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Lattices and cohomological Mackey functors for finite cyclic p-groups

B. Torrecillas[a], Th. Weigel[b,*]

[a] Departamento de Algebra y Análisis Matemático, Universidad de Almería, 04071 Almería, Spain
[b] Università di Milano-Bicocca, U5-3067, Via R.Cozzi, 53, 20125 Milano, Italy

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Abstract

For a finite cyclic p-group G and a discrete valuation domain R of characteristic 0 with maximal ideal pR the $R[G]$-permutation modules are characterized in terms of the vanishing of first degree cohomology on all subgroups (cf. Theorem A). As a consequence any $R[G]$-lattice can be presented by $R[G]$-permutation modules (cf. Theorem C). The proof of these results is based on a detailed analysis of the category of cohomological G-Mackey functors with values in the category of R-modules. It is shown that this category has global dimension 3 (cf. Theorem E). A crucial step in the proof of Theorem E is the fact that a gentle R-order category (with parameter p) has global dimension less than or equal to 2 (cf. Theorem D).

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1. Introduction

For a Dedekind domain R and a finite group G one calls a finitely generated left $R[G]$-module M an $R[G]$-*lattice*, if M – considered as an R-module – is projective. In this paper we

* Corresponding author.
E-mail addresses: btorreci@ual.es (B. Torrecillas), thomas.weigel@unimib.it, tthomas.weigel@gmail.com (Th. Weigel).

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focus on the study of $R[G]$-lattice, where R is a discrete valuation domain of characteristic 0 with maximal ideal pR for some prime number p, and G is a finite cyclic p-group. The study of such lattices has a long history and was motivated by a promising result of F.-E. Diederichsen (cf. [5, Thm. 34:31], [6]) who showed that for the finite cyclic group of order p there are precisely three directly indecomposable such lattices up to isomorphism: the trivial $R[G]$-lattice R, the free $R[G]$-lattice $R[G]$, and the augmentation ideal $\omega_{R[G]} = \ker(R[G] \to R)$. A similar finiteness result holds for cyclic groups of order p^2 (cf. [12]). However, for cyclic p-groups of order larger than p^2 there will be infinitely many isomorphism types of such lattices; even worse, in general this classification problem is "wild" (cf. [7,8,11]). If the $R[G]$-lattice M is isomorphic to $R[\Omega]$ for some finite left G-set Ω, M will be called an $R[G]$-*permutation lattice*. The main purpose of this paper is to establish the following characterization of $R[G]$-permutation lattices for finite cyclic p-groups (cf. Corollary 6.7, Proposition 6.8).

Theorem A. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR for some prime number p, let G be a finite cyclic p-group, and let M be an $R[G]$-lattice. Then the following are equivalent.*

(i) *M is an $R[G]$-permutation lattice,*
(ii) $H^1(U, \mathrm{res}^G_U(M)) = 0$ *for all subgroups U of G,*
(iii) *M_U is R-torsion free for all subgroups U of G,*

where $M_U = M/\omega_{R[U]}M$ denotes the U-coinvariants of M.

By a result of I. Reiner (cf. [5, Theorem 34.31], [15]), one knows that there are $\mathbb{Z}[C_p]$-lattices satisfying (ii), where C_p is the cyclic group of order p, which are not $\mathbb{Z}[C_p]$-permutation lattices. Hence the conclusion of Theorem A does not hold for the ring $R = \mathbb{Z}$.

Theorem A has a number of interesting consequences which we would like to explain in more detail. For a finite p-group G it is in general quite difficult to decide whether a given $R[G]$-lattice M is indeed an $R[G]$-permutation lattice. A sufficient criterion to the just mentioned problem was given by A. Weiss in [27] for an arbitrary finite p-group G and the ring of p-adic integers $R = \mathbb{Z}_p$. He showed that if for a normal subgroup N of G the $\mathbb{Z}_p[G/N]$-module M^N of N-invariants is a $\mathbb{Z}_p[G/N]$-permutation module, and $\mathrm{res}^G_N(M)$ is a free $\mathbb{Z}_p[N]$-module, then M is a $\mathbb{Z}_p[G]$-permutation module (cf. [13, Chap. 8, Theorem 2.6]). Theorem A extends A. Weiss' result for cyclic p-groups in the following way (cf. Proposition 6.12).

Corollary B. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR for some prime number p, let G be a finite cyclic p-group, and let N be a normal subgroup of G. Suppose that the $R[G]$-lattice M is satisfying the following two hypothesis.*

(i) $\mathrm{res}^G_N(M)$ *is an $R[N]$-permutation module, and*
(ii) *M^N is an $R[G/N]$-permutation module.*

Then M is an $R[G]$-permutation module.

Although it seems impossible to describe all isomorphism types of directly indecomposable $R[G]$-lattices, where R is a discrete valuation domain of characteristic 0 with maximal ideal pR and G is a finite cyclic p-group, one can (re)present such lattices in a very natural way (cf. Theorem 6.11).

Theorem C. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR for some prime number p, let G be a finite cyclic p-group, and let M be an $R[G]$-lattice. Then there exist finite G-sets Ω_0 and Ω_1, and a short exact sequence*

$$0 \longrightarrow R[\Omega_1] \longrightarrow R[\Omega_0] \longrightarrow M \longrightarrow 0 \tag{1.1}$$

of $R[G]$-lattices.

The proof of Theorems A and C is based on the theory of *cohomological Mackey functors* for a finite group G. Mackey functors were first introduced by A.W.M. Dress in [9]. Cohomological Mackey functors satisfy an additional identity (cf. [24]). The category of cohomological G-Mackey functors $\mathfrak{cMF}_G({}_R\mathbf{mod})$ with values in the category of R-modules coincides with the category of contravariant functors of an $R^{\circledast}$-order category $\mathcal{M}_R(G)$ (cf. Section 3.2). In case that G is a cyclic p-group of order p^n, one has a *unitary projection functor* (cf. Section 2.10)

$$\pi : \mathfrak{cMF}_R(G) \longrightarrow \mathcal{G}_R(n, p) \tag{1.2}$$

which can be used to analyze the category $\mathfrak{cMF}_G({}_R\mathbf{mod})$. Here $\mathcal{G}_R(n, p)$ denotes the *gentle R-order category* supported on $n + 1$ vertices and parameter p (cf. Section 5.1) which can be seen as an R-order version of the gentle algebra $\mathcal{G}_{\mathbb{F}}(n)$ defined over a field $\mathbb{F}$. The gentle algebra has been subject to intensive investigations (cf. [10]), e.g., it is well known that $\mathcal{G}_{\mathbb{F}}(n)$ is 1-Gorenstein (resp. 0-Gorenstein for $n = 0$ or $n = 1$), but for $n \geq 1$ it is not of finite global dimension. Hence the following property of the gentle R-order category is somehow surprising (cf. Theorem 5.8).

Theorem D. *Let p be a prime number, and let R be a principal ideal domain of characteristic 0 such that $p.1 \in R$ is a prime element. Then* $\mathrm{gldim}(\mathcal{G}_R(n, p)) = 2$ *for $n \geq 2$, and in case that $n = 0$ or 1 one has* $\mathrm{gldim}(\mathcal{G}_R(n, p)) = 1$.

In Section 4 we study *section cohomology groups* which can be associated to any cohomological Mackey functor and any normal section of a finite group. This allows us to introduce the notion of cohomological Mackey functors with the *Hilbert* 90 *property* (cf. Section 4.2). Theorems A and C are a direct consequence of a more general result which states that for a discrete valuation domain R of characteristic 0 and maximal ideal pR every cohomological G-Mackey functor with values in the category of R-lattices and with the Hilbert 90 property is projective (cf. Theorem 6.5). The proof of this more general result is achieved in two steps. The first step is to show that the deflation functor associated to π (cf. (1.2)) maps Hilbert 90 R-lattice functors to projective functors of the gentle R-order category. The second step is to establish injectivity and surjectivity criteria which ensure that a given natural transformation $\phi : \mathbf{X} \to \mathbf{Y}$ involving Hilbert 90 R-lattice functors is indeed an isomorphism (cf. Propositions 4.16 and 6.4).

The first step is based on a sufficient criterion (cf. Theorem 2.16) which guarantees that the deflation functor associated to a unitary projection π is mapping $\circledast$-acyclic R-lattice functors to projective R-lattice functors. Here $\circledast$ denotes the *Yoneda dual* (cf. Section 2.5) which can be seen as the standard dualizing procedure for $R^{\circledast}$-categories. Although this criterion is based on what is usually called “abstract nonsense”, it will turn out to be quite useful: two of the three hypotheses one has to claim can be verified easily for the unitary projection π and involve the Hilbert 90 property, while the third is a direct consequence of Theorem D.

The two main results known to the authors concerning the cohomology of cohomological Mackey functors are due to S. Bouc (cf. [3]) and D. Tambara (cf. [22]), but concern cohomological Mackey functors with values in a field of positive characteristic. Although the just-mentioned results indicate that for cyclic groups the theory of cohomological Mackey functors should be significantly easier (and different) than in the general case, the following consequence is nevertheless surprising.

Theorem E. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR for some prime number p, and let G be a non-trivial finite cyclic p-group. Then* $\mathrm{gldim}_R(\mathcal{M}_R(G)) = 3$.

2. $R^{\circledast}$-categories

Let R be a commutative ring with 1, and let ${}_R\mathbf{mod}$ denote the abelian category of R-modules. An R-module M will be called an *R-lattice*, if M is a finitely generated projective R-module. We denote by ${}_R\mathbf{lat}$ the full subcategory of ${}_R\mathbf{mod}$ the objects of which are R-lattices, and by ${}_R\mathbf{mod}^{\mathrm{f.g.}}$ the full subcategory of ${}_R\mathbf{mod}$ the objects of which are finitely generated R-modules. For certain applications we have to restrict our considerations to Dedekind domains. For such a ring R one has the following property: if $\phi\colon M \to Q$ is a surjective homomorphism of R-lattices, then $\ker(\phi)$ is an R-lattice and the canonical map $\ker(\phi) \to M$ is split-injective.

Following [1, Chap. 2. Section 2] one calls a category $\mathcal{C}$ an *R-category*, if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an R-module for any pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$, and composition

$$\_ \circ \_ \colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C) \tag{2.1}$$

is R-bilinear for any three objects $A, B, C \in \mathrm{ob}(\mathcal{C})$. E.g., ${}_R\mathbf{mod}$ is an R-category. Note that $\mathcal{C}^{\mathrm{op}}$ is an R-category for every R-category $\mathcal{C}$. A (covariant) functor $\phi\colon \mathcal{C} \to \mathcal{D}$ between R-categories $\mathcal{C}$ and $\mathcal{D}$ is called *R-linear*, if

$$\phi_{A,B}\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\phi(A), \phi(B)) \tag{2.2}$$

is a homomorphism of R-modules for every pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$.

2.1. $R^{\circledast}$-order categories

An R-category $\mathcal{C}$ will be called an *R-order category*, if $\mathrm{ob}(\mathcal{C})$ is a finite set and $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an R-lattice for all $A, B \in \mathrm{ob}(\mathcal{C})$. E.g., if μ is an R-order, then $\mu\bullet$, the category with one object $\bullet$ and $\mathrm{Hom}_{\mu\bullet}(\bullet, \bullet) = \mu$, is an R-order category. An R-category $\mathcal{C}$ together with an R-linear functor $\sigma\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ satisfying $\sigma(A) = A$ for all $A \in \mathrm{ob}(\mathcal{C})$ and $\sigma \circ \sigma = \mathrm{id}_{\mathcal{C}}$ will be called an *$R^{\circledast}$-category*. E.g., if μ is an R-algebra with an R-linear antipode $\sigma_\mu\colon \mu \to \mu^{\mathrm{op}}$ of order 2, i.e., $\sigma_\mu \circ \sigma_\mu = \mathrm{id}_\mu$, then $\mu\bullet$ is an $R^{\circledast}$-category. An $R^{\circledast}$-category $(\mathcal{C}, \sigma)$, where $\mathcal{C}$ is an R-order category, will be called an *$R^{\circledast}$-order category*.

2.2. Additive functors

Let $\mathcal{C}$ be an R-category. By $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ we denote the category of R-linear functors from $\mathcal{C}^{\mathrm{op}}$ to ${}_R\mathbf{mod}$, i.e., $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ is a contravariant R-linear functor from $\mathcal{C}$ to ${}_R\mathbf{mod}$. Morphisms in $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ are given by the *R-linear natural transformations*, i.e., $\eta \in \mathrm{nat}_R(\mathbf{F}, \mathbf{G})$ is called *R-linear*, if $\eta_A\colon \mathbf{F}(A) \to \mathbf{G}(A)$ is R-linear for every $A \in \mathrm{ob}(\mathcal{C})$. It is well known that $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ is an abelian category (cf. [14, Chap. IX, Proposition 3.1]).

A functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ will be called an *R-lattice functor* if $\mathbf{F}(A)$ is an R-lattice for every object $A \in \mathrm{ob}(\mathcal{C})$. By $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}) \subseteq \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ we denote the full subcategory of R-lattice functors.

Let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-category, and let $\_^* = \mathrm{Hom}_R(\_, R) \colon {}_R\mathbf{lat} \longrightarrow {}_R\mathbf{lat}^{\mathrm{op}}$ denote the dualizing functor in ${}_R\mathbf{lat}$. Composition of $\_^*$ with σ yields a dualizing functor

$$\_^* \colon \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}) \longrightarrow \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat})^{\mathrm{op}}, \tag{2.3}$$

where $\mathbf{F}^*(A) = \mathbf{F}(A)^*$ and $\mathbf{F}^*(\phi) = \mathbf{F}(\sigma(\phi))^*$ for $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$ and $\phi \colon A \to B \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

2.3. Projectives

Let $\mathcal{C}$ be an R-category, and let $A \in \mathrm{ob}(\mathcal{C})$. Then

$$\mathbf{P}^A = \mathrm{Hom}_{\mathcal{C}}(\_, A) \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})) \tag{2.4}$$

is an R-linear functor from $\mathcal{C}^{\mathrm{op}}$ to ${}_R\mathbf{mod}$. Moreover, if $\mathcal{C}$ is an R-order category, then $\mathbf{P}^A$ is an R-lattice functor. One has the following property (cf. [21, Proposition IV.7.3]).

Fact 2.1. *Let $\mathcal{C}$ be an R-category, let $A \in \mathrm{ob}(\mathcal{C})$ and $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$. Then one has a canonical isomorphism*

$$\theta_{A,\mathbf{F}} \colon \mathrm{nat}_R(\mathbf{P}^A, \mathbf{F}) \longrightarrow \mathbf{F}(A) \tag{2.5}$$

given by $\theta_{A,\mathbf{F}}(\xi) = \xi_A(\mathrm{id}_A)$, $\xi \in \mathrm{nat}_R(\mathbf{P}^A, \mathbf{F})$.

The inverse of $\theta_{A,\mathbf{F}}$ can be given explicit. For $f \in \mathbf{F}(A)$ and $B \in \mathrm{ob}(\mathcal{C})$ one has

$$\theta^{-1}_{A,\mathbf{F}}(f)_B \colon \mathbf{P}^A(B) \to \mathbf{F}(B), \qquad \theta^{-1}_{A,\mathbf{F}}(f)_B(\phi) = \mathbf{F}(\phi)(f), \quad \phi \in \mathrm{Hom}_{\mathcal{C}}(B, A). \tag{2.6}$$

It is straightforward to verify that $\theta^{-1}_{A,\mathbf{F}}(f) \in \mathrm{nat}_R(\mathbf{P}^A, \mathbf{F})$. Let $\chi : \mathbf{F} \to \mathbf{G}$ be an R-linear natural transformation. Then one has a commutative diagram

$$\begin{array}{ccc} \mathbf{F}(A) & \xrightarrow{\theta^{-1}_{A,\mathbf{F}}} & \mathrm{nat}_R(\mathbf{P}^A, \mathbf{F}) \\ \downarrow{\scriptstyle \chi_A} & & \downarrow{\scriptstyle \chi\circ-} \\ \mathbf{G}(A) & \xrightarrow{\theta^{-1}_{A,\mathbf{G}}} & \mathrm{nat}_R(\mathbf{P}^A, \mathbf{G}). \end{array} \tag{2.7}$$

From this fact one concludes the following well known property (see [21, Corollary 7.5]).

Fact 2.2. *Let $\mathcal{C}$ be an R-category, let $\mathbf{F}, \mathbf{G} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$, and let $\phi \in \mathrm{nat}_R(\mathbf{F}, \mathbf{G})$ be a natural transformation such that $\phi_A \colon \mathbf{F}(A) \to \mathbf{G}(A)$ is surjective. Then for every natural transformation $\chi \in \mathrm{nat}_R(\mathbf{P}^A, \mathbf{G})$ there exists $\tilde{\chi} \in \mathrm{nat}_R(\mathbf{P}^A, \mathbf{F})$ making the diagram*

$$\begin{array}{ccccc} & \swarrow^{\tilde{\chi}} & \mathbf{P}^A & & \\ & & \downarrow{\scriptstyle \chi} & & \\ \mathbf{F} & \xrightarrow{\phi} & \mathbf{G} & \longrightarrow & 0 \end{array} \tag{2.8}$$

commute. In particular, $\mathbf{P}^A \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ is projective.

As a consequence one has the following.

Fact 2.3. *Let $\mathcal{C}$ be an R-category such that* $\mathrm{ob}(\mathcal{C})$ *is a set. Then* $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ *is an abelian category with enough projectives.*

If $\mathcal{C}$ is an R-category and $\mathrm{ob}(\mathcal{C})$ is a set, we denote for $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ the right derived functors of $\mathrm{nat}_R(\_, \mathbf{F})$ by $\mathrm{ext}^k_R(\_, \mathbf{F})$, $k \geq 0$.

Let $\mathcal{C}$ be an R-order category. Then by definition $\mathbf{P}^A$ is an R-lattice functor. A projective object $\mathbf{P} \in \mathrm{ob}(\mathfrak{F}(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ which is a lattice functor will be a called a *projective R-lattice functor*. For these functors one concludes the following.

Fact 2.4. *Let R be a Dedekind domain, and let $\mathcal{C}$ be an R-order category. Then every R-lattice functor* $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ *has a projective resolution* $(\mathbf{P}_k, \partial^{\mathbf{P}}_k, \varepsilon_{\mathbf{F}})$*, where* $\mathbf{P}_k$ *is a projective R-lattice functor for every $k \geq 0$.*

Remark 2.5. Let $\mathcal{C}$ be an R-order category such that for all $A, B \in \mathrm{ob}(\mathcal{C})$, $A \neq B$, one has $A \not\simeq B$. Let $\mu_{\mathcal{C}}$ be the R-order given by $\mu_{\mathcal{C}} = \bigoplus_{A,B\in\mathrm{ob}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(A, B)$, where the product is given by

$$\alpha \cdot \beta = \begin{cases} \alpha \circ \beta & \text{for } B_1 = B_2, \\ 0 & \text{for } B_1 \neq B_2 \end{cases} \tag{2.9}$$

for $\alpha \in \mathrm{Hom}_{\mathcal{C}}(B_2, C)$, $\beta \in \mathrm{Hom}_{\mathcal{C}}(A, B_1)$. Then one has a canonical R-linear functor $\rho_{\mathcal{C}} \colon \mathcal{C} \to \mu_{\mathcal{C}}\bullet$ (cf. Section 2.1) induced by the identity on morphisms. Moreover, the category $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ is naturally equivalent to the category of right $\mu_{\mathcal{C}}$-modules $\mathbf{mod}_{\mu_{\mathcal{C}}}$. This equivalence is achieved by assigning a right $\mu_{\mathcal{C}}$-module M the functor $\mathbf{F}_M \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ given by $\mathbf{F}_M(A) = M \cdot \mathrm{id}_A$ for $A \in \mathrm{ob}(\mathcal{C})$. For $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ the mapping $\mathbf{F}_M(\phi) \colon \mathbf{F}_M(B) \to \mathbf{F}_M(A)$ is given by right multiplication with ϕ. A functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ can be made into a right $\mu_{\mathcal{C}}$-module $M_{\mathbf{F}}$, where $M_{\mathbf{F}} = \bigoplus_{A\in\mathrm{ob}(\mathcal{C})} \mathbf{F}(A)$. For $f \in \mathbf{F}(B)$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ one has $f \cdot \phi = \mathbf{F}(\phi)(f)$.

For $A \in \mathrm{ob}(\mathcal{C})$, id_A is an idempotent in $\mu_{\mathcal{C}}$. Moreover, under the identification mentioned above $\mathbf{P}^A$ corresponds to the right $\mu_{\mathcal{C}}$-module $\mathrm{id}_A \cdot \mu_{\mathcal{C}}$.

2.4. Dimensions

Let R be a Dedekind domain, let $\mathcal{C}$ be an R-order category, and let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$. Then $\mathbf{F}$ has *projective R-dimension less than or equal to d* if it has a projective resolution $(\mathbf{P}_k, \partial^{\mathbf{P}}_k, \varepsilon_{\mathbf{F}})$ with $\mathbf{P}_k = 0$ for $k > d$. The minimal such number $d \in \mathbb{N}_0 \cup \{\infty\}$ is called the *projection R-dimension* of $\mathbf{F}$ and will be denoted by $\mathrm{proj.dim}(\mathbf{F})$. The numbers

$$\begin{aligned} \mathrm{gldim}_R(\mathcal{C}) &= \sup\{\, \mathrm{proj.dim}_R(\mathbf{F}) \mid \mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))\,\}, \\ \mathrm{Ldim}_R(\mathcal{C}) &= \sup\{\, \mathrm{proj.dim}_R(\mathbf{F}) \mid \mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))\,\}, \end{aligned} \tag{2.10}$$

will be called the *global R-dimension* and the *global R-lattice dimension* of $\mathcal{C}$, respectively. By a result of M. Auslander, one has

$$\mathrm{gldim}_R(\mathcal{C}) = \sup\{\, \mathrm{proj.dim}_R(\mathbf{F}) \mid \mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}^{\mathrm{f.g.}}))\,\} \tag{2.11}$$

(cf. [16, Theorem 9.12]). In particular,

$$\mathrm{Ldim}_R(\mathcal{C}) \leq \mathrm{gldim}_R(\mathcal{C}) \leq \mathrm{Ldim}_R(\mathcal{C}) + 1. \tag{2.12}$$

E.g., $\mathrm{Ldim}_R(\mathcal{C}) = 0$ if, and only if, every R-lattice functor is projective. An R-order category satisfying $\mathrm{Ldim}_R(\mathcal{C}) \leq 1$ will be called *pseudo-hereditary*. Such a category has the following property: any subfunctor $\mathbf{F}$ of a projective R-lattice functor $\mathbf{P}$ such that $\mathbf{P}/\mathbf{F}$ is an R-lattice functor is projective.

2.5. The Yoneda dual

Let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-category. For $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ one has an R-linear natural transformation $\mathbf{P}(\phi): \mathbf{P}^A \to \mathbf{P}^B$ given by composition with ϕ. Hence one has a functor

$$\_^{\circledast}: \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}) \longrightarrow \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})^{\mathrm{op}}, \tag{2.13}$$

given by $\mathbf{F}^{\circledast}(A) = \mathrm{nat}_R(\mathbf{F}, \mathbf{P}^A)$ for $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ and $A \in \mathrm{ob}(\mathcal{C})$. Moreover, for $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ one has $\mathbf{F}^{\circledast}(\phi) = \mathbf{P}(\sigma(\phi)) \circ \_ : \mathbf{F}^{\circledast}(B) \to \mathbf{F}^{\circledast}(A)$ for $\phi: A \to B \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. We call the functor $\_^{\circledast}$ the *Yoneda dual*.

Remark 2.6. Let μ be an R-algebra with R-linear antipode $\sigma : \mu \to \mu^{\mathrm{op}}$. Then, by Remark 2.5, $\mathfrak{F}_R(\mu\bullet^{\mathrm{op}}, {}_R\mathbf{mod})$ can be identified with the category of right μ-modules. Under this identification, the Yoneda dual satisfies $\_^{\circledast} = \mathrm{Hom}_{\mu}(\_, \mu)^{\times}$. Here we used the symbol $^{\times}$ to express that for a right μ-module M, the left μ-module $\mathrm{Hom}_{\mu}(M, \mu)$ is considered as right μ-module via the map σ.

The Yoneda dual has the following property.

Proposition 2.7. *Let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-category, and let $A \in \mathrm{ob}(\mathcal{C})$. Then one has a canonical natural isomorphism $j_A: (\mathbf{P}^A)^{\circledast} \to \mathbf{P}^A$ which is natural in A, i.e., for $\psi: A \to D \in \mathrm{Hom}_{\mathcal{C}}(A, D)$ one has a commutative diagram*

$$\begin{array}{ccc} (\mathbf{P}^D)^{\circledast} & \xrightarrow{j_D} & \mathbf{P}^D \\ {\scriptstyle \mathbf{P}(\psi)^{\circledast}}\downarrow & & \downarrow{\scriptstyle \mathbf{P}(\sigma(\psi))} \\ (\mathbf{P}^A)^{\circledast} & \xrightarrow{j_A} & \mathbf{P}^A. \end{array} \tag{2.14}$$

In particular, if R is a Dedekind domain and $(\mathcal{C}, \sigma)$ is an $R^{\circledast}$-order category, then $\_^{\circledast}$ maps projective R-lattice functors to projective R-lattice functors, and R-lattice functors to R-lattice functors.

Proof. Let $\phi: B \to C$ be a morphism in $\mathcal{C}$. By the definition of $\mathbf{P}$- and Fact 2.1, one has canonical isomorphisms

$$\begin{array}{ccccccc} (\mathbf{P}^A)^{\circledast}(B) & & \xrightarrow{\quad j_A(B) \quad} & & & & \mathbf{P}^A(B) \\ \| & & & & & & \| \\ \mathrm{nat}_R(\mathbf{P}^A, \mathbf{P}^B) & \xrightarrow{\theta_{A,\mathbf{P}^B}} & \mathbf{P}^B(A) & = & \mathrm{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\sigma} & \mathrm{Hom}_{\mathcal{C}}(B, A) \end{array} \tag{2.15}$$

and the diagram

$$\begin{array}{ccccc} \mathrm{nat}_R(\mathbf{P}^A, \mathbf{P}^C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A, C) & \xrightarrow{\sigma} & \mathrm{Hom}_{\mathcal{C}}(C, A) \\ {\scriptstyle \mathbf{P}(\sigma(\phi))\circ -}\downarrow & & \downarrow{\scriptstyle \sigma(\phi)\circ -} & & \downarrow{\scriptstyle -\circ\phi} \\ \mathrm{nat}_R(\mathbf{P}^A, \mathbf{P}^B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\sigma} & \mathrm{Hom}_{\mathcal{C}}(B, A) \end{array} \tag{2.16}$$

commutes. This shows that j_A is a natural isomorphism. The commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{nat}_R(\mathbf{P}^D, \mathbf{P}^B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(D, B) & \xrightarrow{\sigma} & \mathrm{Hom}_{\mathcal{C}}(B, D) \\ {\scriptstyle -\circ\mathbf{P}(\psi)}\downarrow & & \downarrow{\scriptstyle -\circ\psi} & & \downarrow{\scriptstyle \sigma(\psi)\circ-} \\ \mathrm{nat}_R(\mathbf{P}^A, \mathbf{P}^B) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\sigma} & \mathrm{Hom}_{\mathcal{C}}(B, A) \end{array} \tag{2.17}$$

shows the commutativity of (2.14). The final remark is straightforward. □

Let $A, B \in \mathrm{ob}(\mathcal{C})$, and let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$. For any $\chi \in \mathrm{nat}_R(\mathbf{F}, \mathbf{P}^B)$ one has an R-linear map

$$\sigma \circ \chi_A \colon \ \mathbf{F}(A) \xrightarrow{\chi_A} \mathbf{P}^B(A) \xrightarrow{\sigma_{A,B}} \mathbf{P}^A(B)\ . \tag{2.18}$$

Let $f \in \mathbf{F}(A)$, and let $\eta^{f,B}_{\mathbf{F},A} \colon \mathrm{nat}_R(\mathbf{F}, \mathbf{P}^B) \longrightarrow \mathbf{P}^A(B)$ be given by

$$\eta^{f,B}_{\mathbf{F},A}(\chi) = \sigma(\chi_A(f)). \tag{2.19}$$

For $\phi \colon B \to C \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ one has a commutative diagram

$$\begin{array}{ccc} \mathrm{nat}_R(\mathbf{F}, \mathbf{P}^C) & \xrightarrow{\eta^{f,C}_{\mathbf{F},A}} & \mathbf{P}^A(C) \\ {\scriptstyle \mathbf{P}(\sigma(\phi))\circ-}\downarrow & & \downarrow{\scriptstyle -\circ\phi} \\ \mathrm{nat}_R(\mathbf{F}, \mathbf{P}^B) & \xrightarrow{\eta^{f,B}_{\mathbf{F},A}} & \mathbf{P}^A(B). \end{array} \tag{2.20}$$

Hence $\eta^{f,-}_{\mathbf{F},A} \colon \mathbf{F}^{\circledast} \to \mathbf{P}^A$ is an R-linear natural transformation. The mapping

$$\eta_{\mathbf{F},A} \colon \mathbf{F}(A) \longrightarrow \mathrm{nat}_R(\mathbf{F}^{\circledast}, \mathbf{P}^A) \tag{2.21}$$

is R-linear, and for $\psi \colon D \to A \in \mathrm{Hom}_{\mathcal{C}}(D, A)$, the diagram

$$\begin{array}{ccc} \mathbf{F}(A) & \xrightarrow{\eta_{\mathbf{F},A}} & \mathrm{nat}_R(\mathbf{F}^{\circledast}, \mathbf{P}^A) \\ {\scriptstyle \mathbf{F}(\psi)}\downarrow & & \downarrow{\scriptstyle \mathbf{P}(\sigma(\psi))\circ-} \\ \mathbf{F}(D) & \xrightarrow{\eta_{\mathbf{F},D}} & \mathrm{nat}_R(\mathbf{F}^{\circledast}, \mathbf{P}^D) \end{array} \tag{2.22}$$

commutes. Thus it defines an R-linear, natural transformation $\eta_{\mathbf{F}} \colon \mathbf{F} \to \mathbf{F}^{\circledast\circledast}$. For a natural transformation $\alpha \in \mathrm{nat}_R(\mathbf{F}, \mathbf{G})$ and $A \in \mathrm{ob}(\mathcal{C})$ one has a commutative diagram

$$\begin{array}{ccc} \mathbf{F}(A) & \xrightarrow{\eta_{\mathbf{F},A}} & \mathrm{nat}_R(\mathbf{F}^{\circledast}, \mathbf{P}^A) \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle -\circ\alpha^{\circledast}} \\ \mathbf{G}(A) & \xrightarrow{\eta_{\mathbf{G},A}} & \mathrm{nat}_R(\mathbf{G}^{\circledast}, \mathbf{P}^A). \end{array} \tag{2.23}$$

Hence one has the following.

Proposition 2.8. *Let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-category. Then*

$$\eta\colon \mathrm{id}_{\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})} \longrightarrow {}_-^{\circledast\circledast} \tag{2.24}$$

is a natural transformation. For every $E \in \mathrm{ob}(\mathcal{C})$, $\eta_{\mathbf{P}^E}\colon \mathbf{P}^E \to (\mathbf{P}^E)^{\circledast\circledast}$ is a natural isomorphism. In particular, if R is a Dedekind domain and $(\mathcal{C}, \sigma)$ is an $R^{\circledast}$-order category, then $\eta_{\mathbf{P}}\colon \mathbf{P} \to \mathbf{P}^{\circledast\circledast}$ is an isomorphism for every projective R-lattice functor $\mathbf{P} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$.

Proof. It suffices to show that $\eta_{\mathbf{P}^E}\colon \mathbf{P}^E \to (\mathbf{P}^E)^{\circledast\circledast}$ is a natural isomorphism for every $E \in \mathrm{ob}(\mathcal{C})$. For $A \in \mathrm{ob}(\mathcal{C})$ one has a commutative diagram

$$\begin{array}{ccc} \mathbf{P}^A(E) & \xrightarrow{\theta^{-1}_{E,\mathbf{P}^A}} & \mathrm{nat}_R(\mathbf{P}^E, \mathbf{P}^A) \\ \sigma\big\downarrow & & \big\downarrow -\circ j_E \\ \mathbf{P}^E(A) & \xrightarrow{\eta_{\mathbf{P}^E,A}} & \mathrm{nat}_R((\mathbf{P}^E)^{\circledast}, \mathbf{P}^A), \end{array} \tag{2.25}$$

and all maps apart from $\eta_{\mathbf{P}^E,A}$ are isomorphisms (cf. (2.6), (2.15)). Hence $\eta_{\mathbf{P}^E,A}$ is an isomorphism, and this yields the claim.

2.6. *Derived functors of the Yoneda dual*

Let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-category such that $\mathrm{ob}(\mathcal{C})$ is a set. Then $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})$ is an abelian category with enough projectives (cf. Fact 2.3).

The Yoneda dual ${}_-^{\circledast}\colon \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}) \to \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})^{\mathrm{op}}$ is additive and left-exact. Let $\mathcal{R}^k(\_)^{\circledast}$, $k \geq 1$, denote its right-derived functors, i.e., one has that

$$\mathcal{R}^k(\mathbf{F})^{\circledast}(A) = \mathrm{ext}^k_R(\mathbf{F}, \mathbf{P}^A), \quad \text{for } \mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})), \tag{2.26}$$

and $\mathcal{R}^k(\mathbf{F})^{\circledast}(\phi) = \mathrm{ext}^k_R(\mathbf{F}, \mathbf{P}(\sigma(\phi)))$ for $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. A functor $\mathbf{F}$ will be called $\circledast$-*acyclic*, if $\mathcal{R}^k(\mathbf{F})^{\circledast} = 0$ for all $k > 0$. E.g., every projective functor is $\circledast$-acyclic.

Let R be a Dedekind domain, and let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-order category. An R-lattice functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ will be called $\circledast$-*bi-acyclic*, if $\mathbf{F}$ and $\mathbf{F}^{\circledast}$ are $\circledast$-acyclic. The $R^{\circledast}$-order category $(\mathcal{C}, \sigma)$ will be called $\circledast$-*symmetric*, if every $\circledast$-acyclic R-lattice functor is $\circledast$-bi-acyclic.

2.7. *Gorenstein projective functors*

Let R be a Dedekind domain, and let $(\mathcal{C}, \sigma)$ be an $R^{\circledast}$-order category. For a functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$ a chain complex $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)$ together with a natural transformation $\varepsilon\colon \mathbf{P}_0 \to \mathbf{F}$ will be called a *complete projective R-lattice functor resolution* of $\mathbf{F}$, if

(i) $\mathbf{P}_k$ is a projective R-lattice functor for all $k \in \mathbb{Z}$;
(ii) $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)$ is exact;
(iii) $\varepsilon \circ \partial_1 = 0$ and ε induces a natural isomorphism $\tilde{\varepsilon}\colon \mathrm{coker}(\partial_1) \to \mathbf{F}$.

For an exact chain complex of projective R-lattice functors $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)$ we denote by $(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet) = (\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)^{\circledast}$ the chain complex of projective R-lattice functors given by $\mathbf{Q}_k = \mathbf{P}^{\circledast}_{-k-1}$ and $\partial^{\mathbf{Q}}_k = (\partial^{\mathbf{P}}_{-k})^{\circledast}$. Note that by Proposition 2.8, $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)^{\circledast\circledast}$ is canonically isomorphic to $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)$. The complete projective R-lattice functor resolution $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet, \varepsilon_{\mathbf{F}})$ of $\mathbf{F}$ will be called $\circledast$-*exact* if

(iv) $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)^{\circledast}$ is exact.

An R-lattice functor with a $\circledast$-exact complete projective R-lattice functor resolution is also called a *Gorenstein projective functor*. One has the following property.

Proposition 2.9. *Let R be a Dedekind domain, let $(\mathcal{C},\sigma)$ be an $R^{\circledast}$-order category, and let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$ be an R-lattice functor. Then $\mathbf{F}$ is Gorenstein projective if, and only if, $\mathbf{F}$ is $\circledast$-bi-acyclic.*

Proof. Suppose that $\mathbf{F}$ is $\circledast$-bi-acyclic. By Fact 2.4, $\mathbf{F}$ has a projective resolution $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet, \varepsilon_{\mathbf{F}})$ by projective R-lattice functors. Let $\mathbf{Q}_k = \mathbf{P}^{\circledast}_{-k}$, $\partial^{\mathbf{Q}}_k = (\partial^{\mathbf{P}}_{1-k})^{\circledast}$. Then $(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet)$ is a chain complex of projective R-lattice functors concentrated in non-positive degrees (cf. Proposition 2.7). As $\mathbf{F}$ is $\circledast$-acyclic, one has

$$H_k(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet) \simeq \begin{cases} \mathbf{F}^{\circledast} & \text{for } k = 0; \\ 0 & \text{for } k \neq 0. \end{cases} \tag{2.27}$$

As $\mathbf{F}^{\circledast}$ is an R-lattice functor, it has a projective resolution $(\mathbf{R}_\bullet, \partial^{\mathbf{R}}_\bullet, \mu_{\mathbf{F}^{\circledast}})$ by projective R-lattice functors. Let $(\mathbf{S}_\bullet, \partial^{\mathbf{S}}_\bullet)$ be the chain complex given by $\mathbf{S}_k = \mathbf{R}_k$ for $k \geq 0$ and $\mathbf{S}_k = \mathbf{Q}_{k+1}$ for $k < 0$, and mappings $\partial^{\mathbf{S}}_k = \partial^{\mathbf{R}}_k$ for $k \geq 1$, $\partial^{\mathbf{S}}_k = \partial^{\mathbf{Q}}_{k+1}$ for $k \leq -1$, and $\partial_0 = \varepsilon^{\circledast}_{\mathbf{F}} \circ \mu_{\mathbf{F}^{\circledast}}$. Then $(\mathbf{S}_\bullet, \partial^{\mathbf{S}}_\bullet, \mu_{\mathbf{F}^{\circledast}})$ is a complete projective R-lattice functor resolution of $\mathbf{F}^{\circledast}$.

Let $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet) = (\mathbf{S}_\bullet, \partial^{\mathbf{S}}_\bullet)^{\circledast}$. Then one has $\mathbf{T}_0 = \mathbf{P}_0^{\circledast\circledast}$, $\rho = \varepsilon \circ \eta^{-1}_{\mathbf{P}_0} \colon \mathbf{T}_0 \to \mathbf{F}$ (cf. Proposition 2.8) is satisfying $\rho \circ \partial^{\mathbf{T}}_1 = 0$, and the induced map $\tilde{\rho} \colon \mathrm{coker}(\partial^{\mathbf{T}}_1) \to \mathbf{F}$ is a natural isomorphism. By construction and Proposition 2.8, one has $H_k(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet) = 0$ for $k > 0$. As $\mathbf{F}^{\circledast}$ is $\circledast$-acyclic, one has also $H_k(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet) = 0$ for $k < -1$.

Let $\mathbf{T}^{<0}_\bullet$ and $\mathbf{T}^{\geq 0}_\bullet$ denote the truncated chain complexes, respectively, and consider the short exact sequence of chain complexes $0 \to \mathbf{T}^{<0}_\bullet \to \mathbf{T}_\bullet \to \mathbf{T}^{\geq 0}_\bullet \to 0$. By construction, the connecting homomorphism $H_0(\delta) \colon H_0(\mathbf{T}^{\geq 0}_\bullet) \to H_{-1}(\mathbf{T}^{<0}_\bullet)$ is an isomorphism. The long exact sequence in homology implies that the chain complex $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet)$ has trivial homology, and hence is exact. Thus by Proposition 2.7, $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet, \rho)$ is a complete projective R-lattice functor resolution of $\mathbf{F}$. As $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet)^{\circledast}$ is canonically isomorphic to $(\mathbf{S}_\bullet, \partial^{\mathbf{S}}_\bullet)$, $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet, \rho)$ is also $\circledast$-exact.

Let $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet, \varepsilon_{\mathbf{F}})$ be a $\circledast$-exact complete projective R-lattice functor resolution of $\mathbf{F}$. Then

$$\mathcal{R}^k(\mathbf{F})^{\circledast} = H_{-k-1}((\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet)^{\circledast}) = 0, \quad k > 0, \tag{2.28}$$

i.e., $\mathbf{F}$ is $\circledast$-acyclic. Replacing the chain complex $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet)$ by the chain complex $(\mathbf{T}_\bullet, \partial^{\mathbf{T}}_\bullet)^{\circledast}$ shows that $\mathbf{F}^{\circledast}$ is also $\circledast$-acyclic. □

2.8. Gorenstein $R^{\circledast}$-order categories

Let R be a Dedekind[1] domain, and let $(\mathcal{C},\sigma)$ be an $R^{\circledast}$-order category. For $A \in \mathrm{ob}(\mathcal{C})$ the functors

$$\mathbf{J}^A = (\mathbf{P}^A)^* \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})) \tag{2.29}$$

are R-lattice functors which are *relative injective* in $\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat})$ in the following sense: let $\alpha \colon \mathbf{F} \to \mathbf{G}$ be a split-injective, R-linear transformation of R-lattice functors, and let $\beta \colon \mathbf{F} \to \mathbf{J}^A$

[1] For an arbitrary commutative ring R with 1 the kernel of a surjective homomorphism $\phi \colon M \to Q$ of R-lattices is not necessarily an R-lattice. This is the reason why we restrict all subsequent considerations to R-order categories over a Dedekind domain R.

be any R-linear natural transformation. Then there exists an R-linear natural transformation $\tilde{\beta}\colon \mathbf{G} \to \mathbf{J}^A$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbf{F} & \xrightarrow{\alpha} & \mathbf{G} \\ & & {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\tilde{\beta}} & \\ & & \mathbf{J}^A & & \end{array} \tag{2.30}$$

commutes. Here we called a natural transformation $\alpha\colon \mathbf{F} \to \mathbf{G}$ of R-lattice functors *split injective*, if it is injective and $\operatorname{coker}(\alpha_B)$ is an R-lattice for every $B \in \operatorname{ob}(\mathcal{C})$. The $R^{\circledast}$-order category $(\mathcal{C}, \sigma)$ is called *m-Gorenstein*, $m \geq 0$, if $\operatorname{proj.dim}(\mathbf{J}^A) \leq m$ for all $A \in \operatorname{ob}(\mathcal{C})$. A 0-Gorenstein $R^{\circledast}$-order category is also called *Frobenius*. For m-Gorenstein $R^{\circledast}$-order categories one has the following.

Proposition 2.10. *Let R be a Dedekind domain, and let $(\mathcal{C}, \sigma)$ be an m-Gorenstein $R^{\circledast}$-order category. Then for any $\mathbf{F} \in \operatorname{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$ and $k > m$ one has $\mathcal{R}^k(\mathbf{F})^{\circledast} = 0$.*

Proof. By Fact 2.4, $\mathbf{F}$ has a projective resolution $(\mathbf{P}_i, \partial_i^{\mathbf{P}}, \varepsilon_{\mathbf{F}})$ by projective R-lattice functors. Moreover, by hypothesis, for $A \in \operatorname{ob}(\mathcal{C})$ the functor $\mathbf{J}^A$ has a finite projective resolution $(\mathbf{Q}_j, \partial_j^{\mathbf{Q}}, \varepsilon_A)$ by projective R-lattice functors and $\mathbf{Q}_j = 0$ for $j > m$. Thus $\mathbf{P}^A \simeq (\mathbf{J}^A)^*$ has a finite, relative injective resolution $(\mathbf{I}^j, \delta^j, \mu_A)$, $\mathbf{I}^j = \mathbf{Q}_j^*$, $\delta^{j-1} = (\partial_j^{\mathbf{Q}})^*$, $\mu_A = \varepsilon_A^*$ with $\mathbf{I}^j = 0$ for $j > m$. Consider the double complex $(E_0^{s,t}, \partial_v, \partial_h)$, where $E_0^{s,t} = \operatorname{nat}_R(\mathbf{P}_s, \mathbf{I}^t)$ and ∂_v and ∂_h are the vertical and horizontal differentials induced by $\partial_\bullet^{\mathbf{P}}$ and $\delta^\bullet$, respectively. The cohomology of the total complex $(\operatorname{Tot}^\bullet(E_0^{s,t}), \partial_v + (-1)^\bullet \partial_h)$ can be calculated in two ways.

Applying first the vertical and then the horizontal differential yields a spectral sequence with E_2-term

$$ {}^vE_2^{s,t} = \begin{cases} \operatorname{ext}_R^s(\mathbf{F}, \mathbf{P}^A) & \text{for } t = 0, \\ 0 & \text{for } t \neq 0, \end{cases} \tag{2.31}$$

concentrated on the $(t = 0)$-line. By definition, $\operatorname{nat}_R(\_, \mathbf{I}^j)$ is exact for every short exact sequence of R-lattice functors. Since R is a Dedekind domain, $0 \to \ker(\partial_s^{\mathbf{P}}) \to \mathbf{P}_s \to \operatorname{im}(\partial_s^{\mathbf{P}}) \to 0$ is a short exact sequence of R-lattice functors for every $s \geq 0$. Hence applying first the horizontal and then the vertical differential yields a spectral sequence with E_1-term concentrated on the $(s = 0)$-line, and ${}^hE_1^{0,t} = 0$ for $t > m$. The claim then follows from the fact that both spectral sequences converge to the cohomology of the total complex. $\square$

2.9. *$R^{\circledast}$-order categories with the Whitehead property*

Let R be a Dedekind domain. The Gorenstein property of an $R^{\circledast}$-order category is a quantitative measurement for the failure of being Frobenius. However, for our main purpose another property plays a more important role. We say that an $R^{\circledast}$-order category $(\mathcal{C}, \sigma)$ has the *Whitehead property*,[2] if any $\circledast$-acyclic R-lattice functor is projective. The following property is well known (cf. [4, Proposition VIII.6.7]).

[2] The famous *Whitehead problem*, stated by J. H. Whitehead around 1950, is the question whether every abelian group A satisfying $\operatorname{Ext}_{\mathbb{Z}}^1(A, \mathbb{Z}) = 0$ must be a free abelian group. For finitely generated abelian groups this is easily seen to be true, and K. Stein showed (cf. [20]) that the statement remains valid for countable abelian groups. However, by the extra-ordinary work of S. Shelah (cf. [17–19]) one knows now that this problem is in general undecidable.

Fact 2.11. *Let R be a Dedekind domain, and let $(\mathcal{C},\sigma)$ be an $R^{\circledast}$-order category of finite global R-lattice dimension. Then $(\mathcal{C},\sigma)$ is Gorenstein and has the Whitehead property. Moreover,*

$$\mathrm{Ldim}_R(\mathcal{C}) = \max\{\, k \geq 0 \mid \mathcal{R}^k(\_)^{\circledast} \neq 0\,\}. \tag{2.32}$$

Remark 2.12. Let R be a Dedekind domain, and let $(\mathcal{C},\sigma)$ be an $R^{\circledast}$-order category. For $m \geq 0$ one has the implications

$$\mathrm{gldim}_R(\mathcal{C}) \leq m \implies \mathcal{C}\ m\text{-Gorenstein \& Whitehead} \implies \mathcal{C}\ m\text{-Gorenstein}. \tag{2.33}$$

If G is a finite group, then $(\mathbb{Z}[G]\bullet,\sigma)$, where $\sigma(g) = g^{-1}$ for $g \in G$, is 0-Gorenstein. But $(\mathbb{Z}[G]\bullet,\sigma)$ has the Whitehead property if, and only if, G is the trivial group. Hence the second implication cannot be reversed. For certain values of m one can reverse the first implication. E.g., if $(\mathcal{C},\sigma)$ is 0-Gorenstein, then it has the Whitehead property if, and only if, every R-lattice functor is projective, i.e., $\mathrm{Ldim}_R(\mathcal{C}) = 0$. This is also the case for $m = 1$.

Fact 2.13. *Let R be a Dedekind domain, and let $(\mathcal{C},\sigma)$ be a 1-Gorenstein $R^{\circledast}$-order category. Then $(\mathcal{C},\sigma)$ has the Whitehead property if, and only if, $\mathcal{C}$ is pseudo-hereditary, i.e., one has* $\mathrm{Ldim}_R(\mathcal{C}) \leq 1$.

Proof. By Fact 2.11, it suffices to show the reverse direction of the first implication of (2.33). Suppose that $(\mathcal{C},\sigma)$ is 1-Gorenstein and has the Whitehead property. Let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$. Then there exists a surjective natural transformation $\pi : \mathbf{P} \to \mathbf{F}$ for some projective R-lattice functor $\mathbf{P}$, and $\mathbf{Q} = \ker(\pi)$ is an R-lattice functor. By Proposition 2.10 and the long exact sequence, the sequence

$$\mathcal{R}^1(\mathbf{F})^{\circledast} \longrightarrow \mathcal{R}^1(\mathbf{P})^{\circledast} \longrightarrow \mathcal{R}^1(\mathbf{Q})^{\circledast} \longrightarrow 0 \tag{2.34}$$

is exact. As $\mathcal{R}^1(\mathbf{P})^{\circledast} = 0$, $\mathbf{Q}$ is $\circledast$-acyclic and thus, by hypothesis, projective. □

2.10. Functors between $R^{\circledast}$-categories

Let $(\mathcal{C},\sigma_{\mathcal{C}})$ and $(\mathcal{D},\sigma_{\mathcal{D}})$ be $R^{\circledast}$-categories. An R-linear functor $\phi\colon \mathcal{C} \to \mathcal{D}$ will be called *unitary*, if

$$\sigma_{\mathcal{D}} \circ \phi = \phi \circ \sigma_{\mathcal{C}}. \tag{2.35}$$

If $\mathrm{ob}(\mathcal{C})$ and $\mathrm{ob}(\mathcal{D})$ are sets, the unitary functor $\pi : (\mathcal{C},\sigma_{\mathcal{C}}) \to (\mathcal{D},\sigma_{\mathcal{D}})$ will be called a *unitary projection*, if $\pi : \mathrm{ob}(\mathcal{C}) \longrightarrow \mathrm{ob}(\mathcal{D})$ is a bijection, and

$$\pi_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(\pi(A),\pi(B)) \tag{2.36}$$

is surjective for any pair of objects $A, B \in \mathrm{ob}(\mathcal{C})$. For such a functor composition with π induces an exact inflation functor

$$\mathrm{inf}^{\pi}(\_) = \_ \circ \pi : \mathfrak{F}_R(\mathcal{D}^{\mathrm{op}}, {}_R\mathbf{mod}) \longrightarrow \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}). \tag{2.37}$$

Let $\pi : (\mathcal{C},\sigma_{\mathcal{C}}) \to (\mathcal{D},\sigma_{\mathcal{D}})$ be a unitary projection of $R^{\circledast}$-order categories. Then π induces a surjective homomorphisms of R-orders $\mu(\pi)\colon \mu_{\mathcal{C}} \to \mu_{\mathcal{D}}$ (cf. Remark 2.5). Moreover, the inflation functor $\mathrm{inf}^{\mu_{\mathcal{C}}}_{\mu_{\mathcal{D}}}(\_)\colon \mathbf{mod}_{\mu_{\mathcal{D}}} \to \mathbf{mod}_{\mu_{\mathcal{C}}}$ has a left-adjoint

$$\mathrm{def}^{\mu_{\mathcal{C}}}_{\mu_{\mathcal{D}}}(\_) = \_ \otimes_{\mu_{\mathcal{C}}} \mu_{\mathcal{D}}\colon \mathbf{mod}_{\mu_{\mathcal{C}}} \to \mathbf{mod}_{\mu_{\mathcal{D}}}. \tag{2.38}$$

From this fact one concludes the following.

Fact 2.14. *Let $\pi : (\mathcal{C}, \sigma_{\mathcal{C}}) \to (\mathcal{D}, \sigma_{\mathcal{D}})$ be a unitary projection of $R^{\circledast}$-order categories.*

(a) *There exists a functor*

$$\mathrm{def}^{\pi}(\_) : \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}) \longrightarrow \mathfrak{F}_R(\mathcal{D}^{\mathrm{op}}, {}_R\mathbf{mod}) \tag{2.39}$$

which is left-adjoint to $\mathrm{inf}^{\pi}(\_)$.

(b) *The unit of the adjunction* $\eta : \mathrm{id}_{\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})} \longrightarrow \mathrm{inf}^{\pi} \circ \mathrm{def}^{\pi}$ *is a natural surjection, and the co-unit* $\varepsilon : \mathrm{def}^{\pi} \circ \mathrm{inf}^{\pi} \longrightarrow \mathrm{id}_{\mathfrak{F}_R(\mathcal{D}^{\mathrm{op}}, {}_R\mathbf{mod})}$ *is a natural isomorphism.*

(c) *For all $A \in \mathrm{ob}(\mathcal{C})$ there exists an isomorphism $\xi_A : \mathrm{def}^{\pi}(\mathbf{P}^A) \to \mathbf{P}^{\pi(A)}$ making the diagram*

$$\begin{array}{ccc} \mathrm{def}^{\pi}(\mathbf{P}^A) & \xrightarrow{\xi_A} & \mathbf{P}^{\pi(A)} \\ {\scriptstyle \mathrm{def}^{\pi}(\mathbf{P}(\phi))}\downarrow & & \downarrow{\scriptstyle \mathbf{P}(\pi(\phi))} \\ \mathrm{def}^{\pi}(\mathbf{P}^B) & \xrightarrow{\xi_B} & \mathbf{P}^{\pi(B)} \end{array} \tag{2.40}$$

commute for all $\phi : A \to B \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

For $A \in \mathrm{ob}(\mathcal{C})$ and $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ put $\mathbf{Q}^A = \mathrm{inf}^{\pi}(\mathbf{P}^{\pi(A)})$ and $\mathbf{Q}(\phi) = \mathrm{inf}^{\pi}(\mathbf{P}(\pi(\phi)))$. Then

$$\tau_A : \mathbf{P}^A \xrightarrow{\ \eta_{\mathbf{P}^A}\ } \mathrm{inf}^{\pi}(\mathrm{def}^{\pi}(\mathbf{P}^A)) \xrightarrow{\ \mathrm{inf}^{\pi}(\xi_A)\ } \mathbf{Q}^A \tag{2.41}$$

is a surjection satisfying $\tau_B \circ \mathbf{P}(\phi) = \mathbf{Q}(\phi) \circ \tau_A$ for all $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

For $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ let $\mathbf{F}^{\boxtimes} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ be the functor given by $\mathbf{F}^{\boxtimes}(A) = \mathrm{nat}_R(\mathbf{F}, \mathbf{Q}^A)$ and $\mathbf{F}^{\boxtimes}(\phi) = \mathbf{Q}(\sigma_{\mathcal{C}}(\phi)) \circ \_$ for $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Then $\_^{\boxtimes} : \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}) \to \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod})^{\mathrm{op}}$ is a functor. By (2.35), one has

$$\mathbf{P}(\pi(\sigma_{\mathcal{C}}(\phi))) = \mathbf{P}(\sigma_{\mathcal{D}}(\pi(\phi))) : \mathbf{P}^{\pi(B)} \to \mathbf{P}^{\pi(A)} \tag{2.42}$$

for all $\phi \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. Hence the mapping $\tilde{\tau} : \_^{\circledast} \longrightarrow \_^{\boxtimes}$ induced by τ is a natural transformation. Since

$$\mathbf{F}^{\boxtimes}(A) = \mathrm{nat}_R(\mathbf{F}, \mathbf{Q}^A) \simeq \mathrm{nat}_R(\mathrm{def}^{\pi}(\mathbf{F}), \mathbf{P}^{\pi(A)}) = \mathrm{inf}^{\pi}(\mathrm{def}^{\pi}(\mathbf{F})^{\circledast}), \tag{2.43}$$

$\_^{\boxtimes}$ can be identified with $\mathrm{inf}^{\pi}(\mathrm{def}^{\pi}(\_)^{\circledast})$. Thus by the left-adjointness of $\mathrm{def}^{\pi}(\_)$, $\tilde{\tau}$ induces a natural transformation

$$\hat{\tau} : \mathrm{def}^{\pi}(\_^{\circledast}) \longrightarrow \mathrm{def}^{\pi}(\_)^{\circledast} : \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}) \longrightarrow \mathfrak{F}_R(\mathcal{D}^{\mathrm{op}}, {}_R\mathbf{mod})^{\mathrm{op}}. \tag{2.44}$$

For $A \in \mathrm{ob}(\mathcal{C})$ the mapping $\hat{\tau}_{\mathbf{P}^A} : \mathrm{def}^{\pi}((\mathbf{P}^A)^{\circledast}) \to \mathrm{def}^{\pi}(\mathbf{P}_A)^{\circledast}$ coincides with the isomorphism

$$\hat{\tau}_{\mathbf{P}^A} : \mathrm{Hom}_{\mu_{\mathcal{C}}}(\mathrm{id}_A \cdot \mu_{\mathcal{C}}, \mu_{\mathcal{C}})^{\times} \otimes_{\mu_{\mathcal{C}}} \mu_{\mathcal{D}} \longrightarrow \mathrm{Hom}_{\mu_{\mathcal{D}}}(\mathrm{id}_{\pi(A)} \cdot \mu_{\mathcal{D}}, \mu_{\mathcal{D}})^{\times}. \tag{2.45}$$

From this fact one concludes the following.

Fact 2.15. *Let $\pi : (\mathcal{C}, \sigma_{\mathcal{C}}) \to (\mathcal{D}, \sigma_{\mathcal{D}})$ be a unitary projection of $R^{\circledast}$-order categories. Then $\hat{\tau}_{\mathbf{P}} : \mathrm{def}^{\pi}(\mathbf{P}^{\circledast}) \to \mathrm{def}^{\pi}(\mathbf{P})^{\circledast}$ is an isomorphism for every projective R-lattice functor $\mathbf{P} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$.*

If $\pi : (\mathcal{C}, \sigma_{\mathcal{C}}) \to (\mathcal{D}, \sigma_{\mathcal{D}})$ is a unitary projection of $R^{\circledast}$-order categories, its deflation functor $\mathrm{def}^{\pi}(\_) : \mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}) \to \mathfrak{F}_R(\mathcal{D}^{\mathrm{op}}, {}_R\mathbf{mod})$ is right exact and maps projectives to projectives (cf. [25, Proposition 2.3.10]). We denote by $\mathcal{L}_k\mathrm{def}^{\pi}(\_)$ its left derived functors. Functors $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{mod}))$ satisfying $\mathcal{L}_k\mathrm{def}^{\pi}(\mathbf{F}) = 0$ for all $k > 0$ will be called *π-acyclic*.

Theorem 2.16. *Let R be a Dedekind domain, and let $\pi : (\mathcal{C}, \sigma_{\mathcal{C}}) \to (\mathcal{D}, \sigma_{\mathcal{D}})$ be a unitary projection of $R^{\circledast}$-order categories. Assume further that*

(i) *$(\mathcal{C}, \sigma_{\mathcal{C}})$ is $\circledast$-symmetric (cf. Section 2.6);*
(ii) *$(\mathcal{D}, \sigma_{\mathcal{D}})$ has the Whitehead property (cf. Section 2.9);*
(iii) *if $\mathbf{F}^{\circledast} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$ is $\circledast$-acyclic, $\mathbf{F}$ is also π-acyclic.*

Then $\mathrm{def}^{\pi}(\mathbf{G})$ is a projective R-lattice functor for any $\circledast$-acyclic R-lattice functor $\mathbf{G} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$.

Proof. Suppose that $\mathbf{G} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{C}^{\mathrm{op}}, {}_R\mathbf{lat}))$ is $\circledast$-acyclic. By hypothesis (i), $\mathbf{G}$ is $\circledast$-bi-acyclic and thus Gorenstein projective (cf. Proposition 2.9), i.e., $\mathbf{G}$ admits a $\circledast$-exact complete projective R-lattice functor resolution $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet, \varepsilon_{\mathbf{G}})$. Shifting the chain complex $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)$ appropriately, one concludes that every functor $\mathbf{C}_k = \mathrm{coker}(\partial_{k+1})$ admits a $\circledast$-exact complete projective R-lattice functor resolution for all $k \in \mathbb{Z}$, and hence is Gorenstein projective. Thus by Proposition 2.9, $\mathbf{C}_k$ is $\circledast$-bi-acyclic, i.e., $\mathbf{C}_k$ and $\mathbf{C}_k^{\circledast}$ are π-acyclic for all $k \in \mathbb{Z}$.

Let $(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet) = (\mathrm{def}^{\pi}(\mathbf{P}_\bullet), \mathrm{def}^{\pi}(\partial^{\mathbf{P}}_\bullet))$. As $H_k(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet) \simeq \mathcal{L}_1\mathrm{def}^{\pi}(\mathbf{C}_{k-1}) = 0$, $(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet)$ is exact. By Fact 2.15, one has an isomorphism of chain complexes

$$(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet)^{\circledast} \simeq \mathrm{def}^{\pi}((\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet)^{\circledast}). \tag{2.46}$$

Moreover, as $H_k((\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet)^{\circledast}) \simeq \mathcal{L}_1\mathrm{def}^{\pi}(\mathbf{C}^{\circledast}_{1-k}) = 0$ (cf. Section 2.7), $(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet)^{\circledast}$ is also exact. Hence $(\mathbf{Q}_\bullet, \partial^{\mathbf{Q}}_\bullet, \mathrm{def}^{\pi}(\varepsilon_{\mathbf{G}}))$ is a $\circledast$-exact complete projective R-lattice functor resolution of $\mathrm{def}^{\pi}(\mathbf{G})$. In particular, $\mathrm{def}^{\pi}(\mathbf{G})$ is Gorenstein projective and thus $\circledast$-bi-acyclic (cf. Proposition 2.9). Hence by hypothesis (ii), $\mathrm{def}^{\pi}(\mathbf{G})$ is projective. □

3. Cohomological Mackey functors

Throughout this section G will denote a finite group, and – if not stated otherwise – R will denote a commutative ring with unit $1_R \in R$.

3.1. Cohomological G-Mackey functors

A *cohomological G-Mackey functor* $\mathbf{X}$ with values in the category of R-modules is a family of R-modules $(\mathbf{X}_U)_{U \subseteq G}$ together with homomorphisms of R-modules

$$\begin{aligned} i^{\mathbf{X}}_{U,V} &: \mathbf{X}_U \longrightarrow \mathbf{X}_V, \\ t^{\mathbf{X}}_{V,U} &: \mathbf{X}_V \longrightarrow \mathbf{X}_U, \\ c^{\mathbf{X}}_{g,U} &: \mathbf{X}_U \longrightarrow \mathbf{X}_{{}^gU}, \end{aligned} \tag{3.1}$$

for $U, V \subseteq G$, $V \subseteq U$, $g \in G$, which satisfies the identities:

(cMF$_1$) $i^{\mathbf{X}}_{U,U} = t^{\mathbf{X}}_{U,U} = c^{\mathbf{X}}_{u,U} = \mathrm{id}_{\mathbf{X}_U}$ for all $U \subseteq G$ and all $u \in U$;
(cMF$_2$) $i^{\mathbf{X}}_{V,W} \circ i^{\mathbf{X}}_{U,V} = i^{\mathbf{X}}_{U,W}$ and $t^{\mathbf{X}}_{V,U} \circ t^{\mathbf{X}}_{W,V} = t^{\mathbf{X}}_{W,U}$ for all $U, V, W \subseteq G$ and $W \subseteq V \subseteq U$;

(cMF₃) $c^{\mathbf{X}}_{h,{}^gU} \circ c^{\mathbf{X}}_{g,U} = c^{\mathbf{X}}_{hg,U}$ for all $U \subseteq G$ and $g, h \in G$;
(cMF₄) $i^{\mathbf{X}}_{{}^gU,{}^gV} \circ c^{\mathbf{X}}_{g,U} = c^{\mathbf{X}}_{g,V} \circ i^{\mathbf{X}}_{U,V}$ for all $U, V \subseteq G$, $V \subseteq U$, and $g \in G$;
(cMF₅) $t^{\mathbf{X}}_{{}^gV,{}^gU} \circ c^{\mathbf{X}}_{g,V} = c^{\mathbf{X}}_{g,U} \circ t^{\mathbf{X}}_{V,U}$ for all $U, V \subseteq G$, $V \subseteq U$, and $g \in G$;
(cMF₆) $i^{\mathbf{X}}_{U,W} \circ t^{\mathbf{X}}_{V,U} = \sum_{g\in W\backslash U/V} t^{\mathbf{X}}_{{}^gV\cap W,W} \circ c^{\mathbf{X}}_{g,V\cap W^g} \circ i^{\mathbf{X}}_{V,V\cap W^g}$, where $W^g = g^{-1}Wg$ for all subgroups $U, V, W \subseteq G$ and $V, W \subseteq U$;
(cMF₇) $t^{\mathbf{X}}_{V,U} \circ i^{\mathbf{X}}_{U,V} = |U : V|.\mathrm{id}_{\mathbf{X}_U}$ for all subgroups $U, V \subseteq G$, $V \subseteq U$.

A homomorphism of cohomological Mackey functors $\phi\colon \mathbf{X} \to \mathbf{Y}$ is a family of R-module homomorphisms $\phi_U\colon \mathbf{X}_U \to \mathbf{Y}_U$, $U \subseteq G$ which commute with all the mappings $i_{.,.}$, $t_{.,.}$ and $c_{g,.}$, $g \in G$. By $\mathfrak{cMF}_G({}_R\mathbf{mod})$ we denote the abelian category of all cohomological G-Mackey functors with values in the category of R-modules. For $\mathbf{X}, \mathbf{Y} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$ we denote by $\mathrm{nat}_G(\mathbf{X}, \mathbf{Y})$ the morphisms in the category $\mathfrak{cMF}_G({}_R\mathbf{mod})$. For further details on Mackey functors see [9,23,24].

3.2. The Mackey category

Let $\mathcal{M}(G)$ be the category the objects of which are subgroups of G with morphisms given by

$$\mathrm{Hom}_{\mathcal{M}(G)}(U, V) = \mathrm{Hom}_G(\mathbb{Z}[G/U], \mathbb{Z}[G/V]). \tag{3.2}$$

Then $\mathcal{M}(G)$ is a $\mathbb{Z}$-order category which is generated by the morphisms

$$\rho^U_g : \mathbb{Z}[G/{}^gU] \longrightarrow \mathbb{Z}[G/U], \qquad \rho^U_g(xgUg^{-1}) = xgU; \tag{3.3}$$

$$\mathfrak{i}_{V,U} : \mathbb{Z}[G/V] \longrightarrow \mathbb{Z}[G/U], \qquad \mathfrak{i}_{V,U}(xV) = xU; \tag{3.4}$$

$$\mathfrak{t}_{U,V} : \mathbb{Z}[G/U] \longrightarrow \mathbb{Z}[G/V], \qquad \mathfrak{t}_{U,V}(xU) = \sum_{r\in\mathcal{R}} xrV; \tag{3.5}$$

$g \in G$, $U, V \subseteq G$, $V \subseteq U$, where $\mathcal{R} \subseteq U$ is a set of right V-coset representatives. The assignment

$$\sigma(U) = U, \qquad \sigma(\rho^U_g) = \rho^{{}^gU}_{g^{-1}}, \qquad \sigma(\mathfrak{i}_{V,U}) = \mathfrak{t}_{U,V}, \qquad \sigma(\mathfrak{t}_{U,V}) = \mathfrak{i}_{V,U}, \tag{3.6}$$

for $U, V \subseteq G$, $V \subseteq U$, $g \in G$, defines an antipode $\sigma\colon \mathcal{M}(G) \to \mathcal{M}(G)^{op}$. Let $\mathcal{M}_R(G)$ denote the R-order category obtained from $\mathcal{M}(G)$ by tensoring with R. Assigning to every cohomological G-Mackey functor $\mathbf{X}$ with values in ${}_R\mathbf{mod}$ the contravariant functor $\tilde{\mathbf{X}}$ given by

$$\tilde{\mathbf{X}}(U) = \mathbf{X}_U, \qquad \tilde{\mathbf{X}}(\rho^U_g) = c^{\mathbf{X}}_{g,U}, \qquad \tilde{\mathbf{X}}(\mathfrak{i}_{V,U}) = i^{\mathbf{X}}_{U,V}, \qquad \tilde{\mathbf{X}}(\mathfrak{t}_{U,V}) = t^{\mathbf{X}}_{V,U}, \tag{3.7}$$

yields an identification between $\mathfrak{cMF}_G({}_R\mathbf{mod})$ and $\mathfrak{F}_R(\mathcal{M}_R(G)^{\mathrm{op}}, {}_R\mathbf{mod})$. Note that some authors prefer to identify the category of cohomological Mackey functors with the category of covariant functors of $\mathcal{M}_R(G)$. The existence of the antipode $\sigma\colon \mathcal{M}_G(R) \to \mathcal{M}_G(R)^{\mathrm{op}}$ shows that both approaches are equivalent.

3.3. The cohomological Mackey functors $\mathbf{\Upsilon}$ and $\mathbf{T}$

Let G be a finite group. There are two particular cohomological G-Mackey functors based on the R-module R. Let $\mathbf{\Upsilon} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$ be given by

$$\mathbf{\Upsilon}_U = R, \qquad i^{\mathbf{\Upsilon}}_{U,V} = |U : V|\mathrm{id}_R, \qquad t^{\mathbf{\Upsilon}}_{V,U} = \mathrm{id}_R, \qquad c^{\mathbf{\Upsilon}}_{g,U} = \mathrm{id}_R, \tag{3.8}$$

and $\mathbf{T} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$ be given by

$$\mathbf{T}_U = R, \qquad i^{\mathbf{T}}_{U,V} = \mathrm{id}_R, \qquad t^{\mathbf{T}}_{V,U} = |U : V|\mathrm{id}_R, \qquad c^{\mathbf{T}}_{g,U} = \mathrm{id}_R, \tag{3.9}$$

for $U, V \subseteq G$, $V \subseteq U$. Then $\boldsymbol{\Upsilon}$ and $\mathbf{T}$ are R-lattice functors, and one has $\mathbf{T} \simeq \boldsymbol{\Upsilon}^*$.

Let R be an integral domain of characteristic 0. For such a ring the subfunctor $\boldsymbol{\Sigma} \subseteq \mathbf{T}$ given by $\boldsymbol{\Sigma}_U = |U| \cdot \mathbf{T}_U$ is canonically isomorphic to $\boldsymbol{\Upsilon}$, i.e., there exists a canonical injective natural transformation $j : \boldsymbol{\Upsilon} \to \mathbf{T}$. We denote by $\mathbf{B} = \mathrm{coker}(j)$ the cokernel of this canonical map.

Let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$, and let $\phi : \mathbf{T} \to \mathbf{X}$ be a natural transformation. Then ϕ is uniquely determined by $\phi_G : \mathbf{T}_G \to \mathbf{X}_G$, and every such morphism defines a unique natural transformation $\phi : \mathbf{T} \to \mathbf{X}$. Hence one has a canonical isomorphism

$$\mathrm{nat}_G(\mathbf{T}, \mathbf{X}) \simeq \mathbf{X}_G. \tag{3.10}$$

In a similar fashion one shows that

$$\mathrm{nat}_G(\boldsymbol{\Upsilon}, \mathbf{X}) \simeq \mathbf{X}^G_{\{1\}}. \tag{3.11}$$

3.4. Invariants and coinvariants

There are two standard procedures which turn a left $R[G]$-module M into a cohomological G-Mackey functor with values in the category of R-modules. By $\mathbf{h}^0(M)$ we denote what is called the *fixed-point-functor* in [23]. In more detail, one has $\mathbf{h}^0(M)_U = M^U$, for $U, V \subseteq G$, $V \subseteq U$, $i^{\mathbf{h}^0(M)}_{U,V} : M^U \to M^V$ is the canonical map, $t^{\mathbf{h}^0(M)}_{V,U} : M^V \to M^U$ is given by the transfer, i.e., if $\mathcal{R} \subseteq U$ denotes a system of coset representative of U/V then $t^{\mathbf{h}^0(M)}_{V,U}$ is given by multiplication with $\sum_{r\in\mathcal{R}} r$, and $c^{\mathbf{h}^0(M)}_{g,U} : M^U \to M^{{}^gU}$ is left-multiplication by $g \in G$.

By $\mathbf{h}_0(M)$ we denote the cohomological G-Mackey functors of *coinvariants*. Thus $\mathbf{h}_0(M)_U = M/\omega_{R[U]}M$, where $\omega_{R[U]} = \ker(R[U] \to R)$ is the augmentation ideal in $R[U]$, and for $U, V \subseteq G$, $V \subseteq U$, $t^{\mathbf{h}_0(M)}_{V,U} : M_V \to M_U$ is the canonical map, the map $i^{\mathbf{h}_0(M)}_{U,V} : M_U \to M_V$ is induced by multiplication with $\sum_{r\in\mathcal{R}} r^{-1}$, and the map $c^{\mathbf{h}^0(M)}_{g,U} : M_U \to M_{{}^gU}$ is induced by multiplication with $g \in G$. E.g., one has canonical isomorphisms of cohomological G-Mackey functors $\boldsymbol{\Upsilon} \simeq \mathbf{h}_0(R)$ and $\mathbf{T} \simeq \mathbf{h}^0(R)$, where R denotes the trivial left $R[G]$-module.

3.5. Standard projective cohomological Mackey functors

By Section 2.3, one knows that for $W \subseteq G$ the functor

$$\mathbf{P}^W = \mathrm{Hom}_G(R[G/\_], R[G/W]) \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod})), \tag{3.12}$$

where $\mathbf{P}^W_U = \mathrm{Hom}_G(R[G/U], R[G/W]) = R[G/W]^U$, is projective in $\mathfrak{cMF}_G({}_R\mathbf{mod})$. These functors can be described as follows.

Fact 3.1. *Let G be a finite group, and let $W \subseteq G$. Then one has canonical isomorphisms*

$$\mathbf{P}^W \simeq \mathbf{h}^0(R[G/W]) \simeq \mathbf{h}^0(\mathbf{P}^W_{\{1\}}) \simeq \mathbf{ind}^G_W(\mathbf{T}^W), \tag{3.13}$$

where $\mathbf{ind}^G_W(\_)$ denotes the induction functor in the category of Mackey functors (cf. [23, Section 4]), and $\mathbf{T}^W \in \mathrm{ob}(\mathfrak{cMF}_W({}_R\mathbf{mod}))$ is the cohomological W-Mackey functor described in Section 3.3.

Both descriptions of the standard projective cohomological G-Mackey functors will be useful for our purpose. Note that one has canonical isomorphisms $\mathbf{P}^G \simeq \mathbf{T}$, i.e., $\mathbf{T}$ is projective. We also put $\mathbf{Q} = \mathbf{P}^{\{1\}} = \mathbf{h}^0(R[G])$.

Remark 3.2. Let G be a finite p-group, and let R be a discrete valuation domain of characteristic 0 with maximal ideal pR. For $W \subseteq G$ there exists a simple cohomological G-Mackey functor $\mathbf{S}^W$ with values in the category of R-modules given by

$$\mathbf{S}^W_U = \begin{cases} \mathbb{F} & \text{for } U = {}^g W; \\ 0 & \text{for } U \neq {}^g W. \end{cases} \tag{3.14}$$

In particular, for $U, V \subseteq G$, $V \subsetneq U$, one has $i^{\mathbf{S}^W}_{U,V} = 0$ and $t^{\mathbf{S}^W}_{V,U} = 0$. Moreover, any simple cohomological G-Mackey functor is isomorphic to some $\mathbf{S}^W$, $W \subseteq G$ (cf. [23]). The Nakayama relations and (3.10) show that for $V \subseteq G$ and $V \neq {}^g W$ one has

$$\mathrm{nat}_G(\mathbf{P}^V, \mathbf{S}^W) = \mathrm{nat}_V(\mathbf{T}, \mathrm{res}^G_V(\mathbf{S}^W)) \simeq \mathbf{S}^W_V = 0. \tag{3.15}$$

On the other hand for $V = {}^g W$ one has

$$\mathrm{nat}_G(\mathbf{P}^V, \mathbf{S}^W) = \mathrm{nat}_V(\mathbf{T}, \mathrm{res}^G_V(\mathbf{S}^W)) \simeq \mathbf{S}^W_V = \mathbb{F}. \tag{3.16}$$

Hence $\mathbf{P}^W$ is the (minimal) projective cover of $\mathbf{S}^W$ for all $W \subseteq G$.

3.6. Standard relative injective cohomological Mackey functors

Let G be a finite group, and $W \subseteq G$. The functor $\mathbf{ind}^G_W(\_)$ commutes with the functor $\_^*$ on R-lattice functors, i.e., one has a natural isomorphism

$$\mathbf{ind}^G_W(\_^*) \simeq \mathbf{ind}^G_W(\_)^* \colon \mathfrak{cMF}_W({}_R\mathbf{lat}) \longrightarrow \mathfrak{cMF}_G({}_R\mathbf{lat})^{\mathrm{op}}. \tag{3.17}$$

Thus $\mathbf{J}^W = (\mathbf{P}^W)^* \simeq \mathbf{ind}^G_W(\boldsymbol{\Upsilon}) \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$.

3.7. The Yoneda dual

Let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$. As $\mathbf{P}^W$, $W \subseteq G$, takes values in the category of R-lattices, the Nakayama relations and (3.11) yield canonical isomorphisms

$$\begin{aligned} \mathrm{nat}_G(\mathbf{X}, \mathbf{P}^W) &\simeq \mathrm{nat}_G(\mathbf{J}^W, \mathbf{X}^*) \simeq \mathrm{nat}_G(\mathbf{ind}^G_W(\boldsymbol{\Upsilon}), \mathbf{X}^*) \\ &\simeq \mathrm{nat}_W(\boldsymbol{\Upsilon}, \mathrm{res}^G_W(\mathbf{X}^*)) \simeq (\mathbf{X}^*_{\{1\}})^W. \end{aligned} \tag{3.18}$$

From this one concludes the following property (cf. Fact 3.1).

Fact 3.3. *Let* $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$. *Then* $\mathbf{X}^{\circledast} \simeq \mathbf{h}^0(\mathbf{X}^*_{\{1\}})$.

4. Section cohomology of cohomological Mackey functors

If not stated otherwise R will denote a commutative ring with unit $1_R \in R$. Let G be a finite group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$. For $U, V \subseteq G$, $V \lhd U$, one defines the *section cohomology groups* of $\mathbf{X}$ by

$$\begin{aligned} &\mathbf{k}^0(U/V, \mathbf{X}) = \ker(i^{\mathbf{X}}_{U,V}), && \mathbf{k}^1(U/V, \mathbf{X}) = \mathbf{X}^U_V, \\ &\mathbf{c}_0(U/V, \mathbf{X}) = \mathrm{coker}(t^{\mathbf{X}}_{V,U}), && \mathbf{c}_1(U/V, \mathbf{X}) = \ker(t^{\mathbf{X}}_{V,U})/\omega_{U/V}\mathbf{X}_V. \end{aligned} \tag{4.1}$$

The following properties were established in [26, Section 2.4].

Proposition 4.1. *Let G be a finite group, let U and V be subgroups of G such that V is normal in U, and let $\mathbf{X}$ be a cohomological G-Mackey functor with values in ${}_R\mathbf{mod}$.*

(a) *The canonical maps yield an exact sequence of R-modules*

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbf{c}_1(U/V,\mathbf{X}) & \longrightarrow & \hat{H}^{-1}(U/V,\mathbf{X}_V) & \longrightarrow & \mathbf{k}^0(U/V,\mathbf{X}) & \\ & & & & & \downarrow & \\ 0 \longleftarrow & \mathbf{k}^1(U/V,\mathbf{X}) & \longleftarrow & \hat{H}^0(U/V,\mathbf{X}_V) & \longleftarrow & \mathbf{c}_0(U/V,\mathbf{X}) & \end{array} \tag{4.2}$$

where $\hat{H}^\bullet(U/V,\_)$ denotes the Tate cohomology groups.

(b) *Let $0 \longrightarrow \mathbf{X} \xrightarrow{\phi} \mathbf{Y} \xrightarrow{\psi} \mathbf{Z} \longrightarrow 0$ be a short exact sequence of cohomological G-Mackey functors with values in ${}_R\mathbf{mod}$. Then one has exact sequences*

$$\begin{aligned} 0 \longrightarrow \mathbf{k}^0(U/V,\mathbf{X}) \xrightarrow{\mathbf{k}^0(\phi)} \mathbf{k}^0(U/V,\mathbf{Y}) \xrightarrow{\mathbf{k}^0(\psi)} \mathbf{k}^0(U/V,\mathbf{Z}) \longrightarrow \cdots \\ \cdots \longrightarrow \mathbf{k}^1(U/V,\mathbf{X}) \xrightarrow{\mathbf{k}^1(\phi)} \mathbf{k}^1(U/V,\mathbf{Y}) \xrightarrow{\mathbf{k}^1(\psi)} \mathbf{k}^1(U/V,\mathbf{Z}) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \mathbf{c}_1(U/V,\mathbf{X}) \xrightarrow{\mathbf{c}_1(\phi)} \mathbf{c}_1(U/V,\mathbf{Y}) \xrightarrow{\mathbf{c}_1(\psi)} \mathbf{c}_1(U/V,\mathbf{Z}) \longrightarrow \cdots \\ \cdots \longrightarrow \mathbf{c}_0(U/V,\mathbf{X}) \xrightarrow{\mathbf{c}_0(\phi)} \mathbf{c}_0(U/V,\mathbf{Y}) \xrightarrow{\mathbf{c}_0(\psi)} \mathbf{c}_0(U/V,\mathbf{Z}) \longrightarrow 0. \end{aligned} \tag{4.4}$$

4.1. *Section cohomology for cyclic subgroups*

Let W be a non-trivial cyclic subgroup of the finite group G generated by the element $w \in W$. Taking coinvariants of the chain complex of $R[W]$-modules $R[W] \xrightarrow{w-1} R[W]$ yields an exact sequence

$$0 \longrightarrow \mathbf{T}^W \xrightarrow{\mathbf{P}(\mathfrak{t}_{W,\{1\}})} \mathbf{Q}^W \xrightarrow{\mathbf{h}^0(w-1)} \mathbf{Q}^W \longrightarrow \boldsymbol{\Upsilon}^W \longrightarrow 0\,, \tag{4.5}$$

(cf. Sections 3.3, and 3.5). If R is an integral domain of characteristic 0, one has additionally a short exact sequence

$$0 \longrightarrow \boldsymbol{\Upsilon}^W \longrightarrow \mathbf{T}^W \longrightarrow \mathbf{B}^W \longrightarrow 0\,. \tag{4.6}$$

Splicing together the short exact sequences (4.5) and (4.6) yields a projective resolution of the cohomological W-Mackey functor $\mathbf{B}^W$. Using this projective resolution, Fact 2.1, (3.10) and (3.11) one concludes the following.

Fact 4.2. *Let R be a integral domain of characteristic 0, and let $W \subseteq G$ be cyclic subgroup of the finite group G. Then for $k \in \{0,1\}$ one has canonical isomorphisms*

$$\begin{aligned} \operatorname{ext}_G^k(\mathbf{ind}_W^G(\mathbf{B}^W),\mathbf{X}) &\simeq \mathbf{k}^k(W/\{1\},\mathbf{X}), \\ \operatorname{ext}_G^{3-k}(\mathbf{ind}_W^G(\mathbf{B}^W),\mathbf{X}) &\simeq \mathbf{c}_k(W/\{1\},\mathbf{X}). \end{aligned} \tag{4.7}$$

Note that Fact 4.2 shows also that for a cyclic subgroup $W \subseteq G$ one can consider the groups $\mathbf{k}^\bullet(W/\{1\}, \_)$, $\mathbf{c}_{3-\bullet}(W/\{1\}, \_)$ together with the respective connecting homomorphisms as a cohomological functor (cf. [14, Section XII.8]).

4.2. *Cohomological Mackey functors of type H^0 and H_0*

Let G be a finite group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$. Then $\mathbf{X}$ will be called *i-injective*, if for all $U, V \subseteq G$, $V \subseteq U$, the map $i^{\mathbf{X}}_{U,V}$ is injective; i.e., $\mathbf{X}$ is i-injective if, and only if, for all $U, V \subseteq G$, $V \lhd U$, one has $\mathbf{k}^0(U/V, \mathbf{X}) = 0$. Moreover, $\mathbf{X}$ will be called *of type H^0* (or to satisfy *Galois descent*), if $\mathbf{X}$ is i-injective and $\mathbf{k}^1(U/V, \mathbf{X}) = 0$ for all $U, V \subseteq G$, $V \lhd U$, i.e., $\mathbf{X}$ is of type H^0 if, and only if, one has a canonical isomorphism (induced by i)

$$\mathbf{X} \simeq \mathbf{h}^0(\mathbf{X}_{\{1\}}). \tag{4.8}$$

The cohomological G-Mackey functor $\mathbf{X}$ will be called to be *Hilbert 90*, if it is of type H^0 and $H^1(U, \mathbf{X}_{\{1\}}) = 0$ for every subgroup U of G. One has the following property.

Proposition 4.3. *Let G be a finite group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$ be Hilbert 90. Then for all $U, V \subset G$, $V \lhd U$, one has $H^1(U/V, \mathbf{X}_V) = 0$.*

Proof. By the 5-term exact sequence, inflation $H^1(U/V, \mathbf{X}^V_{\{1\}}) \to H^1(U, \mathbf{X}_{\{1\}})$ is injective. Hence $H^1(U/V, \mathbf{X}^V_{\{1\}}) = 0$. As $\mathbf{X}$ is of type H^0, $\mathbf{X}^V_{\{1\}}$ and $\mathbf{X}_V$ are isomorphic $R[U/V]$-modules. This yields the claim. □

In a similar fashion one calls $\mathbf{X}$ to be *t-surjective*, if for all $U, V \subseteq G$, $V \subseteq U$, the map $t^{\mathbf{X}}_{V,U}$ is surjective; i.e., $\mathbf{X}$ is t-surjective if, and only if, for all $U, V \subseteq G$, $V \lhd U$, one has $\mathbf{c}_0(U/V, \mathbf{X}) = 0$. The cohomological G-Mackey functor $\mathbf{X}$ will be called *of type H_0* (or to satisfy *Galois co-descent*), if $\mathbf{X}$ is t-surjective and $\mathbf{c}_1(U/V, \mathbf{X}) = 0$ for all $U, V \subseteq G$, $V \lhd U$, i.e., $\mathbf{X}$ is of type H_0 if, and only if, one has a canonical isomorphism (induced by t)

$$\mathbf{h}_0(\mathbf{X}_{\{1\}}) \simeq \mathbf{X}. \tag{4.9}$$

Furthermore, $\mathbf{X}$ will be called to be *co-Hilbert 90*, if it is of type H_0 and for every subgroup U of G one has $\hat{H}^{-1}(U, \mathbf{X}_{\{1\}}) = 0$.

Remark 4.4. Every projective cohomological G-Mackey functor $\mathbf{P}$ with values in the category of R-modules is a direct summand of a coproduct of standard projective cohomological G-Mackey functors. Hence, by Fact 3.1, every projective cohomological G-Mackey functor $\mathbf{P}$ is of type H^0. However, if R is an integral domain of characteristic 0, the Nakayama relations imply that $H^1(G, R[\Omega]) = 0$ for any G-set Ω. In particular, $\mathbf{P}$ is even Hilbert 90.

The periodicity of Tate cohomology for finite cyclic groups has the following consequence.

Proposition 4.5. *Let G be a finite group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$ be Hilbert 90. Let $U, V \subseteq G$, $V \lhd U$, be such that U/V is cyclic. Then $\mathbf{c}_1(U/V, \mathbf{X}) = 0$.*

Proof. By Proposition 4.3 and the periodicity of Tate cohomology (of period 2), one has $\hat{H}^{-1}(U/V, \mathbf{X}_V) \simeq H^1(U/V, \mathbf{X}_V) = 0$. Hence (4.2) yields the claim. □

4.3. Tate duality

Let R be a principal ideal domain of characteristic 0, and let $K = \text{quot}(R)$ denote its quotient field. Then $\mathbb{I} = K/R$ is an injective R-module.[3] Let G be a finite group, and let M be a left $R[G]$-lattice. Then one has an exact sequence of left $R[G]$-modules

$$0 \longrightarrow M^* \longrightarrow \text{Hom}_R(M, K) \longrightarrow \text{Hom}_R(M, \mathbb{I}) \longrightarrow 0, \tag{4.10}$$

where $M^* = \text{Hom}_R(M, R)$. The following property is also known as *Tate duality*.

Proposition 4.6. *Let R be a principal ideal domain of characteristic* 0*, let $K = \text{quot}(R)$ be the quotient field of R, and let $\mathbb{I} = K/R$. Let G be a finite group, and let M be an $R[G]$-lattice. Then for all $k \in \mathbb{Z}$ one has natural isomorphisms*

$$\hat{H}^k(G, M^*) \simeq \text{Hom}_R(\hat{H}^{-k}(G, M), \mathbb{I}). \tag{4.11}$$

Proof. It is well known that one has natural isomorphisms

$$\hat{H}^{k-1}(G, \text{Hom}_R(M, \mathbb{I})) \simeq \text{Hom}_R(\hat{H}^{-k}(G, M), \mathbb{I}) \tag{4.12}$$

for all $k \in \mathbb{Z}$ (cf. [4, p. 148, Ex. VI.7.4]). Moreover, as $\hat{H}^k(G, \text{Hom}_R(M, K)) = 0$ for all $k \in \mathbb{Z}$, one has also natural isomorphisms

$$\hat{H}^{k-1}(G, \text{Hom}_R(M, \mathbb{I})) \simeq \hat{H}^k(G, M^*). \tag{4.13}$$

This yields the claim. □

4.4. Section cohomology of R-lattice functors

Let R be an integral domain of characteristic 0, let G be a finite group, and let $\mathbf{X} \in \text{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ be an R-lattice functor. For $U, V \subseteq G$, $V \subseteq U$, the axiom (cMF$_7$) (cf. Section 3.1) implies that $\mathbf{X}$ is i-injective. Hence by (4.2) one has an isomorphism

$$\mathbf{c}_1(U/V, \mathbf{X}) \simeq \hat{H}^{-1}(U/V, \mathbf{X}_V) \tag{4.14}$$

and a short exact sequence

$$0 \longrightarrow \mathbf{c}_0(U/V, \mathbf{X}) \longrightarrow \hat{H}^0(U/V, \mathbf{X}_V) \longrightarrow \mathbf{k}^1(U/V, \mathbf{X}) \longrightarrow 0. \tag{4.15}$$

Let R be a principal ideal domain, and let $\phi\colon A \to B$ be a homomorphism of R-lattices. Then ϕ is split injective if, and only if, $\phi^*\colon B^* \to A^*$ is surjective. From this fact one concludes the following properties.

Proposition 4.7. *Let R be a principal ideal domain of characteristic* 0*, let G be a finite group, and let $\mathbf{X} \in \text{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$.*

(a) *$\mathbf{X}$ is of type H^0 if, and only if, $\mathbf{X}^*$ is t-surjective.*
(b) *The following are equivalent:*

[3] This follows by an argument similar to the proof of [14, Corollary III.7.3].

(i) $\mathbf{X}$ *is Hilbert 90;*
(ii) $\mathbf{X}^*$ *is of type* H_0*;*
(iii) $\mathbf{X}^*$ *is co-Hilbert 90.*

Proof. Let $U, V \subseteq G$, $V \subseteq U$.

(a) The map $i_{U,V}\colon \mathbf{X}_U \to \mathbf{X}_V$ is split-injective if, and only if, $\mathbf{k}^1(U/V, \mathbf{X}) = 0$. Hence the previously mentioned remark yields the claim.

(b) Suppose that $\mathbf{X}$ is Hilbert 90. Then $H^1(U/V, \mathbf{X}_V) = 0$ for all $U, V \subseteq G$, $V \lhd U$ (cf. Proposition 4.3). By Tate duality (cf. Proposition 4.6), one has

$$\hat{H}^{-1}(U/V, \mathbf{X}_V^*) \simeq \mathrm{Hom}_R(H^1(U/V, \mathbf{X}_V), \mathbb{I}_R) = 0 \tag{4.16}$$

whenever V is normal in U. Hence $\mathbf{c}_1(U/V, \mathbf{X}^*) = 0$ for all $U, V \subseteq G$, $V \lhd U$ (cf. (4.14)). Thus by (a), $\mathbf{X}^*$ is of type H_0. If $\mathbf{X}^*$ is of type H_0, (4.14) implies that $\hat{H}^{-1}(U/V, \mathbf{X}_V^*) = 0$ for all $U, V \subseteq G$, $V \lhd U$, i.e., $\mathbf{X}^*$ is co-Hilbert 90. If $\mathbf{X}^*$ is co-Hilbert 90, then (a) implies that $\mathbf{X}$ is of type H^0. By Tate duality (cf. Proposition 4.6), one has

$$H^1(U/V, \mathbf{X}_V) \simeq \mathrm{Hom}_R(\hat{H}^{-1}(U/V, \mathbf{X}_V^*), \mathbb{I}_R) = 0. \tag{4.17}$$

This yields the claim. □

4.5. *Finite cyclic groups*

If G is a finite group and R is any commutative ring with unit 1, one has $\mathbf{P}^{\{1\}} \simeq (\mathbf{P}^{\{1\}})^*$, i.e., $\mathbf{P}^{\{1\}}$ is projective and relative injective. If G is a finite cyclic group, and $W \subseteq G$ is a non-trivial subgroup of G, applying $\mathbf{ind}_W^G(\_)$ to the exact sequence (4.5) yields an exact sequence

$$0 \longrightarrow \mathbf{P}^W \xrightarrow{\mathbf{P}(\mathrm{t}_{W,\{1\}})} \mathbf{P}^{\{1\}} \xrightarrow{\mathbf{ind}_W^G(w-1)} \mathbf{P}^{\{1\}} \longrightarrow \mathbf{J}^W \longrightarrow 0, \tag{4.18}$$

where $w \in W$ is a generating element of W. In particular,

$$\mathrm{proj.dim}(\mathbf{J}^W) \le 2, \quad W \subseteq G,\ W \ne \{1\}, \tag{4.19}$$

and $\mathrm{proj.dim}(\mathbf{J}^{\{1\}}) = 0$. Thus one has the following (cf. Section 2.8).

Proposition 4.8. *Let R be a Dedekind domain, and let G be a finite cyclic group. Then $\mathcal{M}_R(G)$ is* 2*-Gorenstein.*

For $\mathrm{ext}_G^k(\mathbf{J}^W, \mathbf{X}) = \mathcal{R}^k\mathrm{nat}_G(\mathbf{J}^W, \mathbf{X})$, $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$, $W \subseteq G$, one obtains the following.

Proposition 4.9. *Let R be a Dedekind domain, let G be a finite cyclic group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}))$. Then for $W \subseteq G$ one has*

(i) $\mathrm{ext}_G^0(\mathbf{J}^W, \mathbf{X}) = \mathrm{nat}_G(\mathbf{J}^W, \mathbf{X}) \simeq \mathbf{X}_{\{1\}}^W$*;*
(ii) $\mathrm{ext}_G^1(\mathbf{J}^W, \mathbf{X}) \simeq \mathbf{c}_1(W/\{1\}, \mathbf{X})$*;*
(iii) $\mathrm{ext}_G^2(\mathbf{J}^W, \mathbf{X}) \simeq \mathbf{c}_0(W/\{1\}, \mathbf{X})$*;*

and $\mathrm{ext}_G^k(\mathbf{J}^W, \mathbf{X}) = 0$ *for* $k \ge 3$.

Proof. For $W = \{1\}$, one has $\mathrm{ext}^1_G(\mathbf{J}^W, \mathbf{X}) = \mathrm{ext}^2_G(\mathbf{J}^W, \mathbf{X}) = 0$, and from Fact 2.1 one concludes that $\mathrm{ext}^0_G(\mathbf{J}^W, \mathbf{X}) = \mathrm{nat}_G(\mathbf{J}^W, \mathbf{X}) = \mathbf{X}_{\{1\}}$. Hence the claim holds in this case, and we may assume that $W \neq \{1\}$. From the Nakayama relations and (4.18) one concludes that $\mathrm{ext}^k_G(\mathbf{J}^W, \mathbf{X})$ coincides with the k^{th}-cohomology of the cochain complex

$$0 \longrightarrow \mathbf{X}_{\{1\}} \xrightarrow{w-1} \mathbf{X}_{\{1\}} \xrightarrow{t^{\mathbf{X}}_{\{1\},W}} \mathbf{X}_W \longrightarrow 0 \tag{4.20}$$

concentrated in degrees 0, 1 and 2. This yields the claim in case that $W \neq \{1\}$. □

From Proposition 4.9 one obtains the following description of the higher derived functors of the Yoneda dual.

Proposition 4.10. *Let R be a principal ideal domain of characteristic 0, let G be a finite cyclic group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{c}\mathfrak{M}\mathfrak{F}_G({}_R\mathbf{lat}))$ be a cohomological G-Mackey functor with values in the category of R-lattices. Then the following are equivalent.*

(i) $\mathbf{X}$ *is Hilbert 90;*
(ii) $\mathbf{X}^*$ *is co-Hilbert 90;*
(iii) $\mathbf{X}$ *is ⊛-acyclic;*
(iv) $\mathbf{X}^{⊛}$ *is Hilbert 90;*
(v) $\mathbf{X}^{⊛}$ *is ⊛-acyclic.*

In particular, $(\mathcal{M}_R(G), \sigma)$ is a ⊛-symmetric $R^{⊛}$-order category.

Proof. By Proposition 4.7(b), (i) and (ii) are equivalent. For $W \subseteq G$ one has

$$\mathcal{R}^k(\mathbf{X})^{⊛}_W = \mathrm{ext}^k_G(\mathbf{X}, \mathbf{P}^W) \simeq \mathrm{ext}^k_G(\mathbf{J}^W, \mathbf{X}^*). \tag{4.21}$$

Hence Proposition 4.9 implies that (ii) and (iii) are equivalent, and thus also (iv) and (v) are equivalent. By Fact 3.3, $\mathbf{X}^{⊛} \simeq \mathbf{h}^0(\mathbf{X}^*_{\{1\}})$. Let $W \subseteq G$. The periodicity of Tate cohomology (of period 2) and Tate duality (cf. (4.11)) imply that

$$H^1(W, \mathbf{X}^*_{\{1\}}) \simeq \hat{H}^{-1}(W, \mathbf{X}^*_{\{1\}}) \simeq \mathrm{Hom}_R(H^1(W, \mathbf{X}_{\{1\}}), \mathbb{I}_R). \tag{4.22}$$

Hence (i) implies (iv). Replacing $\mathbf{X}$ by $\mathbf{X}^{⊛}$ shows that (iv) implies (i). This yields the claim. □

The following property will allow us to analyze the projective dimensions of cohomological Mackey functors for finite cyclic groups.

Proposition 4.11. *Let R be a Dedekind domain of characteristic 0, let G be a finite cyclic group, and let $\phi\colon \mathbf{P} \to \mathbf{X}$ be a surjective natural transformation in $\mathfrak{c}\mathfrak{M}\mathfrak{F}_G({}_R\mathbf{mod})$, where $\mathbf{P}$ is a projective R-lattice functor. Then*

(a) $\ker(\phi)$ *is an R-lattice functor;*
(b) *if $\mathbf{X}$ is i-injective,* $\ker(\phi)$ *is of type H^0;*
(c) *if $\mathbf{X}$ is of type H^0,* $\ker(\phi)$ *is Hilbert 90.*

Proof. For (a) there is nothing to prove. Put $\mathbf{K} = \ker(\phi)$, and let $U, V \subseteq G$, $V \subseteq U$. By Remark 4.4 and Fact 4.2, one has an exact sequence

$$\begin{array}{ccccccc} \mathbf{c}_1(U/V, \mathbf{X}) & \longleftarrow & 0 & & & & \\ \downarrow & & & & & & \\ \mathbf{c}_0(U/V, \mathbf{K}) & \longrightarrow & \mathbf{c}_0(U/V, \mathbf{P}) & \longrightarrow & \mathbf{c}_0(U/V, \mathbf{X}) & \longrightarrow & 0 \end{array} \tag{4.23}$$

and isomorphisms

$$\mathbf{k}^0(U/V, \mathbf{X}) \simeq \mathbf{k}^1(U/V, \mathbf{K}), \tag{4.24}$$

$$\mathbf{k}^1(U/V, \mathbf{X}) \simeq \mathbf{c}_1(U/V, \mathbf{K}). \tag{4.25}$$

Hence (4.24) implies (b). If $\mathbf{X}$ is of type H^0, (4.25) yields that $\mathbf{c}_1(U/V, \mathbf{K}) = 0$. Thus by (4.14) and the periodicity of Tate cohomology (of period 2), one has

$$H^1(U/V, \mathbf{K}_V) \simeq \hat{H}^{-1}(U/V, \mathbf{K}_V) \simeq \mathbf{c}_1(U/V, \mathbf{K}) = 0. \tag{4.26}$$

This yields the claim. □

The following property will turn out to be useful for our purpose.

Proposition 4.12. *Let R be an integral domain of characteristic 0, and let $p \in R$. Assume further that G is a finite cyclic group, and that $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ is an R-lattice functor with the Hilbert 90 property. Then $\mathbf{Y} = \mathbf{X}/p\mathbf{X}$ is of type H^0.*

Proof. We may suppose that $p \neq 0$. Then $p\mathbf{X} \simeq \mathbf{X}$. Let $U, V \in G$, $V \subseteq U$. By Fact 4.2, one has a long exact sequence

$$\begin{array}{ccccccc}
\mathbf{k}^0(U/V,\mathbf{Y}) & \longleftarrow & \mathbf{k}^0(U/V,\mathbf{X}) & \xleftarrow{\;p\;} & \mathbf{k}^0(U/V,\mathbf{X}) & \longleftarrow & 0 \\
\downarrow & & & & & & \\
\mathbf{k}^1(U/V,\mathbf{X}) & \xrightarrow{\;p\;} & \mathbf{k}^1(U/V,\mathbf{X}) & \longrightarrow & \mathbf{k}^1(U/V,\mathbf{Y}) & & \\
 & & & & \downarrow & & \\
\mathbf{c}_1(U/V,\mathbf{Y}) & \longleftarrow & \mathbf{c}_1(U/V,\mathbf{X}) & \xleftarrow{\;p\;} & \mathbf{c}_1(U/V,\mathbf{X}) & & \\
\downarrow & & & & & & \\
\mathbf{c}_0(U/V,\mathbf{X}) & \xrightarrow{\;p\;} & \mathbf{c}_0(U/V,\mathbf{X}) & \longrightarrow & \mathbf{c}_0(U/V,\mathbf{Y}) & \longrightarrow & 0
\end{array} \tag{4.27}$$

As $\mathbf{k}^0(U/V, \mathbf{X}) = \mathbf{k}^1(U/V, \mathbf{X}) = \mathbf{c}_1(U/V, \mathbf{X}) = 0$, one concludes that $\mathbf{k}^0(U/V, \mathbf{Y}) = \mathbf{k}^1(U/V, \mathbf{Y}) = 0$. This yields the claim. □

4.6. Injectivity criteria

For a finite p-group G there are useful criteria ensuring that a homomorphism $\phi: \mathbf{X} \to \mathbf{Y}$ of cohomological G-Mackey functors is injective. These criteria are based on the following fact.

Fact 4.13. *Let G be a finite p-group, let $\mathbb{F}$ be a field of characteristic p, and let M be a non-trivial, finitely generated left $\mathbb{F}[G]$-module. Let $B \subseteq M$ be an $\mathbb{F}[G]$-submodule satisfying $B \cap M^G = 0$. Then $B = 0$.*

Proof. The $\mathbb{F}$-algebra $\mathbb{F}[G]$ is artinian. Moreover, as every irreducible left $\mathbb{F}[G]$-module is isomorphic to the trivial left $\mathbb{F}[G]$-module, for any finitely generated left $\mathbb{F}[G]$-module B one has $\mathrm{soc}_G(B) = B^G$. Hence the hypothesis implies $\mathrm{soc}_G(B) = 0$. Thus $B = 0$. □

From Fact 4.13 one concludes the following injectivity criterion.

Lemma 4.14. *Let G be a finite p-group, and let $\mathbb{F}$ be a field of characteristic p. Suppose that for $\phi\colon \mathbf{X} \to \mathbf{Y} \in \mathrm{mor}(\mathfrak{cMF}_G({}_{\mathbb{F}}\mathbf{mod}))$ one has that*

(i) *$\phi_G\colon \mathbf{X}_G \to \mathbf{Y}_G$ is injective, and*
(ii) *$\mathbf{X}$ is of type H^0, and $\mathbf{Y}$ is i-injective.*

Then ϕ is injective.

Proof. By hypothesis (i), $i^{\mathbf{Y}}_{G,U} \circ \phi_G\colon \mathbf{X}_G \to \mathbf{Y}_U$ is injective for all $U \subseteq G$. If $V \subseteq G$ is normal in G, one has a commutative diagram

$$\begin{array}{ccc} \mathbf{X}_G & \xrightarrow{\phi_G} & \mathbf{Y}_G \\ {\scriptstyle i^{\mathbf{X}}_{G,V}}\downarrow & & \downarrow{\scriptstyle i^{\mathbf{Y}}_{G,V}} \\ \mathbf{X}_V & \xrightarrow{\phi_V} & \mathbf{Y}_V \end{array} \tag{4.28}$$

As $\mathbf{X}$ is of type H^0, $\mathrm{im}(i^{\mathbf{X}}_{G,V}) = \mathbf{X}_V^{G/V} = \mathrm{soc}_{G/V}(\mathbf{X}_V)$ (cf. (4.1)). Moreover, since $\phi_V \circ i^{\mathbf{X}}_{G,V} = i^{\mathbf{Y}}_{G,V} \circ \phi_G$ is injective, $\phi_V|_{\mathrm{soc}_{G/V}(\mathbf{X}_V)}\colon \mathrm{soc}_{G/V}(\mathbf{X}_V) \to \mathbf{Y}_V$ is injective, i.e., $\ker(\phi_V) \cap \mathrm{soc}_{G/V}(\mathbf{X}_V) = 0$. Thus by Fact 4.13, $\ker(\phi_V) = 0$ and ϕ_V is injective.

Let U be any subgroup of G, and let $V \subseteq U$ be a subgroup of U which is normal in G. By the previously mentioned remark one has a commutative diagram

$$\begin{array}{ccc} \mathbf{X}_U & \xrightarrow{\phi_U} & \mathbf{Y}_U \\ {\scriptstyle i^{\mathbf{X}}_{U,V}}\downarrow & & \downarrow{\scriptstyle i^{\mathbf{Y}}_{U,V}} \\ \mathbf{X}_V & \xrightarrow{\phi_V} & \mathbf{Y}_V \end{array} \tag{4.29}$$

with ϕ_V is injective. By hypothesis (ii), $i^{\mathbf{X}}_{U,V}$ is injective. Hence ϕ_U is injective, and this yields the claim. □

Let R be a discrete valuation domain of characteristic 0 with prime ideal pR for some prime number p, i.e., $\mathbb{F} = R/pR$ is a field of characteristic p. For a finitely generated R-module A let $\mathrm{gr}_\bullet(A)$ denote the graded $\mathbb{F}[t]$-module associated to the p-adic filtration $(p^k.A)_{k\geq 0}$. Then every homogeneous component $\mathrm{gr}_k(A)$ is a finite-dimensional $\mathbb{F}$-vector space. Moreover, A is a free R-module if, and only if, $\mathrm{gr}_\bullet(A)$ is a free $\mathbb{F}[t]$-module. Let $\phi\colon A \to B$ be a homomorphism of finitely generated R-modules. Then ϕ induces a homomorphism of $\mathbb{F}[t]$-modules $\mathrm{gr}_\bullet(\phi)\colon \mathrm{gr}_\bullet(A) \to \mathrm{gr}_\bullet(B)$. Moreover, one has the following.

Fact 4.15. *Let R be a discrete valuation domain of characteristic 0 with prime ideal pR for some prime number p, and let $\phi\colon A \to B$ be a homomorphism of R-lattices. Then the following are equivalent:*

(i) *ϕ is split-injective;*
(ii) *$\mathrm{gr}_\bullet(\alpha)\colon \mathrm{gr}_\bullet(A) \to \mathrm{gr}_\bullet(B)$ is injective;*
(iii) *$\mathrm{gr}_0(\alpha)\colon \mathrm{gr}_0(A) \to \mathrm{gr}_0(B)$ is injective.*

Lemma 4.14 and Fact 4.15 imply the following criterion for injectivity.

Proposition 4.16. *Let G be a finite p-group, let R be a discrete valuation domain of characteristic 0 with prime ideal pR, and let $\phi\colon \mathbf{X} \to \mathbf{Y} \in \mathrm{mor}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ be a natural transformation of cohomological R-lattice functors with the following properties:*

(i) $\mathrm{gr}_0(\phi_G)\colon \mathrm{gr}_0(\mathbf{X}_G) \to \mathrm{gr}_0(\mathbf{Y}_G)$ *is injective;*
(ii) $\mathrm{gr}_0(\mathbf{X})$ *is of type H^0 and* $\mathrm{gr}_0(\mathbf{Y})$ *is i-injective.*

Then $\phi\colon \mathbf{X} \to \mathbf{Y}$ is injective, and $\phi_U : \mathbf{X}_U \to \mathbf{Y}_U$ is split-injective for all $U \subseteq G$.

5. Gentle $R^{\circledast}$-order categories

Throughout this section we fix a prime number p and assume further that R is a principal ideal domain of characteristic 0 such that pR is a prime ideal, i.e., $\mathbb{F} = R/pR$ is a field, the *residue field* of R at pR. By $K = \mathrm{quot}(R)$ we denote the quotient field of R.

5.1. Gentle $R^{\circledast}$-order categories

By $\mathcal{G}_R(n, p)$, $n \geq 0$, we denote the R-order category with objects $\mathrm{ob}(\mathcal{G}_R(n, p)) = \{0, \ldots, n\}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{G}_R(n,p)}(j, k) = \begin{cases} R.t_{j,k} & \text{for } j < k, \\ R.\mathrm{id}_k & \text{for } j = k, \\ R.i_{j,k} & \text{for } j > k, \end{cases} \tag{5.1}$$

for $0 \leq j, k \leq n$ subject to the relations

(i) $i_{l,j} = i_{k,j} \circ i_{l,k}$ for $j \leq k \leq l$;
(ii) $t_{j,l} = t_{k,l} \circ t_{j,k}$ for $j \leq k \leq l$;
(iii) $i_{j+1,j} \circ t_{j,j+1} = p.\mathrm{id}_j$ for $j \in \{0, \ldots, n-1\}$;
(iv) $t_{k-1,k} \circ i_{k,k-1} = p.\mathrm{id}_k$ for $k \in \{1, \ldots, n\}$;

where we put $t_{k,k} = i_{k,k} = \mathrm{id}_k$ for $k \in \{0, \ldots, n\}$. This category equipped with the natural equivalence $\sigma : \mathcal{G}_R(n, p) \to \mathcal{G}_R(n, p)^{\mathrm{op}}$ of order 2, i.e., $\sigma \circ \sigma = \mathrm{id}_{\mathcal{G}_R(n,p)}$, given by

$$\sigma(k) = k, \qquad \sigma(t_{j,k}) = i_{k,j}, \qquad \sigma(i_{k,j}) = t_{j,k}, \quad 0 \leq j \leq k \leq n; \tag{5.2}$$

and thus forms an $R^{\circledast}$-order category.

Remark 5.1. Let $\mu = \mu_{\mathcal{G}_R(n,p)}$ be the R-order representing $\mathcal{G}_R(n, p)$ (cf. Remark 2.5). Then $\mu \otimes_R \mathbb{F}$ is a gentle $\mathbb{F}$-algebra. It is well known that these algebras are 1-Gorenstein (cf. [10]). However, for $n \geq 1$ they are not of finite global dimension, and, therefore, they do not have the Whitehead property (cf. Facts 2.11 and 2.13).

5.2. The unitary projection

Let C_{p^n} be the cyclic group of order p^n. Then

$$\pi : \mathcal{M}_R(C_{p^n}) \longrightarrow \mathcal{G}_R(n, p), \tag{5.3}$$

given by $\pi(U) = \log_p(|G : U|)$, $\pi(\mathrm{i}_{V,U}) = i_{j,i}$, $\pi(\mathrm{t}_{U,V}) = t_{i,j}$, $\rho^U_g = \mathrm{id}_i$, for $U, V \subseteq G$, $|U| = p^{n-i}$, $|V| = p^{n-j}$, $j \geq i$, is a unitary projection. Applying $\inf^{\pi}(\_)$ shows that every functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)))$ can also be considered as a cohomological Mackey functor for the finite group C_{p^n}. The deflation functor $\mathrm{def}^\pi(\_)$ can be described explicitly using the functor of C_{p^n}-coinvariants $\_{C_{p^n}}$, i.e., for $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_{C_{p^n}}(_R\mathbf{mod}))$ one has

$$\mathrm{def}^\pi(\mathbf{X})(k) = (\mathbf{X}_U)_{C_{p^n}}, \quad |U| = p^{n-k}, \tag{5.4}$$

and $\mathrm{def}^\pi(\alpha)(k) = (\alpha_U)_{C_{p^n}} : (\mathbf{X}_U)_{C_{p^n}} \to (\mathbf{Y}_U)_{C_{p^n}}$ for $\alpha \in \mathrm{Hom}_{\mathcal{M}_R(C_{p^n})}(\mathbf{X}, \mathbf{Y})$. Furthermore, by Fact 2.14(c), one has for $W \subseteq G$, $|W| = p^{n-k}$, that

$$\mathrm{def}^\pi(\mathbf{P}^W) \simeq \mathbf{P}^k. \tag{5.5}$$

5.3. *Simple functors*

As every functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {_R\mathbf{mod}}))$ is in particular a cohomological C_{p^n}-Mackey functor, one can use the description given in [23] in order to determine all simple functors in $\mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {_R\mathbf{mod}}^{\mathrm{f.g.}}))$. For every $\ell \in \{0, \dots, n\}$ there exists a simple functor $\mathbf{S}^\ell$ given by

$$\mathbf{S}^\ell(k) = \begin{cases} \mathbb{F}, & \text{for } k = \ell, \\ 0, & \text{for } k \neq \ell; \end{cases} \qquad \mathbf{S}^\ell(t_{j,k}) = 0, \qquad \mathbf{S}^\ell(i_{k,j}) = 0, \quad 0 \le j \le k \le n. \tag{5.6}$$

From Remark 3.2 one concludes that if R is a discrete valuation ring of characteristic 0 with maximal ideal pR, then every simple functor $\mathbf{S} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {_R\mathbf{mod}}^{\mathrm{f.g.}}))$ must be naturally isomorphic to some $\mathbf{S}^\ell$, $0 \le \ell \le n$.

5.4. *R-lattice functors of rank 1*

Let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {_R\mathbf{lat}}))$ be an R-lattice functor. Then $\mathbf{F}(i_{0,k}) \otimes_R K : \mathbf{F}(k) \otimes_R K \to \mathbf{F}(0) \otimes_R K$ is an isomorphism of finite-dimensional K-vector spaces, i.e., $\mathrm{rk}(\mathbf{F}(k)) = \mathrm{rk}(\mathbf{F}(0))$ for all $k \in \{0, \dots, n\}$, where $\mathrm{rk}(\mathbf{F}(0))$ denotes the rank of the free R-module $\mathbf{F}(0)$. We define the *rank of* $\mathbf{F}$ by $\mathrm{rk}(\mathbf{F}) = rk(\mathbf{F}(0))$.

If M is an R-lattice and $B \subseteq M$ is an R-submodule of M, we denote by

$$\mathrm{sat}_M(B) = \{\, b \in M \mid \exists r \in R \setminus \{0\}\colon\ r.b \in B \,\} \tag{5.7}$$

the *saturation* of B in M. It is again an R-submodule of M. Let $\mathbf{G}$ be a subfunctor of the R-lattice functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {_R\mathbf{lat}}))$. Then $\mathrm{sat}_{\mathbf{F}}(\mathbf{G})$ given by

$$\mathrm{sat}_{\mathbf{F}}(\mathbf{G})(k) = \mathrm{sat}_{\mathbf{F}(k)}(\mathbf{G}(k)), \tag{5.8}$$

$0 \le k \le n$, is a subfunctor of $\mathbf{F}$ containing $\mathbf{G}$. The subfunctor $\mathbf{G}$ will be called *saturated*, if $\mathrm{sat}_{\mathbf{F}}(\mathbf{G}) = \mathbf{G}$. The following fact allows us to reduce some considerations to R-lattice functors of rank 1.

Fact 5.2. *Let* $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {_R\mathbf{lat}}))$, $\mathrm{rk}(\mathbf{F}) > 0$. *Then* $\mathbf{F}$ *contains a saturated subfunctor of rank* 1. *In particular, there exists an ascending chain* $(\mathbf{F}_j)_{0 \le j \le \mathrm{rk}(\mathbf{F})}$ *of subfunctors of* $\mathbf{F}$ *satisfying* $\mathbf{F}_0 = 0$, $\mathbf{F}_{j-1} \subseteq \mathbf{F}_j$, $\mathbf{F}_{\mathrm{rk}(\mathbf{F})} = \mathbf{F}$ *and* $\mathbf{F}_j / \mathbf{F}_{j-1}$ *is an R-lattice functor of rank* 1.

Proof. Let $a \in \mathbf{F}(0)$, $a \neq 0$. Then $\mathbf{R} \subseteq \mathbf{F}$ given by $\mathbf{R}(k) = \mathrm{sat}_{\mathbf{F}(k)}(R\mathbf{F}(i_{k,0})(a))$ together with the canonical maps is a saturated subfunctor of $\mathbf{F}$. The final remark follows by induction.

Let $\mathbf{F}$ be an R-lattice functor of rank 1. By (iii) and (iv) of the definition, for $k \in \{0, \dots, n-1\}$ either $\mathbf{F}(t_{k,k+1})$ is an isomorphism, or $\mathbf{F}(i_{k+1,k})$ is an isomorphism. Thus we can represent $\mathbf{F}$ by a diagram $\Delta_{\mathbf{F}}$, where we draw an arrow from $k+1$ to k if $\mathbf{F}(t_{k,k+1})\colon \mathbf{F}(k+1) \to \mathbf{F}(k)$ is an isomorphism, and an arrow from k to $k+1$ if $\mathbf{F}(i_{k+1,k})\colon \mathbf{F}(k) \to \mathbf{F}(k+1)$ is an isomorphism. It is straightforward to verify that the isomorphism type of $\mathbf{F}$ is uniquely determined by $\Delta_{\mathbf{F}}$, and that for every arrow diagram Δ there exists an R-lattice functor $\mathbf{F}_{\Delta}$ which is represented by this diagram.

Remark 5.3. (a) For $\ell \in \{0, \dots, n\}$ let $\mathbf{P}^{\ell} = \mathrm{Hom}_{\mathcal{G}}(\_, \ell)$ be the standard projective R-lattice functor associated to ℓ (cf. Section 2.3). Then $\mathbf{P}^{\ell}$ has rank 1 and is represented by the arrow diagram

$$0 \longleftarrow 1 \longleftarrow \cdots \longleftarrow \ell-1 \longleftarrow \ell \longrightarrow \ell+1 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n\,. \tag{5.9}$$

(b) If $\mathbf{F}$ is represented by the diagram $\Delta_{\mathbf{F}}$, then $\mathbf{F}^*$ is represented by the diagram $\Delta_{\mathbf{F}^*} = \bar{\Delta}_{\mathbf{F}}$, where all arrows are reversed.

(c) Let $\mathbf{J}^{\ell} = (\mathbf{P}^{\ell})^*$, $\ell \in \{0, \dots, n\}$. Then $\mathbf{J}^{\ell}$ is relative injective and, by (a) and (b), $\mathbf{J}^{\ell}$ is represented by the diagram

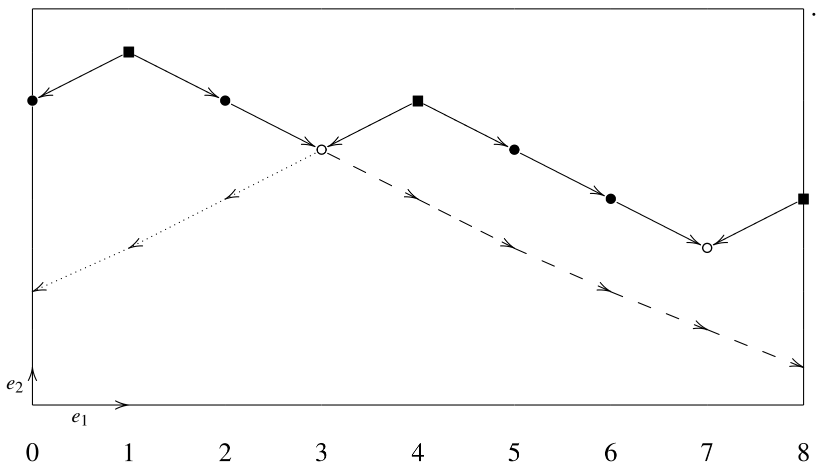
$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow \ell-1 \longrightarrow \ell \longleftarrow \ell+1 \longleftarrow \cdots \longleftarrow n-1 \longleftarrow n\,. \tag{5.10}$$

In particular, $\mathbf{P}^0 \simeq \mathbf{J}^n$ and $\mathbf{P}^n \simeq \mathbf{J}^0$ are relative injective.

Let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ be an R-lattice functor of rank 1. Then $\Delta_{\mathbf{F}}$ defines a connected graph $\Gamma_{\mathbf{F}}$ in the plane $\mathbb{R}^2 = \mathbb{R}e_1 \oplus \mathbb{R}e_2$, where all arrows are diagonal and point in the negative e_2-direction, e.g., for $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(8,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ with $\Delta_{\mathbf{F}}$ given by

$$0 \longleftarrow 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 7 \longleftarrow 8 \tag{5.11}$$

one obtains the graph $\Gamma_{\mathbf{F}}$



. (5.12)

Let $\max(\mathbf{F}) \subset \mathrm{ob}(\mathcal{G}_R(n,p))$ be the set of objects corresponding to the local maxima in the graph $\Gamma_{\mathbf{F}}$, i.e., $k \notin \{0,n\}$ is contained in $\max(\mathbf{F})$ if, and only if, $\Delta_{\mathbf{F}}$ contains a subdiagram of the form $(k-1 \longleftarrow k \longrightarrow k+1)$. Moreover, $0 \in \max(\mathbf{F})$ if $(0 \longrightarrow 1)$ is a subdiagram of $\Delta_{\mathbf{F}}$, while $n \in \max(\mathbf{F})$ if $(n-1 \longleftarrow n)$ is a subdiagram of $\Delta_{\mathbf{F}}$. E.g., for $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(8,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ as in (5.11) one has that $\max(\mathbf{F}) = \{1,4,8\}$. By $\min(\mathbf{F})$ we denote the subset of $\{1,\dots,n-1\}$ corresponding to the local minima in the graph $\Delta_{\mathbf{F}}$, i.e., $\ell \notin \{0,n\}$ is contained in $\min(\mathbf{F})$ if, and only if, $\Delta_{\mathbf{F}}$ contains a subdiagram of the form $(k-1 \longrightarrow k \longleftarrow k+1)$. E.g., for the functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(8,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ as in (5.11) one has $\min(\mathbf{F}) = \{3,7\}$. Thus by construction, one has $|\max(\mathbf{F})| = |\min(\mathbf{F})| + 1$. The following fact is straightforward.

Fact 5.4. *Let* $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ *be an R-lattice functor of rank* 1. *Then* $\mathrm{nat}_R(\mathbf{F}, \mathbf{S}^\ell) \simeq \mathbb{F}$ *if* $\ell \in \max(\mathbf{F})$, *and* $\mathrm{nat}_R(\mathbf{F}, \mathbf{S}^\ell) = 0$ *if* $\ell \notin \max(\mathbf{F})$. *Moreover,* $\mathbf{F}$ *is projective if, and only if,* $\min(\mathbf{F}) = \emptyset$.

Let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ be an R-lattice functor of rank 1 which is not projective. Let $s(\mathbf{F}) \in \max(\mathbf{F})$ be the smallest element in $\max(\mathbf{F})$, and let $t(\mathbf{F})$ be the smallest element in $\min(\mathbf{F})$. The projective R-lattice functor $\mathbf{P}^{s(\mathbf{F})}$ corresponds to the diagram obtained from the diagram $\Delta_{\mathbf{F}}$ by changing all arrows between vertices α and $\alpha+1$, $\alpha \geq t(\mathbf{F})$ to $(\alpha \longrightarrow \alpha+1)$. Let $\mathbf{F}^\wedge \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n)^{\mathrm{op}}, {}_R\mathbf{lat}))$ be the R-lattice functor of rank 1 corresponding to the diagram obtained from the diagram $\Delta_{\mathbf{F}}$ by changing all arrows between vertices $\alpha-1$ and α, $\alpha \leq t(\mathbf{F})$ to $(\alpha - 1 \longleftarrow \alpha)$. E.g., for $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(8,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ as in (5.11), $\Gamma_{\mathbf{F}^\wedge}$ is given by replacing the first segment by the path $\cdots\cdots>$ in (5.12); and $\Delta_{\mathbf{F}^\wedge}$ is given by

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 7 \longleftarrow 8\,. \tag{5.13}$$

5.5. *The global dimension of $\mathcal{G}_R(n,p)$*

The following property will be essential for the subsequent analysis.

Lemma 5.5. *Let* $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ *be an R-lattice functor of rank* 1 *which is not projective. Then one has a short exact sequence of R-lattice functors*

$$0 \longrightarrow \mathbf{P}^{t(\mathbf{F})} \xrightarrow{\ \psi\ } \mathbf{P}^{s(\mathbf{F})} \oplus \mathbf{F}^\wedge \xrightarrow{\ \phi\ } \mathbf{F} \longrightarrow 0. \tag{5.14}$$

Proof. One can identify $\mathbf{P}^{s(\mathbf{F})}$ and $\mathbf{F}^\wedge$ as subfunctors of $\mathbf{F}$ by putting

$$\begin{aligned} \mathbf{P}^{s(\mathbf{F})}(k) &= \begin{cases} \mathbf{F}(k) & \text{for } k \leq t(\mathbf{F}), \\ \mathrm{im}(\mathbf{F}(i_{k,t(\mathbf{F})})) & \text{for } k > t(\mathbf{F}); \end{cases} \\ \mathbf{F}^\wedge(k) &= \begin{cases} \mathrm{im}(\mathbf{F}(t_{k,t(\mathbf{F})})) & \text{for } k \leq t(\mathbf{F}), \\ \mathbf{F}(k) & \text{for } k > t(\mathbf{F}) \end{cases} \end{aligned} \tag{5.15}$$

for $k \in \{0,\dots,n\}$. Let $\phi_1 : \mathbf{P}^{s(\mathbf{F})} \to \mathbf{F}$ and $\phi_2 : \mathbf{F}^\wedge \to \mathbf{F}$ denote the canonical inclusions. By construction, $\phi = \phi_1 \oplus \phi_2 : \mathbf{P}^{s(\mathbf{F})} \oplus \mathbf{F}^\wedge \to \mathbf{F}$ is surjective with kernel $\ker(\phi) \subseteq \mathbf{P}^{s(\mathbf{F})} \oplus \mathbf{F}^\wedge$ given by

$$\ker(\phi)(k) = \{(x,-x) \in \mathbf{P}^{s(\mathbf{F})}(k) \oplus \mathbf{F}^\wedge(k) \mid x \in \mathbf{P}^{s(\mathbf{F})}(k) \cap \mathbf{F}^\wedge(k)\}, \tag{5.16}$$

i.e., $\ker(\phi) \simeq \mathbf{P}^{s(\mathbf{F})} \cap \mathbf{F}^\wedge$. By construction, $\mathbf{X} = \mathbf{P}^{s(\mathbf{F})} \cap \mathbf{F}^\wedge$ is an R-lattice functor of rank 1 with all maps $\mathbf{X}(t_{j,t(\mathbf{F})})$ and $\mathbf{X}(i_{k,t(\mathbf{F})})$ surjective for $0 \leq j < t(\mathbf{F}) < k \leq n$. Hence all maps $\mathbf{X}(t_{j,t(\mathbf{F})})$, $\mathbf{X}(i_{k,t(\mathbf{F})})$, $0 \leq j < t(\mathbf{F}) < k \leq n$, are isomorphisms. Thus $\Delta_{\mathbf{X}} = \Delta_{\mathbf{P}^{t(\mathbf{F})}}$, and this yields the claim. $\square$

The equality $|\max(\mathbf{F}^\wedge)| = |\min(\mathbf{F})| + 1$ has the following consequence.

Proposition 5.6. *Let* $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ *be an* R*-lattice functor of rank* 1. *Then one has a short exact sequence of* R*-lattice functors*

$$0 \longrightarrow \bigoplus_{j\in\min(\mathbf{F})} \mathbf{P}^j \xrightarrow{\ \alpha\ } \bigoplus_{k\in\max(\mathbf{F})} \mathbf{P}^k \xrightarrow{\ \beta\ } \mathbf{F} \longrightarrow 0. \tag{5.17}$$

In particular, $\mathrm{proj.dim}_R(\mathbf{F}) \leq 1$.

Proof. We proceed by induction on $m = |\max(\mathbf{F})|$. If $|\max(\mathbf{F})| = 1$, one has $\min(\mathbf{F}) = \emptyset$, and hence $\mathbf{F}$ is projective. Therefore we may assume that $m > 1$, and that the assertion is true for all R-lattice functors $\mathbf{G}$ of rank 1 satisfying $|\max(\mathbf{G})| < m$. Let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ with $|\max(\mathbf{F})| = m > 1$. Hence $|\max(\mathbf{F}^\wedge)| = m-1$, and, by induction, one has a short exact sequence

$$0 \longrightarrow \bigoplus_{j\in\min(\mathbf{F}^\wedge)} \mathbf{P}^j \xrightarrow{\ \alpha^\wedge\ } \bigoplus_{k\in\max(\mathbf{F}^\wedge)} \mathbf{P}^k \xrightarrow{\ \beta^\wedge\ } \mathbf{F}^\wedge \longrightarrow 0. \tag{5.18}$$

For $s = s(\mathbf{F})$, $t = t(\mathbf{F})$ and $\beta = (\mathrm{id}_{\mathbf{P}^s} \oplus \beta^\wedge) \circ \phi$ one has a commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \vdots & & \downarrow & & & & \\
0 & \dashrightarrow & \ker(\zeta) & \dashrightarrow & \bigoplus_{j\in\min(\mathbf{F}^\wedge)} \mathbf{P}^j & \longrightarrow & 0 & & \\
 & & \vdots & & \downarrow{\scriptstyle \alpha^\wedge} & & \downarrow & & \\
0 & \longrightarrow & \ker(\beta) & \longrightarrow & \mathbf{P}^s \oplus \bigoplus_{k\in\max(\mathbf{F}^\wedge)} \mathbf{P}^k & \xrightarrow{\ \beta\ } & \mathbf{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \mathrm{id}_{\mathbf{P}^s}\oplus\beta^\wedge} & & \| & & \\
0 & \longrightarrow & \mathbf{P}^t & \xrightarrow{\ \psi\ } & \mathbf{P}^s \oplus \mathbf{F}^\wedge & \xrightarrow{\ \phi\ } & \mathbf{F} & \longrightarrow & 0 \\
 & & \vdots & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.19}$$

where ψ and ϕ are as in Lemma 5.5 and ζ is the induced map. By the snake lemma, one may extend this diagram by the arrows " $\dashrightarrow$ ". Hence $\ker(\beta) \simeq \bigoplus_{j\in\min(\mathbf{F})} \mathbf{P}^j$, and this yields the claim. □

Remark 5.7. By Remark 5.3, every functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(1,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ of rank 1 is projective and relative injective.

Finally, one concludes the following theorem which is somehow counterintuitive in view of Remark 5.1.

Theorem 5.8. *Let* p *be a prime number, and let* R *be a principal ideal domain of characteristic* 0 *such that* pR *is a prime ideal. Then*

(a) $\mathrm{Ldim}_R(\mathcal{G}_R(1,p)) = 0$ *and* $\mathrm{gldim}_R(\mathcal{G}_R(1,p)) = 1$*; and*
(b) $\mathrm{Ldim}_R(\mathcal{G}_R(n,p)) = 1$ *and* $\mathrm{gldim}_R(\mathcal{G}_R(n,p)) = 2$ *for* $n \geq 2$.

In particular, $(\mathcal{G}_R(n,p), \sigma)$ *has the Whitehead property.*

Proof. Suppose that $n \geq 2$. By Proposition 5.6, the projective dimension of any R-lattice functor of rank 1 is less than or equal to 1. Hence by Fact 5.2, induction on the rank and the Horseshoe lemma [2, Lemma 2.5.1], $\mathcal{G}_R(n, p)$ is of global R-lattice dimension less than or equal to 1. Since there are R-lattice functors of rank 1 which are not projective, one concludes that $\mathrm{Ldim}_R(\mathcal{G}_R(n, p)) = 1$. For any simple functor $\mathbf{S}^\ell$, $0 \leq \ell \leq n$, one has $\mathrm{proj.dim}(\mathbf{S}^\ell) = 2$. Thus $\mathrm{gldim}_R(\mathcal{G}_R(n, p)) = 2$.

By Remark 5.7, any R-lattice functor of rank 1 of $\mathcal{G}_R(1, p)$ is projective and relative injective. Hence by Fact 5.2, induction on the rank and the Horseshoe lemma, any R-lattice functor is projective, i.e., $\mathrm{Ldim}_R(\mathcal{G}_R(1, p)) = 0$. For the simple functors $\mathbf{S}^\ell$, $\ell \in \{0, 1\}$, one has $\mathrm{proj.dim}(\mathbf{S}^\ell) = 1$. Thus $\mathrm{gldim}_R(\mathcal{G}_R(n, p)) = 1$.

The final remark is a direct consequence of Fact 2.11. □

5.6. *Projective R-lattice functors*

Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR. Then R is a noetherian ring, and every proper subfunctor $\mathbf{G} \subsetneq \mathbf{F}$ of a functor $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {}_R\mathbf{mod}^{\mathrm{f.g.}}))$ must be contained in a maximal subfunctor $\mathbf{M} \subsetneq \mathbf{F}$. Moreover, from the discussion in Section 5.3 one concludes that $\mathbf{F}/\mathbf{M} \simeq \mathbf{S}^\ell$ for some $\ell \in \{0, \dots, n\}$. We define the *radical* of $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {}_R\mathbf{mod}^{\mathrm{f.g.}}))$ by

$$\mathrm{rad}(\mathbf{F}) = \bigcap_{\substack{\mathbf{M} \subsetneq \mathbf{F} \\ \mathbf{M}\ \text{maximal}}} \mathbf{M} \tag{5.20}$$

and the *head* of $\mathbf{F}$ by $\mathrm{hd}(\mathbf{F}) = \mathbf{F}/\mathrm{rad}(\mathbf{F})$. In particular, there exist non-negative integers $f_0, \dots, f_n \in \mathbb{N}_0$ such that

$$\mathrm{hd}(\mathbf{F}) \simeq f_0\mathbf{S}^0 \oplus \cdots \oplus f_n\mathbf{S}^n. \tag{5.21}$$

Here we used the abbreviation $m\mathbf{Z} = \oplus_{1 \leq j \leq m} \mathbf{Z}$. Moreover, $\mathrm{hd}(\mathbf{F}) = 0$ if, and only if, $\mathbf{F} = 0$. Furthermore, the following property holds.

Fact 5.9. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, and let $\phi\colon \mathbf{G} \to \mathbf{F} \in \mathrm{mor}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {}_R\mathbf{mod}^{\mathrm{f.g.}}))$ be a natural transformation of functors with values in the category of finitely generated R-modules. Then ϕ is surjective if, and only if, the induced map $\mathrm{hd}(\phi)\colon \mathrm{hd}(\mathbf{G}) \to \mathrm{hd}(\mathbf{F})$ is surjective.*

From this one concludes the following property.

Fact 5.10. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, and let $\mathbf{F} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {}_R\mathbf{lat}))$. Then $\mathrm{rk}(\mathbf{F}) \leq \dim_{\mathbb{F}}(\mathrm{hd}(\mathbf{F}))$, and equality holds if, and only if, $\mathbf{F}$ is projective.*

Proof. Suppose that $\mathrm{hd}(\mathbf{F}) \simeq f_0\mathbf{S}^0 \oplus \cdots \oplus f_n\mathbf{S}^n$. Put $\mathbf{P} = f_0\mathbf{P}^0 \oplus \cdots \oplus f_n\mathbf{P}^n$. Since $\mathbf{P}$ is projective, there exists a natural transformation $\phi\colon \mathbf{P} \to \mathbf{F}$ such that $\mathrm{hd}(\phi)\colon \mathrm{hd}(\mathbf{P}) \to \mathrm{hd}(\mathbf{F})$ is an isomorphism. By Fact 5.9, ϕ is surjective, and thus

$$\dim_{\mathbb{F}}(\mathrm{hd}(\mathbf{F})) = \dim_{\mathbb{F}}(\mathrm{hd}(\mathbf{P})) = \mathrm{rk}(\mathbf{P}) \geq \mathrm{rk}(\mathbf{F}). \tag{5.22}$$

If $\mathrm{rk}(\mathbf{F}) = \dim_{\mathbb{F}}(\mathrm{hd}(\mathbf{F}))$, then ϕ must be an isomorphism. Assume that $\mathbf{F}$ is projective. Then ϕ is split-surjective, i.e., there exists a projective R-lattice functor $\mathbf{Q} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n, p)^{\mathrm{op}}, {}_R\mathbf{lat}))$

such that $\mathbf{P} \simeq \mathbf{F} \oplus \mathbf{Q}$. As $\mathrm{hd}(\phi)$ is an isomorphism, this yields $\mathrm{hd}(\mathbf{Q}) = 0$. Hence $\mathbf{Q} = 0$, and $\mathbf{F}$ is isomorphic to $\mathbf{P}$. □

The proof of Fact 5.10 has shown also the following.

Corollary 5.11. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, and let $\mathbf{P} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{lat}))$ be a projective R-lattice functor satisfying $\mathrm{hd}(\mathbf{P}) \simeq f_0\mathbf{S}^0 \oplus \cdots \oplus f_n\mathbf{S}^n$. Then $\mathbf{P} \simeq f_0\mathbf{P}^0 \oplus \cdots \oplus f_n\mathbf{P}^n$.*

6. Cohomological Mackey functors for cyclic p-groups

Throughout this section we assume that R is a discrete valuation domain of characteristic 0 with maximal ideal pR and that G is a finite cyclic p-group of order p^n.

6.1. The deflation functor

Let $\pi\colon \mathcal{M}_R(G) \longrightarrow \mathcal{G}_R(n,p)$ denote the unitary projection (cf. Section 5.2), let $\_^{\pi} = \mathrm{inf}^{\pi}(\mathrm{def}^{\pi}(\_))$, and let $\eta\colon \mathrm{id}_{c\mathfrak{M}\mathfrak{F}_G({}_R\mathbf{mod})} \to \_^{\pi}$ denote the unit of the adjunction. In particular, $\eta_{\mathbf{X}}\colon \mathbf{X} \to \mathbf{X}^{\pi}$ is surjective, and $\eta_{\mathbf{X},G}\colon \mathbf{X}_G \to \mathbf{X}^{\pi}_G$ is an isomorphism for all $\mathbf{X} \in \mathrm{ob}(c\mathfrak{M}\mathfrak{F}_G({}_R\mathbf{mod}))$.

Fact 6.1. *Let G be a finite cyclic group of order p^n, let R be an integral domain of characteristic 0, and let $\mathbf{X} \in \mathrm{ob}(c\mathfrak{M}\mathfrak{F}_G({}_R\mathbf{lat}))$ be a cohomological R-lattice functor which is Hilbert 90. Then one has a canonical isomorphism*

$$\mathrm{def}^{\pi}(\mathbf{X})(k) \simeq \mathrm{im}(t^{\mathbf{X}}_{U,G}) \subseteq \mathbf{X}_G, \tag{6.1}$$

for $U \subseteq G$, $|U| = p^{n-k}$.

Proof. Let $g \in G$ be a generator of G, i.e., $\mathrm{def}^{\pi}(\mathbf{X})(k) = \mathbf{X}_U/(1-g)\mathbf{X}_U$. Periodicity of Tate cohomology implies that $\hat{H}^{-1}(G/U, \mathbf{X}_U) = H^1(G/U, \mathbf{X}_U) = 0$. Thus by (4.14), $\ker(t^{\mathbf{X}}_{U,G}) = (1-g)\mathbf{X}_U$. Hence the induced map

$$\mathbf{X}_U/(1-g)\mathbf{X}_U \xrightarrow{\tilde{t}^{\mathbf{X}}_{U,G}} \mathbf{X}_G \tag{6.2}$$

is injective. This yields the claim. □

From Fact 6.1 one concludes the following.

Corollary 6.2. *Let G be a finite cyclic group of order p^n, let R be an integral domain of characteristic 0, and let $\phi\colon \mathbf{X} \to \mathbf{Y} \in \mathrm{mor}(c\mathfrak{M}\mathfrak{F}_G({}_R\mathbf{lat}))$ be a natural transformation of cohomological R-lattice functors with the following properties:*

(i) *$\mathbf{X}$ and $\mathbf{Y}$ are Hilbert 90;*
(ii) *$\phi_G\colon \mathbf{X}_G \to \mathbf{Y}_G$ is injective.*

Then $\mathrm{def}^{\pi}(\phi)\colon \mathrm{def}^{\pi}(\mathbf{X}) \to \mathrm{def}^{\pi}(\mathbf{Y})$ is injective. In particular, if

$$0 \longrightarrow \mathbf{X} \xrightarrow{\alpha} \mathbf{Y} \xrightarrow{\beta} \mathbf{Z} \longrightarrow 0 \tag{6.3}$$

is a short exact sequence of R-lattice functors all of which are Hilbert 90, then

$$0 \longrightarrow \mathrm{def}^{\pi}(\mathbf{X}) \xrightarrow{\mathrm{def}^{\pi}(\alpha)} \mathrm{def}^{\pi}(\mathbf{Y}) \xrightarrow{\mathrm{def}^{\pi}(\beta)} \mathrm{def}^{\pi}(\mathbf{Z}) \longrightarrow 0 \tag{6.4}$$

is exact.

Proposition 6.3. *Let G be a finite cyclic group of order p^n, let R be an Dedekind domain of characteristic* 0*, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ be a cohomological R-lattice functor which is Hilbert 90. Then $\mathbf{X}$ is π-acyclic.*

Proof. Let $(\mathbf{P}_\bullet, \partial^{\mathbf{P}}_\bullet, \varepsilon_{\mathbf{X}})$ be a projective resolution of $\mathbf{X}$ in $\mathfrak{cMF}_G({}_R\mathbf{mod})$ with $\mathbf{P}_k$ projective R-lattice functors. In particular, $\mathbf{Q}_k = \mathrm{im}(\partial^P_k)$, $k \geq 1$, is an R-lattice functor. By construction, one has the short exact sequences

$$\begin{aligned} &0 \longrightarrow \mathbf{Q}_1 \longrightarrow \mathbf{P}_0 \longrightarrow \mathbf{X} \longrightarrow 0, \\ &0 \longrightarrow \mathbf{Q}_{k+1} \longrightarrow \mathbf{P}_k \longrightarrow \mathbf{Q}_k \longrightarrow 0, \end{aligned} \tag{6.5}$$

for $k \geq 1$. Thus by induction and Proposition 4.11, $\mathbf{Q}_k$ is a Hilbert 90 R-lattice functors for all $k \geq 1$. Hence by Corollary 6.2, one has short exact sequences

$$\begin{aligned} &0 \longrightarrow \mathrm{def}^{\pi}(\mathbf{Q}_1) \longrightarrow \mathrm{def}^{\pi}(\mathbf{P}_0) \longrightarrow \mathrm{def}^{\pi}(\mathbf{X}) \longrightarrow 0, \\ &0 \longrightarrow \mathrm{def}^{\pi}(\mathbf{Q}_{k+1}) \longrightarrow \mathrm{def}^{\pi}(\mathbf{P}_k) \longrightarrow \mathrm{def}^{\pi}(\mathbf{Q}_k) \longrightarrow 0, \end{aligned} \tag{6.6}$$

for $k \geq 1$. This implies that $\mathcal{L}_k\mathrm{def}^{\pi}(\mathbf{X}) = 0$ for all $k \geq 1$. □

Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, and let G be a cyclic p-group. As in Section 5.6 one concludes that every proper subfunctor $\mathbf{Y} \subsetneq \mathbf{X}$ of a cohomological G-Mackey functor $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}^{\mathrm{f.g.}}))$ must be contained is a maximal subfunctor $\mathbf{M} \subsetneq \mathbf{X}$. Therefore we define the *radical* of $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}^{\mathrm{f.g.}}))$ by

$$\mathrm{rad}(\mathbf{X}) = \bigcap_{\substack{\mathbf{M} \subsetneq \mathbf{X} \\ \mathbf{M}\ \text{maximal}}} \mathbf{M} \tag{6.7}$$

and the *head* of $\mathbf{X}$ by $\mathrm{hd}(\mathbf{X}) = \mathbf{X}/\mathrm{rad}(\mathbf{X})$. By Remark 3.2, there exist non-negative integers $f_U \in \mathbb{N}_0$, $U \subseteq G$, such that

$$\mathrm{hd}(\mathbf{X}) \simeq \bigoplus_{U \subseteq G} f_U \mathbf{S}^U. \tag{6.8}$$

Since every simple cohomological G-Mackey functor $\mathbf{S} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}^{\mathrm{f.g.}}))$ is isomorphic to $\mathrm{inf}^{\pi}(\boldsymbol{\Sigma})$ for some simple functor $\boldsymbol{\Sigma} \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p)^{\mathrm{op}}, {}_R\mathbf{mod}^{\mathrm{f.g.}}))$, one has $\ker(\eta_{\mathbf{X}}) \subseteq \mathrm{rad}(\mathbf{X})$. This inclusion has the following consequence.

Proposition 6.4. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR, and let G be a finite cyclic p-group. Let $\phi\colon \mathbf{X} \to \mathbf{Y}$ be a natural transformation of cohomological G-Mackey functors with values in the category ${}_R\mathbf{mod}^{\mathrm{f.g.}}$. Then the following are equivalent.*

(i) *ϕ is surjective;*
(ii) *$\phi^{\pi} : \mathbf{X}^{\pi} \to \mathbf{Y}^{\pi}$ is surjective;*
(iii) *$\mathrm{hd}(\phi)\colon \mathrm{hd}(\mathbf{X}) \to \mathrm{hd}(\mathbf{Y})$ is surjective.*

Proof. The natural surjection $\tau\colon \mathrm{id} \to \mathrm{hd}(\_)$ factors through the natural surjection $\eta\colon \mathrm{id} \to \_^{\pi}$, i.e., there exists a natural surjection $\psi\colon \_^{\pi} \to \mathrm{hd}(\_)$ such that $\tau = \psi \circ \eta$. This yields the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose that (iii) holds and that ϕ is not surjective. Then $\mathrm{im}(\phi)$ is contained in a maximal subfunctor $\mathbf{M} \subsetneq \mathbf{Y}$. Thus for $\mathbf{S} = \mathbf{Y}/\mathbf{M}$, the kernel of the map $\phi_*\colon \mathrm{nat}_G(\mathbf{Y}, \mathbf{S}) \to \mathrm{nat}_G(\mathbf{X}, \mathbf{S})$ is non-trivial. However, in the commutative diagram

$$\begin{array}{ccc} \mathrm{nat}_G(\mathrm{hd}(\mathbf{Y}), \mathbf{S}) & \xrightarrow{\mathrm{hd}(\phi)_*} & \mathrm{nat}_G(\mathrm{hd}(\mathbf{X}), \mathbf{S}) \\ \downarrow{\scriptstyle (\tau_{\mathbf{Y}})_*} & & \downarrow{\scriptstyle (\tau_{\mathbf{X}})_*} \\ \mathrm{nat}_G(\mathbf{Y}, \mathbf{S}) & \xrightarrow{\phi_*} & \mathrm{nat}_G(\mathbf{X}, \mathbf{S}) \end{array} \tag{6.9}$$

the vertical maps are isomorphisms, and $\mathrm{hd}(\phi)_*$ is injective forcing $\ker(\phi_*) = 0$, a contradiction. This yields the claim.

We are now ready to prove the following theorem which is one of the key results in this paper.

Theorem 6.5. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR, let G be a finite cyclic p-group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ be a cohomological G-Mackey functor with values in the category of R-lattices which is Hilbert* 90*. Then there exists a finite G-set Ω such that $\mathbf{X} \simeq \mathbf{h}^0(R[\Omega])$. In particular, $\mathbf{X}$ is projective.*

Proof. The deflation functor $\mathrm{def}^{\pi}\colon \mathfrak{cMF}_G({}_R\mathbf{mod}) \longrightarrow \mathfrak{F}_R(\mathcal{G}_R(p,n), {}_R\mathbf{mod})$ associated to the unitary projection $\pi\colon \mathcal{M}_R(G) \longrightarrow \mathcal{G}_R(p,n)$ has the following properties.

(1) $(\mathcal{M}_R(G), \sigma)$ is $\circledast$-symmetric (cf. Proposition 4.10).
(2) $\mathcal{G}_R(n,p)$ has global R-lattice dimension less than or equal to 1 (cf. Theorem 5.8), and thus has the Whitehead property (cf. Facts 2.11 and 2.13).
(3) An R-lattice functor $\mathbf{Y} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ is $\circledast$-acyclic if, and only if, it has the Hilbert 90 property (cf. Proposition 4.10). By Proposition 6.3, such a functor is π-acyclic.

In particular, the hypothesis of Theorem 2.16 are satisfied, and one concludes that $\mathbf{Z} = \mathrm{def}^{\pi}(\mathbf{X}) \in \mathrm{ob}(\mathfrak{F}_R(\mathcal{G}_R(n,p), {}_R\mathbf{mod}))$ is projective. Hence there exist non-negative integers $f_0, \ldots, f_n$ such that $\mathbf{Z} \simeq f_0\mathbf{P}^0 \oplus \cdots f_n\mathbf{P}^n$ (cf. Corollary 5.11).

Let $\eta_{\mathbf{X}}\colon \mathbf{X} \to \mathbf{X}^{\pi}$ be the canonical map (cf. Section 6.1), i.e., $\mathbf{X}^{\pi} \simeq \bigoplus_{0\le k\le n} \inf^{\pi}(f_k\mathbf{P}^k)$. Let $U_k \subseteq G$ denote the unique subgroup of G of index p^k, and let Ω be the G-set $\Omega = \bigsqcup_{0\le k\le n} f_k(G/U_k)$. Put $\mathbf{P} = \mathbf{h}^0(R[\Omega])$. Then $\mathbf{P} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ is projective (cf. Fact 3.1). Since $\mathrm{def}^{\pi}(\mathbf{h}^0(R[G/U_k])) \simeq \mathbf{P}^k$ for all $k \in \{0, \ldots, n\}$, one has an isomorphism $\phi^{\pi}\colon \mathbf{P}^{\pi} \to \mathbf{X}^{\pi}$. Since $\mathbf{P}$ is projective, there exists a homomorphism of cohomological G-Mackey functors such that the diagram

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\phi} & \mathbf{X} \\ \downarrow{\scriptstyle \eta_{\mathbf{P}}} & & \downarrow{\scriptstyle \eta_{\mathbf{X}}} \\ \mathbf{P}^{\pi} & \xrightarrow{\phi^{\pi}} & \mathbf{X}^{\pi} \end{array} \tag{6.10}$$

commutes. By construction, ϕ^{π}_G is an isomorphism, and $\eta_{\mathbf{P},G}$ and $\eta_{\mathbf{X},G}$ are isomorphisms (cf. Section 6.1). Thus ϕ_G is an isomorphism. In particular, with the same notations as used in Section 4.6, the map $\mathrm{gr}_0(\phi_G)\colon \mathrm{gr}_0(\mathbf{P}) \to \mathrm{gr}_0(\mathbf{X})$ is an isomorphism. By hypothesis, $\mathbf{X}$ is an R-lattice functor with the Hilbert 90 property, and the same is true for $\mathbf{P}$ (cf. Remark 4.4). Hence

$\mathrm{gr}_0(\mathbf{X})$ and $\mathrm{gr}_0(\mathbf{P})$ are of type H^0 (cf. Proposition 4.12), and ϕ is injective (cf. Proposition 4.16). Moreover, by Proposition 6.4, ϕ must be surjective. This yields the claim. □

As an immediate consequence of Remark 4.4 we obtain the following.

Corollary 6.6. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, let G be a finite cyclic p-group of order p^n, and let $\mathbf{P} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{lat}))$ be a projective R-lattice functor. Then there exist non-negative integers $f_W \in \mathbb{N}_0$, $W \subseteq G$, such that $\mathbf{P} \simeq \bigoplus_{U\subseteq G} f_W \mathbf{P}^W$, i.e.,*

$$K_0(\mathcal{M}_R(G)) \simeq \mathrm{B}(G) \simeq \mathbb{Z}^n, \tag{6.11}$$

where $\mathrm{B}(G)$ denotes the Burnside ring of G.

In case that the $R[G]$-lattice M satisfies a Hilbert 90 property, one obtains the following.

Corollary 6.7. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, let G be a finite cyclic p-group, and let M be an $R[G]$-lattice such that $H^1(U, \mathrm{res}^G_U(M)) = 0$ for every subgroup U of G. Then there exists a finite G-set Ω such that $M \simeq R[\Omega]$.*

Proof. By hypothesis, $\mathbf{X} = \mathbf{h}^0(M)$ is a cohomological G-Mackey functor with values in the category of R-lattices satisfying the Hilbert 90 property. Thus by Theorem 6.5, there exists a finite G-set Ω such that $\mathbf{X} \simeq \mathbf{h}^0(R[\Omega])$. Hence evaluating the functors $\mathbf{X}$ and $\mathbf{h}^0(R[\Omega])$ on the subgroup $\{1\}$ yields the claim. □

The following property is a direct consequence of Tate duality (cf. Proposition 4.6) and completes the proof of Theorem A.

Proposition 6.8. *Let R be a principal ideal domain of characteristic 0, let G be a finite group, let U be a subgroup of G, and let M be an $R[G]$-lattice. Then the following are equivalent.*

(i) $H^1(U, \mathrm{res}^G_U(M^*)) = 0$;
(ii) $\hat{H}^{-1}(U, \mathrm{res}^G_U(M)) = 0$;
(iii) $M/\omega_{R[U]}M$ *is torsion free.*

Proof. By (4.11), (i) and (ii) are equivalent. Let $N_U : M \to M^U$ be the U-norm map, i.e., for $m \in M$ one has $N_U(m) = \sum_{u\in U} u \cdot m$. As M is an $R[U]$-lattice, M^U is an R-lattice. Hence

$$\mathrm{tor}_R(M/\omega_{R[U]}M) = \ker(N_U)/\omega_{R[U]}M = \hat{H}^{-1}(U, \mathrm{res}^G_U(M)), \tag{6.12}$$

where $\mathrm{tor}_R(\_)$ denotes the R-submodule of R-torsion elements. Thus (ii) and (iii) are equivalent. □

6.2. Projective dimensions

In conjunction with Proposition 4.11, Theorem 6.5 has strong implications on the projective dimension of a cohomological Mackey functor of a cyclic p-group.

Theorem 6.9. *Let R be a discrete valuation domain of characteristic 0 with maximal ideal pR, let G be a finite cyclic p-group, and let $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_R\mathbf{mod}^{\mathrm{f.g.}}))$. Let*

$$\mathbf{P}_2 \xrightarrow{\partial_2} \mathbf{P}_1 \xrightarrow{\partial_1} \mathbf{P}_0 \xrightarrow{\varepsilon_{\mathbf{X}}} \mathbf{X} \longrightarrow 0 \tag{6.13}$$

be a partial projective resolution of $\mathbf{X}$ by projective R-lattice functors. Then

(a) $\ker(\partial_2)$ *is a projective R-lattice functor, i.e.,* $\mathrm{proj.dim}(\mathbf{X}) \leq 3$.
(b) *If* $\mathbf{X}$ *is i-injective, then* $\ker(\partial_1)$ *is a projective R-lattice functor, i.e., one has* $\mathrm{proj.dim}(\mathbf{X}) \leq 2$.
(c) *If* $\mathbf{X}$ *is of type* H^0*, then* $\ker(\varepsilon_{\mathbf{X}})$ *is a projective R-lattice functor, i.e., in this case one has* $\mathrm{proj.dim}(\mathbf{X}) \leq 1$.

In particular, if G is non-trivial then $\mathrm{Ldim}(\mathcal{M}_R(G)) = 2$*, and* $\mathrm{gldim}(\mathcal{M}_R(G)) = 3$.

Proof. (a) By Proposition 4.11(a), (b) and (c), $\ker(\partial_0)$ is an R-lattice functor and thus i-injective, $\ker(\partial_1)$ is of type H^0, and $\ker(\partial_2)$ is Hilbert 90. Hence Theorem 6.5 yields the claim in this case. (b) and (c) follow by a similar argument. From (a) one concludes that $\mathrm{gldim}(\mathcal{M}_R(G)) \leq 3$, and (b) implies $\mathrm{Ldim}(\mathcal{M}_R(G)) \leq 2$. If G is non-trivial, the discussion in Section 4.1 shows that $\mathrm{proj.dim}(\mathbf{B}^G) = 3$. This yields the final remark (cf. (2.12)). □

Remark 6.10. Let $\mathbb{F}$ be a field of characteristic p, and let G be a non-trivial, finite cyclic p-group. Then $\mathcal{M}_{\mathbb{F}}(G)$ is not of finite global dimension, but $\mathcal{M}_{\mathbb{F}}(G)$ is 2-Gorenstein (cf. Proposition 4.8). This phenomenon occurred already for the gentle R-order categories (in dimension 1) (cf. Remark 5.1).

6.3. Lattices

From Theorems 6.5 and 6.9(b), one concludes the following.

Theorem 6.11. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR, let G be a finite cyclic p-group, and let M be an R[G]-lattice. Then there exist finite G-sets* Ω_0 *and* Ω_1*, and a short exact sequence*

$$0 \longrightarrow R[\Omega_1] \longrightarrow R[\Omega_0] \longrightarrow M \longrightarrow 0. \tag{6.14}$$

Proof. Let $\mathbf{X} = \mathbf{h}^0(M)$. As $\mathbf{X}$ is of type H^0, Theorem 6.9(b) implies that $\mathbf{X}$ has a projective resolution

$$0 \longrightarrow \mathbf{P}_1 \xrightarrow{\partial_1} \mathbf{P}_0 \xrightarrow{\varepsilon_{\mathbf{X}}} \mathbf{X} \longrightarrow 0, \tag{6.15}$$

where $\mathbf{P}_0$ and $\mathbf{P}_1$ are projective R-lattice functors. As $\mathbf{P}_0$ and $\mathbf{P}_1$ have the Hilbert 90 property (cf. Remark 4.4), Theorem 6.5 implies that there exist finite G-sets Ω_0 and Ω_1 such that $\mathbf{P}_i = \mathbf{h}^0(R[\Omega_i])$, $i \in \{0, 1\}$. Thus evaluating the functors on $\{1\}$ yields the claim. □

6.4. Extending A. Weiss' theorem

The following property can be seen as an extension of A. Weiss' theorem for finite cyclic p-groups.

Proposition 6.12. *Let R be a discrete valuation domain of characteristic* 0 *with maximal ideal pR for some prime number p, let G be a finite cyclic p-group, and let M be an R[G]-lattice. Suppose that for some subgroup N of G one has*

(i) $\mathrm{res}^G_N(M)$ *is an* $R[N]$*-permutation module;*
(ii) M^N *is an* $R[G/N]$*-permutation module.*

Then M is isomorphic to an R[G]-permutation module, i.e., there exists some finite G-set Ω *such that* $M \simeq R[\Omega]$.

Proof. Let U be a subgroup of G. If $U \subseteq N$, then by (i), $\mathrm{res}^G_U(M)$ is an $R[U]$-permutation module. Thus one has $H^1(U, \mathrm{res}^G_U(M)) = 0$. Suppose that $N \subsetneq U$. Since N is a normal subgroup of U, the 5-term exact sequence in cohomology yields an exact sequence

$$0 \longrightarrow H^1(U/N, M^N) \longrightarrow H^1(U, \mathrm{res}^G_U(M)) \longrightarrow H^1(N, \mathrm{res}^G_N(M))^{G/N} . \quad (6.16)$$

Hence by (i), one has $H^1(N, \mathrm{res}^G_N(M)) = 0$. By (ii), M^N is an $R[U/N]$-permutation module, and therefore $H^1(U/N, M^N) = 0$. Thus by (6.16) one has $H^1(U, \mathrm{res}^G_U(M)) = 0$. The assertion then follows from Theorem A. □

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