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Yetter–Drinfeld modules over bosonizations of dually paired Hopf algebras

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Abstract

Let $(R^\vee, R)$ be a dual pair of Hopf algebras in the category of Yetter–Drinfeld modules over a Hopf algebra H with bijective antipode. We show that there is a braided monoidal isomorphism between rational left Yetter–Drinfeld modules over the bosonizations of R and of $R^\vee$, respectively. As an application of this very general category isomorphism we obtain a natural proof of the existence of reflections of Nichols algebras of semi-simple Yetter–Drinfeld modules over H.

Keywords: Hopf algebras; Quantum groups; Weyl groupoid

0. Introduction

Let H be a Hopf algebra with bijective antipode over the base field $\Bbbk$, and let $(R^\vee, R)$ together with a bilinear form $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ be a dual pair of Hopf algebras in the braided category ${}^H_H\mathcal{YD}$ of left Yetter–Drinfeld modules over H (see Definition 2.2). The smash products or bosonizations $R^\vee \# H$ and $R \# H$ are Hopf algebras in the usual sense. We are interested in

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their braided monoidal categories of left Yetter–Drinfeld modules. By our first main result, Theorem 7.1, there is a braided monoidal isomorphism

$$(\Omega, \omega) : {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}, \tag{0.1}$$

where the index rat means Yetter–Drinfeld modules which are rational over R and over $R^\vee$ (see Definition 2.2). In particular, (Ω, ω) maps Hopf algebras to Hopf algebras. For $X \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$, $\Omega(X) = X$ as a Yetter–Drinfeld module over H.

The origin of the isomorphism (0.1) is the standard correspondence between comodule structures over a coalgebra and module structures over the dual algebra. In Theorem 5.5 we first prove a monoidal isomorphism between right and left relative Yetter–Drinfeld modules, and hence a braided monoidal isomorphism between their Drinfeld centers. Then we show in Theorem 6.5 that this isomorphism preserves the subcategories of right and left Yetter–Drinfeld modules we want. Finally, in Theorem 7.1 we change the sides to left Yetter–Drinfeld modules on both sides. Without this strategy, it would be hard to guess and to prove the correct formulas.

Our motivation to find such an isomorphism of categories comes from the theory of Nichols algebras which in turn are fundamental for the classification of pointed Hopf algebras. If $M \in {}^H_H\mathcal{YD}$, the Nichols algebra $\mathcal{B}(M)$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ which is the unique graded quotient of the tensor algebra $T(M)$ such that M coincides with the space of primitive elements in $\mathcal{B}(M)$.

A basic construction to produce new Nichols algebras is the reflection of semi-simple Yetter–Drinfeld modules $M_1 \oplus \cdots \oplus M_\theta$, where $\theta \in \mathbb{N}$ and $M_1, \ldots, M_\theta$ are finite-dimensional and irreducible objects in ${}^H_H\mathcal{YD}$. For $1 \le i \le \theta$, the i-th reflection of $M = (M_1, \ldots, M_\theta)$ is a certain θ-tuple $R_i(M) = (V_1, \ldots, V_\theta)$ of finite-dimensional irreducible Yetter–Drinfeld modules in ${}^H_H\mathcal{YD}$. It is defined assuming a growth condition of the adjoint action in the Nichols algebra $\mathcal{B}(M)$ of $M_1 \oplus \cdots \oplus M_\theta$. The Nichols algebras $\mathcal{B}(R_i(M))$ of $V_1 \oplus \cdots \oplus V_\theta$ and $\mathcal{B}(M)$ have the same dimension. The reflection operators allow to define the Weyl groupoid of M, an important combinatorial invariant. In this paper we give a natural explanation of the reflection operators in terms of the isomorphism (Ω, ω).

To describe our new approach to the reflection operators, fix $1 \le i \le \theta$, and let K_i^M be the algebra of right coinvariant elements of $\mathcal{B}(M)$ with respect to the canonical projection $\mathcal{B}(M) \to \mathcal{B}(M_i)$ coming from the direct sum decomposition of M. By the theory of bosonization of Radford–Majid, K_i^M is a Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$. To define $R_i(M)$ we have to assume that K_i^M is rational as an R-module. Let $W = \mathrm{ad}\mathcal{B}(M_i)(\oplus_{j\neq i} M_j) \subseteq \mathcal{B}(M)$. Then W is an object in ${}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$, and by Proposition 8.6 its Nichols algebra is isomorphic to K_i^M. This new information on K_i^M is used to prove our second main result, Theorem 8.9, which says that

$$\Omega(K_i^M)\#\mathcal{B}(M_i^*) \cong \mathcal{B}(R_i(M)), \tag{0.2}$$

where the braided monoidal functor (Ω, ω) is defined with respect to the dual pair $(\mathcal{B}(M_i^*), \mathcal{B}(M_i))$. The left-hand side of (0.2) is the bosonization, hence a braided Hopf algebra in a natural way. In [2, Theorem 3.12(1)] a different algebra isomorphism

$$K_i^M\#\mathcal{B}(M_i^*) \cong \mathcal{B}(R_i(M)), \tag{0.3}$$

formally similar to (0.2), was obtained. But there, the left-hand side is not a bosonization, and a priori it is only an algebra and not a braided Hopf algebra. This is the reason why the proof of (0.3) was quite involved. The Hopf algebra structure of $K_i^M\#\mathcal{B}(M_i^*)$ induced from the isomorphism (0.3) was determined in [5, Theorem 4.2].

If $M \in {}^H_H\mathcal{YD}$ is finite-dimensional with finite-dimensional Nichols algebra $\mathcal{B}(M)$, then Theorem 7.1 says that ${}^{R\#H}_{R\#H}\mathcal{YD} \cong {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}$, since all Yetter–Drinfeld-modules are rational over $R = \mathcal{B}(M)$ respectively $R^\vee = \mathcal{B}(M^*)$. Thus the Drinfeld centers of the monoidal categories $\mathcal{C}$ and $\mathcal{D}$ of left modules over $R\#H$ and over $R^\vee\#H$ are equivalent. Note that $R\#H$ is not semisimple if $M \neq 0$. Thus in our situation, $\mathcal{C}$ and $\mathcal{D}$ are not fusion categories. By Etingof et al. [4, Theorem 3.1], the Drinfeld centers of two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if the categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent. We do not know whether it is possible to describe our Theorem 7.1 in a similar way in terms of Morita equivalence.

If R is an algebra and M is a right R-module, we denote its module structure by $\mu_M^R = \mu_M : M \otimes R \to M$. If C is a coalgebra and M is a right C-comodule, we denote by $\delta_M^C = \delta_M : M \to M \otimes C$ the comodule structure map. The same notations μ_M^R and δ_M^C will be used for left modules and left comodules. In the following we assume that H is a Hopf algebra over $\Bbbk$ with comultiplication $\Delta = \Delta_H : H \to H \otimes H,\ h \mapsto h_{(1)} \otimes h_{(2)}$, augmentation $\varepsilon = \varepsilon_H$, and bijective antipode $\mathcal{S}$.

1. Preliminaries on bosonization of Yetter–Drinfeld Hopf algebras

We recall some well-known notions and results (see e.g. [2, Section 1.4]), and note some useful formulas from the theory of Yetter–Drinfeld Hopf algebras.

A left Yetter–Drinfeld module over H is a left H-module and a left H-comodule with H-action and H-coaction denoted by $H \otimes V \to V, h \otimes v \mapsto h \cdot v$, and $\delta = \delta_V : V \to H \otimes V, v \mapsto \delta(v) = v_{(-1)} \otimes v_{(0)}$, such that

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \tag{1.1}$$

for all $v \in V, h \in H$.

The category of left Yetter–Drinfeld modules over H with H-linear and H-colinear maps as morphisms is denoted by ${}^H_H\mathcal{YD}$. It is a monoidal and braided category. If $V, W \in {}^H_H\mathcal{YD}$, then the tensor product is the vector space $V \otimes W$ with diagonal action and coaction given by

$$h \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w, \tag{1.2}$$

$$\delta(v \otimes w) = v_{(-1)} w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}, \tag{1.3}$$

and the braiding is defined by

$$c_{V,W} : V \otimes W \to W \otimes V, \qquad v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}, \tag{1.4}$$

with inverse

$$c_{V,W}^{-1} : W \otimes V \to V \otimes W, \qquad w \otimes v \mapsto v_{(0)} \otimes \mathcal{S}^{-1}(v_{(-1)}) \cdot w, \tag{1.5}$$

for all $h \in H, v \in V, w \in W$.

The category $\mathcal{YD}_H^H$ is defined in a similar way, where the objects are the right Yetter–Drinfeld modules over H, that is, right H modules and right H-comodules V such that

$$\delta(v \cdot h) = v_{(0)} \cdot h_{(2)} \otimes \mathcal{S}(h_{(1)}) v_{(1)} h_{(3)} \tag{1.6}$$

for all $v \in V, h \in H$. The monoidal structure is given by diagonal action and coaction, and the braiding is defined by

$$c_{V,W} : V \otimes W \to W \otimes V, \qquad v \otimes w \mapsto w_{(0)} \otimes v \cdot w_{(1)}, \tag{1.7}$$

for all $V, W \in \mathcal{YD}_H^H$.

We note that for any object $V \in {}^H_H\mathcal{YD}$, there is a linear isomorphism

$$\theta_V : V \xrightarrow{\cong} V, \qquad v \mapsto \mathcal{S}(v_{(-1)}) \cdot v_{(0)}, \tag{1.8}$$

with inverse

$$V \xrightarrow{\cong} V, \qquad v \mapsto \mathcal{S}^{-2}(v_{(-1)}) \cdot v_{(0)}. \tag{1.9}$$

The map θ_V is not a morphism in ${}^H_H\mathcal{YD}$, but

$$\theta_V(h \cdot v) = \mathcal{S}^2(h) \cdot \theta_V(v), \tag{1.10}$$

$$\delta(\theta_V(v)) = \mathcal{S}^2(v_{(-1)}) \otimes \theta_V(v_{(0)}) \tag{1.11}$$

for all $v \in V$, $h \in H$, where $\delta(v) = v_{(-1)} \otimes v_{(0)}$.

If A, B are algebras in ${}^H_H\mathcal{YD}$, then the algebra structure of the tensor product $A \otimes B$ of the vector spaces A, B is defined in terms of the braiding by

$$(a \otimes b)(a' \otimes b') = a(b_{(-1)} \cdot a') \otimes b_{(0)}b' \tag{1.12}$$

for all $a, a' \in A$ and $b, b' \in B$.

Let R be a Hopf algebra in the braided monoidal category ${}^H_H\mathcal{YD}$ with augmentation $\varepsilon_R : R \to \Bbbk$, comultiplication $\Delta_R : R \to R \otimes R$, $r \mapsto r^{(1)} \otimes r^{(2)}$, and antipode $\mathcal{S}_R$. Thus $\varepsilon_R, \Delta_R, \mathcal{S}_R$ are morphisms in ${}^H_H\mathcal{YD}$ satisfying the Hopf algebra axioms. The map $\mathcal{S}_R$ anticommutes with multiplication and comultiplication in the following way.

$$\mathcal{S}_R(rs) = \mathcal{S}_R(r_{(-1)} \cdot s)\mathcal{S}_R(r_{(0)}), \tag{1.13}$$

$$\Delta_R(\mathcal{S}_R(r)) = \mathcal{S}_R(r^{(1)}{}_{(-1)} \cdot r^{(2)}) \otimes \mathcal{S}_R(r^{(1)}{}_{(0)}) \tag{1.14}$$

for all $r, s \in R$.

Let $A = R\#H$ be the bosonization of R. As an algebra, A is the smash product given by the H-action on R with multiplication

$$(r\#h)(r'\#h') = r(h_{(1)} \cdot r')\#h_{(2)}h' \tag{1.15}$$

for all $r, r' \in R$, $h, h' \in H$. We will identify $r\#1$ with r and $1\#h$ with h. Thus we view $R \subseteq A$ and $H \subseteq A$ as subalgebras, and the multiplication map

$$R \otimes H \to A, \qquad r \otimes h \mapsto rh = r\#h,$$

is bijective. Since $\cdot$ denotes the H-action, we will always write ab for the product of elements $a, b \in A$ (and not $a \cdot b$). Note that

$$hr = (h_{(1)} \cdot r)h_{(2)}, \tag{1.16}$$

$$rh = h_{(2)}(\mathcal{S}^{-1}(h_{(1)}) \cdot r) \tag{1.17}$$

for all $r \in R$, $h \in H$. As a coalgebra, A is the cosmash product given by the H-coaction of the coalgebra R. We will denote its comultiplication by

$$\Delta : A \to A \otimes A, \qquad a \mapsto a_{(1)} \otimes a_{(2)}.$$

By definition,

$$(rh)_{(1)} \otimes (rh)_{(2)} = r^{(1)}r^{(2)}{}_{(-1)}h_{(1)} \otimes r^{(2)}{}_{(0)}h_{(2)} \tag{1.18}$$

for all $r \in R, h \in H$. Thus the projection maps

$$\pi : A \to H, \qquad r\#h \mapsto \varepsilon_R(r)h, \tag{1.19}$$
$$\vartheta : A \to R, \qquad r\#h \mapsto r\varepsilon(h), \tag{1.20}$$

are coalgebra maps, and

$$A \to R \otimes H, \qquad a \mapsto \vartheta(a_{(1)}) \otimes \pi(a_{(2)}),$$

is bijective.

Then $A = R\#H$ is a Hopf algebra with antipode $\mathcal{S} = \mathcal{S}_A$, where the restriction of $\mathcal{S}$ to H is the antipode of H, and

$$\mathcal{S}(r) = \mathcal{S}(r_{(-1)})\mathcal{S}_R(r_{(0)}), \tag{1.21}$$

hence

$$\mathcal{S}^2(r) = \mathcal{S}_R^2(\theta_R(r)) \tag{1.22}$$

for all $r \in R$.

The map π is a Hopf algebra projection, and the subalgebra $R \subseteq A$ is a left coideal subalgebra, that is, $\Delta(R) \subseteq A \otimes R$, which is stable under $\mathcal{S}^2$.

The structure of the braided Hopf algebra R can be expressed in terms of the Hopf algebra $R\#H$ and the projection π:

$$R = A^{\mathrm{co}\,H} = \{r \in A \mid r_{(1)} \otimes \pi(r_{(2)}) = r \otimes 1\}, \tag{1.23}$$
$$h \cdot r = h_{(1)} r \mathcal{S}(h_{(2)}), \tag{1.24}$$
$$r_{(-1)} \otimes r_{(0)} = \pi(r_{(1)}) \otimes r_{(2)}, \tag{1.25}$$
$$r^{(1)} \otimes r^{(2)} = r_{(1)} \pi\mathcal{S}(r_{(2)}) \otimes r_{(3)}, \tag{1.26}$$
$$\mathcal{S}_R(r) = \pi(r_{(1)})\mathcal{S}(r_{(2)}) \tag{1.27}$$

for all $h \in H, r \in R$. We list some formulas related to the projection ϑ.

$$\vartheta(a) = a_{(1)} \pi\mathcal{S}(a_{(2)}), \tag{1.28}$$
$$a = \vartheta(a_{(1)})\pi(a_{(2)}), \tag{1.29}$$
$$r^{(1)} \otimes r^{(2)} = \vartheta(r_{(1)}) \otimes r_{(2)}, \tag{1.30}$$
$$\vartheta(a)^{(1)} \otimes \vartheta(a)^{(2)} = \vartheta(a_{(1)}) \otimes \vartheta(a_{(2)}), \tag{1.31}$$
$$\vartheta(a)_{(-1)} \otimes \vartheta(a)_{(0)} = \pi(a_{(1)}\mathcal{S}(a_{(3)})) \otimes \vartheta(a_{(2)}) \tag{1.32}$$

for all $r \in R, a \in A$.

By (1.24), the inclusion $R \subseteq A$ is an H-linear algebra map, where the H-action on A is the adjoint action. By (1.31) and (1.32), the map $\vartheta : A \to R$ is an H-colinear coalgebra map, where the H-coaction of A is defined by

$$A \to H \otimes A, \qquad a \mapsto \pi(a_{(1)} S(a_{(3)})) \otimes a_{(2)}, \tag{1.33}$$

that is, by the coadjoint H-coaction of A.

Finally we note the following useful formulas related to the behavior of ϑ with respect to multiplication.

$$\vartheta(ah) = \varepsilon(h)\vartheta(a), \tag{1.34}$$

$$\vartheta(ha) = h \cdot \vartheta(a), \tag{1.35}$$

for all $h \in H, a \in A$.

Lemma 1.1. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ and $A = R\#H$ its bosonization. Then*

$$\vartheta\mathcal{S}\left(a\pi\mathcal{S}^{-1}(b_{(2)})b_{(1)}\right) = \vartheta\mathcal{S}(b_{(2)})\left(\pi\left(\mathcal{S}(b_{(1)})b_{(3)}\right)\cdot\vartheta\mathcal{S}(a)\right),$$

for all $h \in H$ and $a, b \in A$.

Proof.

$$\begin{aligned}\vartheta\mathcal{S}(b_{(2)})\left(\pi\left(\mathcal{S}(b_{(1)})b_{(3)}\right)\cdot\vartheta\mathcal{S}(a)\right) &= \vartheta\mathcal{S}(b_{(3)})\pi\left(\mathcal{S}(b_{(2)})b_{(4)}\right)\vartheta\mathcal{S}(a)\pi\mathcal{S}\left(\mathcal{S}(b_{(1)})b_{(5)}\right)\\ &= \mathcal{S}(b_{(2)})\pi(b_{(3)})\mathcal{S}(a_{(2)})\pi\mathcal{S}^2(a_{(1)})\pi\mathcal{S}\left(\mathcal{S}(b_{(1)})b_{(4)}\right)\\ &= \vartheta\left(\mathcal{S}(b_{(1)})\pi(b_{(2)})\mathcal{S}(a)\right)\\ &= \vartheta\mathcal{S}\left(a\pi\mathcal{S}^{-1}(b_{(2)})b_{(1)}\right),\end{aligned}$$

where the second equality follows from (1.29) applied to $\mathcal{S}(b_{(2)})$ and (1.28) applied to $\mathcal{S}(a)$, and the third equality follows from (1.28). □

It follows from (1.22) and (1.9) that the antipode $\mathcal{S}_R$ of R is bijective if and only if the antipode $\mathcal{S}$ of R is bijective. In this case the following formulas hold for $\mathcal{S}_R^{-1}$ and $\mathcal{S}^{-1}$.

$$\mathcal{S}_R^{-1}(r) = \mathcal{S}^{-1}(r_{(0)})r_{(-1)} = \vartheta\mathcal{S}^{-1}(r), \tag{1.36}$$

$$\mathcal{S}^{-1}(rh) = \mathcal{S}^{-1}(h)\mathcal{S}_R^{-1}(r_{(0)})\mathcal{S}^{-1}(r_{(-1)}) \tag{1.37}$$

for all $r, s \in R$.

2. Dual pairs of braided Hopf algebras and rational modules

The field $\Bbbk$ will be considered as a topological space with the discrete topology. We denote by $\mathcal{L}_\Bbbk$ the category of *linearly topologized vector spaces* over $\Bbbk$. Objects of $\mathcal{L}_\Bbbk$ are topological vector spaces which have a basis of neighborhoods of 0 consisting of vector subspaces. Morphisms in $\mathcal{L}_\Bbbk$ are continuous $\Bbbk$-linear maps.

Thus an object in $\mathcal{L}_\Bbbk$ is a vector space and a topological space V, where the topology on V is given by a set $\{V_i \subseteq V \mid i \in I\}$ of vector subspaces of V such that for all $i, j \in I$ there is an index $k \in I$ with $V_k \subseteq V_i \cap V_j$. The set $\{V_i \subseteq V \mid i \in I\}$ is a basis of neighborhoods of 0, and a subset $U \subseteq V$ is open if and only if for all $x \in U$ there is an index $i \in I$ such that $x + V_i \subseteq U$.

In particular, a vector subspace $U \subseteq V$ is open if and only if $V_i \subseteq U$ for some $i \in I$.

Let $V, W \in \mathcal{L}_\Bbbk$, and let $\{V_i \subseteq V \mid i \in I\}$ and $\{W_j \subseteq W \mid j \in J\}$ be bases of neighborhoods of 0. Then a linear map $f : V \to W$ is continuous if and only if for all $j \in J$ there is an index $i \in I$ with $f(V_i) \subseteq W_j$. We define the tensor product $V \otimes W$ as an object in $\mathcal{L}_\Bbbk$ with

$$\{V_i \otimes W + V \otimes W_j \mid (i, j) \in I \times J\}$$

as a basis of neighborhoods of 0.

Let R, $R^\vee$ be vector spaces, and let

$$\langle , \rangle : R^\vee \otimes R \to \Bbbk, \qquad \xi \otimes x \mapsto \langle \xi, x\rangle,$$

be a $\Bbbk$-bilinear form. If $X \subseteq R$ and $X' \subseteq R^\vee$ are subsets, we define

$$\begin{aligned} {}^\perp X &= \{\xi \in R^\vee \mid \langle \xi, x\rangle = 0 \text{ for all } x \in X\},\\ X'^\perp &= \{x \in R \mid \langle \xi, x\rangle = 0 \text{ for all } \xi \in X'\}. \end{aligned}$$

We endow $R^\vee$ with the *finite topology* (or the *weak topology*), which is the coarsest topology on $R^\vee$ such that the evaluation maps $\langle\ , x\rangle : R^\vee \to \Bbbk, \xi \mapsto \langle \xi, x\rangle$, for all $x \in R$ are continuous. In the same way we view R as a topological space with the finite topology with respect to the evaluation maps $\langle \xi, \rangle : R \to \Bbbk, x \mapsto \langle \xi, x\rangle$, for all $\xi \in R^\vee$.

Let $\mathcal{E}$ be a cofinal subset of the set of all finite-dimensional subspaces of R (that is, $\mathcal{E}$ is a set of finite-dimensional subspaces of R, and any finite-dimensional subspace $E \subseteq R$ is contained in some $E_1 \in \mathcal{E}$). Let $\mathcal{E}'$ be a cofinal subset of the set of all finite-dimensional subspaces of $R^\vee$. Then $R^\vee$ and R are objects in $\mathcal{L}_\Bbbk$, where

$$\{{}^\perp E \mid E \in \mathcal{E}\} \quad \text{and} \quad \{E'^\perp \mid E' \in \mathcal{E}'\}$$

are bases of neighborhoods of 0 of $R^\vee$ and R, respectively.

The pairing $\langle , \rangle$ is called *non-degenerate* if ${}^\perp R = 0$ and $R^{\vee\perp} = 0$. Let $E \in \mathcal{E}$, and assume that ${}^\perp R = 0$. Then

$$E \to (R^\vee/{}^\perp E)^*, \qquad x \mapsto (\overline{\xi} \mapsto \langle \xi, x\rangle),$$

is injective. Since

$$R^\vee/{}^\perp E \to E^*, \qquad \overline{\xi} \mapsto (x \mapsto \langle \xi, x\rangle),$$

is injective by definition, it follows that

$$R^\vee/{}^\perp E \xrightarrow{\cong} E^*, \qquad \overline{\xi} \mapsto \langle \xi, \rangle, \tag{2.1}$$

is bijective. By the same argument, for all $E' \in \mathcal{E}_{R^\vee}$

$$R/E'^\perp \xrightarrow{\cong} E'^*, \qquad \overline{x} \mapsto \langle , x\rangle, \tag{2.2}$$

is bijective, if $R^{\vee\perp} = 0$.

If V, W are vector spaces, denote by

$$\mathrm{Hom}_{\mathrm{rat}}(R^\vee \otimes V, W) \quad (\text{respectively } \mathrm{Hom}_{\mathrm{rat}}(V \otimes R^\vee, W))$$

the set of all linear maps $g : R^\vee \otimes V \to W$ (respectively $g : V \otimes R^\vee \to W$) such that for all $v \in V$ there is a finite-dimensional subspace $E \subseteq R$ with $g({}^\perp E \otimes v) = 0$ (respectively $g(v \otimes {}^\perp E) = 0$).

Lemma 2.1. *Let $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ be a non-degenerate $\Bbbk$-bilinear form of vector spaces, and let V, W be vector spaces. Then the following hold.*

(1) *The map*

$$D : \mathrm{Hom}(V, R \otimes W) \to \mathrm{Hom}_{\mathrm{rat}}(R^\vee \otimes V, W), \qquad f \mapsto (\langle , \rangle \otimes \mathrm{id})(\mathrm{id} \otimes f),$$

is bijective.

(2) *The map*

$$D' : \mathrm{Hom}(V, R \otimes W) \to \mathrm{Hom}_{\mathrm{rat}}(V \otimes R^\vee, W), \qquad f \mapsto (\mathrm{id} \otimes \langle , \rangle)\tau(f \otimes \mathrm{id}),$$

is bijective, where $\tau : R \otimes W \otimes R^\vee \to W \otimes R^\vee \otimes R$ *is the twist map with* $\tau(x \otimes w \otimes \xi) = w \otimes \xi \otimes x$ *for all* $x \in R, w \in W, \xi \in R^\vee$.

Proof. (1) For completeness we recall the following well-known argument.

Let $f \in \mathrm{Hom}(V, R \otimes W)$, and $g = D(f)$. For all $v \in V$ there is a finite-dimensional subspace $E \subseteq R$ with $f(v) \in E \otimes W$, hence $g({}^\perp E \otimes v) = 0$. Thus $g \in \mathrm{Hom}_{\mathrm{rat}}(R^\vee \otimes V, W)$.

Conversely, let $g \in \mathrm{Hom}_{\mathrm{rat}}(R^\vee \otimes V, W)$. For any finite-dimensional subspace $U \subseteq V$ there is a finite-dimensional subspace $E \subseteq R$ with $g({}^\perp E \otimes U) = 0$. Let $g_{U,E} \in \mathrm{Hom}(R^\vee / {}^\perp E \otimes U, W)$ be the map induced by g, and $f_{U,E} \in \mathrm{Hom}(U, E \otimes W)$ the inverse image of $g_{U,E}$ under the isomorphisms

$$\mathrm{Hom}(U, E \otimes W) \xrightarrow{\cong} \mathrm{Hom}(E^* \otimes U, W) \xrightarrow{\cong} \mathrm{Hom}(R^\vee / {}^\perp E \otimes U, W),$$

where the first map is the canonical isomorphism, and the second map is induced by the isomorphism in (2.1).

If E' is a finite-dimensional subspace of R containing E, then

$$f_{U,E}(v) = f_{U,E'}(v) \quad \text{for all } v \in U.$$

Hence $f_U \in \mathrm{Hom}(U, R \otimes W)$, defined by $f_U(v) = f_{U,E}(v)$ for all $v \in U$, does not depend on the choice of E.

Since $f_{U'} \mid U = f_U$ for all finite-dimensional subspaces $U \subseteq U'$ of V, the inverse image $D^{-1}(g)$ can be defined by the family (f_U).

(2) follows from (1) since the twist map $V \otimes R^\vee \to R^\vee \otimes V$ defines an isomorphism $\mathrm{Hom}_{\mathrm{rat}}(R^\vee \otimes V, W) \cong \mathrm{Hom}_{\mathrm{rat}}(V \otimes R^\vee, W)$. □

Let $R, R^\vee$ be Hopf algebras in the braided monoidal category ${}^H_H\mathcal{YD}$, and let

$$\langle , \rangle : R^\vee \otimes R \to \Bbbk, \qquad \xi \otimes x \mapsto \langle \xi, x \rangle,$$

be a $\Bbbk$-bilinear form of vector spaces.

Definition 2.2. Assume that there are cofinal subsets $\mathcal{E}_R$ (respectively $\mathcal{E}_{R^\vee}$) of the sets of all finite-dimensional vector subspaces of R (respectively of $R^\vee$) consisting of subobjects in ${}^H_H\mathcal{YD}$.

Then the pair $(R, R^\vee)$ together with the bilinear form $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ is called a *dual pair of Hopf algebras* in ${}^H_H\mathcal{YD}$ if

$$\langle , \rangle \text{ is non-degenerate,} \tag{2.3}$$

$$\langle h \cdot \xi, x \rangle = \langle \xi, \mathcal{S}(h) \cdot x \rangle, \tag{2.4}$$

$$\xi_{(-1)} \langle \xi_{(0)}, x \rangle = \mathcal{S}^{-1}(x_{(-1)}) \langle \xi, x_{(0)} \rangle, \tag{2.5}$$

$$\langle \xi, xy \rangle = \langle \xi^{(1)}, y \rangle \langle \xi^{(2)}, x \rangle, \qquad \langle 1, x \rangle = \varepsilon(x), \tag{2.6}$$

$$\langle \xi\eta, x \rangle = \langle \xi, x^{(2)} \rangle \langle \eta, x^{(1)} \rangle, \qquad \langle \xi, 1 \rangle = \varepsilon(\xi), \tag{2.7}$$

$$\Delta_{R^\vee} : R^\vee \to R^\vee \otimes R^\vee \text{ is continuous,} \tag{2.8}$$

$$\Delta_R : R \to R \otimes R \text{ is continuous} \tag{2.9}$$

for all $x, y \in R$, $\xi, \eta \in R^\vee$ and $h \in H$.

A left or right $R^\vee$-module (respectively R-module) M is called *rational* if any element of M is annihilated by ${}^\perp E$ (respectively ${E'}^\perp$) for some finite-dimensional vector subspace $E \subseteq R$ (respectively $E' \subseteq R^\vee$).

Lemma 2.3. *Let $(R, R^\vee)$ together with $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ be a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$. Then for all $x \in R$, $\xi \in R^\vee$ and for all $E \in \mathcal{E}_R$, $E' \in \mathcal{E}_{R^\vee}$,*

$$\langle \mathcal{S}_{R^\vee}(\xi), x\rangle = \langle \xi, \mathcal{S}_R(x)\rangle, \tag{2.10}$$

$${}^\perp E \subseteq R^\vee \text{ and } {E'}^\perp \subseteq R \text{ are subobjects in } {}^H_H\mathcal{YD}. \tag{2.11}$$

Proof. The vector space $\mathrm{Hom}(R^\vee, R^{*\mathrm{op}})$ is an algebra with convolution product. We define linear maps $\varphi_1, \varphi_2, \psi \in \mathrm{Hom}(R^\vee, R^{*\mathrm{op}})$ by

$$\varphi_1(\xi)(x) = \langle \xi, \mathcal{S}_R(x)\rangle, \qquad \varphi_2(\xi)(x) = \langle \mathcal{S}_{R^\vee}(\xi), x\rangle, \qquad \psi(\xi)(x) = \langle \xi, x\rangle,$$

for all $\xi \in R^\vee, x \in R$. Then by (2.6) and (2.7) the unit element in $\mathrm{Hom}(R^\vee, R^{*\mathrm{op}})$ is equal to $\varphi_1 * \psi$ and also to $\psi * \varphi_2$. Hence $\varphi_1 = \varphi_2$.

(2.11) follows from (2.4) and (2.5). $\square$

Note that the bilinear form $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ is a morphism in ${}^H_H\mathcal{YD}$ if and only if (2.4) and (2.5) are satisfied.

The continuity conditions (2.8) and (2.9) are equivalent to the following. For all $E \in \mathcal{E}_R$ and $E' \in \mathcal{E}_{R^\vee}$ there are $E_1 \in \mathcal{E}_R$ and $E'_1 \in \mathcal{E}_{R^\vee}$ such that

$$\Delta_{R^\vee}({}^\perp E_1) \subseteq {}^\perp E \otimes R^\vee + R^\vee \otimes {}^\perp E, \qquad \Delta_R({E'_1}^\perp) \subseteq {E'}^\perp \otimes R + R \otimes {E'}^\perp.$$

By (2.1) and (2.2), rational modules over R or $R^\vee$ are locally finite. Recall that a module over an algebra is *locally finite* if each element of the module is contained in a finite-dimensional submodule.

Example 2.4. Let $R^\vee = \oplus_{n\ge 0} R^\vee(n)$ and $R = \oplus_{n \ge 0} R(n)$ be $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$ with finite-dimensional components $R^\vee(n)$ and $R(n)$ for all $n \ge 0$, and let $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ be a bilinear form of vector spaces such that

$$\langle R^\vee(m), R(n)\rangle = 0 \quad \text{for all } n \ne m \text{ in } \mathbb{N}_0. \tag{2.12}$$

Assume (2.3)–(2.7).

For all integers $n \ge 0$ we define

$$\mathcal{F}_n R = \bigoplus_{i=0}^{n} R(i), \qquad \mathcal{F}_n R^\vee = \bigoplus_{i=0}^{n} R^\vee(i).$$

Then the subspaces $\mathcal{F}_n R \subseteq R$, $n \ge 0$, and $\mathcal{F}_n R^\vee \subseteq R^\vee$, $n \ge 0$, form cofinal subsets of the set of all finite-dimensional subspaces of R and of $R^\vee$ consisting of subobjects in ${}^H_H\mathcal{YD}$. For all $n \ge 0$, let

$$\mathcal{F}^n R = \bigoplus_{i\ge n} R(i), \qquad \mathcal{F}^n R^\vee = \bigoplus_{i \ge n} R^\vee(i).$$

Then by (2.12) and (2.3), for all $n \ge 0$,

$${}^\perp(\mathcal{F}_{n-1} R) = \mathcal{F}^n R^\vee, \qquad (\mathcal{F}_{n-1} R^\vee)^\perp = \mathcal{F}^n R. \tag{2.13}$$

Since the coalgebras $R^\vee$ and R are $\mathbb{N}_0$-graded, it follows that

$$\Delta_{R^\vee}(\mathcal{F}^{2n} R^\vee) \subseteq \mathcal{F}^n R^\vee \otimes R^\vee + R^\vee \otimes \mathcal{F}^n R^\vee,$$
$$\Delta_R(\mathcal{F}^{2n} R) \subseteq \mathcal{F}^n R \otimes R + R \otimes \mathcal{F}^n R$$

for all $n \geq 0$. Thus Δ_R and $\Delta_{R^\vee}$ are continuous.

Hence the pair $(R, R^\vee)$ together with the bilinear form $\langle , \rangle$ is a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$. Moreover, the remaining structure maps of $R^\vee$ and of R, that is multiplication, unit map, augmentation and antipode, are all continuous, since they are $\mathbb{N}_0$-graded. Here, the ground field is graded by $\Bbbk(0) = \Bbbk$, and $\Bbbk(n) = 0$ for all $n \geq 1$.

Since $R(0)$ is a finite-dimensional Hopf algebra in ${}^H_H\mathcal{YD}$, the antipode of $R(0)$ is bijective by Takeuchi [10, Proposition 7.1]. Hence the Hopf subalgebra $\mathcal{F}_0 R\#H$ of $R\#H$ has bijective antipode by (1.22) and (1.9). The filtration

$$\mathcal{F}_0 R\#H \subseteq \mathcal{F}_1 R\#H \subseteq \mathcal{F}_2 R\#H \subseteq \cdots \subseteq R\#H$$

is a coalgebra filtration, and by the argument in [5, Remark 2.1], the antipodes of $R\#H$ and of R are bijective. The same proof shows that the antipodes of $R^\vee\#H$ and of $R^\vee$ are bijective.

Let $(R, R^\vee)$ together with $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ be a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$. We denote by ${}^R({}^H_H\mathcal{YD})$ the category of left R-comodules in the monoidal category ${}^H_H\mathcal{YD}$, and by ${}_{R^\vee}({}^H_H\mathcal{YD})_{\mathrm{rat}}$ the category of left $R^\vee$-modules in ${}^H_H\mathcal{YD}$ which are rational as $R^\vee$-modules.

Proposition 2.5. (1) *For all $M \in {}^R({}^H_H\mathcal{YD})$ let $D(M) = M$ as an object in ${}^H_H\mathcal{YD}$ with $R^\vee$-module structure given by*

$$\xi m = \langle \xi, m_{\langle -1\rangle}\rangle m_{\langle 0\rangle}$$

for all $\xi \in R^\vee, m \in M$, where the left R-comodule structure of M is denoted by $\delta_M(m) = m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}$. Then $D(M) \in {}_{R^\vee}({}^H_H\mathcal{YD})_{\mathrm{rat}}$.

(2) *The functor*

$$D : {}^R({}^H_H\mathcal{YD}) \to {}_{R^\vee}({}^H_H\mathcal{YD})_{\mathrm{rat}}$$

mapping $M \in {}^R({}^H_H\mathcal{YD})$ onto $D(M)$, and with $D(f) = f$ for all morphisms in ${}^R({}^H_H\mathcal{YD})$, is an isomorphism of categories.

Proof. This follows from Lemma 2.1 together with (2.4)–(2.7). □

Lemma 2.6. *The trivial left $R\#H$-module $\Bbbk$ is rational as an R-module (by restriction). Let V, W be left $R\#H$-modules, and $V \otimes W$ the left $R\#H$-module given by diagonal action. If V and W are rational as left R-modules, then $V \otimes W$ is a rational R-module.*

Proof. The trivial R-module $\Bbbk$ is rational since for all $x \in (\Bbbk 1_{R^\vee})^\perp$,

$$x 1_\Bbbk = \varepsilon(x) = \langle 1_{R^\vee}, x\rangle = 0$$

by (2.6).

To prove that $V \otimes W$ is rational as an R-module, let $v \in V, w \in W$. It is enough to show that $E^\perp(v \otimes w) = 0$ for some $E \in \mathcal{E}_{R^\vee}$. Since V and W are rational R-modules, there are $E_1, E_2 \in$

$\mathcal{E}_{R^\vee}$ with $E_1^\perp v = 0$, $E_2^\perp w = 0$. Let $E_3 \in \mathcal{E}_{R^\vee}$ with $E_1 + E_2 \subseteq E_3$. Then $E_3^\perp v = 0$, $E_3^\perp w = 0$. By (2.9) there is a subspace $E \in \mathcal{E}_{R^\vee}$ such that

$$\Delta_R(E^\perp) \subseteq E_3^\perp \otimes R + R \otimes E_3^\perp. \tag{2.14}$$

Let $r \in E^\perp$. Then by (1.18),

$$r(v \otimes w) = r^{(1)} r^{(2)}{}_{(-1)} v \otimes r^{(2)}{}_{(0)} w. \tag{2.15}$$

We rewrite the first tensorand on the right-hand side in (2.15) according to the multiplication rule (1.17) for elements in $R\#H$. Then the equality $r(v \otimes w) = 0$ follows from (2.14), (2.15) and (2.11). □

Lemma 2.6 also holds for $R^\vee$ instead of R using (2.7) and (2.8) instead of (2.6) and (2.9).

Lemma 2.7. *Assume that the antipodes of R and of $R^\vee$ are bijective. Define $\langle , \rangle' : R \otimes R^\vee \to \Bbbk$ by*

$$\langle x, \xi \rangle' = \langle \xi, \mathcal{S}^2(x) \rangle \tag{2.16}$$

for all $x \in R, \xi \in R^\vee$, where $\mathcal{S}$ is the antipode of $R\#H$. Then $(R^\vee, R)$ together with $\langle , \rangle' : R \otimes R^\vee \to \Bbbk$ is a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$.

Proof. Using (1.22) and (2.3)–(2.7) for $\langle , \rangle'$ are easily checked.

We denote by $\perp$ (respectively $\perp'$) the complements with respect to $\langle , \rangle$ (respectively to $\langle , \rangle'$).

To prove (2.8) for $\langle , \rangle'$, we note that by (2.16) for all finite-dimensional subspaces $E \subseteq R$, $E^{\perp'} = {}^\perp(\mathcal{S}^2(E))$. By assumption and (1.22), $\mathcal{S}^2$ induces an isomorphism on R. Hence the weak topologies of $R^\vee$ defined with respect to $\langle , \rangle$ and to $\langle , \rangle'$ coincide, and (2.8) for $\langle , \rangle'$ follows.

To prove (2.9) for $\langle , \rangle'$, we again show that the weak topologies of R defined with respect to $\langle , \rangle$ and to $\langle , \rangle'$ coincide. For all $x \in R, \xi \in R^\vee$,

$$\begin{aligned}\langle x, \xi \rangle' &= \langle \xi, \mathcal{S}^2(x) \rangle \\ &= \langle \xi, \mathcal{S}_R^2(\mathcal{S}(x_{(-1)}) \cdot x_{(0)}) \rangle \quad (\text{by } (1.22)) \\ &= \langle \mathcal{S}_{R^\vee}^2(\xi), \mathcal{S}(x_{(-1)}) \cdot x_{(0)} \rangle \quad (\text{by } (2.10)).\end{aligned}$$

Hence for all $E_1 \in \mathcal{E}_{R^\vee}$,

$$\begin{aligned}{}^{\perp'}E_1 &= \{x \in R \mid \mathcal{S}(x_{(-1)}) \cdot x_{(0)} \in (\mathcal{S}_{R^\vee}^2(E_1))^\perp\} \\ &= (\mathcal{S}_{R^\vee}^2(E_1))^\perp,\end{aligned}$$

where the second equality follows from (1.9) and (2.11). This proves our claim, since $\{\mathcal{S}_{R^\vee}^2(E_1) \mid E_1 \in \mathcal{E}_{R^\vee}\}$ is a cofinal subset of $\mathcal{E}_{R^\vee}$ by the bijectivity of $\mathcal{S}_{R^\vee}$. □

3. Review of monoidal categories and their centers

Our reference for monoidal categories is [7], where the term tensor categories is used. Let $\mathcal{C}$ and $\mathcal{D}$ be strict monoidal categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. We assume that $F(I)$ is the unit object in $\mathcal{D}$. Let

$$\varphi = (\varphi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y))_{X,Y \in \mathcal{C}}$$

be a family of natural isomorphisms. Then (F, φ) is a *monoidal functor* if for all $U, V, W \in \mathcal{C}$

$$\varphi_{I,U} = \mathrm{id}_{F(U)} = \varphi_{U,I}, \tag{3.1}$$

and the diagram

$$\begin{array}{ccc} F(U)\otimes F(V)\otimes F(W) & \xrightarrow{\mathrm{id}\otimes\varphi_{V,W}} & F(U)\otimes F(V\otimes W) \\ \varphi_{U,V}\otimes\mathrm{id}\big\downarrow & & \varphi_{U,V\otimes W}\big\downarrow \\ F(U\otimes V)\otimes F(W) & \xrightarrow{\varphi_{U\otimes V,W}} & F(U\otimes V\otimes W) \end{array} \tag{3.2}$$

commutes. A monoidal functor (F, φ) is called *strict* if $\varphi = \mathrm{id}$. If $\mathcal{C}$ and $\mathcal{D}$ are strict braided monoidal categories, then a monoidal functor (F, φ) is *braided* if for all $X, Y \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} F(X)\otimes F(Y) & \xrightarrow{\varphi_{X,Y}} & F(X\otimes Y) \\ c_{F(X),F(Y)}\big\downarrow & & F(c_{X,Y})\big\downarrow \\ F(Y)\otimes F(X) & \xrightarrow{\varphi_{Y,X}} & F(Y\otimes X) \end{array} \tag{3.3}$$

commutes. A *monoidal equivalence* (respectively *isomorphism*) is a monoidal functor (F, φ) such that F is an equivalence (respectively an isomorphism) of categories. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called an isomorphism if there is a functor $G : \mathcal{D} \to \mathcal{C}$ with $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$. A *braided monoidal equivalence* (respectively *isomorphism*) is a monoidal equivalence (respectively isomorphism) (F, φ) such that (F, φ) is a braided monoidal functor.

If $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ and $(G, \psi) : \mathcal{D} \to \mathcal{E}$ are monoidal (respectively braided monoidal) functors, then the composition

$$(GF, \lambda) : \mathcal{C} \to \mathcal{E}, \qquad \lambda_{X,Y} = G(\varphi_{X,Y})\psi_{F(X),F(Y)}, \quad \text{for all } X, Y \in \mathcal{C}, \tag{3.4}$$

is a monoidal (respectively braided monoidal) functor.

Let $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ be a monoidal isomorphism of categories with inverse functor $G : \mathcal{D} \to \mathcal{C}$. Then (G, ψ) is a monoidal functor with

$$\psi_{U,V} = G(\varphi_{G(U),G(V)})^{-1} : G(U) \otimes G(V) \to G(U \otimes V) \tag{3.5}$$

for all $U, V \in \mathcal{D}$.

For later use we note the following lemma.

Lemma 3.1. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be strict monoidal and braided categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Let $(G, \psi) : \mathcal{D} \to \mathcal{E}$ and $(GF, \lambda) : \mathcal{C} \to \mathcal{E}$ be braided monoidal functors. Assume that the functor G is fully faithful. Then there is exactly one family $\varphi = (\varphi_{X,Y})_{X,Y\in\mathcal{C}}$ such that (F, φ) is a braided monoidal functor and*

$$(GF, \lambda) = (\mathcal{C} \xrightarrow{(F,\varphi)} \mathcal{D} \xrightarrow{(G,\psi)} \mathcal{E}).$$

Proof. Since G is fully faithful, for all $X, Y \in \mathcal{C}$ there is exactly one morphism $\varphi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ with $\lambda_{X,Y} = G(\varphi_{X,Y})\psi_{F(X),F(Y)}$. Then one checks that (F, φ) is a braided monoidal functor. □

We recall the notion of the (left) *center* $\mathcal{Z}(\mathcal{C})$ of a strict monoidal category $\mathcal{C}$ with tensor product $\otimes$ and unit object I (see [7, XIII.4], where the right center is discussed). Objects of $\mathcal{Z}(\mathcal{C})$ are pairs (M, γ), where $M \in \mathcal{C}$, and

$$\gamma = (\gamma_X : M \otimes X \to X \otimes M)_{X\in\mathcal{C}}$$

is a family of natural isomorphisms such that

$$\gamma_{X\otimes Y} = (\mathrm{id}_X \otimes \gamma_Y)(\gamma_X \otimes \mathrm{id}_Y) \tag{3.6}$$

for all $X, Y \in \mathcal{C}$.

Note that by (3.6)

$$\gamma_I = \mathrm{id}_M \tag{3.7}$$

for all $(M, \gamma) \in \mathcal{Z}(\mathcal{C})$.

A morphism $f : (M, \gamma) \to (N, \lambda)$ between objects (M, γ) and (N, λ) in $\mathcal{Z}(\mathcal{C})$ is a morphism $f : M \to N$ in $\mathcal{C}$ such that

$$(\mathrm{id}_X \otimes f)\gamma_X = \lambda_X (f \otimes \mathrm{id}_X) \tag{3.8}$$

for all $X \in \mathcal{C}$. Composition of morphisms is given by the composition of morphisms in $\mathcal{C}$. The category $\mathcal{Z}(\mathcal{C})$ is braided monoidal. For objects $(M, \gamma), (N, \lambda)$ in $\mathcal{Z}(\mathcal{C})$ the tensor product is defined by

$$(M, \gamma) \otimes (N, \lambda) = (M \otimes N, \sigma), \tag{3.9}$$

$$\sigma_X = (\gamma_X \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \lambda_X) \tag{3.10}$$

for all $X \in \mathcal{C}$. The pair (I, id), where $\mathrm{id}_X = \mathrm{id}_{I\otimes X}$ for all $X \in \mathcal{C}$, is the unit in $\mathcal{Z}(\mathcal{C})$. The braiding is defined by

$$\gamma_N : (M, \gamma) \otimes (N, \lambda) \to (N, \lambda) \otimes (M, \gamma). \tag{3.11}$$

We note that a monoidal isomorphism $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ defines in the natural way a braided monoidal isomorphism between the centers of $\mathcal{C}$ and $\mathcal{D}$. For all objects $(M, \gamma) \in \mathcal{C}$ let

$$F^{\mathcal{Z}}(M, \gamma) = (F(M), \widetilde{\gamma}), \tag{3.12}$$

and for all $X \in \mathcal{C}$, the isomorphism $\widetilde{\gamma}_{F(X)}$ is defined by the commutative diagram

$$\begin{array}{ccc} F(M) \otimes F(X) & \xrightarrow{\widetilde{\gamma}_{F(X)}} & F(X) \otimes F(M) \\ \varphi_{M,X}\big\downarrow & & \varphi_{X,M}\big\downarrow \\ F(M \otimes X) & \xrightarrow{F(\gamma_X)} & F(X \otimes M). \end{array} \tag{3.13}$$

For morphisms f in $\mathcal{Z}(\mathcal{C})$ we define $F^{\mathcal{Z}}(f) = F(f)$. For $(M, \gamma), (N, \lambda) \in \mathcal{Z}(\mathcal{C})$ let

$$\varphi^{\mathcal{Z}}_{(M,\gamma),(N,\lambda)} = \varphi_{M,N}. \tag{3.14}$$

Then the next lemma follows by carefully writing down the definitions.

Lemma 3.2. *Let $(F, \varphi) : \mathcal{C} \to \mathcal{D}$ be a monoidal isomorphism. Then*

$$(F^{\mathcal{Z}}, \varphi^{\mathcal{Z}}) : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{D})$$

is a braided monoidal isomorphism.

Finally we note that we may view the categories of vector spaces and of modules or comodules over a Hopf algebra as strict monoidal categories since the associativity and unit constraints are given by functorial maps.

4. Relative Yetter–Drinfeld modules

In this section we assume that B, C are Hopf algebras with bijective antipode, $\rho : B \to C$ is a Hopf algebra homomorphism, and $\mathcal{R} \subseteq {}_B\mathcal{M}$ is a full subcategory of the category of left B-modules closed under tensor products and containing the trivial left B-module $\Bbbk$.

Definition 4.1. We denote by ${}^C_B\mathcal{YD}_\mathcal{R}$ the following monoidal category (depending on the map ρ). Objects of ${}^C_B\mathcal{YD}_\mathcal{R}$ are left B-modules and left C-comodules M with comodule structure $\delta : M \to C \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}$, such that $M \in \mathcal{R}$ as a module over B and

$$\delta(bm) = \rho(b_{(1)})m_{(-1)}\rho\mathcal{S}(b_{(3)}) \otimes b_{(2)}m_{(0)} \tag{4.1}$$

for all $m \in M$ and $b \in B$. Morphisms are left B-linear and left C-colinear maps.

The tensor product $M \otimes N$ of $M, N \in {}^C_B\mathcal{YD}_\mathcal{R}$ is the tensor product of the vector spaces M, N with diagonal action of B and diagonal coaction of C.

We define ${}^C_B\mathcal{YD} = {}^C_B\mathcal{YD}_\mathcal{R}$, when $\mathcal{R} = {}_B\mathcal{M}$ is the category of all B-modules. The full subcategory of ${}^B_B\mathcal{YD}$ consisting of all objects $M \in {}^B_B\mathcal{YD}$ with $M \in \mathcal{R}$ as a B-module is denoted by ${}^B_B\mathcal{YD}_\mathcal{R}$.

The Hopf algebra map $\rho : B \to C$ defines a functor

$${}^\rho() : {}^B_B\mathcal{YD} \to {}^C_B\mathcal{YD}, \tag{4.2}$$

mapping an object $M \in {}^B_B\mathcal{YD}$ onto ${}^\rho M$, where ${}^\rho M = M$ as a B-module, and where ${}^\rho M$ is a C-comodule by $M \xrightarrow{\delta_M} B \otimes M \xrightarrow{\rho\otimes \mathrm{id}_M} C \otimes M$.

Let

$$\Phi : {}^B_B\mathcal{YD}_\mathcal{R} \to \mathcal{Z}({}^C_B\mathcal{YD}_\mathcal{R}) \tag{4.3}$$

be the functor defined on objects $M \in {}^B_B\mathcal{YD}_\mathcal{R}$ by

$$\begin{aligned} &\Phi(M) = ({}^\rho M, c_M), \qquad c_{M,X} : M \otimes X \to X \otimes M, \\ &m \otimes x \mapsto m_{(-1)}x \otimes m_{(0)}, \end{aligned} \tag{4.4}$$

for all $X \in {}^C_B\mathcal{YD}_\mathcal{R}$, where $M \to B \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}$, denotes the B-comodule structure of M. We let $\Phi(f) = f$ for morphisms f in ${}^B_B\mathcal{YD}_\mathcal{R}$. It is easy to see that Φ is a well-defined functor.

We need the existence of enough objects in ${}^C_B\mathcal{YD}_\mathcal{R}$.

Definition 4.2. The category ${}^C_B\mathcal{YD}_\mathcal{R}$ is called *B-faithful* if the following conditions are satisfied.

$$\text{For any } 0 \neq b \in B, bX \neq 0 \quad \text{for some } X \in {}^C_B\mathcal{YD}_\mathcal{R}. \tag{4.5}$$

$$\text{For any } 0 \neq t \in B \otimes B, t(X \otimes Y) \neq 0 \quad \text{for some } X, Y \in {}^C_B\mathcal{YD}_\mathcal{R}. \tag{4.6}$$

Examples 4.3. (1) Let B be the left B-module with the regular representation, and the left C-comodule with the coadjoint coaction

$$B \to C \otimes B, \qquad b \mapsto \rho(b_{(1)}\mathcal{S}(b_{(3)})) \otimes b_{(2)}. \tag{4.7}$$

Then B is an object in ${}^C_B\mathcal{YD}$. Since $bB \neq 0$, $t(B \otimes B) \neq 0$ for all $0 \neq b \in B$, $0 \neq t \in B \otimes B$, the category ${}^C_B\mathcal{YD}$ is B-faithful.

(2) Let

$$R = \bigoplus_{n \in \mathbb{N}_0} R(n)$$

be an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$, and $A = R\#H$ the bosonization. We define ${}^H_A\mathcal{YD}$ with respect to the Hopf algebra map $\pi : A \to H$. As in (1), A with the regular representation and the coadjoint coaction with respect to π defined in (4.7), is an object in ${}^H_A\mathcal{YD}$. The H-coaction $\delta_A : A \to H \otimes A$ can be computed explicitly as

$$\delta_A(rh) = r_{(-1)}h_{(1)}\mathcal{S}(h_{(3)}) \otimes r_{(0)}h_{(2)}$$

for all $r \in R$, $h \in H$. Hence it follows that for all $n \geq 0$,

$$\mathcal{F}^n A = \bigoplus_{i \geq n} R(i) \otimes H \subseteq A$$

is an ideal and a left H-subcomodule of $A \in {}^H_A\mathcal{YD}$. Note that

$$\bigcap_{n \geq 0} \mathcal{F}^n A = 0, \qquad \bigcap_{n \geq 0} (\mathcal{F}^n A \otimes A + A \otimes \mathcal{F}^n A) = 0. \tag{4.8}$$

Hence for any $0 \neq a \in A$, $0 \neq t \in A \otimes A$ there is an integer $n \geq 0$ with

$$a(A/\mathcal{F}^n A) \neq 0, \qquad t(A/\mathcal{F}^n A \otimes A/\mathcal{F}^n A) \neq 0.$$

Thus ${}^H_A\mathcal{YD}_\mathcal{R}$ is A-faithful for all full subcategories $\mathcal{R}$ of ${}_A\mathcal{M}$ such that $A/\mathcal{F}^n A \in {}^H_A\mathcal{YD}_\mathcal{R}$ for all $n \geq 0$. Note that for all $n \geq 0$, $A/\mathcal{F}^n A$ as an R-module is annihilated by $\oplus_{i \geq n} R(i)$.

Proposition 4.4. *Assume that ${}^C_B\mathcal{YD}_\mathcal{R}$ is B-faithful.*

(1) *The functor $\Phi : {}^B_B\mathcal{YD}_\mathcal{R} \to \mathcal{Z}({}^C_B\mathcal{YD}_\mathcal{R})$ is fully faithful, strict monoidal and braided.*
(2) *Let $(M, \gamma) \in \mathcal{Z}({}^C_B\mathcal{YD}_\mathcal{R})$ with comodule structure $\delta_M : M \to C \otimes M$. Assume that there is a $\Bbbk$-linear map $\widetilde{\delta}_M : M \to B \otimes M$, denoted by $\widetilde{\delta}_M(m) = m_{[-1]} \otimes m_{[0]}$ for all $m \in M$, with*

$$\gamma_X(m \otimes x) = m_{[-1]}x \otimes m_{[0]} \tag{4.9}$$

$$\delta_M = (\rho \otimes \mathrm{id}_M)\widetilde{\delta}_M, \tag{4.10}$$

for all $X \in {}^C_B\mathcal{YD}_\mathcal{R}$, $x \in X$ and $m \in M$. Then the map $\widetilde{\delta}_M$ is uniquely determined. Let $\widetilde{M} = M$ as a B-module. Then $\widetilde{M} \in {}^B_B\mathcal{YD}_\mathcal{R}$ with B-comodule structure $\widetilde{\delta}_M$, and $\Phi(\widetilde{M}) = (M, \gamma)$.

Proof. (1) It is clear from the definitions that Φ is strict monoidal and braided, see (1.7), (3.11) and (4.4). To prove that Φ is fully faithful, let $M, N \in {}^B_B\mathcal{YD}$, and $f : \Phi(M) \to \Phi(N)$ a morphism in $\mathcal{Z}({}^C_B\mathcal{YD}_\mathcal{R})$. In particular, $f : M \to N$ is a left B-linear and left C-colinear homomorphism. We have to show that f is left B-colinear. Let $X \in {}^C_B\mathcal{YD}_\mathcal{R}$, $m \in M$ and $x \in X$. Then

$$f(m)_{(-1)}x \otimes f(m)_{(0)} = m_{(-1)}x \otimes f(m_{(0)}), \tag{4.11}$$

since f is a morphism in $\mathcal{Z}({}^C_B\mathcal{YD}_\mathcal{R})$. It follows from (4.11) and (4.5) that

$$f(m)_{(-1)} \otimes f(m)_{(0)} = m_{(-1)} \otimes f(m_{(0)})$$

in $B \otimes M$ for all $m \in M$, that is, f is B-colinear.

(2) The map $\widetilde{\delta}_M$ is uniquely determined by (4.5) and (4.9). We have to show that $\widetilde{M}$ is a B-comodule with structure map $\widetilde{\delta}_M$, and that $\widetilde{M} \in {}^B_B\mathcal{YD}_\mathcal{R}$ with comodule structure $\widetilde{\delta}_M$ and the given B-module structure.

Let $X, Y \in {}^C_B\mathcal{YD}_\mathcal{R}$, $x \in X$, $y \in Y$ and $m \in M$. By (3.6),

$$\Delta(m_{[-1]})(x \otimes y) \otimes m_{[0]} = m_{[-1]}x \otimes m_{[0][-1]}y \otimes m_{[0][0]}.$$

Hence $\widetilde{\delta}_M$ is coassociative by (4.6). Let $\Bbbk \in {}^C_B\mathcal{YD}_\mathcal{R}$ be the trivial object. Then by (3.7),

$$1 \otimes m = \gamma_\Bbbk(m \otimes 1) = m_{[-1]}1 \otimes m_{[0]} = 1 \otimes \varepsilon(m_{[-1]})m_{(0)}$$

for all $m \in M$. Hence the comultiplication $\widetilde{\delta}_M$ is counitary.

For all $X \in {}^C_B\mathcal{YD}_\mathcal{R}$, the map γ_X is B-linear. Hence

$$(b_{(1)}m)_{[-1]}b_{(2)}x \otimes (b_{(1)}m)_{[0]} = b_{(1)}m_{[-1]}x \otimes b_{(2)}m_{[0]}$$

for all $b \in B$, $m \in M$ and $x \in X$. Hence $\widetilde{M} \in {}^B_B\mathcal{YD}_\mathcal{R}$ by (4.5).

Finally $\Phi(\widetilde{M}) = (M, \gamma)$ by (4.9) and (4.10). □

Remark 4.5. In general, $\Phi : {}^B_B\mathcal{YD}_\mathcal{R} \to \mathcal{Z}({}^C_B\mathcal{YD}_\mathcal{R})$ is not an equivalence. However, in the case when $C = \Bbbk$ and $\rho = \varepsilon$, hence ${}^C_B\mathcal{YD} = {}_B\mathcal{M}$, it is well-known (compare [7], XIII.5) that $\Phi : {}^B_B\mathcal{YD} \to \mathcal{Z}({}_B\mathcal{M})$ is an equivalence. Indeed, let $(M, \gamma) \in \mathcal{Z}({}_B\mathcal{M})$. Define $m_{[-1]} \otimes m_{[0]} = \gamma_B(m \otimes 1)$ for all $m \in M$, where the B-module structure of $B \in {}_B\mathcal{M}$ is given by multiplication. Then for any $X \in {}_B\mathcal{M}$ and $x \in X$ there is a B-linear map $f : B \to X$ with $f(1) = x$, and $\gamma_X(m \otimes x) = m_{[-1]}x \otimes m_{[0]}$ by the naturality of γ. This proves (4.9). Similarly, (4.10) follows by considering the trivial B-module $\Bbbk$ and the B-linear map ε. Moreover, ${}_B\mathcal{M}$ is B-faithful by Examples 4.3(1). Thus in this case the assumption in Proposition 4.4(2) is always satisfied.

Definition 4.6. We denote by $\mathcal{YD}^C_B$ the monoidal category whose objects are right B-modules and right C-comodules M with comodule structure denoted by $\delta : M \to M \otimes C, m \mapsto m_{(0)} \otimes m_{(-1)}$, such that

$$\delta(mb) = m_{(0)}b_{(2)} \otimes \mathcal{S}(\rho(b_{(1)}))m_{(1)}\rho(b_{(3)}) \tag{4.12}$$

for all $m \in M$ and $b \in B$. Morphisms are right B-linear and right C-colinear maps.

The tensor product $M \otimes N$ of $M, N \in \mathcal{YD}^C_B$ is the tensor product of the vector spaces M, N with diagonal action of B and diagonal coaction of C. The monoidal category $\mathcal{YD}^C_C$ is braided by (1.7).

We define a functor

$$\Psi : \mathcal{YD}^C_C \to \mathcal{Z}(\mathcal{YD}^C_B) \tag{4.13}$$

on objects $M \in \mathcal{YD}^C_C$ by

$$\Psi(M) = (M_\rho, c_M), \qquad c_{M,X} : M \otimes X \to X \otimes M, \qquad m \otimes x \mapsto x_{(0)} \otimes mx_{(1)}, \tag{4.14}$$

for all $X \in \mathcal{YD}^C_B$, where M_ρ is M as a B-module via ρ. We let $\Psi(f) = f$ for morphisms f in $\mathcal{YD}^C_C$.

Example 4.7. Let C be the regular corepresentation with right C-comodule structure given by the comultiplication Δ_C of C. We define a right B-module structure on C by the adjoint action, that is

$$c \lhd b = \rho\mathcal{S}(b_{(1)})c\rho(b_{(2)}) \tag{4.15}$$

for all $c \in C, b \in B$. Then C is an object in ${\mathcal{YD}}^C_B$.

Proposition 4.8. (1) *The functor $\Psi : {\mathcal{YD}}^C_C \to \mathcal{Z}({\mathcal{YD}}^C_B)$ is fully faithful, strict monoidal and braided.*

(2) *Let $(M, \gamma) \in \mathcal{Z}({\mathcal{YD}}^C_B)$ with module structure $\mu_M : M \otimes B \to M$. Assume that there is a $\Bbbk$-linear map $\widetilde{\mu}_M : M \otimes C \to M$ such that*

$$\gamma_X(m \otimes x) = x_{(0)} \otimes \widetilde{\mu}_M(m \otimes x_{(1)}), \tag{4.16}$$

$$\mu_M = \widetilde{\mu_M}(\mathrm{id} \otimes \rho) \tag{4.17}$$

for all $X \in {\mathcal{YD}}^C_B$, $x \in X$ and $m \in M$. Then the map $\widetilde{\mu}_M$ is uniquely determined. Let $\widetilde{M} = M$ as a C-comodule. Then $\widetilde{M} \in {\mathcal{YD}}^C_C$ with C-module structure $\widetilde{\mu}_M$, and $\Psi(\widetilde{M}) = (M, \gamma)$.

Proof. (1) Again it is clear that Ψ is strict monoidal and braided. To see that Ψ is fully faithful, let $M, N \in {\mathcal{YD}}^C_C$ and $f : \Psi(M) \to \Psi(N)$ a morphism in $\mathcal{Z}({\mathcal{YD}}^C_B)$. We have to show that f is right C-linear. Let $X = C \in {\mathcal{YD}}^C_B$ in Example 4.7. Since f is a morphism in $\mathcal{Z}({\mathcal{YD}}^C_B)$,

$$x_{(1)} \otimes f(mx_{(2)}) = x_{(1)} \otimes f(m)x_{(2)}$$

for all $x \in C, m \in M$. By applying $\varepsilon \otimes \mathrm{id}$ to this equation it follows that f is right C-linear.

(2) Let $C \in {\mathcal{YD}}^C_B$ as in Example 4.7. Then $(\varepsilon \otimes \mathrm{id})\gamma_C = \widetilde{\mu}_M$. Hence $\widetilde{\mu}_M$ is uniquely determined. Let $X = Y = C \in {\mathcal{YD}}^C_B$. By (3.6)

$$x_{(1)} \otimes y_{(1)} \otimes \widetilde{\mu}_M(m \otimes x_{(2)}y_{(2)}) = x_{(1)} \otimes y_{(1)} \otimes \widetilde{\mu}_M(\widetilde{\mu}_M(m \otimes x_{(2)}) \otimes y_{(2)})$$

for all $x, y \in C, m \in M$. By applying $\varepsilon \otimes \varepsilon \otimes \mathrm{id}$ it follows that $\widetilde{\mu}_M$ is associative. By (3.7), $\widetilde{\mu}_M$ is unitary. We will write $mc = \widetilde{\mu}_M(m \otimes c)$ for all $m \in M, c \in C$.

Since γ_C is right C-colinear,

$$x_{(1)} \otimes (mx_{(3)})_{(0)} \otimes x_{(2)}(mx_{(3)})_{(1)} = x_{(1)} \otimes m_{(0)}x_{(2)} \otimes m_{(1)}x_{(3)}$$

for all $x \in C, m \in M$. By applying $\varepsilon \otimes \mathrm{id}$ it follows that $\widetilde{M} \in {\mathcal{YD}}^C_C$.

Finally $\Psi(\widetilde{M}) = (M, \gamma)$ by (4.16) and (4.17). □

We fix an odd integer l, and assume that the antipodes of B and C are bijective.

Let $M \in {\mathcal{YD}}^C_B$ with C-comodule structure $\delta_M : M \to M \otimes C$, $m \mapsto m_{(0)} \otimes m_{(1)}$. We define an object $S_l(M) \in {}^C_B\mathcal{YD}$ by $S_l(M) = M$ as a vector space with left B-action and left C-coaction given by

$$bm = m\mathcal{S}^{-l}(b), \tag{4.18}$$

$$\delta_{S_l(M)}(m) = \mathcal{S}^l(m_{(1)}) \otimes m_{(0)} \tag{4.19}$$

for all $b \in B, m \in M$. For morphisms f in ${\mathcal{YD}}^C_B$ we set $S_l(f) = f$.

Let $M \in {}^C_B\mathcal{YD}$ with comodule structure $\delta_M : M \to C \otimes M,\ m \mapsto m_{(-1)} \otimes m_{(0)}$. We define $S_l^{-1}(M) = M$ as a vector space with right B-action and right C-coaction given by

$$mb = \mathcal{S}^l(b)m, \tag{4.20}$$

$$\delta_{S_l^{-1}(M)}(m) = m_{(0)} \otimes \mathcal{S}^{-l}(m_{(-1)}) \tag{4.21}$$

for all $b \in B, m \in M$. For morphisms f in ${}^C_B\mathcal{YD}$ we set $S_l^{-1}(f) = f$.

Lemma 4.9. *Let l be an odd integer, and assume that the antipodes of B and C are bijective.*

(1) *The functor $S_l : \mathcal{YD}^C_B \to {}^C_B\mathcal{YD}$ mapping an object $M \in \mathcal{YD}^C_B$ onto $S_l(M)$, and a morphism f onto f, is an isomorphism of categories with inverse S_l^{-1}.*

(2) *Let $B = C = H$, and $\rho = \mathrm{id}_H$. Then $(S_l, \varphi) : \mathcal{YD}^H_H \to {}^H_H\mathcal{YD}$ is a braided monoidal isomorphism, where φ is defined by*

$$\varphi_{M,N} : S_l(M) \otimes S_l(N) \to S_l(M \otimes N),$$
$$m \otimes n \mapsto m\mathcal{S}^{-1}(n_{(1)}) \otimes n_{(0)} = \mathcal{S}^{-1}(n_{(-1)})m \otimes n_{(0)},$$

for all $M, N \in \mathcal{YD}^H_H$.

The inverse braided monoidal isomorphism is $(S_l^{-1}, \psi) : {}^H_H\mathcal{YD} \to \mathcal{YD}^H_H$, where ψ is defined by

$$\psi_{M,N} : S_l^{-1}(M) \otimes S_l^{-1}(N) \to S_l^{-1}(M \otimes N),$$
$$m \otimes n \mapsto n_{(-1)}m \otimes n_{(0)} = mn_{(1)} \otimes n_{(0)},$$

for all $M, N \in {}^H_H\mathcal{YD}$.

Proof. (1) Let $M \in \mathcal{YD}^C_B$. Then $S_l(M) \in {}^C_B\mathcal{YD}$ since for all $m \in M, b \in B$,

$$\begin{aligned}\delta_{S_l(M)}(bm) &= \delta_{S_l(M)}(m\mathcal{S}^{-l}(b)) = \mathcal{S}^l\left(\rho\mathcal{S}\mathcal{S}^{-l}(b_{(3)})m_{(1)}\rho\mathcal{S}^{-l}(b_{(1)})\right) \otimes m_{(0)}\mathcal{S}^{-l}(b_{(2)}) \\ &= \rho(b_{(1)})\mathcal{S}^l(m_{(1)})\mathcal{S}\rho(b_{(3)}) \otimes b_{(2)}m_{(0)}.\end{aligned}$$

Thus S_l is a well-defined functor. Similarly it follows that S_l^{-1} is a well-defined functor.

(2) is shown in [1, Proposition 2.2.1, 1.] for $l = -1$. □

Remark 4.10. In general, it is not clear whether the functor S_l in Lemma 4.9 is monoidal. This is one of the reasons why in the proof of our braided monoidal isomorphism of left Yetter–Drinfeld modules given in Theorem 7.1 we have to change sides starting in Theorem 5.5 with a monoidal isomorphism between relative right and left Yetter–Drinfeld modules.

5. The first isomorphism

Definition 5.1. Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$.

We denote by ${}^{R\#H}_H\mathcal{YD}$ and $\mathcal{YD}^{R\#H}_H$ the categories ${}^{R\#H}_H\mathcal{YD}$ and $\mathcal{YD}^{R\#H}_H$ in Definitions 4.1 and 4.6 with respect to the inclusion $H \subseteq R\#H$ as the Hopf algebra map ρ.

We denote by ${}^H_{R\#H}\mathcal{YD}$ the category ${}^H_{R\#H}\mathcal{YD}$ in Definition 4.1 where ρ is the Hopf algebra projection $\pi : R\#H \to H$ of $R\#H$.

Assume that $(R, R^\vee)$ together with $\langle , \rangle$ is a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$ with bijective antipodes. Then the antipodes of $R\#H$ and of $R^\vee\#H$ are bijective by (1.22) and (1.9).

We denote by ${}_{R^\vee\#H}^{\;H}\mathcal{YD}_{\mathrm{rat}}$ (respectively ${}_{R^\vee\#H}^{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$) the full subcategory of objects of ${}_{R^\vee\#H}^{\;H}\mathcal{YD}$ (respectively of ${}_{R^\vee\#H}^{R^\vee\#H}\mathcal{YD}$) which are rational as $R^\vee$-modules by restriction. The full subcategories of ${}_{R\#H}^{R\#H}\mathcal{YD}$ (respectively of $\mathcal{YD}_{R\#H}^{R\#H}$) consisting of objects which are rational over R will be denoted by ${}_{R\#H}^{R\#H}\mathcal{YD}_{\mathrm{rat}}$ (respectively ${}_{\mathrm{rat}}\mathcal{YD}_{R\#H}^{R\#H}$).

Lemma 5.2. *Let R be a Hopf algebra in ${}_H^H\mathcal{YD}$, and let ${}_R({}_H^H\mathcal{YD})$ be the category of left R-modules in the monoidal category ${}_H^H\mathcal{YD}$.*

(1) *Let $M \in {}_{R\#H}^{\;H}\mathcal{YD}$. Define $V_1(M) = M$ as a vector space and as a left H- and a left R-module by restriction of the $R\#H$-module structure. Then $V_1(M) \in {}_H^H\mathcal{YD}$ with the given H-comodule structure, and the multiplication map $R \otimes M \to M$ is a morphism in ${}_H^H\mathcal{YD}$.*
(2) *The functor*

$$V_1 : {}_{R\#H}^{\;H}\mathcal{YD} \to {}_R({}_H^H\mathcal{YD})$$

mapping objects $M \in {}_H^H\mathcal{YD}$ to $V_1(M)$ and morphisms f to f, is an isomorphism of categories. The inverse functor V_1^{-1} maps an object $M \in {}_R({}_H^H\mathcal{YD})$ onto the vector space M with given H-comodule structure and $R\#H$-module structure $R\#H\otimes M \xrightarrow{\mathrm{id}_R\otimes\mu_M^H} R\otimes M \xrightarrow{\mu_M^R} M$.

Proof. It follows from the definition of the smash product that M is a left $R\#H$-module if and only if μ_M^R is H-linear.

The set of all elements $a \in R\#H$ satisfying the following Yetter–Drinfeld condition

$$\delta^H(am) = \pi(a_{(1)})m_{(-1)}\pi\mathcal{S}(a_{(3)}) \otimes a_{(2)}m_{(0)} \tag{5.1}$$

for all $m \in M$ and $a \in R\#H$, is a subalgebra of $R\#H$. Hence (5.1) holds for all $a \in R\#H$ and $m \in M$ if and only if (5.1) holds for all $m \in M$ and $a \in R \cup H$. Note that (5.1) for all $m \in M$ and $a \in H$ is the Yetter–Drinfeld condition of ${}_H^H\mathcal{YD}$, and (5.1) for all $m \in M$ and $a \in R$ says that μ_M^R is H-colinear, since for all $a \in R$, $a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \in R\#H \otimes R\#H \otimes R$, hence

$$a_{(1)} \otimes a_{(2)} \otimes \pi\mathcal{S}(a_{(3)}) = a_{(1)} \otimes a_{(2)} \otimes 1.$$

This proves the Lemma. □

Lemma 5.3. *Let R be a Hopf algebra in ${}_H^H\mathcal{YD}$, and let ${}^R({}_H^H\mathcal{YD})$ be the category of left R-comodules in the monoidal category ${}_H^H\mathcal{YD}$.*

(1) *Let $M \in {}_{H}^{R\#H}\mathcal{YD}$ with comodule structure $\delta_M : M \to R\#H \otimes M$. Define $V_2(M) = M$ as a vector space with left H-comodule structure δ_M^H and left R-comodule structure δ_M^R given by*

$$\delta_M^H = (\pi \otimes \mathrm{id}_M)\delta_M, \qquad \delta_M^R = (\vartheta \otimes \mathrm{id}_M)\delta_M.$$

Then $V_2(M) \in {}_H^H\mathcal{YD}$ with H-comodule structure δ_M^H and the given H-module structure, and $\delta_M^R : M \to R \otimes M$ is a morphism in ${}_H^H\mathcal{YD}$.
(2) *The functor*

$$V_2 : {}_{H}^{R\#H}\mathcal{YD} \to {}^R({}_H^H\mathcal{YD})$$

mapping objects $M \in {}_H^H\mathcal{YD}$ to $V_2(M)$ and morphisms f to f, is an isomorphism of categories. The inverse functor V_2^{-1} maps an object $M \in {}^R({}_H^H\mathcal{YD})$ onto the vector space M with given H-module structure and $R\#H$-comodule structure $M \xrightarrow{\delta_M^R} R \otimes M \xrightarrow{\mathrm{id}_R\otimes\delta_M^H} R\#H \otimes M$.

Proof. This is shown similarly to the proof of Lemma 5.2. □

For later use we note a formula for the right $R\#H$-comodule structure of a left $R\#H$-comodule defined via $\mathcal{S}^{-1}$.

Lemma 5.4. *Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$ with bijective antipode, M a left H-comodule with H-coaction $\delta^H : M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$, and*

$$\delta^R : M \to R \otimes M, \qquad m \mapsto m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}$$

a linear map. Define $\delta : M \to R\#H \otimes M$, $m \mapsto m_{[-1]} \otimes m_{[0]}$, by $\delta = (\mathrm{id} \otimes \delta^H)\delta^R$. Then

$$\vartheta\mathcal{S}^{-1}(m_{[-1]}) \otimes m_{[0]} = \mathcal{S}_R^{-1}\left(\mathcal{S}^{-1}(m_{\langle 0\rangle(-1)}) \cdot m_{\langle -1\rangle}\right) \otimes m_{\langle 0\rangle(0)} \tag{5.2}$$

for all $m \in M$.

Proof. Let $m \in M$. Then $m_{[-1]} \otimes m_{[0]} = m_{\langle -1\rangle}m_{\langle 0\rangle(-1)} \otimes m_{\langle 0\rangle(0)}$, and

$$\begin{aligned}\mathcal{S}^{-1}(m_{\langle 0\rangle(-1)}) \cdot m_{\langle -1\rangle} \otimes m_{\langle 0\rangle(0)} &= \mathcal{S}^{-1}(m_{\langle 0\rangle(-1)})m_{\langle -1\rangle}m_{\langle 0\rangle(-2)} \otimes m_{\langle 0\rangle(0)}\\ &= \mathcal{S}^{-1}(m_{[0](-1)})m_{[-1]} \otimes m_{[0](0)}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathcal{S}_R^{-1}\left(\mathcal{S}^{-1}(m_{\langle 0\rangle(-1)}) \cdot m_{\langle -1\rangle}\right) \otimes m_{\langle 0\rangle(0)} = \mathcal{S}_R^{-1}\left(\mathcal{S}^{-1}(m_{[0](-1)})m_{[-1]}\right) \otimes m_{[0](0)}\\ &\quad = \vartheta\mathcal{S}^{-1}\left(\mathcal{S}^{-1}(m_{[0](-1)})m_{[-1]}\right) \otimes m_{[0](0)} \quad \text{(by (1.36))}\\ &\quad = \vartheta\mathcal{S}^{-1}(m_{[-1]}) \otimes m_{[0]} \quad \text{(by (1.34))}. \quad \square\end{aligned}$$

Theorem 5.5. *Let $(R, R^\vee)$ be a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$ with bijective antipodes and with bilinear form $\langle , \rangle$.*

A monoidal isomorphism

$$(F, \varphi) : \mathcal{YD}_H^{R\#H} \to {}_{R^\vee\#H}^{\ \ \ \ H}\mathcal{YD}_{\mathrm{rat}}$$

is defined as follows.

For any object $M \in \mathcal{YD}_H^{R\#H}$ with right $R\#H$-comodule structure denoted by

$$\delta_M : M \to M \otimes R\#H, \qquad m \mapsto m_{[0]} \otimes m_{[1]},$$

let $F(M) = M$ as a vector space and $F(M) \in {}_{R^\vee\#H}^{\ \ \ \ H}\mathcal{YD}$ with left H-action, H-coaction $\delta^H_{F(M)}$ and $R^\vee$-action, respectively, given by

$$hm = m\mathcal{S}^{-1}(h), \tag{5.3}$$

$$\delta^H_{F(M)}(m) = \pi\mathcal{S}(m_{[1]}) \otimes m_{[0]}, \tag{5.4}$$

$$\xi m = \langle \xi, \vartheta\mathcal{S}(m_{[1]})\rangle m_{[0]} \tag{5.5}$$

for all $h \in H, m \in M, \xi \in R^\vee$. For any morphism f in $\mathcal{YD}_H^{R\#H}$ let $F(f) = f$. The natural transformation φ is defined by

$$\varphi_{M,N} : F(M) \otimes F(N) \to F(M \otimes N), \qquad m \otimes n \mapsto m\pi\mathcal{S}^{-1}(n_{[1]}) \otimes n_{[0]}, \tag{5.6}$$

for all $M, N \in \mathcal{YD}_H^{R\#H}$.

Proof. The functor F is the composition of the isomorphisms

$$\mathcal{YD}_H^{R\#H} \xrightarrow{S} {}^{R\#H}_{H}\mathcal{YD} \xrightarrow{V_2} {}^{R}({}^{H}_{H}\mathcal{YD}) \xrightarrow{D} {}_{R^\vee}({}^{H}_{H}\mathcal{YD})_{\mathrm{rat}} \xrightarrow{V_1^{-1}} {}^{H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}},$$

where $S = S_1$ is the isomorphism of Lemma 4.9, V_2 is the isomorphism of Lemma 5.3, D is the isomorphism of Proposition 2.5, and where the last isomorphism is the restriction of V_1^{-1} for $R^\vee$ of Lemma 5.2 to rational objects.

Let $M, N \in \mathcal{YD}_H^{R\#H}$. The map

$$\varphi = \varphi_{M,N} : F(M) \otimes F(N) \to F(M \otimes N)$$

is a linear isomorphism with $\varphi^{-1}(m \otimes n) = m\pi(n_{[1]}) \otimes n_{[0]}$ for all $m \in M, n \in N$. It follows from the Yetter–Drinfeld condition (4.12) that φ is an H-linear and H-colinear map, since for all $m \in M, n \in N$ and $h \in H$,

$$\begin{aligned}
\varphi(h(m \otimes n)) &= \varphi(m\mathcal{S}^{-1}(h_{(1)}) \otimes n\mathcal{S}^{-1}(h_{(2)}))\\
&= m\mathcal{S}^{-1}(h_{(1)})\pi\mathcal{S}^{-1}(h_{(4)}n_{[1]}\mathcal{S}^{-1}(h_{(2)})) \otimes n_{[0]}\mathcal{S}^{-1}(h_{(3)})\\
&= m\mathcal{S}^{-1}(h_{(1)})\mathcal{S}^{-2}(h_{(2)})\pi\mathcal{S}^{-1}(n_{[1]})\mathcal{S}^{-1}(h_{(4)}) \otimes n_{[0]}\mathcal{S}^{-1}(h_{(3)})\\
&= h\varphi(m \otimes n),\\
\delta^H_{F(M\otimes N)}\varphi(m \otimes n) &= \pi\mathcal{S}(\pi(n_{[4]})m_{[1]}\pi\mathcal{S}^{-1}(n_{[2]})n_{[1]}) \otimes m_{[0]}\pi\mathcal{S}^{-1}(n_{[3]}) \otimes n_{[0]}\\
&= \pi\mathcal{S}(n_{[2]}m_{[1]}) \otimes m_{[0]}\pi\mathcal{S}^{-1}(n_{[1]}) \otimes n_{[0]}\\
&= (\mathrm{id}_H \otimes \varphi)\delta^H_{F(M)\otimes F(N)}(m \otimes n).
\end{aligned}$$

To prove that φ is a left $R^\vee$-linear map, let $\xi \in R^\vee$, $m \in M$ and $n \in N$. We first show that

$$\xi_{(-2)} \otimes \xi_{(-1)}\langle \xi_{(0)}, \vartheta\mathcal{S}(a)\rangle = \pi(\mathcal{S}(a_{(2)})a_{(4)}) \otimes \pi(\mathcal{S}(a_{(1)})a_{(5)})\langle \xi, \vartheta\mathcal{S}(a_{(3)})\rangle \tag{5.7}$$

for all $a \in R\#H$.

By (1.32),

$$\begin{aligned}
&(\vartheta\mathcal{S}(a))_{(-2)} \otimes (\vartheta\mathcal{S}(a))_{(-1)} \otimes (\vartheta\mathcal{S}(a))_{(0)}\\
&\quad = \Delta(\pi(S(a_{(3)})\mathcal{S}^2(a_{(1)}))) \otimes \vartheta\mathcal{S}(a_{(2)})\\
&\quad = \pi(\mathcal{S}(a_{(5)})\mathcal{S}^2(a_{(1)})) \otimes \pi(\mathcal{S}(a_{(4)})\mathcal{S}^2(a_{(2)})) \otimes \vartheta\mathcal{S}(a_{(3)}).
\end{aligned}$$

Hence (5.7) follows from (2.5).

Then

$$\begin{aligned}
\varphi(\xi(m \otimes n)) &= \varphi(\xi_{(1)}m \otimes \xi_{(2)}n)\\
&= \varphi(\xi^{(1)}\xi^{(2)}{}_{(-1)}m \otimes \xi^{(2)}{}_{(0)}n)\\
&= \varphi\left(\xi^{(1)}\left(m\mathcal{S}^{-1}(\xi^{(2)}{}_{(-1)})\right) \otimes \xi^{(2)}{}_{(0)}n\right)\\
&= \varphi\left(\left\langle \xi^{(1)}, \vartheta\mathcal{S}\left(\xi^{(2)}{}_{(-1)}m_{[1]}\mathcal{S}^{-1}(\xi^{(2)}{}_{(-3)})\right)\right\rangle m_{[0]}\mathcal{S}^{-1}(\xi^{(2)}{}_{(-2)})\right.\\
&\qquad \left.\otimes \langle \xi^{(2)}{}_{(0)}, \vartheta\mathcal{S}(n_{[1]})\rangle n_{[0]}\right)\\
&= \varphi(m_{[0]}\mathcal{S}^{-1}(\xi^{(2)}{}_{(-1)}) \otimes n_{[0]}) \quad \text{(by (1.34))}\\
&\qquad \times \left\langle \xi^{(1)}, \vartheta\left(\xi^{(2)}{}_{(-2)}\mathcal{S}(m_{[1]})\right)\right\rangle \langle \xi^{(2)}{}_{(0)}, \vartheta\mathcal{S}(n_{[1]})\rangle.
\end{aligned}$$

Hence by (5.7) we obtain

$$\begin{aligned}\varphi(\xi(m\otimes n)) &= \varphi(m_{[0]}\mathcal{S}^{-1}(\pi(\mathcal{S}(n_{[1]})n_{[5]}))\otimes n_{[0]})\\ &\quad\times\langle\xi^{(1)},\vartheta(\pi(\mathcal{S}(n_{[2]})n_{[4]})\mathcal{S}(m_{[1]}))\rangle\langle\xi^{(2)},\vartheta\mathcal{S}(n_{[3]})\rangle\\ &= m_{[0]}\pi(\mathcal{S}^{-1}(n_{[6]})n_{[2]})\pi\mathcal{S}^{-1}(n_{[1]})\otimes n_{[0]}\quad(\text{by (2.6)})\\ &\quad\times\langle\xi,\vartheta\mathcal{S}(n_{[4]})\vartheta(\pi(\mathcal{S}(n_{[3]})n_{[5]})\mathcal{S}(m_{[1]}))\rangle\\ &= m_{[0]}\pi\mathcal{S}^{-1}(n_{[4]})\otimes n_{[0]}\\ &\quad\times\left\langle\xi,\vartheta\mathcal{S}(n_{[2]})\vartheta(\pi(\mathcal{S}(n_{[1]})n_{[3]})\mathcal{S}(m_{[1]}))\right\rangle\\ &= m_{[0]}\pi\mathcal{S}^{-1}(n_{[3]})\otimes n_{[0]}\langle\xi,\vartheta\mathcal{S}(m_{[1]}\pi\mathcal{S}^{-1}(n_{[2]})n_{[1]})\rangle,\end{aligned}$$

where the last equality follows from Lemma 1.1 and from (1.35).

On the other hand

$$\begin{aligned}\xi\varphi(m\otimes n) &= \xi(m\pi\mathcal{S}^{-1}(n_{[1]})\otimes n_{[0]})\\ &= \langle\xi,\vartheta\mathcal{S}(\pi(n_{[4]})m_{[1]}\pi\mathcal{S}^{-1}(n_{[2]})n_{[1]})\rangle m_{[0]}\pi\mathcal{S}^{-1}(n_{[3]})\otimes n_{[0]}\\ &= m_{[0]}\pi\mathcal{S}^{-1}(n_{[3]})\otimes n_{[0]}\langle\xi,\vartheta\mathcal{S}(m_{[1]}\pi\mathcal{S}^{-1}(n_{[2]})n_{[1]})\rangle\quad(\text{by (1.34)}).\end{aligned}$$

Hence $\varphi(\xi(m\otimes n))=\xi\varphi(m\otimes n)$.

It is easy to check that the diagrams (3.2) commute for (F,φ). Hence (F,φ) is a monoidal functor. □

6. The second isomorphism

In this section we assume that $(R,R^\vee)$ is a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$ with bijective antipodes and bilinear form $\langle,\rangle$. The monoidal isomorphism $(F,\varphi):\mathcal{YD}^{R\#H}_H\to{}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}}$ of Theorem 5.5 induces by Lemma 3.2 a braided monoidal isomorphism between the centers

$$(F^{\mathcal{Z}},\varphi^{\mathcal{Z}}):\mathcal{Z}(\mathcal{YD}^{R\#H}_H)\to\mathcal{Z}({}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}}).$$

Assume that ${}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}}$ is $R^\vee\#H$-faithful. By Propositions 4.8 and 4.4, the functors

$$\begin{aligned}&\Psi:{}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}\to\mathcal{Z}(\mathcal{YD}^{R\#H}_H),\\ &\Phi:{}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}\to\mathcal{Z}({}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}})\end{aligned}$$

are fully faithful, strict monoidal and braided. The functor Ψ is defined with respect to the Hopf algebra inclusion $\iota:H\to R\#H$. We denote the image of $M\in{}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$ in $\mathcal{YD}^{R\#H}_H$ defined by restriction by M_{res}. The functor Φ is defined with respect to the Hopf algebra projection $\pi:R^\vee\#H\to H$, and we denote the image of $M\in{}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ in ${}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}}$ by ${}^\pi M$.

Our goal is to show in Theorem 6.5 that (F,φ) induces a braided monoidal isomorphism

$${}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}\to{}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}.$$

Let $G:{}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}}\to\mathcal{YD}^{R\#H}_H$ be the inverse functor of the isomorphism F of Theorem 5.5. Then $(G,\psi):{}_{R^\vee\#H}{}^H\mathcal{YD}_{\mathrm{rat}}\to\mathcal{YD}^{R\#H}_H$ is a monoidal isomorphism, where ψ is defined by (3.5). We first construct functors

$$\widetilde{F}:{}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}\to{}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}},\qquad\widetilde{G}:{}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}\to{}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$$

such that the diagrams

$$\begin{array}{ccc} {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H} & \xrightarrow{\widetilde{F}} & {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}} \\ \Psi\big\downarrow & & \Phi\big\downarrow \\ \mathcal{Z}(\mathcal{YD}^{R\#H}_{H}) & \xrightarrow{F^{\mathcal{Z}}} & \mathcal{Z}({}^{H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}) \end{array} \tag{6.1}$$

and

$$\begin{array}{ccc} {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}} & \xrightarrow{\widetilde{G}} & {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H} \\ \Phi\big\downarrow & & \Psi\big\downarrow \\ \mathcal{Z}({}^{H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}) & \xrightarrow{G^{\mathcal{Z}}} & \mathcal{Z}(\mathcal{YD}^{R\#H}_{H}) \end{array} \tag{6.2}$$

commute.

The existence of $\widetilde{F}$ will follow from the next two lemmas.

Lemma 6.1. *Let* $(F^{\mathcal{Z}}, \varphi^{\mathcal{Z}}) : \mathcal{Z}(\mathcal{YD}^{R\#H}_{H}) \to \mathcal{Z}({}^{H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}})$ *be the monoidal isomorphism induced by the isomorphism* (F, φ) *of Theorem 5.5. Let* $M \in {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$*, and* $\Psi(M) = (M_{\mathrm{res}}, \gamma)$*, where* $\gamma = c_M$ *is defined in (4.14). Then*

$$F^{\mathcal{Z}}\Psi(M) = (F(M_{\mathrm{res}}), \widetilde{\gamma}),$$

and $\widetilde{\gamma}_{F(X)} : F(M_{\mathrm{res}}) \otimes F(X) \to F(X) \otimes F(M_{\mathrm{res}})$ *is given by*

$$\widetilde{\gamma}_{F(X)}(m \otimes x) = x_{[0]}\pi\left(\mathcal{S}(x_{[1]})x_{[4]}m_{[1]}\right) \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[3]})x_{[2]} \tag{6.3}$$

for all $X \in \mathcal{YD}^{R\#H}_{H}$*,* $x \in X$ *and* $m \in M$*.*

Proof. Let $X \in \mathcal{YD}^{R\#H}_{H}$ with comodule structure

$$X \to X \otimes R\#H, \qquad x \mapsto x_{[0]} \otimes x_{[1]}.$$

Recall that $\widetilde{\gamma}_{F(X)} = {\varphi_{X,M_{\mathrm{res}}}}^{-1} F(c_{M,X})\varphi_{M_{\mathrm{res}},X}$ by (3.13). It follows from the definition of $\varphi_{X,M_{\mathrm{res}}}$ in Theorem 5.5 that

$${\varphi_{X,M_{\mathrm{res}}}}^{-1}(x \otimes m) = x\pi(m_{[1]}) \otimes m_{[0]} \tag{6.4}$$

for all $x \in X, m \in M$. Hence

$$\begin{aligned} \widetilde{\gamma}_{F(X)}(m \otimes x) &= {\varphi_{X,M_{\mathrm{res}}}}^{-1} F(c_{M,X})\varphi_{M_{\mathrm{res}},X}(m \otimes x) \\ &= {\varphi_{X,M_{\mathrm{res}}}}^{-1} F(c_{M,X})(m\pi\mathcal{S}^{-1}(x_{[1]}) \otimes x_{[0]}) \\ &= {\varphi_{X,M_{\mathrm{res}}}}^{-1}\left(x_{[0]} \otimes m\pi\mathcal{S}^{-1}(x_{[2]})x_{[1]}\right) \\ &= x_{[0]}\pi\left(\mathcal{S}\left(\left(\pi\mathcal{S}^{-1}(x_{[2]})x_{[1]}\right)_{[1]}\right) m_{[1]}\left(\pi\mathcal{S}^{-1}(x_{[2]})x_{[1]}\right)_{[3]}\right) \\ &\quad \otimes m_{[0]}\left(\pi\mathcal{S}^{-1}(x_{[2]})x_{[1]}\right)_{[2]} \\ &= x_{[0]}\pi\left(\mathcal{S}(\mathcal{S}^{-1}(x_{[6]})x_{[1]})m_{[1]}\mathcal{S}^{-1}(x_{[4]})x_{[3]}\right) \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[5]})x_{[2]} \\ &= x_{[0]}\pi\left(\mathcal{S}(x_{[1]})x_{[4]}m_{[1]}\right) \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[3]})x_{[2]}. \qquad \square \end{aligned}$$

In the next lemma we define a map $\delta_{\widetilde{F}(M)}$ which will be the coaction of $R\#H$ on $\widetilde{F}(M)$ in Theorem 6.5.

Lemma 6.2. *Let $M \in {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$. We denote the left H-comodule structure of $F(M_{\mathrm{res}})$ by $M \to H \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}$. Define a linear map*

$$\delta_M^{R^\vee} : M \to R^\vee \otimes M, \qquad m \mapsto m^{\langle -1\rangle} \otimes m^{\langle 0\rangle},$$

by the equation

$$mr = \langle r, \mathcal{S}^{-1}_{R^\vee}(\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle})\rangle' {m^{\langle 0\rangle}}_{(0)} \tag{6.5}$$

for all $r \in R, m \in M$. Let

$$\delta_{\widetilde{F}(M)} : M \to R^\vee\#H \otimes M, \qquad m \mapsto m^{[-1]} \otimes m^{[0]} = m^{\langle -1\rangle}{m^{\langle 0\rangle}}_{(-1)} \otimes {m^{\langle 0\rangle}}_{(0)}. \tag{6.6}$$

Then the following hold.

(1) *For all $m \in M, a \in R\#H$,*
$$\langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}, \vartheta\mathcal{S}(a)\rangle {m^{\langle 0\rangle}}_{(0)} = m\pi\mathcal{S}^{-1}(a_{(2)})a_{(1)}.$$
(2) *Let $X \in \mathcal{YD}^{R\#H}_H$, and let $\widetilde{\gamma}_{F(X)} : F(M_{\mathrm{res}}) \otimes F(X) \to F(X) \otimes F(M_{\mathrm{res}})$ be the isomorphism in ${}^{H}_{R^\vee\#H}\mathcal{YD}$ defined in Lemma 6.1. Then for all $x \in X$ and $m \in M$, $m^{[-1]}x \otimes m^{[0]} = \widetilde{\gamma}_{F(X)}(m \otimes x)$.*
(3) *For all $m \in M$, $\pi(m^{[-1]}) \otimes m^{[0]} = m_{(-1)} \otimes m_{(0)}$.*

Proof. (1) The map $\delta_M^{R^\vee}$ is well-defined since M is a rational right R-module, $\langle , \rangle$ is non-degenerate, and the maps $\mathcal{S}_{R^\vee}$ and

$$R^\vee \otimes M \to R^\vee \otimes M, \qquad \xi \otimes m \mapsto \mathcal{S}^{-1}(m_{(-1)}) \cdot \xi \otimes m_{(0)},$$

are bijective.

Note that if (1) holds for $a \in R\#H$ then it holds for ha for all $h \in H$. Thus it is enough to assume in (1) that $a \in \mathcal{S}^{-1}(R)$. For all $r \in R$ and $a = \mathcal{S}^{-1}(r)$,

$$\pi\mathcal{S}^{-1}(a_{(2)})a_{(1)} = \pi\mathcal{S}^{-2}(r_{(1)})\mathcal{S}^{-1}(r_{(2)}) = \mathcal{S}_R(\mathcal{S}^{-2}(r))$$

by (1.27). Therefore (1) is equivalent to

$$\langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}, r\rangle {m^{\langle 0\rangle}}_{(0)} = m\mathcal{S}_R(\mathcal{S}^{-2}(r))$$

for all $r \in R, m \in M$. This last equation holds by our definition of $\delta_M^{R^\vee}$ since

$$\begin{aligned} m\mathcal{S}_R(\mathcal{S}^{-2}(r)) &= \langle \mathcal{S}_R(\mathcal{S}^{-2}(r)), \mathcal{S}^{-1}_{R^\vee}(\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle})\rangle' {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (6.5))} \\ &= \langle \mathcal{S}^{-2}(r), \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}\rangle' {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (2.10))} \\ &= \langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}, r\rangle {m^{\langle 0\rangle}}_{(0)}. \end{aligned}$$

Here, we used that by Lemma 2.7, $(R^\vee, R)$ together with $\langle , \rangle' : R \otimes R^\vee \to \Bbbk$ is a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$.

(2) Let $X \in \mathcal{YD}^{R\#H}_H$. By Lemma 6.1 we have to show that

$$m^{[-1]}x \otimes m^{[0]} = x_{[0]}\pi(\mathcal{S}(x_{[1]})x_{[4]}m_{[1]}) \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[3]})x_{[2]} \tag{6.7}$$

for all $x \in X, m \in M$.

By (6.6) and (6.5), the left-hand side of (6.7) can be written as

$$\begin{aligned} m^{[-1]}x \otimes m^{[0]} &= m^{\langle -1\rangle}({m^{\langle 0\rangle}}_{(-1)}x) \otimes {m^{\langle 0\rangle}}_{(0)} \\ &= m^{\langle -1\rangle}(x\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)})) \otimes {m^{\langle 0\rangle}}_{(0)} \\ &= \langle m^{\langle -1\rangle}, \vartheta\mathcal{S}({m^{\langle 0\rangle}}_{(-1)}x_{[1]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-3)}))\rangle x_{[0]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-2)}) \otimes {m^{\langle 0\rangle}}_{(0)} \\ &= \langle m^{\langle -1\rangle}, \vartheta\mathcal{S}(x_{[1]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-2)}))\rangle x_{[0]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \otimes {m^{\langle 0\rangle}}_{(0)}, \end{aligned}$$

where the last equality follows from (1.34). Thus (6.7) is equivalent to the equation

$$\begin{aligned} &\langle m^{\langle -1\rangle}, \vartheta\mathcal{S}(x_{[1]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-2)}))\rangle x_{[0]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \otimes {m^{\langle 0\rangle}}_{(0)} \\ &\quad = x_{[0]}\pi(\mathcal{S}(x_{[1]})x_{[4]}m_{[1]}) \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[3]})x_{[2]} \end{aligned} \tag{6.8}$$

for all $x \in X, m \in M$.

To simplify (6.8) we apply the isomorphism

$$X \otimes M \to X \otimes M, \qquad x \otimes m \mapsto x\mathcal{S}^{-2}(m_{(-1)}) \otimes m_{(0)}. \tag{6.9}$$

Under the isomorphism (6.9) the left-hand side of (6.8) becomes

$$\begin{aligned} &\langle m^{\langle -1\rangle}, \vartheta\mathcal{S}(x_{[1]}\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}))\rangle x_{[0]} \otimes {m^{\langle 0\rangle}}_{(0)} \\ &\quad = \langle m^{\langle -1\rangle}, \vartheta({m^{\langle 0\rangle}}_{(-1)}\mathcal{S}(x_{[1]}))\rangle x_{[0]} \otimes {m^{\langle 0\rangle}}_{(0)} \\ &\quad = \langle m^{\langle -1\rangle}, {m^{\langle 0\rangle}}_{(-1)} \cdot \vartheta\mathcal{S}(x_{[1]})\rangle x_{[0]} \otimes {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (1.35))} \\ &\quad = \langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}, \vartheta\mathcal{S}(x_{[1]})\rangle x_{[0]} \otimes {m^{\langle 0\rangle}}_{(0)}, \quad \text{(by (2.4))} \end{aligned}$$

and the right-hand side equals

$$\begin{aligned} &x_{[0]}\pi\left(\mathcal{S}(x_{[1]})x_{[4]}m_{[1]}\right)\mathcal{S}^{-2}\mathcal{S}\pi\left((m_{[0]}\pi\mathcal{S}^{-1}(x_{[3]})x_{[2]})_{[1]}\right) \otimes (m_{[0]}\pi\mathcal{S}^{-1}(x_{[3]})x_{[2]})_{[0]} \\ &\quad = x_{[0]}\pi\left(\mathcal{S}(x_{[1]})x_{[8]}m_{[2]}\right)\mathcal{S}^{-1}\pi\left(\mathcal{S}\left(\pi\mathcal{S}^{-1}(x_{[7]})x_{[2]}\right)m_{[1]}\pi\mathcal{S}^{-1}(x_{[5]})x_{[4]}\right) \\ &\qquad \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[6]})x_{[3]} \\ &\quad = x_{[0]}\pi\left(\mathcal{S}(x_{[1]})x_{[8]}m_{[2]}\right)\mathcal{S}^{-1}\pi\left(\mathcal{S}(x_{[2]})x_{[7]}m_{[1]}\mathcal{S}^{-1}(x_{[5]})x_{[4]}\right) \otimes m_{[0]}\pi\mathcal{S}^{-1}(x_{[6]})x_{[3]} \\ &\quad = x_{[0]} \otimes m\pi\mathcal{S}^{-1}(x_{[2]})x_{[1]}. \end{aligned}$$

Thus the claim follows from (1).

(3) Let $m \in M$. By (6.5) and (2.6), $m = m1 = \varepsilon(m^{\langle -1\rangle})m^{\langle 0\rangle}$. Hence

$$\pi(m^{[-1]}) \otimes m^{[0]} = \varepsilon(m^{\langle -1\rangle}){m^{\langle 0\rangle}}_{(-1)} \otimes {m^{\langle 0\rangle}}_{(0)} = m_{(-1)} \otimes m_{(0)}. \quad \square$$

The existence of $\widetilde{G}$ will follow from the next two lemmas.

Let $M \in {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$. We denote the left $R^\vee\#H$-comodule structure of M by

$$M \to R^\vee\#H \otimes M, \qquad m \mapsto m^{[-1]} \otimes m^{[0]} = m^{\langle -1\rangle}{m^{\langle 0\rangle}}_{(-1)} \otimes {m^{\langle 0\rangle}}_{(0)},$$

where $M \to R^\vee \otimes M, m \mapsto m^{\langle -1\rangle} \otimes m^{\langle 0\rangle}$, is the $R^\vee$-comodule structure of M. For all objects $X \in {}_{R^\vee\#H}^{H}\mathcal{YD}_{\mathrm{rat}}$ the right $R\#H$-comodule structure of $G(X)$ is denoted by

$$X \to X \otimes R\#H, \qquad x \mapsto x_{[0]} \otimes x_{[1]}.$$

Note that $G(X) = X$ as a vector space. The right H-module structure of $G(X)$ is defined by

$$xh = \mathcal{S}(h)x \tag{6.10}$$

for all $x \in X, h \in H$. Since $FG(^\pi M) =^\pi M$, it follows that

$$\pi\mathcal{S}(m_{[1]}) \otimes m_{[0]} = \pi(m^{[-1]}) \otimes m^{[0]} \tag{6.11}$$

for all $m \in M$.

Lemma 6.3. *Let $(G^{\mathcal{Z}}, \psi^{\mathcal{Z}}) : \mathcal{Z}({}^H_{R^\vee\#H}\mathcal{YD}_{\text{rat}}) \to \mathcal{Z}(\mathcal{YD}^{R\#H}_H)$ be the monoidal isomorphism induced by the monoidal isomorphism (G, ψ). Let $M \in {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\text{rat}}$, and $\Phi(M) = ({}^\pi M, \gamma)$, where $\gamma = c_M$ is defined in (4.4). Then*

$$G^{\mathcal{Z}}\Phi(M) = (G({}^\pi M), \widetilde{\gamma}),$$

and $\widetilde{\gamma}_{G(X)} : G({}^\pi M) \otimes G(X) \to G(X) \otimes G({}^\pi M)$ is given by

$$\widetilde{\gamma}_{G(X)}(m \otimes x) = \left(\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}\pi\mathcal{S}^2(x_{[1]})) \cdot m^{\langle -1\rangle}\right) x_{[0]} \otimes \pi\mathcal{S}(x_{[2]}){m^{\langle 0\rangle}}_{(0)} \tag{6.12}$$

for all $X \in {}^H_{R^\vee\#H}\mathcal{YD}_{\text{rat}}$, $x \in X$ and $m \in M$.

Proof. Let $X \in {}^H_{R^\vee\#H}\mathcal{YD}_{\text{rat}}$. By (4.4), $\gamma_X : {}^\pi M \otimes X \to X \otimes {}^\pi M$ is defined by

$$\gamma_X(m \otimes x) = m^{[-1]}x \otimes m^{[0]}$$

for all $x \in X, m \in M$.

By (3.5) and (3.13), the isomorphism $\widetilde{\gamma}_{G(X)}$ is defined by the equation

$$\widetilde{\gamma}_{G(X)}G(\varphi_{G({}^\pi M),G(X)}) = G(\varphi_{G(X),G({}^\pi M)})G(\gamma_X). \tag{6.13}$$

We apply both sides of (6.13) to an element $m \otimes x$, $m \in M$, $x \in X$. Then

$$\widetilde{\gamma}_{G(X)}G(\varphi_{G({}^\pi M),G(X)})(m \otimes x) = \widetilde{\gamma}_{G(X)}(m\pi\mathcal{S}^{-1}(x_{[1]}) \otimes x_{[0]}),$$

and

$$\begin{aligned} G(\varphi_{G(X),G({}^\pi M)})G(\gamma_X)(m \otimes x) &= (m^{[-1]}x)\pi\mathcal{S}^{-1}({m^{[0]}}_{[1]}) \otimes {m^{[0]}}_{[0]} \\ &= \pi({m^{[0]}}_{[1]})(m^{[-1]}x) \otimes {m^{[0]}}_{[0]} \quad \text{(by (6.10))} \\ &= \pi\mathcal{S}^{-1}(m^{[-1]})m^{[-2]}x \otimes m^{[0]} \quad \text{(by (6.11))} \\ &= \left(\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}\right) x \otimes {m^{\langle 0\rangle}}_{(0)}, \end{aligned}$$

where in the proof of the last equality the following formula in $R^\vee\#H$ is used for $a = m^{[-1]} = m^{\langle -1\rangle}{m^{\langle 0\rangle}}_{(-1)}$. Let $\xi \in R^\vee$, $h \in H$ and $a = \xi h \in R^\vee\#H$. Then

$$\begin{aligned} \pi\mathcal{S}^{-1}(a_{(2)})a_{(1)} &= \pi\mathcal{S}^{-1}({\xi^{(2)}}_{(0)}h_{(2)})\xi^{(1)}{\xi^{(2)}}_{(-1)}h_{(1)} \\ &= \mathcal{S}^{-1}(h_{(2)})\xi h_{(1)} \\ &= \mathcal{S}^{-1}(h) \cdot \xi. \end{aligned}$$

We have shown that

$$\widetilde{\gamma}_{G(X)}(m\pi\mathcal{S}^{-1}(x_{[1]}) \otimes x_{[0]}) = \left(\mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}\right) x \otimes {m^{\langle 0\rangle}}_{(0)}. \tag{6.14}$$

Since $m \otimes x = m\pi(x_{[2]})\pi\mathcal{S}^{-1}(x_{[1]}) \otimes x_{[0]} = (\pi\mathcal{S}(x_{[2]})m)\pi\mathcal{S}^{-1}(x_{[1]}) \otimes x_{[0]}$, we obtain from (6.14) and the Yetter–Drinfeld condition for M

$$\begin{aligned}\widetilde{\gamma}_{G(X)}(m \otimes x) &= \left(\mathcal{S}^{-1}\left((\pi\mathcal{S}(x_{[1]})m)^{\langle 0\rangle}{}_{(-1)}\right) \cdot \left(\pi\mathcal{S}(x_{[1]})m\right)^{\langle -1\rangle}\right) x_{[0]} \\ &\quad \otimes (\pi\mathcal{S}(x_{[1]})m)^{\langle 0\rangle}{}_{(0)} \\ &= \left(\mathcal{S}^{-1}\left((\pi\mathcal{S}(x_{[1]})m^{\langle 0\rangle})_{(-1)}\right) \cdot \pi\mathcal{S}(x_{[2]})m^{\langle -1\rangle}\right) x_{[0]} \\ &\quad \otimes \left(\pi\mathcal{S}(x_{[1]})m^{\langle 0\rangle}\right)_{(0)} \\ &= \left(\mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}\pi\mathcal{S}^2(x_{[1]})) \cdot m^{\langle -1\rangle}\right) x_{[0]} \otimes \pi\mathcal{S}(x_{[2]})m^{\langle 0\rangle}{}_{(0)}. \quad \square\end{aligned}$$

Lemma 6.4. *Let $M \in {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$, and $G^{\mathcal{Z}}\Phi(M) = (G({}^\pi M), \widetilde{\gamma})$ as in Lemma 6.3. Let $\langle , \rangle' : R \otimes R^\vee \to \Bbbk$ be the form defined in (2.16). Define a linear map $\mu_{\widetilde{G}(M)} : M \otimes R\#H \to M$ by*

$$\mu_{\widetilde{G}(M)}(m \otimes a) = \langle m^{\langle -1\rangle}, m^{\langle 0\rangle}{}_{(-1)} \cdot \vartheta(\pi\mathcal{S}^2(a_{(2)})\mathcal{S}(a_{(1)}))\rangle \pi\mathcal{S}(a_{(3)})m^{\langle 0\rangle}{}_{(0)} \tag{6.15}$$

for all $m \in M$, $a \in R\#H$. Then the following hold.

(1) *For all $X \in {}_{R^\vee\#H}^{H}\mathcal{YD}_{\mathrm{rat}}$, $x \in X$ and $m \in M$,*

$$\widetilde{\gamma}_{G(X)}(m \otimes x) = x_{[0]} \otimes \mu_{\widetilde{G}(M)}(m \otimes x_{[1]}).$$

(2) *For all $m \in M$ and $h \in H$, $\mu_{\widetilde{G}(M)}(m \otimes h) = mh$.*

(3) *For all $m \in M$ and $r \in R$, $\mu_{\widetilde{G}(M)}(m \otimes r) = \langle r, \vartheta\mathcal{S}^{-1}(m^{[-1]})\rangle' m^{[0]}$.*

Proof. (1) Let $X \in {}_{R^\vee\#H}^{H}\mathcal{YD}_{\mathrm{rat}}$, $x \in X$ and $m \in M$. Then by (6.12),

$$\begin{aligned}\widetilde{\gamma}_{G(X)}(m \otimes x) &= (\mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}\pi\mathcal{S}^2(x_{[1]})) \cdot m^{\langle -1\rangle})x_{[0]} \otimes \pi\mathcal{S}(x_{[2]})m^{\langle 0\rangle}{}_{(0)} \\ &= \langle \mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}\pi\mathcal{S}^2(x_{[2]})) \cdot m^{\langle -1\rangle}, \vartheta\mathcal{S}(x_{[1]})\rangle x_{[0]} \otimes \pi\mathcal{S}(x_{[3]})m^{\langle 0\rangle}{}_{(0)} \\ &= x_{[0]} \otimes \langle m^{\langle -1\rangle}, m^{\langle 0\rangle}{}_{(-1)} \cdot \vartheta(\pi\mathcal{S}^2(x_{[2]})\mathcal{S}(x_{[1]}))\rangle \pi\mathcal{S}(x_{[3]})m^{\langle 0\rangle}{}_{(0)} \\ &= x_{[0]} \otimes \mu_{\widetilde{G}(M)}(m \otimes x_{[1]}),\end{aligned}$$

where we used (5.5) and the equality $X = FG(X)$ together with (1.35) and (2.4).

(2) Let $m \in M$ and $h \in H$. Then

$$\begin{aligned}\mu_{\widetilde{G}(M)}(m \otimes h) &= \langle m^{\langle -1\rangle}, m^{\langle 0\rangle}{}_{(-1)} \cdot \vartheta(\pi\mathcal{S}^2(h_{(2)})\mathcal{S}(h_{(1)}))\rangle \pi\mathcal{S}(h_{(3)})m^{\langle 0\rangle}{}_{(0)} \\ &= \langle m^{\langle -1\rangle}, m^{\langle 0\rangle}{}_{(-1)} \cdot 1\rangle \pi\mathcal{S}(h)m^{\langle 0\rangle}{}_{(0)} \\ &= \langle m^{\langle -1\rangle}, 1\rangle \pi\mathcal{S}(h)m^{\langle 0\rangle} \quad \text{(by (2.7))} \\ &= \pi\mathcal{S}(h)m \\ &= mh.\end{aligned}$$

(3) Let $m \in M$ and $r \in R$. Then $r_{(1)} \otimes \pi(r_{(2)}) = r \otimes 1$. Hence

$$\begin{aligned}
\mu_{\widetilde{G}(M)}(m \otimes r) &= \langle m^{\langle -1\rangle}, {m^{\langle 0\rangle}}_{(-1)} \cdot \vartheta(\mathcal{S}(r))\rangle {m^{\langle 0\rangle}}_{(0)} \\
&= \langle m^{\langle -1\rangle}, {m^{\langle 0\rangle}}_{(-1)} \cdot \vartheta(\mathcal{S}(r_{(-1)})\mathcal{S}_R(r_{(0)}))\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (1.21))} \\
&= \langle m^{\langle -1\rangle}, ({m^{\langle 0\rangle}}_{(-1)}\mathcal{S}(r_{(-1)})) \cdot \mathcal{S}_R(r_{(0)})\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (1.35))} \\
&= \langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot m^{\langle -1\rangle}, \mathcal{S}(r_{(-1)}) \cdot \mathcal{S}_R(r_{(0)})\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (2.4))} \\
&= \langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot \mathcal{S}_{R^\vee}^{-1}(m^{\langle -1\rangle}), \mathcal{S}(r_{(-1)}) \cdot \mathcal{S}_R^2(r_{(0)})\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (2.10))} \\
&= \langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot \mathcal{S}_{R^\vee}^{-1}(m^{\langle -1\rangle}), \mathcal{S}^2(r)\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (1.22))} \\
&= \langle \mathcal{S}^{-1}({m^{\langle 0\rangle}}_{(-1)}) \cdot \vartheta\mathcal{S}^{-1}(m^{\langle -1\rangle}), \mathcal{S}^2(r)\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (1.36))} \\
&= \langle \vartheta\mathcal{S}^{-1}(m^{\langle -1\rangle}{m^{\langle 0\rangle}}_{(-1)}), \mathcal{S}^2(r)\rangle {m^{\langle 0\rangle}}_{(0)} \quad \text{(by (1.35))} \\
&= \langle r, \vartheta\mathcal{S}^{-1}(m^{[-1]})\rangle' m^{[0]}. \quad \square
\end{aligned}$$

Theorem 6.5. *Let $(R, R^\vee)$ be a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$ with bijective antipodes and bilinear form $\langle , \rangle : R^\vee \otimes R \to \Bbbk$. Let $\langle , \rangle' : R \otimes R^\vee \to \Bbbk$ be the form defined in (2.16). Assume that ${}_{R^\vee\#H}^{H}\mathcal{YD}_{\mathrm{rat}}$ is $R^\vee\#H$-faithful.*

Then the functor

$$(\widetilde{F}, \widetilde{\varphi}) : {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$$

as defined below is a braided monoidal isomorphism.

For any object $M \in {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$ with right $R\#H$-comodule structure denoted by

$$\delta_M : M \to M \otimes R\#H, \qquad m \mapsto m_{[0]} \otimes m_{[1]},$$

let $\widetilde{F}(M) = M$ as a vector space and $\widetilde{F}(M) \in {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ with left H-action, H-coaction $\delta^H_{\widetilde{F}(M)}$, $R^\vee$-action, and $R^\vee\#H$-coaction

$$\delta_{\widetilde{F}(M)} : M \to R^\vee\#H \otimes M, \qquad m \mapsto m^{[-1]} \otimes m^{[0]},$$

respectively, given by

$$hm = m\mathcal{S}^{-1}(h), \tag{6.16}$$

$$\delta^H_{\widetilde{F}(M)}(m) = \pi\mathcal{S}(m_{[1]}) \otimes m_{[0]}, \tag{6.17}$$

$$\xi m = \langle \xi, \vartheta\mathcal{S}(m_{[1]})\rangle m_{[0]}, \tag{6.18}$$

$$mr = \langle r, \vartheta\mathcal{S}^{-1}(m^{[-1]})\rangle' m^{[0]} \tag{6.19}$$

for all $h \in H, m \in M, \xi \in R^\vee$ and $r \in R$. For any morphism f in ${}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$ let $\widetilde{F}(f) = f$. The natural transformation $\widetilde{\varphi}$ is defined by

$$\widetilde{\varphi}_{M,N} : \widetilde{F}(M) \otimes \widetilde{F}(N) \to \widetilde{F}(M \otimes N), \tag{6.20}$$

$$m \otimes n \mapsto m\pi\mathcal{S}^{-1}(n_{[1]}) \otimes n_{[0]} = \pi\mathcal{S}^{-1}(n^{[-1]})m \otimes n^{[0]}, \tag{6.21}$$

for all $M, N \in {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$.

Proof. Let $M \in {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$. As in Lemma 6.1 we write $\Psi(M) = (M_{\mathrm{res}}, \gamma)$. Then

$$F^{\mathcal{Z}}\Psi(M) = (F(M_{\mathrm{res}}), \widetilde{\gamma}).$$

By Lemma 5.4, the definitions of $\delta_{\widetilde{F}(M)}$ in Lemma 6.2 and in (6.19) coincide. Thus, by Lemma 6.2(2), for all $X \in \mathcal{YD}^{R\#H}_{H}$, the isomorphism

$$\widetilde{\gamma}_{F(X)} : F(M_{\mathrm{res}}) \otimes F(X) \to F(X) \otimes F(M_{\mathrm{res}})$$

has the form

$$\widetilde{\gamma}_{F(X)}(m \otimes x) = m^{[-1]}x \otimes m^{[0]}$$

for all $m \in M, x \in X$, where $\delta_{\widetilde{F}(M)}(m) = m^{[-1]} \otimes m^{[0]}$ is defined in Lemma 6.2. By Lemma 6.2(3), the left H-comodule structure of $F(M_{\mathrm{res}})$ is $(\pi \otimes \mathrm{id})\delta_{\widetilde{F}(M)}$. The left H-action, H-coaction and $R^\vee\#H$-action of $\widetilde{F}(M)$ are those of $F(M_{\mathrm{res}})$, see Theorem 5.5.

We now conclude from Proposition 4.4 that $\widetilde{F}(M)$ with $R^\vee\#H$-comodule structure $\delta_{\widetilde{F}(M)}$ is an object in ${}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$, and $\Phi(\widetilde{F}(M)) = F^{\mathcal{Z}}\Psi(M)$.

Thus we have defined a functor $\widetilde{F} : {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ such that the diagram (6.1) commutes. By Lemma 3.1 there is a uniquely determined family $\widetilde{\varphi}$ such that $(\widetilde{F}, \widetilde{\varphi})$ is a braided monoidal functor with

$$(F^{\mathcal{Z}}, \varphi^{\mathcal{Z}})(\Psi, \mathrm{id}) = (\Phi, \mathrm{id})(\widetilde{F}, \widetilde{\varphi}).$$

Let $M, N \in {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$. Then $\Phi(\widetilde{\varphi}_{M,N}) = \varphi^{\mathcal{Z}}_{\Psi(M),\,\Psi(N)}$ by (3.4), that is, for all $m \in M, n \in N$,

$$\widetilde{\varphi}_{M,N}(m \otimes n) = \varphi_{M_{\mathrm{res}},N_{\mathrm{res}}}(m \otimes n) = m\pi\mathcal{S}^{-1}(n_{[1]}) \otimes n_{[0]}$$

by Theorem 5.5. To define the inverse functor of $\widetilde{F}$ let $M \in {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$. Let $\widetilde{G}(M) = M$ as a vector space with right $R\#H$-comodule structure and H-module structure given by ${}^{\pi}M$, and with right $R\#H$-module structure $\mu_{\widetilde{G}}$ defined in (6.15). Then $\widetilde{G}(M) \in \mathcal{YD}^{R\#H}_{R\#H}$ by Proposition 4.8 and Lemma 6.4(1), (2). It follows from Lemma 6.4(3) that $\widetilde{G}(M)$ is rational as an R-module. We let $\widetilde{G}(f) = f$ for morphisms in ${}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$.

Thus we have defined a functor $\widetilde{G} : {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}} \to {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}$, and it is clear form the explicit definitions of $\widetilde{F}$ and $\widetilde{G}$ that $\widetilde{F}\widetilde{G} = \mathrm{id}$ and $\widetilde{G}\widetilde{F} = \mathrm{id}$. □

7. The third isomorphism

Finally we compose the isomorphism in Theorem 6.5 with the isomorphism in Lemma 4.9.

We recall from Lemmas 5.2 and 5.3 the description of left modules and left comodules over $R\#H$, where R is a Hopf algebra in ${}^H_H\mathcal{YD}$. In particular, the restriction of an object $M \in {}^{R\#H}_{R\#H}\mathcal{YD}$ with $R\#H$-comodule structure δ_M is an object in ${}^H_H\mathcal{YD}$, where the H-action is defined by restriction and the H-coaction is $(\pi \otimes \mathrm{id})\delta_M$.

Theorem 7.1. *Let $(R, R^\vee)$ be a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$ with bijective antipodes and with bilinear form $\langle\,,\rangle : R^\vee \otimes R \to \Bbbk$. Assume that ${}^{\;H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ is $R^\vee\#H$-faithful.*

Then the functor

$$(\Omega, \omega) : {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$$

as defined below is a braided monoidal isomorphism.

Let $M \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$ *with left* R*-comodule structure denoted by*

$$\delta_M^R : M \to R \otimes M, \qquad m \mapsto m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}.$$

Let $\Omega(M) = M$ *as an object in* ${}^H_H\mathcal{YD}$ *by restriction, and* $\Omega(M) \in {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ *with* $R^\vee$*-action and* $R^\vee$*-coaction* $\delta^{R^\vee}_{\Omega(M)}$*, respectively, given by*

$$\xi m = \langle \xi, m_{\langle -1\rangle}\rangle m_{\langle 0\rangle}, \tag{7.1}$$

$$\delta^{R^\vee}_{\Omega(M)}(m) = c^2_{R^\vee,M}(m^{\langle\langle -1\rangle\rangle} \otimes m^{\langle\langle 0\rangle\rangle}), \tag{7.2}$$

where

$$rm = \langle m^{\langle\langle -1\rangle\rangle}, \theta_R(r)\rangle m^{\langle\langle 0\rangle\rangle} \tag{7.3}$$

for all $m \in M, \xi \in R^\vee$ *and* $r \in R$*. For any morphism* f *in* ${}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$ *let* $\Omega(f) = f$*. The natural transformation* ω *is defined by*

$$\omega_{M,N} : \Omega(M) \otimes \Omega(N) \to \Omega(M \otimes N), \qquad m \otimes n \mapsto \mathcal{S}^{-1}\mathcal{S}_R(n_{\langle -1\rangle})m \otimes n_{\langle 0\rangle}, \tag{7.4}$$

for all $M, N \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$*.*

Proof. Let $(S_1^{-1}, \psi) : {}^{R\#H}_{R\#H}\mathcal{YD} \to \mathcal{YD}^{R\#H}_{R\#H}$ be the braided monoidal isomorphism defined in Lemma 4.9(2). Let $M \in {}^{R\#H}_{R\#H}\mathcal{YD}$, and assume that M is rational as a left R-module. By definition, $S_1^{-1}(M) = M$ as a vector space, and $mr = \mathcal{S}(r)m$ for all $m \in M, r \in R$, where $\mathcal{S}$ is the antipode of $R\#H$. Let $m \in M$. Since M is a rational left R-module, $E'^{\perp}m = 0$ for some $E' \in \mathcal{E}_{R^\vee}$. Choose a subspace $E'' \in \mathcal{E}_{R^\vee}$ with $\mathcal{S}_{R^\vee}(E') \subseteq E''$. Then $\mathcal{S}(r)m = \mathcal{S}(r_{(-1)})\mathcal{S}_R(r_{(0)})m = 0$ for all $r \in E''^{\perp}$ by (1.21) and (2.10). Hence $S_1^{-1}(M)$ is rational as a right R-module.

Thus (S_1^{-1}, ψ) induces a functor on the rational objects. We denote the induced functor again by

$$(S_1^{-1}, \psi) : {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}} \to {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H}.$$

Let

$$(\widetilde{F}, \widetilde{\varphi}) : {}_{\mathrm{rat}}\mathcal{YD}^{R\#H}_{R\#H} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$$

be the braided monoidal isomorphism of Theorem 6.5. Then the composition

$$(\Omega, \omega) = (\widetilde{F}, \widetilde{\varphi})(S_1^{-1}, \psi) \tag{7.5}$$

is a braided monoidal isomorphism.

Let $M \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$. The $R^\vee\#H$-coaction denoted by

$$\delta_{\Omega(M)} : M \to R^\vee\#H \otimes M, \qquad m \mapsto m^{[-1]} \otimes m^{[0]},$$

is given by

$$\mathcal{S}(r)m = \langle r, \vartheta\mathcal{S}^{-1}(m^{[-1]})\rangle' m^{[0]} \tag{7.6}$$

for all $m \in M$ and $r \in R$.

Let

$$\delta^{R^\vee}_{\Omega(M)} = (\vartheta \otimes \mathrm{id})\delta_{\Omega(M)} : M \to R^\vee \otimes M, \qquad m \mapsto m^{\langle -1\rangle} \otimes m^{\langle 0\rangle},$$

be the $R^\vee$-coaction of $\Omega(M)$.

To prove (7.2), let $m \in M$, $r \in R$. Then by (7.6) and (1.22),

$$\mathcal{S}(r)m = \langle \vartheta \mathcal{S}^{-1}(m^{[-1]}), \mathcal{S}_R^2(\theta_R(r))\rangle m^{[0]},$$

hence

$$\begin{aligned}\mathcal{S}_R(r)m &= \langle \vartheta \mathcal{S}^{-1}(m^{[-1]}), \mathcal{S}_R^2(\theta_R(r_{(0)}))\rangle r_{(-1)} m^{[0]} \quad \text{(by (1.21))}\\ &= \langle \mathcal{S}_{R^\vee}^2 \vartheta \mathcal{S}^{-1}(m^{\langle -1\rangle} m^{\langle 0\rangle}{}_{(-1)}), \theta_R(r_{(0)})\rangle r_{(-1)} m^{\langle 0\rangle}{}_{(0)} \quad \text{(by (2.10))}\\ &= \langle \mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}) \cdot \mathcal{S}_{R^\vee}^2 \vartheta \mathcal{S}^{-1}(m^{\langle -1\rangle}), \theta_R(r_{(0)})\rangle r_{(-1)} m^{\langle 0\rangle}{}_{(0)} \quad \text{(by (1.35))}\\ &= \langle \mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}) \cdot \mathcal{S}_{R^\vee}(m^{\langle -1\rangle}), \theta_R(r_{(0)})\rangle r_{(-1)} m^{\langle 0\rangle}{}_{(0)} \quad \text{(by (1.36))}\\ &= \langle \mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}) \cdot \mathcal{S}_{R^\vee}(m^{\langle -1\rangle}), \theta_R(r)_{(0)}\rangle \mathcal{S}^{-2}(\theta_R(r)_{(-1)}) m^{\langle 0\rangle}{}_{(0)} \quad \text{(by (1.11)).}\end{aligned}$$

Since $\theta_R \mathcal{S}_R^{-1} = \mathcal{S}_R^{-1}\theta_R$, we obtain by (2.10)

$$rm = \langle \mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}) \cdot m^{\langle -1\rangle}, \theta_R(r)_{(0)}\rangle \mathcal{S}^{-2}(\theta_R(r)_{(-1)}) m^{\langle 0\rangle}{}_{(0)}. \tag{7.7}$$

Note that $c_{R^\vee,M}^{-1}(m^{\langle -1\rangle} \otimes m^{\langle 0\rangle}) = m^{\langle 0\rangle}{}_{(0)} \otimes \mathcal{S}^{-1}(m^{\langle 0\rangle}{}_{(-1)}) \cdot m^{\langle -1\rangle}$. Hence by (7.7) and (2.5),

$$rm = \langle m^{\langle\langle -1\rangle\rangle}, \theta_R(r)\rangle m^{\langle\langle 0\rangle\rangle},$$

where $m^{\langle\langle -1\rangle\rangle} \otimes m^{\langle\langle 0\rangle\rangle} = c_{M,R^\vee}^{-1} c_{R^\vee,M}^{-1}(m^{\langle -1\rangle} \otimes m^{\langle 0\rangle})$.

Finally, by (7.5) and (3.4) the natural transformation ω is given by

$$\omega_{M,N} : \Omega(M) \otimes \Omega(N) \to \Omega(M \otimes N), \qquad m \otimes n \mapsto n_{[-1]}\pi\mathcal{S}^{-1}(n_{[-2]})m \otimes n_{[0]}, \tag{7.8}$$

for all $M, N \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$, where

$$N \to R\#H \otimes N, \qquad n \mapsto n_{[-1]} \otimes n_{[0]} = n_{\langle -1\rangle} n_{\langle 0\rangle(-1)} \otimes n_{\langle 0\rangle(0)},$$

denotes the $R\#H$-coaction of N. Let $r \in R$, $h \in H$ and $a = rh \in R\#H$. Then

$$\begin{aligned}a_{(2)}\pi\mathcal{S}^{-1}(a_{(1)}) &= \varepsilon(h) r_{(2)}\pi\mathcal{S}^{-1}(r_{(1)})\\ &= \varepsilon(h) r^{(2)}{}_{(0)}\pi\mathcal{S}^{-1}(r^{(1)} r^{(2)}{}_{(-1)})\\ &= \varepsilon(h) r_{(0)}\mathcal{S}^{-1}(r_{(-1)})\\ &= \varepsilon(h)\mathcal{S}^{-1}\mathcal{S}_R(r) \quad \text{by (1.27).}\end{aligned}$$

Hence (7.4) follows from (7.8). □

We specialize the last theorem to the case of $\mathbb{N}_0$-graded dual pairs of braided Hopf algebras in ${}^H_H\mathcal{YD}$.

Let $R = \oplus_{n\ge 0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$. We view the bosonization $R\#H$ as an $\mathbb{N}_0$-graded Hopf algebra with $\deg R(n) = n$ for all $n \ge 0$, and $\deg H = 0$.

For any Yetter–Drinfeld module $W \in {}^{R\#H}_{R\#H}\mathcal{YD}$ we define two ascending filtrations of Yetter–Drinfeld modules in ${}^H_H\mathcal{YD}$ by

$$\mathcal{F}_n^\delta W = \{w \in W \mid \delta_W^R(w) \in \bigoplus_{i=0}^n R(i) \otimes W\}, \tag{7.9}$$

$$\mathcal{F}_n^\mu W = \{w \in W \mid R(i)w = 0 \text{ for all } i > n\} \tag{7.10}$$

for all $n \ge 0$. Then $\cup_{n\ge 0}\mathcal{F}_n^\delta W = W$. But in general, $\cup_{n\ge 0}\mathcal{F}_n^\mu W \ne W$.

Given an abelian monoid Γ and a Γ-graded Hopf algebra A with bijective antipode, we say that $M \in {}^A_A\mathcal{YD}$ is *Γ-graded* if $M = \oplus_{\gamma \in \Gamma} M(\gamma)$ is a vector space grading and if the module and comodule maps of M are Γ-graded of degree 0.

Corollary 7.2. *Let $R^\vee = \oplus_{n\geq 0} R^\vee(n)$ and $R = \oplus_{n\geq 0} R(n)$ be $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$ with finite-dimensional components $R^\vee(n)$ and $R(n)$ for all $n \geq 0$, and let $\langle , \rangle : R^\vee \otimes R \to \Bbbk$ be a bilinear form of vector spaces satisfying (2.3)–(2.7) and (2.12). Then the functor*

$$(\Omega, \omega) : {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$$

as defined in Theorem 7.1 is a braided monoidal isomorphism.

Moreover, the following hold.

(1) *A left R– (respectively $R^\vee$) – module M is rational if and only if for any $m \in M$ there is a natural number n_0 such that $R(n)m = 0$ (respectively $R^\vee(n)m = 0$) for all $n \geq n_0$.*
(2) *Let $M \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$ be $\mathbb{Z}$-graded. Then $\Omega(M)$ is a $\mathbb{Z}$-graded object in ${}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ with $\Omega(M)(n) = M(-n)$ for all $m \in \mathbb{Z}$.*
(3) *For any $M \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$ and $n \geq 0$,*

$$\mathcal{F}^\mu_n \Omega(W) = \mathcal{F}^\delta_n W, \qquad \mathcal{F}^\delta_n \Omega(W) = \mathcal{F}^\mu_n W.$$

Proof. By Example 2.4, the antipodes of R and of $R^\vee$ are bijective, and $(R, R^\vee)$ together with $\langle , \rangle$ is a dual pair of Hopf algebras in ${}^H_H\mathcal{YD}$. By Examples 4.3(2), the category ${}_{R^\vee\#H}{}^H_H\mathcal{YD}_{\mathrm{rat}}$ is $R^\vee\#H$-faithful. Thus (Ω, ω) is a braided monoidal isomorphism by Theorem 7.1.

(1) is clear from Example 2.4, and (2) and (3) can be checked using (7.1) and (7.2). □

Proposition 7.3. *Let $R = \oplus_{n\geq 0} R(n)$ be an $\mathbb{N}_0$-graded Hopf algebra in ${}^H_H\mathcal{YD}$ with finite-dimensional components $R(n)$ for all $n \geq 0$. Let W be an irreducible object in the category of $\mathbb{Z}$-graded left Yetter–Drinfeld modules over $R\#H$. Assume that W is locally finite as an R-module, or equivalently finite-dimensional. Let $n_0 \leq n_1$ in $\mathbb{Z}$, and $W = \oplus_{i=n_0}^{n_1} W(i)$ be the decomposition into homogeneous components such that $W(n_0) \neq 0$, $W(n_1) \neq 0$. Then*

$$\mathcal{F}^\delta_n W = \bigoplus_{i=n_0}^{n_0+n} W(i), \qquad \mathcal{F}^\mu_n W = \bigoplus_{i=n_1-n}^{n_1} W(i) \tag{7.11}$$

for all $n \geq 0$. Moreover, $W(n_0)$ and $W(n_1)$ are irreducible Yetter–Drinfeld modules over $R(0)\#H$, where the action and coaction arise from the action and coaction of $R\#H$ on W by restriction and projection, respectively.

Proof. The inclusions $\supseteq$ in (7.11) follow from the definitions since W is a $\mathbb{Z}$-graded Yetter–Drinfeld module. On the other hand, assume that $\mathcal{F}^\delta_n W \neq \oplus_{i=n_0}^{n_0+n} W(i)$ for some $n \geq 0$. Then there exist $l > n_0+n$ and $w \in W(l) \cap \mathcal{F}^\delta_n(W)$ with $w \neq 0$, since W is a $\mathbb{Z}$-graded Yetter–Drinfeld module. Then the Yetter–Drinfeld submodule of W generated by w is contained in $\oplus_{n>n_0} W(n)$. This is a contradiction to $W(n_0) \neq 0$ and the irreducibility of W. The proof of the second equation in (7.11) is similar. By degree reasons, $W(n_0)$ is a Yetter–Drinfeld module over $R(0)\#H$ in the way explained in the claim. It is irreducible, since W is irreducible and hence it is the $R\#H$-module generated by any nonzero Yetter–Drinfeld submodule over $R(0)\#H$ of $W(n_0)$. Similarly, $W(n_1)$ is an irreducible Yetter–Drinfeld module over $R(0)\#H$, since W is the $R\#H$-comodule generated by any nonzero Yetter–Drinfeld submodule over $R(0)\#H$ of $W(n_1)$. □

Let R be a braided Hopf algebra in ${}^H_H\mathcal{YD}$, and let K be a Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$. Then

$$K\#R := (K\#(R\#H))^{\mathrm{co}\,H}$$

denotes the braided Hopf algebra in ${}^H_H\mathcal{YD}$ of H-coinvariant elements with respect to the canonical projection $K\#(R\#H) \to R\#H \to H$.

Corollary 7.4. *In the situation of Theorem 7.1 assume that R is a Hopf subalgebra of a Hopf algebra B in ${}^H_H\mathcal{YD}$ with a Hopf algebra projection onto R, and let $K := B^{\mathrm{co}\,R}$.*

(1) *$K = (B\#H)^{\mathrm{co}\,R\#H}$ is a Hopf algebra in ${}^{R\#H}_{R\#H}\mathcal{YD}$, and the multiplication map $K\#R \to B$ is an isomorphism of Hopf algebras in ${}^H_H\mathcal{YD}$.*
(2) *Assume that K is rational as an R-module. Then $\Omega(K)\#R^\vee$ is a Hopf algebra in ${}^H_H\mathcal{YD}$ with a Hopf algebra projection onto $R^\vee$.*

Proof. (1) is shown in [2, Lemma 3.1]. By Theorem 7.1, $\Omega(K)$ is a Hopf algebra in ${}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}$. This proves (2). □

8. An application to Nichols algebras

In the last section we want to apply the construction in Corollary 7.4 to Nichols algebras. We show in Theorem 8.9 that if B is a Nichols algebra of a semi-simple Yetter–Drinfeld module, then the Hopf algebra $\Omega(K)\#R^\vee$ constructed in Corollary 7.4 is again a Nichols algebra. The advantage of the construction is that the new Nichols algebra is usually not twist equivalent to the original one.

We start with some general observations.

Remark 8.1. Let $R = \oplus_{n\in\mathbb{N}_0} R(n)$ be an $\mathbb{N}_0$-graded bialgebra in ${}^H_H\mathcal{YD}$.

(1) The space

$$P(R) = \{x \in R \mid \Delta_R(x) = 1 \otimes x + x \otimes 1\}$$

of primitive elements of R is an $\mathbb{N}_0$-graded subobject of R in ${}^H_H\mathcal{YD}$, since it is the kernel of the graded, H-linear and H-colinear map

$$R \to R \otimes R, \qquad x \mapsto \Delta_R(x) - 1 \otimes x - x \otimes 1.$$

(2) Assume that $R(0) = \Bbbk$. Then $R(1) \subseteq P(R)$. Moreover, R is an $\mathbb{N}_0$-graded braided Hopf algebra in ${}^H_H\mathcal{YD}$.

Let $M \in {}^H_H\mathcal{YD}$. A *pre-Nichols algebra* [8] of M is an $\mathbb{N}_0$-graded braided bialgebra $R = \oplus_{n\geq\mathbb{N}_0} R(n)$ in ${}^H_H\mathcal{YD}$ such that

(N1) $R(0) = \Bbbk$,
(N2) $R(1) = M$,
(N3) R is generated as an algebra by M.

The Nichols algebra of M is a pre-Nichols algebra R of M such that

(N4) $P(R) \cap R(n) = 0$ for all $n \geq 2$.

It is denoted by $\mathcal{B}(M)$. Up to isomorphism, $\mathcal{B}(M)$ is uniquely determined by M. By Remark 8.1, our definition of $\mathcal{B}(M)$ coincides with [3, Definition 2.1]. The Nichols algebra $\mathcal{B}(M)$ has the following *universal property*:

For any pre-Nichols algebra R of M there is exactly one map

$$\rho : R \to \mathcal{B}(M), \qquad \rho \mid M = \mathrm{id},$$

of $\mathbb{N}_0$-graded braided bialgebras in ${}^H_H\mathcal{YD}$. Thus $\mathcal{B}(M)$ is the smallest pre-Nichols algebra of M.

In the situation of Theorem 7.1, the functor

$$(\Omega, \omega) : {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}} \to {}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$$

is a braided monoidal isomorphism. Hence for any $\mathbb{N}_0$-graded braided bialgebra B in ${}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$ with multiplication μ_B and comultiplication Δ_B, the image $\Omega(B)$ is an $\mathbb{N}_0$-graded braided bialgebra in ${}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$ with multiplication

$$\Omega(B) \otimes \Omega(B) \xrightarrow{\omega_{B,B}} \Omega(B \otimes B) \xrightarrow{\Omega(\mu_B)} \Omega(B)$$

and comultiplication

$$\Omega(B) \xrightarrow{\Omega(\Delta_B)} \Omega(B \otimes B) \xrightarrow{\omega^{-1}_{B,B}} \Omega(B) \otimes \Omega(B).$$

The unit elements and the augmentations in B and $\Omega(B)$ coincide.

Corollary 8.2. *Under the assumptions of Theorem 7.1, let $M \in {}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$. Then*

$$\Omega(\mathcal{B}(M)) \cong \mathcal{B}(\Omega(M))$$

as $\mathbb{N}_0$-graded braided Hopf algebras in ${}^{R^\vee\#H}_{R^\vee\#H}\mathcal{YD}_{\mathrm{rat}}$.

Proof. By (N3) and (2.9), $\mathcal{B}(M)$ is rational as an R-module, since M is rational. By Theorem 7.1, (Ω, ω) is a braided monoidal isomorphism. Hence $\mathcal{B}(M)$ is an $\mathbb{N}_0$-graded braided bialgebra in ${}^{R\#H}_{R\#H}\mathcal{YD}_{\mathrm{rat}}$. Since Ω is the identity on morphisms, (N1)–(N4) hold for $\Omega(\mathcal{B}(M))$. This proves the Corollary. □

Let B be a coalgebra. An $\mathbb{N}_0$-filtration $\mathcal{F} = (\mathcal{F}_n B)_{n\in\mathbb{N}_0}$ of B is a family of subspaces $\mathcal{F}_n B$, $n \geq 0$, of B such that

$\mathcal{F}_n B$ is a subspace of $\mathcal{F}_m B$ for all $m, n \in \mathbb{N}_0$ with $n \leq m$,
$B = \bigcup_{n\in\mathbb{N}_0} \mathcal{F}_n B$, and
$\Delta_B(x) \in \sum_{i=0}^{n} \mathcal{F}_i B \otimes \mathcal{F}_{n-i} B$ for all $x \in \mathcal{F}_n B$, $n \in \mathbb{N}_0$.

Lemma 8.3. *Let B be a coalgebra having an $\mathbb{N}_0$-filtration $\mathcal{F}$. Let $U \in {}^B\mathcal{M}$ be a non-zero object. Then there exists $u \in U \setminus \{0\}$ such that $\delta(u) \in \mathcal{F}_0 B \otimes U$.*

Proof. The coradical B_0 of B is contained in $\mathcal{F}_0 B$ by [9, Lemma 5.3.4]. Hence $\delta^{-1}(\mathcal{F}_0 B \otimes U) \neq 0$, since for any irreducible subcomodule $U' \subseteq U$ there is a simple subcoalgebra C' with $\delta(U') \subseteq C' \otimes U'$.

We give an alternative and more explicit proof. Let $x \in U \setminus \{0\}$. Then there exists $n \in \mathbb{N}_0$ with $\delta(x) \in \mathcal{F}_n B \otimes U$. If $n = 0$, we are done. Assume now that $n \geq 1$ and let $\pi_0 : B \to B/\mathcal{F}_0 B$ be the canonical linear map. Since $\mathcal{F}$ is a coalgebra filtration, there is a maximal $m \in \mathbb{N}_0$ such that

$$\pi_0(x_{(-m)}) \otimes \cdots \otimes \pi_0(x_{(-1)}) \otimes x_{(0)} \neq 0,$$

where $\delta(x) = x_{(-1)} \otimes x_{(0)}$. Let $f_1, \dots, f_m \in B^*$ with $f_i|_{B_0} = 0$ for all $i \in \{1, \dots, m\}$ such that

$$y := f_1(x_{(-m)}) \cdots f_m(x_{(-1)}) x_{(0)} \neq 0.$$

Then $\delta(y) = f_1(x_{(-m-1)}) \cdots f_m(x_{(-2)}) x_{(-1)} \otimes x_{(0)} \in \mathcal{F}_0 B \otimes U$ by the maximality of m. □

Lemma 8.4. *Let Γ be an abelian group with neutral element 0, and A a Γ-graded Hopf algebra.*

(1) *Let K be a Nichols algebra in ${}^A_A\mathcal{YD}$, and $K(1) = \oplus_{\gamma \in \Gamma} K(1)_\gamma$ a Γ-graded object in ${}^A_A\mathcal{YD}$. Then there is a unique Γ-grading on K extending the grading on $K(1)$. Moreover, $K(n)$ is Γ-graded in ${}^A_A\mathcal{YD}$ for all $n \geq 0$.*
(2) *Let K be a Γ-graded braided Hopf algebra in ${}^A_A\mathcal{YD}$. Then the bosonization $K\#A$ is a Γ-graded Hopf algebra with $\deg K(\gamma)\#A(\lambda) = \gamma + \lambda$ for all $\gamma, \lambda \in \Gamma$.*
(3) *Let $H \subseteq A$ be a Hopf subalgebra of degree 0, and $\pi : A \to H$ a Hopf algebra map with $\pi \mid H = \mathrm{id}$. Define $R = A^{\mathrm{co}\, H}$. Then R is a Γ-graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with $R(\gamma) = R \cap A(\gamma)$ for all $\gamma \in \Gamma$.*

Proof. (1) The module and comodule maps of $K(1)$ are Γ-graded and hence the infinitesimal braiding $c \in \mathrm{Aut}(K(1) \otimes K(1))$, being determined by the module and comodule maps, is Γ-graded. Now the claim of the lemma follows from the fact that $K(n)$ for $n \in \mathbb{N}$ as well as the structure maps of K as a braided Hopf algebra are determined by c and $K(1)$.

(2) and (3) are easily checked. □

We now study the projection of H-Yetter–Drinfeld Hopf algebras in Corollary 7.4 in the case of Nichols algebras. Recall that for any $M, N \in {}^H_H\mathcal{YD}$ there is a canonical surjection

$$\pi_{\mathcal{B}(N)} : \mathcal{B}(M \oplus N) \to \mathcal{B}(N), \qquad \pi_{\mathcal{B}(N)} \mid N = \mathrm{id}, \qquad \pi_{\mathcal{B}(N)} \mid M = 0,$$

of braided Hopf algebras in ${}^H_H\mathcal{YD}$. It defines a canonical projection

$$\pi_{\mathcal{B}(N)\#H} = \pi_{\mathcal{B}(N)}\#\mathrm{id} : \mathcal{B}(M \oplus N)\#H \to \mathcal{B}(N)\#H$$

of Hopf algebras. Let $K = (\mathcal{B}(M \oplus N)\#H)^{\mathrm{co}\, \mathcal{B}(N)\#H}$ be the space of right $\mathcal{B}(N)\#H$-coinvariant elements with respect to the projection $\pi_{\mathcal{B}(N)\#H}$. Thus K is a braided Hopf algebra in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$ with $\mathcal{B}(N)\#H$-action

$$\mathrm{ad} : \mathcal{B}(N)\#H \otimes K \to K, \qquad a \otimes x \mapsto (\mathrm{ad}a)x = a_{(1)} x \mathcal{S}(a_{(2)}),$$

and $\mathcal{B}(N)\#H$-coaction

$$\delta_K : K \to \mathcal{B}(N)\#H \otimes K, x \mapsto \pi_{\mathcal{B}(N)\#H}(x_{(1)}) \otimes x_{(2)}.$$

Then by [2, Lemma 3.1], $K = \mathcal{B}(M \oplus N)^{\mathrm{co}\, \mathcal{B}(N)}$, the space of right $\mathcal{B}(N)$-coinvariant elements with respect to $\pi_{\mathcal{B}(N)}$.

The bosonization $\mathcal{B}(N)\#H$ is a $\mathbb{Z}$-graded Hopf algebra with $\deg N = 1$ and $\deg H = 0$. We always view the bosonizations of Nichols algebras in ${}^H_H\mathcal{YD}$ as graded Hopf algebras in this way.

Lemma 8.5. *Let $M, N \in {}^H_H\mathcal{YD}$ and $K = (\mathcal{B}(M \oplus N)\#H)^{\mathrm{co}\, \mathcal{B}(N)\#H}$.*

(1) *The standard $\mathbb{N}_0$-grading of $\mathcal{B}(M \oplus N)$ induces an $\mathbb{N}_0$-grading on*

$$W = (\mathrm{ad}\mathcal{B}(N))(M) = \bigoplus_{n \in \mathbb{N}_0} (\mathrm{ad}N)^n(M)$$

with $\deg(\mathrm{ad}N)^n(M) = n+1$*. Then W is a $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$, where $W \subseteq K$ is a subobject in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$.*

(2) *Assume that $M = \oplus_{i\in I} M_i$ is a direct sum of irreducible objects in ${}^H_H\mathcal{YD}$. Let $W_i = (\mathrm{ad}\mathcal{B}(N))(M_i)$ for all $i \in I$. Then $W = \oplus_{i\in I} W_i$ is a decomposition into irreducible subobjects W_i in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. For all $i \in I$, $W_i = \oplus_{n\geq 0}(\mathrm{ad}N)^n(M_i)$ is a $\mathbb{Z}$-graded object in the category of left Yetter–Drinfeld modules over $\mathcal{B}(N)\#H$.*

Proof. (1) Let $a \in N$ and $x \in \mathcal{B}(M \oplus N)$ a homogeneous element. Then $\Delta_{\mathcal{B}(M\oplus N)\#H}(a) = a \otimes 1 + a_{(-1)} \otimes a_{(0)}$, since a is primitive in $\mathcal{B}(N)$. Hence

$$(\mathrm{ad}a)(x) = ax - (a_{(-1)} \cdot x)a_{(0)}$$

is of degree $\deg x + 1$ in $\mathcal{B}(M \oplus N)$. This implies the decomposition of W. Moreover, $W \subseteq K$, since $M \subseteq K$.

Since $W = (\mathrm{ad}\mathcal{B}(N)\#H)(M)$, it is clear that W is stable under the adjoint action of $\mathcal{B}(N)\#H$, and that

$$\mathrm{ad} : \mathcal{B}(N)\#H \otimes W \to W$$

is $\mathbb{Z}$-graded. To see that $W \subseteq K$ is a $\mathcal{B}(N)\#H$-subcomodule, and that the comodule structure

$$W \to \mathcal{B}(N)\#H \otimes W$$

is $\mathbb{Z}$-graded, we compute δ_K on elements of W. For all $a \in \mathcal{B}(N)\#H$ and $x \in M$,

$$\begin{aligned}\delta_K(\mathrm{ad}a)(x) &= (\pi_{\mathcal{B}(N)\#H} \otimes \mathrm{id})\Delta_{\mathcal{B}(M\oplus N)\#H}(\mathrm{ad}a)(x)\\ &= \pi_{\mathcal{B}(N)\#H}\left(a_{(1)}x_{(1)}\mathcal{S}(a_{(4)})\right) \otimes a_{(2)}x_{(2)}\mathcal{S}(a_{(3)})\\ &= \pi_{\mathcal{B}(N)\#H}\left(a_{(1)}x\mathcal{S}(a_{(4)})\right) \otimes a_{(2)}\mathcal{S}(a_{(3)})\\ &\quad + \pi_{\mathcal{B}(N)\#H}\left(a_{(1)}x_{(-1)}\mathcal{S}(a_{(4)})\right) \otimes a_{(2)}x_{(0)}\mathcal{S}(a_{(3)})\\ &= a_{(1)}x_{(-1)}\mathcal{S}(a_{(3)}) \otimes (\mathrm{ad}a_{(2)})(x_{(0)}).\end{aligned}$$

Thus the $\mathcal{B}(N)\#H$-costructure of W is well-defined and $\mathbb{Z}$-graded.

(2) is shown in [2, Propositions 3.4 and 3.5]. □

Proposition 8.6. *Let $M, N \in {}^H_H\mathcal{YD}$ and $K = (\mathcal{B}(M \oplus N)\#H)^{\mathrm{co}\,\mathcal{B}(N)\#H}$. Then there is a unique isomorphism*

$$K \cong \mathcal{B}\left((\mathrm{ad}\mathcal{B}(N))(M)\right)$$

of braided Hopf algebras in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$ which is the identity on $(\mathrm{ad}\mathcal{B}(N))(M)$.

Proof. Since $M \oplus N$ is a $\mathbb{Z}$-graded object in ${}^H_H\mathcal{YD}$ with $\deg M = 1$ and $\deg N = 0$, the Nichols algebra $\mathcal{B}(M \oplus N)$ is a $\mathbb{Z}$-graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ by Lemma 8.4(1). Hence the bosonization $\mathcal{B}(M\oplus N)\#H$ is a $\mathbb{Z}$-graded Hopf algebra with $\deg M = 1$, $\deg N = 0$, $\deg H = 0$. By Lemma 8.4(3), K is a $\mathbb{Z}$-graded Hopf algebra in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. By [2, Proposition 3.6], K is generated as an algebra by $K(1) = (\mathrm{ad}\mathcal{B}(N))(M)$. Hence $K(n) = K(1)^n$ for all $n \geq 1$, and $K(0) = \Bbbk$.

It remains to prove that all homogeneous primitive elements of K are of degree one. Let $n \in \mathbb{N}_{\geq 2}$ and let $U \subseteq K(n)$ be a subspace of primitive elements. We have to show that $U = \{0\}$. By Remark 8.1(1) we may assume that $U \in {}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Since $\mathcal{B}(N)\#H$ has a coalgebra filtration

$\mathcal{F}$ with $\mathcal{F}_0 = H$ and $\mathcal{F}_1 = NH + H$, Lemma 8.3 implies that there exists a nonzero primitive element $u \in U$ with $\delta(u) \in H \otimes U$. Then u is primitive in $\mathcal{B}(M \oplus N)$. Indeed,

$$\Delta_{K\#(\mathcal{B}(N)\#H)} = u \otimes 1 + 1u_{[-1]} \otimes u_{[0]} = u \otimes 1 + u_{(-1)} \otimes u_{(0)},$$

and hence $\Delta_{K\#\mathcal{B}(N)}(u) = (\vartheta \otimes \mathrm{id})\Delta_{K\#(\mathcal{B}(N)\#H)}(u) = u \otimes 1 + 1 \otimes u$.

Since $K(n) = (\mathrm{ad}\mathcal{B}(N)(M))^n$, u is an element of degree at least n in the usual grading of $\mathcal{B}(M \oplus N)$. This contradicts the assumption that $\mathcal{B}(M \oplus N)$ is a Nichols algebra. □

Next we prove the converse of the above proposition under additional restrictions, see Proposition 8.8.

Let C be a coalgebra, $D \subseteq C$ a subcoalgebra, and W a left C-comodule with comodule structure $\delta : W \to C \otimes W$. We denote the largest D-subcomodule of W by

$$W(D) = \{w \in W \mid \delta(w) \in D \otimes W\}.$$

Lemma 8.7. *Let $N \in {}^H_H\mathcal{YD}$ and $W \in {}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Assume that $\oplus_{i\in I} W_i$ is a decomposition of W into irreducible objects in the category of $\mathbb{Z}$-graded left Yetter–Drinfeld modules over $\mathcal{B}(N)\#H$. Let $M = W(H)$, and $M_i = M \cap W_i$ for all $i \in I$.*

(1) *$M = \oplus_{i\in I} M_i$ is a decomposition into irreducible objects in ${}^H_H\mathcal{YD}$.*
(2) *For all $i \in I$, M_i is the $\mathbb{Z}$-homogeneous component of W_i of minimal degree, and $W_i = B(N) \cdot M_i = \oplus_{n\geq 0} N^n \cdot M_i$.*

Proof. Let $W = \oplus_{n\in\mathbb{Z}} W(n)$ be the $\mathbb{Z}$-grading of W in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Then M is a $\mathbb{Z}$-graded object in ${}^H_H\mathcal{YD}$ with homogeneous components $M(n) = M \cap W(n)$ for all $n \in \mathbb{Z}$. It is clear that $M = \oplus_{i\in I} M_i$, where $M_i = M \cap W_i = W_i(H)$ for all i.

Let $i \in I$. By Lemma 8.3, $M_i \neq 0$. Let $0 \neq M_i'$ be a homogeneous subobject of M_i in ${}^H_H\mathcal{YD}$, and let n be its degree. Then the $\mathcal{B}(N)\#H$-module $W_i' := \mathcal{B}(N) \cdot M_i'$ is a $\mathbb{Z}$-graded subobject of W_i in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$, the homogeneous components of W_i' have degrees $\geq n$, and the degree n component of W_i' coincides with M_i' since $\mathcal{B}(N)(0) = \Bbbk$ and $\deg N = 1$. Thus the irreducibility of W_i implies that $M_i = M_i'$ is irreducible and it is the homogeneous component of W_i of minimal degree.

Finally, for all $i \in I$ and $n \in \mathbb{N}_0$,

$$\deg(N^n \cdot M_i) = n + \deg M_i,$$

since the multiplication map $\mathcal{B}(N)\#H \otimes W_i \to W_i$ is graded. It follows that $W_i = \oplus_{n\geq 0} N^n \cdot M_i$ for all i. □

Proposition 8.8. *Let $N \in {}^H_H\mathcal{YD}$ and $W \in {}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Assume that W is a semi-simple object in the category of $\mathbb{Z}$-graded left Yetter–Drinfeld modules over $\mathcal{B}(N)\#H$. Let $K = \mathcal{B}(W)$ be the Nichols algebra of W in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$, and define $M = W(H)$. Then there is a unique isomorphism*

$$K\#\mathcal{B}(N) \cong \mathcal{B}(M \oplus N)$$

of braided Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $M \oplus N$.

Proof. Let $\oplus_{i\in I} W_i$ be a decomposition of W into irreducible objects in the category of $\mathbb{Z}$-graded left Yetter–Drinfeld modules over $\mathcal{B}(N)\#H$. For all $i \in I$, let $M_i = W_i \cap M$. By Lemma 8.7(2), we can define a new $\mathbb{Z}$-grading on W by

$$\deg(N^n \cdot M_i) = n + 1$$

for all $n \in \mathbb{N}_0$, $i \in I$. Then W is a $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Because of Lemma 8.4(1), and since $W = K(1)$, we know that K is a $\mathbb{Z}$-graded braided Hopf algebra with this new $\mathbb{Z}$-grading on $K(1)$. Thus by Lemma 8.4(2) and (3), $K\#(\mathcal{B}(N)\#H)$ is a $\mathbb{Z}$-graded Hopf algebra, and

$$R := K\#\mathcal{B}(N) = (K\#(\mathcal{B}(N)\#H))^{\mathrm{co}\,H}$$

is a $\mathbb{Z}$-graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with $\Bbbk 1$ as degree 0 part and with $M \oplus N$ as degree 1 part.

Let $m \in M$ and $b \in \mathcal{B}(N)$. Then

$$b \cdot m = b^{(1)}(b^{(2)}{}_{(-1)} \cdot m)\mathcal{S}_{\mathcal{B}(N)}(b^{(2)}{}_{(0)}) \tag{8.1}$$

in the algebra $R = K\#\mathcal{B}(N)$. Since K is generated as an algebra by $K(1)$, and since $K(1) = \mathcal{B}(N) \cdot M$, we conclude from (8.1) that R is generated as an algebra by $R(1) = M \oplus N$. Thus R is a pre-Nichols algebra of $M \oplus N$.

By the universal property of the Nichols algebra $\mathcal{B}(M \oplus N)$, there is a surjective homomorphism

$$\rho : R \to \mathcal{B}(M \oplus N), \qquad \rho \mid M \oplus N = \mathrm{id},$$

of $\mathbb{N}_0$-graded Hopf algebras in ${}^H_H\mathcal{YD}$. Then

$$\rho\#\mathrm{id} : R\#H \to \mathcal{B}(M \oplus N)\#H$$

is a surjective map of Hopf algebras. Let $K' = (\mathcal{B}(M\oplus N)\#H)^{\mathrm{co}\,\mathcal{B}(N)\#H}$. Since the multiplication maps

$$R\#H \to K\#(\mathcal{B}(N)\#H), \qquad K'\#(\mathcal{B}(N)\#H) \to \mathcal{B}(M \oplus N)\#H$$

are bijective maps of Hopf algebras, the map $\rho\#\mathrm{id}$ defines a surjective map of Hopf algebras

$$\rho' : K\#(\mathcal{B}(N)\#H) \to K'\#(\mathcal{B}(N)\#H), \qquad \rho' \mid (M \oplus N) = \mathrm{id}.$$

The action of $\mathcal{B}(N)\#H$ on K is the adjoint action in $K\#(\mathcal{B}(N)\#H)$. Since the algebras K and K' are generated by $(\mathrm{ad}\mathcal{B}(N))(M)$ on both sides, ρ' induces a map

$$\rho_1 : K \to K', \qquad \rho_1 \mid M = \mathrm{id},$$

of $\mathbb{N}_0$-graded braided Hopf algebras in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$, and a map

$$\rho_2 : \mathcal{B}(N) \cdot M \to (\mathrm{ad}\mathcal{B}(N))(M), \qquad \rho_2 \mid M = \mathrm{id},$$

in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Since $(\mathrm{ad}\mathcal{B}(N))(M_i)$ is irreducible in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$ for all $i \in I$, it follows that ρ_2 is bijective. Hence ρ_1 is bijective by the universal property of the Nichols algebra $K = \mathcal{B}(W)$. Thus $\rho = \rho_1\#\mathrm{id}_{\mathcal{B}(N)}$ is bijective. □

We now apply Corollary 7.2 to Nichols algebras. Let $N \in {}^H_H\mathcal{YD}$ be finite-dimensional. Then the dual vector space $N^* = \mathrm{Hom}(N, \Bbbk)$ is an object in ${}^H_H\mathcal{YD}$ with

$$\langle h \cdot \xi, x\rangle = \langle \xi, \mathcal{S}(h) \cdot x\rangle,$$
$$\xi_{(-1)}\langle \xi_{(0)}, x\rangle = \mathcal{S}^{-1}(x_{(-1)})\langle \xi, x_{(0)}\rangle$$

for all $\xi \in N^*$, $x \in N$, $h \in H$, where $\langle , \rangle : N^* \otimes N \to \Bbbk$ is the evaluation map. The Nichols algebras of the finite-dimensional Yetter–Drinfeld modules N^* and N have finite-dimensional $\mathbb{N}_0$-homogeneous components, and there is a canonical pairing $\langle , \rangle : \mathcal{B}(N^*) \otimes \mathcal{B}(N) \to \Bbbk$

extending the evaluation map such that the conditions (2.3)–(2.7) and (2.12) hold, see for example [2, Proposition 1.10]. Let

$$(\Omega_N, \omega_N) : {}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}_{\mathrm{rat}} \to {}^{\mathcal{B}(N^*)\#H}_{\mathcal{B}(N^*)\#H}\mathcal{YD}_{\mathrm{rat}}$$

be the functor of Corollary 7.2 with respect to the canonical dual pairing.

Theorem 8.9. *Let $n \geq 1$, and let $M_1, \dots, M_n, N$ be finite-dimensional objects in ${}^H_H\mathcal{YD}$. Assume that for all $1 \leq i \leq n$, M_i is irreducible in ${}^H_H\mathcal{YD}$, and that $(\mathrm{ad}\mathcal{B}(N))(M_i)$ is a finite-dimensional subspace of $\mathcal{B}(\oplus_{i=1}^n M_i \oplus N)$. For all i let $V_i = \mathcal{F}_0^\mu(\mathrm{ad}\mathcal{B}(N))(M_i)$, and let $K = \mathcal{B}(\oplus_{i=1}^n M_i \oplus N)^{\mathrm{co}\,\mathcal{B}(N)}$.*

(1) *The modules $V_1, \dots, V_n$ are irreducible in ${}^H_H\mathcal{YD}$, $\Omega_N(K)$ is a braided Hopf algebra in ${}^{\mathcal{B}(N^*)\#H}_{\mathcal{B}(N^*)\#H}\mathcal{YD}_{\mathrm{rat}}$, and there is a unique isomorphism*

$$\Omega_N(K)\#\mathcal{B}(N^*) \cong \mathcal{B}(\bigoplus_{i=1}^n V_i \oplus N^*)$$

of braided Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $\oplus_{i=1}^n V_i \oplus N^*$.

(2) *For all $1 \leq i \leq n$, let $m_i = \max\{m \in \mathbb{N}_0 \mid (\mathrm{ad}N)^m(M_i) \neq 0\}$, and $W_i = (\mathrm{ad}\mathcal{B}(N))(M_i)$. Then*

$$W_i = \bigoplus_{n=0}^{m_i} (\mathrm{ad}N)^n(M_i), \qquad V_i = (\mathrm{ad}N)^{m_i}(M_i),$$

$$\Omega_N(W_i) \cong \bigoplus_{n=0}^{m_i} (\mathrm{ad}N^*)^n(V_i), \qquad M_i \cong (\mathrm{ad}N^*)^{m_i}(V_i)$$

for all i.

Proof. (1) Let $W = (\mathrm{ad}\mathcal{B}(N))(M)$. By Lemma 8.5(2), $W_1, \dots, W_n$ are irreducible objects in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$, $W = \oplus_{i=1}^n W_i$, and for all $1 \leq i \leq n$, M_i is the $\mathbb{Z}$-homogeneous component of W_i of minimal degree. By Proposition 7.3, the Yetter–Drinfeld modules $V_1, \dots, V_n \in {}^H_H\mathcal{YD}$ are irreducible. By Proposition 8.6, K is isomorphic to the Nichols algebra of W in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$. Since $(\mathrm{ad}\mathcal{B}(N))(M)$ is a finite-dimensional and $\mathbb{Z}$-graded object in ${}^{\mathcal{B}(N)\#H}_{\mathcal{B}(N)\#H}\mathcal{YD}$, it is a rational $\mathcal{B}(N)$-module. Therefore $\Omega_N(\mathcal{B}(W)) \cong \mathcal{B}(\Omega_N(W))$ by Corollary 8.2.

Hence there is a unique isomorphism $\Omega_N(K) \cong \mathcal{B}(\Omega_N(W))$ of braided Hopf algebras in ${}^{\mathcal{B}(N^*)\#H}_{\mathcal{B}(N^*)\#H}\mathcal{YD}_{\mathrm{rat}}$ which is the identity on $\Omega_N(W)$. Recall that

$$\Omega_N(W)(H) = \mathcal{F}_0^\delta \Omega_N(W) = \mathcal{F}_0^\mu W = \bigoplus_{i=1}^n V_i$$

by Corollary 7.2(3). Then by Proposition 8.8 there is a unique isomorphism

$$\Omega_N(K)\#\mathcal{B}(N^*) \cong \mathcal{B}(\bigoplus_{i=1}^n V_i \oplus N^*)$$

of braided Hopf algebras in ${}^H_H\mathcal{YD}$ which is the identity on $\oplus_{i=1}^n V_i \oplus N^*$ which proves (1). For the last conclusion we have to check the assumptions of Proposition 8.8, that is, $\Omega_N(W) \in {}^{\mathcal{B}(N^*)\#H}_{\mathcal{B}(N^*)\#H}\mathcal{YD}_{\mathrm{rat}}$ is a semi-simple $\mathbb{Z}$-graded Yetter–Drinfeld module. By Corollary 7.2(2), $\Omega_N(W)$ is $\mathbb{Z}$-graded, and it is semi-simple since W is semi-simple by Lemma 8.5 and Ω_N is an isomorphism by Corollary 7.2.

(2) Let $i \in \{1, \dots, n\}$. The first equation follows from the definition of W_i and the second from Proposition 7.3 for $R = \mathcal{B}(N)$ with $\deg N = 1$ and $\deg(\mathrm{ad}N)^n(M_i) = 1 + n$ for all $n \geq 0$. By Corollary 7.2, $\Omega_N(W_i) = W_i$ is $\mathbb{Z}$-graded with homogeneous components $(\mathrm{ad}N)^n(M_i)$ of degree $-n-1$. Moreover, $V_i = \mathcal{F}_0^\delta \Omega_N(W_i)$ by the proof of (1), and hence $\Omega_N(W_i) = \oplus_{n=0}^{m_i}(N^*)^n V_i$ since $\Omega_N(W_i)$ is irreducible. In particular, $M_i = (N^*)^{m_i} V_i$. These equations imply the remaining claims of (2). □

Remark 8.10. Theorem 8.9 still holds if we replace the canonical pairing in the definition of (Ω_N, ω_N) by any dual pairing $\langle , \rangle : \mathcal{B}(N^*) \otimes \mathcal{B}(N) \to \Bbbk$ satisfying (2.3)–(2.7) and (2.12).

The definition of the Weyl groupoid of a Nichols algebra of a semi-simple Yetter–Drinfeld module over H is based on [2, Theorem 3.12], see also [2, Section 3.5] and [6, Theorem 6.10, Section 5]. To see that Theorem 8.9 can be considered as an alternative approach to the definition of the Weyl groupoid, we introduce some notations.

Let $\theta \geq 1$ be a natural number. Let $\mathcal{F}_\theta$ denote the class of all families $M = (M_1, \dots, M_\theta)$, where $M_1, \dots, M_\theta \in {}_H^H\mathcal{YD}$ are finite-dimensional irreducible Yetter–Drinfeld modules. If $M \in \mathcal{F}_\theta$, we define

$$\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta).$$

For families $M, M' \in \mathcal{F}_\theta$, we write $M \cong M'$, if $M_j \cong M'_j$ in ${}_H^H\mathcal{YD}$ for all j.

For $1 \leq i \leq \theta$ and $M \in \mathcal{F}_\theta$, we say that the i-th *reflection* $R_i(M)$ *is defined* if for all $j \neq i$ there is a natural number $m_{ij}^M \geq 0$ such that $(\mathrm{ad}M_i)^{m_{ij}^M}(M_j)$ is a non-zero finite-dimensional subspace of $\mathcal{B}(M)$, and $(\mathrm{ad}M_i)^{m_{ij}^M+1}(M_j) = 0$. Assume that $R_i(M)$ is defined. Then we set $R_i(M) = (V_1, \dots, V_\theta)$, where

$$V_j = \begin{cases} V_i^*, & \text{if } j = i, \\ (\mathrm{ad}M_i)^{m_{ij}^M}(M_j), & \text{if } j \neq i. \end{cases}$$

For all $j \neq i$, let $a_{ij}^M = -m_{ij}^M$. By [2, Lemma 3.22], (a_{ij}^M) with $a_{ii}^M = 2$ for all i is a generalized Cartan matrix.

The next Corollary follows from a restatement of Theorem 8.9. Thus we obtain a new proof of [2, Theorem 3.12(2)] which allows to define the Weyl groupoid of $M \in \mathcal{F}_\theta$.

Corollary 8.11 *([2, Theorem 3.12(2)]). Let $M \in \mathcal{F}_\theta$, and $1 \leq i \leq \theta$. Assume that $R_i(M)$ is defined. Then $R_i(M) \in \mathcal{F}_\theta$, $R_i^2(M)$ is defined, $R_i^2(M) \cong M$, and $a_{ij}^M = a_{ij}^{R_i(M)}$ for all $1 \leq j \leq \theta$.*

In the situation of the last Corollary, let $K_i^M = \mathcal{B}(M)^{\mathrm{co}\,\mathcal{B}(M_i)}$ with respect to the projection $\mathcal{B}(M) \to \mathcal{B}(M_i)$. Then

$$K_i^M \# \mathcal{B}(M_i) \cong \mathcal{B}(M)$$

by bosonization, and

$$\Omega(K_i^M) \# \mathcal{B}(M_i^*) \cong \mathcal{B}(R_i(M))$$

by Theorem 8.9.

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