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Ideal approximation theory

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Abstract

Let $(\mathcal{A}; \mathcal{E})$ be an exact category and $\mathcal{F} \subseteq \mathrm{Ext}$ a subfunctor. A morphism φ in $\mathcal{A}$ is an $\mathcal{F}$-phantom if the pullback of an $\mathcal{E}$-conflation along φ is a conflation in $\mathcal{F}$. If the exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects and projective morphisms, it is proved that an ideal $\mathcal{I}$ of $\mathcal{A}$ is special precovering if and only if there is a subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ with enough injective morphisms such that $\mathcal{I}$ is the ideal of $\mathcal{F}$-phantom morphisms. A crucial step in the proof is a generalization of Salce's Lemma for ideal cotorsion pairs: if $\mathcal{I}$ is a special precovering ideal, then the ideal cotorsion pair $(\mathcal{I}, \mathcal{I}^{\perp})$ generated by $\mathcal{I}$ in $(\mathcal{A}; \mathcal{E})$ is complete. This theorem is used to verify: (1) that the ideal cotorsion pair cogenerated by the pure-injective modules of R-Mod is complete; (2) that the ideal cotorsion pair cogenerated by the contractible complexes in the category of complexes $\mathrm{Ch}(R\text{-Mod})$ is complete; and, using Auslander and Reiten's theory of almost split sequences, (3) that the ideal cotorsion pair cogenerated by the Jacobson radical $\mathrm{Jac}(\Lambda\text{-mod})$ of the category Λ-mod of finitely generated representations of an Artin algebra is complete.

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1. Introduction

Let $(\mathcal{A}; \mathcal{E})$ be an exact category [9,14,22]. Given a subcategory $\mathcal{C} \subseteq \mathcal{A}$, define ${}^{\perp}\mathcal{C} \subseteq \mathcal{A}$ to be the subcategory of objects F such that $\mathrm{Ext}(F, C) = 0$ for every $C \in \mathcal{C}$, and define $\mathcal{C}^{\perp}$ dually. A *cotorsion pair* in $(\mathcal{A}; \mathcal{E})$ is a pair $(\mathcal{F}, \mathcal{C})$ of subcategories of $\mathcal{A}$ satisfying $\mathcal{F} = {}^{\perp}\mathcal{C}$ and $\mathcal{C} = \mathcal{F}^{\perp}$. The notion of a cotorsion pair was introduced by Salce ([27], but see [17, Lemma 2.2.6]) in the setting of Ab, the abelian category of abelian groups. In the general setting of an exact category, this notion provides the proper context for the study of precovers and preenvelopes (approximation theory). Expositions of approximation theory for categories of modules may be found in the monographs of Beligiannis and Reiten [4] and Göbel and Trlifaj [17], but cotorsion pairs have also been used to study approximation theory in sheaf categories [13], general Grothendieck categories [15,20], and more abstract exact categories [28].

While the present state of approximation theory tends to stress the importance of objects and subcategories, the purpose of this article is to give morphisms and ideals of categories equal status. If $\mathcal{J}$ is an ideal of $\mathcal{A}$, define ${}^{\perp}\mathcal{J}$ to be the ideal of morphisms i such that $\mathrm{Ext}(i, j) = 0$ for every $j \in \mathcal{J}$, and define $\mathcal{J}^{\perp}$ dually. An *ideal cotorsion pair* in $(\mathcal{A}; \mathcal{E})$ is a pair $(\mathcal{I}, \mathcal{J})$ of ideals of $\mathcal{A}$ satisfying $\mathcal{I} = {}^{\perp}\mathcal{J}$ and $\mathcal{J} = \mathcal{I}^{\perp}$. Our aim is to develop approximation theory for ideal cotorsion pairs in analogy with the approximation theory of cotorsion pairs in an exact category. In contrast to the references above, no completeness assumptions are made on an exact category.

Examples of preenvelopes are the injective and pure-injective envelopes of a module. The existence of flat precovers was conjectured by Enochs [11] and proved in [8]. These are all examples (cf. [12]) of approximations relative to a *subcategory* of the category R-Mod of left R-modules over an associative ring R. But there also exist approximations relative to an ideal. For example, if Λ is an Artin algebra, and Λ-mod denotes the category of finitely presented left Λ-modules, the work of Auslander and Reiten [1] shows that every object $M \in \Lambda$-mod has a cover (resp., envelope) with respect to the ideal $\mathrm{Jac}(\Lambda\text{-mod})$ given by the *Jacobson radical* of Λ-mod. Another example, one that provides the prototype for the present theory, is given by the ideal of phantom morphisms in a module category R-Mod; the existence of phantom covers was proved by the third author [19].

Let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal and A an object of $\mathcal{A}$. An *$\mathcal{I}$-precover* of A is a morphism $i \in \mathcal{I}$, $i : X \to A$, such that any other morphism $i' : X' \to A$ in $\mathcal{I}$ factors through i,

$$\begin{tikzcd} & X' \arrow[dl, dotted] \arrow[d, "i'"] \\ X \arrow[r, "i"] & A. \end{tikzcd}$$

If the category is equipped with an exact structure $(\mathcal{A}; \mathcal{E})$, then an $\mathcal{I}$-precover $i : X \to A$ of A in $\mathcal{A}$ is *special* if it is obtained as the pushout of a conflation η along a morphism $j : Y \to B$ in $\mathcal{I}^{\perp}$:

$$\begin{tikzcd} \eta: & Y \arrow[r] \arrow[d, "j"] & Z \arrow[r] \arrow[d] & A \arrow[d, equal] \\ & B \arrow[r] & X \arrow[r, "i"] & A. \end{tikzcd}$$

The condition $\mathrm{Ext}^1(\mathcal{I}, j) = 0$ implies (Proposition 11) that a special $\mathcal{I}$-precover is an $\mathcal{I}$-precover. The ideal $\mathcal{I}$ is called *(special) precovering* if every object $A \in \mathcal{A}$ has an (a special) $\mathcal{I}$-precover $i : X \to A$. Given an ideal $\mathcal{J} \subseteq \mathcal{A}$, a special $\mathcal{J}$-preenvelope is defined in a dual manner and an ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ in $(\mathcal{A}; \mathcal{E})$ is called *complete* if every object in $\mathcal{A}$ has a special $\mathcal{J}$-preenvelope and a special $\mathcal{I}$-precover.

The main contribution of this paper is to establish the connection between special precovering ideals and phantom morphisms in exact categories. Theorem 1 below asserts that every special precovering ideal can be represented, under suitable hypotheses, as an ideal of phantoms. Given a subfunctor $\mathcal{F}$ of Ext [2,10] a morphism φ will be called an *$\mathcal{F}$-phantom morphism* if the pullback of any conflation along φ is a conflation in $\mathcal{F}$. The $\mathcal{F}$-phantom morphisms form an ideal, denoted by $\Phi(\mathcal{F})$.

The notion of a phantom morphism arises in topology, in the study of maps between CW-complexes [23]. In the context of triangulated categories [26], phantom morphisms were first studied by Neeman [25]. In the stable category of a finite group ring, the theory of phantom morphisms was developed in a series of papers [16,6,7,5] by Benson and Gnacadja; the triangulated version of this notion of a phantom morphism was studied in [3,24]. Their definition of a phantom morphism was generalized by the third author [19] to the category of left R-modules over an associative ring R. It is proved in the sequel (Proposition 36) that a morphism $\varphi : M \to N$ in R-Mod is a phantom morphism if and only if the induced natural transformation

$$\mathrm{Tor}_1^R(-, \varphi) : \mathrm{Tor}_1^R(-, M) \to \mathrm{Tor}_1^R(-, N)$$

of additive functors from R-Mod to Ab is 0. In the terminology of the present paper, these phantoms are defined relative to the subfunctor Pext of Ext_R^1, whose conflations are the pure exact sequences in R-Mod.

An additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ *has enough injective morphisms* if for every object $B \in \mathcal{A}$, there is an $\mathcal{F}$-conflation

$$\eta_B : \quad B \xrightarrow{e} C \longrightarrow A,$$

where $e : B \to C$ is an $\mathcal{F}$-injective morphism. The subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ is said to *have enough special injective morphisms* if for every object $B \in \mathcal{A}$, there is an $\mathcal{F}$-conflation η_B as above that arises as the pullback along an $\mathcal{F}$-phantom morphism (see Proposition 15).

Theorem 1. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough injective objects and projective morphisms. The following statements regarding an ideal $\mathcal{I}$ of $\mathcal{A}$ are equivalent:*

(1) *there is an additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ with enough injective morphisms and $\mathcal{I} = \Phi(\mathcal{F})$;*
(2) *the ideal $\mathcal{I}$ is special precovering;*
(3) *the ideal cotorsion pair $(\mathcal{I}, \mathcal{I}^{\perp})$ is complete; and*

(4) *the additive subfunctor $PB(\mathcal{I}) \subseteq Ext$, whose conflations are obtained by pulling back arbitrary conflations along morphisms in $\mathcal{I}$, has enough special injective morphisms and $\mathcal{I} = \Phi(PB(\mathcal{I}))$.*

The hypothesis that $(\mathcal{A}; \mathcal{E})$ have enough projective morphisms is only used in the proof of (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is Salce's Lemma [17, Lemma 2.2.6] for ideal cotorsion pairs; it relies only on the hypothesis that there exist enough injective objects. The implication (4) $\Rightarrow$ (1) is trivial and (3) $\Rightarrow$ (4) holds in general. An immediate consequence (Corollary 20) is that if $(\mathcal{A}; \mathcal{E})$ is an exact category with enough injective objects and projective morphisms, and $\mathcal{F} \subseteq \mathrm{Ext}$ is an additive subfunctor with enough injective morphisms, then the ideal cotorsion pair cogenerated by the ideal of $\mathcal{F}$-injective morphisms is complete. There are three well-known examples of the statement Theorem 1(1).

(1) Let $(R\text{-Mod}, \mathrm{Ext}^1_R)$ be the abelian category of left R-modules over an associative ring R with identity. It has enough injective objects and projective objects and the exact substructure

$$(R\text{-Mod}, \mathrm{Pext}^1_R) \subseteq (R\text{-Mod}, \mathrm{Ext}^1_R),$$

whose conflations are the pure exact sequences, provides an additive subfunctor of Ext with enough injective objects. The phantom morphisms in this case will be called *pure phantom* morphisms.

(2) The abelian category $\mathrm{Ch}(R\text{-Mod})$ of complexes of left R-modules has enough injective objects and projective morphisms. The exact substructure $(\mathrm{Ch}(R\text{-Mod}); \mathcal{E}_0)$, whose conflations are the semisplit sequences, provides an additive subfunctor of Ext with enough injective objects. The phantom morphisms in this case will be called *semisplit* phantom morphisms.

(3) If Λ is an Artin algebra, then the abelian category $(\Lambda\text{-mod}, \mathrm{Ext}^1_\Lambda)$ of finitely presented left modules over Λ has enough injective objects and projective objects. By the work of Auslander and Reiten, the additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}^1_\Lambda$ generated by the almost split sequences in Λ-mod has enough injective morphisms. The phantom morphisms in this case will be called *Auslander–Reiten (AR) phantom* morphisms.

There is an object version of Theorem 1 that pertains to the first two examples, where the additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ satisfies the stronger condition of having enough injective *objects.* This means that for every object $B \in \mathcal{A}$, there is an $\mathcal{F}$-conflation

$$\eta_B : \quad B \longrightarrow E \longrightarrow A,$$

where E is an $\mathcal{F}$-injective object.

Theorem 2. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough injective objects and projective morphisms. The following statements regarding an ideal $\mathcal{I}$ of $\mathcal{A}$ are equivalent:*

(1) *there is an additive subfunctor $\mathcal{F} \subseteq Ext$ with enough injective objects and $\mathcal{I} = \Phi(\mathcal{F})$;*

(2) *for every object $A \in \mathcal{A}$, there exists a conflation*

$$B \longrightarrow C \xrightarrow{\ i\ } A,$$

where $i : C \to A$ belongs to $\mathcal{I}$ and B is an object in $\mathcal{I}^{\perp}$;

(3) *the ideal $\mathcal{I}$ is special precovering and $\mathcal{I}^{\perp}$ is an object ideal, i.e., there exists an additive category $\mathcal{X} \subseteq \mathcal{A}$ such that $\mathcal{I}^{\perp} = \mathcal{I}(\mathcal{X})$ is the ideal of morphisms that factor through an object in $\mathcal{X}$; and*
(4) *the additive subfunctor $PB(\mathcal{I}) \subseteq Ext$ has enough special injective objects and $\mathcal{I} = \Phi(PB(\mathcal{I}))$.*

An example similar to Condition (3) arises in the work of Šaroch and Šťovíček [29].

Condition (4) of Theorem 1 establishes a bijective correspondence (Corollary 19) between special precovering ideals $\mathcal{I}$ and subfunctors $\mathcal{F} = \mathrm{PB}(\mathcal{I})$ that have enough special injective morphisms. The additive subfunctor $\mathrm{PB}(\mathcal{I}) \subseteq \mathrm{Ext}$ that arises in Condition (4) embodies the creative aspect of the theorem. Corollary 21 offers an explicit description of the ideal $\mathcal{I}^{\perp}$, whose morphisms are the $\mathrm{PB}(\mathcal{I})$-injective morphisms. The latter portion of the article details how this phenomenon manifests itself in the three examples. In the first example, of the inclusion $(R\text{-Mod}; \mathrm{Pext}^1_R) \subseteq (R\text{-Mod}; \mathrm{Ext}^1_R)$ of exact structures, the ideal of $\mathcal{I}^{\perp}$ consists of the morphisms that factor through those left R-modules Z that arise in a short exact sequence

$$0 \longrightarrow M \longrightarrow Z \longrightarrow E \longrightarrow 0,$$

where M is a pure-injective left R-module and $E \in R\text{-Mod}$ is injective. In the second example, of the semisplit exact category $(\mathrm{Ch}(R\text{-Mod}); \mathcal{E}_0)$ of the category of complexes $\mathrm{Ch}(R\text{-Mod})$, the objects of $\mathcal{I}^{\perp}$ can be described similarly as those complexes Z^* that appear in a short exact sequence of complexes

$$0 \longrightarrow C^* \longrightarrow Z^* \longrightarrow E^* \longrightarrow 0,$$

where C^* is contractible and E^* is injective. Using the Frobenius property of $(\mathrm{Ch}(R\text{-Mod}); \mathcal{E}_0)$ these complexes may be alternatively characterized as those complexes M^* for which the canonical morphism $\xi_{M^*} : \Omega\Sigma(M^*) \to M^*$ (Definition 43) is null-homotopic (Theorem 44). In the third example, of the socle subfunctor of Ext_{Λ}, where Λ is an Artin algebra, we call an almost split sequence *left special* if it belongs to the subfunctor $\mathrm{PB}(\mathcal{I})$, that is, if it arises as the pullback along an Auslander–Reiten phantom morphism. In the last section of the article, it is proved (Corollary 49) that if U is an indecomposable Λ-module that is not injective, then there exists a left special almost split sequence with U as its left term if and only if the canonical morphism $\xi_U : \Omega\Sigma(U) \to U$ is not a split epimorphism.

The classical situation of a complete cotorsion pair $(\mathcal{F}, \mathcal{C})$ in an exact category $(\mathcal{A}; \mathcal{E})$ yields another example of a complete ideal cotorsion pair, given by the pair $(\mathcal{I}(\mathcal{F}), \mathcal{I}(\mathcal{C}))$ of the corresponding object ideals (Theorem 28). This example shows that the study of complete ideal cotorsion pairs subsumes that of complete cotorsion pairs, and therefore, that the approximation theory developed here for ideals generalizes the classical approximation theory for the subcategories of an exact category.

2. Preliminaries

By an *ideal* $\mathcal{I}$ of an additive category $\mathcal{A}$ we mean an additive subfunctor of the additive bifunctor $\mathrm{Hom} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$. The ideal $\mathcal{I}$ associates to every pair A and B of objects in $\mathcal{A}$ a subgroup $\mathcal{I}(A, B) \subseteq \mathrm{Hom}(A, B)$ so that if $f : X \to A$ and $g : B \to Y$ are morphisms in $\mathcal{A}$, then the natural transformation $\mathrm{Hom}(f, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(X, Y)$ that assigns to $i \in \mathrm{Hom}(A, B)$, the composition

$$\mathrm{Hom}(f, g)(i) : \; X \xrightarrow{\quad f \quad} A \xrightarrow{\quad i \quad} B \xrightarrow{\quad g \quad} Y$$

respects $\mathcal{I}$. In other words, if $i \in \mathcal{I}(A, B)$, then $gif = \mathrm{Hom}(f, g)(i) \in \mathcal{I}(X, Y)$. If $A \in \mathcal{A}$ is an object, we say that A belongs to $\mathcal{I}$ if the identity morphism 1_A belongs to $\mathcal{I}(A, A)$, and we denote by $\mathrm{Ob}(\mathcal{I}) \subseteq \mathcal{A}$ the full subcategory of objects of $\mathcal{I}$. This is an additive subcategory of $\mathcal{A}$. In the other direction, every additive subcategory $\mathcal{X} \subseteq \mathcal{A}$ gives rise to the ideal $\mathcal{I}(\mathcal{X})$ generated by morphisms of the form 1_X, where $X \in \mathcal{X}$; this is the ideal of morphisms that factor through an object in $\mathcal{X}$. An ideal $\mathcal{I}$ of $\mathcal{A}$ is called an *object* ideal if it is generated by its objects, $\mathcal{I} = \mathcal{I}(\mathrm{Ob}(\mathcal{I}))$.

Exact Categories. We rely exclusively on Bühler [9] as the reference for exact categories, but we use some of the terminology of Keller [14,22]. An exact category $(\mathcal{A}; \mathcal{E})$ consists of an additive category $\mathcal{A}$, together with a distinguished collection $\mathcal{E}$ of composable[1] pairs of morphisms (i, p) such that i is the kernel of p and p the cokernel of i. Such a pair is depicted by

$$\eta : \quad B \xrightarrow{\;i\;} C \xrightarrow{\;p\;} A$$

and is called a *conflation.* The kernel $i : B \to C$ that appears in the conflation η is an *inflation;* the cokernel $p : C \to A$ a *deflation.* The collection $\mathcal{E}$ of conflations is closed under isomorphism and satisfies closure properties that ensure that, given objects A and B of $\mathcal{A}$, the isomorphism classes of conflations η as above form an abelian group[2] $\mathrm{Ext}(A, B)$ with respect to the *Baer sum* operation. Furthermore, the rule $(A, B) \to \mathrm{Ext}(A, B)$ is an additive bifunctor

$$\mathrm{Ext} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}.$$

Because the collection $\mathcal{E}$ of conflations is the union of the elements, up to isomorphism, of the $\mathrm{Ext}(A, B)$, with $A, B \in \mathcal{A}$, we may refer to the exact structure in terms of the associated functor Ext, $(\mathcal{A}; \mathcal{E}) = (\mathcal{A}; \mathrm{Ext})$. An abelian category $\mathcal{A}$ acquires the structure of an exact category $(\mathcal{A}; \mathcal{E})$, whose collection $\mathcal{E}$ of conflations consists of the short exact sequences in $\mathcal{A}$.

If $f : X \to A$ is a morphism in $\mathcal{A}$, then the pullback of η along f is a conflation in $(\mathcal{A}; \mathcal{E})$. This operation induces a morphism

$$\mathrm{Ext}(f, B) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(X, B)$$

of abelian groups. Similarly, if $g : B \to Y$ is a morphism in $\mathcal{A}$, then the pushout of η along g is a conflation in $(\mathcal{A}, \mathcal{E})$ and yields the morphism

$$\mathrm{Ext}(A, g) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, Y).$$

When these operations are applied in succession, the composition – denoted by $\mathrm{Ext}(f, g)$ – is independent of the order. The proof rests on the notion of a *pushout–pullback factorization* [9, Prop. 3.1] of a morphism of conflations

[1] A pair of morphisms (i, j) is *composable* if $\mathrm{Dom}(j)$, the domain of j, is equal to $\mathrm{Codom}(i)$, the codomain of i.

[2] To avoid set-theoretic complications, we assume throughout that the class $\mathrm{Ext}(A, B)$ is a set.

$$\begin{tikzcd}
\eta: & B \arrow[r, "i"] \arrow[d, "g"'] & C \arrow[r, "p"] \arrow[d, "h"'] & A \arrow[d, "f"'] \\
\eta': & Y \arrow[r, "i'"] & Z \arrow[r, "p'"] & X.
\end{tikzcd}$$

This morphism factors uniquely through the pushout of η along g as

$$\begin{tikzcd}
\eta: & B \arrow[r, "i"] \arrow[d, "g"'] & C \arrow[r, "p"] \arrow[d, "h_1"'] & A \arrow[d, equal] \\
 & Y \arrow[r, "i_1"] \arrow[d, equal] & N \arrow[r, "j"] \arrow[d, "h_2"'] & A \arrow[d, "f"'] \\
\eta': & Y \arrow[r, "i'"] & Z \arrow[r, "p'"] & X.
\end{tikzcd}$$

The conflation in the middle row is then the pullback of η' along f, which yields the equation

$$\mathrm{Ext}(A, g)(\eta) = \mathrm{Ext}(f, Y)(\eta').$$

The following proposition actually follows from the definition of an additive bifunctor Ext : $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$, but we give an explicit proof, because it will be invoked often and is characteristic of the general manner of thinking used in the sequel.

Proposition 3. *If $f : X \to A$ and $g : B \to Y$ are morphisms in an exact category $(\mathcal{A}; \mathcal{E})$, then the diagram*

$$\begin{tikzcd}
\mathrm{Ext}(A, B) \arrow[r, "\mathrm{Ext}(f, B)"] \arrow[d, "\mathrm{Ext}(A, g)"] & \mathrm{Ext}(X, B) \arrow[d, "\mathrm{Ext}(X, g)"] \\
\mathrm{Ext}(A, Y) \arrow[r, "\mathrm{Ext}(f, Y)"] & \mathrm{Ext}(X, Y)
\end{tikzcd}$$

of abelian groups is commutative. Thus $Ext(f, g) = Ext(X, g)Ext(f, B) = Ext(f, Y)Ext(A, g)$.

Proof. Suppose that a conflation $\eta : B \to C \to A$ is given. Taking the pullback along $f : X \to A$ and the pushout along $g : B \to Y$ yields a morphism of conflations given by the composition

$$\begin{tikzcd}
& B \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\
\eta : & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "g'"'] & A \arrow[d, equal] \\
& Y \arrow[r, "i'"] & Y' \arrow[r] & A.
\end{tikzcd}$$

The pushout–pullback factorization of this morphism yields a commutative square of conflations

$$\begin{tikzcd}
& & B \arrow[rr] \arrow[dl, "g"'] \arrow[dd, equal] & & X' \arrow[rr] \arrow[dl] \arrow[dd, "f'"] & & X \arrow[dl, equal] \arrow[dd, "f"] \\
\gamma : & Y \arrow[rr, crossing over] \arrow[dd, equal] & & Z \arrow[rr, crossing over] \arrow[dd] & & X \arrow[dd, "f"] & \\
& \eta : & B \arrow[rr] \arrow[dl, "g"] & & C \arrow[rr] \arrow[dl, "g'"] & & A \arrow[dl, equal] \\
& Y \arrow[rr] & & B' \arrow[rr] & & A. &
\end{tikzcd}$$

The top front row is the conflation γ, expressible as

$$\mathrm{Ext}(X, g)\mathrm{Ext}(f, B)(\eta) = \gamma = \mathrm{Ext}(f, Y)\mathrm{Ext}(A, g)(\eta). \quad \square$$

Example 4. Let $A, B \in \mathcal{A}$ and suppose that η and γ belong to $\mathrm{Ext}(A, B)$. The *Baer sum* of η and γ is obtained by constructing the direct sum $\eta \oplus \gamma \in \mathrm{Ext}(A \oplus A, B \oplus B)$ and applying the morphism $\mathrm{Ext}(\Delta, \nabla)$ to $\eta \oplus \gamma$, where $\Delta : A \to A \oplus A$ (resp., $\nabla : B \oplus B \to B$) is the morphism induced on the product (resp., coproduct) by the identity morphism 1_A (resp., 1_B).

Notation. Throughout the article, we assume that $(\mathcal{A}; \mathcal{E})$ is an exact category in the sense of [9]. Any mention of Ext refers to the additive bifunctor associated to $(\mathcal{A}; \mathcal{E})$.

Subfunctors of Ext (*[2,10]*)**.** If $(\mathcal{A}; \mathcal{E})$ is an exact category, an additive subfunctor of $\mathrm{Ext} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$ is defined in analogy with the way that an ideal of $\mathcal{A}$ was defined to be an additive subfunctor of Hom. For every pair A and B of objects in $\mathcal{A}$, an additive subfunctor $\mathcal{F}$ of Ext associates a subgroup $\mathcal{F}(A, B) \subseteq \mathrm{Ext}(A, B)$ so that if $f : X \to A$ and $g : B \to Y$ are morphisms in $\mathcal{A}$, then the natural transformation $\mathrm{Ext}(f, g) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(X, Y)$ respects $\mathcal{F}$. This means that if $\eta \in \mathcal{F}(A, B)$, then $\mathrm{Ext}(f, g)(\eta) \in \mathcal{F}(X, Y)$. A subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ yields a substructure

$$(\mathcal{A}; \mathcal{F}) \subseteq (\mathcal{A}; \mathcal{E}),$$

whose conflations, the $\mathcal{F}$*-conflations,* are those conflations $\eta : B \to C \to A$ that belong to $\mathcal{F}(A, B)$. The subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ may thus be considered as a collection of conflations closed under isomorphism, containing all the trivial conflations – and so satisfying Axioms [E0] and [E0$^{\mathrm{op}}$] of [9] for an exact category – and satisfying Axioms [E2] and [E2$^{\mathrm{op}}$] of [9]. Furthermore, the collection of $\mathcal{F}$-conflations satisfies the axiom

E1′: The class of conflations is closed under direct sums.

On the other hand, if $\mathcal{F} \subseteq \mathcal{E}$ is a collection of conflations satisfying the properties above, a result of Auslander and Solberg [2, Lemma 1.1] asserts that the $\mathcal{F}$-conflations constitute an additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$.

The Axiom [E1′] for an additive subfunctor of Ext follows [9, Prop. 2.16] from the axioms for an exact category. In particular, an exact substructure $(\mathcal{A}; \mathcal{E}') \subseteq (\mathcal{A}; \mathcal{E})$ constitutes a subfunctor of Ext; an *exact substructure* of $(\mathcal{A}; \mathcal{E})$ is an exact structure $\mathcal{E}'$ on $\mathcal{A}$ such that every $\mathcal{E}'$-conflation is a conflation. The chief difference (see [10]) between a subfunctor of Ext and an exact substructure is that in an exact substructure a composition of inflations (resp., deflations) is an inflation (resp., a deflation), and if a composition fg is an inflation (resp., a deflation), then g (resp., f) is an inflation (resp., a deflation).

Notation. Throughout the article, we assume that $\mathcal{F} \subseteq \mathcal{E}$ is an additive subfunctor of Ext.

The next lemma is used twice in the sequel; it is a special case of the Obscure Axiom [9, Prop. 2.16].

Lemma 5. *If an $\mathcal{F}$-inflation $i_0 : B \to C_0$ factors through an $\mathcal{E}$-inflation $i : B \to C$*

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[dr, "i_0"'] & C \arrow[d, "g"] \\ & C_0, \end{tikzcd}$$

then i too is an $\mathcal{F}$-inflation.

Proof. The morphism $g : C \to C_0$ induces a morphism of $\mathcal{E}$-conflations

$$\begin{tikzcd} \eta : & B \arrow[r, "i"] \arrow[d, equal] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "h"] \\ \eta_0 : & B \arrow[r, "i_0"] & C_0 \arrow[r] & A_0. \end{tikzcd}$$

But η_0 is an $\mathcal{F}$-conflation, so that the pullback η along h is also an $\mathcal{F}$-conflation. Thus the morphism i is an $\mathcal{F}$-inflation. □

Definition 6. Given a conflation $\eta : B \to C \to A$ and a morphism $\varphi : X \to A$, the pullback gives rise to a commutative diagram

$$\begin{tikzcd}
\eta_0: & B \arrow[r, "i"] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X \arrow[d, "\varphi"'] \\
\eta: & B \arrow[r] & C \arrow[r] & A
\end{tikzcd}$$

with top row $\eta_0 = \mathrm{Ext}(\varphi, B)(\eta)$. The morphism $\varphi : X \to A$ is an $\mathcal{F}$*-phantom* morphism if the top row η_0 of every such pullback is an $\mathcal{F}$-conflation.

Thus a morphism $\varphi : X \to A$ is an $\mathcal{F}$-phantom provided that for every $B \in \mathcal{A}$, the morphism

$$\mathrm{Ext}(\varphi, B) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(X, B)$$

of abelian groups takes values in the subgroup $\mathcal{F}(X, B)$. It follows that the collection $\Phi(\mathcal{F})$ of $\mathcal{F}$-phantom morphisms forms an ideal of $\mathcal{A}$.

A morphism $f : X \to A$ in $\mathcal{A}$ is $\mathcal{F}$*-projective* if $\mathcal{F}(f, B) = 0$ for every $B \in \mathcal{A}$. This means that the pullback along f of any $\mathcal{F}$-conflation is split. An object A in $\mathcal{A}$ is $\mathcal{F}$*-projective* if the identity morphism 1_A is an $\mathcal{F}$-projective morphism. The ideal of $\mathcal{F}$-projective morphisms is denoted by $\mathcal{F}$-proj; the subcategory of $\mathcal{F}$-projective objects by $\mathcal{F}$-Proj. An $\mathcal{F}$*-injective morphism* (resp., $\mathcal{F}$*-injective object*) is defined dually. The ideal of $\mathcal{F}$-injective morphisms is denoted by $\mathcal{F}$-inj; the subcategory of $\mathcal{F}$-injective objects by $\mathcal{F}$-Inj. In case $\mathcal{F} = \mathcal{E}$, the prefix is dropped.

If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathrm{Ext}$ are subfunctors, then evidently $\Phi(\mathcal{F}_1) \subseteq \Phi(\mathcal{F}_2)$. Given an ideal $\mathcal{I}$ of the exact category $(\mathcal{A}; \mathcal{E})$, the next proposition describes the minimum additive subfunctor $\mathcal{F}$ of Ext for which $\mathcal{I} \subseteq \Phi(\mathcal{F})$.

Proposition 7. *Let $\mathcal{I}$ be an ideal $\mathcal{A}$. The collection of conflations that arise as pullbacks by morphisms in $\mathcal{I}$ comprise an additive subfunctor $PB(\mathcal{I}) \subseteq \mathcal{E}$.*

Proof. The task is to verify that the collection $\mathrm{PB}(\mathcal{I})$ of conflations that are pullbacks along $\mathcal{I}$ includes the trivial conflations, and is closed under the direct sum, pullback and pushout operations. If $\eta : B \to C \to A$ is a trivial conflation, then the pullback of η along the zero morphism $0_A : A \to A$ is isomorphic to η and so implies that $\eta \in \mathrm{PB}(\mathcal{I})(A, B)$. A direct sum of conflations η_1 and η_2 that arise as pullbacks of γ_1 and γ_2 along the morphisms i_1 and i_2 of $\mathcal{I}$, respectively, is the pullback of $\gamma_1 \oplus \gamma_2$ along the diagonal morphism $\begin{pmatrix} i_1 & 0 \\ 0 & i_2 \end{pmatrix}$. If $\eta : B \to C \to A$ is a conflation obtained as the pullback of η' along the morphism $i : A \to A'$, then the pullback of η along any morphism $f : X \to A$ is a pullback of η' along the composition if, which belongs to $\mathcal{I}$. Finally, to see that the collection $\mathrm{PB}(\mathcal{I})$ is closed under pushouts, suppose that $\eta : B \to C \to A$ is a pullback along a morphism $i : A \to A'$ in $\mathcal{I}$ of the conflation $\eta' : B \to C' \to A'$. By Proposition 3, the pushout of η along the morphism $g : B \to Y$ is given by the conflation

$$\mathrm{Ext}(A, g)(\eta) = \mathrm{Ext}(A, g)\mathrm{Ext}(i, B)(\eta') = \mathrm{Ext}(i, Y)\mathrm{Ext}(A', g)(\eta'),$$

which is the pullback along i of the conflation $\mathrm{Ext}(A', g)(\eta')$. $\square$

Orthogonality. A pair (f, g) of morphisms in $\mathcal{A}$ with $f : X \to A$ and $g : B \to Y$ is *orthogonal* if the morphism

$$\mathrm{Ext}(f, g) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(X, Y)$$

of abelian groups is zero.

For example, if $f : A \to A$ is the identity morphism 1_A, then Proposition 3 implies

$$\mathrm{Ext}(1_A, g) = \mathrm{Ext}(A, g)\mathrm{Ext}(1_A, B) = \mathrm{Ext}(A, g),$$

so that the pair $(1_A, g)$ is orthogonal if and only if $\mathrm{Ext}(A, g)$ is zero. Similarly, the morphism of abelian groups

$$\mathrm{Ext}(1_A, 1_B) : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B)$$

is the identity map, so that the pair $(1_A, 1_B)$ of morphisms is orthogonal if and only if $\mathrm{Ext}(A, B) = 0$.

The bifunctor Ext is *half-exact* in each variable. We will use a very special case of this property, which we verify for completeness.

Lemma 8. *If $B \xrightarrow{j} C \xrightarrow{q} E$ is a conflation with E an injective object, then for every $A \in \mathcal{A}$, the morphism of abelian groups*

$$Ext(A, j) : Ext(A, B) \to Ext(A, C)$$

is onto.

Proof. The statement of the lemma asserts that every conflation $\eta : C \to Z \to A$ arises as a pushout of along $j : B \to C$. Let us explain how the given information yields a commutative diagram

$$\begin{tikzcd}
\gamma : & B \arrow[r, "i'"] \arrow[d, "j"] & Z' \arrow[r] \arrow[d, "j'"] & A \arrow[d, equal] \\
\eta : & C \arrow[r, "i"] \arrow[d, "q"] & Z \arrow[r] \arrow[d, "p"] & A \\
 & E \arrow[r, equal] & E, &
\end{tikzcd}$$

all of whose rows and columns are conflations. The morphism $p : Z \to E$ exists, because E is an injective object; it is a deflation, because $q = pi$ is. It induces the conflation in the middle column, and the morphism $i : C \to Z$ induces the morphism $i' : B \to Z'$ on kernels. Now $j'i' = ij$ is a composition of inflations and therefore itself an inflation. This $i' : B \to Z'$ is an inflation that induces the conflation γ in the top row. The equality $E = E$ in the bottom row implies that the commutative square in the top left is a pushout/pullback diagram, and that yields the equality $A = A$ in the right column. The conflation η is thus attained as the pushout of the conflation γ along the morphism $j : B \to C$. $\square$

Suppose that a morphism of conflations is given, as in the commutative diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "k"] & A \arrow[d, "f"] \\ Y \arrow[r, "h"] & Z \arrow[r] & X. \end{tikzcd}$$

Then the morphism $k : C \to Z$ is called an *extension of g by A*.

Proposition 9. *If $\mathcal{M}$ is a collection of morphisms in $\mathcal{A}$, then*

$$\mathcal{M}^{\perp} := \{g : Ext(m, g) = 0 \ \ for\ all\ m \in \mathcal{M}\}$$

is an ideal closed under extension by injective objects.

Proof. Let $\eta : B \to C \to A$ be a conflation, and suppose that $f : X \to A$ belongs to $\mathcal{M}$. If g_1, $g_2 : B \to Y$ belong to $\mathcal{M}^{\perp}$, then Proposition 3 implies

$$\begin{aligned} \mathrm{Ext}(f, g_1 + g_2) &= \mathrm{Ext}(X, g_1 + g_2)\mathrm{Ext}(f, B) = [\mathrm{Ext}(X, g_1) + \mathrm{Ext}(X, g_2)]\mathrm{Ext}(f, B) \\ &= \mathrm{Ext}(X, g_1)\mathrm{Ext}(f, B) + \mathrm{Ext}(X, g_2)\mathrm{Ext}(f, B) \\ &= \mathrm{Ext}(f, g_1) + \mathrm{Ext}(f, g_2) = 0. \end{aligned}$$

If $h : Y \to N$ is a morphism in $\mathcal{A}$, another application of Proposition 3 yields

$$\begin{aligned} \mathrm{Ext}(f, hg_1) &= \mathrm{Ext}(X, hg_1)\mathrm{Ext}(f, B) = \mathrm{Ext}(X, h)\mathrm{Ext}(X, g_1)\mathrm{Ext}(f, B) \\ &= \mathrm{Ext}(X, h)\mathrm{Ext}(f, g_1) = 0. \end{aligned}$$

Similarly, if $k : M \to B$, then Proposition 3 implies that $\mathrm{Ext}(f, g_1k) = \mathrm{Ext}(f, g_1)\mathrm{Ext}(A, k) = 0$.

Finally let us consult the diagram above, and prove that if g belongs to $\mathcal{M}^{\perp}$, and A is an injective object, then so does k. The foregoing implies that the composition $hg = kj$ also belongs to $\mathcal{M}^{\perp}$. If $m : W \to G$ belongs to $\mathcal{M}$, then Proposition 3 implies that $\mathrm{Ext}(m, k)\mathrm{Ext}(G, j) = \mathrm{Ext}(m, kj) = 0$. By the previous lemma, $\mathrm{Ext}(G, j)$ is onto, so that $\mathrm{Ext}(m, k) = 0$. □

The dual of Proposition 9 implies that if $\mathcal{M}$ is a collection of morphisms in $\mathcal{A}$, then

$${}^{\perp}\mathcal{M} := \{f : \mathrm{Ext}(f, m) = 0 \ \ \text{for all } m \in \mathcal{M}\}$$

is an ideal in $\mathcal{A}$. A pair of ideals $(\mathcal{I}, \mathcal{J})$ is *orthogonal* if every pair (f, g) of morphisms, $f \in \mathcal{I}$ and $g \in \mathcal{J}$, is orthogonal.

Example 10. Let $\mathcal{F}$ be a subfunctor of Ext, then the pair of ideals $(\Phi(\mathcal{F}), \mathcal{F}\text{-inj})$ is orthogonal. If i is an $\mathcal{F}$-phantom and j an $\mathcal{F}$-injective morphism, then $\mathrm{Ext}(i, j)$ acts on any conflation η by pullback along i, which yields an $\mathcal{F}$-conflation, followed by pushout along j, which is necessarily trivial. Thus $\mathrm{Ext}(i, j) = 0$.

In particular, suppose that $\mathcal{I}$ is an ideal of $\mathcal{A}$ and $\mathcal{F} = \mathrm{PB}(\mathcal{I})$. Then $\mathcal{F}\text{-inj} = \mathcal{I}^{\perp}$, because a morphism j is $\mathrm{PB}(\mathcal{I})$-injective if and only if $\mathrm{Ext}(i, j) = 0$ for every $i \in \mathcal{I}$ if and only if $j \in \mathcal{I}^{\perp}$.

Given an ideal $\mathcal{I} \subseteq \mathcal{A}$, the following proposition uses the exact structure $\mathcal{E}$ on $\mathcal{A}$ to provide a criterion for a morphism $i : C \to A$ in $\mathcal{I}$ to be an $\mathcal{I}$-precover of A.

Proposition 11. *Let $\mathcal{I}$ be an ideal of $\mathcal{A}$ and consider the pushout*

$$\begin{tikzcd} \eta' : & Y \arrow[r] \arrow[d, "j"'] & Z \arrow[r] \arrow[d] & A \arrow[d, equal] \\ \eta : & B \arrow[r] & C \arrow[r, "i"] & A \end{tikzcd}$$

of a conflation η' along a morphism $j : Y \to B$ in $\mathcal{I}^{\perp}$. If $i : C \to A$ belongs to $\mathcal{I}$, then it is an $\mathcal{I}$-precover of A.

Proof. The pushout of η' along j is the conflation $\eta = \mathrm{Ext}(A, j)(\eta')$. If $i' : C' \to A$ is a morphism in $\mathcal{I}$, the pullback of η along i' is

$$\mathrm{Ext}(i', B)(\eta) = \mathrm{Ext}(i', B)\mathrm{Ext}(A, j)(\eta') = \mathrm{Ext}(i', j)(\eta') = 0,$$

by hypothesis. Whence the factorization

$$\begin{tikzcd} & & C' \arrow[dl, dotted] \arrow[d, "i'"] \\ B \arrow[r] & C \arrow[r, "i"] & A. \end{tikzcd} \qquad \square$$

An $\mathcal{I}$-precover $i : C \to A$ that arises from a pushout along a morphism $j \in \mathcal{I}^{\perp}$ as in Proposition 11 was defined in the introduction to be a *special $\mathcal{I}$-precover.* Recall that if every object $A \in \mathcal{A}$ has a special $\mathcal{I}$-precover, then the ideal $\mathcal{I}$ is *special precovering.*

Definition 12. An orthogonal pair $(\mathcal{I}, \mathcal{J})$ of ideals is an *ideal cotorsion pair* in $(\mathcal{A}; \mathcal{E})$ if $\mathcal{I} = {}^{\perp}\mathcal{J}$ and $\mathcal{J} = \mathcal{I}^{\perp}$.

Theorem 13. *If $\mathcal{I}$ is a special precovering ideal of $\mathcal{A}$, then the orthogonal pair of ideals $(\mathcal{I}, \mathcal{I}^{\perp})$ is an ideal cotorsion pair.*

Proof. It must shown that $\mathcal{I} = {}^{\perp}[\mathcal{I}^{\perp}]$. It is clear that $\mathcal{I} \subseteq {}^{\perp}[\mathcal{I}^{\perp}]$, so suppose that $i' : C' \to A$ belongs to ${}^{\perp}[\mathcal{I}^{\perp}]$. By hypothesis, there is a special $\mathcal{I}$-precover of A, which occurs as the deflation of a conflation

$$\eta : \quad B \longrightarrow C \xrightarrow{\;i\;} A$$

obtained by pushout along a morphism $j : Y \to B$ in $\mathcal{I}^{\perp}$: $\eta = \mathrm{Ext}(A, j)(\eta')$ for some conflation η'. Since $i' \in {}^{\perp}[\mathcal{I}^{\perp}]$, the pullback of η along $i' : C' \to A$ is found to be $\mathrm{Ext}(i', B)(\eta) = \mathrm{Ext}(i', B)\mathrm{Ext}(A, j)(\eta') = \mathrm{Ext}(i', j)(\eta') = 0$. As in the proof of the previous proposition, one obtains a factorization

$$\begin{tikzcd} & & & C' \arrow[dl, "g"'] \arrow[d, "i'"] \\ \eta: & B \arrow[r] & C \arrow[r, "i"] & A, \end{tikzcd}$$

which implies that $i' = ig \in \mathcal{I}$. □

Corollary 14. *If $\mathcal{I}$ is a special precovering ideal, then $\mathcal{I} = \Phi(PB(\mathcal{I}))$ is the ideal of $PB(\mathcal{I})$-phantom morphisms.*

Proof. By definition of $\mathrm{PB}(\mathcal{I})$, every morphism in $\mathcal{I}$ is a $\mathrm{PB}(\mathcal{I})$-phantom morphism, so suppose that i is a $\mathrm{PB}(\mathcal{I})$-phantom. In other words, every pullback of a conflation along i is a $\mathrm{PB}(\mathcal{I})$-conflation. If $j \in \mathcal{I}^{\perp}$, then j is $\mathrm{PB}(\mathcal{I})$-injective, so that $\mathrm{Ext}(i, j) = 0$. Thus $i \in {}^{\perp}[\mathcal{I}^{\perp}] = \mathcal{I}$, by the theorem. □

If $\mathcal{J}$ is an ideal of $\mathcal{A}$, then the notion of a *special $\mathcal{J}$-preenvelope* is defined as the dual of a special precover; a *special preenveloping ideal* as the dual of a special precovering ideal.

Recall from the Introduction what it means for an additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ to have enough (special) injective morphisms.

Proposition 15. *Let $\mathcal{F} \subseteq Ext$ be an additive subfunctor with enough injective morphisms. Then $\Phi(\mathcal{F}) = {}^{\perp}(\mathcal{F}\text{-}inj)$.*

Proof. By Example 10, $\Phi(\mathcal{F}) \subseteq {}^{\perp}(\mathcal{F}\text{-inj})$. To show the converse inclusion ${}^{\perp}(\mathcal{F}\text{-inj}) \subseteq \Phi(\mathcal{F})$, suppose that $f : X \to A$ belongs to ${}^{\perp}(\mathcal{F}\text{-inj})$. It must be shown that the pullback of any conflation $\eta : B \to C \to A$ along f is an $\mathcal{F}$-conflation.

By hypothesis, there is an $\mathcal{F}$-injective $\mathcal{F}$-inflation $e : B \to Y$ that gives rise to the following commutative square of conflations

$$\begin{tikzcd}
 & \eta': & B \arrow[rr, "i"] \arrow[dl, "e"'] \arrow[dd, equal] & & Z \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[dl, equal] \arrow[dd, "f"] \\
\gamma: & Y \arrow[rr] \arrow[dd, equal] & & Z' \arrow[rr] \arrow[dd] & & X \arrow[dd, "f" near start] & \\
 & \eta: & B \arrow[rr] \arrow[dl, "e"] & & C \arrow[rr] \arrow[dl] & & A \arrow[dl, equal] \\
 & Y \arrow[rr] & & C' \arrow[rr] & & A. &
\end{tikzcd}$$

Since $\mathrm{Ext}(f, e) = 0$, the conflation γ is trivial. Thus the $\mathcal{F}$-inflation e factors as $e = gi$, for some $g : Z \to Y$. By Lemma 5, the morphism i is an $\mathcal{F}$-inflation and η' an $\mathcal{F}$-conflation. □

This proposition implies that if an additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ has enough special injective morphisms, then the ideal $\mathcal{F}$-inj of $\mathcal{F}$-injective morphisms is a special preenveloping ideal.

3. The proof of Theorem 1

Let us first dispense with the easy implication (3) $\Rightarrow$ (4) of Theorem 1; this implication does not use any of the additional hypotheses on the exact category $(\mathcal{A}; \mathcal{E})$. Recall from Example 10 that $\mathcal{I}^{\perp}$ is the ideal of $\mathrm{PB}(\mathcal{I})$-injective morphisms. If the ideal cotorsion pair $(\mathcal{I}, \mathcal{I}^{\perp})$ is complete, then $\mathcal{I}^{\perp}$ is a special preenveloping ideal. Because $\mathcal{I} = {}^{\perp}[\mathcal{I}^{\perp}]$, this implies that $\mathrm{PB}(\mathcal{I})$ has enough special injective morphisms. That $(\mathcal{I}, \mathcal{I}^{\perp})$ is complete also implies that $\mathcal{I}$ is a special precovering ideal, so that Corollary 14 implies that $\mathcal{I} = \Phi(\mathrm{PB}(\mathcal{I}))$.

Proof of (1) $\Rightarrow$ (2). An additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ is given. The ambient exact category $(\mathcal{A}; \mathcal{E})$ has enough projective morphisms and there exist enough $\mathcal{F}$-injective morphisms.

Lemma 16. *Let $A \in \mathcal{A}$ and consider a conflation $\gamma : K \to P \xrightarrow{p} A$ where $p : P \to A$ is a projective morphism. A morphism $\varphi : X \to A$ is an $\mathcal{F}$-phantom morphism if and only if the pullback of γ along φ is an $\mathcal{F}$-conflation.*

Proof. Let $\eta : B \to C \to A$ be a conflation. Because the morphism $p : P \to A$ is projective, it induces a morphism of conflations

$$\begin{tikzcd}
\gamma: & K \arrow[r] \arrow[d, "g"] & P \arrow[r, "p"] \arrow[d] & A \arrow[d, equal] \\
\eta: & B \arrow[r] & C \arrow[r] & A.
\end{tikzcd}$$

The conflation η is the pushout of γ along the morphism $g : K \to B$, $\eta = \mathrm{Ext}(A, g)(\gamma)$. This morphism of conflations is part of a commutative diagram of conflations given by

$$\begin{tikzcd}
& \gamma': & K \arrow[rr, "i"] \arrow[dl, "g"'] \arrow[dd, equal] & & Z \arrow[rr] \arrow[dl] \arrow[dd] & & X \arrow[dl, equal] \arrow[dd, "\varphi"] \\
\eta': & B \arrow[rr] \arrow[dd, equal] & & Z' \arrow[rr] \arrow[dd] & & X \arrow[dd, "\varphi"'] & \\
& \gamma: & K \arrow[rr] \arrow[dl, "g"] & & P \arrow[rr] \arrow[dl] & & A \arrow[dl, equal] \\
\eta: & B \arrow[rr] & & C \arrow[rr] & & A. &
\end{tikzcd}$$

By hypothesis, the pullback γ' of the conflation γ along $\varphi : X \to A$ is an $\mathcal{F}$-conflation. Consulting the diagram, or Proposition 3, one concludes that the pullback η' of η along φ may be represented as

$$\begin{aligned}
\eta' &= \mathrm{Ext}(\varphi, B)(\eta) = \mathrm{Ext}(\varphi, B)\mathrm{Ext}(A, g)(\gamma) \\
&= \mathrm{Ext}(X, g)\mathrm{Ext}(\varphi, K)(\gamma) = \mathrm{Ext}(X, g)(\gamma'),
\end{aligned}$$

the pushout along g of the $\mathcal{F}$-conflation γ'. It is therefore itself an $\mathcal{F}$-conflation. $\square$

The following theorem yields the implication (1) ⇒ (2) of Theorem 1, but also gives additional information that is useful when treating examples.

Theorem 17. *Let $(\mathcal{A}; \mathcal{E})$ be an exact structure with enough projective morphisms, and suppose that $\mathcal{F} \subseteq Ext$ is a subfunctor with enough injective morphisms. Given an object A, consider a conflation $\gamma : K \to P \overset{p}{\to} A$, where $p : P \to A$ is a projective deflation, and take the pushout*

$$\begin{tikzcd} \gamma: & K \arrow[r] \arrow[d, "e"] & P \arrow[r, "p"] \arrow[d] & A \arrow[d, equal] \\ & C \arrow[r] & X \arrow[r, "\varphi"] & A \end{tikzcd}$$

along a morphism $e : K \to C$ that is an $\mathcal{F}$-injective $\mathcal{F}$-inflation. The morphism $\varphi : X \to A$ is then a special $\mathcal{F}$-phantom precover of A.

Proof. It suffices, by the lemma, to prove that the pullback of γ along $\varphi : X \to A$ is an $\mathcal{F}$-conflation. For then the morphism $\varphi : X \to A$ is an $\mathcal{F}$-phantom morphism obtained by pushout along a morphism $e \in \Phi(\mathcal{F})^{\perp}$. Compose this pullback with the morphism of conflations given in the statement of the theorem to obtain

$$\begin{tikzcd} \gamma': & K \arrow[r, "i"] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\ \gamma: & K \arrow[r] \arrow[d, "e"] & P \arrow[r, "p"] \arrow[d] & A \arrow[d, equal] \\ & C \arrow[r] & X \arrow[r, "\varphi"] & A. \end{tikzcd}$$

This composition is clearly homotopic to 0,

$$\begin{tikzcd} \gamma': & K \arrow[r, "i"] \arrow[d, "e"] & Z \arrow[r] \arrow[d] \arrow[dl, dotted] & X \arrow[d, "\varphi"] \arrow[dl, "1_X"'] \\ & C \arrow[r] & X \arrow[r, "\varphi"] & A. \end{tikzcd}$$

The dotted arrow is given by a morphism $g : Z \to C$ that satisfies $e = gi$. Because $e : K \to C$ is an $\mathcal{F}$-inflation, Lemma 5 implies that $i : K \to Z$ is an $\mathcal{F}$-inflation, and therefore that γ' is an $\mathcal{F}$-conflation. $\square$

Proof of (2) ⇒ (3). For the proof of this implication, the hypothesis that the ambient exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects is used. A special precovering ideal $\mathcal{I}$ is given, and the task is to prove that the orthogonal ideal $\mathcal{I}^{\perp}$ is special preenveloping.

Theorem 18 (*Salce's Lemma*)**.** *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough injective objects and suppose that $\mathcal{I}$ is a special precovering ideal of $\mathcal{A}$. Given an object $A \in \mathcal{A}$, consider a conflation $\eta : A \overset{e}{\to} E \to X$, where E is an injective object, and take the pullback*

$$\begin{tikzcd}
\eta' : & A \arrow[r, "a"] \arrow[d, equal] & J \arrow[r] \arrow[d] & I \arrow[d, "i"] \\
\eta : & A \arrow[r, "e"] & E \arrow[r] & X
\end{tikzcd}$$

along a special $\mathcal{I}$-precover $i : I \to X$. The morphism $a : A \to J$ is then a special $\mathcal{I}^{\perp}$-preenvelope of A. Consequently, the orthogonal ideal $\mathcal{I}^{\perp}$ is a special preenveloping ideal.

Proof. As $i \in \mathcal{I} \subseteq {}^{\perp}[\mathcal{I}^{\perp}]$, the dual of Proposition 11 implies that it is enough to prove $a \in \mathcal{I}^{\perp}$. Then $a : A \to J$ is a special $\mathcal{I}^{\perp}$-preenvelope of A. The special $\mathcal{I}$-precover $i : I \to X$ arises from a pushout along a morphism $g : Y \to W$ in $\mathcal{I}^{\perp}$ as in the commutative diagram

$$\begin{tikzcd}
 & & & & & & Y \arrow[dl, "g"'] \arrow[dd] \\
 & & & W \arrow[rr, equal] \arrow[dd] & & W \arrow[dd] & \\
 & & & & & & Z \arrow[dl, "h"'] \arrow[dd, "ih"] \\
\eta' : & A \arrow[rr, "a"] \arrow[dd, equal] & & J \arrow[rr] \arrow[dd] & & I \arrow[dd, "i"] & \\
 & & B \arrow[rr] \arrow[dl, equal] & & E \arrow[rr] \arrow[dl, equal] & & X \arrow[dl, equal] \\
\eta : & A \arrow[rr, "e"] & & E \arrow[rr] & & X. &
\end{tikzcd}$$

The pullback of η along ih is also the pullback along h of the conflation $\eta' : A \overset{a}{\to} J \to I$. Thus we obtain the commutative diagram

$$\begin{tikzcd}
 & & & & Y \arrow[rr, equal] \arrow[dl, "g"'] \arrow[dd] & & Y \arrow[dl, "g"'] \arrow[dd] \\
 & & & W \arrow[rr, equal, crossing over] \arrow[dd] & & W \arrow[dd] & \\
 & & A \arrow[rr, "f"] \arrow[dl, equal] \arrow[dd, equal] & & Z' \arrow[rr] \arrow[dl, "k"] \arrow[dd] & & Z \arrow[dl, "h"] \arrow[dd, "ih"] \\
\eta' : & A \arrow[rr, "a", crossing over] \arrow[dd, equal] & & J \arrow[rr, crossing over] \arrow[dd] & & I \arrow[dd, "i"] & \\
 & & A \arrow[rr] \arrow[dl, equal] & & E \arrow[rr] \arrow[dl, equal] & & X \arrow[dl, equal] \\
\eta : & A \arrow[rr, "e"] & & E \arrow[rr] & & X &
\end{tikzcd}$$

in which every row and column is a conflation. Since $g \in \mathcal{I}^{\perp}$, and $k : Z' \to J$ is an extension of g by the injective object E, Proposition 9 implies that $k \in \mathcal{I}^{\perp}$ and therefore that $a = kf \in \mathcal{I}^{\perp}$. □

Because the implication (4) ⇒ (1) is trivial, this marks the end of the proof of Theorem 1. The next observation is a direct consequence of Condition (4) of Theorem 1.

Corollary 19. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough injective objects and projective morphisms. The rule $\mathcal{I} \mapsto PB(\mathcal{I})$ is a bijective correspondence between special precovering ideals of $(\mathcal{A}; \mathcal{E})$ and subfunctors $\mathcal{F} \subseteq Ext$ that have enough special injective morphisms. The inverse rule is given by $\mathcal{F} \mapsto \Phi(\mathcal{F})$.*

Let us inspect the details of the proof of Salce's Lemma more closely and glean some observations.

Corollary 20. *Suppose that the exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects and projective morphisms. If the additive subfunctor $\mathcal{F} \subseteq Ext$ has enough injective morphisms, then the ideal cotorsion pair cogenerated by $\mathcal{F}$-inj is complete.*

Proof. By Proposition 15, the ideal ${}^{\perp}(\mathcal{F}\text{-inj})$ is the ideal $\Phi = \Phi(\mathcal{F})$ of $\mathcal{F}$-phantom morphisms. By Theorem 17, this ideal Φ is special precovering. By Theorem 18, the ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is complete. □

Corollary 21. *Suppose that the exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects and projective morphisms, and that the additive subfunctor $\mathcal{F} \subseteq Ext$ has enough injective morphisms. If $\Phi = \Phi(\mathcal{F})$ is the ideal of $\mathcal{F}$-phantom morphisms, then the ideal $\Phi^{\perp}$ of $PB(\Phi)$-injective morphisms is the least ideal of $\mathcal{A}$ containing $\mathcal{F}$-inj and closed under extension by injective objects.*

Proof. Let $\mathcal{J}$ be an ideal of $\mathcal{A}$ that contains the ideal $\mathcal{F}$-inj and is closed under extensions by injective objects. Toward a proof of $\Phi^{\perp} \subseteq \mathcal{J}$, let $b' : B \to J'$ be a morphism in $\Phi^{\perp}$ and consider the last commutative diagram in the proof of Theorem 18. By Theorem 17, we may take the morphism $g : Y \to W$ to belong to the ideal $\mathcal{F}$-inj $\subseteq \mathcal{J}$. Because $\mathcal{J}$ is closed under extensions by injective objects, and the object E in the commutative diagram is injective, the morphism $k : Z' \to J$ also belongs to $\mathcal{J}$. Thus the $\Phi^{\perp}$-preenvelope $b = kf : B \to J$ also belongs to the ideal $\mathcal{J}$. Because b is a $\Phi^{\perp}$-preenvelope of B, the morphism $b' : B \to J'$ factors through b, and so belongs to $\mathcal{J}$. $\square$

The next proposition describes the precise relationship between the additive subfunctors

$$\mathrm{PB}(\Phi) \subseteq \mathcal{F} \subseteq \mathrm{Ext}.$$

Proposition 22. *Suppose, as above, that the exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects and projective morphisms, and that the additive subfunctor $\mathcal{F} \subseteq$ Ext has enough injective objects. The additive subfunctor PB(Φ) $\subseteq$ Ext is the maximum additive subfunctor of $\mathcal{F}$ with enough special injective morphisms.*

Proof. Let $\mathcal{F}' \subseteq \mathcal{F}$ be an additive subfunctor with enough special injective morphisms. Our task is to prove that every $\mathcal{F}'$-conflation $\eta : B \to C \to A$ is a PB(Φ)-conflation, i.e., η is obtained as the pullback of some conflation along a morphism in Φ.

Let $e : B \to Z$ be a special $\mathcal{F}'$-injective $\mathcal{F}'$-inflation. The injective property of e induces a morphism

$$\begin{tikzcd} \eta: & B \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ \gamma: & B \arrow[r, "e"] & Z \arrow[r] & X, \end{tikzcd}$$

of $\mathcal{F}'$-conflations. In addition, e is special, so that the $\mathcal{F}'$-conflation γ arises as the pullback along a morphism $i : X \to A'$ in ${}^{\perp}(\mathcal{F}'\text{-inj})$,

$$\begin{tikzcd} \gamma: & B \arrow[r, "e"] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X, \arrow[d, "i"] \\ \eta': & B \arrow[r] & C' \arrow[r] & A'. \end{tikzcd}$$

Composing these morphisms of conflations shows that η arises as the pullback along if of the conflation η'. By Proposition 15, ${}^{\perp}(\mathcal{F}'\text{-inj}) = \Phi(\mathcal{F}') \subseteq \Phi(\mathcal{F}) = \Phi$. Since $i \in \Phi$ so does if. $\square$

This proposition allows us to characterize the situation when $\mathcal{F} = \mathrm{PB}(\Phi)$ in Theorem 1.

Corollary 23. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough injective objects and projective morphisms and suppose that $\mathcal{F} \subseteq$ Ext is a subfunctor with enough injective morphisms. If $\Phi = \Phi(\mathcal{F})$, then the following conditions are equivalent:*

(1) *the subfunctor $\mathcal{F} \subseteq$ Ext has enough special injective morphisms;*
(2) *$\mathcal{F} = PB(\Phi)$; and*
(3) *$\Phi^{\perp} = \mathcal{F}$-inj.*

Proof. Proposition 22 yields (1) ⇒ (2). The implication (2) ⇒ (3) was verified in Example 10. The implication (3) ⇒ (1) follows from Condition (3) of Theorem 1, which implies that $\mathcal{F}^{\perp}$ is special preenveloping. □

4. Object-orthogonal ideals

An ideal $\mathcal{I}$ of $(\mathcal{A}; \mathcal{E})$ is called *object-orthogonal* if the orthogonal ideal $\mathcal{I}^{\perp}$ is an object ideal. This section is devoted to a proof of Theorem 2, which is a version of Theorem 1 for the case when the ideal $\mathcal{I}$ is object-orthogonal. The strategy is to first prove the implication (1) ⇒ (2), and then establish the equivalence of (2), (3), and (4). Because the implication (4) ⇒ (1) is trivial, this will complete the proof of Theorem 2.

The following theorem is the object version of Theorem 17. It yields the implication (1) ⇒ (2) of Theorem 2.

Theorem 24. *Let $\mathcal{F} \subseteq \mathrm{Ext}$ be an additive subfunctor. If $(\mathcal{A}; \mathcal{E})$ has enough projective morphisms and there exist enough $\mathcal{F}$-injective objects, then for every object A of $\mathcal{A}$ there is an $\mathcal{F}$-phantom morphism $\varphi : X \to A$ that occurs as part of a conflation*

$$E \longrightarrow X \xrightarrow{\varphi} A,$$

where E is an $\mathcal{F}$-injective object.

Proof. This proof is the same as that of Theorem 17. In this case, we may take the $\mathcal{F}$-inflation $e : K \to C$ with C an $\mathcal{F}$-injective object. Now let $E = C$ to obtain the statement of the theorem. □

To see that the implication (4) ⇒ (3) of Theorem 2 holds, recall from Example 10 that $\mathcal{I}^{\perp}$ is the ideal of $\mathrm{PB}(\mathcal{I})$-injective morphisms. The hypothesis that there exist enough $\mathrm{PB}(\mathcal{I})$-injective objects implies that every object $A \in \mathcal{A}$ admits a $\mathrm{PB}(\mathcal{I})$-inflation $e_A : A \to E(A)$ with $E(A)$ a $\mathrm{PB}(\mathcal{I})$-injective object. It follows that any morphism $e : A \to B$ in $\mathcal{I}^{\perp}$ factors through the object $E(A)$, and that $\mathcal{I}^{\perp}$ is an object ideal.

The next proposition yields a proof of (3) ⇒ (2). Call a morphism $i : C \to A$ in $\mathcal{I}$ an *object-special* $\mathcal{I}$-precover of A, if it is the deflation of a conflation

$$\eta : \quad B \longrightarrow C \xrightarrow{i} A,$$

where $B \in \mathcal{I}^{\perp}$. An object-special $\mathcal{I}$-precover is obviously a special $\mathcal{I}$-precover, because η is the pushout of itself by the identity morphism $1_B \in \mathcal{I}^{\perp}$.

Proposition 25. *If a special precovering ideal $\mathcal{I}$ of $(\mathcal{A}; \mathcal{E})$ is object-orthogonal, then every object $A \in \mathcal{A}$ has an object-special $\mathcal{I}$-precover.*

Proof. Suppose that the ideal $\mathcal{I}$ is an object-orthogonal special precovering ideal. Every $A \in \mathcal{A}$ has a special $\mathcal{I}$-precover $i' : C' \to A$ that arises as the pushout,

$$\begin{tikzcd} \eta' : & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] \arrow[d] & A \arrow[d, equal] \\ & B' \arrow[r] & C' \arrow[r, "i'"] & A, \end{tikzcd}$$

of a conflation η', along a morphism $g : Y \to B'$ in $\mathcal{I}^{\perp}$. By hypothesis, the ideal $\mathcal{I}^{\perp}$ is an object ideal, so that the morphism $g : Y \to B'$ factors through an object $B \in \mathcal{I}^{\perp}$ as $g = g_1 g_0 : Y \to B \to B'$. The pushout of η' along g then factors as a composition of pushouts

$$\begin{tikzcd} \eta' : & Y \arrow[r] \arrow[d, "g_0"'] & Z \arrow[r] \arrow[d] & A \arrow[d, equal] \\ \eta : & B \arrow[r] \arrow[d, "g_1"'] & C \arrow[r, "i"] \arrow[d, "k"] & A \arrow[d, equal] \\ & B' \arrow[r] & C' \arrow[r, "i'"] & A. \end{tikzcd}$$

The pushout of η' along g_0 is a conflation η whose deflation $i = i'k : C \to A$ belongs to $\mathcal{I}$. The morphism $i : C \to A$ is therefore an object-special $\mathcal{I}$-precover of A. $\square$

All that remains is to verify (2) $\Rightarrow$ (4). We shall need to cite the following special case of Proposition 9.

Lemma 26. *If $\mathcal{I}$ is an ideal of $(\mathcal{A}; \mathcal{E})$, then the subcategory $Ob(\mathcal{I}^{\perp})$ is closed under extension by injective objects. This means that if there exists a conflation*

$$B \longrightarrow C \xrightarrow{\ e\ } E$$

with $B \in \mathcal{I}^{\perp}$ and E is an injective object, then $C \in \mathcal{I}^{\perp}$.

Proof. The identity morphism 1_C is an extension of 1_B by the injective object E, so that Proposition 9 applies. $\square$

Theorem 27. *Suppose that the exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects and $\mathcal{I}$ is a special precovering ideal of $\mathcal{A}$. Assume, furthermore, that $\mathcal{J} \subseteq \mathcal{I}^{\perp}$ is an object ideal of $\mathcal{A}$ with the property that every object $A \in \mathcal{A}$ has an $\mathcal{I}$-precover $i : C \to A$ that is the deflation of a conflation*

$$J \longrightarrow C \xrightarrow{\ i\ } A,$$

where J belongs to $\mathcal{J}$. Then $\mathcal{I}$ is object-orthogonal; the ideal $\mathcal{I}^{\perp}$ is generated by those objects Z that appear in some conflation

$$J \longrightarrow Z \longrightarrow E,$$

where J belongs to $\mathcal{J}$ and E is an injective object.

Proof. Suppose that $B \in \mathcal{A}$ is given, and let us show that there exists an $\mathcal{I}^{\perp}$-preenvelope of B of the form Z as prescribed. There is a conflation $\eta : B \to E \to A$, with E an injective object. Consider the pullback of η along the morphism $i : C \to A$ given by the hypotheses,

$$\begin{tikzcd}
 & & J \arrow[r, equal] \arrow[d] & J \arrow[d] \\
\eta' : & B \arrow[r, "b"] \arrow[d, equal] & Z \arrow[r] \arrow[d] & C \arrow[d, "i"] \\
\eta : & B \arrow[r] & E \arrow[r] & A,
\end{tikzcd}$$

where the object J belongs to $\mathcal{J}$. The lemma implies that Z belongs to $\mathcal{I}^{\perp}$, and, as in the proof of Theorem 18, the morphism $b : B \to Z$ is a special $\mathcal{I}^{\perp}$-preenvelope. Because it factors through an object Z as described in the statement of the theorem, the ideal $\mathcal{I}^{\perp}$ is generated by such objects. $\square$

Complete Cotorsion Pairs. Recall from the introduction the definition of a cotorsion pair $(\mathcal{F}, \mathcal{C})$. If $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair and $F \in \mathcal{F}$, $C \in \mathcal{C}$, then $\mathrm{Ext}(1_F, 1_C) = 0$, so that the associated pair $(\mathcal{I}(\mathcal{F}), \mathcal{I}(\mathcal{C}))$ of object ideals is orthogonal. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is *special precovering* if for every object $A \in \mathcal{A}$, there is a conflation

$$C \longrightarrow F \longrightarrow A,$$

with $F \in \mathcal{F}$ and $C \in \mathcal{C}$; a *special preenveloping* cotorsion pair is defined dually. A cotorsion pair is *complete* if it is both special preenveloping and special precovering.

Theorem 28. *If $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair, then the pair $(\mathcal{I}(\mathcal{F}), \mathcal{I}(\mathcal{C}))$ of object ideals is a complete ideal cotorsion pair.*

Proof. All that needs to be shown is that $\mathcal{I}(\mathcal{C}) = \mathcal{I}(\mathcal{F})^{\perp}$ and $\mathcal{I}(\mathcal{F}) = {}^{\perp}\mathcal{I}(\mathcal{C})$. For then the definition of a complete cotorsion pair implies that every object in $\mathcal{A}$ has an object-special $\mathcal{I}(\mathcal{C})$-preenvelope and an object-special $\mathcal{I}(\mathcal{F})$-precover. Let us at least verify the equality ${}^{\perp}(\mathcal{I}(\mathcal{C})) = \mathcal{I}(\mathcal{F})$; the dual is proved similarly.

Suppose that $f : X \to A$ belongs to ${}^{\perp}(\mathcal{I}(\mathcal{C}))$. By hypothesis, there is a conflation $C \to F \xrightarrow{i} A$ such that $F \in \mathcal{F}$ and $C \in \mathcal{C}$. Since $\mathrm{Ext}(f, C) = 0$, the morphism f lifts to F as in the diagram

$$\begin{tikzcd} & & X \arrow[dl, dotted] \arrow[d, "f"] \\ C \arrow[r] & F \arrow[r, "i"] & A. \end{tikzcd}$$

Thus f factors through F and so belongs to $\mathcal{I}(\mathcal{F})$. $\square$

Question 29. *Suppose that* $(\mathcal{I}, \mathcal{J})$ *is a complete ideal cotorsion pair such that both of the ideals* $\mathcal{I}$ *and* $\mathcal{J}$ *are object ideals. Is the cotorsion pair* $(Ob(\mathcal{I}), Ob(\mathcal{J}))$ *complete?*

5. Introduction to examples

The next three sections of the paper are devoted to three examples: pure phantom morphisms of modules over an associative ring, semisplit phantom morphisms of complexes, and Auslander–Reiten phantoms in the category Λ-mod of finitely presented representations of an Artin algebra Λ. In all three cases, the ambient exact structure $(\mathcal{A}; \mathcal{E})$ is an abelian category with enough projective objects and injective objects. In the first two cases the associated subfunctor $\mathcal{F} \subseteq \text{Ext}$ also has enough injective objects and projective objects, while in the third case, there exist enough $\mathcal{F}$-injective morphisms and $\mathcal{F}$-projective morphisms. We may therefore apply Theorem 1 in all three cases, and even Theorem 2 in the first two. Moreover, all three examples satisfy properties that are dual to the hypotheses of these theorems. We may therefore invoke the dual theory of *cophantom* morphisms, which is introduced in this section. The next two observations are useful when treating examples.

Proposition 30. *Let* $(\mathcal{A}; \mathcal{E})$ *be an exact category and suppose that* $\mathcal{F} \subseteq \text{Ext}$ *is a subfunctor with enough projective morphisms. A morphism* $\varphi : X \to A$ *is a phantom morphism if for every* $\mathcal{F}$*-projective morphism* $p : P \to X$*, the composition* $\varphi p : P \to X \to A$ *is projective.*

Proof. The condition is necessary, for if $\varphi : X \to A$ is a phantom morphism and a conflation $\eta : B \to C \xrightarrow{g} A$ is given, then the pullback of η along φ is an $\mathcal{F}$-conflation η'. If $p : P \to X$ is an $\mathcal{F}$-projective morphism, then it lifts to the middle term Z of η' as indicated by the dotted arrow:

$$\begin{tikzcd} & & & P \arrow[dl, dotted] \arrow[d, "p"] \\ \eta' : & B \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & X \arrow[d, "\varphi"] \\ \eta : & B \arrow[r] & C \arrow[r, "g"] & A. \end{tikzcd}$$

Thus φp factors through g as needed.

To prove that the condition is sufficient, assume that φ satisfies the condition, and let η be a conflation as above. We must prove that the pullback η' of η along φ is an $\mathcal{F}$-conflation. Let

$p : P \to X$ be an $\mathcal{F}$-projective $\mathcal{F}$-deflation. Since φp factors through $g : C \to A$, the hypothesis on $\mathcal{F}$ and the universal property of the pullback Z imply that there is morphism of conflations given by

$$\begin{tikzcd}
\gamma' : & K \arrow[r] \arrow[d, "f"] & P \arrow[r, "p"] \arrow[d, dotted] & X \arrow[d, equal] \\
\eta' : & B \arrow[r] & Z \arrow[r] & X.
\end{tikzcd}$$

But then η' is the pushout of γ' along f, and is therefore itself an $\mathcal{F}$-conflation. $\square$

The following criterion is a kind of dual to Lemma 16.

Lemma 31. *Suppose that $(\mathcal{A}; \mathcal{E})$ has enough injective morphisms and let $\mathcal{I}$ be an ideal of $\mathcal{A}$. Given an object $B \in \mathcal{A}$, consider a conflation*

$$\eta_B : \quad B \xrightarrow{e} E \longrightarrow \Sigma,$$

where $e : B \to E$ is an injective morphism. Then $B \in \mathcal{I}^{\perp}$ if and only if $Ext(i, B) = 0$ for every $i : X \to \Sigma$ in $\mathcal{I}$.

Proof. The condition is clearly necessary. To see that it is sufficient, let $i' : X' \to A$ be an arbitrary morphism in $\mathcal{I}$, and consider a conflation

$$\begin{tikzcd}
 & & & X' \arrow[d, "i'"] \arrow[dl, dotted] \\
\gamma : & B \arrow[r, "m"] & C \arrow[r] & A.
\end{tikzcd}$$

We must verify that the morphism $i' : X' \to A$ factors through C as indicated by the dotted arrow. Because the morphism $e : B \to E$ is injective, it extends to C along m to provide a morphism of conflations

$$\begin{tikzcd}
 & & & X' \arrow[d, "i'"] \\
\gamma : & B \arrow[r, "m"] \arrow[d, equal] & C \arrow[r] \arrow[d] & A \arrow[d, "s"] \\
 & B \arrow[r, "e"] & E \arrow[r] & \Sigma.
\end{tikzcd}$$

Now $i = si' : X' \to \Sigma$ belongs to $\mathcal{I}$, so the condition implies that this morphism factors through E. The right commutative square is a pullback diagram, which implies that $i' : X' \to A$ factors through C as desired. $\square$

Cophantom Morphisms. If $(\mathcal{A}; \mathcal{E})$ is an exact category and $\mathcal{F} \subseteq \mathrm{Ext}$ an additive subfunctor, then a morphism $\psi : Y \to B$ in $\mathcal{A}$ is an $\mathcal{F}$-*cophantom* morphism if the pushout along ψ of any conflation belongs to the subfunctor $\mathcal{F}$. The ideal of $\mathcal{F}$-cophantom morphisms is denoted by $\Psi = \Psi(\mathcal{F})$. The main results of this article may be dualized as follows so that they apply to cophantom morphisms.

Theorem 32. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough projective objects and injective morphisms. The following statements regarding an ideal $\mathcal{J}$ of $\mathcal{A}$ are equivalent:*

(1) *there is an additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ with enough projective morphisms and $\mathcal{J} = \Psi(\mathcal{F})$;*
(2) *the ideal $\mathcal{J}$ is special preenveloping;*
(3) *the ideal cotorsion pair $({}^{\perp}\mathcal{J}, \mathcal{J})$ is complete; and*
(4) *the additive subfunctor $PO(\mathcal{J}) \subseteq \mathrm{Ext}$, whose conflations are obtained by pushout along morphisms in $\mathcal{J}$, has enough special projective morphisms and $\mathcal{J} = \Psi(PO(\mathcal{J}))$.*

Theorem 33. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category with enough projective objects and injective morphisms. The following statements regarding an ideal $\mathcal{J}$ of $\mathcal{A}$ are equivalent:*

(1) *there is an additive subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$ with enough projective objects and $\mathcal{J} = \Psi(\mathcal{F})$;*
(2) *for every object $B \in \mathcal{A}$, there exists a conflation*

$$B \xrightarrow{j} C \longrightarrow A,$$

where $j : B \to C$ belongs to $\mathcal{J}$ and A is an object in ${}^{\perp}\mathcal{J}$;
(3) *the ideal $\mathcal{J}$ is special preenveloping and ${}^{\perp}\mathcal{J}$ is an object ideal; and*
(4) *the additive subfunctor $PO(\mathcal{J}) \subseteq \mathrm{Ext}$ has enough special projective objects and $\mathcal{J} = \Psi(PO(\mathcal{J}))$.*

Hereditary Ideal Cotorsion Pairs. In this section, we assume that the exact category $(\mathcal{A}; \mathcal{E})$ has enough projective objects: given an object $A \in \mathcal{A}$, there is a conflation

$$\Omega(A) \longrightarrow P \longrightarrow A,$$

where P is a projective object, and $\Omega(A)$ is called a *syzygy* of A. The syzygy of A is not well-defined, but the functor $\mathrm{Ext}^{n+1}(A, -)$, defined by recursion as $\mathrm{Ext}^{n}(\Omega(A), -)$ is.

If $f : A \to B$ is a morphism in $\mathcal{A}$, a syzygy morphism $\Omega(f) : \Omega(A) \to \Omega(B)$ is induced by f and yields a morphism of conflations

$$\begin{tikzcd}
\Omega(A) \arrow[r] \arrow[d, "\Omega(f)"] & P(A) \arrow[r] \arrow[d, "P(f)"] & A \arrow[d, "f"] \\
\Omega(B) \arrow[r] & P(B) \arrow[r] & B.
\end{tikzcd}$$

As above, the morphism $\Omega(f)$ is not well-defined, but the natural transformation of functors

$$\mathrm{Ext}^{n+1}(f,-) : \mathrm{Ext}^{n+1}(B,-) \to \mathrm{Ext}^{n+1}(A,-),$$

given by $\mathrm{Ext}^n(\Omega(f),-)$, is well-defined. Also, if $\Omega(f)$ and $\Omega'(f) : \Omega(A) \to \Omega(B)$ are syzygy morphisms induced by $f : A \to B$, they are equivalent modulo the ideal $\mathcal{I}(\mathcal{E}\text{-Proj})$ of morphisms that factor through a projective object. So if $(\mathcal{I}, \mathcal{J})$ is an ideal cotorsion pair in $(\mathcal{A}; \mathcal{E})$, then $\mathcal{I}$ contains all the projective objects of $\mathcal{A}$, and therefore, the induced syzygy morphisms must also be equivalent modulo $\mathcal{I}$. In particular, if some syzygy $\Omega(f)$ belongs to the ideal $\mathcal{I}$, then every syzygy of f does.

The definition of $\mathrm{Ext}^n(-,-)$ is also natural in the left variable, so that a bifunctor

$$\mathrm{Ext}^n(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$$

arises for all natural numbers $n \geq 1$. An ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ is *hereditary* if for every $i \in \mathcal{I}$, $j \in \mathcal{J}$, $\mathrm{Ext}^n(i, j) = 0$ for all $n \geq 1$.

Proposition 34. *If the exact category $(\mathcal{A}; \mathcal{E})$ has enough projective objects, then an ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ is hereditary if and only if $\Omega(\mathcal{I}) \subseteq \mathcal{I}$.*

Proof. In the same way that we described the case $n = 1$ in Proposition 3, this bifunctor satisfies the equation

$$\begin{aligned}\mathrm{Ext}^n(f, g) &= \mathrm{Ext}^n(\mathrm{Dom}(f), g)\mathrm{Ext}^n(f, \mathrm{Dom}(g))\\ &= \mathrm{Ext}^n(f, \mathrm{Codom}(g))\mathrm{Ext}^n(\mathrm{Codom}(f), g)\end{aligned}$$

for morphisms $f : \mathrm{Dom}(f) \to \mathrm{Codom}(f)$ and $g : \mathrm{Dom}(g) \to \mathrm{Codom}(g)$. Thus

$$\begin{aligned}\mathrm{Ext}^{n+1}(f, g) &= \mathrm{Ext}^{n+1}(\mathrm{Dom}(f), g)\mathrm{Ext}^{n+1}(f, \mathrm{Dom}(g))\\ &= \mathrm{Ext}^n(\Omega[\mathrm{Dom}(f)], g)\mathrm{Ext}^n(\Omega(f), \mathrm{Dom}(g))\\ &= \mathrm{Ext}^n(\mathrm{Dom}(\Omega(f)), g)\mathrm{Ext}^n(\Omega(f), \mathrm{Dom}(g))\\ &= \mathrm{Ext}^n(\Omega(f), g).\end{aligned}$$

To prove the proposition, assume that the ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ is hereditary. If $i \in \mathcal{I}$, then for every $j \in \mathcal{J}$, $\mathrm{Ext}(\Omega(i), j) = \mathrm{Ext}^2(i, j) = 0$, which implies that $\Omega(i) \in {}^{\perp}\mathcal{J} = \mathcal{I}$. Conversely, if $\Omega(\mathcal{I}) \subseteq \mathcal{I}$, then we proceed, by induction on n, to verify that $\mathrm{Ext}^n(i, j) = 0$. If the result holds for n, then for every $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $\mathrm{Ext}^{n+1}(i, j) = \mathrm{Ext}^n(\Omega(i), j) = 0$, because $\Omega(i) \in \mathcal{I}$. $\square$

If the exact category $(\mathcal{A}; \mathcal{E})$ has enough injective objects, the preceding considerations may be dualized to define a *cosyzygy* $\Sigma(B)$ of an object $B \in \mathcal{A}$ and *cosyzygy* $\Sigma(f)$ of a morphism $f : A \to B$. Higher Ext functors may then be defined using cosyzygies: if $B \in \mathcal{A}$, let

$$\mathrm{Ext}^{n+1}(-, B) = \mathrm{Ext}^n(-, \Sigma(B)).$$

This definition is natural in both variables, and so gives rise to a bifunctor $\mathrm{Ext}^n(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$, for every $n \geq 1$. As above, an ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$ may be defined to be hereditary if for every $i \in \mathcal{I}$, $j \in \mathcal{J}$, $\mathrm{Ext}^n(i, j) = 0$ for all $n \geq 1$, and a proposition dual to Proposition 34 holds. The material fact here (see, for example [21, Theorem 6.9]) is that if the exact category $(\mathcal{A}; \mathcal{E})$ has enough projective objects and enough injective objects, then these two definitions of Ext coincide, and there is no ambiguity regarding the definition of a hereditary ideal cotorsion pair $(\mathcal{I}, \mathcal{J})$. In the sequel, we shall need the following criterion for an ideal cotorsion pair to be hereditary.

Proposition 35. *Suppose that $(\mathcal{A}; \mathcal{E})$ is an exact category with enough projective objects and enough injective objects. If $\mathcal{M}$ is an ideal of $\mathcal{A}$ such that $\Omega(\mathcal{M}) \subseteq \mathcal{M}$, then the ideal cotorsion theory $({}^{\perp}(\mathcal{M}^{\perp}), \mathcal{M}^{\perp})$ generated by $\mathcal{M}$ is hereditary.*

Proof. By the dual of Proposition 34, it suffices to verify that $\Sigma(\mathcal{M}^{\perp}) \subseteq \mathcal{M}^{\perp}$. But if $g \in \mathcal{M}^{\perp}$, then for every $m \in \mathcal{M}$, $\mathrm{Ext}(m, \Sigma(g)) = \mathrm{Ext}^2(m, g) = \mathrm{Ext}(\Omega(m), g) = 0$. □

6. Pure phantom morphisms

Let R be an associative ring with identity. The ambient exact category $(\mathcal{A}; \mathcal{E}) = (R\text{-Mod}, \mathrm{Ext}_R)$ is the abelian category R-Mod of left R-modules. It is a classical result that the abelian category $(R\text{-Mod}; \mathrm{Ext}_R)$ has enough injective objects and enough projective objects. A short exact sequence $0 \to B \to C \to A \to 0$ is *pure-exact* if for every finitely presented left R-module M and morphism $f : M \to A$, there is a lifting, indicated by the dotted arrow, that makes the diagram

$$\begin{tikzcd} & & & M \arrow[dl, dotted] \arrow[d, "f"] & \\ 0 \arrow[r] & B \arrow[r] & C \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commutative. The rule that associates to a pair of objects (A, B) the subgroup $\mathrm{Pext}(A, B) \subseteq \mathrm{Ext}(A, B)$ of pure-exact sequences as above constitutes a subfunctor $\mathcal{F} \subseteq \mathrm{Ext}$. The Pext-injective objects are called *pure-injective* modules; the Pext-projective objects *pure-projective* modules. It is well-known [30] that there exist enough pure injective and pure projective objects in the sense of the present theory. Therefore, Theorems 2 and 33 both apply.

A Pext-phantom morphism is called a *pure phantom* morphism. A left R-module M is pure projective if and only if it is a direct summand of a direct sum (perhaps infinite) of finitely presented modules. It follows from Proposition 30, that a morphism $\varphi : X \to A$ of left R-modules is a pure phantom morphism provided that for every finitely presented module M and morphism $p : M \to X$, the composition $\varphi p : M \to X \to A$ factors through a projective module. For the representations of a finite group ring $k[G]$, this is the definition of a phantom morphism given by Benson and Gnacadja [6]. Condition (4) of the next proposition is the definition of a phantom morphism given in [19].

Proposition 36. *The following are equivalent for a morphism $\varphi : X \to A$ of left R-modules:*

(1) *φ is a pure phantom morphism;*
(2) *the morphism φ is a direct limit of morphisms $p_i : X_i \to A$, each of which factors through a projective module P_i;*
(3) *for every $n \geq 1$ and right R-module Z, $\mathrm{Tor}_n(Z, \varphi) = 0$;* and
(4) *$\mathrm{Tor}_1(Z, \varphi) = 0$, for every right R-module Z.*

Proof. (1) ⇒ (2). Every module $X = \varinjlim X_i$ is a direct limit of finitely presented modules. If the structural morphisms of this limit are denoted by $x_i : X_i \to X$, then $\varphi = \varinjlim \varphi x_i$ is the colimit of the projective morphisms $p_i = \varphi x_i$.
(2) ⇒ (3). For every $n \geq 1$, the functor $\mathrm{Tor}_n(Z, -)$ commutes with direct limits. Since $\mathrm{Tor}_n(Z, p_i) = 0$, it follows that $\mathrm{Tor}_n(Z, \varphi) = 0$.
(4) ⇒ (1). Let $\eta : 0 \to K \to P \to A \to 0$ be a short exact sequence of left R-modules, with P projective. By Proposition 30, it suffices to verify that the pullback

$$\begin{tikzcd}
\eta' : & 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, equal] & Q \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, "\varphi"] & 0 \\
\eta : & 0 \arrow[r] & K \arrow[r] & P \arrow[r, "g"] & A \arrow[r] & 0
\end{tikzcd}$$

of η along φ is a pure exact sequence. Let us verify that the morphism $i : K \to Q$ is a pure monomorphism. If Z is a right R-module, then this morphism of short exact sequences gives rise to a morphism of long exact sequences, part of which is given by

$$\begin{tikzcd}
\mathrm{Tor}_1(Z, X) \arrow[r, "\Delta"] \arrow[d, "\mathrm{Tor}_1(Z, \varphi)"] & Z \otimes_R K \arrow[r, "Z \otimes i"] \arrow[d, equal] & Z \otimes_R Q \arrow[d, "Z \otimes f"] \\
\mathrm{Tor}_1(Z, A) \arrow[r] & Z \otimes_R K \arrow[r] & Z \otimes_R P.
\end{tikzcd}$$

Because $\mathrm{Tor}_1(Z, \varphi) = 0$, it follows that $\Delta = 0$, and therefore, that $Z \otimes i$ is a monomorphism of abelian groups. □

A Pext-cophantom morphism is called a *pure cophantom* morphism. These are the morphisms ψ such that $\mathrm{Ext}^1_R(M, \psi) = 0$ for every pure projective (resp., finitely presented) left R-module M. The pure cophantom left R-modules are therefore the FP-injective modules. By the dual of Proposition 30, a morphism $\psi : B \to Y$ is a pure cophantom morphism if and only if the composition $B \xrightarrow{\psi} Y \to \mathrm{PE}(Y)$ with the pure injective envelope of Y factors through an injective left R-module.

Proposition 37. *Let Φ (resp., Ψ) denote the ideal in R-Mod of pure phantom (resp., pure cophantom) morphisms. The ideal cotorsion pair $(\Phi, \Phi^{\perp})$ in R-Mod is hereditary; if R is left coherent, then so is the ideal cotorsion pair $({}^{\perp}\Psi, \Psi)$.*

Proof. The ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is cogenerated by the pure injective left R-modules. By Lemma 3.2.10 of [17], the cosyzygy of a pure injective module is itself pure injective. By the dual of Proposition 35, the ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is hereditary. Similarly, the ideal cotorsion pair $({}^{\perp}\Psi, \Psi)$ is generated by the pure projective left R-modules. If R is left coherent, then the syzygy of a pure projective left R-module is itself pure projective, so that Proposition 35 implies that the ideal cotorsion pair $({}^{\perp}\Psi, \Psi)$ is hereditary. □

Tor-Orthogonal Ideal Pairs. The ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is a special case of the morphism version of a standard construction [17, Lemma 2.2.3] by which cotorsion pairs in the category R-Mod are constructed. To state the morphism version of this lemma, recall the definition of the *character module* of a right R-module A_R. It is the left R-module of $\mathbb{Z}$-morphisms

$$_R(A^c) := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}),$$

where $\mathbb{Q}/\mathbb{Z}$ is the minimal injective cogenerator in the category of $\mathbb{Z}$-modules. If $\chi : A \to \mathbb{Q}/\mathbb{Z}$ is a character on A, then the action of $r \in R$ is given by $(r\chi)(a) := \chi(ar)$. If $f : A_R \to B_R$ is a morphism of right R-modules, then $f^c : B^c \to A^c$ denotes the induced morphism of character modules, so that the rule $A \mapsto A^c$ defines an exact contravariant functor $(-)^c$: Mod-$R \to$ R-Mod, which induces an isomorphism of abelian groups

$$(A \otimes_R X)^c = \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(X, A^c)$$

for every right R-module A and left R-module X, natural in both variables. This isomorphism induces an isomorphism

$$\mathrm{Tor}(A, X)^c \cong \mathrm{Ext}^1_R(X, A^c)$$

of the higher derived functors, also natural in both variables. Naturality implies that if $f : A_R \to B_R$ is a morphism of right R-modules and $g : {}_RX \to {}_RY$ is a morphism of left R-modules, then the morphism

$$\mathrm{Tor}(f, g) : \mathrm{Tor}(A, X) \to \mathrm{Tor}(B, Y)$$

of abelian groups is zero if and only if the morphism $\mathrm{Ext}(g, f^c) : \mathrm{Ext}(Y, B^c) \to \mathrm{Ext}(X, A^c)$ is zero.

An *ideal* Tor *pair* $(\mathcal{M}, \mathcal{N})$ is a pair of ideals $\mathcal{M} \subseteq$ Mod-R and $\mathcal{N} \subseteq$ R-Mod such that (1) $f \in \mathcal{M}$ if and only if $\mathrm{Tor}(f, g) = 0$ for every $g \in \mathcal{N}$; and (2) $g \in \mathcal{N}$ if and only $\mathrm{Tor}(f, g) = 0$ for every $f \in \mathcal{M}$.

Proposition 38 (*Cf. [17, Lemma 2.2.3]*). *If $(\mathcal{M}, \mathcal{N})$ is an ideal Tor pair, then $\mathcal{N} = {}^{\perp}(\mathcal{M}^c)$, where $\mathcal{M}^c$ denotes the collection of morphisms f^c, where $f \in \mathcal{M}$. Consequently, $(\mathcal{N}, \mathcal{N}^{\perp})$ is an ideal cotorsion pair in R-Mod.*

Proof. A morphism g belongs to $\mathcal{N}$ iff for every $f \in \mathcal{M}$, $\mathrm{Tor}(f, g) = 0$ iff $\mathrm{Ext}(g, f^c) = 0$ iff $g \in {}^{\perp}\mathcal{M}^c$. □

Clearly, the pair (Mod-R, Φ) is an ideal Tor pair. The proposition implies that $\Phi = {}^{\perp}[(\text{Mod-}R)^c]$. Now every character module is pure injective, and every pure injective right R-module is a direct summand of a character module (just apply the character construction twice). The ideal in R-Mod of morphisms generated by $(\text{Mod-}R)^c$ is therefore the ideal of morphisms that factor through a pure injective left R-module.

Phantomless Rings. A left R-module F is an object of the ideal Φ of pure phantom morphisms if and only $\mathrm{Tor}_1(-, 1_F) = \mathrm{Tor}_1(-, F) = 0$. The subcategory $\mathrm{Ob}(\Phi)$ of objects in Φ is therefore the subcategory R-Flat of flat left R-modules. The ring R is said to be *left phantomless* if $\Phi = \mathcal{I}(R\text{-Flat})$, that is, if every pure phantom morphism factors through a flat module.

While the ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is cogenerated by the ideal $\mathcal{I}(R\text{-Pinj})$ of morphisms that factor through a pure injective module, the cotorsion pair cogenerated by the *subcategory* R-Pinj of pure injective left R-modules is given by $(R\text{-Flat}, R\text{-Flat}^{\perp})$. The modules in $R\text{-Flat}^{\perp}$

are called *cotorsion* modules and the notation R-Cotor $= R$-Flat$^{\perp}$ is used. By [8], the cotorsion pair (R-Flat, R-Cotor) is complete, so that Theorem 28 implies that the ideal cotorsion pair $(\mathcal{I}(R\text{-Flat}), \mathcal{I}(R\text{-Cotor}))$ is itself complete. These considerations imply that a ring R is left phantomless if and only if the equality

$$(\Phi, \Phi^{\perp}) = (\mathcal{I}(R\text{-Flat}), \mathcal{I}(R\text{-Cotor}))$$

of ideal cotorsion pairs holds.

In general, every pure injective module is cotorsion, and the subcategory $R\text{-Cotor} \subseteq R\text{-Mod}$ is closed under extensions. By Theorem 27, we obtain the inclusions

$$\mathcal{I}(R\text{-Pinj}) \subseteq \Phi^{\perp} \subseteq \mathcal{I}(R\text{-Cotor})$$

of ideals.

Example 39. [31] The ring R is left *Xu* if every left cotorsion R-module is pure injective, that is, if all the above inclusions are equalities. Such a ring is certainly left phantomless.

The ring R is *left perfect* if $R\text{-Cotor} = R\text{-Mod}$; it is *left pure semisimple* if it satisfies the stronger equality $R\text{-Pinj} = R\text{-Mod}$. If R is left pure semisimple, then the inclusions above are clearly equalities, and so R is necessarily a left Xu ring.

Proposition 40. *A left perfect ring R is left Xu if and only if it is left pure semisimple.*

Proof. If R is a left perfect, left Xu ring, then $\mathcal{I}(R\text{-Pinj}) = \mathcal{I}(R\text{-Cotor}) = \mathcal{I}(R\text{-Mod})$, so that R is left pure semisimple. $\square$

The ring R is *Quasi-Frobenius (QF)* if it is left Artinian, and R, considered as a left module over itself, is injective. This is a left–right symmetric condition and implies that R is left perfect. By Theorem 27, the object ideal $\Phi^{\perp}$ is generated by modules Z that arise as extensions

$$0 \longrightarrow M \longrightarrow Z \longrightarrow E \longrightarrow 0,$$

where M is pure injective and E is injective. If R is QF, then E is also projective, so that the short exact sequence is split exact and $Z = M \oplus E$ is itself pure injective. Whence the following.

Proposition 41. *If R is a QF ring, then $\Phi^{\perp} = \mathcal{I}(R\text{-}Pinj)$. If R is, furthermore, left phantomless, then it is of finite representation type.*

Proof. If R is left phantomless, then $\mathcal{I}(R\text{-Cotor}) = \Phi^{\perp} = \mathcal{I}(R\text{-Pinj})$, and so R is left Xu. Because R is also left perfect, Proposition 40 implies that R is left pure semisimple. By [18, Corollary 5.3], every left pure semisimple QF ring is of finite representation type. $\square$

A ring R is *left semi-hereditary* if every finitely generated submodule of a projective left R-module is itself projective. Equivalently, the ring R is left coherent and of global flat dimension at most 1.

Proposition 42. *Every left semi-hereditary ring is left phantomless.*

Proof. Let us verify that the flat cover of a left R-module M is its phantom cover. To do so, it suffices to show that the kernel C of the flat cover

$$0 \longrightarrow C \longrightarrow \mathrm{FC}(M) \xrightarrow{\varphi} M \longrightarrow 0$$

is pure injective. Because R is of global flat dimension at most 1, the module C is itself a flat module. The kernel C of the flat cover of M is a cotorsion module. Over a left coherent ring, every flat cotorsion module is pure injective. □

If R is a left Artinian, left hereditary ring, then it is left phantomless. On the other hand, the left Artinian property implies that R is left perfect, so that Proposition 40 implies that R is left Xu if and only if it is left pure semisimple. In this way, one finds many examples of left phantomless rings that are *not* left Xu.

7. Semisplit phantom morphisms

Let $(\mathcal{A}; \mathcal{E})$ be the abelian category Ch(R-Mod) of complexes of left R-modules. The object M^* of Ch(R-Mod) is depicted as

$$\cdots \longrightarrow {}_RM^0 \xrightarrow{d^0} {}_RM^1 \xrightarrow{d^1} \cdots \longrightarrow {}_RM^n \xrightarrow{d^n} {}_RM^{n+1} \xrightarrow{d^{n+1}} \cdots,$$

where ${}_RM^n$ is a left R-module for every $n \in \mathbb{Z}$, and the boundary morphisms $d_n : M^n \to M^{n+1}$ are R-linear morphisms that satisfy $d^{n+1}d^n = 0$. For $n \in \mathbb{Z}$, the rule $\mathrm{Deg}_n : M^* \mapsto M^n$ that assigns to the complex M^* its component of degree n defines an exact functor Deg_n : Ch(R-Mod) $\to$ R-Mod. In the other direction, the functor D^n : R-Mod $\to$ Ch(R-Mod) associates to a module $M \in R$-Mod the complex $D^n(M)$, whose objects are 0 in all degrees except n and $n+1$, and $d_n = 1_M : M \to M$. This complex is depicted as

$$\cdots \longrightarrow 0 \xrightarrow{0} M \xrightarrow{1_M} M \xrightarrow{0} 0 \longrightarrow \cdots.$$

It is readily verified that D^n is the left adjoint of Deg_n i.e., for every left R-module M and complex X^*, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Ch}}(D^n(M), X^*) \cong \mathrm{Hom}_R(M, X^n), \tag{1}$$

natural in both M and X^*. If $P \in R$-Mod is a projective module, then it follows from the exactness of Deg_n that $D^n(P)$ is a projective object of Ch(R-Mod). Given a complex X^*, we may find for every X^n, $n \in \mathbb{Z}$, an epimorphism $p^n : P^n \to X^n$ in R-Mod from a projective module. Let $\pi^n : D^n(P^n) \to X^*$ be the corresponding morphism of complexes given by the isomorphism (1). The coproduct $\oplus_n \pi^n : \oplus_n D^n(P^n) \to X^*$ is then an epimorphism in Ch(R-Mod) with a projective domain. The abelian category Ch(R-Mod) therefore has enough projective objects.

The functor D^{n-1} : R-Mod $\to$ Ch(R-Mod) is the right adjoint of Deg_n, and the exactness of Deg_n may be used similarly to prove that if $E \in R$-Mod is an injective left R-module, then the complex $D^{n-1}(E)$ is an injective object in the category Ch(R-Mod). An argument dual to the one given above then shows that the abelian category Ch(R-Mod) also has enough injective objects.

An exact substructure (Ch(R-Mod); $\mathcal{E}_0$) of the abelian category of complexes is given by the exact category whose conflations are those short exact sequences

$$\eta : \quad 0 \longrightarrow X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \longrightarrow 0$$

of complexes such that for every $n \in \mathbb{Z}$, the short exact sequence

$$\mathrm{Deg}_n(\eta):\ 0 \longrightarrow X^n \xrightarrow{f^n} Y^n \xrightarrow{g^n} Z^n \longrightarrow 0$$

is split exact; such exact sequences of complexes are called *semisplit*.

The exact category (Ch(R-Mod); $\mathcal{E}_0$) is *Frobenius* [9, Section 13.4]. This means that: (1) there exist enough injective objects in (Ch(R-Mod); $\mathcal{E}_0$); (2) there exist enough projective objects in (Ch(R-Mod); $\mathcal{E}_0$); and (3) $\mathcal{E}_0$-Proj $=$ $\mathcal{E}_0$-Inj. The projective/injective objects of this exact category are the *contractible* complexes, those of the form $\oplus_n\ D^n(A^n)$. We will consider the exact substructure (Ch(R-Mod); $\mathcal{E}_0$) as a subfunctor $\mathcal{F} \subseteq$ Ext satisfying $\mathcal{F}$-inj $=$ $\mathcal{F}$-proj. To see that the subfunctor $\mathcal{F} \subseteq$ Ext has enough projective objects, let X^* be a complex of left R-modules and set $M = X^n$ in the isomorphism (1). Choose the morphism $\iota_n : D^n(X^n) \to X^*$ of complexes associated to the identity morphism. The coproduct of the morphisms ι_n, $n \in \mathbb{Z}$, is then a semisplit epimorphism $\oplus_n\ \iota_n : \oplus_n\ D^n(X^n) \to X^*$ of complexes with a contractible domain. A similar argument uses the right adjoint property of D^{n-1} to show that the subfunctor $\mathcal{F} \subseteq$ Ext has enough injective objects.

Theorems 2 and 33 imply that the subcategory $\mathcal{F}$-inj $=$ $\mathcal{F}$-proj of contractible complexes cogenerates (resp., generates) a complete ideal cotorsion pair. A morphism that belongs to the ideal $\Phi = \Phi(\mathcal{F})$ of $\mathcal{F}$-phantom morphisms is called a *semisplit phantom* morphism. By Proposition 30, a morphism $f^* : M^* \to X^*$ of complexes is a semisplit phantom if for every contractible complex C^* and morphism $c^* : C^* \to M^*$, the composition f^*c^* factors through a projective complex. This implies that each of the morphisms $f^n c^n : C^n \to X^n$ factors through a projective module.

The complete ideal cotorsion theory cogenerated by the contractible complexes is given by $(\Phi, \Phi^{\perp})$; and the one generated by them by $({}^{\perp}\Psi, \Psi)$. Proposition 35 implies that the ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is hereditary, because the syzygy of a contractible complex C^* is itself contractible. Indeed, the complex C^* is of the form $\oplus_n\ D^n(M^n)$, so if

$$0 \longrightarrow \Omega(M^n) \longrightarrow P^n \longrightarrow M^n \longrightarrow 0$$

is a short exact sequence in R-Mod with P^n a projective module, then one obtains a short exact sequence of complexes

$$0 \longrightarrow \oplus_n\ D^n(\Omega(M^n)) \longrightarrow \oplus_n D^n(P^n) \longrightarrow C^* \longrightarrow 0,$$

where the middle term is projective and the syzygy of C^* is given by the contractible complex $\oplus_n\ D^n(\Omega(M^n))$. A similar argument proves that the ideal cotorsion pair $({}^{\perp}\Psi, \Psi)$ is also hereditary.

Because $\mathcal{F}$-inj is an object ideal, Theorem 27 implies that $\Phi^{\perp}$ is also an object ideal, generated by those complexes Z^* that appear in a conflation

$$C^* \longrightarrow Z^* \longrightarrow E^*,$$

where C^* is a contractible complex and E^* an injective one. In what follows, we will use the Frobenius property of the exact category $(\mathrm{Ch}(R\text{-Mod}), \mathcal{E}_0)$ to obtain a more constructive description of a semisplit phantom precover of a complex, and the objects of $\Phi^\perp$, but we need to slightly weaken the property in order to include the example of Auslander–Reiten phantom morphisms, treated in the next section.

Let $(\mathcal{A}; \mathcal{E})$ be an exact category and $\mathcal{F} \subseteq \mathrm{Ext}$ a subfunctor with enough injective morphisms such that for every object $A \in \mathcal{A}$, there exists a conflation

$$\Omega(A) \xrightarrow{e} P \xrightarrow{p} A, \tag{2}$$

where $p : P \to A$ is a projective morphism and $e : \Omega(A) \to P$ is an $\mathcal{F}$-injective morphism. This property is clearly satisfied by $(\mathrm{Ch}(R\text{-Mod}); \mathcal{E}_0) \subseteq (\mathrm{Ch}(R\text{-Mod}); \mathcal{E})$, because every projective complex in $\mathrm{Ch}(R\text{-Mod})$ is contractible.

There is an $\mathcal{F}$-injective $\mathcal{F}$-inflation $e' : \Omega(A) \to E$ given in the top row of the diagram below. Because $e : \Omega(A) \to P$ is an $\mathcal{F}$-injective morphism, a morphism of conflations

$$\begin{tikzcd}
\Omega(A) \arrow[r, "e'"] \arrow[d, equal] & E' \arrow[r, "\sigma'"] \arrow[d, "f"] & \Sigma' \arrow[d, "\varphi"] \\
\Omega(A) \arrow[r, "e"] & P \arrow[r, "p"] & A
\end{tikzcd}$$

arises, where the top row is actually an $\mathcal{F}$-conflation and the object $\Sigma' = \Sigma'(\Omega(A))$ is an $\mathcal{F}$-cosyzygy of a syzygy of A. Lemma 16 implies that $\varphi : \Sigma' \to A$ is an $\mathcal{F}$-phantom morphism. This commutative diagram yields another morphism of conflations

$$\begin{tikzcd}
\Omega(A) \arrow[r, "e"] \arrow[d, "e'"] & P \arrow[r, "g"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & A \arrow[d, equal] \\
E' \arrow[r, "\begin{pmatrix} f \\ -\sigma' \end{pmatrix}"] & P \oplus \Sigma' \arrow[r, "(p, \varphi)"] & A.
\end{tikzcd}$$

Now $p : P \to A$ is also a phantom morphism, because it is projective, so that the morphism $(p, \varphi) : P \oplus \Sigma' \to A$ is itself phantom. The choice of $e' : \Omega(A) \to E'$ implies that it belongs to $\Phi^\perp$. By Theorem 17, the morphism (p, φ) is a special phantom precover of A.

Definition 43. Given an object $M \in \mathcal{A}$, define $\xi_M : \Omega(\Sigma(M)) \to M$ to be any morphism that makes the diagram

$$\begin{tikzcd}
\Omega\Sigma(M) \arrow[r, "\omega"] \arrow[d, "\xi_M"] & P \arrow[r, "p"] \arrow[d] & \Sigma(M) \arrow[d, equal] \\
M \arrow[r, "e"] & E \arrow[r, "\sigma"] & \Sigma(M)
\end{tikzcd}$$

commute, where $e : M \to E$ is an injective inflation, and $p : P \to \Sigma(M)$ is a projective deflation with $\omega : \Omega(\Sigma(M)) \to P$ an $\mathcal{F}$-injective morphism.

Theorem 44. *Let $(\mathcal{A}; \mathcal{E})$ be an exact category and $\mathcal{F} \subseteq \mathrm{Ext}$ a subfunctor with enough injective morphisms. Suppose furthermore, that for every object $A \in \mathcal{A}$ there exists a conflation*

$$\Omega(A) \xrightarrow{\omega} P \xrightarrow{p} A$$

with p projective and ω $\mathcal{F}$-injective. If Φ is the ideal of $\mathcal{F}$-phantom morphisms, then an object $M \in \mathcal{A}$ belongs to $\Phi^{\perp}$ if and only if some (any) morphism $\xi_M : \Omega\Sigma(M) \to M$ is $\mathcal{F}$-injective.

Proof. Apply the condition to $A = \Sigma(M)$. As above, there is a morphism of conflations

$$\begin{tikzcd}
\Omega\Sigma(M) \arrow[r, "e'"] \arrow[d, equal] & E' \arrow[r, "\sigma'"] \arrow[d] & \Sigma' \arrow[d, "\varphi"] \\
\Omega\Sigma(M) \arrow[r, "\omega"] & P \arrow[r, "p"] & \Sigma(M),
\end{tikzcd}$$

where the top row is an $\mathcal{F}$-conflation, $e' : \Omega\Sigma(M) \to E'$ is an $\mathcal{F}$-injective morphism, and $\varphi \in \Phi$. Composing this morphism of conflations with the one that appears in the definition of ξ_M yields the commutative diagram

$$\begin{tikzcd}
\Omega\Sigma(M) \arrow[r, "e'"] \arrow[d, "\xi_M"] & E' \arrow[r] \arrow[d] \arrow[dl, dotted] & \Sigma' \arrow[d, "\varphi"] \arrow[dl, dotted] \\
M \arrow[r, "e"] & E \arrow[r, "\sigma"] & \Sigma(M).
\end{tikzcd}$$

If $M \in \Phi^{\perp}$, then $\mathrm{Ext}(\varphi, M) = 0$, so that one obtains the homotopy indicated by the dotted arrows. The morphism ξ_M then factors through $e' : \Omega\Sigma(M) \to E'$, and is therefore itself $\mathcal{F}$-injective.

To prove the converse, suppose that $\xi_M : \Omega\Sigma(M) \to M$ is $\mathcal{F}$-injective. Then ξ_M factors through e' and the homotopy indicated by the dotted arrows arises. It follows that the morphism $\varphi : \Sigma' \to \Sigma(M)$ factors through $\sigma : E \to \Sigma(M)$. The morphism $p : P \to \Sigma(M)$ is projective so that it too factors through $\sigma : E \to \Sigma(M)$. The special phantom precover $(p, \varphi) : P \oplus \Sigma' \to \Sigma(M)$ therefore factors through σ, and we may infer that every phantom morphism $\varphi' : X \to \Sigma(M)$ factors through $\sigma : E \to \Sigma(M)$. By Lemma 31, $M \in \Phi^{\perp}$. $\square$

A consequence of the theorem is that a complex M^* belongs to $\Phi^\perp$ if and only if some (any) morphism $\xi_{M^*} : \Omega\Sigma(M^*) \to M^*$ factors through a contractible complex.

8. Auslander–Reiten phantom morphisms

Let Λ be an Artin algebra and denote by Λ-mod the abelian category of finitely presented left Λ-modules. The *Jacobson radical* Jac(Λ-mod) of Λ-mod is the intersection of all left (resp., right) maximal ideals of Λ-mod. Let $\mathcal{F} \subseteq \mathrm{Ext}_\Lambda$ be the subfunctor defined by the condition that $\eta \in \mathcal{F}(A, B)$ provided that every morphism $j : B \to Y$ that belongs to the Jacobson radical $j \in \mathrm{Jac}(B, Y)$ factors through the inflation of η as indicated by the dotted arrow

$$\begin{tikzcd} \eta: & 0 \arrow[r] & B \arrow[r] \arrow[d, "j"] & C \arrow[r] \arrow[dl, dotted] & A \arrow[r] & 0 \\ & & Y & & & \end{tikzcd}$$

The collection of short exact sequences that satisfy this property constitutes a subfunctor $\mathcal{F} \subseteq \mathrm{Ext}_\Lambda$, and every morphism in the Jacobson radical of Λ-mod is $\mathcal{F}$-injective. The subfunctor $\mathcal{F}$ is the largest subfunctor of Ext_Λ for which Jac(Λ-mod) $\subseteq \mathcal{F}$-inj. In this section, we will apply the theory of almost split sequences [1] to prove that $\mathcal{F}$ has enough injective morphisms. A morphism that belongs to the ideal $\Phi = \Phi(\mathcal{F})$ is called an *Auslander–Reiten (AR) phantom* morphism. Theorem 1 implies that the ideal cotorsion pair $(\Phi, \Phi^\perp)$ generated by the AR phantom morphisms is complete.

Every object $M \in \Lambda$-mod is of finite length, and therefore admits a direct sum decomposition

$$M = \bigoplus_{j=1}^{m} U_j,$$

where each summand U_i is an indecomposable module with a local endomorphism ring. If the decomposition of $N \in \Lambda$-mod as a direct sum of indecomposables is given by $N = \oplus_{i=1}^n V_i$, then a morphism $f : M \to N$ belongs to $\mathrm{Jac}(M, N)$ if and only if each of the induced morphisms $f_{ij} : U_j \to V_i$ belongs to $\mathrm{Jac}(U_j, V_i)$. For this reason, the Jacobson radical is determined by the subgroups $\mathrm{Jac}(U, V)$, where U and V are indecomposable. Indeed, if U and V are both indecomposable, but not isomorphic, then $\mathrm{Jac}(U, V) = \mathrm{Hom}_\Lambda(U, V)$ while $\mathrm{Jac}(U, U) = \mathrm{Jac}(\mathrm{End}(U))$, where $\mathrm{Jac}(\mathrm{End}(U))$ denotes the *Jacobson radical* of the local endomorphism ring of U.

Proposition 45. *If $V \in \Lambda$-mod is an indecomposable module, then a morphism $f : M \to V$ in Λ-mod belongs to $\mathrm{Jac}(M, V)$ if and only if it is not a split epimorphism. Dually, if $U \in \Lambda$-mod is indecomposable, then a morphism $f : U \to N$ in Λ-mod belongs to $\mathrm{Jac}(U, N)$ if and only if it is not a split monomorphism.*

Proof. Decompose $M = \oplus_{j=1}^m U_j$ as a direct sum of indecomposables. Then $f : M \to V$ is *not* in $\mathrm{Jac}(M, V)$ if and only if one of the induced morphisms $f_j : U_j \to V$ is an isomorphism if and only if there is a section $g : V \to M$ such that $fg = 1_V$. $\square$

An *almost split,* or *Auslander–Reiten,* sequence in Λ-mod is a short exact sequence

$$\alpha: \quad 0 \longrightarrow U \xrightarrow{i} W \xrightarrow{p} V \longrightarrow 0$$

that is (1) not split exact; (2) every morphism $j : U \to Y$ that is not a split monomorphism factors through $i : U \to W$; and (3) every morphism $j' : X \to V$ that is not a split epimorphism factors through $p : W \to V$. Condition (1) implies that the Λ-module U is not injective; Condition (2) that it is indecomposable. By the proposition, Condition (2) is equivalent to the property that every morphism $j : U \to Y$ that belongs to the Jacobson radical of Λ-mod factors though $i : U \to W$. It follows that every almost split sequence in Λ-mod belongs to $\mathcal{F}$. Because U is indecomposable, another application of Proposition 45 shows that the $\mathcal{F}$-inflation $i : U \to W$ itself belongs to the Jacobson radical, and is therefore $\mathcal{F}$-injective. The following result is then part of Auslander and Reiten's theory of almost split sequences.

Theorem 46. *There exist enough $\mathcal{F}$-injective morphisms. Furthermore,*

$$\mathcal{F}\text{-}inj = Jac(\Lambda\text{-}mod) + \mathcal{I}(\Lambda\text{-}Inj),$$

where $\mathcal{I}(\Lambda\text{-}Inj)$ denotes the ideal of morphisms that factor through an injective object in Λ-mod. Dually, there exist enough $\mathcal{F}$-projective morphisms and $\mathcal{F}\text{-}proj = Jac(\Lambda\text{-}mod) + \mathcal{I}(\Lambda\text{-}Proj)$.

Proof. Decompose a finitely presented Λ-module $M = \oplus_{j=1}^{m} U_j$ as a direct sum of indecomposables. If U_j is injective, then the trivial sequence

$$0 \longrightarrow U_j \xrightarrow{1} U_j \longrightarrow 0 \longrightarrow 0$$

belongs to $\mathcal{F}$ and $1 : U_j \to U_j$ is an injective morphism, which is therefore $\mathcal{F}$-injective. If U_j is not injective, then [1, Theorem 4.2] implies that there exists an almost split sequence

$$\alpha_j: \quad 0 \longrightarrow U_j \xrightarrow{\iota_j} W_j \longrightarrow V_j \longrightarrow 0.$$

It follows from the foregoing considerations that the direct sum of these short exact sequences gives a short exact sequence in $\mathcal{F}$ whose inflation $i : M \to W = \oplus_{j=1}^{m} W_j$ is $\mathcal{F}$-injective.

To prove the equality, it suffices to establish the inclusion $\mathcal{F}\text{-inj} \subseteq \text{Jac}(\Lambda\text{-mod}) + \mathcal{I}(\Lambda\text{-inj})$, so let $f : M \to N$ be a morphism in Λ-mod that belongs to $\mathcal{F}$-inj and decompose $M = \oplus_{j=1}^{m} U_j$ as a direct sum of indecomposables. If the summand U_j is injective, then the restriction $f_j : U_j \to N$ belongs to $\mathcal{I}(\Lambda\text{-inj})$. If not, then the restriction $f_j : U_j \to N$ also belongs to $\mathcal{F}$-inj, and therefore factors through the inflation $\iota_j : U_j \to W_j$ of the almost split sequence α_j. Because ι_j belongs to Jac(Λ-mod), so does f_j. It follows that $f = \sum_j f_j$ belongs to $\text{Jac}(\Lambda\text{-mod}) + \mathcal{I}(\Lambda\text{-Inj})$.

Let us define the subfunctor $\mathcal{F}' \subseteq \text{Ext}_\Lambda$ by the condition that $\eta' \in \mathcal{F}'(A, B)$ provided that every morphism $j' : X \to A$ that belongs to the Jacobson radical $j' \in \text{Jac}(X, A)$ factors through the deflation of η as indicated by the dotted arrow

$$\begin{tikzcd}
 & & & X \arrow[dl, dotted] \arrow[d, "j'"] & \\
\eta : 0 \arrow[r] & B \arrow[r] & C \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

The definition of $\mathcal{F}'$ is dual to that of $\mathcal{F}$ and there is a well-known duality [1, Section 1] of categories $D : (\Lambda\text{-mod})^{\mathrm{op}} \to \text{mod-}\Lambda$ that may now be applied to infer, using dual reasoning, that this subfunctor $\mathcal{F}' \subseteq \mathrm{Ext}_\Lambda$ also has enough projective morphisms, and that $\mathcal{F}'\text{-proj} = \mathrm{Jac}(\Lambda\text{-mod})+\mathcal{I}(\Lambda\text{-Proj})$, where $\mathcal{I}(\Lambda\text{-Proj})$ is the ideal of morphisms in Λ-mod that factor through a projective module. Condition (3) of an almost split sequence implies that it belongs to $\mathcal{F}'$.

To finish the proof, we will show that $\mathcal{F} = \mathcal{F}'$. It suffices to verify that $\mathcal{F} \subseteq \mathcal{F}'$, for the duality D may then be used to obtain the other inclusion $\mathcal{F}' \subseteq \mathcal{F}$. To that end, let $\eta \in \mathcal{F}(N, M)$ and decompose these two Λ-modules as direct sums of indecomposable modules $M = \oplus_{j=1}^{m} U_j$ and $N = \oplus_{i=1}^{n} V_i$. Then $\mathcal{F}(N, M) = \mathcal{F}(\oplus_i V_i, \oplus_j U_j) = \oplus_{i,j} \mathcal{F}(V_i, U_j)$ so that we may decompose $\eta = \sum \eta_{ij}$ as a sum of the short exact sequences

$$\eta_{ij} : \quad 0 \longrightarrow U_i \xrightarrow{m_{ij}} W_{ij} \longrightarrow V_j \longrightarrow 0,$$

where U_i and V_j are indecomposable. Each of the sequences η_{ij} belongs to $\mathcal{F}$ because it is obtained by pullback along V_j and pushout along U_i. If some sequence η_{ij} is not split exact, $\eta_{ij} \neq 0$, then U_i is not injective and the morphism $m_{ij} : U_i \to W_{ij}$ belongs to $\mathcal{F}'$. Because the morphisms in $\mathrm{Jac}(\Lambda\text{-mod})$ are $\mathcal{F}$-injective, one obtains the following composition of morphisms of short exact sequences

$$\begin{tikzcd}
\alpha_i : 0 \arrow[r] & U_i \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, "w"] & V \arrow[r] \arrow[d, "v"] & 0 \\
\eta_{ij} : 0 \arrow[r] & U_i \arrow[r] \arrow[d, equal] & W_{ij} \arrow[r] \arrow[d, "w'"] & V_j \arrow[r] \arrow[d, "v'"] & 0 \\
\alpha_i : 0 \arrow[r] & U_i \arrow[r] & W \arrow[r] & V \arrow[r] & 0,
\end{tikzcd}$$

where the short exact sequence α_i is almost split. By [1, Proposition 5.2], the composition $w'w : W \to W$ is an automorphism, and therefore the composition $vv' : V \to V$ is also an automorphism. Condition (3) of an almost split sequence implies that V is indecomposable. Because V_j is also indecomposable, the morphism $v : V \to V_j$ is an isomorphism. It follows that the short exact sequence η_{ij} is almost split, and therefore belongs to the subfunctor $\mathcal{F}'$. Consequently, so does $\eta = \sum \eta_{ij}$. $\square$

Because $\mathcal{F} = \mathcal{F}'$, the subfunctor $\mathcal{F} \subseteq \mathrm{Ext}_\Lambda$ is called the *socle* of Ext_Λ. By Theorem 1, the ideal $\Phi = \Phi(\mathcal{F})$ of Auslander–Reiten phantom morphisms generates a complete ideal cotorsion pair $(\Phi, \Phi^\perp)$.

Corollary 47. *A morphism $\varphi : X \to A$ of finitely presented left Λ-modules is an Auslander–Reiten phantom morphism if and only if, for every $j \in Jac(Y, X)$, the composition φj factors through a projective Λ-module.*

Proof. By Proposition 30, it is enough to prove that for every $\mathcal{F}$-projective morphism $j' : Y \to X$, the composition $\varphi j'$ factors through a projective Λ-module. By the theorem, the morphism j' may be expressed as a sum $j' = j + p$, where $j \in \mathrm{Jac}(\Lambda\text{-mod})$ and p factors through a projective module. It is therefore sufficient to verify that the composition φj factors through a projective Λ-module. □

A Λ-module C that belongs to Φ is called an *AR phantom object.* According to the next consequence of Theorem 46, the indecomposable AR phantom objects have been characterized in [1, Theorem 5.5].

Corollary 48. *Let C be an indecomposable Λ-module that is not projective. Then C is an AR phantom object if and only if the short exact sequence*

$$\eta : \quad 0 \longrightarrow \Omega(C) \longrightarrow P(C) \xrightarrow{p} C \longrightarrow 0$$

where $p : P(C) \to C$ is the projective cover of C is almost split.

Proof. That the Λ-module C belongs to Φ means that $1_C \in \Phi$. By Corollary 47, this is equivalent to the condition that every morphism $j : X \to C$ in the Jacobson radical factors through a projective module. Because C is indecomposable, this is equivalent to the condition that every morphism $j : X \to C$ that is not a split epimorphism factors through $p : P(C) \to C$. □

An almost split sequence η in Λ-mod is called *left special* if it arises as the pullback along an AR phantom morphism $\varphi : V \to V'$,

$$\begin{tikzcd}
\eta : & 0 \arrow[r] & U \arrow[r, "j"] \arrow[d, equal] & W \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "\varphi"] & 0 \\
\eta' : & 0 \arrow[r] & U \arrow[r] & W' \arrow[r] & V' \arrow[r] & 0.
\end{tikzcd}$$

Equivalently, the almost split sequence η belongs to the subfunctor $\mathrm{PB}(\Phi)$. For example, if $C \in \Lambda$-mod is an AR phantom object, then the almost split sequence that appears in Corollary 48 is left special.

Corollary 49. *Let U be an indecomposable, noninjective left Λ-module. There exists a left special almost split sequence η as above if and only if $U \notin \Phi^{\perp}$.*

Proof. If $U \in \Phi^{\perp}$, then it is obvious that any short exact sequence with left term U that arises by an AR phantom morphism is split exact. On the other hand, if $U \notin \Phi^{\perp}$, then there exists a special $\Phi^{\perp}$-preenvelope $j : U \to W$ of U that arises as the monomorphism in a short exact sequence arising as the pullback along an AR phantom morphism $\varphi : V \to V'$, as in the diagram above. Because $j : U \to W$ is a special $\Phi^{\perp}$-preenvelope, and U is not $\mathcal{F}$-injective, the morphism j is not a split monomorphism. The short exact sequence η is therefore a nonzero element of $\mathcal{F}(V, U)$. Another application of [1, Proposition 5.2] implies that there exists an indecomposable direct summand $i : \tau^{-1}(U) \to V$ of V such that the pullback of η along i is an almost split sequence whose left term is U. This almost split sequence is special, because it is the pullback of η' along the AR phantom morphism φi. $\square$

By Proposition 15 and that fact that $\mathcal{F}\text{-inj} = \mathrm{Jac}(\Lambda\text{-mod}) + \mathcal{I}(\Lambda\text{-inj})$, we conclude that

$$\Phi = {}^{\perp}(\mathcal{F}\text{-inj}) = {}^{\perp}(\mathrm{Jac}(\Lambda\text{-mod})),$$

and therefore that the complete ideal cotorsion pair $(\Phi, \Phi^{\perp})$ is cogenerated by $\mathrm{Jac}(\Lambda\text{-mod}) \subseteq \Phi^{\perp}$. Because $\mathrm{Jac}(\Lambda\text{-mod})$ is the maximum ideal of Λ-mod that contains no *nonzero* objects, the ideals $\mathcal{I}$ that contain $\mathrm{Jac}(\Lambda\text{-mod})$ are characterized by their objects. Indeed, if $\mathcal{I} \supseteq \mathrm{Jac}(\Lambda\text{-mod})$ is an ideal of Λ-mod, then an indecomposable module $U \in \Lambda$-mod is an object in $\mathcal{I}$ if and only if the inclusion $\mathrm{Jac}(U, U) \subseteq \mathcal{I}(U, U)$ is proper. Consequently, $\mathcal{I} = \mathrm{Jac}(\Lambda\text{-mod}) + \mathcal{I}[\mathrm{Ob}(\mathcal{I})]$.

To characterize the objects of $\Phi^{\perp}$, let us verify that the hypotheses of Theorem 44 are satisfied: if A is a finitely presented left Λ-module, then there is a short exact sequence in Λ-mod

$$0 \longrightarrow \Omega(A) \xrightarrow{\ j\ } P \xrightarrow{\ p\ } A \longrightarrow 0,$$

where P is a projective module, and $j : \Omega(A) \to P$ belongs to the Jacobson radical. This only requires taking the morphism $p : P \to A$ to be the projective cover of A. Then the kernel $\Omega(A) \subseteq P$ does not contain a nonzero direct summand of P. This inclusion $j : \Omega(A) \to P$ therefore belongs to the Jacobson radical of Λ-mod. The following is then a consequence of Theorem 44.

Corollary 50. *An indecomposable left Λ-module U that is not injective belongs to $\Phi^{\perp}$ if and only if the morphism $\xi_U : \Omega\Sigma(U) \to U$ is not a split epimorphism.*

Proof. By Theorem 44, the module U belongs to $\Phi^{\perp}$ if and only if $\xi_U : \Omega\Sigma(U) \to U$ belongs to $\mathcal{F}$-inj. Because U is indecomposable, this holds if and only if U is injective or if $\xi_U \in \mathrm{Jac}(\Omega\Sigma(U), U)$, which is equivalent to the condition that the morphism $\xi_U : \Omega\Sigma(U) \to U$ is not a split epimorphism. $\square$

If the Artin algebra Λ is *Quasi-Frobenius,* then $\Lambda\text{-proj} = \Lambda\text{-inj}$, so if an indecomposable left Λ-module U is not projective/injective, we may take $\Omega\Sigma(U) = U$ and $\xi_U : U \to U$ to be the identity. The corollary implies that U does not belong to $\Phi^{\perp}$. If, on the other hand, the indecomposable left Λ-module U is projective/injective, then it certainly belongs to $\Phi^{\perp}$. Because $\Phi^{\perp}$ contains the Jacobson radical, it is determined by the objects that belong to it, $\Phi^{\perp} = \mathrm{Jac}(\Lambda\text{-mod}) + \mathcal{I}(\Lambda\text{-inj}) = \mathcal{F}\text{-inj}$. By Corollary 23, the socle $\mathcal{F} \subseteq \mathrm{Ext}$ is the subfunctor $\mathrm{PB}(\Phi)$ that consists of those conflations that arise as pullbacks along AR phantom morphisms. In particular, every almost split sequence in Λ-mod is left special.

Corollary 51. *If Λ is a Quasi-Frobenius Artin algebra, and U is an indecomposable left Λ-module that is not injective, then the almost split sequence with left term U is left special.*

Example 52 (*Cf. [1, Section 6]*)**.** Let k be a field and suppose that Λ is the hereditary Artin algebra given by

$$\Lambda = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

There are three projective indecomposable left Λ-modules, given by $P_i = \Lambda e_{ii}$, for $1 \leq i \leq 3$. There are three simple left Λ-modules, given by $S_i = \mathrm{top}(P_i)$, with $S_1 = P_1$. Finally, there are three injective left Λ-modules $E_i = E(S_i)$, with $E(S_2) = S_2$ and $E(S_3) = S_3$. It is not difficult to verify that these six indecomposable left Λ-modules, the three projective indecomposables and the three injective indecomposables, constitute a complete list of the indecomposable left modules over Λ.

The three noninjective indecomposables are therefore given by the projectives P_1, P_2 and P_3. The almost split sequence with left term P_1 is given by

$$0 \longrightarrow P_1 \longrightarrow P_2 \oplus P_3 \longrightarrow E(S_1) \longrightarrow 0.$$

Because the middle term is projective, the right term $E(S_1)$ is an AR phantom object and this almost split sequence is therefore left special. The almost split sequence with left term P_2 is given by

$$0 \longrightarrow P_2 \longrightarrow E(S_1) \longrightarrow S_3 \longrightarrow 0.$$

This almost split sequence is not left special, because the morphism $\xi : \Omega\Sigma(P_2) \to P_2$ is not a split epimorphism. Indeed, taking the projective cover of S_3, one obtains the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S_1 \arrow[r] \arrow[d, "\xi"'] & P(S_3) \arrow[r] \arrow[d] & S_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_2 \arrow[r] & E(S_1) \arrow[r] & S_3 \arrow[r] & 0
\end{tikzcd}$$

with exact rows, and sees that $\Omega\Sigma(P_2) = S_1$ is simple. Similar considerations apply to the third projective indecomposable P_3: the almost split sequence with left term P_3 is not left special.

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