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Spectra of combinatorial Laplace operators on simplicial complexes

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Abstract

We first develop a general framework for Laplace operators defined in terms of the combinatorial structure of a simplicial complex. This includes, among others, the graph Laplacian, the combinatorial Laplacian on simplicial complexes, the weighted Laplacian, and the normalized graph Laplacian. This framework then allows us to define the normalized Laplace operator Δ_i^{up} on simplicial complexes which we then systematically investigate. We study the effects of a wedge sum, a join and a duplication of a motif on the spectrum of the normalized Laplace operator and identify some of the combinatorial features of a simplicial complex that are encoded in its spectrum.

Keywords: Normalized graph Laplacian; Combinatorial Laplacian; Hypergraph Laplacian; Graph Laplacian; Simplicial complex

1. Introduction

The study of graph Laplacians has a long and prolific history. It first appeared in a paper by Kirchhoff [25], where he analysed electrical networks and stated the celebrated matrix tree

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theorem. The Laplace operator L of [25] operates on a real valued function f on the vertices of a graph as

$$Lf(v_i) = \deg v_i\, f(v_i) - \sum_{v_i \sim v_j} f(v_j). \tag{1.1}$$

In spite of its rather early beginnings this topic did not gain much attention among scientists until the early 1970s and the work of Fiedler [14], and his results on correlation among the smallest non-zero eigenvalue and the connectivity of a graph. Before Fiedler drew attention to the graph Laplacian, graphs were usually characterized by means of the spectrum of its adjacency matrix, but in the wake of [14], there has been a number of papers (e.g. [19]) arguing in favour of the graph Laplacian and its spectrum. For good survey articles on the graph Laplacian the reader is referred to [28] or [30].

In a different tradition, the graph Laplacian was generalized to simplicial complexes by Eckmann [13], who formulated and proved the discrete version of the Hodge theorem; this can be formulated as

$$\ker(\delta_i^* \delta_i + \delta_{i-1}\delta_{i-1}^*) \cong \tilde{H}^i(K, \mathbb{R})$$

where

$$L_i = \delta_i^* \delta_i + \delta_{i-1}\delta_{i-1}^*$$

is the higher order combinatorial Laplacian. Many subsequent papers then studied properties of the *higher order combinatorial Laplacian* (see [10,15,11]), building upon properties of the graph Laplacian. In particular, this operator has been employed extensively in investigating the features of networks related to dynamics and coverings (see [31,33]). More recently, in [23], the combinatorial Laplacian is systematically studied in a context of a discrete exterior calculus.

While the graph Laplacian introduced by Kirchhoff naturally appears in his work on electrical flows, for other processes on graphs, like random walks or diffusion, a different operator appears. This was first investigated almost a century after Kirchhoff's work by Bottema [5] who studied a transition probability operator on graphs that is equivalent to the following version of the graph Laplace operator:

$$\Delta f(v_i) = f(v_i) - \frac{1}{\deg v_i} \sum_{v_i \sim v_j} f(v_j). \tag{1.2}$$

It took, however, almost another one hundred years until a significant advance in the study of this operator Δ, which got to be known by the name *normalized graph Laplacian* to distinguish it from the graph Laplacian L and to emphasize the fact that its eigenvalues are in the interval $[0, 2]$. In contrast to L, Δ is well suited for problems related to random walks on graphs and graph expanders. For a good introduction to this topic the reader is invited to consult [6] or [18].

The main goals of this paper are a systematic framework that can be used as a starting point for a study of any of the above mentioned versions of the Laplace operator, and the definition and investigation of the normalized Laplacian on simplicial complexes. The latter is based on the simple observation that the form of the combinatorial Laplacian is tightly connected to the choice of the scalar product on the coboundary vector spaces. On the other hand, the scalar products can be viewed in terms of weight functions. Thus, by controlling the weights, we control the range of the eigenvalues of the Laplace operator. Most importantly, for the normalized Laplacian, the eigenvalues are confined to the range $[0, i + 2]$, where i is the order of the Laplacian. This generalizes the fact that the eigenvalues of the normalized graph Laplacian Δ are in the interval $[0, 2]$. We shall analyse the spectrum of this normalized Laplacian and its connection with the combinatorial structure of the simplicial complex. Perhaps somewhat surprisingly, this generality also permits us to gain new insights for its special case, the already extensively studied normalized graph Laplacian.

There have already been several attempts towards the normalization of the combinatorial Laplace operator. In particular, Chung in [7] defined a normalized Laplacian on simplicial complexes as $\delta^*\delta+\rho\delta\delta^*$, where ρ is a positive constant. However, the spectrum of this operator is not bounded by a constant. Recently, Taszus [34] suggested a normalization of the combinatorial Laplacian via its matrix form $D^{-1/2}L_i D^{-1/2}$, where L_i is the matrix corresponding to the operator $\delta^*\delta + \delta\delta^*$, where the adjoints are defined with respect to the standard scalar product and D is a diagonal matrix of L_i. This operator, too, does not have a bounded spectrum. Lu and Peng in [27] considered random walks on hypergraphs and, to that end, defined a normalized Laplacian on a uniform hypergraph H as $\mathcal{L}_s(H) = \Delta(G^s_H)$, where G^s_K is a $(s - 1)$-dual graph (see Definition 5.2) of a simplicial complex (hypergraph) H. The drawback of this definition is that it fails to fit into general theory and does not take into account higher order relations among edges of a hypergraph. In contrast to this and from a very different direction, Garland [16] considered a normalized Laplacian on simplicial complexes that arise as Bruhat–Tits buildings for simple p-adic groups and their quotients by discrete subgroups.

As is already clear from Eckmann's seminal work [13], the Laplacians of a simplicial complex encode its basic topology, that is, its homology groups. In terms of the spectrum, they are given by the dimensions of the eigensets for the eigenvalue 0. This is the same for all the Laplace operators investigated here. These operators, however, differ in the nonzero part of the spectrum and, thereby, encode specific combinatorial or geometric features of a (perhaps weighted) simplicial complex in addition to its topological aspects. Many combinatorial operations that one can perform on a simplicial complex do not affect its homology; nevertheless, they typically leave characteristic traces in the spectrum of a suitable Laplace operator, and that is what we are trying to explore. In the weighted case, there is additional geometric information that likewise influences the spectrum. Let us try to explain this aspect from the following perspective. As is well known, from a covering of a set, one can construct a simplicial complex, by letting an i-dimensional simplex corresponds to every intersection of i members of the covering. The Čech cohomology of the covering then is isomorphic to the simplicial cohomology of the resulting complex. When, in addition, the set that is covered carries a measure, then we can assign to every simplex in this construction a weight equal to the measure of the corresponding intersection. We thus obtain a weighted simplicial complex, and we can define a corresponding Laplacian. Its spectrum then reflects the geometry of the intersection pattern, and not only its topology. Since such intersection patterns arise in many areas of application, for instance as colocalization patterns of proteins in a cell [32,4] or for many geographical data sets, we wish to propose this Laplacian spectrum as a new tool in data analysis. This will be developed elsewhere, on the mathematical basis of the present paper.

This paper is organized as follows. In Section 2 we give the basic definitions for simplicial complexes and recall Eckmann's discrete version of the Hodge theorem. We define the combinatorial Laplace operator in its full generality and provide explicit expressions. Section 3 starts with the theorem about the number of zeros in the spectrum of the general Laplace operator. We then discuss the effect of the scalar products on the spectrum and obtain the upper and lower bound on the maximum eigenvalue of the Laplacian. Finally, we state the definition of the *normalized combinatorial Laplace operator*, which will be the main object of the remainder of the paper. We calculate the spectrum of the normalized combinatorial Laplacian for some special classes of simplicial complexes in Section 4. In particular, we discuss the spectrum of an i-simplex, of an orientable and a non-orientable circuit, of a path and of a star. In Section 5 we analyse regular, pure simplicial complexes. In Section 6 we discuss the effect of wedges, joins and duplication of motifs on the spectrum of the normalized combinatorial Laplace operator. In Section 7 we identify the combinatorial features of simplicial complexes that cause the appearance of certain integer eigenvalues in the spectrum of Δ_i^{up}. We discuss the occurrence of the eigenvalue $i+2$ in the spectrum of Δ_i^{up}, and its connection to the chromatic number of the underlying graph of a complex. Furthermore, the relation among the eigenvalue $i+1$ and the duplication of vertices is established.

2. Notations, definitions and the combinatorial Laplace operator

An *abstract simplicial complex* K on a finite set V is a collection of subsets of V, which is closed under inclusion. An i-face or an i-simplex of K is an element of cardinality $i+1$. 0-faces are usually called *vertices* and 1-faces *edges*. The collection of all i-faces of simplicial complex K is denoted by $S_i(K)$. The *dimension* of an i-face is i, and the dimension of a complex K is the maximum dimension of a face in K. The faces which are maximal under inclusion are called *facets*. We say that a simplicial complex K is *pure* if all facets have the same dimension. Note that a simplicial complex can also be considered as a hypergraph (a facet of a simplicial complex corresponds to an edge in a hypergraph). For two $(i+1)$-simplices sharing an i-face we use the term i*-down neighbours*, and for two i-simplices which are faces of an $(i+1)$-simplex, we say that they are $(i+1)$*-up neighbours*. We say that a face F is *oriented* if we choose an ordering on its vertices and write $[F]$. Two orderings of the vertices are said to determine *the same orientation* if there is an even permutation transforming one ordering into the other. If the permutation is odd, then the orientations are opposite.

In the remainder, K will be an abstract simplicial complex on a vertex set $[n] = \{1, 2, \dots, n\}$, when not stated otherwise. The i-th chain group $C_i(K, \mathbb{R})$ of a complex K with coefficients in $\mathbb{R}$ is a vector space over the field $\mathbb{R}$ with basis $B_i(K, \mathbb{R}) = \{[F] \mid F \in S_i(K)\}$.

The cochain groups $C^i(K, \mathbb{R})$ are defined as duals of the chain groups, i.e. $C^i(K, \mathbb{R}) := \hom(K, \mathbb{R})$. The basis of $C^i(K, \mathbb{R})$ is given by the set of functions $\{e_{[F]} \mid [F] \in B_i(K, \mathbb{R})\}$ such that

$$e_{[F]}([F']) = \begin{cases} 1 & \text{if } [F'] = [F] \\ 0 & \text{otherwise.} \end{cases}$$

The functions $e_{[F]}$ are also known as *elementary cochains*. Traditionally, $C^i(K, G)$ for an arbitrary group G is called a cochain group. Therefore, we shall also refer to the $C^i(K, \mathbb{R})$ as cochain *groups*, although the $C^i(K, \mathbb{R})$ have the structure of *vector spaces*, in fact. Note that the one-dimensional vector space $C^{-1}(K, \mathbb{R})$ is generated by the identity function on the empty simplex. We define the *simplicial coboundary maps*

$$(\delta_i f)([v_0, \dots, v_{i+1}]) = \sum_{j=0}^{i+1} (-1)^j f([v_0, \dots, \hat{v}_j \cdots v_{i+1}]),$$

where $\hat{v}_j$ denotes that the vertex v_j has been omitted. The δ_i are the connecting maps in the *augmented cochain complex* of K with coefficients in $\mathbb{R}$, i.e., the sequence of vector spaces and linear transformations

$$\cdots \xleftarrow{\delta_{i+1}} C^{i+1}(K,\mathbb{R}) \xleftarrow{\delta_i} C^i(K,\mathbb{R}) \xleftarrow{\delta_{i-1}} \cdots \leftarrow C^{-1}(K,\mathbb{R}) \leftarrow 0. \tag{2.1}$$

Alternatively, δ_i can be viewed as the dual of the boundary map ∂_{i+1}. For a systematic treatment of simplicial homology and cohomology the reader is referred to [22]. It is straightforward to check that $\delta_i\delta_{i-1} = 0$, ergo the image of δ_{i-1} is contained in the kernel of δ_i and the reduced cohomology group for every $i \geq 0$ is

$$\tilde{H}^i(K,\mathbb{R}) := \ker \delta_i / \operatorname{im} \delta_{i-1}.$$

After choosing inner products $(,)_{C^i}$ and $(,)_{C^{i+1}}$ on $C^i(K,\mathbb{R})$ and $C^{i+1}(K,\mathbb{R})$, respectively, the adjoint $\delta_i^* : C^{i+1}(K,\mathbb{R}) \to C^i(K,\mathbb{R})$ of the coboundary operator δ_i is defined by

$$(\delta_i f_1, f_2)_{C^{i+1}} = (f_1, \delta_i^* f_2)_{C^i},$$

for every $f_1 \in C^i(K,\mathbb{R})$ and $f_2 \in C^{i+1}(K,\mathbb{R})$.

Definition 2.1. We define the following three operators on $C^i(K,\mathbb{R})$:

(i) *i-dimensional combinatorial up Laplace operator* or simply i-up Laplace operator

$$\mathcal{L}_i^{up}(K) := \delta_i^*\delta_i,$$

(ii) *i-dimensional combinatorial down Laplace operator* or i-down Laplace operator

$$\mathcal{L}_i^{down}(K) := \delta_{i-1}\delta_{i-1}^*,$$

(iii) *i-dimensional combinatorial Laplace operator* or i-Laplace operator

$$\mathcal{L}_i(K) := \mathcal{L}_i^{up}(K) + \mathcal{L}_i^{down}(K) = \delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*.$$

Since

$$C^{i+1}(K,\mathbb{R}) \underset{\delta_i^*}{\overset{\delta_i}{\leftrightarrows}} C^i(K,\mathbb{R}) \underset{\delta_{i-1}^*}{\overset{\delta_{i-1}}{\leftrightarrows}} C^{i-1}(K,\mathbb{R}),$$

all three operators are well defined. Moreover, it follows directly from the definition that $\mathcal{L}_i^{up}(K)$, $\mathcal{L}_i^{down}(K)$ and $\mathcal{L}_i(K)$ are self-adjoint, non-negative and compact operators. Hence their eigenvalues are real, non-negative, and admit a variational characterization as expressed by the Courant–Fischer–Weyl min–max principle that we state here.

Theorem 2.1 (*Min–Max Theorem*)**.** *Let V_k and $\mathcal{V}_k$ denote a k-dimensional subspace of V and a family of k-dimensional subspaces of V, respectively, and assume that $A : V \to V$ is a compact, self-adjoint operator of a Hilbert space V. Then*

$$\lambda_k = \min_{V_k\in\mathcal{V}_k} \max_{g\in V_k} \frac{(Ag,g)}{(g,g)} = \max_{V_k\in\mathcal{V}_{m-k+1}} \min_{g\in V_{m-k+1}} \frac{(Ag,g)}{(g,g)}, \tag{2.2}$$

where $\lambda_1 \leq \cdots \leq \lambda_m$ *are the eigenvalues of* A. *The* g *realizing such a* min–max *or* max–min *then are corresponding eigenfunctions, and the* min–max *spaces* V_k *are spanned by the eigenfunctions for the eigenvalues* $\lambda_1, \ldots, \lambda_k$, *and analogously, the* max–min *spaces* V_{m-k+1} *are spanned by the eigenfunctions for the eigenvalues* $\lambda_k, \ldots, \lambda_m$.

For any operator A acting on a Hilbert space, we denote the weakly increasing rearrangement of its eigenvalues by $\mathbf{s}(A) = (\lambda_0, \ldots, \lambda_m)$ and write $\mathbf{s}(A) \stackrel{\circ}{=} \mathbf{s}(B)$, when the multisets $\mathbf{s}(A)$ and $\mathbf{s}(B)$ differ only in their multiplicities of zero (this is an equivalence relation). We denote a union of multisets by $\stackrel{\circ}{\cup}$.

We now state the discrete version of the Hodge theorem and provide its proof for the sake of completeness.

Theorem 2.2 (*Eckmann 1944*). *For an abstract simplicial complex* K,

$$\ker \mathcal{L}_i(K) \cong \tilde{H}^i(K, \mathbb{R}).$$

Proof. Since $\delta_i \delta_{i-1} = 0$ and $\delta_{i-1}^* \delta_i^* = 0$, then

$$\operatorname{im} \mathcal{L}_i^{down}(K) \subset \ker \mathcal{L}_i^{up}(K), \tag{2.3}$$

$$\operatorname{im} \mathcal{L}_i^{up}(K) \subset \ker \mathcal{L}_i^{down}(K). \tag{2.4}$$

Thus,

$$\begin{aligned} \ker \mathcal{L}_i(K) &= \ker \delta_i^* \delta_i \cap \ker \delta_{i-1} \delta_{i-1}^* \\ &= \ker \delta_i \cap \ker \delta_{i-1}^* \\ &= \ker \delta_i \cap (\operatorname{im} \delta_{i-1})^\perp \\ &\cong \tilde{H}^i(K, \mathbb{R}). \quad \square \end{aligned}$$

Due to (2.3) and (2.4), λ is a non-zero eigenvalue of $\mathcal{L}_i(K)$ if and only if it is an eigenvalue of $\mathcal{L}_i^{up}(K)$ or $\mathcal{L}_i^{down}(K)$. Therefore,

$$\mathbf{s}(\mathcal{L}_i(K)) \stackrel{\circ}{=} \mathbf{s}(\mathcal{L}_i^{up}(K)) \stackrel{\circ}{\cup} \mathbf{s}(\mathcal{L}_i^{down}(K)). \tag{2.5}$$

As a direct consequence of the fact that $\mathbf{s}(AB) \stackrel{\circ}{=} \mathbf{s}(BA)$, for operators A and B on suitably chosen Hilbert spaces, we get the following equality, which was pointed out to us by Johannes Rauh:

$$\mathbf{s}(\mathcal{L}_i^{up}(K)) \stackrel{\circ}{=} \mathbf{s}(\mathcal{L}_{i+1}^{down}(K)). \tag{2.6}$$

From (2.5) and (2.6) we conclude that each of the three families of multisets

$$\{\mathbf{s}(\mathcal{L}_i(K)) \mid 0 \leq i \leq d\}, \qquad \{\mathbf{s}(\mathcal{L}_i^{up}(K)) \mid 0 \leq i \leq d\} \quad \text{or} \quad \{\mathbf{s}(\mathcal{L}_i^{down}(K)) \mid 0 \leq i \leq d\}$$

determines the other two. Therefore, it suffices to consider only one of them. In the remainder of the paper, we will omit the argument K in $\mathbf{s}(\mathcal{L}_i(K))$, $\mathbf{s}(\mathcal{L}_i^{up}(K))$, $\mathcal{L}_i^{up}(K)$, $\mathcal{L}_i^{down}(K)$, $S_i(K)$, etc., when it is clear which simplicial complex we investigate or when we state our results for a general simplicial complex K.

For explicit expressions for up and down Laplacians, we have to fix scalar products on the cochain groups. To that end, we introduce the weight function and additional notation.

Definition 2.2. The *weight function* w is an evaluation function on the set of all faces of K

$$w : \bigcup_{i=-1}^{\dim K} S_i(K) \to \mathbb{R}^+.$$

The *weight of a face* F is $w(F)$.

For any choice of the inner product on the space $C^i(K,\mathbb{R})$, where elementary cochains form an orthogonal basis, there exists a weight function w, such that

$$(f,g)_{C^i} = \sum_{F\in S_i(K)} w(F) f([F]) g([F]).$$

Furthermore, there is a one-to-one correspondence between weight functions and possible scalar products on cochain groups $C^i(K,\mathbb{R})$, such that elementary cochains are orthogonal. In the remainder we will interchangeably use the terms weights, weight function and scalar product.

Definition 2.3. Let $\bar{F} = \{v_0,\ldots,v_{i+1}\}$ be an $(i+1)$-face of a complex K and let $F = \{v_0,\ldots,\hat{v_k},\ldots,v_{i+1}\}$ be an i-face of $\bar{F}$. Then *the boundary of the oriented face* $[\bar{F}]$ is

$$\partial[\bar{F}] = \sum_k (-1)^k [v_0,\ldots,\hat{v_k},\ldots,v_{i+1}],$$

and the sign of $[F]$ in the boundary of $[\bar{F}]$ is denoted by $\mathrm{sgn}([F],\partial[\bar{F}])$ and is equal to $(-1)^k$.

By abuse of notation, we will write $\partial\bar{F}$ to denote the set of all i-faces of $\bar{F}$. The i-up Laplace operator is given by

$$(\mathcal{L}_i^{up} f)([F]) = \sum_{\substack{\bar{F}\in S_{i+1}:\\ F\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)} f([F]) + \sum_{\substack{F'\in S_i: F\neq F',\\ F,F'\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)} \mathrm{sgn}([F],\partial[\bar{F}]) \times \mathrm{sgn}([F'],\partial[\bar{F}]) f([F']),$$

and the expression for the i-down Laplace operator is

$$(\mathcal{L}_i^{down} f)([F]) = \sum_{E\in\partial F} \frac{w(F)}{w(E)} f([F]) + \sum_{F': F\cap F'=E} \frac{w(F')}{w(E)} \mathrm{sgn}([E],\partial[F]) \times \mathrm{sgn}([E],\partial[F']) f([F']).$$

When dealing with linear operators it is often more convenient to study their matrix form. Hence we give the following expressions for the $(e_{[F]}, e_{[F']})$-th and the $(e_{[F]}, e_{[F]})$-th entry of $\mathcal{L}_i^{up}$ and $\mathcal{L}_i^{down}$, where $F \neq F'$

$$(\mathcal{L}_i^{up})_{(e_{[F]},e_{[F']})} = \mathrm{sgn}([F], \partial[\bar{F}])\mathrm{sgn}([F'], \partial[\bar{F}])\frac{w(\bar{F})}{w(F)},$$

$$(\mathcal{L}_i^{up})_{(e_{[F]},e_{[F]})} = \sum_{\substack{\bar{F}\in S_{i+1},\\ F\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)},$$

$$(\mathcal{L}_i^{down})_{(e_{[F]},e_{[F']})} = \mathrm{sgn}([E], \partial[F])\mathrm{sgn}([E], \partial[F'])\frac{w(F')}{w(E)},$$

$$(\mathcal{L}_i^{down})_{(e_{[F]},e_{[F]})} = \sum_{E\in\partial F} \frac{w(F)}{w(E)}.$$

The Laplace operator $\mathcal{L}$ of a simplicial complex K is uniquely determined by a weight function w_K on the faces of K. Thus, we write $\mathcal{L}(K, w_k)$.

Remark 2.1. From the explicit expressions of Laplace operators it is evident that $\mathcal{L}_i^{up}$ is uniquely determined by its restriction on the $(i+1)$-skeleton of K, whereas $\mathcal{L}_i^{down}$ is determined by its i-skeleton. Therefore, when studying $\mathcal{L}_i^{up}$ (or $\mathcal{L}_i^{down}$), it suffices to observe pure $(i+1)$ (or i)-simplicial complexes.

Let D_i be the matrix corresponding to the operator δ_i, D_i^T its transpose and W_i the diagonal matrix representing the scalar product on C^i, then the $\mathcal{L}_i^{up}$ and $\mathcal{L}_i^{down}$ operators are expressed as

$$\mathcal{L}_i^{up} = W_i^{-1} D_i^T W_{i+1} D_i,$$

and

$$\mathcal{L}_i^{down} = D_{i-1} W_{i-1}^{-1} D_{i-1}^T W_i,$$

respectively. Therefore, the combinatorial Laplace operator analysed by Duval, Reiner [12], Friedmann [15] and others [31,10] is the combinatorial Laplace operator $\mathcal{L}_i$ for the identity matrices weight matrices W_i $(-1 \leq i \leq \dim K)$, i.e. $\mathcal{L}_i(K, w_K)$, where $w_K \equiv 1$. In the remainder of the paper, this version of the Laplace operator will be denoted by L_i. The graph Laplacian (1.1) studied by Kirchhoff [25], Fiedler [14], Grone and Merris [19] and many others is a special case of L_i; in fact it is equal to L_0^{up}. The *normalized graph Laplace operator* (1.2) investigated by Chung, Yau, Grigoryan and others, see [8,2], is equal to $\mathcal{L}_0^{up}$ for W_1 being the matrix with diagonal entries equal to the edge weights and W_0 the diagonal degree matrix, that is the weight function on a vertex v is $w(v) = \deg v$.

Therefore, the combinatorial Laplacian $\mathcal{L}(K, w_K)$, as defined here, unifies all Laplace operators studied so far and provides a general framework for a systematic study of different versions of Laplacians.

Our goal in this paper is to define the higher dimensional analogue of the normalized graph Laplacian and to investigate its properties. However, we will state our results in full generality whenever possible and emphasize which results do not depend on the choice of the scalar products and which are the consequence of suitably chosen weights.

3. The normalized combinatorial Laplacian: definition and basic properties

In this section we derive an upper and a lower bound for the maximal eigenvalue of $\mathcal{L}_i^{up}$, introduce the normalized combinatorial Laplacian Δ_i^{up}, and state and prove its basic properties. We emphasize its advantages compared to the other choices of weights.

Let λ_m and λ_0 be the maximal and the minimal eigenvalue of $\mathcal{L}_i^{up}(K, w_K)$, respectively. As the Laplace operator is positive definite, λ_0 is always larger or equal to zero. The exact number of zero eigenvalues in the spectrum of $\mathcal{L}_i^{up}$ and $\mathcal{L}_i^{down}$ is given in the following theorem.

Theorem 3.1. *The multiplicity of the eigenvalue zero in*

(i) $\mathbf{s}(\mathcal{L}_i^{up})$ *is*

$$\dim C^i - \sum_{j=0}^{i} (-1)^{i+j} (\dim C^j - \dim \tilde{H}^j),$$

or equivalently

$$\dim C^i + \sum_{j=1}^{d-i} (-1)^{j} (\dim C^{i+j} - \dim \tilde{H}^{i+j}).$$

(ii) $\mathbf{s}(\mathcal{L}_i^{down})$ *is*

$$\dim \tilde{H}^i - \sum_{j=0}^{i-1} (-1)^{i+j-1} (\dim C^j - \dim \tilde{H}^j).$$

Proof. The following are short exact sequences that split

$$0 \to \ker \delta_i \to C^i \to \operatorname{im} \delta_i \to 0,$$
$$0 \to \operatorname{im} \delta_{i-1} \to \ker \delta_i \to \tilde{H}^i \to 0.$$

This is a direct consequence of the fact that $\operatorname{im} \delta_i$ and $\tilde{H}^i$ are projective modules (for details on projective modules and splitting exact sequences the reader is referred to [9]). Therefore,

$$\dim C^i = \dim \ker \delta_i + \dim \operatorname{im} \delta_i, \tag{3.1}$$

and

$$\dim \ker \delta_i = \dim \tilde{H}^i + \dim \operatorname{im} \delta_{i-1}. \tag{3.2}$$

From (3.1) and (3.2)

$$\dim \operatorname{im} \delta_i = \sum_{j=-1}^{i} (-1)^{i+j} (\dim C^j - \dim \tilde{H}^j).$$

The number of zeros in the spectrum of $\mathcal{L}_i^{up}$ is equal to the dimension of its kernel; thus,

$$\begin{aligned} \dim \ker \mathcal{L}_i^{up} &= \dim \ker \delta_i \\ &= \dim C^i - \sum_{j=-1}^{i} (-1)^{j+i} (\dim C^j - \dim \tilde{H}^j). \end{aligned}$$

The expression (3.1) for the number of zeros in $\mathbf{s}(\mathcal{L}_i^{up})$ is easily obtained by using the Euler characteristic and the equality $\chi = \sum_{i=-1}^{d} (-1)^i \dim C^i = \sum_{i=-1}^{d} (-1)^i \dim \tilde{H}^i$. As for $\mathcal{L}_i^{down}$, the following holds:

$$\begin{aligned} \dim \ker \mathcal{L}_i^{down} &= \dim \ker \delta_{i-1}^* = \dim C^i - \dim \operatorname{im} \delta_{i-1} \\ &= \dim C^i - \sum_{j=-1}^{i-1} (-1)^{j+i-1} (\dim C^j - \dim \tilde{H}^j). \quad \square \end{aligned}$$

The number of zero eigenvalues in spectra of various Laplace operators, as expected, does not depend on a choice of the scalar products on the cochain vector spaces.

Remark 3.1. If a simplicial complex is $(i+1)$-dimensional, then the number of zero eigenvalues in the spectrum of $\mathcal{L}_i^{up}(K)$ is $\dim C^i - \dim C^{i+1} + \dim \tilde{H}^{i+1}$, whereas there are exactly $\dim C^{i+1} - \dim \tilde{H}^{i+1} + \dim \tilde{H}^i$ zeros in the spectrum of $\mathcal{L}_{i+1}^{down}$.

Next we introduce the degree of a simplex F.

Definition 3.1. *The degree* of an i-face F of K is equal to the sum of the weights of all simplices that contain F in its boundary, i.e.

$$\deg F = \sum_{\bar{F} \in S_{i+1}(K): F \in \partial \bar{F}} w(\bar{F}).$$

The upper bound on $\mathbf{s}(\mathcal{L}_i^{up})$ follows from the subsequent discussion. We have

$$(\mathcal{L}_i^{up} f, f) = (\delta_i f, \delta_i f) \tag{3.3a}$$

$$= \left(\sum_{\bar{F} \in S_{i+1}(K)} f(\partial[\bar{F}]) \mathrm{e}_{[\bar{F}]}, \sum_{\bar{F} \in S_{i+1}(K)} f(\partial[\bar{F}]) \mathrm{e}_{[\bar{F}]} \right) \tag{3.3b}$$

$$= \sum_{\bar{F} \in S_{i+1}(K)} f(\partial[\bar{F}])^2 w(\bar{F}) \tag{3.3c}$$

$$\leq (i+2) \sum_{F \in S_i(K)} f([F])^2 \sum_{\bar{F} \in S_{i+1}(K): F \in \partial \bar{F}} w(\bar{F}), \tag{3.3d}$$

where (3.3d) is obtained by using the Cauchy–Schwarz inequality. In terms of degrees the last inequality can be restated as

$$(\mathcal{L}_i^{up} f, f) \leq (i+2) \sum_{F \in S_i(K)} f([F])^2 \deg F. \tag{3.4}$$

By dividing (3.4) by(f, f) we get

$$\frac{(\mathcal{L}_i^{up} f, f)}{(f, f)} \leq (i+2) \frac{\sum_{F \in S_i(K)} f([F])^2 \deg F}{\sum_{F \in S_i(K)} f([F])^2 w(F)}. \tag{3.5}$$

Replacing f in (3.5) with the eigenfunction f_m, corresponding to the largest eigenvalue $\lambda_{\max}$ of $\mathcal{L}_i^{up}$ gives

$$\lambda_{\max} \leq (i+2) \frac{\sum_{F \in S_i(K)} f_m([F])^2 \deg F}{\sum_{F \in S_i(K)} f_m([F])^2 w(F)}. \tag{3.6}$$

Therefore, if

$$w(F) = \deg F, \tag{3.7}$$

for every $F \in S_i(K)$, then $\lambda_{\max} \leq i+2$ and the eigenvalues of $\mathcal{L}_i^{up}$ are in the interval $[0, i+2]$.

Definition 3.2. Let w be a weight function on K which satisfies (3.7) for every face of the simplicial complex K, which is not a facet ($\dim F < \dim K$), then the Laplace operator defined on the cochain complex of K is called the *weighted normalized combinatorial Laplace operator*. If additionally, the weights of the facets of K are equal to 1, then the obtained operator is called the *normalized combinatorial Laplace operator* and is denoted by Δ_i^{up}. We will keep the same notation for the weighted normalized combinatorial Laplacian, emphasizing that we are considering its weighted version.

If (3.7) does not hold, we derive a bound on the maximal eigenvalue of the Laplacian $\mathcal{L}_i^{up}$ from the inequality (3.6), i.e.

$$\lambda_m \le (i+2)\frac{\max\limits_{F\in S_i(K)} \deg F}{\min\limits_{F\in S_i(K)} w(F)}. \tag{3.8}$$

Here $\min_{F\in S_i(K)} w(F)$ stands for the minimal *non-zero* weight over all i-faces F of K. The inequality (3.8) in the case of the combinatorial Laplacian L_i^{up} reduces to

$$\lambda_m \le (i+2) \max_{F\in S_i(K)} \deg F, \tag{3.9}$$

which for $i = 0$ becomes exactly

$$\lambda_m \le 2 \max_{v\in S_0(G)} \deg v.$$

This is the well-known bound on the maximal eigenvalue of L_0^{up} (see [1]). Another upper bound of the spectrum of L_i^{up} was obtained by Duval and Reiner in [12] as a part of more general study, i.e.

$$\lambda_m \le n, \tag{3.10}$$

where n is the number of vertices of the complex K. The inequality (3.9) is sharper than (3.10) for large values of n and small values of i. In particular, if $\max_{F\in S_i} \deg F < n/(i+2)$, then the estimate (3.9) is sharper, otherwise (3.10) is. We sum up our results in the following theorem.

Theorem 3.2. *The spectrum of $\mathcal{L}_i^{up}$ is bounded from above by*

(i) $i+2$, *if* $\mathcal{L}_i^{up} = \Delta_i^{up}$,
(ii) $(i+2)\max_{F\in S_i(K)} \deg F$, *if* $\mathcal{L}_i^{up} = L_i^{up}$,
(iii) $(i+2)\max_{F\in S_i(K)} \deg F / \min_{F\in S_i(K)} w(F)$, *for all other choices of scalar products.*

In the following theorems we present some lower bounds on $\lambda_{\max}$.

Theorem 3.3. *Without loss of generality, let K be a pure simplicial complex of dimension $i+1$, $\mathcal{L}(K, w_K)$ the Laplace operator with the weight function w_K, $\lambda_{\max}$ the maximal eigenvalue in the spectrum of $\mathcal{L}_i^{up}(K, w_K)$, and* $\mathrm{vol}_i(K) = \sum_{F\in S_i} \deg F$, *then*

(i) $\dim C^i/(\dim C^{i+1} - \dim H^{i+1}) \le \lambda_{\max}$, *if* $\mathcal{L}_i^{up} = \Delta_i^{up}$,
(ii) $\mathrm{vol}_i(K)/(\dim C^{i+1} - \dim H^{i+1}) \le \lambda_{\max}$, *if* $\mathcal{L}_i^{up} = L_i^{up}$,
(iii) $\mathrm{vol}_i(K)/(\max_{F\in S_i} w(F)(\dim C^{i+1} - \dim H^{i+1})) \le \lambda_{\max}$, *for all other choices of scalar products.*

Proof. The sum of all eigenvalues is equal to the trace of the Laplace matrix, i.e. $\sum_{F\in S_i}\sum_{\bar{F}:F\in\bar{F}}\frac{w(\bar{F})}{w(F)}$. Together with Theorem 3.1, this yields the inequality

$$\frac{\sum_{F\in S_i}\sum_{\bar{F}:F\in\bar{F}}\frac{w(\bar{F})}{w(F)}}{\dim C^{i+1}-\dim H^{i+1}}\leq\lambda_{\max},\tag{3.11}$$

which proves the theorem. □

Theorem 3.4. *Let K be a pure $(i+1)$-dimensional simplicial complex and let $\lambda_{\max}$ denote the maximum eigenvalue of the operator $\mathcal{L}_i^{up}(K,w)$, then*

$$\frac{D}{d}+\frac{(i+1)D}{Nd}\leq\lambda_{\max},\tag{3.12}$$

where D, d are maximal degree, weight, respectively over all i-simplices and N is the minimal number of $(i+1)$-faces which are incident to an i-simplex of degree D.

Proof. Assume that F is an i-simplex of maximal degree with the minimal number of incident $(i+1)$-faces, i.e. there exist exactly $N(i+1)$-simplices which contain F as a facet and $\sum_{\bar{F}:F\in\bar{F}}w(\bar{F})=D$. Let $f=\sum_{k=1}^{N}\operatorname{sgn}([F],\partial[\bar{F}_k])e_{[\bar{F}_k]}$, then we obtain

$$\begin{aligned}
\lambda_{\max} &\geq \frac{(\delta_i^* f,\delta_i^* f)}{(f,f)}\\
&\geq \frac{1}{D}\Bigg(\sum_{k=1}^{N}\operatorname{sgn}([F],\partial[\bar{F}_k])\sum_{E\in\partial\bar{F}_k}\operatorname{sgn}([E],\partial[\bar{F}_k])\frac{w(\bar{F}_k)}{w(E)}\mathrm{e}_{[E]},\\
&\qquad \sum_{k=1}^{N}\operatorname{sgn}([F],\partial[\bar{F}_k])\sum_{E\in\partial\bar{F}_k}\operatorname{sgn}([E],\partial[\bar{F}_k])\frac{w(\bar{F}_k)}{w(E)}\mathrm{e}_{[E]}\Bigg)\\
&= \frac{1}{D}(e_{[F]},e_{[F]})+\frac{1}{D}\Bigg(\sum_{k=1}^{N}\operatorname{sgn}([F],\partial[\bar{F}_k])\sum_{\substack{E\in\partial\bar{F}_k:\\E\neq F}}\operatorname{sgn}([E],\partial[\bar{F}_k])\frac{w(\bar{F}_k)}{w(E)}\mathrm{e}_{[E]},\\
&\qquad \sum_{k=1}^{D}\operatorname{sgn}([F],\partial[\bar{F}_k])\sum_{\substack{E\in\partial\bar{F}_k:\\E\neq F}}\operatorname{sgn}([E],\partial[\bar{F}_k])\frac{w(\bar{F}_k)}{w(E)}\mathrm{e}_{[E]}\Bigg)\\
&= \frac{D}{w(F)}+\frac{1}{D}\sum_{k=1}^{N}\sum_{\substack{E\in\partial\bar{F}_k:\\E\neq F}}\frac{w^2(\bar{F}_k)}{w(E)}\\
&\geq \frac{D}{w(F)}+\frac{i+1}{dD}\sum_{k=1}^{N}w^2(\bar{F}_k)\\
&\geq \frac{D}{d}+\frac{i+1}{dD}\frac{D^2}{N}.
\end{aligned}$$

The inequalities above are a consequence of the variational characterization of eigenvalues (Theorem 2.1) and of the Cauchy–Schwarz inequality. □

The previous result for $\mathcal{L}(K, w_K) = L$ generalizes Proposition 8.2 from [12] and its proof. As another special case of Theorem 3.4 we obtain the following lower bounds for the maximal eigenvalue of the normalized Laplacian.

Corollary 3.5.

$$1 + \frac{i+1}{D} \leq \lambda_{\max}, \tag{3.13}$$

where D is the maximal degree over all i simplices and $\lambda_{\max}$ the maximal eigenvalue of Δ_i^{up}.

Remark 3.2 (*Negative Weights*)**.** If negative weights in the definition of the weight function are allowed, then bilinear forms (inner products) on cochain vector spaces are no longer positive definite. With arbitrary weights, $\mathcal{L}_i^{up}$ acts on functions on i-simplices

$$\Delta_i^{up} f([F]) = \frac{1}{w(F)} \sum_{\substack{\bar{F} \in S_{i+1} \\ F \in \partial \bar{F}}} \operatorname{sgn}([F], \partial[\bar{F}]) f(\partial[\bar{F}]).$$

This approach enables us to use negative weights, but it also deprives us of the structure of the cohomology of simplicial complexes. The eigenvalues need no longer be real nor non-negative. Here, however, we do not pursue the study of Laplacians with negative weights.

4. Circuits, paths, stars and their spectrum

In this section we calculate the spectrum of the up (down) normalized Laplace operator for some classes of simplicial complexes.

Theorem 4.1. *Let K be an $(n-1)$-dimensional simplex. Then $\mathbf{s}(\Delta_i^{up}(K))$ consists of the eigenvalue $n/(n-i-1)$ with multiplicity $\binom{n-1}{i+1}$ and the eigenvalue zero with multiplicity $\binom{n-1}{i}$.*

Proof. We will prove that the function $f \in C^i(K, \mathbb{R})$,

$$f_{[\bar{F}]}([F]) = \begin{cases} \operatorname{sgn}([F], \partial[\bar{F}]) & \text{if } F \text{ is facet of } (i+1)\text{-face } \bar{F} \\ 0 & \text{otherwise,} \end{cases}$$

is an eigenfunction of $\Delta_i^{up}(K)$ for the eigenvalue $n/(n-i-1)$.

It is not difficult to see that there are exactly $\binom{n-1}{i+1}$ linearly independent functions of this form. We have to check that the equality

$$(\Delta_i^{up} f_{[\bar{F}]})[F] = \frac{n}{n-i-1} f([F])$$

holds for every i-dimensional face F of K. We distinguish three cases:

(i) F is an arbitrary facet of $\bar{F}$. Then,

$$\begin{aligned}
(\Delta_i^{up} f_{[\bar{F}]})([F]) &= \sum_{\substack{\bar{E}\in S_{i+1}:\\ F\in\partial\bar{E}}} \frac{w(\bar{E})}{w(F)} f_{[\bar{F}]}([F]) \\
&\quad + \sum_{\substack{F'\in S_i(L):\\ (\exists\bar{E}\in S_{i+1}(L))F,F'\in\partial\bar{E}}} \frac{w(\bar{E})}{w(F)}\mathrm{sgn}([F],\partial[\bar{E}])\mathrm{sgn}([F'],\partial[\bar{E}]) f_{[\bar{F}]}([F']) \\
&= \frac{1}{n-i-1}\sum_{\substack{\bar{E}\in S_{i+1}:\\ F\in\partial\bar{E}}} f_{[\bar{F}]}([F]) + \frac{1}{n-i-1} \\
&\quad \times \sum_{\substack{F'\in S_i(L):\\ (\exists\bar{E}\in S_{i+1}(L))F,F'\in\partial\bar{E}}} \mathrm{sgn}([F],\partial[\bar{E}])\mathrm{sgn}([F'],\partial[\bar{E}]) f_{[\bar{F}]}([F']) \\
&= f_{[\bar{F}]}([F]) + \frac{i+1}{n-i-1}\mathrm{sgn}([F],\partial[\bar{F}]) \\
&= \frac{n}{n-i-1} f([F]).
\end{aligned}$$

(ii) F and $\bar{F}$ have i vertices in common, i.e. they intersect in a face of dimension $i-1$.

Then by definition $f([F]) = 0$. Let $v_0, v_1, \ldots, v_{i+2} \in [n]$ be arbitrary vertices of L ordered increasingly. Without loss of generality, assume $0 \le j < k < l \le i+2$, $\bar{F} = [v_0, \ldots, \hat{v}_l, \ldots, v_{i+2}]$ and $[F] = [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_k, \ldots, v_{i+2}]$. Then there exist exactly two i-faces F_1 and F_2 in the boundary of $\bar{F}$ and two $(i+1)$-simplices $\bar{F}_1$ and $\bar{F}_2$ of L, such that $F, F_1 \in \partial\bar{F}_1$ and $F, F_2 \in \partial\bar{F}_2$. In particular, $F_1 = [v_0, \ldots, \hat{v}_k, \ldots, \hat{v}_l, \ldots, v_{i+2}]$, $F_2 = [v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_l, \ldots, v_{i+2}]$ and $\bar{F}_1 = [v_0, \ldots, \hat{v}_k, \ldots, v_{i+2}]$, $\bar{F}_2 = [v_0, \ldots, \hat{v}_j, \ldots, v_{i+2}]$. Now it is straightforward to calculate

$$\begin{aligned}
(\Delta_i^{up} f_{[\bar{F}]})([F]) &= 0 + \mathrm{sgn}([F],\partial[\bar{F}_1])\mathrm{sgn}([F_1],\partial[\bar{F}_1]) f_{[\bar{F}]}([F_1]) \\
&\quad + \mathrm{sgn}([F],\partial[\bar{F}_2])\mathrm{sgn}([F_2],\partial[\bar{F}_2]) f_{[\bar{F}]}([F_2]) \\
&= \mathrm{sgn}([F],\partial[\bar{F}_1])\mathrm{sgn}([F_1],\partial[\bar{F}_1])\mathrm{sgn}([F_1],\partial[\bar{F}]) \\
&\quad + \mathrm{sgn}([F],\partial[\bar{F}_2])\mathrm{sgn}([F_2],\partial[\bar{F}_2])\mathrm{sgn}([F_2],\partial[\bar{F}]) \\
&= (-1)^j(-1)^{l-1}(-1)^k + (-1)^{k-1}(-1)^{l-1}(-1)^j \\
&= 0.
\end{aligned}$$

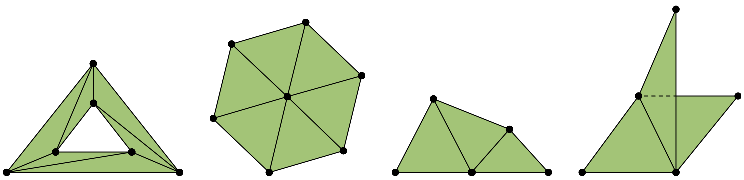
(iii) F and $\bar{F}$ have less than i vertices in common.

Then there are no faces in the boundary of $\bar{F}$ which are $(i+1)$-up neighbours of F. This implies that $\Delta_i^{up} f([F]) = 0$, which completes the proof. □

In the remainder of this section, we calculate the spectrum of circuits, paths and stars.

Definition 4.1. A pure simplicial complex L of dimension i is called an *i-path of length m* iff there is an ordering of its i-simplices $F_1 < F_2 < \cdots < F_m$, such that F_i and F_j are $(i-1)$-down neighbours iff $|j-l| = 1$. When F_m coincides with F_1, we say that L is an *i-circuit* of length $(m-1)$. The vertices in the intersection $\bigcap_{j=1}^{m-1} F_j$ are called *centres* of L.

Note that the simplicial complexes in Fig. 1(b) and (c) have one central vertex, i.e. a centre.



(a) 2-circuit. (b) 2-circuit. (c) 2-path. (d) 2-star.

Fig. 1. Examples of circuits, paths and stars.

Before we proceed to calculate $\mathbf{s}(\Delta_i^{up})$ of these complexes, we recall the definition of orientability.

Definition 4.2. Let K be a pure $(i+1)$-dimensional simplicial complex. We say that K is *orientable* iff it is possible to assign an orientation to all $(i+1)$-faces of K in such a way that any two simplices that intersect in an i-face induce a different orientation on that face. We say that such simplices are oriented *coherently*.

Note that if an $i+1$-dimensional simplicial complex is orientable, then any of its $i+1$-faces has at most one i-down neighbour.

Choosing an orientation on $(i+1)$-faces of the orientable simplicial complex K is equivalent to choosing a basis $B_{i+1}(K)$ of the vector space $C_{i+1}(K,\mathbb{R})$ consisting of elementary $(i+1)$-chains $[\bar{F}]$ that are oriented *coherently*.

For the subsequent calculations, the following obvious result (see e.g. [17]) will be useful.

Lemma 4.2. *If two matrices M and P commute, i.e., $MP = PM$, and if λ is a simple eigenvalue of P, then its corresponding eigenvector v is also an eigenvector of M.*

Let $\tilde{p}$ be a permutation of the elements of a basis $B_i(K)$ of $C_i(K,\mathbb{R})$, for an arbitrary simplicial complex K, and let $\bar{p}$ be the permutation of elementary cochains of dimension i induced by $\tilde{p}$. Denote the linear extension of $\bar{p}$ on $C^i(K,\mathbb{R})$ by p. Then we have the following equivalences

$$\tilde{p}([F]) = [F] \Leftrightarrow \bar{p}(e_{[F]}) = e_{[F]} \Leftrightarrow p(e_{[F]}) = e_{[F]}.$$

To simplify the notation, we will designate any of the maps $\tilde{p}$, $\bar{p}$, p by p. It will be clear from the argument of p which one is used. Furthermore, we will write $p(F)$ to denote the i-face which is uniquely determined by the mapping $p([F])$. To prove that p and Δ_i^{down} commute, it is necessary to check if $p\Delta_i^{down} e_{[F]} = \Delta_i^{down} p e_{[F]}$ holds for every i-face F. Since

$$p\Delta_i^{down} e_{[F]} = \sum_{E\in\partial F} \frac{w(F)}{w(E)} p(e_{[F]}) + \sum_{\substack{F'\in S_i(K):\\ (\exists E\in S_{i-1}(K)) F\cap F'=E}} \frac{w(F)}{w(E)} \operatorname{sgn}([E],\partial[F]) \times \operatorname{sgn}([E],\partial[F'])\, p(e_{[F']}),$$

and

$$\Delta_i^{down} p e_{[F]} = \sum_{p(E)\in\partial p(F)} \frac{w(p(F))}{w(p(E))} e_{p([E])} + \sum_{\substack{p(F')\in S_i(K):\\ (\exists p(E)\in S_{i-1}(K))\\ p(F)\cap p(F')=p(E)}} \frac{w(p(F))}{w(p(E))} \times \operatorname{sgn}(p([E]),\partial p([F]))\operatorname{sgn}(p([E]),\partial p([F']))\, e_{p([F'])},$$

it suffices to show

$$\sum_{E\in\partial F}\frac{w(F)}{w(E)}=\sum_{p(E)\in\partial p(F)}\frac{w(p(F))}{w(p(E))}\tag{4.1}$$

and

$$\begin{aligned}&\frac{w(p(F))}{w(p(E))}\operatorname{sgn}(p([E]),\partial p([F]))\operatorname{sgn}(p([E]),\partial p([F']))\\&\quad=\frac{w(F)}{w(E)}\operatorname{sgn}([E],\partial[F])\operatorname{sgn}([E],\partial[F'])\end{aligned}\tag{4.2}$$

for every F and F' which are $(i-1)$-down neighbours in K and every elementary i-cochain $e_{[F]}$.

Theorem 4.3. *Let K be an orientable i-circuit of length m. Then the eigenvalues of $\Delta_i^{down}(K)$ are $i-\cos(2\pi j/m)$, $j=0,1,\ldots,m-1$.*

Proof. Let $F_1<F_2<\cdots<F_m$ be the ordering of i-simplices of K satisfying the conditions of Definition 4.1. Moreover, let $[F_1],[F_2],\ldots,[F_m]$ be a choice of coherent orientation on them. Let $p:C^i(K,\mathbb{R})\to C^i(K,\mathbb{R})$ be a map given by $p([F_k])=[F_{k+1}]$, for $1\le k<m$ and $p([F_m])=[F_1]$. It is not difficult to check that

$$p\Delta_i^{down}=\Delta_i^{down}p.\tag{4.3}$$

In particular, equality (4.1) is satisfied since the weights of all i-faces are equal to 1 and $w(F)/w(E)=w(pF)/w(pE)$. Equality (4.2) holds because i-faces of K are coherently oriented, which gives the equalities $\operatorname{sgn}([E],\partial[F])\operatorname{sgn}([E],\partial[F'])=-1$ and $\operatorname{sgn}([pE],\partial[pF])\operatorname{sgn}([pE],\partial[pF'])=-1$, where F and F' are $(i-1)$-down neighbours of K and E is their intersecting face. Hence (4.3) is true.

Let P be the matrix associated with the mapping p. P is a permutation matrix and its characteristic polynomial is $\lambda^m-1=0$. Eigenvectors of P are $U_\theta=(1,\theta,\theta^2,\ldots,\theta^{m-1})^T$, where θ is the m-th root of unity. Thus, the eigenfunctions of the map p are $u_\theta([F_k])=\theta^{k-1}$.

With Lemma 4.2, we can now easily calculate the eigenvalues of Δ_i^{down}.

Let $E_k:=F_{k-1}\cap F_k$ for $2\le k\le m-1$ and let $E_m:=F_m\cap F_1$. We have

$$\begin{aligned}\Delta_i^{down}u_\theta([F_k])&=\sum_{\substack{E\in S_{i-1}(L):\\E\in\partial F_k}}\frac{w(F_k)}{w(E)}\theta^{k-1}\\&\quad+\frac{w(F_k)}{w(E_k)}\operatorname{sgn}([E_k],\partial[F_k])\operatorname{sgn}([E_k],\partial F_{k-1})\theta^{k-2}\\&\quad+\frac{w(F_k)}{w(E_{k+1})}\operatorname{sgn}([E_{k+1}],\partial[F_k])\operatorname{sgn}([E_{k+1}],\partial[F_{k+1}])\theta^{k}\\&=\left(\frac{2}{2}+i-1\right)\theta^{k-1}-\frac{1}{2}\theta^{k-2}-\frac{1}{2}\theta^{k}\\&=\theta^{k-1}\left(i-\frac{\theta^{-1}+\theta}{2}\right)\\&=\theta^{k-1}\left(i-\cos\left(\frac{2\pi j}{m}\right)\right).\end{aligned}$$

It is straightforward to check that a similar equality holds for $k = 1$ and $k = m$. Thus, $\lambda_j = i - \cos(2\pi j/n)$, where $j = 0, 1, \ldots, m-1$ are the eigenvalues of $\Delta_i^{down}(K)$. $\square$

Remark 4.1. The eigenvalues of an orientable i-circuit depend only on its length; thus there are different combinatorial structures which give the same eigenvalues of Δ_i^{down}. For example, 1, 1.5, 1.5, 2.5, 2.5, 3 are the eigenvalues of Δ_2^{down} of both simplicial complexes, in Fig. 1(b), and the simplicial complex in Fig. 1(a).

A similar analysis can be carried out for a non-orientable i-circuit of length m. In that case we define p to be $p([F_k]) = [F_{k+1}]$, for $1 \le k < m$ and $p([F_m]) = -[F_1]$. The remaining calculations are carried out as in Theorem 4.3. Thus, we have the following theorem.

Theorem 4.4. *Let K be a non-orientable i-circuit of length m. Then the eigenvalues of $\Delta_i^{down}(K)$ are $i - \sin(2\pi j/m)$ for m even and $i + \cos(2\pi j/m)$ for m odd, where $j = 0, 1, \ldots, m-1$.*

Corollary 4.5. *Eigenvalues of $\Delta_i^{down}(K)$ of an i-path K of length m are $\lambda_k = i - \cos(\pi k/m)$, for $k = 0, \ldots, m-1$.*

Proof. Since there are no self-intersections of dimension $(i-1)$ in an i-path, every path is orientable. From Theorem 4.3, we conclude that in the spectrum of the i-th down Laplacian of an i-circuit of length $2m$, all eigenvalues appear twice, except $(i-1)$ and $(i+1)$. In particular, $\lambda_k = i - \cos(k\pi/m) = i - \cos((2m-k)\pi/m) = \lambda_{2m-k}$, for $k \neq 0$ and $k \neq m$. Let $\phi = \exp(ik\pi/m)$ (where here $i = \sqrt{-1}$ should not be confused with the same symbol i for the order of the Laplace operator); then the eigenvector corresponding to λ_k is $u_k = (1, \exp(ik\pi/m), \ldots, \exp(i(2m-1)k\pi/m))^T$.

The function $v_k = u_k + u_{2m-k}$ is the eigenvector for the eigenvalue λ_k as well

$$v_k(m) = \mathrm{e}^{i\frac{\pi k}{m}} + \mathrm{e}^{i\frac{\pi(2m-k)}{m}} = \mathrm{e}^{i\frac{\pi k}{m}} + \mathrm{e}^{-i\frac{\pi k}{m}}.$$

It is now a straightforward calculation to see that the first m entries of v_k, for every $k = 0, 1, \ldots, m-1$, constitute the eigenvectors of K for the eigenvalue $i - \cos(\pi k/m)$. $\square$

This idea generalizes to paths with self-intersections of dimension $(i-1)$, but then it is necessary to distinguish among orientable and non-orientable paths. The eigenvalues of a star are described in the following theorem.

Theorem 4.6. *Let K be a simplicial complex consisting of m i-simplices assembled in a star like formation, i.e., all simplices have one $(i-1)$-face in common. Then the non-zero eigenvalues of $\Delta_i^{down}(K)$ are i with multiplicity $(m-1)$ and $(i+1)$ with multiplicity 1.*

Proof. Let F_k, $k \in \{1, \ldots, m\}$, be an i-dimensional face of K and let $\bigcap_k F_k = E$. Let $p : B_i(K, \mathbb{R}) \to B_i(K, \mathbb{R})$ be a permutation, such that $p([F_k]) = [F_{k+1}]$. Since $F_k \cap F_j = E$, for any two i-faces of K, we can fix the orientations on the F_k such that they induce the same orientation on E. Now it is easy to check that

$$p\Delta_i^{down} = \Delta_i^{down} p.$$

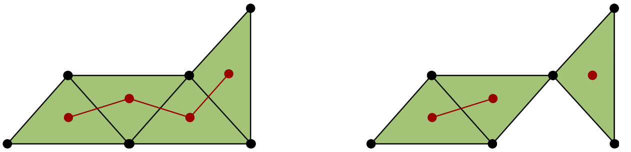


Fig. 2. Examples of i-path connected simplicial complexes and their dual graphs.

Let θ denote an m-th root of unity different from 1 and u the eigenvector of p corresponding to it. Then we obtain

$$\begin{aligned}\Delta_i^{down} u_\theta([F_k]) &= \sum_{E, E\in\partial F_k} \frac{w(F_k)}{w(E)}\theta^{k-1} + \sum_{F, F\neq F_k} \frac{w(F)}{w(E)} u_\theta([F]) \\ &= i\theta^{k-1} + \frac{1}{m}(1+\theta+\cdots+\theta^{m-1}) \\ &= i\theta^{k-1}.\end{aligned}$$

Thus, u_θ is an eigenfunction of $\Delta_i^{down}(K)$ corresponding to the eigenvalue i. The case when $\theta = 1$ results in the eigenvalue $k+1$. $\square$

5. Regular simplicial complexes

In this section we analyse the spectrum of the normalized Laplacian of a regular simplicial complex, as defined in [29].

Definition 5.1. A simplicial complex K is i-regular iff all its i-faces have the same degree.

Note that a regular graph is a 0-regular simplicial complex. To characterize the eigenvalues of regular simplicial complexes, we introduce the notion of i-dual graph and i-path connected simplicial complexes.

Definition 5.2. Let K be a simplicial complex. Then a graph G_K with the vertex set $V = \{F_j \mid F_j \in S_i(K)\}$ and the edge set $E = \{(F_j, F_l) \mid F_j \cap F_l \in S_{i-1}(K)\}$ is called an *i-dual graph* of K. Note that in graph theory dual graphs are called *line graphs*.

Definition 5.3. A simplicial complex K is i-path connected iff for any two i-faces F_1, F_2 of K there exists an i-path connecting them.

Remark 5.1. The definition of i-path connectedness is different from the definition of i-connected simplicial complexes in [26].

From now on, until the end of this section, we assume K to be $i+1$-path connected (see Fig. 2).

Theorem 5.1. *Let K be an orientable i-regular simplicial complex, with i-simplices of degree r, and G_K its $(i+1)$-dual graph. Then for $r \neq 1 (r = 2)$*

$$\lambda_k = \frac{(i+2)}{2}\mu_k,$$

where the λ's are the eigenvalues of Δ_{i+1}^{down} and the μ's the eigenvalues of $\Delta_0^{up}(G_K)$, both ordered non-decreasingly. If $r = 1$, then the only eigenvalue of Δ_{i+1}^{down} is $\lambda_1 = (i + 2)$.

Proof. Assume that $r > 1$. Since the complex K is orientable, we can choose an orientation on the $(i + 1)$-simplices of K, s.t. $\operatorname{sgn}([E], \partial[F])\operatorname{sgn}([E], \partial[F']) = -1$, where F and F' are $(i + 1)$-simplices and E their intersecting face of dimension i. Such oriented simplices uniquely determine a basis B^{i+1} of C^{i+1}. Hence, the matrix of the operator Δ_{i+1}^{down} with respect to B^i is equal to $(i+2)/rI - 1/rA$, where $A = (a_{ij})$ and $a_{ij} = 1$ if the $(i + 1)$-simplices F_i and F_j are i-down neighbours. Assume that G_K is the $(i + 1)$-dual graph of K; then G_K is regular as well, and the degree of its vertices is $(r - 1)(i + 2)$. Furthermore, the adjacency matrix of G_K equals A. Thus

$$\Delta_{i+1}^{down} = \frac{(2-r)(i+2)}{r} I + \frac{(r-1)(i+2)}{r} \Delta_0^{up}(G_K);$$

therefore

$$\lambda_k = \frac{(2-r)(i+2)}{r} + \frac{(r-1)(i+2)}{r} \mu_k.$$

The eigenvalue 0 is in $\mathbf{s}(\Delta_0^{up}(G_K))$; thus $(2-r)(i+2)/r$ must be in the spectrum of $\Delta_{i+1}^{down}(K)$. Since the operator Δ_{i+1}^{down} is positive definite $(2-r)(i+2)/r \geq 0$, then $2 \geq r$. Together with the assumption at the beginning $r > 1$, we conclude that r must be equal to 2 (another way to see that $r \leq 2$ is from a definition of orientable simplicial complexes). Finally,

$$\lambda_k = \frac{(i+2)}{2} \mu_k.$$

If $r = 1$, then $\Delta_{i+1}^{down} = (i + 2)I$ and its only eigenvalue is $i + 2$. □

In other words, the i-up spectrum of the normalized Laplacian of orientable $(i + 1)$-dimensional pseudomanifolds is uniquely determined by the normalized spectrum of its $(i + 1)$-dual graph.

From the previous theorem we obtain the following corollary.

Corollary 5.2. *Let K be an i-regular, orientable, simplicial complex, with eigenvalue $i+2$; then the spectrum of Δ_{i+1}^{down} is symmetric about $(i + 2)/2$.*

Theorem 5.3. *Let K be an i-regular simplicial complex, with i-simplices of degree r, and let G_K be its $(i + 1)$-dual graph and $i + 2 \in \mathbf{s}(\Delta_{i+1}^{down})$. Then*

$$\lambda_k = i + 2 - \frac{(r-1)(i+2)}{r} \mu_{n-k},$$

where the λ's are the eigenvalues of Δ_{i+1}^{down} and the μ's the eigenvalues of $\Delta_0^{up}(G_K)$ both ordered non-decreasingly, and n is the number of vertices of G_K.

Proof. Since $i + 2 \in \mathbf{s}(\Delta_{i+1}^{down}(K))$, according to Theorem 7.1 we can choose an orientation on the $(i+1)$-simplices of K, s.t. $\operatorname{sgn}([E], \partial[F])\operatorname{sgn}([E], \partial[F']) = 1$, for every i-down neighbours F and F', where $F \cap F' = E$ and $\dim E = i$. The matrix of the operator Δ_{i+1}^{down} is

$$\Delta_{i+1}^{down} = \frac{i+2}{r} I + \frac{1}{r} A, \tag{5.1}$$

where $A = (a_{ij})$, and

$$a_{ij} = \begin{cases} 1 & \text{if } F \text{ and } F' \text{ are } i\text{-down neighbours} \\ 0 & \text{otherwise.} \end{cases}$$

Since the degree of every vertex in the dual graph G_K is $(r-1)(i+2)$, then

$$\Delta_{i+1}^{down} = (i+2)I - \frac{(r-1)(i+2)}{r} \Delta_0^{up}(G_K). \quad \square$$

Remark 5.2. The eigenvalues of Δ_{i+1}^{down} are non-negative; hence $(i+2)-(r-1)(i+2)/r\,\mu_{n-k} \geq 0$, and

$$\frac{r}{r-1} \geq \mu_n, \tag{5.2}$$

where μ_n is the maximal eigenvalue of $\Delta_0^{up}(G_K)$. Inequality (5.2) is always satisfied for $r = 2$.

6. Constructions and their effect on the spectrum: wedges, joins and duplication of motifs

6.1. Wedges

Let $(X_i)_{i \in I}$ be a family of topological spaces and $x_i \in X_i$; then the wedge sum $\bigvee_i X_i$ is the quotient of their disjoint union by the identification $x_i \sim x_j$, for all $i, j \in I$, i.e.

$$\bigvee_i X_i := \bigsqcup_i X_i / \{x_i \sim x_j \mid i, j \in I\}.$$

For the purposes of this paper we define a combinatorial wedge sum, which is in many ways similar to the above wedge sum.

Definition 6.1. For simplicial complexes K_1 and K_2 with vertex sets $[n]$ and $[m]$, respectively, and k-simplices $F_1 = \{v_0, \ldots, v_k\}$ in $S_k(K_1)$ and $F_2 = \{u_0, \ldots, u_k\}$ in $S_k(K_2)$, the *combinatorial k-wedge sum* of K_1 and K_2 is an abstract simplicial complex on the vertex set $[m+n-k-1]$, such that

$$K_1 \vee_k K_2 := \{\{v_{i_0}, \ldots, v_{i_k}\} \mid \{v_{i_0}, \ldots, v_{i_k}\} \in K_1 \text{ or if } \{u_{i_0}, \ldots, u_{i_k}\} \in K_2\},$$

where $u_{i_j} := u_l$ if $v_{i_j} = v_l$, $u_{i_j} := v_{i_j} + k + 1$ if $v_{i_j} > n$ and $u_{i_j} := v_{i_j}$ for the other values of v_{i_j}. This definition generalizes in an obvious way to the k-wedge sum of arbitrary many simplicial complexes.

It is not difficult to check that $K_1 \vee_k K_2$ is a simplicial complex, too.

Remark 6.1. The combinatorial wedge sum $K_1 \vee_k K_2$ can also be viewed as

$$K_1 \sqcup K_2 / \{F_1 \sim F_2\},$$

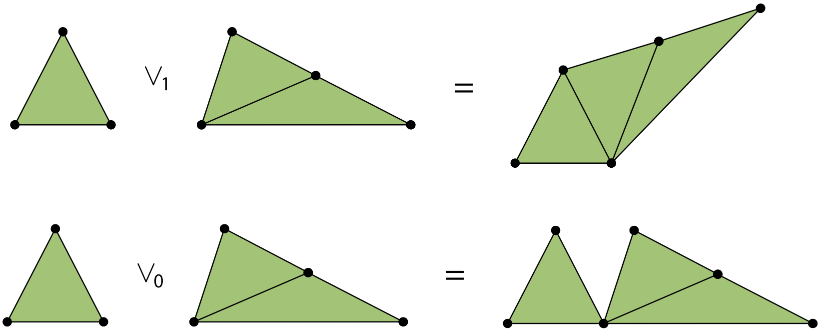


Fig. 3. Combinatorial wedge.

where $\sim$ is an equivalence relation which identifies the faces F_1 and F_2. The combinatorial k-wedge sum among graphs is a common notion in graph theory, although it is called by many different names: the combinatorial 0-wedge sum of graphs is also known as vertex amalgamation [21], coalescence [20] and join [2], whereas the combinatorial 1-wedge sum of graphs is called edge amalgamation.

Note that $K_1 \vee_k K_2$, for arbitrary k, and the wedge sum of K_1 and K_2 as topological spaces have isomorphic homology groups. From the homological point of view it is impossible to distinguish among k-wedge sums for different values of k as well as among different choices of the base points. However, combinatorially, they are clearly different; see e.g. the two wedge sums in Fig. 3. Consequently, in a combinatorial k-wedge sum of simplicial complexes, it is important which complexes are identified as well as the dimension of these complexes. The following theorem gives the first characterization of the effect of the wedge sum on the spectrum of the Laplacian.

Theorem 6.1.

$$\mathbf{s}(\Delta_i^{up}(K_1 \vee_k K_2)) \overset{\circ}{=} \mathbf{s}(\Delta_i^{up}(K_1)) \overset{\circ}{\cup} \mathbf{s}(\Delta_i^{up}(K_2))$$

for all i, k with $0 \leq k < i$.

Proof. Since K_1 and K_2 are identified by a face of dimension k, then obviously, $C^i(K_1 \vee_k K_2, \mathbb{R}) = C^i(K_1, \mathbb{R}) \oplus C^i(K_2, \mathbb{R})$ for every $i > k$. Thus, the coboundary mapping $\delta_i : C^i(K_1 \vee_k K_2, \mathbb{R}) \to C^{i+1}(K_1 \vee_k K_2, \mathbb{R})$ will map $C^i(K_j, \mathbb{R})$ to $C^{i+1}(K_j, \mathbb{R})$, $j = 1, 2$ and the same holds for the adjoint δ_i^*. $\square$

The operator Δ_i^{up} is uniquely determined by the i and $(i+1)$-faces of K. Hence its non-zero eigenvalues depend only on the structure of the $(i+1)$-faces of K. By abuse of notation, let $S_{i+1}(K)$ determine a pure $(i+1)$-dimensional subcomplex of K, whose facet set is $S_{i+1}(K)$. Then, there exist $k_1, \ldots, k_{m-1} < i$, and simplicial complexes $K_1, \ldots, K_m$, such that

$$S_{i+1}(K) = K_1 \vee_{k_1} K_2 \vee_{k_2} \cdots \vee_{k_{m-1}} K_m, \tag{6.1}$$

i.e.

$$\mathbf{s}(\Delta_i^{up}(K)) \overset{\circ}{=} \mathbf{s}(\Delta_i^{up}(K_1)) \overset{\circ}{\cup} \cdots \overset{\circ}{\cup} \mathbf{s}(\Delta_i^{up}(K_m)).$$

Therefore, when studying Δ_i^{up}, it is useful to determine if K can be represented as a combinatorial k-wedge sum of simplicial complexes and if so, how many of them there are. One possible way to answer this question is via the $(i+1)$-dual graph of K. The number of complexes in the wedge sum (6.1) is exactly the number of connected components of the $(i+1)$-dual graph of K. It is also equal to the number of $(i+1)$-path connected components.

Remark 6.2. If K is an $(i+1)$-path connected simplicial complex, it cannot be decomposed into a combinatorial k-wedge ($k < i$) of simplicial complexes.

We collect the above observations in the following proposition.

Proposition 6.2. *The following statements are equivalent.*

(i) $S_{i+1}(K) \cong K_1 \vee_{k_1} K_2 \vee_{k_2} \cdots \vee_{k_{m-1}} K_m$, *where* $k_1, \ldots, k_{m-1} < i$ *and* $K_1, \ldots, K_m$ *are simplicial complexes.*

(ii) *The* $(i+1)$*-dual graph* G_K *of* K *has at least* m *connected components.*

(iii) *The number of* $(i+1)$*-path connected components of* K *is at least* m.

The analysis on the combinatorial wedge sum above does not depend on the choice of the scalar products. Hence Theorem 6.1 and Proposition 6.2 hold for the general Laplace operator $\mathcal{L}$ as well. In the remainder of this section we investigate the effect of the k-wedge sum for $i = k$ on the spectrum of the (weighted) normalized combinatorial Laplacian Δ_i^{up}.

Theorem 6.3. *Let* K_1 *and* K_2 *be simplicial complexes, for which the spectra of* $\Delta_i^{up}(K_1)$ *and* $\Delta_i^{up}(K_2)$ *both contain the eigenvalue* λ, *and let* f_1, f_2 *be their corresponding eigenfunctions. If an* i*-wedge* $K := (K_1 \vee_i K_2)$ *is obtained by identifying* i*-faces* F_1 *and* F_2, *for which* $f_1([F_1]) = f_2([F_2])$, *then the spectrum of* $\Delta_i^{up}(K)$ *contains the eigenvalue* λ, *too.*

Proof. We will prove that

$$g([F]) = \begin{cases} f_1([F]) & \text{for every } F \text{ which is an } i\text{-face of } K_1 \text{ different from } F_1 \\ f_2([F]) & \text{for every } F \text{ which is an } i\text{-face of } K_2 \end{cases}$$

is an eigenfunction of $\Delta_i^{up}(K)$ corresponding to the eigenvalue λ. For an i-dimensional face F of K_1 different from F_1, the following equality holds:

$$\Delta_i^{up}(K)|_{K_1 - F_1} f_1([F]) = \lambda f_1([F]).$$

Similar is true when $F \in S_i(K_2)$, $F \neq F_2$, i.e.

$$\Delta_i^{up}(K)|_{K_2 - F_2} f_2([F]) = \lambda f_2([F]).$$

Let w_{K_1} and w_{K_2} denote the weight functions on the complexes K_1, K_2 respectively. Since we investigate Δ_i^{up}, the weights of the i-simplices are uniquely determined by the weights of the $(i+1)$-simplices and the incidence relations among them. Thus, for the weight (degree) of the simplex $F = F_1 = F_2$, we have $w_K(F) = w_{K_1}(F_1) + w_{K_1}(F_2)$, whereas the weights of all other simplices from K_1 or K_2 will remain the same in K. Hence

$$
\begin{aligned}
\Delta_i^{up}(K)f([F]) &= \frac{1}{w_{K_1}(F_1)+w_{K_2}(F_2)} \sum_{\bar F\in S_{i+1}(K_1)} w_{K_1}(\bar F)\mathrm{sgn}([F],\partial[\bar F])f(\partial[\bar F]) \\
&\quad + \frac{1}{w_{K_1}(F_1)+w_{K_2}(F_2)} \sum_{\bar F\in S_{i+1}(K_2)} w_{K_2}(\bar F)\mathrm{sgn}([F],\partial[\bar F])f_2(\partial[\bar F]) \\
&= \frac{w_{K_1}(F_1)}{w_{K_1}(F_1)+w_{K_2}(F_2)}\frac{1}{w_{K_1}(F_1)} \\
&\quad \times \sum_{\bar F\in S_{i+1}(K_1)} w_{K_1}(\bar F)\mathrm{sgn}([F],\partial[\bar F])f(\partial[\bar F]) \\
&\quad + \frac{w_{K_2}(F_2)}{w_{K_1}(F_1)+w_{K_2}(F_2)}\frac{1}{w_{K_2}(F_2)} \\
&\quad \times \sum_{\bar F\in S_{i+1}(K_2)} w_{K_2}(\bar F)\mathrm{sgn}([F],\partial[\bar F])f_2(\partial[\bar F]) \\
&= \frac{w_{K_1}(F_1)}{w_{K_1}(F_1)+w_{K_2}(F_2)}\lambda f_1([F]) + \frac{w_{K_2}(F_2)}{w_{K_1}(F_1)+w_{K_2}(F_2)}\lambda f_2([F]) \\
&= \lambda f([F]). \quad \square
\end{aligned}
$$

This also includes the case when either f_1 or f_2 is identically equal to zero.

Remark 6.3. The previous theorem will hold for the weighted normalized Laplacian if the weight function $w_K : \bigcup_k S_k(K) \to \mathbb{R}^+$ is

$$
w_K(F) = \begin{cases} w_{K_1}(F) & \text{if } F \text{ is a face of } K_1 \text{ and } \dim F > i \\ w_{K_2}(F) & \text{if } F \text{ is a face of } K_2 \text{ and } \dim F > i \\ \sum\limits_{\substack{\bar F_1\in K_1: \\ F\in\partial\bar F_1}} w_{K_1}(\bar F_1) + \sum\limits_{\substack{\bar F_2\in K_2: \\ F\in\partial\bar F_2}} w_{K_2}(\bar F_2) & \text{if } F \text{ is a face of } K \text{ and } \dim F \le i. \end{cases}
$$

Example 6.1. Let σ_1 be an i-simplex, then $\mathbf{s}(\Delta_i^{down}(\sigma)) \stackrel{\circ}{=} \mathbf{s}(\Delta_{i-1}^{up}(\sigma)) \stackrel{\circ}{=} \{i+1\}$. A function which is equal to 1 on every oriented simplex in the boundary of $[\sigma]$ will be an eigenfunction of Δ_i^{down} corresponding to $(i+1)$.

According to Theorem 6.3, an $(i-1)$-wedge of any number of i-simplices will possess the eigenvalue $(i+1)$, as long as we are able to orient them such that any two simplices whose intersection is of dimension i induce the same orientation on their intersecting face. For an alternative proof of this claim, see Theorem 7.2.

Theorem 6.3 identifies some of the eigenvalues of the combinatorial wedge sum. However, the results obtained by using the interlacing theorem for simplicial maps, as shown in the next theorem, are more comprehensive.

Theorem 6.4. *Let $\mu_1, \ldots, \mu_m$ be the eigenvalues of $\Delta_i^{up}(K_1 \cup K_2)$ and $\lambda_1, \ldots, \lambda_{m-1}$ the eigenvalues of $\Delta_i^{up}(K)$, where $K := (K_1 \vee_i K_2)$, then*

$$\mu_i \le \lambda_i \le \mu_{i+1}$$

for every $0 \le i \le m-1$.

Proof. Let F_1 and F_2 be i-faces which are identified in an i-wedge sum K, and let $f : K_1 \cup K_2 \to K_1 \vee_{F_1 \sim F_2} K_2$ be a map, which identifies the vertices of F_1 with the vertices of F_2, and is the identity on the remaining vertices of $K_1 \cup K_2$. Furthermore, f is a simplicial map. The interlacing theorem for simplicial maps (see [24]) gives

$$\mu_i \leq \lambda_i \leq \mu_{i+k},$$

where $k = |S_i(K_1 \cup K_2)| - |S_i(K)|$. □

Thus the spectrum of Δ_i^{up} of the union of two simplicial complexes majorizes the spectrum of their i-wedge sum.

Remark 6.4. The wedge sums of graphs and its effect on the spectrum of the normalized graph Laplacian have already been analysed in [2], and the spectrum of the combinatorial graph Laplacian was analysed in [20]. These are special cases of the general theory presented here.

6.2. Joins

Let K_1 and K_2 be simplicial complexes on the vertex sets $[n]$ and $[m]$, respectively. The *join* $K_1 * K_2$ is a simplicial complex on the vertex set $[m+n]$, whose faces are $F_1 * F_2 := \{v_0, \ldots, v_k, n + u_0, \ldots, n + u_l\}$, where $F_1 = \{v_0, \ldots, v_k\}$ is a simplex in K_1 and $F_2 = \{u_0, \ldots, u_l\}$ a simplex in K_2. The cochain groups of $K_1 * K_2$ are

$$C^i(K_1 * K_2, \mathbb{R}) = \bigoplus_{i_1+i_2+1=i} C^{i_1}(K_1, \mathbb{R}) \otimes C^{i_2}(K_2, \mathbb{R}),$$

and the coboundary map δ_i is defined as the graded derivation

$$\delta_i(f \otimes g) = \delta f \otimes g + (-1)^{|f|} f \otimes \delta g,$$

where $f \otimes g \in C^i(K_1 * K_2, \mathbb{R})$ and $|f|$ denotes the order of a cochain group which contains f.

A natural scalar product on a tensor product of Hilbert spaces is

$$(f_1 \otimes g_1, f_2 \otimes g_2) = (f_1, f_2)_{C^{i_1}(K_1)} (g_1, g_2)_{C^{i_2}(K_2)}, \tag{6.2}$$

where $f_1, f_2 \in C^{i_1}(K_1)$, $g_1, g_2 \in C^{i_2}(K_2)$. A more general product is

$$(f_1 \otimes g_1, f_2 \otimes g_2) = p(i_1)(f_1, f_2)_{C^{i_1}(K_1)} q(i_2)(g_1, g_2)_{C^{i_2}(K_2)}, \tag{6.3}$$

where $p : \{0, 1, \ldots, \dim K_1\} \to \mathbb{R}^+$ and $q : \{0, 1, \ldots, \dim K_2\} \to \mathbb{R}^+$ are positive, real valued functions. In terms of the weight functions, this is

$$w_{K_1 * K_2}(F_1 \otimes F_2) = p(\dim F_1) w_{K_1}(F_1) q(\dim F_2) w_{K_2}(F_2). \tag{6.4}$$

An elementary calculation yields

$$\delta_i^*(f \otimes g) = \frac{p(|f|)}{p(|f|-1)} \delta^* f \otimes g + \frac{q(|g|)}{q(|g|-1)} (-1)^{|f|} f \otimes \delta^* g. \tag{6.5}$$

Then the following result holds.

Theorem 6.5.

$$\mathbf{s}((\delta_i^* \delta_i + \delta_{i-1} \delta_{i-1}^*)(K_1 * K_2)) \stackrel{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}((\delta_{i_1}^* \delta_{i_1} + \delta_{i_1-1} \delta_{i_1-1}^*)(K_1)) \\ \mu_j \in \mathbf{s}((\delta_{i_2}^* \delta_{i_2} + \delta_{i_2-1} \delta_{i_2-1}^*)(K_2))}} P_{\lambda_i} \lambda_i + Q_{\mu_j} \mu_j, \tag{6.6}$$

where $i_1 + i_2 + 1 = i$*, and*

$$P_{\lambda_i} = \begin{cases} p(i_1+1)/p(i_1) & \text{if } \lambda_i \in \mathbf{s}(\mathcal{L}^{up}_{i_1}(K_1)), \\ p(i_1)/p(i_1-1) & \text{if } \lambda_i \in \mathbf{s}(\mathcal{L}^{down}_{i_1}(K_1)), \end{cases}$$

and

$$Q_{\mu_j} = \begin{cases} q(i_2+1)/q(i_2) & \text{if } \mu_j \in \mathbf{s}(\mathcal{L}^{up}_{i_2}(K_2)), \\ q(i_2)/q(i_2-1) & \text{if } \mu_j \in \mathbf{s}(\mathcal{L}^{down}_{i_2}(K_2)). \end{cases}$$

Proof.

$$\begin{aligned} \delta_i^*\delta_i(f\otimes g) = {} & \frac{p(|f|+1)}{p(|f|)}\delta^*\delta f\otimes g + (-1)^{|f|+1}\frac{q(|g|)}{q(|g|-1)}\delta f\otimes \delta^* g \\ & + (-1)^{|f|}\frac{p(|f|)}{p(|f|-1)}\delta^* f\otimes \delta g + \frac{q(|g|+1)}{q(|g|)} f\otimes \delta^*\delta g \end{aligned} \tag{6.7}$$

$$\begin{aligned} \delta_i\delta_i^*(f\otimes g) = {} & \frac{p(|f|)}{p(|f|-1)}\delta\delta^* f\otimes g + (-1)^{|f|}\frac{q(|g|)}{q(|g|-1)}\delta f\otimes \delta^* g \\ & + (-1)^{|f|-1}\frac{p(|f|)}{p(|f|-1)}\delta^* f\otimes \delta g + \frac{q(|g|)}{q(|g|-1)} f\otimes \delta\delta^* g. \end{aligned} \tag{6.8}$$

The addition of (6.7) and (6.8) gives

$$\begin{aligned} (\delta_i^*\delta_i + \delta_i\delta_i^*)(f\otimes g) = {} & \left(\frac{p(|f|+1)}{p(|f|)}\mathcal{L}^{up}_{|f|}(K_1) + \frac{p(|f|)}{p(|f|-1)}\mathcal{L}^{down}_{|f|}(K_1)\right) f\otimes g \\ & + f\otimes \left(\frac{q(|g|+1)}{q(|g|)}\mathcal{L}^{up}_{|g|}(K_2) + \frac{q(|g|)}{q(|g|-1)}\mathcal{L}^{down}_{|g|}(K_2)\right) g. \end{aligned} \tag{6.9}$$

From the last equation we immediately deduce

$$\mathbf{s}((\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*)(K_1 * K_2)) \overset{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}(\mathcal{L}_{i_1}(K_1)) \\ \mu_j \in \mathbf{s}(\mathcal{L}_{i_2}(K_2))}} P_{\lambda_i}\lambda_i + Q_{\mu_j}\mu_j. \quad \square \tag{6.10}$$

Remark 6.5. Proposition 4.9. in [12] treats the special case of Theorem 6.5 where the functions p and q are identically equal to 1. In that case, the eigenvalues of these complexes satisfy

$$\mathbf{s}((\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*)(K_1 * K_2)) \overset{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}((\delta_{i_1}^*\delta_{i_1} + \delta_{i_1-1}\delta_{i_1-1}^*)(K_1)) \\ \mu_j \in \mathbf{s}((\delta_{i_2}^*\delta_{i_2} + \delta_{i_2-1}\delta_{i_2-1}^*)(K_2))}} \lambda_i + \mu_j. \tag{6.11}$$

In [12], it is assumed that the weight functions on the cochain spaces of K_1 and K_2 are equal to the identity, which yields the combinatorial Laplacian.

The next theorem provides necessary conditions on p and q for the Laplace operator defined on $K_1 * K_2$ to be normalized.

Theorem 6.6. *Let* w_{K_1} *and* w_{K_2} *be the weight functions on* K_1 *and* K_2*, resp., such that* $\mathcal{L}(K_1, w_{K_1})$ *and* $\mathcal{L}(K_2, w_{K_2})$ *are the normalized Laplace operators. Without loss of generality, assume that* $\dim K_1 \le \dim K_2$*. If* $p(i+1)/p(i) + q(j+1)/q(j) = 1$ *for every* $i < \dim K_1$ *and* $j < \dim K_2$*, then* $\mathbf{s}(\mathcal{L}^{up}_i(K_1 * K_2, p w_{K_1} q w_{K_2})) \subset [0, i+2]$*, or in other words* $\mathcal{L}(K_1 * K_2, p w_{K_1} q w_{K_2})$ *is the normalized Laplacian.*

Proof. We check for which values of p and q the weight function of a join $K_1 * K_2$ satisfies the normalizing condition (3.7). For arbitrary $F_1 \in K_1$ and $F_2 \in K_2$, we have

$$\begin{aligned}
\deg F_1 \otimes F_2 &= \sum_{\substack{F \in S_{i+1}(K_1 * K_2):\\ F_1 \otimes F_2 \in \partial F}} w(F)\\
&= \sum_{\bar{F}_1 : F_1 \in \partial \bar{F}_1} w_{K_1 * K_2}(\bar{F}_1 \otimes F_2) + \sum_{\bar{F}_2 : F_2 \in \partial \bar{F}_2} w_{K_1 * K_2}(F_1 \otimes \bar{F}_2)\\
&= \sum_{\bar{F}_1 : F_1 \in \partial \bar{F}_1} p(\dim \bar{F}_1) w_{K_1}(\bar{F}_1) q(\dim F_2) w_{K_2}(F_2)\\
&\quad + \sum_{\bar{F}_2 : F_2 \in \partial \bar{F}_2} p(\dim F_1) w_{K_1}(F_1) q(\dim \bar{F}_2) w_{K_2}(\bar{F}_2)\\
&= p(\dim F_1 + 1) q(\dim F_2) \deg F_1 w_{K_2}(F_2)\\
&\quad + p(\dim F_1) q(\dim F_2 + 1) \deg F_2 w_{K_1}(F_1)\\
&= (p(\dim F_1 + 1) q(\dim F_2)\\
&\quad + p(\dim F_1) q(\dim F_2 + 1)) w_{K_1}(F_1) w_{K_2}(F_2).
\end{aligned}$$

Thus, the weight function $w_{K_1 * K_2}$ satisfies (3.7) iff

$$(p(i+1)q(j) + p(i)q(j+1)) = p(i)q(j),$$

for every i, j. □

The following corollary is a direct consequence of Theorems 6.5 and 6.6.

Corollary 6.7. *Let* $\dim K_1 = d_1$ *and* $\dim K_2 = d_2$ *and let* $\mathcal{L}(K_1, w_{K_1})$ *and* $\mathcal{L}(K_2, w_{K_2})$ *be normalized Laplace operators. Assume* $w_{K_1 * K_2} := w_{K_1} w_{K_2}$ *and denote* $\mathcal{L}(K_1 * K_2, w_{K_1 * K_2})$ *by* $\Delta(K_1 * K_2)$*, then*

$$\mathbf{s}(\Delta^{down}_{d_1+d_2+1}(K_1 * K_2)) \mathrel{\mathring{=}} \bigcup_{\substack{\lambda_i \in \mathbf{s}(\Delta^{down}_{d_1}(K_1))\\ \mu_j \in \mathbf{s}(\Delta^{down}_{d_2}(K_2))}} \lambda_i + \mu_j, \tag{6.12}$$

or equivalently

$$\mathbf{s}(\Delta^{up}_{d_1+d_2}(K_1 * K_2)) \mathrel{\mathring{=}} \bigcup_{\substack{\lambda_i \in \mathbf{s}(\Delta^{up}_{d_1-1}(K_1))\\ \mu_j \in \mathbf{s}(\Delta^{up}_{d_2-1}(K_2))}} \lambda_i + \mu_j.$$

Proof. For $F_1 \in K_1$ and $F_2 \in K_2$ we have

$$\deg F_1 \otimes F_2 = \sum_{\bar{F}_1 : F_1 \in \partial \bar{F}_1} w_{K_1}(\bar{F}_1) w_{K_2}(F_2) + \sum_{\bar{F}_2 : F_2 \in \partial \bar{F}_2} w_{K_1}(F_1) w_{K_2}(\bar{F}_2).$$

If neither F_1 nor F_2 is a facet of K_1, K_2, then the degree of $F_1 \otimes F_2$ is $2w_{K_1}(F_1)w_{K_2}(F_2)$. Therefore, (3.7) does not hold. Consequently, the Laplace operator determined by this function will not be the normalized Laplace operator of the join $K_1 * K_2$. However, if F_1 or F_2 is a facet, then $\deg F_1 \otimes F_2 = w_{K_1}(F_1)w_{K_2}(F_2)$. Thus, $w_{K_1 * K_2}$ coincides with the weight function determining $\Delta^{up}_i(K_1 * K_2)$, for $i = d_1 + d_2 + 1$. Together with (6.11), this yields (6.12). □

As a direct consequence of Corollary 6.7 and the fact that $\mathbf{s}(\Delta_{-1}^{up}(K)) = \{1\}$ we get the following corollary.

Corollary 6.8. *If K is simplicial complex of dimension d, and $v * K$ a cone over K, then*

$$\mathbf{s}(\Delta_d^{up}(v * K)) \mathrel{\overset{\circ}{=}} \bigcup_{\lambda_i \in \mathbf{s}(\Delta_{d-1}^{up}(K))} 1 + \lambda_i .$$

Remark 6.6 (*Direct Product of Graphs*)**.** Direct products of graphs can be treated similarly as joins of simplicial complexes. The direct product of two graphs G_1 and G_2 is the simplicial complex G of dimension 1 with $C^1(G) = C^1(G_1) \otimes C^0(G_2) \oplus C^0(G_1) \otimes C^1(G_2)$ and $C^0(G) = C^0(G_1) \otimes C^0(G_2)$. Then, by applying the same principle as in Theorem 6.5, we obtain

$$\mathbf{s}(\mathcal{L}_0^{up}(G_1 \times G_2)) \mathrel{\overset{\circ}{=}} \bigcup_{\substack{\lambda_i \in \mathbf{s}(\mathcal{L}_0^{up}(G_1)) \\ \mu_j \in \mathbf{s}(\mathcal{L}_0^{up}(G_2))}} \frac{p(1)}{p(0)}\lambda_i + \frac{q(1)}{q(0)}\mu_j ,$$

where $p(0)$, $p(1)$ and $q(0)$, $q(1)$ are, as before, parameters of a scalar product. This was proven by Fiedler [14] for the special case when $p = q \equiv 1$ and by Grigoryan in [18] for the case of the normalized graph Laplacian for p, q with $p(1)/p(0) + q(1)/q(0) = 1$.

Note that the extension of the direct product to higher dimensions would lead to a cubical (instead of simplicial) complexes.

6.3. Duplication of motifs

Let K be a simplicial complex on the vertex set $[n]$ and S a collection of simplices in K. The *closure* $\mathrm{Cl}\, S$ of S is the smallest subcomplex of K that contains each simplex in S. The *star* $\mathrm{St}\, S$ of S is the set of all simplices in K that have a face in S. The *link* $\mathrm{lk}\, S$ of S is $\mathrm{Cl}\,\mathrm{St}\, S - \mathrm{St}\,\mathrm{Cl}\, S$.

If the subcomplex Σ of K on the vertices $v_0, \ldots, v_k$ contains all of K's faces on those vertices, then it is called a *motif*.

Definition 6.2. A subcomplex Σ of a simplicial complex K is a k-motif iff

(i) $(\forall F_1, F_2 \in \Sigma)\ F_1, F_2 \subset F \in K \Rightarrow F \in \Sigma$
(ii) $\dim \mathrm{lk}\, \Sigma = k$.

In fact, as a consequence of Theorem 6.1 for $i < k$ we obtain

$$\mathbf{s}(\Delta_i^{up}(K)) \mathrel{\overset{\circ}{=}} \mathbf{s}(\Delta_i^{up}(K - \mathrm{St}\,\Sigma)) \mathbin{\overset{\circ}{\cup}} \mathbf{s}(\mathrm{Cl}\,\mathrm{St}\,\Sigma).$$

Therefore, it is meaningful to investigate the effect of the duplication of a k-motif on the spectrum of Δ_i^{up} only if $i = k$.

Remark 6.7. If K is an $(i + 1)$-path connected simplicial complex, then any motif satisfying (i) in Definition 6.2 will have a link of dimension i.

Let $u_0, \ldots, u_m$ be vertices of $\mathrm{lk}\, \Sigma$. By the definition of the link, these vertices are different from those in the motif Σ ($u_i \neq v_j$, for every $0 \le i \le m$ and $0 \le j \le k$). Let Σ' denote a simplicial complex on the vertices $v'_0, \ldots, v'_k$, which is isomorphic to Σ. And let $f : v'_i \mapsto v_i$ be a simplicial isomorphism among these complexes. Then $K^{\Sigma} := K \cup \{\{v'_{i_0}, \ldots, v'_{i_l}, u_{j_1}, \ldots, u_{j_s}\} \mid \{v_{i_0}, \ldots, v_{i_l}, u_{j_1}, \ldots, u_{j_s}\} \in K\}$.

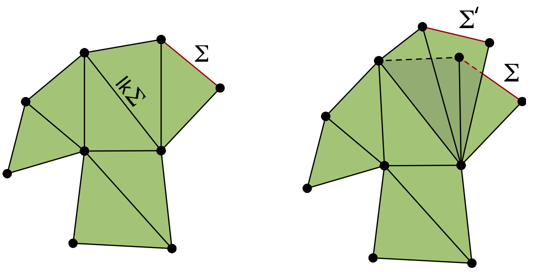


Fig. 4. Duplication of motif Σ.

Proposition 6.9. *K^{Σ} is a simplicial complex and* $\mathrm{Cl}\,\mathrm{St}\,\Sigma$ *is isomorphic to* $\mathrm{Cl}\,\mathrm{St}\,\Sigma'$.

Proof. Elementary. $\square$

Definition 6.3. We say that the simplicial complex K^{Σ} is obtained from the simplicial complex K by the *duplication of the i-motif Σ*.

Remark 6.8. It could be argued that it is $\mathrm{Cl}\,\mathrm{St}\,\Sigma$ that we duplicate rather than Σ alone. This point of view will be very helpful in the sequel, but we will refer to duplication as the duplication of the motif Σ, since this is consistent with the previous work on the duplication of motifs of graphs (see [2]) (see Fig. 4).

Theorem 6.10. *Let n be the number of i-simplices in* $\mathrm{St}\,\Sigma$. *Then there exist n linearly independent functions $f_1, \ldots, f_n$, satisfying*

$$\Delta_i^{up}(K) f_j([F]) = \lambda_j f_j([F]),$$

for every $F \in S_i(\mathrm{St}\,\Sigma)$ and some real values λ_j. The doubling of the motif Σ produces a simplicial complex K^{Σ} with the eigenvalues λ_j and the eigenfunctions g_j which agree with f_j on $\mathrm{St}\,\Sigma$ *and $-f_j$ on* $\mathrm{St}\,\Sigma'$ *and are zero elsewhere.*

Proof. It is trivial to check that $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)$ and $\Delta_i^{up}(K^{\Sigma})$ coincide on $\mathrm{St}\,\Sigma$. Let $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}$ be the restriction of the operator $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)$ on $\mathrm{St}\,\Sigma$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}$ and $f_1, \ldots, f_n$ the corresponding eigenfunctions. Then

$$g_j([F]) = \begin{cases} f_j([F]) & \text{for } F \text{ in } \mathrm{St}\,\Sigma \\ -f_j([F]) & \text{for } F \text{ in } \mathrm{St}\,\Sigma' \\ 0 & \text{otherwise} \end{cases}$$

is an eigenfunction of $\Delta_i^{up}(K^{\Sigma})$ with eigenvalue λ_j. Without loss of generality, assume that the labelling of the vertices of Σ is $v_0, \ldots, v_k$ and the vertices of Σ' is $v'_0, \ldots, v'_k$, and they are chosen such that $v_0 < \cdots < v_k < v'_0, < \cdots < v'_k$. Enumerate the vertices of $\mathrm{lk}\,\Sigma$ with $u_1, \ldots, u_m$ such that

$v_0 < \cdots < v_k < v'_0, < \cdots < v'_k < u_1 < \cdots < u_m$. Then,

$$\Delta_i^{up} f_j([F]) = \Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma} f_j([F]) = \lambda_j f_j([F]),$$

and

$$\Delta_i^{up}(-f_j)([F']) = \Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma} - f_j([F']) = -\lambda_j f_j([F']),$$

for all $F \in S_i(\Sigma)$ and $F' \in S_i(\Sigma')$.

Furthermore, assume that $[u_1, \dots, u_{i+1}]$ is a face of $\mathrm{lk}\,\Sigma$, then

$$\begin{aligned}\Delta_i^{up} f_j([u_1, \dots, u_{i+1}]) &= \sum_{v_j,[v_j,u_1,\dots,u_{i+1}]\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} (-1)^1 f_j(\partial[v_j, u_1, \dots, u_{i+1}]) \\ &\quad + \sum_{v'_j,[v'_j,u_1,\dots,u_{i+1}]\in S_{i+1}\mathrm{Cl}\,\mathrm{St}\,\Sigma'} (-1)^1(-f_j)(\partial[v'_j, u_1, \dots, u_{i+1}]) \\ &= 0.\end{aligned}$$

Since the functions f_j are 0 on the boundary of those $(i+1)$-simplices that are neither in $\mathrm{Cl}\,\mathrm{St}\,\Sigma$ nor in $\mathrm{Cl}\,\mathrm{St}\,\Sigma'$, we omit them from the discussion. Hence the λ_j's are the eigenvalues of $\Delta_i^{up}(K^{\Sigma})$. □

As a simple consequence of Theorem 6.10 we have the following corollary.

Corollary 6.11. *If the spectrum of the simplicial complex* $\mathrm{Cl}\,\mathrm{St}\,\Sigma$ *contains the eigenvalue* λ*, with an eigenfunction* f *that is identically equal to zero on* $\mathrm{lk}\,\Sigma$*, then the spectrum of* K^{Σ} *will contain the eigenvalue* λ *as well.*

Theorem 6.10 is an improved and generalized version of Theorem 2.3 from [2], which was stated for the case of the normalized graph Laplacian Δ_0^{up}. The duplication of the motif Σ will leave a specific trace in the spectrum of the resulting simplicial complex K^{Σ}. In particular, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}$, then after duplicating the motif Σ, m times, the spectrum of the resulting complex will contain $(m-1)$ instances of every eigenvalue λ_j.

It is not always straightforward to calculate the eigenvalues of $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}$; therefore we prove a theorem about interlacing of the λ_j and the eigenvalues μ_j of $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)$. With the notation of Theorem 6.10, we have the following theorem.

Theorem 6.12. *The following inequality holds*

$$\mu_i \le \lambda_i \le \mu_{i+|S_i(\mathrm{lk}\,\Sigma)|},$$

where $|S_i(\mathrm{lk}\,\Sigma)|$ *denotes the number of* i*-simplices in the link of a motif* Σ.

Proof. The matrix $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}$ is obtained from the matrix $\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)$ by deleting $|S_i(\mathrm{lk}\,\Sigma)|$ rows and columns. Thus, the interlacing inequality follows directly from the Cauchy interlacing theorem. □

Remark 6.9. Theorem 6.10 and Corollary 6.11 will hold for any choice of the weight function satisfying (3.7).

7. Eigenvalues in the spectrum of Δ_i^{up} and the combinatorial properties they encode

One of the main advantages of the *normalized* combinatorial Laplace operator is the fact that the spectrum of any simplicial complex K is bounded from above by a constant. The eigenvalues of $\Delta_i^{up}(K)$ are in the interval $[0, i+2]$. As this is not the case for the spectrum of the combinatorial Laplacian L, or for any other known type of the combinatorial Laplace operator $\mathcal{L}$, it seems impossible to assign combinatorial properties to the presence of a particular eigenvalue in the spectrum of L and $\mathcal{L}$. Nonetheless, the global properties of the spectrum of L_i relate to the combinatorial properties of the complex. For instance, the spectrum of certain combinatorially suitable complexes is proved to be an integer (see [10,12]).

Returning to the normalized Laplacian, the appearance of the eigenvalue 2 in the spectrum of the normalized graph Laplacian Δ_0^{up} means that the underlying graph is bipartite (see [6]), while the eigenvalue 1 is produced by the duplication of motifs (see [2]). In the following, we characterize some of the integer eigenvalues in the spectrum of Δ_i^{up}.

7.1. Eigenvalue $i + 2$

Without loss of generality assume that K is an $(i+1)$-path connected simplicial complex on the vertex set $[n]$. As shown earlier, the following inequality holds:

$$(\Delta_i^{up}(K)f, f) = \sum_{\bar{F}\in S_{i+1}(K)} f(\partial[\bar{F}])^2 w(\bar{F}) \tag{7.1a}$$

$$\leq (i+2) \sum_{F\in S_i(K)} f([F])^2 w(F). \tag{7.1b}$$

The equality in (7.1b) is reached iff there exists a function $f \in C^i(K,\mathbb{R})$, which satisfies

$$\mathrm{sgn}([F_j], \partial[\bar{F}]) f([F_j]) = \mathrm{sgn}([F_k], \partial[\bar{F}]) f([F_k]),$$

for every $\bar{F}$ in S_{i+1} and $F_j, F_k \in \partial\bar{F}$. Thus $|f([F])|$ must be constant for every $F \in S_i(K)$. Assume further that $|f([F])| = 1$, then for every $F \in \partial\bar{F}$, $f([F])$ is equal either to $\mathrm{sgn}([F], \partial[\bar{F}])$ or to $-\mathrm{sgn}([F], \partial[\bar{F}])$. Now it is possible to consider f as a choice of orientation on the $(i+1)$-faces of K.

Theorem 7.1. *The existence of a function f satisfying the equality in (7.1b) is equivalent to the existence of an orientation on the $(i+1)$-simplices of K, for which any two $(i+1)$-simplices intersecting in a common i-face induce the same orientation on the intersecting simplex. (This condition is opposite to the condition of coherently oriented simplices.)*

Theorem 7.2. *For an i-connected simplicial complex K the following statements are equivalent:*

1. *spectrum $\Delta_i^{up}(K)$ contains the eigenvalue $i+2$;*
2. *there are no $(i+1)$-orientable circuits of odd length nor $(i+1)$-non orientable circuits of even length in K.*

Proof. (1) $\Rightarrow$ (2) proceeds by contradiction. Assume that there exists an $(i+1)$-orientable circuit of odd length, whose i-simplices $F_1, \ldots, F_{2n+1}$ are ordered increasingly, as suggested in Definition 4.1. Then it is possible to orient these simplices in such a way that every two neighbouring simplices induce different orientations on their intersecting face. Denote these oriented simplices by $[F_1], \ldots, [F_{2n+1}]$. In order to have the same orientation induced on the intersecting face, we reverse the orientation of every simplex $[F_k]$, for k even. Thus, $[F_l]$ and $-[F_{l+1}]$ induce the same orientation on $[F_l \cap F_{l+1}]$, for every $1 \leq l \leq 2n$. However, $[F_1]$ and $[F_{2n+1}]$ remain coherently oriented, which contradicts Theorem 7.1. The analysis for the case of $(i+1)$-non-orientable circuits is analogous.

(2) $\Rightarrow$ (1). Let F_1 be an arbitrary $(i+1)$-face of K. Consider its positive orientation $[F_1]$ and call it an *initial* oriented face. Let $[F_{i_1 i_2 \cdots i_n}]$ be an $(i+1)$-face of K which shares an i-face with $[F_{i_1 i_2 \cdots i_{n-1}}]$ and both faces induce the same orientation on their intersecting face. Now, assume the opposite: the eigenvalue $i+2$ is not in the spectrum of Δ_i^{up}, i.e. it is not possible to choose an orientation on the $(i+1)$-faces of K, which satisfies the conditions of Theorem 7.1. This means

that after some number of steps in the construction above, two faces $[F_{i_1 i_2 \cdots i_n}]$, $[F_{i_1 i_2 \cdots i_m}]$ which are the same, but differently oriented, are obtained. Obviously, there exists a circuit containing $[F_{i_1 i_2 \cdots i_n}]$, which does not admit an orientation as in Theorem 7.1. This is possible only in the case when a circuit is orientable and odd or non-orientable and even. This is a contradiction; hence, $i+2$ is contained in the spectrum of Δ_i^{up}. $\square$

The spectrum of the normalized graph Laplacian contains the eigenvalue 2 iff the chromatic number of the underlying graph is 2. However, in general, such a connection between the chromatic number and the boundary eigenvalue in the spectrum of the normalized combinatorial Laplace operator only holds in one direction.

Theorem 7.3. *If the chromatic number of the* 1*-skeleton of the simplicial complex* K *is* $i+2$*, then* $i+2$ *is contained in* $\mathbf{s}(\Delta_i^{up}(K))$.

Proof. Let $I_0, \ldots, I_{i+1}$ be disjoint sets of vertices of K, such that every simplex of K contains at most one point of each set. Thus, there are no vertices of $\bar{F} \in S_{i+1}(K)$ which are contained in the same I_j. To avoid notational complications we relabel the vertices of K: instead of $v \in I_j$ ($v \in \{1, \ldots, n\}$) we write $jn + v$. Therefore, we have

$$v \in I_j, \qquad u \in I_k \quad \text{and} \quad j < k \Rightarrow v < u.$$

The function f, defined as $f([v_0, \ldots, \hat{v_j}, \ldots, v_{i+1}]) = (-1)^j$ ($[v_0, \ldots, v_{i+1}]$, is an $(i+1)$-simplex of K whose vertices are ordered increasingly, i.e. $v_0 < \cdots < v_{i+1}$) is the eigenfunction of $\Delta_i^{up}(K)$ corresponding to the eigenvalue $i+2$, i.e.

$$\begin{aligned}\Delta_i^{up} f([F]) &= \frac{\sum\limits_{\bar{F}: F \in \partial \bar{F}} \operatorname{sgn}([F], \partial[\bar{F}]) f(\partial[\bar{F}])}{\deg F} \\ &= (i+2) f([F]). \quad \square\end{aligned}$$

7.2. *Eigenvalues* $(i+1)$ *and 1*

As a special case of Theorem 6.10 we consider a motif Σ consisting of only one vertex.

Corollary 7.4. *When we duplicate an* i*-motif* Σ *consisting of one vertex which is the centre of neither an* $(i+1)$*-orientable odd circuit nor an* $(i+1)$*-non-orientable even circuit, then we produce the eigenvalue* $(i+1)$ *in the spectrum of* K^{Σ}.

Proof. Let $v_0 = \Sigma$ and let the 0-simplices of $\operatorname{lk} \Sigma$ be $u_1, \ldots, u_k$. In $\operatorname{Cl}\operatorname{St} \Sigma$ all $(i+1)$-simplices must contain v_1. Since v_1 is neither a centre of an $(i+1)$-orientable odd circuit nor a centre of an $(i+1)$-non-orientable even circuit, by Theorem 7.2, $i+2 \in \mathbf{s}(\operatorname{Cl}\operatorname{St}\Sigma)$. From Theorem 7.1, it follows that there is a function $f \in C^i(\operatorname{Cl}\operatorname{St}\Sigma, \mathbb{R})$, s.t.

$$\operatorname{sgn}([F_1], \partial[\bar{F}]) f([F_1]) = \cdots = \operatorname{sgn}([F_{i+2}], \partial[\bar{F}]) f([F_{i+2}])$$

for every $\bar{F} \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)$ and each of its i-faces. Let g be a function which coincides with f on i-faces of $\operatorname{St}\Sigma$, with $-f$ on i-faces of $\operatorname{St}\Sigma'$ and is zero elsewhere. We will now show that g is an eigenfunction of $\Delta_i^{up}(K^{\Sigma})$ associated with the eigenvalue $(i+1)$. Let F be an arbitrary i-face of $\operatorname{St}\Sigma$, then

$$\begin{aligned}
\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}\, g([F]) &= \frac{1}{w(F)} \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} \mathrm{sgn}([F], \partial[\bar{F}])g(\partial[\bar{F}]) \\
&= \frac{1}{w(F)} \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} \mathrm{sgn}([F], \partial[\bar{F}]) \\
&\quad \times \sum_{\substack{F_j\in\partial\bar{F}\\ F_j\notin \mathrm{lk}\,\Sigma}} \mathrm{sgn}([F_j], \partial[\bar{F}]) f([F_j]) \\
&= \frac{1}{w(F)} \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} \mathrm{sgn}([F], \partial[\bar{F}])(i+1) \\
&\quad \times \mathrm{sgn}([F], \partial[\bar{F}]) f([F]) \\
&= (i+1)\frac{1}{w(F)} \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} f([F]) \\
&= (i+1).
\end{aligned}$$

The same analysis holds for i-faces of $\mathrm{St}\,\Sigma'$. Let F be an i-faces of $\mathrm{Cl}\,\mathrm{St}\,\Sigma - \mathrm{St}\,\Sigma$, then

$$\begin{aligned}
&\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)|_{\mathrm{St}\,\Sigma}\, f([F]) = \frac{1}{w(F)} \left(\sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} \mathrm{sgn}([F], \partial[\bar{F}])g(\partial[\bar{F}]) \right.\\
&\quad \left. + \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma')} \mathrm{sgn}([F], \partial[\bar{F}])g(\partial[\bar{F}]) \right) \\
&= \frac{1}{w(F)}(i+1) \left(\sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} g([F_j]) + \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma')} g([F_j]) \right) \\
&= \frac{1}{w(F)}(i+1) \left(\sum_{F_j\in S_i(\mathrm{St}\,\Sigma)} f([F_j]) + \sum_{F_j'\in S_i(\mathrm{St}\,\Sigma')} -f([F_j]) \right) \\
&= 0,
\end{aligned}$$

where F_j is a face of $\bar{F}$. □

This theorem is a generalization of the vertex doubling effect on the normalized graph Laplacian Δ_0^{up} discussed in [2].

In the graph case, the eigenvalue 1 plays a very important role, since its multiplicity is usually significantly higher than other eigenvalues in graphs obtained from real world data; see [3]. For the Laplace operator on higher dimensional simplicial complexes, the role of the eigenvalue 1 is partially transferred to the eigenvalue $(i+1)$ in higher dimensions, as shown above. Nevertheless, the next theorem gives a characterization of the eigenvalue 1 in the spectrum of Δ_i^{up}.

Theorem 7.5. *Let K be a simplicial complex with an eigenvalue $i+2$ in the spectrum of Δ_i^{up} and let G_K^i be its i-dual graph. Then,*

$$1 \in \mathbf{s}(\Delta_0^{up}(G_K^i)) \Leftrightarrow 1 \in \mathbf{s}(\Delta_i^{up}(K)).$$

Proof. The multiplicity of the eigenvalue 1 in the spectrum of $\Delta_i^{up}(K)$ is equal to the dimension of the kernel of the adjacency matrix A_i^{up} of the i-faces of K. Its entries are

$$(A_i^{up})_{[F],[F']} = \begin{cases} \operatorname{sgn}([F], \partial[\bar{F}])\operatorname{sgn}([F'], \partial[\bar{F}]) & F, F' \text{ are } (i+1)\text{-up neighbours} \\ 0 & \text{otherwise.} \end{cases}$$

Due to Theorem 7.1, it is possible to orient the $(i+1)$-simplices of K such that $\operatorname{sgn}([F], \partial[\bar{F}])\operatorname{sgn}([F'], \partial[\bar{F}])$ is always positive. Consequently, all entries of the matrix A_i^{up} will be positive. The adjacency matrices of G_K^i and A_i^{up} are the same; hence, the dimension of the kernel of A_i^{up} is equal to the multiplicity of the eigenvalue 1 in the spectrum of the normalized graph Laplacian of the graph G_K^i. □

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References

[1] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, Linear Multilinear Algebra 18 (2) (1985) 141–145.
[2] A. Banerjee, J. Jost, On the spectrum of the normalized graph Laplacian, Linear Algebra Appl. 428 (11–12) (2008) 3015–3022.
[3] A. Banerjee, J. Jost, Graph spectra as a systematic tool in computational biology, Discrete Appl. Math. 157 (10) (2009) 2425–2431.
[4] A. Barysenka, A. Dress, W. Schubert, An information theoretic thresholding method for detecting protein colocalizations in stacks of fluorescence images, J. Biotech. 149 (2010) 127–131.
[5] O. Bottema, Über die Irrfahrt in einem Straßennetz, Math. Z. (ISSN: 1432-1823) 39 (1935) 137–145.
[6] F. Chung, Spectral Graph Theory, in: CBMS Regional Conference Series in Mathematics, No. 92, American Mathematical Society, 1996.
[7] F. Chung, The Laplacian of a hypergraph, in: Expanding Graphs: Proc. DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 1993, pp. 21–36.
[8] F. Chung, A combinatorial Laplacian with vertex weights, J. Combin. Theory Ser. A 75 (2) (1996) 316–327. http://dx.doi.org/10.1006/jcta.1996.0080.
[9] P.M. Cohn, Basic Algebra, Springer, 2002.
[10] X. Dong, M. Wachs, Combinatorial Laplacian of the matching complex, Electron. J. Combin. 9 (1) (2002) 1–11.
[11] A. Duval, C. Klivans, J. Martin, Simplicial matrix-tree theorems, Trans. Amer. Math. Soc. 361 (11) (2009) 6073–6114.
[12] A. Duval, V. Reiner, Shifted simplicial complexes are Laplacian integral, Trans. Amer. Math. Soc. 354 (11) (2002) 4313–4344.
[13] B. Eckmann, Harmonische Funktionen und Randwertaufgaben in einem Komplex, Comment. Math. Helv. 17 (1) (1944) 240–255.
[14] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (2) (1973) 298–305.
[15] J. Friedman, Computing betti numbers via combinatorial laplacians, in: Proc. 28th Ann. ACM Sympos. Theory Comput., 1996, pp. 386–391.
[16] H. Garland, p-adic curvature and the cohomology of discrete subgroups of p-adic groups, Ann. of Math. 97 (1973) 375–423.
[17] J.V. Greenman, Symmetry, graphs and eigenvalues, Math. Gaz. 61 (417) (1977) 195–200.
[18] A. Grigor'yan, Analysis on Graphs, Lecture Notes, University of Bielefeld.
[19] R. Grone, R. Merris, Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (2) (1990) 218–238.

[20] R. Grone, R. Merris, Coalescence, majorization, edge valuations and the Laplacian spectra of graphs, Linear Multilinear Algebra 27 (2) (1990) 139–146.
[21] J. Gross, T. Tucker, Topological Graph Theory, Courier Dover Publications, 2001.
[22] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
[23] A. Hirani, Discrete exterior calculus, Ph.D. Thesis, California Institute of Technology, Pasadena, California, May 2003.
[24] D. Horak, J. Jost, Interlacing inequalities for eigenvalues of discrete Laplace operators, Ann. Global Anal. Geom. (2012).
[25] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem. 72 (1847) 497–508.
[26] D. Kozlov, Combinatorial Algebraic Topology, Springer, 2007.
[27] L. Lu, X. Peng, High-ordered random walks and generalized Laplacians on hypergraphs, in: Proc. Algorithms and Models for the Web-Graph, 2011.
[28] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[29] F. Meunier, Combinatorial Stokes formulae, European J. Combin. 29 (1) (2008) 286–297.
[30] B. Mohar, The Laplacian spectrum of graphs, Graph Theory Combin. Appl. 2 (1991) 871–898.
[31] A. Muhammad, M. Egerstedt, Control using higher order Laplacians in network topologies, in: Mathematical Theory of Networks and Systems, Kyoto, Japan, 2006, pp. 1024–1038.
[32] W. Schubert, B. Bonnekoh, A. Pommer, L. Philipsen, R. Bockelmann, Y. Malykh, H. Gollnick, M. Friedenberger, M. Bode, A. Dress, Analyzing proteome topology and function by automated multidimensional fluorescence microscopy, Nat. Biotech. 24 (2006) 1270–1278.
[33] A. Tahbaz-Salehi, A. Jadbabaie, Distributed coverage verification in sensor networks without location information, IEEE Trans. Automat. Control 55 (8) (2010) 1837–1849.
[34] C. Taszus, Higher order Laplace Beltrami spectra of networks, Master's Thesis, Friedrich Schiller Universität Jena, 2010.