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The doodle of a finitely determined map germ from $\mathbb{R}^2$ to $\mathbb{R}^3$ ☆

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Abstract

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a representative of a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$. Consider the curve obtained as the intersection of the image of the mapping f with a sufficiently small sphere S^2_ϵ centered at the origin in $\mathbb{R}^3$, call this curve the associated doodle of the map germ f. For a large class of map germs the associated doodle has many transversal self-intersections. The topological classification of such map germs is considered from the point of view of the associated doodles.

MSC: primary 58K15; secondary 58K40, 58K65

Keywords: Doodle; Topological classification; Double point curve

1. Introduction

Two map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are topologically equivalent if there are germs of homeomorphisms $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $g = \psi \circ f \circ \phi$, in other words f and g are equal up to continuous change of coordinates in the source and target.

This is not an easy equivalence relation to work with, for example, the mappings $(x, y) \to (x, xy + y^3, xy^2 + cy^4)$, where $c = 0.4, 0.9, 1.1$, are all topologically distinct. If one looks at the

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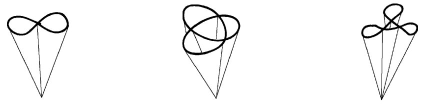
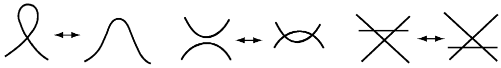


c = 0.4 c = 0.9 c = 1.1

Fig. 1.



i) ii) iii)

Fig. 2.

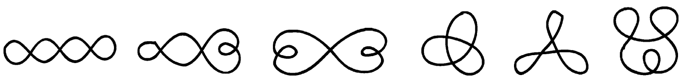


Fig. 3.

image of a neighborhood around the origin in $\mathbb{R}^2$ by these three mappings, one suspects they are not topologically equivalent (Fig. 1). Indeed, the first one has only one line of self-intersection while the other two have three. To distinguish the last two, just notice that the removal of one self-intersection line splits the last one into two components while the other remains connected.

What calls attention in the examples above is that they all have a cone like image and their intersection with a sufficiently small 2-sphere S^2_ϵ around the origin in $\mathbb{R}^3$ is a curve with transversal self-intersection (crossings). Changes in the topological type of $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ correspond to local transitions on the curve in S^2_ϵ (Fig. 2). In fact, the three transitions are related to the failure of the finite determinacy condition of f: (i) there is a line of cross-caps, (ii) there is a line of non-transverse self-intersection and (iii) there is a line of triple points. Here, finite determinacy means with respect to smooth changes of coordinates in the source and the target.

These curves with transversal intersection on S^2 are called *doodles* and have been systematically studied since Gauss' time [10]. The reader can find in Scott Carter's book [6] an introduction to the theme. Carter lists the topological types of all such curves with up to four crossings: there are two curves with two crossings, six with three crossings (Fig. 3) and nineteen with four crossings.

According to a result by Takuo Fukuda, finitely determined map germs $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, $n\leqslant p$, are always topologically conelike [8]. In particular, when $n=2$ and $p=3$, the link of the cone is a doodle and we call it the *associated doodle* of f. Our aim is to describe the beginning of the topological classification of finitely determined map germs $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ by means of their associated doodles. For a large class of these mappings our classification is complete (Sections 4 and 5). In Section 6 we address the realization problem, that is, given a certain doodle find a finitely determined map germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ whose associated doodle is the given one.

Similar problems have been considered by the second named author and R. Martins in [14] for finitely determined singularities of ruled surfaces in $\mathbb{R}^3$. There one finds a complete classification with only eleven possible topological classes of map germs. See also [12].

All map germs considered are real analytic except when otherwise stated. We adopt the notation and basic definitions that are usual in singularity theory (e.g., $\mathcal{A}$-equivalence, stability, finite determinacy, etc.), as the reader can find in Wall's survey paper [21].

2. Finite determinacy of map germs from $(\mathbb{R}^2, 0)$ to $(\mathbb{R}^3, 0)$

We start by recalling Whitney's results about a stable map from a n-manifold to $\mathbb{R}^{2n-1}$ (see [22]). To simplify the notation, we restrict ourselves to the case $n = 2$. Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth map, where U is open. The map f is stable if and only if it is *semiregular*. This means there is a discrete subset $\Sigma \subset U$ such that:

(1) $f : U \setminus \Sigma \to \mathbb{R}^3$ is an immersion with normal crossings.
(2) For each $x_0 \in \Sigma$, $f^{-1}(f(x_0)) = \{x_0\}$ and the map germ $f : (\mathbb{R}^2, x_0) \to (\mathbb{R}^3, f(x_0))$ is $\mathcal{A}$-equivalent to the *Whitney umbrella* (also called *cross-cap* or *pinch point*), which is the map germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ given by $(x, y) \mapsto (x, y^2, xy)$.

The normal crossings condition means that f either presents transverse double points, which appear along a smooth curve in U, or triple transverse points, which appear as isolated points. We denote by $D^3(f) \subset U$ the set of triple points and by $D^2(f) \subset U$ the closure of the double point curve (so that it also includes $D^3(f)$ and Σ).

Finite determinacy is a very desirable property in the study of function and map germs. However, to test finite determinacy for map germs using the definition, under $\mathcal{A}$-equivalence, can be quite hard. In the complex analytic case there is a geometrical characterization due to Mather–Gaffney [21]. Roughly speaking, it says that a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is finitely determined if and only if it has an isolated instability at the origin. In the real analytic case, $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is finitely determined if and only if its complexification $\hat{f} : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is also finitely determined. In particular, if $n = 2$ and $p = 3$, there is the following immediate consequence:

Lemma 2.1. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then there is a representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $f^{-1}(0) = \{0\}$ and the restriction $f|_{U \setminus \{0\}}$ is an immersion with only transverse double points.*

It is due to the work of D. Mond [17] on the geometry of map germs from surfaces to 3-space that finite determinacy of $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is equivalent to the finiteness of three analytic invariants, namely $C(f)$ the number of pinch points, $T(f)$ the number of triple points and $N(f)$ that measures, in some sense (sic) the non-transverse self-intersections.

On the other hand, it is also very convenient to give an analytic structure to the double point curve $D^2(f)$. In fact, if $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a finite map, then we consider the scheme $D^2(f) = f^{-1}(V(\mathcal{F}_1(f)))$, where $\mathcal{F}_1(f)$ denotes the first Fitting ideal of a presentation matrix of the induced map $f^* : \mathcal{O}_3 \to \mathcal{O}_2$. It follows that the underlying set germ coincides with the double point set as defined previously [18]. We will denote by $\mu(D^2(f))$ the Milnor number of the plane curve $D^2(f)$ and call it the *Mond number* of $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. The first named author and Mond show in [13] that f is finitely determined if and only if $D^2(f)$ has isolated singularity:

Lemma 2.2. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finite map germ. Then it is finitely determined if and only if its Mond number is finite.*

If f has corank 1 (which will be always the case in here), we can compute the analytic structure of $D^2(f)$ by using a simpler method, which is described in [13]. After changes of coordinates in the source and target, f assumes the normal form $f(x, y) = (x, p(x, y), q(x, y))$. Now consider $\widetilde{D}^2(f)$ the curve in $(\mathbb{R}^3, 0)$ (or $(\mathbb{C}^3, 0)$ in the complex case) defined by equations

$$\frac{p(x, y) - p(x, u)}{y - u} = \frac{q(x, y) - q(x, u)}{y - u} = 0.$$

Thus $D^2(f)$ is obtained from $\widetilde{D}^2(f)$ by taking the projection onto the (x, y)-plane. Therefore, the corresponding equation for $D^2(f)$ follows by the elimination of the u variable in the above equations.

We finish this section with the following result due to Mond [16], which gives a partition of all corank 1 map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ according to its 2-jet. We will denote by $J^2(2, 3)$ the space of 2-jets $j^2 f(0)$ of map germs $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ and by $\Sigma^1 J^2(2, 3)$ the subset of 2-jets of corank 1. Moreover, $\mathcal{A}^2$ denotes the space of 2-jets of diffeomorphisms in the source and target.

Proposition 2.3 *(Classification of 2-jets). There exist four orbits in $\Sigma^1 J^2(2, 3)$ under the action of $\mathcal{A}^2$, which are*

$$\big(x, y^2, xy\big), \qquad \big(x, y^2, 0\big), \qquad (x, xy, 0), \qquad (x, 0, 0).$$

Note that the class (x, y^2, xy) corresponds to the Whitney umbrella. Moreover, the first and second classes can be labeled by the Boardman symbol $\Sigma^{1,0}$.

Here we will restrict ourselves to the classes (x, y^2, xy), $(x, y^2, 0)$ or $(x, xy, 0)$. We are interested in studying the topological structure of these map germs. We postpone the study of map germs in the $(x, 0, 0)$ class and map germs of corank 2. The reader should be warned they can present very complicated configurations.

3. The doodle of a finitely determined singularity

We use Fukuda's result [8], which implies that any finitely determined map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, with $n \leqslant p$ has a cone structure over its link. This link is obtained by intersecting the image of a representative of f with a sufficiently small sphere S_ϵ^{p-1} centered at the origin in $\mathbb{R}^p$.

The conic structure of the singularities is a classic theme. For isolated complex hypersurface singularities, it follows from the Milnor fibration theorem [15]. It was generalized by Burghelea and Verona [5] for arbitrary complex analytic set germs. Fukuda's technique can be applied to smooth map germs.

Theorem 3.1. *(See [8].) Suppose $n \leqslant p$. Then given a semi-algebraic subset W of $J^r(n, p)$, there exist an integer s, depending only on n, p and r, and a closed semi-algebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$ having codimension $\geqslant 1$ such that for any C^∞ mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s f(0)$ belonging to $(\pi_r^s)^{-1}(W) \setminus \Sigma_W$, there exists a positive number ϵ_0 such that for any number ϵ with $0 < \epsilon \leqslant \epsilon_0$ we have*

(1) *$\tilde{S}_\epsilon^{n-1} = f^{-1}(S_\epsilon^{p-1})$ is a homotopy $(n-1)$-sphere which, if $n \neq 4, 5$ is diffeomorphic to the natural $(n-1)$-sphere S^{n-1},*
(2) *the restricted mapping $f|\tilde{S}_\epsilon^{n-1} : \tilde{S}_\epsilon^{n-1} \to S_\epsilon^{p-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair),*

(3) *letting* $\tilde{D}_\epsilon^{n-1} = f^{-1}(D_\epsilon^{p-1})$, *the restricted mapping* $f|\tilde{D}_\epsilon^{n-1} \setminus \{0\} : \tilde{D}_\epsilon^{n-1} \setminus \{0\} \to D_\epsilon^{p-1} \setminus \{0\}$ *is proper, topologically stable* (C^∞ *stable if* (n, p) *is nice*) *and topologically equivalent* (C^∞ *equivalent if* (n, p) *is nice*) *to the product mapping*

$$\left(f|\tilde{S}_\epsilon^{n-1}\right) \times \mathrm{id}_{(0,\epsilon)} : \tilde{S}_\epsilon^{n-1} \times (0,\epsilon) \to S_\epsilon^{p-1} \times (0,\epsilon)$$

defined by $(x,t) \mapsto (f(x),t)$, *and*

(4) *consequently,* $f|\tilde{D}_\epsilon^{n-1} : \tilde{D}_\epsilon^{n-1} \to D_\epsilon^{p-1}$ *is topologically equivalent to the cone*

$$C\left(f|\tilde{S}_\epsilon^{n-1}\right) : \tilde{S}_\epsilon^{n-1} \times [0,\epsilon)/\tilde{S}_\epsilon^{n-1} \times \{0\} \to S_\epsilon^{p-1} \times [0,\epsilon)/S_\epsilon^{p-1} \times \{0\}$$

of the stable mapping $f|\tilde{S}_\epsilon^{n-1} : \tilde{S}_\epsilon^{n-1} \to S_\epsilon^{p-1}$ *defined by* $C(f|\tilde{S}_\epsilon^{n-1})(x,t) = (f(x),t)$.

Corollary 3.2. *Suppose* $n \leqslant p$ *and let* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *be a finitely determined map germ. Then there is* $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ $\mathcal{A}$*-equivalent to* f *and there exist a representative* $g : U \subset \mathbb{R}^n \to \mathbb{R}^p$ *and a positive number* ϵ_0 *such that* (1)–(4) *of Theorem* 3.1 *hold, for any* ϵ *with* $0 < \epsilon \leqslant \epsilon_0$.

Proof. Assume that f is r-determined for some r and let $W = \{j^r f(0)\}$. By the above theorem there is an s, and a closed semi-algebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$ having codimension $\geqslant 1$ such that for any C^∞ mapping $g : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s g(0)$ belonging to $(\pi_r^s)^{-1}(W) \setminus \Sigma_W$, there exists $\epsilon_0 > 0$ such that (1), (2), (3) and (4) of Theorem 3.1 hold, for any ϵ with $0 < \epsilon \leqslant \epsilon_0$.

Since $(\pi_r^s)^{-1}(W) \setminus \Sigma_W \neq \emptyset$, we can take a map $g : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s g(0) \in (\pi_r^s)^{-1}(W) \setminus \Sigma_W$. This implies that $j^r g(0) = j^r f(0)$ and g is $\mathcal{A}$-equivalent to f. $\square$

Definition 3.3. Suppose $n \leqslant p$ and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a finitely determined map germ. We say that the stable map $f|\tilde{S}_\epsilon^{n-1} : \tilde{S}_\epsilon^{n-1} \to S_\epsilon^{p-1}$ is the *link* of f, where f is a representative such that (1), (2), (3) and (4) of Theorem 3.1 hold, for any ϵ with $0 < \epsilon \leqslant \epsilon_0$. This is well defined up to $\mathcal{A}$-equivalence.

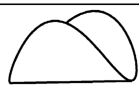
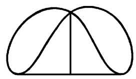
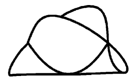
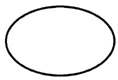
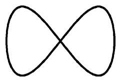
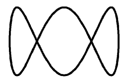
From now on we turn our attention to the case $(n, p) = (2, 3)$. By the property (4) of Theorem 3.1, any finitely determined map germ is topologically equivalent to the cone of its link. In particular, we have the following immediate consequence, which implies that the topological classification of finitely determined maps germs from $(\mathbb{R}^2, 0)$ into $(\mathbb{R}^3, 0)$ can be reduced to the topological classification of a certain subset of doodles.

Corollary 3.4. *Two finitely determined map germs* $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *are topologically equivalent if and only if their associated doodles are topologically equivalent.*

Example 3.5. In Table 1 we present some simple examples of images of finitely determined mappings and their associated doodles.

In order to describe the topology of a doodle we use the Gauss word. This concept was introduced by Gauss [10] who studied the problem of “realizability” in the plane of Gauss words, a problem similar in nature to the “planarity” problem for graphs.

Table 1

	Regular	Cross-cap	S_1^-
Map germ	(x, y^2, y)	(x, y^2, yx)	$(x, y^2, y(y^2 - x^2))$
Surface			
Doodle			
Gauss word	$\emptyset$	aa^{-1}	$ab^{-1}ba^{-1}$

Definition 3.6. Let $\gamma : S^1 \to S^2$ be a doodle with r crossings. We choose r letters $a_1, \dots, a_r$ which label them. We also fix orientations on both S^1 and S^2 and choose a base point $z_0 \in S^1$. We consider a permutation

$$\sigma : \{1, \dots, 2r\} \to \{a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1}\},$$

constructed as follows: We denote by $z_1, \dots, z_{2r} \in S^1$ the source double points ordered such that $z_0 \leqslant z_1 < \cdots < z_{2r}$. Assume that $\gamma(z_i) = \gamma(z_j) = a_k$ with $i < j$. Then we put $\sigma(i) = a_k$ and $\sigma(j) = a_k^{-1}$ if the pair of tangent vectors $(\gamma'(z_i), \gamma'(z_j))$ is positively oriented in S^2 or $\sigma(i) = a_k^{-1}$ and $\sigma(j) = a_k$ otherwise.

As usual, when working with permutations, in order to simplify the notation, we will identify the permutation σ with the sequence $\sigma(1) \dots \sigma(2r)$. This sequence is called the *signed Gauss word* of the doodle γ. The *non-signed Gauss word* is defined by just forgetting the exponents in the signed Gauss word. In Section 5 below, this non-signed version will be more appropriate to describe some topological properties of the doodle. For simplicity, we will use Gauss word for the signed version, unless otherwise specified.

It is obvious that the Gauss word is not uniquely determined, since it depends on the labels $a_1, \dots, a_r$, the chosen orientations on both S^1 and S^2 and on the base point $z_0 \in S^1$. Different choices will produce the following changes in the Gauss word:

(1) permuting the alphabet set $a_1, \dots, a_r$;
(2) cyclically permuting the sequence which defines the Gauss word;
(3) reversing the sequence;
(4) changing all the exponents from $+1$ to -1 and vice versa.

We say that two Gauss words are *equivalent* if they are related by means of these four operations. Up to this equivalence, the Gauss word is well defined and has the following property: two doodles are topologically equivalent if and only if their Gauss words are equivalent.

There are known classifications of doodles with a low number of crossings. On the other hand, given a Gauss word, it is not always possible to immerse the corresponding curve on the 2-sphere. For instance, $aba^{-1}b^{-1}$ requires a genus 1 surface to be immersed. We refer to [6] for details and pictures.

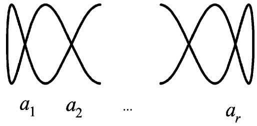


Fig. 4. The doodle μ_r.

Example 3.7. Let us consider the curve $\mu_r : [0, 2\pi] \to \mathbb{R}^2$ given by $\mu_r(t) = (\sin(t), \sin(rt))$, if r is even, or $\mu_r(t) = (\sin(t), \cos(rt))$ if r is odd. This corresponds to a closed immersed plane curve with r crossings (Fig. 4).

Embedding this plane curve on the sphere S^2 (for instance, through the inverse of the stereographic projection), we obtain a doodle which also will be denoted by μ_r. Its Gauss word is equal to

$$\begin{cases} a_1 a_2^{-1} a_3 a_4^{-1} \dots a_r^{-1} a_r \dots a_2 a_1^{-1}, & \text{if } r \text{ is even,} \\ a_1 a_2^{-1} a_3 a_4^{-1} \dots a_r a_r^{-1} \dots a_2 a_1^{-1}, & \text{if } r \text{ is odd.} \end{cases}$$

For instance, the three doodles given in Table 1 correspond to μ_0, μ_1 and μ_2, respectively.

Now we can reformulate Corollary 3.4 in terms of the Gauss words.

Corollary 3.8. *Two finitely determined map germs $f, g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ are topologically equivalent if and only if the Gauss words of their associated doodles are equivalent.*

Remark 3.9. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a smooth map germ and assume that there is a representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $f^{-1}(0) = \{0\}$ and the restriction $f|_{U\setminus\{0\}}$ is an immersion with only transverse double points. Then it is possible to associate a Gauss word which coincides with the Gauss word of the doodle in the case that f is finitely determined (see Lemma 2.1).

In fact, our construction is purely topological. Hence, the assertion is also true if f is just continuous and $f|_{U\setminus\{0\}}$ is a topological immersion with only transverse double points.

By shrinking the neighbourhood U if necessary, we can assume that $D^2(f) \subset U$ is simply connected. Then $f(D^2(f))$ has a tree structure with one vertex at the origin of $\mathbb{R}^3$ and r adjacent edges labeled by r letters $a_1, \dots, a_r$. Analogously, $D^2(f)$ also has a tree structure with one vertex at the origin of $\mathbb{R}^2$ and $2r$ adjacent edges labeled by $X_1, \dots, X_{2r}$. We assume that the components are ordered $X_1 < \dots < X_{2r}$ according to the orientation of the plane $\mathbb{R}^2$. Of course, if $f(X_i) = f(X_j) = a_k$, then we will write the letters a_k and a_k^{-1} in positions i and j, but we have two possibilities.

Assume $i < j$. We orient all the connected components X_i, X_j, a_k from the origin. We choose points $y \in a_k$, $x_i \in X_i$ and $x_j \in X_j$ such that $f(x_i) = f(x_j) = y$. Let B a small ball around y in $\mathbb{R}^3$ such that $f^{-1}(B) \setminus D^2(f)$ has four connected components: U_i^+ and U_i^- are on the left and right of x_i respectively and U_j^+ and U_j^- are on the left and right of x_j, respectively. We look at the orientation of $(f(U_i^+), f(U_j^+))$ with respect to a_k (Fig. 5). If they are positively oriented we put a_k in position i and a_k^{-1} in position j. Otherwise, we take the opposite positions.

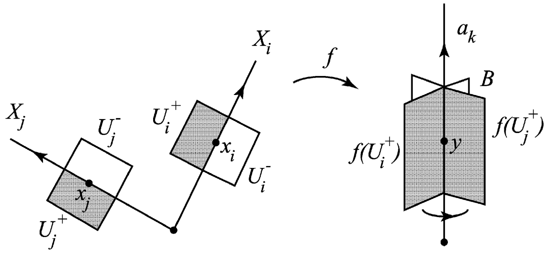


Fig. 5.

4. The doodle of a map germ of type $\Sigma^{1,0}$

The aim of this and the next sections is to describe the topology of a large class of finitely determined map germs by means of their associated doodles. As it was mentioned in Section 2, we are only interested in corank 1 map germs whose 2-jet belong to the orbits (x, y^2, xy), $(x, y^2, 0)$ or $(x, xy, 0)$ (see Proposition 2.3). That is, we will not consider map germs in the orbit $(x, 0, 0)$ nor corank 2 map germs.

Here we study the topological classification of all finitely determined map germs in the orbits (x, y^2, xy) or $(x, y^2, 0)$. Both orbits can be labeled together under the Boardman symbol $\Sigma^{1,0}$ and they correspond to corank 1 map germs with no triple points (i.e., $T(f) = 0$).

Note that the (x, y^2, xy) orbit correspond to a single $\mathcal{A}$-class, namely, the cross-cap. It has the simplest non-trivial doodle with Gauss word aa^{-1}. In general, if f has type $\Sigma^{1,0}$ and its double point curve $D^2(f)$ has r real branches, its associated doodle is equivalent to μ_r (see Example 3.7). To see this, we need the following lemma, which gives a very convenient prenormal form for f.

Lemma 4.1. *(See [16].) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a map germ. Then f has type $\Sigma^{1,0}$ if and only if it has corank* 1 *and $T(f) = 0$. Moreover, if it has type $\Sigma^{1,0}$ then f is $\mathcal{A}$-equivalent to a germ of the form*

$$(x, y) \mapsto \big(x, y^2, yp\big(x, y^2\big)\big),$$

for some function germ $p \in \mathcal{E}_2$.

Theorem 4.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ of type $\Sigma^{1,0}$. Then its doodle its equivalent to μ_r, where r is the number of branches of $D^2(f)$.*

Proof. By the above lemma we can assume that $f(x, y) = (x, y^2, yp(x, y^2))$, for some function germ p such that $p(0, 0) = 0$. This implies that the double point curve is given by $p(x, y^2) = 0$. In particular, such a curve is symmetric with respect to the x axis and the pairs of points sharing the same image are of the form $z = (x, y)$ and $\bar{z} = (x, -y)$.

On the other hand, the link of f is $f|\tilde{S}^1_\epsilon : \tilde{S}^1_\epsilon \to S^2_\epsilon$ and the domain $\tilde{S}^1_\epsilon$ is given in the plane (x, y) by equation

$$x^2 + y^4 + y^2 p\big(x, y^2\big)^2 = \epsilon^2.$$

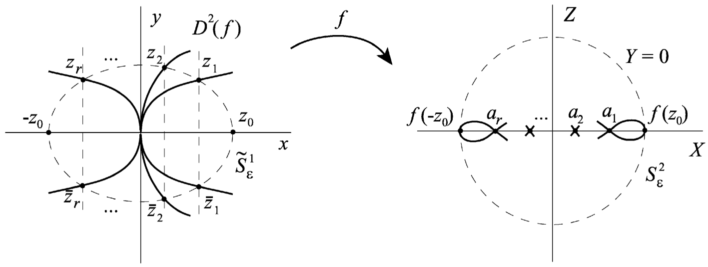


Fig. 6.

This determines a simple closed curve which is also symmetric with respect to the x axis and which meets transversely the double point curve at points $z_1, \dots, z_r$ and $\bar{z}_1, \dots, \bar{z}_r$. We assume that $z_i = (x_i, y_i)$ and $\bar{z}_i = (x_i, -y_i)$, with $x_1 \geqslant \cdots \geqslant x_k$ and $y_i > 0$, $i = 1, \dots, r$. We choose $z_0 = (\epsilon, 0)$ as the base point. We deduce that the non-signed Gauss word is equal to

$$a_1 a_2 \dots a_r a_r \dots a_2 a_1,$$

where each a_i is the label of the crossing $f(z_i)$ (Fig. 6).

Moreover, we also see that the doodle $f(\tilde{S}^1_\epsilon)$ has the following properties:

(1) The doodle is contained in the hemisphere $Y \geqslant 0$ of S^2_ϵ and intersects the equator $Y = 0$ at the base point $f(z_0)$ and its opposite $f(-z_0)$.
(2) It is symmetric with respect to the meridian $Z = 0$.
(3) The doodle intersects the meridian $Z = 0$ only at the double points $a_1, \dots, a_r$, together with $f(z_0)$ and $f(-z_0)$. Moreover, they present the following relative position on the meridian:

$$f(-z_0) < a_r < \cdots < a_1 < f(z_0).$$

The right-hand side of Fig. 6 shows the projection of the doodle into the XZ-plane, so that the interior of the dashed circle corresponds to the projection of the hemisphere $Y \geqslant 0$ of S^2_ϵ. It is easy to see that the only doodle which verifies these conditions is the doodle μ_r of Example 3.7. This concludes the proof. $\square$

Remark 4.3. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a continuous map germ. We assume there is a representative $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $f^{-1}(0) = \{0\}$ and $f|_{U \setminus \{0\}} : U \setminus \{0\} \to \mathbb{R}^3$ is a topological immersion with only transverse double points. We have seen in Remark 3.9 that f has a well-defined Gauss word which coincides with that of the doodle if it is smooth and finitely determined.

We claim that it is possible to extend the above theorem for this class of map germs: Let f be a map germ defined by $f(x, y) = (x, y^2, p(x, y))$ where p is even in y (i.e., $p(x, y) = p(x, -y)$). Then the Gauss word of f coincides with that of the doodle μ_r.

In fact, the double point curve $D^2(f)$ is given in $U \setminus \{0\}$ as the set of points (x, y) such that $p(x, y) = 0$ and $y \neq 0$. Hence, it is symmetric with respect to the x axis. Moreover, if (x, y) is one of these points, then $f(x, y) = f(x, -y)$.

On the other hand, the image of the double point curve $f(D^2(f))$ also presents the required symmetries. Note that it is defined outside the origin by equations $Z = 0$, $p(X, \sqrt{|Y|}) = 0$ and

$Y \neq 0$, where (X, Y, Z) denote the coordinates of $\mathbb{R}^3$. Therefore, all the arguments used in the proof of the theorem are also valid here.

Example 4.4. Let M_r the map germ given by

$$M_r(x, y) = \left(x, y^2, \Im\left((x+iy)^{r+1}\right)\right),$$

for $r \geqslant 0$ and where $\Im(z)$ denotes the imaginary part of $z \in \mathbb{C}$. In this case, the equation for the double point curve is given by

$$p\left(x, y^2\right) = \frac{1}{y}\Im\left((x+iy)^{r+1}\right),$$

which is a homogeneous polynomial of degree r. It is the product of r linear forms which are pairwise non-collinear. This implies that $p(x, y^2)$ has isolated singularity and hence, the map germ M_r is finitely determined. Moreover, the double point curve is the union of r distinct lines through the origin. Therefore, the associated doodle of M_r is μ_r.

Corollary 4.5. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ of type $\Sigma^{1,0}$. Then it is topologically equivalent to the map germ M_r, where r is the number of branches of $D^2(f)$.*

Remark 4.6. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ of type $\Sigma^{1,0}$ given by $f(x, y) = (x, y^2, yp(x, y^2))$. Then the number r of crossings of the associated doodle is equal to the number of branches of the plane curve germ $p(x, y^2) = 0$. As a consequence,

$$r = 1 - \operatorname{ind}\left(\nabla p\left(x, y^2\right)\right),$$

where $\operatorname{ind}(\nabla p(x, y^2))$ denotes the topological index of the gradient vector field of $p(x, y^2)$ (see [3]).

5. The doodle of a map germ of type $(x, xy, 0)$

In this section, map germs whose 2-jet is in the $(x, xy, 0)$ orbit are considered. We will distinguish two types of finitely determined map germs in this orbit, namely, fold and cusp types. For the fold type we obtain the complete topological classification. In fact, they are topologically equivalent to that of type $\Sigma^{1,0}$ (see Theorem 4.2). For the map germs of cusp type, we assume it is non-degenerate in some sense that we will make precise later, in order to obtain a topological classification. Moreover, all the results in this section are described by the non-signed Gauss word.

Definition 5.1. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ whose 2-jet belongs to the $(x, xy, 0)$ orbit. We can assume that f is written in the form

$$f(x, y) = \left(x, xy + g(x, y), h(x, y)\right),$$

for some function germs $g, h \in m_2^3$. Since f is finitely determined, it follows that g must have finite order in y, that is, $g(0, y) = a_k y^k + a_{k+1} y^{k+1} + \cdots$ with $a_k \neq 0$, $k \geqslant 3$. We say that f has *fold type* if k is even or *cusp type* if k is odd.

Lemma 5.2. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a finitely determined map germ whose 2-jet belongs to the $(x,xy,0)$ orbit. Then f is topologically $\mathcal{A}$-equivalent to a map germ of the form $(x,y^2,p(x,y))$, if f has fold type, or $(x,xy+y^3,p(x,y))$, if f has cusp type, being $p(x,y)$ a continuous function germ.*

Proof. We consider the map germ from the plane to the plane $\tilde{f}:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$, given by $\tilde{f}(x,y)=(x,xy+g(x,y))$, where g has order k in y. We also consider weights $(k-1,1)$ for the variables (x,y) and denote by $\tilde{f}_0$ the initial part of $\tilde{f}$, that is, $\tilde{f}_0(x,y)=(x,xy+a_ky^k)$. Since $\tilde{f}_0$ is finitely determined, we can use a result of Damon [7] which implies that $\tilde{f}$ is topologically $\mathcal{A}$-equivalent to $\tilde{f}_0$.

The singular set of $\tilde{f}_0$ is defined by the smooth curve $x+ka_ky^{k-1}=0$. Moreover, by analyzing the number of preimages of a generic value one concludes that $\tilde{f}_0$ is topologically $\mathcal{A}$-equivalent to either the fold (x,y^2), if k is even, or the cusp $(x,xy+y^3)$, if k is odd. Finally, we extend these topological equivalences to $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ in order to obtain the desired map germs. □

Theorem 5.3. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a finitely determined map germ whose 2-jet belongs to the $(x,xy,0)$ orbit and has fold type. Then it is topologically $\mathcal{A}$-equivalent to the germ M_r, where r is the number of branches of $D^2(f)$.*

Proof. By the above lemma, f is topologically $\mathcal{A}$-equivalent to a map germ of the form $\tilde{f}(x,y)=(x,y^2,p(x,y))$, where p is a continuous function germ. We write $p=p_1+p_2$, where

$$p_1(x,y)=\frac{p(x,y)-p(x,-y)}{2},\qquad p_2(x,y)=\frac{p(x,y)+p(x,-y)}{2},$$

so that p_1,p_2 are respectively odd and even functions in y. Then we consider the homeomorphism $\psi:(\mathbb{R}^3,0)\to(\mathbb{R}^3,0)$ given by

$$\psi(X,Y,Z)=\bigl(X,Y,Z-p_2\bigl(X,\sqrt{|Y|}\bigr)\bigr),$$

which give us $\psi(\tilde{f}(x,y))=(x,y^2,p_1(x,y))$. Now, we use the same argument of the proof of Theorem 4.2 which is also valid in this situation. □

Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a finitely determined map germ whose 2-jet belongs to the $(x,xy,0)$ orbit and has cusp type. We can assume without loss of generality that f is written in the form

$$f(x,y)=\bigl(x,xy+y^k+\cdots,h(x,y)+\cdots\bigr),$$

where $k\geqslant 3$ is odd, $h\in m_2^3$ is a weighted homogeneous polynomial with weights $(k-1,1)$ and the dots denote higher order terms with respect to the weighted grading. We want to find the defining equations for the double set curves $\tilde{D}^2(f)$ and $D^2(f)$. Following [13], we represent a point of $\tilde{D}^2(f)$ as a triple (x,y,u), so that $f(x,y)=f(x,u)$ and the equations of $\tilde{D}^2(f)$ have the initial part given by

$$x+y^{k-1}+y^{k-2}u+\cdots+u^{k-1}=0,\qquad \frac{h(x,y)-h(x,u)}{y-u}=0.$$

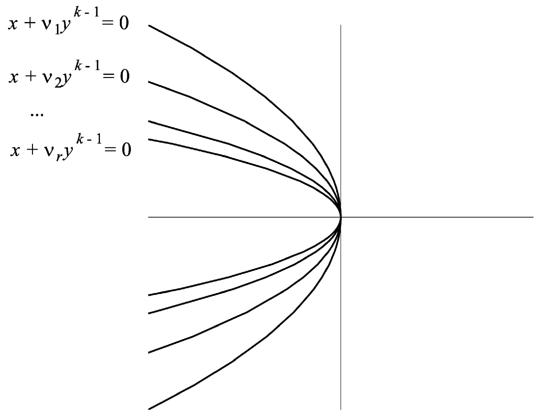


Fig. 7.

Eliminating x in the second equation give us a homogeneous equation $H(y,u)=0$, which is also symmetric in (y,u). We decompose this equation in a product of real linear factors to obtain:

$$x + y^{k-1} + y^{k-2}u + \cdots + u^{k-1} = 0, \qquad (u - \lambda_1 y)\ldots(u - \lambda_r y) = 0,$$

for some $\lambda_i \in \mathbb{R} \cup \{\infty\}$ (in order to simplify the notation we also include the case $\lambda_i = \infty$ which corresponds to the factor y). Moreover, for each factor with slope λ_i there is also its symmetric factor which has slope $\lambda_j = 1/\lambda_i$.

The other double point curve is $D^2(f)$, which is defined as the projection of $\tilde{D}^2(f)$ onto the (x,y)-plane. The equation for $D^2(f)$ is obtained by eliminating the u variable in the equations of $\tilde{D}^2(f)$. Again, the initial part of this equation can be computed easily:

$$\bigl(x + \nu_1 y^{k-1}\bigr)\ldots\bigl(x + \nu_r y^{k-1}\bigr) = 0, \tag{1}$$

where $\nu_i = \lambda_i^{k-1} + \lambda_i^{k-2} + \cdots + 1$, for any $i = 1,\ldots,r$. Since k is odd, we always have $\nu_i > 0$ (except the case $\lambda_i = \infty$, which again corresponds to the factor y). Thus, each one of the branches of $D^2(f)$ has a defining equation with initial part $x + \nu_i y^{k-1} = 0$, that is, a parabola tangent to the y axis in the left half-plane.

Definition 5.4. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ whose 2-jet belongs to the $(x, xy, 0)$ orbit and has cusp type. We will say it is *non-degenerate* if $\nu_i \neq \nu_j$ for any $i \neq j$, where $\nu_1, \ldots, \nu_r$ are defined by Eq. (1). This guarantees that all branches of $D^2(f)$ have a relative position as in Fig. 7. Because of the finite determinacy of f, $D^2(f)$ must have a reduced structure (see [17]). In particular, if f is weighted homogeneous then it is non-degenerate, since in this case the equation $(x + \nu_1 y^{k-1})\ldots(x + \nu_r y^{k-1}) = 0$ is exactly the defining equation of the real part of $D^2(f)$. More generally, this is also true if f is semi-weighted homogeneous.

Notice that the non-degeneracy condition (together with the fact that f is in the $(x, xy, 0)$ orbit and of cusp type) restricts on the possible doodles associated to f. In fact, the (non-signed) Gauss word must verify, up to equivalence, the following compatibility conditions:

C_1: Assume r is odd. There is a unique branch $m \in \{1, \dots, r\}$ with slope $\lambda_m = -1$. This means that it is its own symmetric. The assigned letter a_i to the ith crossing will appear in the symmetric positions m and $2r - m$

$$\dots a_i \dots | \dots a_i \dots .$$

(The vertical bar | in the middle is used here to separate the two halves of the Gauss word.)

C_2: Otherwise, assume r is even. Then, for any $m \in \{1, \dots, r\}$ we have $\lambda_m \neq -1$ and we cannot have the same letter a_i in the symmetric positions m and $2r - m$.

C_3: Let us consider a branch with positive slope $0 < \lambda_m < 1$, where $m \in \{1, \dots, r\}$. Its symmetric branch has slope $\lambda_\ell = 1/\lambda_m$, with $\ell \in \{1, \dots, r\}$, $\ell \neq m$. So, if a_i is in the position m, it must also be in the position ℓ and if a_j is in the position $2r - m$ it must also be in the position $2r - \ell$

$$\dots a_i \dots a_i \dots | \dots a_j \dots a_j \dots .$$

C_4: Let us consider two branches with positive slope $0 < \lambda_{m_1} < \lambda_{m_2} < 1$ and their symmetric branches $\lambda_{\ell_1} > \lambda_{\ell_2} > 1$, with $\lambda_{\ell_k} = 1/\lambda_{m_k}$, $m_k, \ell_k \in \{1, \dots, r\}$. We denote the corresponding letters in the Gauss word by

$$\begin{aligned} &a_{i_1} \text{ in positions } m_1, \ell_1,\\ &a_{i_2} \text{ in positions } m_2, \ell_2,\\ &a_{j_1} \text{ in positions } 2r - m_1, 2r - \ell_1,\\ &a_{j_2} \text{ in positions } 2r - m_2, 2r - \ell_2. \end{aligned}$$

Then, the relative position of these letters in the Gauss word will be as follows:

$$\dots a_{i_1} \dots a_{i_2} \dots a_{i_2} \dots a_{i_1} \dots | \dots a_{j_1} \dots a_{j_2} \dots a_{j_2} \dots a_{j_1} \dots .$$

C_5: Finally, we consider a branch with negative slope $-1 < \lambda_m < 0$, with $m \in \{1, \dots, r\}$. Its symmetric branch has slope $\lambda_\ell = 1/\lambda_m$, with $\ell \in \{1, \dots, r\}$, $\ell \neq m$. So, if a_i is in the position m, it must also be in the position $2r - \ell$ and if a_j is in the position ℓ it must also be in the position $2r - m$

$$\dots a_i \dots a_j \dots | \dots a_i \dots a_j \dots .$$

Definition 5.5. Given a doodle we say it has a loop if a letter a_i appears in the Gauss word in two consecutive positions $\dots a_i a_i \dots$ (we also include here the case that a_i appears in the first and last positions). For instance, the doodle μ_r has exactly 2 loops. The doodles with Gauss word *aabbcc* have 3 loops and the trefoil *abcabc* does not have any loop.

In the next theorem we show that the above conditions $C_1, \dots, C_5$ will imply restrictions on the maximum number of loops of a doodle associated to a non-degenerate map germ of cusp type. We also give a classification of all the possible (non-signed) Gauss words with at most 4 crossings.

Theorem 5.6. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a finitely determined map germ in the $(x, xy, 0)$ orbit, of cusp type and non-degenerate. We denote by r the number of branches of $D^2(f)$. Then:*

(1) *The number of loops in the associated doodle of f is $\leqslant 3$.*
(2) *If r is even, the number of loops is equal to 0 or 2.*
(3) *For $r \leqslant 4$, there are only seven possible Gauss words, up to equivalence:*

$$\emptyset, \quad aa, \quad aabb, \quad aabbcc, \quad abcabc, \quad abbacddc, \quad abcdbadc.$$

Proof. Because of condition C_1, we will have at most one loop corresponding to the slope $\lambda_m = -1$ and appearing either in positions $a_i \dots | \dots a_i$ or $\dots a_i \mid a_i \dots$. Moreover, this can happens only if the number of branches of $D^2(f)$ is odd.

Apart from this, the remaining loops will have necessarily positive slope. By condition C_3, if we have one of them, we will have one loop in the first half of the Gauss word and the symmetric loop in the second half. Finally, condition C_4 implies that we cannot have more that one loop in each half of the Gauss word. This concludes the proof of (1) and (2).

Let us prove (3). If $r \leqslant 2$, our list includes, up to equivalence, all the possibilities. For $r = 3$, we have three possibilities, up to equivalence, namely $abccba$, $aabbcc$ and $abcabc$. We see that the first one is not a possible Gauss word, according to the above properties. In fact, in $abccba$ the three branches are their own symmetric, in contradiction with C_1. Moreover, the equivalent forms of this Gauss word are $bccbaa$ and $ccbaab$. They also violate C_1, since none of the branches is its own symmetric.

We assume now that $r = 4$, which give us five possible Gauss words, up to equivalence,

$$aabbccdd, \quad abbaccdd, \quad abbacddc, \quad abdcbadc, \quad aabcdbcd.$$

We start with $aabbccdd$, which is not possible to occur because of (1). The second Gauss word $abbaccdd$ clearly gives a contradiction with C_3. Analogously, the equivalent forms are again not possible: $bbaccdda$, $baccddab$, $accddabb$, $cddabbac$, $ddabbacc$ and $dabbaccd$ by C_2 and $ccddabba$ by C_3.

Finally, we use a similar analysis for $aabcdbcd$. This word and its equivalent forms $bcdbcdaa$, $cdbcdaab$, $dbcdaabc$, $bcdaabcd$, $cdaabcdb$, $daabcdbc$ are in contradiction with C_3. There are two more variants of this Gauss word, $abcdbcda$, $bcdaabcd$, which are not possible to occur because of C_2. $\square$

Example 5.7. For each one of the seven Gauss words presented in (3) above, we can find a finitely determined map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ in the $(x, xy, 0)$ orbit, of cusp type and non-degenerate which realizes such a Gauss word, that is, the associated doodle of f has the given Gauss word. Consider the P_3 singularity, from Mond's classification [16], defined by the map germ $f(x, y) = (x, xy + y^3, xy^2 + cy^4)$, which is finitely determined if $c \neq 0, \frac{1}{2}, 1, \frac{3}{2}$. It is not difficult to see that f has Gauss word

$$\begin{cases} aa, & \text{if } c < \frac{1}{2} \text{ or } c > \frac{3}{2}, \\ abcabc, & \text{if } \frac{1}{2} < c < 1, \\ aabbcc, & \text{if } 1 < c < \frac{3}{2}. \end{cases}$$

Analogously, we can also consider the map germ $f(x,y)=(x,xy+y^3,xy^3+cy^5)$. A simple computation shows that its Mond number is finite if $c\neq 0,\frac{4}{5},1,\frac{9}{5}$ and hence, f is finitely determined. Now the Gauss word of f is equal to

$$\begin{cases} \emptyset, & \text{if } c<\frac{4}{5} \text{ or } c>\frac{9}{5},\\ aabb, & \text{if } \frac{4}{5}<c<1,\\ abcdbadc, & \text{if } 1<c<\frac{9}{5}. \end{cases}$$

Finally, we find the finitely determined map germ

$$f(x,y)=\bigl(x,xy+y^3,y\bigl(x+1.1y^2\bigr)\bigl(x+2y^2\bigr)\bigl(x+4y^2\bigr)\bigr),$$

which realizes the Gauss word $abbacddc$.

The above theorem shows that for a general r, it can be very difficult to find the complete classification of all the Gauss words of finitely determined map germs f in the $(x,xy,0)$ orbit, of cusp type and non-degenerate with r branches in $D^2(f)$. However, we will see that if we add some extra restrictions we can obtain some results. In the next theorem, we will consider the case that all the slopes λ_i have the same sign.

Theorem 5.8. *Let $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ be a finitely determined map germ in the $(x,xy,0)$ orbit, of cusp type and non-degenerate. We assume that $D^2(f)$ has r branches and set $t=[r/2]$. We also denote by $\lambda_1<\cdots<\lambda_r$ the slopes of the corresponding real linear factors of $\tilde{D}^2(f)$.*

(1) *If r is even and $\lambda_i>0$, $\forall i$, the Gauss word is equivalent to*

$$a_1\dots a_t a_t\dots a_1 \mid b_1\dots b_t b_t\dots b_1.$$

(2) *If r is odd, $\lambda_1=-1$ and $\lambda_i>0$, $\forall i>1$, the Gauss word is equivalent to*

$$a_0a_1\dots a_t a_t\dots a_1 \mid b_1\dots b_t b_t\dots b_1 a_0.$$

(3) *If r is even and $\lambda_i<0$, $\forall i$, the Gauss word is equivalent to one of the form*

$$a_1\dots a_t b_1\dots b_t \mid a_{\sigma(1)}\dots a_{\sigma(t)} b_{\sigma(1)}\dots b_{\sigma(t)},$$

for some permutation $\sigma\in\Sigma_t$.

(4) *If r is odd and $\lambda_i<0$, $\forall i$, the Gauss word is equivalent to one of the form*

$$a_1\dots a_t a_0 b_1\dots b_t \mid a_{\sigma(1)}\dots a_{\sigma(t)} a_0 b_{\sigma(1)}\dots b_{\sigma(t)},$$

for some permutation $\sigma\in\Sigma_t$.

Proof. In the first case, since $\lambda_i>0$, $\forall i$, all the letters in the Gauss word are located according to condition C_4, which gives

$$a_1\dots a_t a_t\dots a_1 b_1\dots b_t b_t\dots b_1.$$

In the second case, we must add a letter a_0 corresponding to $\lambda_1 = -1$. By condition C_1, this letter will appear in the symmetric positions. Moreover, since $\lambda_i > 0$, $\forall i > 1$, we will have $\nu_i > \nu_1 = 1$, $\forall i > 1$. This implies that a_0 will appear necessarily in the first and last positions:

$$a_0 a_1 \dots a_t a_t \dots a_1 \mid b_1 \dots b_t b_t \dots b_1 a_0.$$

In the third case, we have that r is even and the slopes are organized as follows:

$$\lambda_1 < \cdots < \lambda_t < -1 < \frac{1}{\lambda_t} < \cdots < \frac{1}{\lambda_1} < 0.$$

We will denote the corresponding coefficients of the parabolas by

$$\nu_i = \lambda_i^{k-1} + \lambda_i^{k-2} + \cdots + 1, \qquad \tilde{\nu}_i = \frac{1}{\lambda_i^{k-1}} + \frac{1}{\lambda_i^{k-2}} + \cdots + 1, \quad i = 1, \dots, t.$$

Since k is odd, these coefficients will verify

$$\nu_1 > \cdots > \nu_t > 1 > \tilde{\nu}_{\sigma(1)} > \cdots > \tilde{\nu}_{\sigma(t)},$$

for some permutation $\sigma \in \Sigma_t$. It follows from this and condition C_5 that the Gauss word will have the form

$$a_1 \dots a_t b_1 \dots b_t \mid a_{\sigma(1)} \dots a_{\sigma(t)} b_{\sigma(1)} \dots b_{\sigma(t)}.$$

Finally, the case (4) is analogous to (3), just by adding one more letter a_0 corresponding to $\lambda_{t+1} = -1$. $\square$

Depending on the number of branches r, we can have more restrictions on the possible Gauss words and sometimes it is possible to improve parts (3) and (4) of 5.8 above. For instance, if $r = 4$ and $\lambda_i < 0$, $\forall i$, by (3) we could have two possibilities: *abcdabcd* and *abcdbadc*. However, the first one cannot be constructed in the sphere, that is, there is no doodle immersed on the sphere with such Gauss word. Hence, only the second Gauss word is realizable. The same happens with $r = 5$. According to (4), we could have again two possibilities, *abcdeabcde* and *abcdebaced*, but the second one is not realizable.

However, in general we can have more than one possibility for the Gauss word in the case $\lambda_i < 0$, $\forall i$. For instance, if we consider $r = 7$, we find two Gauss words:

$$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_1 a_2 a_3 a_4 a_5 a_6 a_7, \qquad a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_3 a_2 a_1 a_4 a_7 a_6 a_5.$$

Both are realizable in the sphere and compatible with conditions $C_1, \dots, C_5$.

6. Doodles from deformations

For a given doodle, is there a finitely determined map germ whose associated doodle is the given one? In this section we show how to construct such map germs for doodles obtained as deformation of some plane curve singularities.

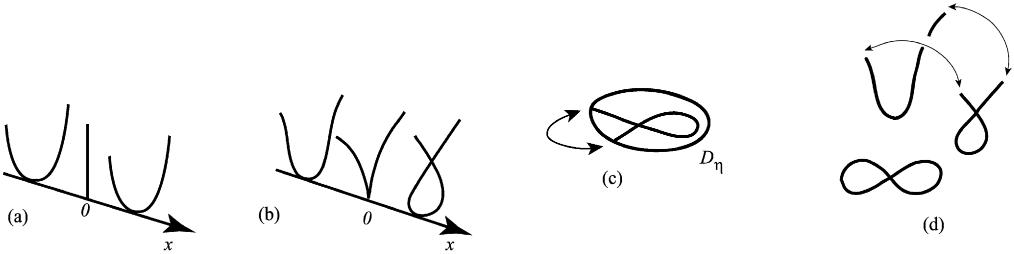


Fig. 8.

6.1. An example

The mapping $(x, y) \to (x, y^2, xy)$ can be seen as a one-parameter family of parabolas parametrized by x and degenerated at $x = 0$ (Fig. 8(a)). This mapping is topologically equivalent to $(x, y) \to (x, y^2, y^3 - xy)$, that is, a one-parameter family of deformations of the plane curve singularity $y \to (y^2, y^3)$. Here, when $x > 0$ the curves $y \to (y^2, y^3 - xy)$ are morsifications (i.e. stabilizations), while for $x < 0$ it is an embedded curve (with no self-intersection) (Fig. 8(b)). The doodle of the mapping $(x, y) \to (x, y^2, y^3 - xy)$, that is, the intersection of the image of f with a small sphere S_ϵ centered at the origin can alternatively be obtained as follows: firstly we enclose a morsification of the curve $y \to (y^2, y^3)$ in a appropriately small disc D_η (following N. A'Campo [2] we call it a *divide* of the plane curve singularity) and then we connect by an arc the two points that the divide meets the boundary of D_η (Fig. 8(c)). This arc connection is precisely what happens when we intersect the image of the mapping $(x, y) \to (x, y^2, y^3 - xy)$ with S_ϵ (Fig. 8(d)).

Indeed, the bifurcation set of the unfolding of the curve $y \to (y^2, y^3)$ is just the origin $x = 0$ and for any $x < 0$ the real trace of the curve $y \to (y^2, y^3 - xy)$ has no self-intersection (Fig. 8(b)).

6.2. The general case

We consider a real irreducible algebraic plane curve singularity $y \to (\alpha(y), \beta(y))$, whose complexification has finite Milnor number, which is the same to say that the defining equation $F(X, Y) = 0$ of the complexified curve has isolated singularity. For this set of curves the work of N. A'Campo [1] or S. Guseĭn-Zade [11] applies and hence there exist real morsifications for this plane curve singularities with $\mu(F)/2$ transverse self-intersections.

The values of y mapped by $y \to (\alpha(y), \beta(y))$ to the self-intersection points are closely related to the roots of the equations $\alpha(y) = 0$ and $\beta(y) = 0$. For this reason the bifurcation set of a versal unfolding of the curve $y \to (\alpha(y), \beta(y))$ is closely related to the discriminants of algebraic equations (Fig. 9(a)). We refer to Arnold's book [4] for an extensive treatment on the topic.

In particular, the space of parameters of a versal unfolding of a simple plane curve singularity $y \to (\alpha(y), \beta(y))$ is partitioned by the bifurcation set into path connected components, all of them being algebraic submanifolds of dimension greater than zero and having the origin in their adherence.

Another property of the discriminant of an algebraic equation of degree n, say, $x^n + a_{n-2}x^{n-2} + \cdots + a_1x + a_0 = 0$ is that it splits the $(n - 1)$-space of the coefficients into path connected components, at least one of which corresponding to the coefficients a_i of the equations with n distinct real roots and another component corresponding to the equations with n non-real roots, the origin being on the adherence of this two components (Fig. 9(a)).

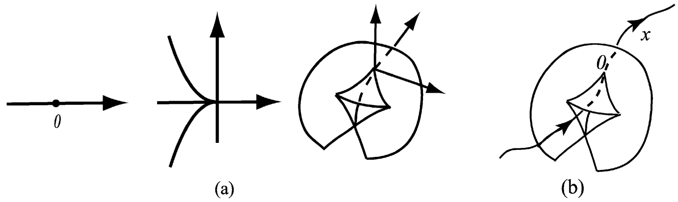


Fig. 9.

Analogously, the bifurcation set of a versal deformation of $y \to (\alpha(y), \beta(y))$ splits the parameter space of the versal unfolding into path connected components, at least one of them (cf. A'Campo or Guseĭn-Zade) corresponding to real morsifications ($\mathbb{R}$cc) and another to non-real morsifications ($\mathbb{C}$cc) having the origin on the adherence. Also, the bifurcation set itself is stratified and each stratum has the origin on its adherence.

Thus, we can choose a one-parameter unfolding of $y \to (\alpha(y), \beta(y))$ with the unfolding parameter x varying on a curve belonging to one component $\mathbb{R}$cc, passing through the origin and into one component $\mathbb{C}$cc (Fig. 9(b)).

Thus, the corank 1 mapping $(x, y) \to (x, p(x, y), q(x, y))$, where $p(x, y)$ is a deformation of $\alpha(y)$ with deformation parameter x such that $p(0, y) = \alpha(y)$ and the same for $q(x, y)$ and $\beta(y)$, is a one-parameter family of morsifications of the curve $y \to (\alpha(y), \beta(y))$, when x varies on a component of type $\mathbb{R}$cc and embedded curves for $x = 0$ or x in a component of type $\mathbb{C}$cc. A small sphere intersect the image of this mapping in a doodle topologically equivalent to the one obtained by connecting by an arc the two points that a divide of $y \to (\alpha(y), \beta(y))$ intersects the boundary of a sufficiently small disc.

In general, we can consider one-parameter families of deformations of the same curve $y \to (\alpha(y), \beta(y))$, having the unfolding parameter x varying on many distinct path connected components of the complement of the bifurcation set in the parameter space, or on one of the strata of the bifurcation set itself, so long we choose on one side a connected component that realizes at least one self-intersection and on the other side x varies on the component $\mathbb{C}$cc, to accomplish the arc connection of the boundary points of the divide. So, these choices can give rise to distinct doodles associated to the same plane curve singularity, in particular its adjacent singularities.

Lets us see this in two examples. The first one, the singularity A_{2k} has only one component $\mathbb{R}$cc, while the other example below, namely the singularity E_6 has two.

So let us start with the curve A_4, that is, $y \to (y^2, y^5)$. Here the $\mathcal{A}$-versal unfolding is $(y; a, b) \to (y^2, y^5 + by^3 + ay; a, b)$ and the bifurcation set in the parameter space of coordinates (a, b) is composed by the curves $a = 0$ and $b^2 - 4a = 0$, with $a < 0$ (Fig. 10(a)). If we make the deformation parameters vary on a curve within the region $\mathbb{R}$cc, passing through the origin and into the region $\mathbb{C}$cc we obtain the desired mapping whose doodle is the divide of the plane curve singularity $y \to (y^2, y^5)$ connecting the two boundary points by an arc. For instance, if we take $b = -x$ and $a = \frac{x^2}{5}$, that is, if we make the unfolding parameters (a, b) to vary over the parabola $x \to (\frac{x^2}{5}, -x)$ (Fig. 10(b)) then the mapping $(x, y) \to (x, y^2, y^5 - xy^3 + \frac{x^2}{5} y)$ satisfies the requirement, that is, the associated doodle has Gauss word $aa^{-1}bb^{-1}$.

Now, let us make the unfolding parameters (a, b) to travel on another curve in the (a, b)-plane, going from the third quadrant, through the origin and into the first quadrant. Say, on the

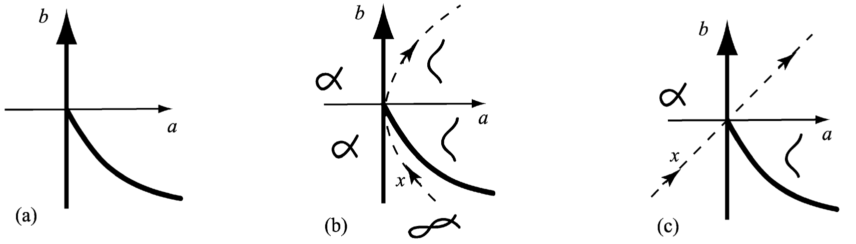


Fig. 10.

Fig. 11.

line $x \to (x, x)$ (Fig. 10(c)). So, the mapping $(x, y) \to (x, y^2, y^5 + xy^3 + xy)$ has doodle whose Gauss word is aa^{-1}.

More generally, $y \to (y^2, y^{2k+1})$ has $\mathcal{A}$-versal unfolding $(y; a_1, \ldots, a_k) \to (y^2, y^{2k+1} + a_1 y^{2k-1} + a_2 y^{2k-3} + \cdots + a_k y; a_1, \ldots, a_k)$. So, if $y \to (y^2, y^{2k+1} + b_1 y^{2k-1} + b_2 y^{2k-3} + \cdots + b_k y)$ is a morsification of the plane curve singularity $y \to (y^2, y^{2k+1})$ then making $(a_1, a_2, \ldots, a_k)$ to vary over the curve $x \to (b_1 x, b_2 x^2, \ldots, b_k x^k)$, in the k-space of unfolding parameters then the mapping $(x, y) \to (x, y^2, y^{2k+1} + b_1 x y^{2k-1} + b_2 x^2 y^{2k-3} + \cdots + b_k x^k y)$ is one whose doodle is μ_k (see Fig. 4).

Now we consider the plane curve singularity E_6, that is, $y \to (y^3, y^4)$. In this case, the bifurcation set of the $\mathcal{A}$-versal unfolding $(y; a, b, c) \to (y^3 + ay, y^4 + by^2 + cy; a, b, c)$ provides two connected components of type $\mathbb{R}$cc, that is, there are two distinct real morsifications (Fig. 11). It should be remarked that the work of A'Campo or Guseĭn-Zade does not describe all possible real deformations of a given plane curve singularity. We learn from F. Scalco Dias [9] that if we make $c = 0$, $a = -1$ and $b = \beta - 1$ then for sufficiently small absolute values of β we obtain the two morsifications taking $\beta < 0$ or $\beta > 0$. More precisely, the mapping $(x, y) \to (x, y^3 - xy, y^4 + (\beta - 1)xy^2)$, or equivalently the mapping $(x, y) \to (x, y^3 + xy, y^4 + (1 - \beta)xy^2)$ is one whose doodle has Gauss word $ab^{-1}ca^{-1}bc^{-1}$ when $\beta < 0$ while for $\beta > 0$ the corresponding Gauss word is $aa^{-1}bb^{-1}cc^{-1}$.

6.3. Chebyshev doodles

The monomial curves $y \to (y^p, y^q)$, with $(p, q) = 1$ provide an interesting set of examples. As A'Campo said [2], the Chebyshev polynomials *miraculously* are real morsifications of such curves, up to affine transformations. A'Campo attributes this remark to R. Thom [20] and to Guseĭn-Zade [11]. We call this particular type of real morsification a Chebyshev morsification.

Recall the Chebyshev polynomials of the first kind are: $T_1(y) = y$, $T_2(y) = 2y^2 - 1$, $T_3 = 4y^3 - 3y$, $T_4(y) = 8y^4 - 8y^2 + 1$, $T_5(y) = 16y^5 - 20y^3 + 5y$, $\ldots$.

Using this remark and the procedure of the previous section, we obtain finitely $\mathcal{A}$-determined mappings associated to an ample set of given doodles, among them are the ones obtained from Chebyshev morsifications connecting the end points of a divide of it.

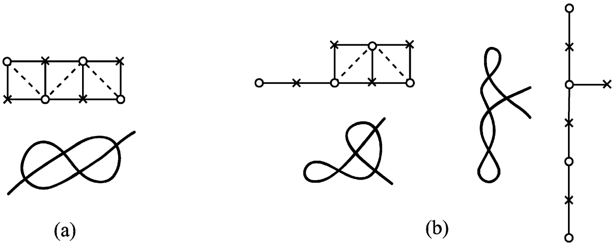


Fig. 12.

Notice that the Chebyshev morsification is a very symmetric real morsification of a given curve singularity $y \to (y^p, y^q)$, with $(p,q) = 1$. The reason behind this is the fact that the Chebyshev polynomials are interpolation polynomials for points in the plane distributed uniformly. This implies that the Dynkin diagram of the morsification is the most symmetric one (cf. [11]).

For instance, the E_8 singularity $y \to (y^3, y^5)$, has Chebyshev morsification $y \to (y^3 - \frac{3}{4}y, y^5 - \frac{5}{4}y^3 + \frac{5}{16}y)$ (Fig. 12(a)). The other two real morsifications of E_8 (Fig. 12(b)) cannot be obtained using Chebyshev polynomial.

So, the mapping $(x,y) \to (x, y^3 - \frac{3}{4}xy, y^5 - \frac{5}{4}xy^3 + \frac{5}{16}x^2y)$ is an example of a finitely $\mathcal{A}$-determined mapping whose doodle has Gauss word $abc^{-1}a^{-1}dcb^{-1}d^{-1}$ (Fig. 12(a)).

The other two two real morsifications of $y \to (y^3, y^5)$ give rise, by arc connecting the two end points of the divide, to doodles whose Gauss words are $ab^{-1}ba^{-1}cc^{-1}dd^{-1}$ and $aa^{-1}b^{-1}cd^{-1}bc^{-1}d$, respectively. It seems that map germs with these associated doodles have 2-jet $(x, 0, 0)$ and shall be studied in another opportunity.

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