 5.2 and 5.6] and [19, Theorem 1.3], one only has to show that the two actions are orbit equivalent. More concretely, it is enough to prove, by a result of Feldman and Moore [4], that B is unitarily conjugate to A inside M.

The main idea of the proof, due to Ioana, is to exploit the information given by the isomorphism $M \simeq B \rtimes \Lambda$ via the dual co-action[1]

$$\begin{aligned} \Delta : M &\to M \mathbin{\overline{\otimes}} M \\ bv_s &\mapsto bv_s \otimes v_s, \end{aligned}$$

$b \in B, s \in \Lambda$ ($v_s, s \in \Lambda$, denote the canonical unitaries corresponding to the action of Λ). This morphism Δ allows us to play against each other two data of the single action $\Gamma \curvearrowright X$: the rigidity of $\Delta(L\Gamma)$, and the malleability of the algebra $M \mathbin{\overline{\otimes}} M = (A \mathbin{\overline{\otimes}} A) \rtimes (\Gamma \times \Gamma)$.

Assume that B is not unitarily conjugate to A, or equivalently that $B \nprec_M A$ by [15, Theorem A.1]. We refer to Section 2.1 for the definition of Popa's intertwining symbol "$\prec$". The rest of the proof can be cut into four steps, which leads to a contradiction.

Step (1). One shows that there exists a unitary $u \in M \mathbin{\overline{\otimes}} M$ such that

$$u\Delta(L\Gamma)u^* \subset L\Gamma \mathbin{\overline{\otimes}} L\Gamma.$$

Step (2). One can deduce that the algebra $C := \Delta(A)' \cap (M \mathbin{\overline{\otimes}} M)$ satisfies

$$C \prec_{M \mathbin{\overline{\otimes}} M} A \mathbin{\overline{\otimes}} A.$$

Step (3). The previous steps, and an enhanced version of Popa's conjugacy criterion [14, Theorem 5.2] roughly imply that there exist a unitary $v \in M \mathbin{\overline{\otimes}} M$, a group homomorphism $\delta : \Gamma \to \Gamma \times \Gamma$, and a character $\omega : \Gamma \to \mathbb{C}$ such that

$$v\Delta(C)v^* = A \mathbin{\overline{\otimes}} A \quad \text{and} \quad v\Delta(u_g)v^* = \omega(g)u_{\delta(g)}, \quad \forall g \in \Gamma.$$

Step (4). Using Step (3), one can now show that if a sequence (x_n) in M has Fourier coefficients (with respect to the decomposition $M = A \rtimes \Gamma$) which tend to zero pointwise in norm $\|\cdot\|_2$, then this is also the case of the sequence $\Delta(x_n)$, with respect to the decomposition $M \mathbin{\overline{\otimes}} M = (M \mathbin{\overline{\otimes}} A) \rtimes \Gamma$. This easily contradicts the fact that $B \nprec_M A$.

What has to be adapted. First note that Popa's orbit equivalence superrigidity results ([17, 5.2 and 5.6] and [19, Theorem 1.3]) are still valid for Gaussian actions as in Theorem A or Theorem B. Thus we only have to prove Steps (1)–(4) for such Gaussian actions.

[1] This morphism was also introduced by Popa and Vaes in [20, Lemma 3.2].

Steps (3) and (4) are very general, and will work for any mixing action satisfying the conclusions of steps (1) and (2).

Step (1) is the result of Popa's deformation/rigidity strategy so it should not be specific to Bernoulli shifts. In [9], it was a direct consequence of [9, Corollary 4.3]. Using the results in [3], one can easily get the Gaussian counterpart of [9, Corollary 4.3], namely Corollary 2.8.

Finally, Step (2) relies on a beautiful localization theorem due to Ioana, [8, Theorem 6.1]. That theorem states that if D is an abelian subalgebra of $M \overline{\otimes} M$ which is normalized by "enough" unitaries in $L(\Gamma \times \Gamma)$, then either $D' \cap (M \overline{\otimes} M) \prec A \overline{\otimes} A$, or $D \prec M \overline{\otimes} L\Gamma$, or $D \prec L\Gamma \overline{\otimes} M$. This theorem is applied to $D = \Delta(A)$. Using mixing properties, and the fact that $B \not\prec A$, the last two cases cannot hold and Step (2) follows.

So the point of the whole proof of Theorems A *and* B *is to generalize Ioana's localization theorem* [8, Theorem 6.1]. It will be done in Section 3, Theorem 3.1. We explain below the main difficulties to obtain such a generalization.

Main difficulties in the generalization. Unlike Bernoulli shifts, general mixing Gaussian actions do not satisfy the following properties, which were crucial in Ioana's argument.

- *Cylinder structure*: If $\Gamma \curvearrowright [0,1]^\Gamma$ is a Bernoulli action, we call *finite cylinder subalgebra* a subalgebra of $A = L^\infty([0,1]^\Gamma)$ of the form $A_F = L^\infty([0,1]^F)$, for some finite subset $F \subset \Gamma$. Then the union of all finite cylinder subalgebras is a strongly dense $*$-subalgebra A_0 of A, which is stable under the action of Γ. In fact, A_0 is a graded $\mathbb{C}\Gamma$-module;
- *Finitely supported coefficients*: If $\Gamma \curvearrowright [0,1]^\Gamma$ is a Bernoulli action, there exists a strongly dense $*$-subalgebra A_0 of $L^\infty([0,1]^\Gamma)$ such that for any $a, b \in A_0 \ominus \mathbb{C}$, $\langle \sigma_g(a), b\rangle = 0$ if $g \in \Gamma$ large enough.

The use of the second property above can be avoided using ε-orthogonality and a trick involving convex combinations. To avoid using the cylinder structure, the idea is to replace cylinders by general finite dimensional subsets of $L^\infty(X)$, and use a multiple mixing property automatically enjoyed by mixing Gaussian actions.

Definition 1.2. A trace-preserving action $\Gamma \curvearrowright^\sigma A$ of a countable group on an abelian von Neumann algebra is 2-*mixing* if for any $a, b, c \in A$, the quantity $\tau(a\sigma_g(b)\sigma_h(c))$ tends to $\tau(a)\tau(b)\tau(c)$ as $g, h, g^{-1}h$ tend to infinity.

In fact, every steps of the proof still hold for general s-malleable actions (in the sense of Popa [19]) which are 2-mixing.

Definition 1.3 (*[19]*). A measure preserving action $\Gamma \curvearrowright (X, \mu)$ is said to be s-*malleable* if there exists a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms of $L^\infty(X \times X, \mu \otimes \mu)$, and an automorphism $\beta \in \mathrm{Aut}(L^\infty(X \times X, \mu \otimes \mu))$ such that:

- the map $t \mapsto \alpha_t(x)$ is strongly continuous for any $x \in L^\infty(X \times X)$;
- the automorphisms α_t, $t \in \mathbb{R}$ and β commute with the double action of Γ on $X \times X$;
- $\alpha_1(L^\infty(X) \otimes 1) = 1 \otimes L^\infty(X)$, where we identify $L^\infty(X \times X) \simeq L^\infty(X) \overline{\otimes} L^\infty(X)$;
- for any $t \in \mathbb{R}$, one has $\alpha_t \circ \beta = \beta \circ \alpha_{-t}$;
- β acts trivially on $L^\infty(X) \otimes 1$ and $\beta^2 = \mathrm{id}$.

Such a pair $((\alpha_t)_t, \beta)$ is called an s-*malleable deformation*.

Theorem C. *Let Γ be an ICC group and $\Gamma \curvearrowright^\sigma (X, \mu)$ be a free ergodic action of Γ. Assume that σ is 2-mixing and s-malleable, and that one of the following two conditions holds.*

- *Γ is w-rigid or*
- *Γ is non-amenable and is isomorphic to the product of two infinite groups, and some tensor power of the Koopman representation $\sigma_{|L^2(X)\ominus\mathbb{C}}$ is weakly contained in the regular representation of Γ.*

Then σ is W^*-superrigid.

This result is theoretically satisfying compared to the existing orbit equivalence superrigidity results about malleable actions from [17,19]. However, the only known examples of actions satisfying the assumptions are Gaussian actions. Therefore, in this article we will only focus on the concrete example of Gaussian actions: we will not provide an explicit proof of Theorem C. As we mentioned above, Theorem C can be proved in the same way than Theorems A and B.

An application to group von Neumann algebras. As another application of Theorem 3.1, we construct a large class of II_1 factors which are not stably isomorphic to group von Neumann algebras. These factors are the crossed-product von Neumann algebras of Gaussian actions associated to representations π as in Theorem A or Theorem B, with the extra-assumption that π is not weakly contained in the regular representation.

In [3, Proposition 2.8], such Gaussian actions were shown not to be conjugate to generalized Bernoulli shifts. Using Theorems A and B, we get that the associated factors are not isomorphic to crossed-product factors of Bernoulli actions, and in particular, to von Neumann algebras of certain wreath-product groups. However, showing that such factors are not isomorphic to algebras $L\Lambda$, with no assumptions on the group Λ is much harder, and will require the work of Ioana, Popa and Vaes [9].

Theorem D. *Let Γ be an ICC group and $\pi : \Gamma \to \mathcal{O}(H)$ a mixing orthogonal representation of Γ such that one of the following two conditions holds.*

- *Γ is w-rigid or*
- *Γ is non-amenable and is isomorphic to the product of two infinite groups, and some tensor power of π is weakly contained in the regular representation of Γ.*

Assume moreover that π itself is not contained in a direct sum $\lambda^{\oplus\infty}$ of copies of the left-regular representation. Let $\Gamma \curvearrowright^{\sigma} A$ be the Gaussian action associated to π and put $M = A \rtimes \Gamma$. Then M is not stably isomorphic to a group von Neumann algebra.

By [3, Proposition 2.9], we know that for each $n \geq 3$, $\mathrm{SL}(n, \mathbb{Z})$ admits a representation as in Theorem C. Thus we obtain the existence of a II_1 factor M_n, which is not stably isomorphic to a group von Neumann algebra. But using Theorem 4.1, we get that the M_n's are pairwise non-stably isomorphic : $M_n \ncong (M_m)^t$, $\forall t > 0$, $\forall n \neq m$.

Organization of the article. Apart from the Introduction, this article contains three other sections. Section 2 is the preliminary section, in which we recall Popa's intertwining techniques, and several facts on Gaussian actions that we proved in [3]. Section 3 is devoted to proving Theorem 3.1, which generalizes [8, Theorem 6.1]. It is the technical heart of the article. Finally, we prove in Section 4 Theorems A, B, D stated above.

2. Preliminaries

2.1. Intertwining by bimodules

We recall here an essential tool introduced by Popa, the so-called intertwining by bimodules' lemma.

Theorem 2.1 (*Popa, [13,16]*). *Let $P, Q \subset M$ be finite von Neumann algebras (with possibly non-unital inclusions). Then the following are equivalent.*

- *There exist projections $p \in P, q \in Q$, a normal $*$-homomorphism $\psi : pPp \to qQq$, and a non-zero partial isometry $v \in pMq$ such that $xv = v\psi(x)$, for all $x \in pPp$;*
- *There exists a P–Q subbimodule H of $L^2(1_P M 1_Q)$ which has finite index when regarded as a right Q-module;*
- *There is no sequence of unitaries $(u_n) \in \mathcal{U}(P)$ such that for all $x, y \in M$,*

$$\|E_Q(1_Q x^* u_n y 1_Q)\|_2 \to 0.$$

Following [13], if $P, Q \subset M$ satisfy these conditions, we say that *a corner of P embeds into Q inside M*, and we write $P \prec_M Q$.

Assume that we are in the concrete situation where M is of the form $M = B \rtimes \Gamma$ for some trace preserving action of Γ on a finite von Neumann algebra and $Q = B$. Denote by u_g, $g \in \Gamma$ the canonical unitaries in M implementing the action of Γ. Then it is easy to check that a subalgebra $P \subset M$ satisfies $P \nprec B$ if and only if there exists a sequence of unitaries $v_n \in \mathcal{U}(P)$ such that

$$\|E_B(v_n u_g^*)\|_2 \to 0, \quad \forall g \in \Gamma.$$

This result can be improved as follows.

Lemma 2.2 (*Ioana, [8, Theorem 1.3.2]*). *Let $\Gamma \curvearrowright B$ be a trace preserving action on a finite von Neumann algebra (B, τ). Put $M = B \rtimes \Gamma$, and let $P \subset M$ be a von Neumann subalgebra. Then $P \nprec B$ if and only if there exists a sequence of unitaries $v_n \in \mathcal{U}(P)$ such that*

$$\lim_n \left(\sup_{g \in \Gamma} \|E_B(v_n u_g^*)\|_2 \right) = 0.$$

One may ponder another natural question: What does it mean to embed into the group algebra $L\Gamma$ inside a crossed-product algebra $M = A \rtimes \Gamma$? In some specific circumstances, this implies the unitary conjugacy into $L\Gamma$, as the following standard result shows.

We denote by $\mathcal{Q} = \{u \in \mathcal{U}(M),\ uQu^* = Q\}$ the *normalizer* of a subalgebra Q of a von Neumann algebra M. The *quasi-normalizer* $\mathcal{QN}_M(Q)$ of Q in M is the *-subalgebra of M formed by Q–Q finite elements. We recall that an element $x \in M$ is Q–Q finite if there exist $x_1, \ldots, x_k \in M$ such that

$$xQ \subset \sum_{i=1}^k Qx_i \quad \text{and} \quad Qx \subset \sum_{i=1}^k x_i Q.$$

Proposition 2.3. *Let $\Gamma \curvearrowright A$ be a free mixing action of an ICC group Γ on an abelian von Neumann algebra, and let N be a type II_1 factor. Put $M = (A \rtimes \Gamma) \overline{\otimes} N$, and assume that $Q \subset pMp$ is a von Neumann subalgebra such that $Q \nprec_M 1 \otimes N$. Put $P = \mathcal{QN}_{pMp}(Q)''$.*

(1) *If $Q \prec_M L\Gamma \overline{\otimes} N$ then there exists a non-zero partial isometry $v \in pM$ such that $vv^* \in \mathcal{Z}(P)$ and $v^* P v \subset L\Gamma \overline{\otimes} N$.*

(2) *If $rQ \prec_M L\Gamma \overline{\otimes} N$ for all $r \in Q' \cap pMp$ then there exists a unitary $u \in \mathcal{U}(M)$ such that $uPu^* \subset L\Gamma \overline{\otimes} N$.*

Proof. (1) By assumption, there exist projections $p_0 \in Q, q \in L\Gamma \overline{\otimes} N$, a non-zero partial isometry $v \in p_0 M q$ and a *-homomorphism $\varphi : p_0 Q p_0 \to q(L\Gamma \overline{\otimes} N)q$ such that for all $x \in p_0 Q p_0$, one has $xv = v\varphi(x)$.

By [21, Remark 3.8], one can assume that $\varphi(p_0Qp_0) \not\prec_M 1 \otimes N$. Hence [13, Theorem 3.1] implies that $\mathcal{QN}_{qMq}(\varphi(p_0Qp_0))'' \subset L\Gamma \overline{\otimes} N$. But we see that $v^*Pv \subset \mathcal{QN}_{qMq}(\varphi(p_0Qp_0))''$. Moreover $vv^* \in p_0(Q' \cap M) \subset P$. However vv^* is not necessarily in $\mathcal{Z}(P)$ but one can modify v as follows to obtain such a condition.

Take partial isometries $v_1, \ldots, v_k \in P$ such that $v_i^*v_i \leq vv^*$, $i = 1, \ldots, k$ and $\sum_{i=1}^k v_iv_i^*$ is a central projection in P. Since $L\Gamma \overline{\otimes} N$ is a factor, there exist partial isometries $w_1, \ldots, w_k \in L\Gamma \overline{\otimes} N$ such that $w_iw_i^* = v^*v_i^*v_iv$ and $w_iw_j^* = 0$, for all $1 \leq i \neq j \leq k$. Define a non-zero partial isometry by $w = \sum_i v_ivw_i \in pM$. We get

- $ww^* = \sum_i v_ivw_iw_i^*v^*v_i^* = \sum_i v_iv_i^* \in \mathcal{Z}(P)$;
- $w^*Pw \subset \sum_i w_i^*v^*Pvw_i \subset L\Gamma \overline{\otimes} N$.

(2) Consider a maximal projection $r_0 \in Q' \cap pMp$ for which there exists a unitary $u \in \mathcal{U}(M)$ such that $u(r_0Pr_0)u^* \subset L\Gamma \overline{\otimes} N$. One has to show that $r_0 = p$. Otherwise we can cut by $r = p - r_0$, and we obtain an algebra $rQ \subset rMr$ such that $rQ \prec_M L\Gamma \overline{\otimes} N$ and $rQ \not\prec_M 1 \otimes N$. Remark that $rPr \subset \mathcal{QN}_{rMr}(rQ)''$. Applying (1), we get that there exists a non-zero partial isometry $v \in rM$, such that $vv^* \in (rPr)' \cap rMr \subset (rQ)' \cap rMr$ and $v^*(rPr)v \subset L\Gamma \overline{\otimes} N$.

Since $L\Gamma \overline{\otimes} N$ is a factor, modifying v if necessary, one can assume that $v^*v \perp ur_0u^*$. Now the following "cutting and pasting" argument contradicts the maximality of r_0. The partial isometry $w_0 = ur_0 + v^*$ satisfies $w_0^*w_0 = r_0 + vv^* \in Q' \cap pMp$ and $w_0(r_0 + vv^*)Qw_0^* \subset L\Gamma$. Extending w_0 into a unitary, we obtain a $w \in \mathcal{U}(M)$ satisfying $w(r_0 + vv^*)Qw^* \subset L\Gamma$. □

2.2. *Gaussian actions*

To any orthogonal representation $\pi : \Gamma \to \mathcal{O}(H)$ of a discrete countable group, one can associate a trace preserving action $\sigma_\pi : \Gamma \curvearrowright A$ on an abelian von Neumann algebra, called the Gaussian action associated to π. This Gaussian action can be constructed as follows. For more explicit constructions, see [1, Appendix A.7] or [12]. Consider the unique abelian tracial von Neumann algebra (A, τ) generated by unitaries $(w(\xi))_{\xi \in H_{\mathbb{R}}}$ such that:

- $w(0) = 1$ and $w(\xi + \eta) = w(\xi)w(\eta)$, $w(\xi)^* = w(-\xi)$, for all $\xi, \eta \in H_{\mathbb{R}}$;
- $\tau(w(\xi)) = \exp(-\|\xi\|^2)$, for all $\xi \in H_{\mathbb{R}}$.

It is easy to check that these conditions imply that the vectors $(w(\xi))_{\xi \in H_{\mathbb{R}}}$ are linearly independent and span a weakly dense *-subalgebra of A. Then the Gaussian action σ_π is defined by $(\sigma_\pi)_g(w(\xi)) = w(\pi(g)\xi)$, for all $g \in \Gamma$, $\xi \in H$.

As explained in [5] or [12], Gaussian actions are s-malleable (Definition 1.3): the rotation operators θ_t, $t \in \mathbb{R}$ on $H \oplus H$ and the symmetry ρ defined by

$$\theta_t = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}, \quad \text{and} \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

give rise to a one-parameter group of automorphisms α_t and an automorphism β of $A \overline{\otimes} A$, which are easily seen to satisfy the conditions of Definition 1.3.

Now consider the von Neumann algebras $M = A \rtimes \Gamma$ and $\tilde{M} = (A \overline{\otimes} A) \rtimes_{\sigma \otimes \sigma} \Gamma$. View M as a subalgebra of $\tilde{M}$ using the identification $M \simeq (A \overline{\otimes} 1) \rtimes \Gamma$. The automorphisms defined above then extend to automorphisms of $\tilde{M}$ still denoted (α_t) and β, in such a way that $\alpha_t(u_g) = \beta(u_g) = u_g$, for all $g \in \Gamma$.

Since these automorphisms come from an s-malleable deformation, they satisfy the so-called transversality property.

Lemma 2.4 (*[19, Lemma 2.1]*). *For any $x \in M$ and $t \in \mathbb{R}$ one has*

$$\|x - \alpha_{2t}(x)\|_2 \leq 2\|\alpha_t(x) - E_M \circ \alpha_t(x)\|_2.$$

With more conditions on the representation π, we also get the spectral gap property.

Lemma 2.5 (*Spectral Gap, [3]*). *Denote by λ the left regular representation of Γ. Assume that the representation π is such that $\pi^{\otimes l} \prec \lambda$ for some $l \geq 1$. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$.*

Then for every von Neumann subalgebra $Q \subset M$ with no amenable direct summand, one has $Q' \cap \tilde{M}^\omega \subset M^\omega$.

In fact this lemma admits a relative version.

Recall from [10] that if (M, τ) is a finite von Neumann algebra, $p \in M$ a projection, and $Q \subset M$ and $P \subset pMp$ are subalgebras, one says that P is *amenable relative to Q* inside M if there exists a P-central state φ on $p\langle M, e_Q\rangle p$ such that $\varphi(pxp) = \tau(pxp)/\tau(p)$ for any $x \in M$. Here $\langle M, e_Q\rangle$ denotes Jones' basic construction associated to the inclusion $Q \subset M$. Following [9, Section 2.4], P is said to be *strongly non-amenable relative to Q* if for all non-zero projection $p_1 \in P' \cap pMp$, Pp_1 is not amenable relative to Q.

Lemma 2.6. *Denote by λ the left regular representation of Γ. Assume that the representation π is such that $\pi^{\otimes l} \prec \lambda$ for some $l \geq 1$. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$.*

Then for any finite von Neumann algebra N and any von Neumann subalgebra $Q \subset M \overline{\otimes} N$ which is strongly non-amenable relative to $1 \otimes N$, one has $Q' \cap (\tilde{M} \overline{\otimes} N)^\omega \subset Q' \cap (M \overline{\otimes} N)^\omega$.

2.3. Deformation/rigidity results on Gaussian actions

We mention here different versions of statements that we proved in [3] using deformation/rigidity arguments.

The following result is a variation of [3, Theorem 3.4], with a formulation closer to [9, Theorem 4.2].

Theorem 2.7. *Assume that $\Gamma \curvearrowright A$ is the Gaussian action associated to a mixing representation of an ICC group Γ. Let N be a II_1 factor. Put $M = A \rtimes \Gamma$ and define (α_t) as in Section 2.2.*

Let $p \in M \overline{\otimes} N$, and $Q \subset p(M \overline{\otimes} N)p$ be a von Neumann subalgebra such that there exist $t_0 = 1/2^n$, $z \in \tilde{M} \overline{\otimes} N$ and $c > 0$ satisfying

$$|\tau((\alpha_{t_0} \otimes \mathrm{id})(u^*)zu)| \geq c, \quad \text{for all } u \in \mathcal{U}(Q).$$

Put $P = \mathcal{QN}_{p(M \overline{\otimes} N)p}(Q)''$. Then at least one of the following assertions occurs.

(1) *$Q \prec 1 \otimes N$;*
(2) *$P \prec A \overline{\otimes} N$;*
(3) *There exists a non-zero partial isometry $v \in pM$ such that $vv^* \in \mathcal{Z}(P)$, and $v^*Pv \subset L\Gamma \overline{\otimes} N$.*

Proof. Assume that (2) is not satisfied. Using the fact that π is mixing, the same proof as the one of [3, Theorem 3.4] gives that $Q \prec L\Gamma \overline{\otimes} N$. Now if $Q \nprec 1 \otimes N$, Proposition 2.3(1) implies that (3) holds true. $\square$

Now one can deduce the Gaussian version of [9, Corollary 4.3], which implies Step (1) of the proof of Theorems A and B.

Corollary 2.8. *Assume that Γ is ICC and let $\Gamma \curvearrowright A$ be a mixing Gaussian action. Put $M = A \rtimes \Gamma$. Let N be a II_1 factor and $Q \subset p(M \overline{\otimes} N)p$ a von Neumann subalgebra, for some $p \in M \overline{\otimes} N$. Assume that we are in one of the following situations:*

- *$Q \subset p(M \overline{\otimes} N)p$ has the relative property (T);*
- *$Q' \cap p(M \overline{\otimes} N)p$ is strongly non-amenable relative to $1 \otimes N$ (see end of Section 2.2), and some tensor power of π is weakly contained in the regular representation of Γ.*

Denote by $P = \mathcal{QN}_{p(M \overline{\otimes} N)p}(Q)''$. Then one of the following assertions is true.

(1) *$Q \prec 1 \otimes N$;*
(2) *$P \prec A \overline{\otimes} N$;*
(3) *There exists a unitary $v \in M \overline{\otimes} N$ such that $v^* P v \subset L\Gamma \overline{\otimes} N$.*

Proof. The assumptions imply that the deformation $\alpha_t \otimes \mathrm{id}$ converges to identity uniformly on Q. Indeed, if $Q \subset p(M \overline{\otimes} N)p$ has relative property (T), this is almost by definition. If $Q' \cap p(M \overline{\otimes} N)p$ is strongly non-amenable relative to $1 \otimes N$, then this is a consequence of spectral gap Lemma 2.6, and transversality property 2.4 (see the proof of [19, Lemma 5.2]).

Hence, for all $r \in Q' \cap pMp$, the subalgebra $rQ \subset rMr$ satisfies the assumptions of Theorem 2.7. Then if $Q \not\prec 1 \otimes N$ and $P \not\prec A \overline{\otimes} N$, Theorem 2.7 applied to all such rQ's implies in particular that for all $r \in Q' \cap pMp$, $rQ \prec L\Gamma \overline{\otimes} N$. Now (3) follows from Proposition 2.3(2). $\square$

In [3, Theorem 3.8], we also obtained a localization result for subalgebras of M that commute inside M^ω with rigid subalgebras of M^ω, for some free ultrafilter ω on $\mathbb{N}$. In fact, the same proof leads to the following improvement. We include a sketch of the proof for convenience.

Theorem 2.9. *Let $\Gamma \curvearrowright A$ be a mixing Gaussian action. Put $M = A \rtimes \Gamma$ and consider a II_1 factor N. Assume that (v_n) is a bounded sequence of elements in $M \overline{\otimes} N$ such that $\alpha_t \otimes \mathrm{id}$ converges to identity uniformly on the set $\{v_n, n \in \mathbb{N}\}$. Choose a free ultrafilter ω on $\mathbb{N}$, and denote by $D \subset M \overline{\otimes} N \subset (M \overline{\otimes} N)^\omega$ the subalgebra of elements that commute with the element $(v_n)_n \in (M \overline{\otimes} N)^\omega$. Put $P = \mathcal{QN}_{M \overline{\otimes} N}(D)''$.*

Then one of the following is true.

(1) *$(v_n)_n \in (A \overline{\otimes} N)^\omega \rtimes \Gamma$;*
(2) *$D \prec L\Gamma \overline{\otimes} N$;*
(3) *$P \prec_M A \overline{\otimes} N$.*

Sketch of proof. Assume that $(v_n)_n \notin (A^\omega \rtimes \Gamma) \overline{\otimes} N^\omega$. We will show that the D satisfies the assumptions of Theorem 2.7.

Define $x = (x_n) = (v_n) - E_{(A \overline{\otimes} N)^\omega \rtimes \Gamma}((v_n)) \neq 0$. Dividing x if necessary by $\|x\|_2$, one can assume that $\|x\|_2 \leq 1$. For $F \subset \Gamma$ finite, denote by $P_F : L^2(M) \to L^2(M)$ the projection onto the closed linear span of elements of the form xu_g, $x \in A$, $g \in F$. One checks that:

- $\alpha_t \otimes \mathrm{id}$ converges to identity uniformly on $\{x_n, n \in \mathbb{N}\}$;
- $\lim_{n \to \omega} \|[x_n, d]\|_2 \to 0$, for any $d \in D$;
- $\lim_{n \to \omega} \|(P_F \otimes \mathrm{id})(x_n)\|_2 \to 0$, for any finite subset $F \subset \Gamma$.

Using [24, Lemma 3.8], one can show that this last condition implies that

$$\lim_{n\to\omega} \langle x_n \xi x_n^*, \eta\rangle = 0, \quad \forall \xi, \eta \in (L^2(\tilde{M}) \ominus L^2(M)) \otimes L^2(N). \tag{2.1}$$

Fix $\varepsilon > 0$. Then there exists a $t = 1/2^k$ such that $\|(\alpha_t \otimes \mathrm{id})(x_n) - x_n\|_2 < \varepsilon, \forall n$.

Fix $u \in \mathcal{U}(D)$ and put $\delta_t(u) = (\alpha_t \otimes \mathrm{id})(u) - E_{M \overline{\otimes} N}((\alpha_t \otimes \mathrm{id})(u))$. Then note that $\delta_t(u) \in (L^2(\tilde{M}) \ominus L^2(M)) \otimes L^2(N)$, and that for all n,

$$\begin{aligned}\|\delta_t(u)x_n - \delta_t(ux_n)\|_2 &= \|(1 - E_{M \overline{\otimes} N})((\alpha_t \otimes \mathrm{id})(u)x_n - (\alpha_t \otimes \mathrm{id})(ux_n))\|_2 \\ &\leq \|(\alpha_t \otimes \mathrm{id})(u)x_n - (\alpha_t \otimes \mathrm{id})(ux_n)\|_2 \\ &\leq \|x_n - (\alpha_t \otimes \mathrm{id})(x_n)\|_2 < \varepsilon.\end{aligned}$$

Hence we get

$$\begin{aligned}\lim_{n\to\omega} \|\delta_t(u)x_n\|_2^2 &\leq \lim_{n\to\omega} \langle \delta_t(ux_n), \delta_t(u)x_n\rangle + \varepsilon \\ &= \lim_{n\to\omega} \langle \delta_t(x_n u), \delta_t(u)x_n\rangle + \varepsilon \\ &\leq \lim_{n\to\omega} \langle x_n \delta_t(u) x_n^*, \delta_t(u)\rangle + 2\varepsilon.\end{aligned}$$

Combining this last inequality with (2.1), we obtain

$$\lim_{n\to\omega} \|\delta_t(u)x_n\|_2^2 \leq 2\varepsilon. \tag{2.2}$$

But exactly as in the proof of Popa's transversality lemma [19, Lemma 2.1], one shows that for all $n \in \mathbb{N}$,

$$\begin{aligned}\|(\alpha_{2t} \otimes \mathrm{id})(u)x_n - ux_n\|_2 &\leq \|(\alpha_t \otimes \mathrm{id})(u)x_n - (\alpha_{-t} \otimes \mathrm{id})(u)x_n\|_2 + 2\varepsilon \\ &\leq \|\delta_t(u)x_n\|_2 + \|E_{M \overline{\otimes} N}((\alpha_t \otimes \mathrm{id})(u))x_n \\ &\quad - (\alpha_{-t} \otimes \mathrm{id})(u)x_n\|_2 + 2\varepsilon \\ &= 2\|\delta_t(u)x_n\|_2 + 2\varepsilon,\end{aligned} \tag{2.3}$$

where the last equality is obtained by applying the relation $\|y\|_2 = \|(\beta \otimes \mathrm{id})(y)\|_2$ to the element $y = E_{M \overline{\otimes} N}((\alpha_t \otimes \mathrm{id})(u))x_n - (\alpha_{-t} \otimes \mathrm{id})(u)x_n$, and using the properties of β (see Definition 1.3).

Hence, if $\varepsilon < 1$, Eqs. (2.2) and (2.3) imply that

$$\lim_{n\to\omega} \|(\alpha_{2t} \otimes \mathrm{id})(u)x_n - ux_n\|_2 < 6\sqrt{\varepsilon}.$$

Put $z = E_{M \overline{\otimes} N}((x_n x_n^*)_n)$. We have

$$2 \lim_{n\to\omega} \|x_n\|_2^2 - 2\Re(\tau((\alpha_{2t} \otimes \mathrm{id})(u^*)zu)) = \lim_{n\to\omega} \|(\alpha_{2t} \otimes \mathrm{id})(u)x_n - ux_n\|_2^2 < 36\varepsilon.$$

If ε was chosen to be small enough, this implies the result. $\square$

Corollary 2.10. *For $i = 1, 2$, consider mixing Gaussian actions $\Gamma_i \curvearrowright A_i$ and put $M_i = A_i \rtimes \Gamma_i$, $A = A_1 \overline{\otimes} A_2$, $\Gamma = \Gamma_1 \times \Gamma_2$ and $M = M_1 \overline{\otimes} M_2 = A \rtimes \Gamma$.*

For $i = 1, 2$, define $\tilde{M}_i$ and (α_t^i), as in Section 2.2, and denote by $\tilde{M} = \tilde{M}_1 \overline{\otimes} \tilde{M}_2$ equipped with the deformation $(\alpha_t) = (\alpha_t^1 \otimes \alpha_t^2)$.

Assume that (v_n) is a bounded sequence of elements in M such that α_t converges uniformly to the identity on the set $\{v_n, n \in \mathbb{N}\}$. Choose a free ultrafilter ω on $\mathbb{N}$, and denote $D \subset M \subset M^\omega$ the subalgebra of elements that commute with the element $(v_n)_n \in M^\omega$. Put $P = \mathcal{QN}_M(D)''$.

Then one of the following is true.

(1) $(v_n)_n \in A^\omega \rtimes \Gamma$;
(2) $D \prec_M L\Gamma_1 \overline{\otimes} M_2$ *or* $D \prec_M M_1 \overline{\otimes} L\Gamma_2$;
(3) $P \prec_M A_1 \overline{\otimes} M_2$ *or* $P \prec_M M_1 \overline{\otimes} A_2$.

Proof. Exactly as in the proof of [8, Theorem 3.2, Claim 2], we get that if α_t converges uniformly on $\{v_n, n \in \mathbb{N}\}$, then so do $\alpha_t^1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes \alpha_t^2$. Thus if (2) and (3) are not satisfied, Theorem 2.9 implies that $(v_n) \in (A_1^\omega \rtimes \Gamma_1) \overline{\otimes} (A_2^\omega \rtimes \Gamma_2) = A^\omega \rtimes \Gamma$. □

2.4. 2-mixing property

Definition 2.11. A trace-preserving action $\Gamma \curvearrowright^\sigma A$ of a countable group on an abelian von Neumann algebra is said to be 2-*mixing* if for any $a, b, c \in A$, the quantity $\tau(a\sigma_g(b)\sigma_h(c))$ tends to $\tau(a)\tau(b)\tau(c)$ as $g, h, g^{-1}h$ tend to infinity.

Proposition 2.12. *An action $\Gamma \curvearrowright^\sigma A$ is* 2*-mixing if and only if for all $a, b, c \in A$, one has*

$$|\tau(a\sigma_g(b)\sigma_h(c)) - \tau(a)\tau(\sigma_g(b)\sigma_h(c))| \to 0,$$

when $g \to \infty, h \to \infty$.

Proof. The *if* part is straightforward. For the converse, assume that σ is 2-mixing. It is sufficient to show that if $a, b, c \in A$, with $\tau(a) = 0$, then $\tau(a\sigma_g(b)\sigma_h(c)) \to 0$, as $g, h \to \infty$.

Assume by contradiction that there exist sequences $g_n, h_n \in \Gamma$ going to infinity, and $\delta > 0$ such that $|\tau(a\sigma_{g_n}(b)\sigma_{h_n}(c))| \geq \delta$, for all n. Then two cases are possible:

Case 1. The sequence $g_n^{-1}h_n$ is contained in a finite set. Then taking a subsequence if necessary, one can assume that $g_n^{-1}h_n = k$ is constant. Then for all n, we get

$$\tau(a\sigma_{g_n}(b)\sigma_{h_n}(c)) = \tau(a\sigma_{g_n}(b)\sigma_k(c)).$$

But since σ is mixing this quantity tends to 0 as n tends to infinity.

Case 2. The sequence $g_n^{-1}h_n$ is not contained in a finite set. Then taking a subsequence if necessary, one can assume that $g_n^{-1}h_n \to \infty$ when $n \to \infty$. Then the 2-mixing implies that $\tau(a\sigma_{g_n}(b)\sigma_{h_n}(c)) \to 0$.

In both cases, we get a contradiction. □

Of course any 2-mixing action is mixing. The converse holds for Gaussian actions.

Proposition 2.13. *If $\Gamma \curvearrowright^\sigma A$ is the Gaussian action associated to a mixing representation π on H, then σ is* 2*-mixing.*

Proof. By a linearity/density argument, it is enough to prove that for all $\xi, \eta, \delta \in H$, and all sequences $g_n, h_n \in \Gamma$ tending to infinity, one has

$$\lim_n \left[\tau(\omega(\xi)\sigma_{g_n}(\omega(\eta))\sigma_{h_n}(\omega(\delta))) - \tau(\omega(\xi))\tau(\sigma_{g_n}(\omega(\eta))\sigma_{h_n}(\omega(\delta)))\right] = 0.$$

But one checks that:

- $\tau(\omega(\xi)\sigma_{g_n}(\omega(\eta))\sigma_{h_n}(\omega(\delta))) = \exp(-\|\xi + \pi(g_n)\eta + \pi(h_n)\delta\|^2)$;
- $\tau(\omega(\xi))\tau(\sigma_{g_n}(\omega(\eta))\sigma_{h_n}(\omega(\delta))) = \exp(-\|\xi\|^2 - \|\pi(g_n)\eta + \pi(h_n)\delta\|^2)$.

The difference is easily seen to tend to 0. □

3. The key step

We now state the key theorem from which Theorems A and B follow as explained in the Introduction.

Theorem 3.1. *For $i = 1, 2$, consider mixing Gaussian actions $\Gamma_i \curvearrowright A_i$ of discrete countable groups Γ_i, and put $M_i = A_i \rtimes \Gamma_i$, $A = A_1 \overline{\otimes} A_2$, $\Gamma = \Gamma_1 \times \Gamma_2$ and*

$$M = M_1 \overline{\otimes} M_2 = A \rtimes \Gamma.$$

Let $t > 0$. Realize $(L\Gamma)^t \subset M^t$ by fixing an integer $n \geq t$ and a projection $p \in L\Gamma \otimes M_n(\mathbb{C})$ with trace t/n. Let $D \subset M^t$ be an abelian von Neumann subalgebra, and denote by Λ'' the von Neumann algebra generated by the group of unitaries $\Lambda = \mathcal{N}_{M^t}(D) \cap \mathcal{U}((L\Gamma)^t)$. Make the following assumptions:

(i) *$\Lambda'' \nprec_M L\Gamma_1 \otimes 1$ and $\Lambda'' \nprec_M 1 \otimes L\Gamma_2$;*
(ii) *$D \nprec L\Gamma_1 \otimes M_2$ and $D \nprec_M M_1 \otimes L\Gamma_2$.*

Denote by $C = D' \cap M^t$. Then for all projections $q \in \mathcal{Z}(C)$, $Cq \prec_M A$.

Proof. Exactly as in the proof of [9, Theorem 5.1], we first show that it is sufficient to prove that $C \prec_M A$. Indeed, assume that we have shown that the assumptions of the theorem imply that $C \prec_M A$.

Consider the set of projections

$$\mathcal{P} = \{q_1 \in \mathcal{Z}(C) \mid Cq \prec_M A, \text{ for all non-zero subprojections } q \in \mathcal{Z}(C)q_1\}.$$

Then $\mathcal{P}$ admits a unique maximal element $p_1 \in \mathcal{Z}(C)$. By uniqueness, p_1 commutes with the normalizer of C, and in particular with Λ''. Using [24, Lemma 3.8] and assumption (i), we get that $p_1 \in (L\Gamma)^t$. We want to show that $p_1 = p(= 1_C)$. Otherwise, we can cut by $p - p_1$ and we see that $(p - p_1)D \subset (p - p_1)(M \otimes M_n(\mathbb{C}))(p - p_1)$ satisfies the assumptions of the theorem. Thus $(p - p_1)C \prec_M A$. This contradicts the maximality of p_1.

So the rest of the proof is devoted to showing that $C \prec_M A$. As in the proof of [9, Theorem 5.1], we assume that $t \leq 1$, so that n can be chosen to be equal to 1. This assumption largely simplifies notations, and does not hide any essential part of the proof.

Note that the assumption (i) implies that there exists a sequence of unitary elements $v_n \in \mathcal{U}(pL\Gamma p)$ that normalize D and such that

$$\|E_{L\Gamma_1 \otimes 1}(av_n b)\|_2 \to 0 \quad \text{and} \quad \|E_{1 \otimes L\Gamma_2}(av_n b)\|_2 \to 0, \quad \forall a, b \in M. \tag{3.1}$$

We will proceed in two steps to prove that $C \prec_M A$. In a first step we collect properties regarding the sequence (v_n) or sequences of the form $(v_n a v_n^*)$, $a \in D$. In the second step we show the result, reasoning by contradiction. Before moving on to these two steps, we introduce some notations:

- We denote by u_g, $g \in \Gamma$ the canonical unitaries in M implementing the action of Γ;
- For any element $x \in M$, we denote by $x = \sum_{g \in \Gamma} x_g u_g$ ($x_g \in A$ for all $g \in \Gamma$) its Fourier decomposition.
- If $S \subset \Gamma$ is any subset, denote by $P_S : L^2(M) \to L^2(M)$ the projection onto the closed linear span of the vectors au_g, $a \in A$, $g \in S$.
- If $K \subset A$ is a closed subspace, we denote by $Q_K : L^2(M) \to L^2(M)$ the projection onto the closed linear span of the vectors au_g, $a \in K$, $g \in \Gamma$.

Warning for the sequel: $g, h \in \Gamma$ does not mean $(g, h) \in \Gamma$!

Step 1: *Properties of the sequences* $(v_n a v_n^*)$, $a \in D$.

Lemma 3.2. *For any free ultrafilter ω on $\mathbb{N}$, and any $a \in D$, the element $(v_n a v_n^*)_n \in M^\omega$ belongs to $A^\omega \rtimes \Gamma$.*

Proof. We will apply Corollary 2.10. Fix $a \in D$. Since the v_n's are in $L\Gamma$, the deformation α_t introduced in the statement of Corollary 2.10 converges uniformly on the set $\{v_n a v_n^*, n \in \mathbb{N}\}$. Thus Corollary 2.10 implies that one of the following holds true:

- $(v_n a v_n^*)_n \in A^\omega \rtimes \Gamma$;
- $D \prec_M L\Gamma_1 \overline{\otimes} M_2$ or $D \prec_M M_1 \overline{\otimes} L\Gamma_2$;
- $P \prec_M A_1 \overline{\otimes} M_2$ or $P \prec_M M_1 \overline{\otimes} A_2$, where $P = \mathcal{N}_{pMp}(D)''$.

The second case is excluded by assumption, so we are left to show that the third case is not possible. By symmetry, it is sufficient to show that $P \nprec_M A_1 \overline{\otimes} M_2$. But we claim that for all $x, y \in M$, $\|E_{A_1 \overline{\otimes} M_2}(x v_n y)\|_2 \to 0$. Since $v_n \in \mathcal{U}(P)$, this claim implies the result.

By Kaplansky's density theorem, and by linearity, it is sufficient to prove the claim for x and y of the form $u_g \otimes 1$, $g \in \Gamma_1$. In particular $x v_n y$ lies in $L\Gamma$. So using the fact that

$$\begin{array}{ccc} A_1 \overline{\otimes} M_2 & \subset & M \\ \cup & & \cup \\ 1 \otimes L\Gamma_2 & \subset & L\Gamma \end{array}$$

is a commuting square, (3.1) directly implies that $\|E_{A_1 \overline{\otimes} M_2}(x v_n y)\|_2 \to 0$. $\square$

For an element $x \in M = L\Gamma$, denote by $h(x)$ the height of x: $h(x) = \sup_{g \in \Gamma} |x_g|$, where $x = \sum x_g u_g$ is the Fourier decomposition of x.

Lemma 3.3. *There exists $\delta > 0$ such that $h(v_n) > \delta$ for all n.*

Proof. Assume that the result is false. Taking a subsequence if necessary, we get that $h(v_n) \to 0$. Then we claim that for all finite subset $S \subset \Gamma$, and all $a \in (M \ominus (L\Gamma_1 \overline{\otimes} M_2)) \cap (M \ominus (M_1 \overline{\otimes} L\Gamma_2))$,

$$\lim_n \|P_S(v_n a v_n^*)\|_2 = 0.$$

Note that $(M \ominus (L\Gamma_1 \overline{\otimes} M_2)) \cap (M \ominus (M_1 \overline{\otimes} L\Gamma_2))$ is the subset of elements in M whose Fourier coefficients lie in the weak closure of $(A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$.

By a linearity/density argument, to prove this claim it is sufficient to show that for any bounded sequence of elements $w_n \in L\Gamma$ and $a \otimes b \in (A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$,

$$\|E_A(v_n(a \otimes b) w_n)\|_2 \to 0.$$

We can assume that $\sup_n \|w_n\|_2 \leq 1$. Write $v_n = \sum_{g \in \Gamma} v_{n,g} u_g$ and $w_n = \sum_{g \in \Gamma} w_{n,g} u_g$. We have

$$E_A(v_n(a \otimes b) w_n) = \sum_{g \in \Gamma} v_{n,g} w_{n,g^{-1}} \sigma_g(a \otimes b),$$

which leads to the formula:

$$\|E_A(v_n(a \otimes b) w_n)\|_2^2 = \sum_{g, g' \in \Gamma} v_{n,g} w_{n,g^{-1}} \overline{v_{n,g'} w_{n,g'^{1}}} \tau(\sigma_g(a \otimes b) \sigma_{g'}(a^* \otimes b^*)). \tag{3.2}$$

Fix $\varepsilon > 0$. Since the action $\Gamma_i \curvearrowright A_i$ is mixing for $i = 1, 2$, there exist finite sets $F_i \subset \Gamma_i$ such that $|\tau((a \otimes b)\sigma_{(s,t)}(a^* \otimes b^*))| = |\tau(a\sigma_s(a^*))\tau(b\sigma_t(b^*))| < \varepsilon$, if $(s, t) \notin F = F_1 \times F_2$. Now (3.2) and the Cauchy–Schwarz inequality imply

$$\begin{aligned}\|E_A(v_n(a \otimes b)w_n)\|_2^2 &\leq \sum_{g\in\Gamma} \sum_{g'\in gF} |v_{n,g} w_{n,g^{-1}} \overline{v_{n,g'} w_{n,g'^{-1}}} \tau(\sigma_g(a \otimes b)\sigma_{g'}(a^* \otimes b^*))| + \varepsilon. \\ &\leq \|a\|_2^2\|b\|_2^2 h(v_n)|F| \sum_{g\in\Gamma} |v_{n,g} w_{n,g^{-1}}| + \varepsilon \\ &\leq \|a\|_2^2\|b\|_2^2 h(v_n)|F| + \varepsilon.\end{aligned}$$

Hence, $\limsup_n \|E_A(v_n(a \otimes b)w_n)\|_2^2 \leq \varepsilon$. Since ε was arbitrary, we get the claim.

Now take $\varepsilon' < \|p\|_2/3$. By assumption, $D \not\prec_M L\Gamma_1 \overline{\otimes} M_2$ and $D \not\prec_M M_1 \overline{\otimes} L\Gamma_2$, so there exists $a \in \mathcal{U}(D)$ such that

$$\|E_{L\Gamma_1 \overline{\otimes} M_2}(a)\|_2 < \varepsilon' \quad \text{and} \quad \|E_{M_1 \overline{\otimes} L\Gamma_2}(a)\|_2 < \varepsilon'.$$

By Lemma 3.2, the sequence $(v_n a v_n^*)_n$ belongs to $A^\omega \rtimes \Gamma$, so that there exists a finite subset $F \subset \Gamma$ such that $\|P_F(v_n a v_n^*)\|_2 \geq \|p\|_2 - \varepsilon'$. Thus if we define $a_0 = a - E_{L\Gamma_1 \overline{\otimes} M_2}(a) - E_{M_1 \overline{\otimes} L\Gamma_2}(a - E_{L\Gamma_1 \overline{\otimes} M_2}(a))$, we get

$$\begin{aligned}\|p\|_2 - \varepsilon' &\leq \|P_F(v_n a v_n^*)\|_2 \\ &\leq \|P_F(v_n(a - E_{L\Gamma_1 \overline{\otimes} M_2}(a))v_n^*)\|_2 + \varepsilon' \\ &\leq \|P_F(v_n a_0 v_n^*)\|_2 + 2\varepsilon'.\end{aligned}$$

But a_0 is orthogonal to $L\Gamma_1 \overline{\otimes} M_2$ and $M_1 \overline{\otimes} L\Gamma_2$, because the conditional expectations $E_{L\Gamma_1 \overline{\otimes} M_2}$ and $E_{M_1 \overline{\otimes} L\Gamma_2}$ commute. Therefore, when n goes to infinity, the claim implies that $\|P_F(v_n a_0 v_n^*)\|_2 \to 0$ which leads to the absurd statement that $\|p\|_2 \leq 3\varepsilon' < \|p\|_2$. □

We end this paragraph with a lemma that localizes the Fourier coefficients of elements $v_n a v_n^*$ inside A, for a particular (fixed) $a \in D$. In fact, this lemma will be the starting point of our reasoning by contradiction in Step 2 below, being the initialization of an induction process.

Lemma 3.4. *There exists an $a \in \mathcal{U}(D)$, a $\delta_0 > 0$, a finite dimensional subspace $K \subset (A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$, and a sequence $(g_n, h_n) \in \Gamma$ such that:*

- *$g_n, h_n \to \infty$, as $n \to \infty$;*
- $\liminf \|Q_{\sigma_{(g_n,h_n)}(K)}(v_n a v_n^*)\|_2 > \delta_0$.

Proof. Put $\delta_1 = \liminf h(v_n) > 0$ and consider a sequence $(g_n, h_n) \in \Gamma$ such that $|v_{n,(g_n,h_n)}| = h(v_n)$ for all n. Now (3.1) implies that the sequences (g_n) and (h_n) go to infinity with n. Moreover, we have

$$\limsup_n \|v_n - v_{n,(g_n,h_n)} u_{(g_n,h_n)}\|_2 = \sqrt{\|p\|_2^2 - \delta_1^2}.$$

Take $\varepsilon > 0$ such that $\sqrt{\|p\|_2^2 - \delta_1^2} + 4\varepsilon < \|p\|_2$. By assumption (ii), there exists $a \in \mathcal{U}(D)$ such that $\|E_{L\Gamma_1 \overline{\otimes} M_2}(a)\|_2 < \varepsilon$ and $\|E_{M_1 \overline{\otimes} L\Gamma_2}(a)\|_2 < \varepsilon$. Thus the element $a_1 = a - E_{L\Gamma_1 \overline{\otimes} M_2}(a) - E_{M_1 \overline{\otimes} L\Gamma_2}(a - E_{L\Gamma_1 \overline{\otimes} M_2}(a))$ satisfies $\|a - a_1\|_2 < 3\varepsilon$, and its Fourier coefficients are in $(A_1 \ominus \mathbb{C}1) \overline{\otimes} (A_2 \ominus \mathbb{C}1)$. We conclude that there exists a finite dimensional $K \subset (A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$ such that, $\|a - Q_K(a)\|_2 < 4\varepsilon$.

Finally, we get that

$$\|v_n a v_n^* - v_{n,(g_n,h_n)} u_{(g_n,h_n)} Q_K(a) v_n^*\|_2 < \sqrt{\|p\|_2^2 - \delta_1^2} + 4\varepsilon.$$

Since $v_{n,(g_n,h_n)} u_{(g_n,h_n)} Q_K(a) v_n^*$ belongs to the image of the projection $Q_{\sigma_{(g_n,h_n)}(K)}$, we get the result with $\delta_0 > 0$ defined by $\|p\|_2^2 - \delta_0^2 = \left(\sqrt{\|p\|_2^2 - \delta_1^2} + 4\varepsilon\right)^2$. □

Step 2: *We show that* $C \prec_M A$.

Notation. For a finite subset $G \subset \Gamma$, finite dimensional subspaces $K_1, K_2 \subset A$ and $\lambda > 0$, define

$$[K_1 \times \sigma_G(K_2)]^\lambda = \operatorname{conv}\{\lambda a \sigma_g(b) \mid a \in K_1, b \in K_2, g \in G, \|a\|_2 \leq 1, \|b\|_2 \leq 1\}.$$

We have that $[K_1 \times \sigma_G(K_2)]^\lambda$ is a closed convex subset $\mathcal{C}$ of A (being the convex hull of a compact subset in a finite dimensional vector space). Then the set $\tilde{\mathcal{C}}$ consisting of vectors $\xi \in L^2(M)$ whose Fourier coefficients $\xi_g = \langle \xi, u_g \rangle$ $(g \in \Gamma)$ belong to $\mathcal{C}$ is a closed convex subset of $L^2(M)$. Hence one can define the "orthogonal projection onto this set" $Q_{\mathcal{C}} : L^2(M) \to L^2(M)$ as follows. For $x \in L^2(M)$, $Q_{\mathcal{C}}(x)$ is the unique point of $\tilde{\mathcal{C}}$ such that

$$\|x - Q_{\mathcal{C}}(x)\| = \inf_{y \in \tilde{\mathcal{C}}} \|x - y\|.$$

Note that the restriction of $Q_{\mathcal{C}}$ to $L^2(A)$ is equal to the orthogonal projection onto $\mathcal{C}$, and that $Q_{\mathcal{C}}\left(\sum_{g\in\Gamma} x_g u_g\right) = \sum_{g\in\Gamma} Q_{\mathcal{C}}(x_g) u_g$.

Remark 3.5. This notation is consistent with the previous notation Q_K: If $K \subset A$ is a finite dimensional subspace, then $Q_K(a) = Q_{\mathcal{C}}(a)$, where $\mathcal{C} = [\mathbb{C}1 \times \sigma_{\{e\}}(K)]^\lambda$ as soon as $\lambda \geq \|a\|_2$.

Before getting into the heart of the proof, we check some easy properties of these convex sets.

Lemma 3.6. *Fix* $\lambda > 0$ *and finite dimensional subspaces* $K_1, K_2 \subset A$. *Then there exists a constant* $\kappa > 0$ *such that for all finite* $G \subset \Gamma$, *and all* $x \in [K_1 \times \sigma_G(K_2)]^\lambda$,

$$\|x\|_\infty \leq \kappa.$$

Proof. Since K_1 and K_2 are finite dimensional, there exists a constant $c > 0$ such that $\|a\|_\infty \leq c\|a\|_2$ for all $a \in K_1$ or $a \in K_2$. One sees that $\kappa = \lambda c^2$ satisfies the conclusion of the lemma. □

Lemma 3.7. *For finite subsets* $F, G \subset \Gamma$, *and finite dimensional subspaces* $K_1, K_2, K_1', K_2' \subset A$ *and* $\lambda_1, \lambda_2 > 0$, *we have*

$$[K_1 \times \sigma_F(K_2)]^{\lambda_1} + [K_1' \times \sigma_G(K_2')]^{\lambda_2} \subset [(K_1 + K_1') \times \sigma_{G\cup F}(K_2 + K_2')]^{\lambda_1+\lambda_2}.$$

Proof. This is straightforward. □

From now on, we assume by contradiction that $C \not\prec_M A$. The contradiction we are looking for is then a direct consequence of the following lemma. Indeed, using Lemma 3.4, and iterating Lemma 3.8 enough times, we get the absurd statement that there exist unitaries $a_n = v_n a v_n^*$ and elements b_n of the form $Q_{\mathcal{C}_n}(a_n)$ such that $\liminf_n \|a_n - b_n\|_2^2$ is negative.

Lemma 3.8. *Fix $a \in \mathcal{U}(D)$ and put $a_n = v_n a v_n^*$ for all n. Assume that there exists a sequence of finite subsets $F_n \times G_n \subset \Gamma = \Gamma_1 \times \Gamma_2$, finite dimensional subspaces $K_1 \subset A$, $K_2 \subset (A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$, $\lambda > 0$ and $\delta > 0$ such that:*

- $\sup_n |F_n||G_n| < \infty$;
- $F_n \to \infty$ *(meaning that for all $g \in \Gamma_1$, $g \notin F_n$ for n large enough)*, $G_n \to \infty$;
- $\limsup_n \|a_n - Q_{\mathcal{C}_n}(a_n)\|_2^2 < \|p\|_2^2 - \delta^2$, *where* $\mathcal{C}_n = [K_1 \times \sigma_{F_n \times G_n}(K_2)]^\lambda$.

Then there exists a sequence of finite subsets $F_n' \times G_n' \subset \Gamma$, finite dimensional subspaces $K_1' \subset A$, $K_2' \subset (A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$, and $\lambda' > 0$ such that:

- $\sup_n |F_n'||G_n'| < \infty$;
- $F_n' \to \infty$, $G_n' \to \infty$;
- $\limsup_n \|a_n - Q_{\mathcal{C}_n'}(a_n)\|_2^2 < \|p\|_2^2 - 3\delta^2/2$, *where* $\mathcal{C}_n' = [K_1' \times \sigma_{F_n' \times G_n'}(K_2')]^{\lambda'}$.

The multiple mixing property will be used in the proof of this lemma through the following lemma.

Lemma 3.9. *Let $x, y, z \in A_1 \otimes A_2$. For any sequences $g_n = (g_n^1, g_n^2) \in \Gamma$ and $h_n = (h_n^1, h_n^2) \in \Gamma$ such that $g_n^1, g_n^2, h_n^1, h_n^2 \to \infty$, we have*

$$|\tau(x\sigma_{g_n}(y)\sigma_{h_n}(z)) - \tau(x)\tau(\sigma_{g_n}(y)\sigma_{h_n}(z))| \to 0.$$

Proof. Without loss of generality, one can assume that $x = x_1 \otimes x_2$, $y = y_1 \otimes y_2$, $z = z_1 \otimes z_2$. We have

- $\tau(x\sigma_{g_n}(y)\sigma_{h_n}(z)) = \tau(x_1\sigma_{g_n^1}(y_1)\sigma_{h_n^1}(z_1))\tau(x_2\sigma_{g_n^2}(y_2)\sigma_{h_n^2}(z_2))$;
- $\tau(x)\tau(\sigma_{g_n}(y)\sigma_{h_n}(z)) = \tau(x_1)\tau(\sigma_{g_n^1}(y_1)\sigma_{h_n^1}(z_1))\tau(x_2)\tau(\sigma_{g_n^2}(y_2)\sigma_{h_n^2}(z_2))$.

So the result follows directly from the multiple mixing property of the Gaussian actions $\Gamma_i \curvearrowright A_i$, $i = 1, 2$. $\square$

Proof of Lemma 3.8. Let $a, F_n, G_n, K_1, K_2, \lambda, \delta$ and $\mathcal{C}_n$ be as in the implication. Fix $\varepsilon > 0$, with $\varepsilon \ll \delta$. By Lemma 3.2 one can find $S \subset \Gamma$ finite such that $\|a_n - P_S(a_n)\|_2 \leq \varepsilon$, for all n. Hence we get that $\limsup_n \|a_n - P_S \circ Q_{\mathcal{C}_n}(a_n)\|_2 < \sqrt{\|p\|_2^2 - \delta^2} + \varepsilon$.

Now following Ioana's idea (see the proof of [8, Theorem 5.2] and also the end of the proof of [23, Theorem 14.1] for a more clear exposition of this idea), we will consider an element $d \in \mathcal{U}(C)$ with sufficiently spread out Fourier coefficients so that for n large enough, $d(P_S \circ Q_{\mathcal{C}_n}(a_n))d^*$ is almost orthogonal to $P_S \circ Q_{\mathcal{C}_n}(a_n)$, while it is still close to a_n. Then the sum $d(P_S \circ Q_{\mathcal{C}_n}(a_n))d^* + P_S \circ Q_{\mathcal{C}_n}(a_n)$ should be even closer to a_n.

Let $\alpha > 0$ be a (finite) constant such that $\|x\|_\infty \leq \alpha\|x\|_2$, for all $x \in K_1$. Since $K_2 \subset (A_1 \ominus \mathbb{C}1) \otimes (A_2 \ominus \mathbb{C}1)$ is finite dimensional, the set

$$L = \{g \in \Gamma \mid \exists a, b \in K_2, \|a\|_2 \leq 1, \|b\|_2 \leq 1 : |\langle \sigma_g(a), b\rangle| \geq \varepsilon/|S|^2\lambda^2\alpha^2\}$$

is finite. Hence for all n, $L_n = \cup_{g,h \in F_n \times G_n} gLh^{-1}$ is finite, with cardinality smaller or equal to $|F_n|^2|G_n|^2|L|$, which is itself majorized by some N, not depending on n.

Since $C \not\prec A$, Ioana's intertwining criterion (Lemma 2.2) implies that there exists $d \in \mathcal{U}(C)$ such that $\|P_F(d)\|_2 \leq \varepsilon/\kappa|S|$, whenever $|F| \leq N$, where κ is given by Lemma 3.6 applied to K_1, K_2 and λ.

By Kaplansky's density theorem, one can find $d_0, d_1 \in M$, and $T = T_1 \times T_2 \subset \Gamma$ finite such that:

- $d_i = P_T(d_i)$, $i = 1, 2$;
- $\|d_0 - d\|_2 \leq \min(\varepsilon, \varepsilon/\kappa|S|)$, $\|d_1 - d^*\|_2 \leq \varepsilon$;
- $\|d_i\|_\infty \leq 1$, $i = 1, 2$.

Since $a_n \in D$ for all n and $d \in C = D' \cap M$, we have $da_nd^* = a_n$. Thus for all n, $\|a_n - d_0a_nd_1\|_2 \leq 2\varepsilon$, and so

$$\limsup_n \|a_n - d_0(P_S \circ Q_{\mathcal{C}_n}(a_n))d_1\|_2 \leq \sqrt{\|p\|_2^2 - \delta^2} + 3\varepsilon.$$

Now, for all n, put $T_n = T \setminus L_n$. By definition of d, $\|P_T(d) - P_{T_n}(d)\|_2 \leq \varepsilon/\kappa|S|$, hence $\|d_0 - P_{T_n}(d_0)\|_2 \leq 3\varepsilon/\kappa|S|$. Notice that $\|P_S \circ Q_{\mathcal{C}_n}(a_n)\|_\infty \leq \kappa|S|$, which implies that

$$\limsup_n \|a_n - P_{T_n}(d_0)P_S \circ Q_{\mathcal{C}_n}(a_n)d_1\|_2 \leq \sqrt{\|p\|_2^2 - \delta^2} + 6\varepsilon.$$

Denote by $x_n = P_S \circ Q_{\mathcal{C}_n}(a_n)$ and $y_n = P_{T_n}(d_0)P_S \circ Q_{\mathcal{C}_n}(a_n)d_1$.

We want to show that $\limsup_n |\langle x_n, y_n\rangle|$ is small.

Write $d_0 = \sum_{g\in T} d_{0,g}u_g$, $a_n = \sum_h a_{n,h}u_h$, and $d_1 = \sum_{k\in T} d_{1,k}u_k$. We get

$$\begin{aligned}\langle y_n, x_n\rangle &= \sum_{\substack{g\in T_n, h\in S, k\in T\\ ghk\in S}} \tau(d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h}))Q_{\mathcal{C}_n}(a_{n,ghk})^*)\\ &= \sum_{\substack{g\in T, h\in S, k\in T\\ ghk\in S}} \mathbf{1}_{\{g\in T_n\}}\tau(d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h}))Q_{\mathcal{C}_n}(a_{n,ghk})^*).\end{aligned}$$

Claim. *For all fixed $x, y \in A$, and $g \in T$, there exists n_0 such that for all $n \geq n_0$, and all $a, b \in \mathcal{C}_n$,*

$$|\mathbf{1}_{\{g\in T_n\}}\tau(xy\sigma_g(a)b^*)| \leq 2\varepsilon\|x\|_2\|y\|_2/|S|^2.$$

To prove this claim, first recall that for all n, $\mathcal{C}_n = [K_1 \times \sigma_{F_n\times G_n}(K_2)]^\lambda$. Denote by $\tilde{K}_1 = \operatorname{span}\{xy\sigma_g(a)b^*, a, b \in K_1\}$. Since $\tilde{K}_1$ and K_2 have finite dimension and since $F_n, G_n \to \infty$, Lemma 3.9 implies that there exists n_0 such that for $n \geq n_0$, and for all $s, t \in F_n \times G_n$ one has

$$\sup_{\substack{a\in\tilde{K}_1, \|a\|_2\leq 1\\ b,c\in K_2, \|b\|_2\leq 1, \|c\|_2\leq 1}} |\tau(a\sigma_{gs}(b)\sigma_t(c^*)) - \tau(a)\tau(\sigma_{gs}(b)\sigma_t(c^*))| \leq \varepsilon\|x\|_2\|y\|_2/|S|^2\lambda^2. \quad (3.3)$$

Thus take $n \geq n_0$. By definition of $\mathcal{C}_n$, it is sufficient to prove that for all $a, b \in K_1$, $c, d \in K_2$, with $\|a\|_2, \|b\|_2, \|c\|_2, \|d\|_2 \leq 1$, and all $s, t \in F_n \times G_n$,

$$|\mathbf{1}_{\{g\in T_n\}}\tau(xy\sigma_g(\lambda a\sigma_s(c))\lambda b^*\sigma_t(d^*))| \leq 2\varepsilon\|x\|_2\|y\|_2/|S|^2.$$

We can assume that $g \in T_n$. An easy calculation gives

$$\begin{aligned}|\tau(xy\sigma_g(\lambda a\sigma_s(c))\lambda b^*\sigma_t(d^*))| &\leq \varepsilon\|x\|_2\|y\|_2/|S|^2 + \lambda^2|\tau(xy\sigma_g(a)b^*)\tau(\sigma_{gs}(c)\sigma_t(d^*))|\\ &\leq \varepsilon\|x\|_2\|y\|_2/|S|^2 + \lambda^2\|x\|_2\|y\|_2\|a\|_\infty\|b\|_\infty\varepsilon/|S|^2\lambda^2\alpha^2\\ &\leq 2\varepsilon\|x\|_2\|y\|_2/|S|^2,\end{aligned}$$

where the first inequality is deduced from (3.3), while the second is because $g \notin L_n$. So the claim is proven.

Now we can estimate $|\langle x_n, y_n\rangle|$, for n large enough.

$$\begin{aligned}|\langle x_n, y_n\rangle| &\leq \sum_{g\in T, h\in S, k'\in S} |\mathbf{1}_{\{g\in T_n\}}\tau(d_{0,g}\sigma_{gh}(d_{1,h^{-1}g^{-1}k'})\sigma_g(Q_{C_n}(a_{n,h}))Q_{C_n}(a_{n,k'})^*)| \\ &\leq \sum_{g\in T, h\in S, k'\in S} 2\varepsilon\|d_{0,g}\|_2\|d_{1,h^{-1}g^{-1}k'}\|_2/|S|^2 \\ &\leq 2\varepsilon\|d_0\|_2\|d_1\|_2 \leq 2\varepsilon.\end{aligned}$$

Therefore, we obtain:

- $\limsup_n \|a_n - x_n\|_2 < \sqrt{\|p\|_2^2 - \delta^2} + \varepsilon$;
- $\limsup_n \|a_n - y_n\|_2 < \sqrt{\|p\|_2^2 - \delta^2} + 6\varepsilon$;
- $\limsup_n |\langle x_n, y_n\rangle| \leq 2\varepsilon$.

Thus using the formula

$$\|x - (y+z)\|_2^2 = \|x - y\|_2^2 + \|x - z\|_2^2 - \|x\|_2^2 + 2\Re\langle y, z\rangle,$$

one checks that $\limsup_n \|a_n - (x_n + y_n)\|_2^2 \leq \|p\|_2^2 - 3\delta^2/2$, if ε is small enough.

Now observe that

$$y_n = \sum_{g\in T_n, h\in S, k\in T} d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h}))u_{ghk}.$$

So let us check that y_n has its Fourier coefficients in $[K_0 \times \sigma_{(T_1F_n)\times(T_2G_n)}(K_2)]^{\lambda|S||T|}$, where $K_0 = \operatorname{span}\{d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(c), c \in K_1, g, k \in T, h \in S\}$.

Fix $n \in \mathbb{N}$, and $s \in \Gamma$. Denote by $y_{n,s} = E_A(y_n u_s^*)$. We have

$$y_{n,s} = \sum_{\substack{g\in T_n, h\in S, k\in T\\ ghk=s}} d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(Q_{\mathcal{C}_n}(a_{n,h})).$$

Thus it is a convex combination of terms of the form

$$\begin{aligned}\mathcal{T} &= \sum_{\substack{g\in T, h\in S, k\in T\\ ghk=s}} d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(\lambda a_h\sigma_{t_h}(b_h)) \\ &= \frac{1}{|S||T|}\sum_{\substack{g\in T, h\in S, k\in T\\ ghk=s}} |S||T|d_{0,g}\sigma_{gh}(d_{1,k})\sigma_g(\lambda a_h\sigma_{t_h}(b_h)),\end{aligned}$$

for elements $a_h \in K_1, b_h \in K_2$, with $\|a_h\|_2, \|b_h\|_2 \leq 1$ and $t_h \in F_n \times G_n$, for all $h \in S$. But such terms $\mathcal{T}$ are themselves convex combinations of elements of the form $\lambda|S||T|x\sigma_{gt}(y)$, with $x \in K_0$, $y \in K_2$, $\|x\|_2, \|y\|_2 \leq 1$ and $gt \in T(F_n \times G_n) = (T_1F_n) \times (T_2G_n)$.

Therefore, as pointed out in Lemma 3.7, $x_n + y_n$ has Fourier coefficients in $\mathcal{C}'_n = [K'_1 \times \sigma_{F'_n\times G'_n}(K'_2)]^{\lambda'}$, with $K'_1 = K_1 + K_0$, $K'_2 = K_2$, $\lambda' = \lambda + \lambda|S||T|$, and $F'_n = F_n \cup T_1F_n$, $G'_n = G_n \cup T_2G_n$.

We conclude that:

$$\|a_n - Q_{C'_n}(a_n)\|_2^2 \leq \|p\|_2^2 - 3\delta^2/2,$$

which proves the lemma. □

The proof of Theorem 3.1 is complete. □

Taking $\Gamma_2 = \{e\}$ and $A_2 = \mathbb{C}$ we obtain a similar statement for a single mixing action $\Gamma \curvearrowright A$.

Corollary 3.10. *Assume that $\Gamma \curvearrowright A$ is a mixing Gaussian action. Denote by $M = A \rtimes \Gamma$. Consider an abelian von Neumann subalgebra $D \subset pMp$, $p \in L\Gamma$, which is normalized by a sequence of unitaries $(v_n) \in \mathcal{U}(pL\Gamma p)$ with $v_n \to 0$ weakly. Put $C = D' \cap pMp$. Then one of the following is true:*

- $D \prec_M L\Gamma$
- *For all $q \in \mathcal{Z}(C)$, $qC \prec_M A$.*

In fact, S. Vaes asked during his series of lectures at the IHP in Paris (spring 2011) whether such a corollary could hold for any mixing action. A. Ioana showed that this is true for Bernoulli shifts [8, Theorem 6.2], and as we just showed, the proof can be adapted to Gaussian actions. In our proof, we only used the following properties of Gaussian actions:

- The 2-mixing property;
- The malleability property.

Moreover, the malleability of Gaussian actions is only used to prove Lemma 3.2 (*i.e.* to show that the sequences $(v_n a v_n^*)$, $a \in D$ lie in $A^\omega \rtimes \Gamma$). We suspect that this lemma might be shown only using multiple mixing properties, but we were not able to reach this conclusion.

4. Proof of the results

We will prove the following generalization of Theorems A and B that considers *some* amplifications. We will follow closely the proof of [9, Theorem 10.1].

Theorem 4.1. *Let Γ be an ICC countable discrete group, and $\pi : \Gamma \to \mathcal{O}(H_\mathbb{R})$ an orthogonal representation of Γ. Make one of the following two assumptions:*

- *Γ is w-rigid and ICC, and π is mixing;*
- *Γ is an ICC non-amenable product of two infinite groups and π is mixing and admits a tensor power which is weakly contained in the regular representation.*

Let $\Gamma \curvearrowright A$ be the Gaussian action associated to π and put $M = A \rtimes \Gamma$. Let $\Lambda \curvearrowright B$ be another free ergodic action on an abelian von Neumann algebra, and put $N = B \rtimes \Lambda$.

If for some $t \geq 1$, $M \simeq N^t$, then $t = 1$, $\Gamma \simeq \Lambda$ and the actions $\Gamma \curvearrowright A$ and $\Lambda \curvearrowright B$ are conjugate.

Proof. Take a projection $p_0 \in M$ with trace equal to $1/t \leq 1$. By assumption we have an isomorphism $\theta : N \to p_0 M p_0$. For notational simplicity, we will omit the θ, and just write $N = p_0 M p_0$.

Claim. *To prove the theorem, it is sufficient to show that either there exists $u \in \mathcal{U}(M)$ such that $uL\Lambda u^* \subset L\Gamma$, or $B \prec_M A$.*

Indeed, if $uL\Lambda u^* \subset L\Gamma$, then Corollary 3.10 easily implies that $B \prec_M A$ ($B \prec L\Gamma$ is excluded because of Proposition 2.3). If $B \prec_M A$, then by [15, Theorem A.1] (and using an amplification B^t of B into a Cartan subalgebra of M), there exists a unitary $v \in M$ such that $vBv^* \subset A$. Thus $(B \subset N)$ and $(qA \subset qMq)$ follow isomorphic, where $q = vp_0v^* \in A$. Hence $\mathcal{R}(\Lambda \curvearrowright B) \simeq \mathcal{R}(\Gamma \curvearrowright A)^{1/t}$. Popa's orbit equivalence superrigidity theorems ([17, Theorems 5.2 and 5.6] and [19, Theorem 1.3]) imply that $t = 1$, $\Gamma \simeq \Lambda$ and the actions are conjugate. This proves the claim.

Assume that $B \nprec_M A$ and that no unitary of M conjugates $L\Lambda$ into $L\Gamma$. We will follow the procedure described in the Introduction (steps (1)–(4)) to get a contradiction.

Denote by $\Delta_0 : N \to N \overline{\otimes} N$ the comultiplication defined by $\Delta(bv_s) = bv_s \otimes v_s$ for $b \in B, s \in \Lambda$. Amplifying the $*$-morphism Δ_0, we obtain a (possibly non-unital) $*$-morphism $\Delta : M \to M \overline{\otimes} M$, which satisfies the following properties.

Lemma 4.2. *Let $P \subset M$ a von Neumann subalgebra.*

(1) *If $P \nprec_M B$ then $\Delta(P) \nprec_{M \overline{\otimes} M} M \otimes 1$;*
(2) *If P is diffuse, then $\Delta(P) \nprec 1 \otimes M$;*
(3) *If $\Delta(M) \prec M \overline{\otimes} P$, then $L\Lambda \prec P$. Moreover, if $\Delta(M)s \prec M \overline{\otimes} P$ for every non-zero projection $s \in \Delta(M)' \cap \Delta(1)(M \overline{\otimes} M)\Delta(1)$, then $L\Lambda r \prec P$ for all non-zero projections $r \in L\Lambda' \cap M$;*
(4) *If $\Delta(M) \prec P \overline{\otimes} M$, then there exists a projection $q \in P' \cap M$ such that $Pq \subset qMq$ has finite index;*
(5) *If P has no amenable direct summand, then $\Delta(P)$ is strongly non-amenable relative to $M \otimes 1$ or $1 \otimes M$. In particular, if $N \subset M$ is an amenable subalgebra, then $\Delta(P) \nprec M \overline{\otimes} N$ or $N \overline{\otimes} M$.*

Proof. These properties are true for Δ_0 by [9, Lemma 10.2] and [8, Lemma 9.2(4)]. One easily checks that they are still true for Δ (see also [2, Proposition 4.1]). □

Since we assumed that no unitary of M conjugates $L\Lambda$ into $L\Gamma$, then moreover part of statement (3) above combined with Proposition 2.3.ii provides a projection $s \in \Delta(M)' \cap \Delta(1)(M \overline{\otimes} M)\Delta(1)$ such that $s\Delta(M) \nprec M \overline{\otimes} L\Gamma$.

Step (1) There exists a unitary $v \in \mathcal{U}(M \overline{\otimes} M)$ such that

$$vs\Delta(L\Gamma)v^* \subset L(\Gamma \times \Gamma).$$

Proof. Put $Q = s\Delta(L\Gamma) \subset s(M \overline{\otimes} M)s$. The assumptions of the theorem imply that either $Q \subset s(M \overline{\otimes} M)s$ has relative property (T) (case 1), or that $Q = Q_1 \overline{\otimes} Q_2$ with Q_1 diffuse and Q_2 being strongly non-amenable relative to $M \otimes 1$ and $1 \otimes M$ (Lemma 4.2.5) (case 2).

Applying Corollary 2.8 either to Q (in case 1) or to Q_1 (in case 2), we get that one of the following assertions holds true (note that Q is contained in the normalizer of Q_1).

(a) $Q \prec 1 \otimes M$ (in case 1) or $Q_1 \prec 1 \otimes M$ (in case 2).
(b) $Q \prec A \overline{\otimes} M$.
(c) There exists $v_1 \in \mathcal{U}(M \overline{\otimes} M)$ such that $v_1 Q v_1^* \subset L\Gamma \overline{\otimes} M$.

But assertions (a) and (b) are excluded because of items (2) and (5) in Lemma 4.2. Thus (c) holds true.

Now one can apply again Corollary 2.8 to the subalgebra $v_1 Q v_1^* \subset (v_1 s v_1^*)(L\Gamma \overline{\otimes} M)(v_1 s v_1^*)$ (in case 1) or to $v_1 Q_1 v_1^*$ (in case 2), to deduce that one of the following is true.

(a′) $Q \prec L\Gamma \otimes 1$ (in case 1) or $Q_1 \prec L\Gamma \otimes 1$ (in case 2).
(b′) $Q \prec L\Gamma \otimes A$.
(c′) Step (1) is true.

Again (b′) is false. By Lemma 4.2.1, (a′) would imply that $Q_0 \prec B$ where Q_0 is a diffuse von Neumann subalgebra of $L\Gamma$. Passing to relative commutants, it further implies that $B \prec_M Q_0' \cap M$ the later being contained in $L\Gamma$ since our Gaussian action is mixing (see for instance [3, Lemma 3.5]). But then $M \prec_M L\Gamma$ (by Proposition 2.3), which is impossible. So (a′) is excluded. □

Now define $q = vsv^* \in L(\Gamma \times \Gamma)$ and $C = (vs\Delta(A)v^*)' \cap q(M \overline{\otimes} M)q$.

Step (2) For every projection $p \in \mathcal{Z}(C)$, we have that $Cp \prec A \overline{\otimes} A$. Moreover there exists a unitary $u \in \mathcal{U}(M \overline{\otimes} M)$ such that $u\mathcal{Z}(C)u^* \subset A \overline{\otimes} A$.

Proof. Note that the normalizer of the abelian von Neumann subalgebra $vs\Delta(A)v^* \subset q(M \overline{\otimes} M)q$ contains $vs\Delta(L\Gamma)v^* \subset qL(\Gamma \times \Gamma)q$. So we apply Theorem 3.1 to $vs\Delta(A)v^*$ and we get that one of the following is true:

- $vs\Delta(L\Gamma)v^* \prec L\Gamma \otimes 1$ or $1 \otimes L\Gamma$;
- $vs\Delta(A)v^* \prec L\Gamma \otimes M$ or $M \otimes L\Gamma$;
- For every projection $p \in \mathcal{Z}(C)$, we have that $Cp \prec A \overline{\otimes} A$.

Since Γ is non-amenable, Lemma 4.2.5 implies that the first case is impossible. Using Proposition 2.3.i, the second case implies one of the following impossible situations:

- $s\Delta(A) \prec 1 \otimes M$ (excluded because A is diffuse);
- $s\Delta(M) \prec L\Gamma \overline{\otimes} M$ (excluded because $L\Gamma \subset M$ has infinite index);
- $s\Delta(A) \prec M \otimes 1$ (contradicts our assumption $B \not\prec A$);
- $s\Delta(M) \prec M \overline{\otimes} L\Gamma$ (contradicts the definition of s).

Thus the first part of Step 2 is true. In particular C is of type I. The second part is then deduced exactly as in the proof of [8, Theorem 6.2, Claim 3]. □

Now the conclusion follows from the end of the proof of [9, Theorem 10.1] (Steps 3 and 4). □

Proof of Theorem D. Let π be an orthogonal representation as in the statement of the theorem. Assume by contradiction that there exists a countable group Λ such that $M \simeq (L\Lambda)^t$ for some $t > 0$. Then adapting the proof of [9, Theorem 8.2], we get that $t = 1$, and $\Lambda \simeq \Sigma \rtimes \Gamma$, for some infinite abelian group Σ and some action $\Gamma \curvearrowright \Sigma$ by automorphisms. Moreover, the initial Gaussian action σ is conjugate to the action of Γ on $L\Sigma$.

Now, since σ is mixing, the action $\Gamma \curvearrowright \Sigma \setminus \{e\}$ has finite stabilizers. But then the representation $\Gamma \curvearrowright \ell^2(\Sigma \setminus \{e\})$ is a direct sum of quasi-regular representations of the form $\Gamma \curvearrowright \ell^2(\Gamma/\Gamma_0)$, where Γ_0 is a finite subgroup of Γ. But such quasi-regular representations are all contained in the regular representation.

So we conclude that the Koopman representation $\Gamma \curvearrowright L^2(A) \ominus \mathbb{C}1$ is contained in a direct sum of copies of the regular representation. Thus, Proposition 1.7 in [12] implies this is also the case of the representation π, which is excluded by assumption. □

Acknowledgments

We warmly thank Cyril Houdayer and Adrian Ioana for their valuable advice and comments about this work. We also thank Stefaan Vaes and the mysterious referees for useful remarks.

References

[1] B. Bekka, P. de la Harpe, A. Valette, Kazhdan's Property (T), in: New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.
[2] M. Berbec, S. Vaes, W^*-superrigidity for group von Neumann algebras of left–right wreath products, 2012, Preprint.
[3] R. Boutonnet, On solid ergodicity for Gaussian actions, J. Funct. Anal. 263 (2012) 1040–1063.
[4] J. Feldmann, C.C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras, II, Trans. Amer. Math. Soc. 234 (1977) 325–359.
[5] A. Furman, On Popa's cocycle superrigidity theorem, Int. Math. Res. Not. IMRN 2007 (2007) 1–46. Art. ID rnm073.

[6] D. Gaboriau, Orbit equivalence and measured group theory, in: Proceedings of the ICM (Hyderabad, India, 2010), Vol. III, Hindustan Book Agency, 2010, pp. 1501–1527.
[7] C. Houdayer, S. Popa, S. Vaes, A class of groups for which every action is W^*-superrigid, Groups Geom. Dyn. (2013) in press. arXiv:1010.5077.
[8] A. Ioana, W^*-superrigidity for Bernoulli actions of property (T) groups, J. Amer. Math. Soc. 24 (2011) 1175–1226.
[9] A. Ioana, S. Popa, S. Vaes, A class of superrigid group von Neumann algebras, Ann. of Math. (2013) in press.
[10] N. Ozawa, S. Popa, On a class of II_1 factors with at most one Cartan subalgebra, Ann. of Math. (2) 172 (2010) 713–749.
[11] J. Peterson, Examples of groups which are virtually W^*-superrigid, 2009, Preprint. arXiv:1002.1745.
[12] J. Peterson, T. Sinclair, On cocycle superrigidity for Gaussian actions, Ergodic Theory Dynam. Systems 32 (2012) 249–272.
[13] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups I, Invent. Math. 165 (2006) 369–408.
[14] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups II, Invent. Math. 165 (2006) 409–453.
[15] S. Popa, On a class of type II_1 factors with Betti numbers invariants, Ann. of Math. 163 (2006) 809–899.
[16] S. Popa, Some rigidity results for non-commutative Bernoulli shifts, J. Funct. Anal. 230 (2006) 273–328.
[17] S. Popa, Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups, Invent. Math. 170 (2007) 243–295.
[18] S. Popa, Deformation and rigidity for group actions and von Neumann algebras, in: Proceedings of the ICM (Madrid, 2006), Vol. I, European Mathematical Society Publishing House, 2007, pp. 445–477.
[19] S. Popa, On the superrigidity of malleable actions with spectral gap, J. Amer. Math. Soc. 21 (2008) 981–1000.
[20] S. Popa, S. Vaes, Group measure space decomposition of II_1 factors and W^*-superrigidity, Invent. Math. 182 (2010) 371–417.
[21] S. Vaes, Explicit computations of all finite index bimodules for a family of II_1 factors, Ann. Sc. Éc. Norm. Supér. 41 (2008) 743–788.
[22] S. Vaes, Rigidity for von Neumann algebras and their invariants, in: Proceedings of the ICM (Hyderabad, 2010), Vol. III, Hindustan Book Agency, 2010, pp. 1624–1650.
[23] S. Vaes, W^*-superrigidity for Bernoulli actions and wreath product group von Neumann algebras, in: Lecture Notes From the Trimester at the Institut Henri Poincaré Von Neumann Algebras and Ergodic Theory of Group Actions, Paris, Spring, 2011. https://perswww.kuleuven.be/~u0018768/slides/vng2011-vaes.pdf.
[24] S. Vaes, One-cohomology and the uniqueness of the group measure space of a II_1 factor, Math. Ann. 355 (2013) 661–696.