



Wreath products and representations of p -local finite groups [☆]

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Abstract

Given two finite p -local finite groups and a fusion preserving morphism between their Sylow subgroups, we study the question of extending it to a continuous map between their classifying spaces. The results depend on the construction of the wreath product of p -local finite groups which is also used to study p -local permutation representations.

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1. Introduction

A fusion system \mathcal{F} on a finite p -group P is a small category whose objects are the subgroups of P and whose morphisms are group monomorphisms which include all those homomorphisms obtained from conjugation by the elements of P . The idea of *saturated* fusion systems was formulated in the early 1980s by Puig [19] who studied representations of finite groups. Every block b of the group algebra kG , where k is an algebraically closed field of characteristic p , gives rise to a *saturated* fusion system on its defect group $P \leq G$. The principal block of kG

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gives rise to the fusion system $\mathcal{F}_S(G)$ whose objects are the subgroups of a Sylow p -subgroup S of G and its morphisms are induced by conjugation in G . Not all fusion systems have the form $\mathcal{F}_S(G)$, see, e.g., [6, Examples 9.3, 9.4] or [11].

The significance of $\mathcal{F}_S(G)$ in topology was recognized by Martino and Priddy in [14]. In [16, 17], Oliver shows that $\mathcal{F}_S(G)$ determines the homotopy type of the p -completion (in the sense of Bousfield and Kan [2]) of $BG = K(G, 1)$.

In order to understand self-homotopy equivalences of BG_p^\wedge , Broto, Levi and Oliver considered in [5] a category $\mathcal{L}_S(G)$ closely related to $\mathcal{F}_S(G)$. This category was studied earlier by Puig. Abstraction of this construction led them in [6] to the notion of a *centric linking system* \mathcal{L} associated to a saturated fusion system (S, \mathcal{F}) . The triple $(S, \mathcal{F}, \mathcal{L})$ is called a p -local finite group. Its *classifying space* is by definition the space $|\mathcal{L}|_p^\wedge$, a terminology justified by the fact that $|\mathcal{L}_S(G)|_p^\wedge \simeq BG_p^\wedge$ [5, Lemma 1.2]. The spaces $|\mathcal{L}|_p^\wedge$ have many properties in common with p -completed classifying spaces of finite groups. Thus, p -local finite groups provide an important connection between group theory and topology via their linking systems.

This paper focuses on the following fundamental problem. In what way, if any, a fusion preserving map $(S, \mathcal{F}) \rightarrow (S', \mathcal{F}')$, see details below, gives rise to a map $|\mathcal{L}|_p^\wedge \rightarrow |\mathcal{L}'|_p^\wedge$ between the classifying spaces? A step forward is given in Theorem B below. It is related to the yet open problem of defining the concept of morphisms between p -local finite groups in a way which is compatible with maps between their classifying spaces. It also gives a new insight to the study of maps between p -completed classifying spaces.

We will define a permutation representation of a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ as a homotopy class of a map $|\mathcal{L}| \rightarrow (B\Sigma_n)_p^\wedge$ where Σ_n is a symmetric-group. In Theorem C below we will prove a p -local form of Cayley’s theorem, namely the existence of p -local regular representations. We will then approach the notion of the homotopy-index of the Sylow subgroup S in $(S, \mathcal{F}, \mathcal{L})$ through the regular representation. The index of a subgroup S in a finite group G is the number of the orbits of S in its action by translation on G . In other words, restriction of the regular representation of G to S results in $|G : S|$ copies of the regular representation of S . From the homotopy point of view, one could define the homotopy-index of S in \mathcal{L} as the minimal n for which there is a map $|\mathcal{L}| \rightarrow (B\Sigma_{n \cdot |S|})_p^\wedge$ whose restriction to BS is homotopic to the map $BS \xrightarrow{n \cdot \text{reg}_S} (B\Sigma_{n \cdot |S|})_p^\wedge$ induced by n copies of the regular representation of S . But this number is very difficult to compute, even for a p -local finite group associated to a finite group. Instead, we will define the lower homotopy-index of S in \mathcal{L} as the smallest number p^k such that the map $BS \rightarrow (B\Sigma_{p^k \cdot |S|})_p^\wedge$ induced by $p^k \cdot \text{reg}_S$ can be extended up to homotopy to a map $|\mathcal{L}| \rightarrow (B\Sigma_{p^k \cdot |S|})_p^\wedge$. This is a new invariant of p -local finite groups.

Let us now describe our results in greater detail. Suppose that $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ are p -local finite groups. Given a group homomorphism $\rho : S \rightarrow S'$ it is natural to ask if $B\rho : BS \rightarrow BS'$ can be extended, up to homotopy, to a map $\tilde{f} : |\mathcal{L}|_p^\wedge \rightarrow |\mathcal{L}'|_p^\wedge$ whose restriction to BS , namely $\tilde{f} \circ \Theta$, is homotopic to the composite $BS \xrightarrow{B\rho} BS' \xrightarrow{\Theta'} |\mathcal{L}'|_p^\wedge$ where Θ and Θ' are the maps described in 2.9.

Recall that ρ is called *fusion preserving* if for every $\varphi \in \mathcal{F}(P, Q)$ there exists some $\varphi' \in \mathcal{F}'(\rho(P), \rho(Q))$ such that $\rho \circ \varphi = \varphi' \circ \rho$. Ragnarsson shows in [21] that stably, namely in the homotopy category of spectra, the morphism \tilde{f} above exists if and only if ρ is fusion preserving. Unstably this is unknown.

The content of Theorem B below is that \tilde{f} exists provided the target \mathcal{L}' is replaced with its wreath product with Σ_n for some n , a construction which we now describe.

The *wreath product* of a space X with a subgroup $G \leq \Sigma_n$, denoted $X \wr G$, is the homotopy orbit space $(X^n)_{hG}$ where G acts by permuting the factors (see Definition 3.4). This construction is equipped with a map $\Delta: X \rightarrow X \wr G$ which factors through the diagonal map $X \rightarrow X^n$. We prove in 3.6 below that if H is a discrete groups then there is a homotopy equivalence $(BH) \wr G \simeq B(H \wr G)$ such that $\Delta: BH \rightarrow (BH) \wr G$ is induced by the diagonal inclusion $H \leq H \wr G$. The next result should be compared with [3, Theorems D and E].

Theorem A. *Fix a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ where $S \neq 1$. Let K be a subgroup of Σ_n and let S' be a Sylow p -subgroup of $S \wr K$. Then there exists a p -local finite group $(S', \mathcal{F}', \mathcal{L}')$ which is equipped with a homotopy equivalence $|\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ such that the composite*

$$BS' \xrightarrow{B \text{ incl}} B(S \wr K) \simeq (BS) \wr K \xrightarrow{\Theta \wr K} |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$$

is homotopic to the natural map $\Theta': BS' \rightarrow |\mathcal{L}'|$. Moreover, $(S', \mathcal{F}', \mathcal{L}')$ satisfying these properties is unique up to an isomorphism of p -local finite groups.

In Remark 5.3 we show that when Theorem A is applied to a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ of a finite group G then $(S', \mathcal{F}', \mathcal{L}')$ is the p -local finite group of $G \wr K$. If $S = 1$ then $|\mathcal{L}| = *$ and we choose $(S', \mathcal{F}', \mathcal{L}')$ to be the p -local finite group associated to K and the map $\Delta: |\mathcal{L}| \rightarrow |\mathcal{L}'|$ is any map $* \rightarrow |\mathcal{L}'|$.

We prove Theorem A in Section 5 which is a technical section, however the remainder of the paper is completely independent of its proof.

1.1. Definition. We call the p -local finite group $(S', \mathcal{F}', \mathcal{L}')$ in the theorem above the *wreath product* of $(S, \mathcal{F}, \mathcal{L})$ with K and denote its fusion system and linking system by $\mathcal{F} \wr K$ and $\mathcal{L} \wr K$, respectively. Let $\Delta: |\mathcal{L}| \rightarrow |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ denote the diagonal inclusion followed by the homotopy equivalence in Theorem A.

Theorem B. *Let $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ be p -local finite groups. Suppose that $\rho: S \rightarrow S'$ is a fusion preserving homomorphism. Then there exist some $m \geq 0$ and a map $\tilde{f}: |\mathcal{L}|_p^\wedge \rightarrow |\mathcal{L}' \wr \Sigma_{p^m}|_p^\wedge$ such that the diagram below commutes up to homotopy*

$$\begin{array}{ccc}
 BS & \xrightarrow{\eta \circ \Theta} & |\mathcal{L}|_p^\wedge \\
 \downarrow B\rho & & \searrow \tilde{f} \\
 BS' & \xrightarrow{\eta \circ \Theta'} & |\mathcal{L}'|_p^\wedge \xrightarrow{\Delta_p^\wedge} |\mathcal{L}' \wr \Sigma_{p^m}|_p^\wedge
 \end{array}$$

In Theorem 7.3 below we prove a more elaborate result.

A *permutation representation* of a finite group G is a homomorphism $\rho: G \rightarrow \Sigma_n$. The *rank* of ρ is n . Throughout, we will call ρ simply a “representation.” Clearly G acts on itself by left (or right) translations giving rise to Cayley’s embedding $\text{reg}_G: G \rightarrow \Sigma_{|G|}$ which is called the *regular permutation representation* of G .

Two representations $\rho_1, \rho_2: G \rightarrow \Sigma_n$ are *equivalent* if they are conjugate in Σ_n , that is, if they differ by an inner automorphism of Σ_n . The set of equivalence classes of representations of G of

rank n is denoted $\text{Rep}_n(G)$. There are obvious inclusions $\Sigma_n \times \Sigma_m \leq \Sigma_{n+m}$ and $\Sigma_n \times \Sigma_m \leq \Sigma_{nm}$ obtained by taking the disjoint union and the product of sets of cardinality n and m . They give rise to commutative, associative and unital binary operations $+$ and \times on the set $\coprod_{n \geq 0} \text{Rep}_n(G)$. We shall write $k \cdot \rho$ for the k -fold sum $\rho + \dots + \rho$.

Let \mathcal{F} be a fusion system on S . A representation $\rho : S \rightarrow \Sigma_n$ is called \mathcal{F} -invariant if for every $P \leq S$ and every $\varphi \in \mathcal{F}(P, S)$ the representations $\rho|_P$ and $\rho \circ \varphi$ of P are equivalent. Let $\text{Rep}_n(\mathcal{F})$ denote the subset of $\text{Rep}_n(S)$ of all the equivalence classes of the \mathcal{F} -invariant representations of S of rank n . It is easy to see that $\coprod_{n \geq 0} \text{Rep}_n(\mathcal{F})$ is closed under the operations $+$ and \times on $\coprod_{n \geq 0} \text{Rep}_n(S)$.

We define the set of representations at p of rank n of a space X as the set $\text{Rep}_{n,p}(X) = [X, (B\Sigma_n)_p^\wedge]$ of unpointed homotopy classes of unpointed maps. Since $(B\Sigma_m)_p^\wedge \times (B\Sigma_n)_p^\wedge \simeq (B(\Sigma_m \times \Sigma_n))_p^\wedge$ (see [2, Theorem I.7.2]), the following maps $(B(\Sigma_m \times \Sigma_n))_p^\wedge \rightarrow (B\Sigma_{m+n})_p^\wedge$ and $(B(\Sigma_m \times \Sigma_n))_p^\wedge \rightarrow (B\Sigma_{mn})_p^\wedge$ induced by the inclusions equip $\coprod_{n \geq 0} \text{Rep}_{n,p}(X)$ with commutative and associative binary operations $+$ and \times such that $+$ is distributive over \times .

If P is a finite p -group then there are bijections

$$\text{Rep}_n(P) \xrightarrow[\approx]{\rho \mapsto B\rho} [BP, B\Sigma_n] \xrightarrow[\approx]{f \mapsto \eta \circ f} [BP, (B\Sigma_n)_p^\wedge]$$

where $\eta : B\Sigma_n \rightarrow (B\Sigma_n)_p^\wedge$ is the completion map. The first bijection is a classical result going back to Hurewicz and the second was first shown by Mislin in [15, proof of the main theorem]. In light of these bijections we make the following definition.

1.2. Definition. Fix a p -local finite group $(S, \mathcal{F}, \mathcal{L})$. We say that a permutation representation $f : |\mathcal{L}| \rightarrow (B\Sigma_n)_p^\wedge$ is S -regular if $n = m \cdot |S|$ for some $m \geq 0$ and the composite $BS \xrightarrow{\Theta} |\mathcal{L}| \xrightarrow{f} (B\Sigma_n)_p^\wedge$ is homotopic to $BS \xrightarrow{B(m\text{-reg}_S)_p^\wedge} (B\Sigma_n)_p^\wedge$.

We will deduce from Theorem B the following p -local form of Cayley’s theorem.

Theorem C. Every p -local finite group $(S, \mathcal{F}, \mathcal{L})$ admits an S -regular permutation representation $f : |\mathcal{L}| \rightarrow (B\Sigma_{p^m})_p^\wedge$.

Recall from [5, Definition 2.2] that a continuous map $f : X \rightarrow Y$ is a homotopy monomorphism at p if $H^*(X; \mathbb{F}_p)$ is a finitely generated module over $H^*(Y; \mathbb{F}_p)$ via f^* . In Proposition 7.13 we show that every S -regular permutation representation is a homotopy monomorphism at p .

The reason we did not define permutation representations as maps $|\mathcal{L}| \rightarrow B\Sigma_n$ (without p -completing the target) is that Theorem C would fail completely. For example, the nerve of the linking system of the Solomon p -local finite group, constructed by Levi and Oliver in [11], was shown to be simply connected in [8] and therefore [23, Theorem 8.1.11] implies that $[|\mathcal{L}_{\text{Sol}}|, B\Sigma_n] = *$. In particular, the restriction of any $f : |\mathcal{L}_{\text{Sol}}| \rightarrow B\Sigma_n$ to BS via Θ is induced by the trivial representation $\rho : S \rightarrow \Sigma_n$.

1.3. Definition. The ring $\text{Rep}(|\mathcal{L}|)$ of the virtual permutation representations of a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ is the Grothendieck group completion of the commutative monoid $(\coprod_{n \geq 0} \text{Rep}_{n,p}(|\mathcal{L}|), +)$.

The ring $\text{Rep}(\mathcal{F})$ of the virtual \mathcal{F} -invariant representations of S of a saturated fusion system \mathcal{F} on S is the Grothendieck group completion of the commutative monoid $(\coprod_{n \geq 0} \text{Rep}_n(\mathcal{F}), +)$.

Clearly $\text{Rep}(\mathcal{F})$ is a subring of $\text{Rep}(S)$. In Section 8 we will construct a ring homomorphism $\Phi : \text{Rep}(\mathcal{L}) \rightarrow \text{Rep}(\mathcal{F})$ which sends a map $f : |\mathcal{L}| \rightarrow (B\Sigma_n)_p^\wedge$ to the representation $\rho : S \rightarrow \Sigma_n$ such that $f \circ \Theta \simeq \eta \circ B\rho$ where f and Θ are as in Definition 1.2. We shall also see that $\text{reg}_S : S \rightarrow \Sigma_{|S|}$ generates an ideal $\text{Rep}^{\text{reg}}(\mathcal{F})$ in $\text{Rep}(\mathcal{F})$ whose underlying group is isomorphic to \mathbb{Z} .

The idea behind the next definition is that if H is a subgroup of index n in a finite group G then $\text{reg}_G|_H \simeq n \cdot \text{reg}_H$. Therefore the image of the restriction map $\text{Rep}(G) \rightarrow \text{Rep}(H)$ intersects $\text{Rep}^{\text{reg}}(H) := \{k \cdot \text{reg}_H\}_{k \in \mathbb{Z}}$ in a subgroup of index divisible by n .

1.4. Definition. The lower p -local index of S in \mathcal{L} , denoted $\text{Lind}_p(\mathcal{L} : S)$, is the index of $\text{Im}(\Phi) \cap \text{Rep}^{\text{reg}}(\mathcal{F})$ in $\text{Rep}^{\text{reg}}(\mathcal{F})$.

We will prove in Lemma 8.5 that $\text{Lind}_p(\mathcal{L} : S)$ is always a p -power. We conjecture that it is always equal to 1. A partial result is the theorem below.

Theorem D. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Then $\text{Lind}_p(\mathcal{L} : S) = 1$ if either*

- (1) $(S, \mathcal{F}, \mathcal{L})$ is associated with a finite group.
- (2) $(S, \mathcal{F}, \mathcal{L})$ is one of the exotic examples in [6, Examples 9.3 and 9.4] or in [22] or in [9] or in [7, Example 5.3].

Notation. The following notation will be used through the paper:

- $\eta : X \rightarrow X_p^\wedge$ is the Bousfield–Kan p -completion.
- If X is a G -space, $\kappa : X \rightarrow (X)_{hG} = EG \times_G X$ is the map from X into the Borel construction.
- Given a map $f : X \rightarrow Y$ of spaces, let $\text{map}^f(X, Y)$ denote the path component of f in $\text{map}(X, Y)$. By convention f is the basepoint of this space.
- If $f : X \times Y \rightarrow Z$, the adjoint map is denoted by $f^\sharp : X \rightarrow \text{map}(Y, Z)$.
- $\Theta : BS \rightarrow |\mathcal{L}|$ is the map from the Sylow subgroup introduced in 2.9.

2. Preliminaries on p -local finite groups

We start with the notion of a saturated fusion system which is due to Puig [19] (see also [6]).

2.1. Definition. A fusion system \mathcal{F} on a finite p -group S is a category whose objects are the subgroups of S and the set of morphisms $\mathcal{F}(P, Q)$ between two subgroups P, Q , satisfies the following conditions:

- (a) $\mathcal{F}(P, Q)$ consists of group monomorphisms and contains the set $\text{Hom}_S(P, Q)$ of all the homomorphisms $c_s : P \rightarrow Q$ which are induced by conjugation by elements $s \in S$.
- (b) Every morphism in \mathcal{F} factors as an isomorphism in \mathcal{F} followed by an inclusion.

In a fusion system \mathcal{F} over a p -group S , we say that two subgroups $P, Q \leq S$ are \mathcal{F} -conjugate if there is an isomorphism between them in \mathcal{F} . Let $\text{Syl}_p(G)$ be the set of the Sylow p -subgroups

of a group G . Given $P \leq G$ and $g \in G$, $c_g \in \text{Hom}(P, G)$ is the monomorphism $c_g(x) = gxg^{-1}$. We write $\text{Out}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$.

2.2. Definition. Let \mathcal{F} be a fusion system on a p -group S . A subgroup $P \leq S$ is *fully centralized* in \mathcal{F} if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P . A subgroup $P \leq S$ is *fully normalized* in \mathcal{F} if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is \mathcal{F} -conjugate to P .

A fusion system \mathcal{F} on S is *saturated* if:

- (I) Each fully normalized subgroup $P \leq S$ is fully centralized and $\text{Aut}_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(P))$.
- (II) For $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$ set

$$N_{\varphi} = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \text{Aut}_S(\varphi(P))\}.$$

If $\varphi(P)$ is fully centralized then there is $\bar{\varphi} \in \mathcal{F}(N_{\varphi}, S)$ such that $\bar{\varphi}|_P = \varphi$.

2.3. Definition. Let \mathcal{F} be a fusion system on a p -group S . A subgroup $P \leq S$ is *\mathcal{F} -centric* if P and all its \mathcal{F} -conjugates contain their S -centralizers. A subgroup $P \leq S$ is *\mathcal{F} -radical* if $\text{Out}_{\mathcal{F}}(P)$ has no non-trivial normal p -subgroup.

2.4. Definition. (See [6].) Let \mathcal{F} be a fusion system on a p -group S . A *centric linking system associated to \mathcal{F}* is a category \mathcal{L} whose objects are the \mathcal{F} -centric subgroups of S , together with a functor $\pi : \mathcal{L} \rightarrow \mathcal{F}^c$ and monomorphisms $P \xrightarrow{\delta_P} \text{Aut}_{\mathcal{L}}(P)$ for each \mathcal{F} -centric subgroup $P \leq S$, which satisfy the following conditions:

- (A) π is the identity on objects. For each pair of objects $P, Q \in \mathcal{L}$, the action of $Z(P)$ on $\mathcal{L}(P, Q)$ via precomposition and $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}}(P)$ is free and π induces a bijection $\mathcal{L}(P, Q)/Z(P) \xrightarrow{\cong} \mathcal{F}(P, Q)$.
- (B) If $P \leq S$ is \mathcal{F} -centric then $\pi(\delta_P(g)) = c_g \in \text{Aut}_{\mathcal{F}}(P)$ for all $g \in P$.
- (C) For each $f \in \mathcal{L}(P, Q)$ and each $g \in P$, the following square commutes in \mathcal{L} :

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \delta_P(g) \downarrow & & \downarrow \delta_Q(\pi(f)(g)) \\ P & \xrightarrow{f} & Q \end{array}$$

A *p -local finite group $(S, \mathcal{F}, \mathcal{L})$* consists of a saturated fusion systems \mathcal{F} on S together with an associated linking system.

2.5. Definition. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. A *system of lifts* in \mathcal{L} consists of functions $\delta_{P,Q} : N_S(P, Q) \rightarrow \mathcal{L}(P, Q)$, one for each pair $P, Q \leq S$ of \mathcal{F} -centric subgroups, such that:

- (a) $\pi(\delta_{P,Q}(g)) = c_g \in \mathcal{F}(P, Q)$ for all $g \in N_S(P, Q)$.
- (b) $\delta_{P,P}(g) = \delta_P(g)$ for all $g \in P$, namely $\delta_{P,P}$ extends the structure map δ_P .
- (c) If $g \in N_S(P, Q)$ and $h \in N_S(Q, R)$ then $\delta_{P,R}(hg) = \delta_{Q,R}(h) \circ \delta_{P,Q}(g)$.

For any $P \leq Q$ set $\iota_P^Q = \delta_{P,Q}(e)$ where e is the identity of S .

2.6. Remark. Any p -local finite group admits a system of lifts by [6, Proposition 1.11].

We will write \hat{g} for $\delta_{P,Q}(g)$. In this notation conditions (a) and (c) become $\pi(\hat{g}) = c_g$ and $\widehat{hg} = \hat{h} \circ \hat{g}$. Also $\iota_Q^R \circ \iota_P^Q = \iota_P^R$.

2.7. Remark. Every morphism in \mathcal{L} is both a monomorphism and an epimorphism (but not necessarily an isomorphism). This is shown in [6, remarks after Lemma 1.10] and [3, Corollary 3.10]. We shall use this fact repeatedly throughout.

The orbit category of a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ is denoted by $\mathcal{O}(\mathcal{F})$. This is the category whose objects are the subgroups of S and whose morphisms are

$$\mathcal{O}(\mathcal{F})(P, Q) = \text{Rep}_{\mathcal{F}}(P, Q) \stackrel{\text{def}}{=} \text{Inn}(Q) \backslash \mathcal{F}(P, Q).$$

Also, $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the \mathcal{F} -centric subgroups of S .

2.8. Proposition. (See [6, Proposition 2.2].) *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. There exists a functor $\tilde{B} : \mathcal{O}(\mathcal{F}^c) \rightarrow \mathbf{Top}$ which is isomorphic in the homotopy category of spaces to the functor $P \mapsto BP$, and such that there is a homotopy equivalence*

$$\text{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B} \xrightarrow{\simeq} |\mathcal{L}|.$$

2.9. Notation. For a finite group G , let $\mathcal{B}G$ denote the category with one object \bullet_G and G as its set of automorphisms. For an \mathcal{F} -centric $P \leq S$ the monomorphism δ_P gives rise to a functor $BP \rightarrow \mathcal{L}$ which, by abuse of notation, we denote by δ_P . For $P = S$, upon taking nerves of categories, we obtain a map

$$\Theta : BS \rightarrow |\mathcal{L}|$$

and we write $\Theta|_{BQ}$ for $\Theta \circ B \text{incl}_Q^S$.

If Q is \mathcal{F} -centric, then the natural isomorphism of functors in Proposition 2.8 shows that $\Theta|_{BQ}$ is homotopic to $BQ \simeq \tilde{B}(Q) \rightarrow \text{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B} = |\mathcal{L}|$. Therefore, for any \mathcal{F} -centric $Q \leq S$ and any morphism $\rho : Q \rightarrow S$ in \mathcal{F} we have $\Theta \circ B\rho \simeq \Theta|_{BQ}$. In particular, $\Theta|_{BQ'} \circ B\psi \simeq \Theta|_{BQ}$ for any $\psi \in \text{Iso}_{\mathcal{F}}(Q, Q')$. It follows from Alperin’s fusion theorem for saturated fusion systems [6, Theorem A.10] that:

2.10. Proposition. *For any $Q, Q' \leq S$ and any $\rho \in \mathcal{F}(Q, Q')$ the maps $\Theta|_{BQ}$ and $\Theta|_{BQ'} \circ B\rho$ are homotopic.*

The following proposition on mapping spaces will be needed in Section 7. Here and elsewhere in this paper we use the letter η for the p -completion map $X \rightarrow X_{\hat{p}}$.

2.11. Proposition. *Fix a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ and let P be a finite p -group. Given a homomorphism $\rho : P \rightarrow S$, set $Q = \rho(P) \leq S$. Then:*

(a) *There is a homotopy equivalence*

$$\text{map}^{\eta \circ \Theta \circ B\rho}(BP, |\mathcal{L}|_p^\wedge) \simeq \text{map}^{\eta \circ \Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge),$$

and this space is the p -completed classifying space of a p -local finite group.

(b) *After p -completion, the map*

$$\text{map}^{\Theta|_{BQ}}(BQ, |\mathcal{L}|) \xrightarrow{\eta_*} \text{map}^{\eta \circ \Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge)$$

induces a split surjection on homotopy groups.

Proof. (a) First of all, we can choose a fully centralized subgroup $Q' \leq S$ in \mathcal{F} and an isomorphism $\psi : Q \rightarrow Q'$ in \mathcal{F} . Let $\rho' : P \rightarrow S$ denote the composite $P \xrightarrow{\rho} Q \xrightarrow{\psi} Q' \leq S$. By Proposition 2.10 observe that

$$(1) \quad \Theta|_{BQ} \simeq \Theta|_{BQ'} \circ B\psi.$$

Hence, $\Theta \circ B\rho \simeq \Theta \circ B\rho'$. It follows from [6, Theorem 6.3] that there are homotopy equivalences

$$\begin{aligned} \text{map}^{\eta \circ \Theta \circ B\rho}(BP, |\mathcal{L}|_p^\wedge) &\simeq \text{map}^{\eta \circ \Theta \circ B\rho'}(BP, |\mathcal{L}|_p^\wedge) \simeq \text{map}^{\eta \circ \Theta|_{BQ'}}(BQ', |\mathcal{L}|_p^\wedge) \\ &\simeq \text{map}^{\eta \circ \Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge) \end{aligned}$$

where the first equivalence is implied by Eq. (1) and the third one follows since $B\psi : BQ \rightarrow BQ'$ is a homotopy equivalence. Also by [6, Theorem 6.3], this space is homotopy equivalent to the classifying space of a p -local finite group $|C_{\mathcal{L}}(Q')|_p^\wedge$.

(b) We can assume from (1), by replacing Q with Q' if necessary, that Q is fully centralized in \mathcal{F} . In [6, p. 822] a functor

$$\Gamma : C_{\mathcal{L}}(Q) \times BQ \rightarrow \mathcal{L}$$

is constructed where $C_{\mathcal{L}}(Q)$ is the centralizer linking system [6, Definition 2.4] of Q in \mathcal{F} . By p -completing the geometric realization of Γ and taking adjoints we obtain a commutative square in which the bottom row is a homotopy equivalence by [6, Theorem 6.3]

$$(2) \quad \begin{array}{ccc} |C_{\mathcal{L}}(Q)| & \xrightarrow{|\Gamma|^\#} & \text{map}^{\Theta|_{BQ}}(BQ, |\mathcal{L}|) \\ \eta \downarrow & & \downarrow \eta_* \\ |C_{\mathcal{L}}(Q)|_p^\wedge & \xrightarrow[\simeq]{(|\Gamma|_p^\wedge)^\#} & \text{map}^{\eta \circ \Theta|_{BQ}}(BQ, |\mathcal{L}|_p^\wedge) \end{array}$$

Since $|C_{\mathcal{L}}(Q)|$ is p -good by [6, Proposition 1.12], upon p -completion of the diagram (2), we see that the vertical arrow on the left becomes an equivalence and therefore the composite $(\eta_*)_p^\wedge \circ (|\Gamma|^\#)_p^\wedge$ is a homotopy equivalence. In particular $(\eta_*)_p^\wedge$ is split surjective on homotopy groups. \square

We end this section with a description of the product of p -local finite groups.

2.12. Let \mathcal{F}_i be a saturated fusion system on a finite p -group S_i for $i = 1, \dots, n$. Define $S = \prod_{i=1}^n S_i$ and consider the product category $\prod_{i=1}^n \mathcal{F}_i$. Its objects are the subgroups of S of the form $\prod_i P_i$ where $P_i \leq S_i$, and morphisms have the form $\prod_i P_i \xrightarrow{\prod_i \varphi_i} \prod_i Q_i$ where $\varphi_i \in \mathcal{F}_i(P_i, Q_i)$.

2.13. Notation. For $P \leq S = \prod_{i=1}^n S_i$, we denote by $P^{(i)}$ the image of P under the projection $p^{(i)} : S \rightarrow S_i$. Clearly $P \leq \prod_{i=1}^n P^{(i)}$.

Let \mathcal{F} be the fusion system on S generated by $\prod_i \mathcal{F}_i$. Thus, every morphism $\varphi \in \mathcal{F}(P, Q)$ is given by the restriction of a morphism $\prod_i P^{(i)} \xrightarrow{\prod_i \varphi_i} \prod_i Q^{(i)}$ in $\prod_i \mathcal{F}_i$. The φ_i 's are unique in the sense that they are completely determined by φ because $p^{(i)}|_P : P \rightarrow P^{(i)}$ are by definition surjective and $p^{(i)}|_Q \circ \varphi = \varphi_i \circ p^{(i)}|_P$. We see that $\varphi \mapsto (\varphi_i)_{i=1}^n$ induces an inclusion $\mathcal{F}(P, Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})$. In particular, $\prod_i \mathcal{F}_i$ is a full subcategory of \mathcal{F} .

We shall write $\times_{i=1}^n \mathcal{F}_i$ for the fusion system \mathcal{F} just defined and we call it the product fusion system of the \mathcal{F}_i 's.

2.14. Lemma. *With the notation above, (S, \mathcal{F}) is a saturated fusion system. If $P \leq S$ is \mathcal{F} -centric then all the groups $P^{(i)}$ are \mathcal{F}_i -centric for $i = 1, \dots, n$.*

The assignment $P \mapsto \prod_i P^{(i)}$ and the inclusions $\mathcal{F}(P, Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})$ give rise to a functor $r : \mathcal{F}^c \rightarrow \prod_i \mathcal{F}_i^c$ which is a retraction of the inclusion $\prod_i \mathcal{F}_i^c \subseteq \mathcal{F}^c$.

Proof. It is shown in [6, Lemma 1.5] that $\mathcal{F} = \times_i \mathcal{F}_i$ is a saturated fusion system on S .

The assignments $P \mapsto \prod_i P^{(i)}$ and $\varphi \mapsto \prod_i \varphi_i$ give rise to a functor $r : \mathcal{F} \rightarrow \prod_i \mathcal{F}_i$ which by inspection is a retraction to the inclusion $j : \prod_i \mathcal{F}_i \rightarrow \mathcal{F}$. It remains to show that j and r restrict to $\prod_i \mathcal{F}_i^c$ and \mathcal{F}^c .

Observe that $C_S(P) = \prod_i C_{S_i}(P^{(i)})$ for any $P \leq S$. If P is \mathcal{F} -centric then

$$(1) \quad \prod_{i=1}^n C_{S_i}(P^{(i)}) = C_S(P) \leq P \leq \prod_{i=1}^n P^{(i)}.$$

Therefore $C_{S_i}(P^{(i)}) \leq P^{(i)}$ for all i . Now, if Q_i are \mathcal{F}_i -conjugate to $P^{(i)}$ via isomorphisms $\varphi_i \in \mathcal{F}_i(P^{(i)}, Q_i)$ then $(\varphi_1 \times \dots \times \varphi_n)|_P$ is an \mathcal{F} -isomorphism onto some $Q \leq S$ such that $Q^{(i)} = Q_i$. By definition Q is also \mathcal{F} -centric and applying (1) to Q we obtain that $C_{S_i}(Q_i) \leq Q_i$ for all i . We deduce that $P^{(i)}$ are \mathcal{F}_i -centric.

Assume now that $P_i \leq S_i$ are \mathcal{F}_i -centric for all $i = 1, \dots, n$. Then $P = \prod_i P_i$ is \mathcal{F} -centric because if Q is \mathcal{F} -conjugate to P then it has the form $\prod_i Q_i$ where Q_i are \mathcal{F}_i -conjugate to P_i and therefore $C_S(Q) = \prod_i C_{S_i}(Q_i) \leq Q$. \square

The construction of the product of saturated fusion systems appears in [6], but we were unable to find a reference for the product of p -local finite groups.

2.15. Definition. Let $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be p -local finite groups for $i = 1, \dots, n$. Their product $\times_{i=1}^n (S_i, \mathcal{F}_i, \mathcal{L}_i)$ is the p -local finite group $(S, \mathcal{F}, \mathcal{L})$ where $S = \prod_{i=1}^n S_i$ and $\mathcal{F} = \times_{i=1}^n \mathcal{F}_i$.

The centric linking system $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$ is defined as the following pullback of small categories where r is defined in Lemma 2.14

$$\begin{array}{ccc} \times_{i=1}^n \mathcal{L}_i & \xrightarrow{r_{\mathcal{L}}} & \prod_{i=1}^n \mathcal{L}_i \\ \pi \downarrow & & \downarrow \prod_{i=1}^n \pi_i \\ (\times_{i=1}^n \mathcal{F}_i)^c & \xrightarrow{r} & \prod_{i=1}^n \mathcal{F}_i^c \end{array}$$

The functor $\pi : \mathcal{L} \rightarrow \mathcal{F}$ is defined by the pullback and the monomorphisms $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}}(P)$ are defined by the composites

$$P \leq \prod_i P^{(i)} \xrightarrow{\prod_i \delta_{P^{(i)}}} \prod_i \text{Aut}_{\mathcal{L}_i}(P^{(i)}).$$

We need to prove that axioms (A)–(C) of Definition 2.4 hold.

Proof. For any \mathcal{F} -centric subgroups $P, Q \leq S$ the set $\mathcal{L}(P, Q)$ is the pullback

$$(1) \quad \begin{array}{ccc} \mathcal{L}(P, Q) & \hookrightarrow & \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)}) \\ \pi \downarrow & & \downarrow \prod_i \pi \\ \times_{i=1}^n \mathcal{F}_i(P, Q) & \hookrightarrow^r & \prod_{i=1}^n \mathcal{F}_i(P^{(i)}, Q^{(i)}) \end{array}$$

We start by proving that the monomorphisms δ_P are well defined. That is, given $g = (g_i) \in P \leq S$ where P is \mathcal{F} -centric, $\prod_i \delta_{P^{(i)}}(g_i) \in \text{Aut}_{\mathcal{L}}(P)$. The pullback diagram (1) shows that it is enough to check that $\prod \pi_i(\delta_{P^{(i)}}(g_i)) \in r((\times_{i=1}^n \mathcal{F}_i)^c)$. It follows from the fact that $\pi_i(\delta_{P^{(i)}}(g_i)) = c_{g_i} \in \text{Aut}_{\mathcal{F}_i}(P^{(i)})$ and $r(c_g) = \prod c_{g_i}$. This also shows that axiom (B) holds since $\pi(\delta_P(g)) = \prod \pi_i(\delta_{P^{(i)}}(g_i))|_P = c_g|_P$.

We continue to prove that $(S, \mathcal{F}, \mathcal{L})$ satisfies axioms (A) and (C). It follows from the definition that π is the identity on objects. Observe that $\prod_i C_{S_i}(P^{(i)})$ acts transitively and freely on the fibre of the right-hand arrow in (1) because axiom (A) holds in $(S_i, \mathcal{F}_i, \mathcal{L}_i)$. Now, axiom (A) for $(S, \mathcal{F}, \mathcal{L})$ follows from the fact that $C_S(P) = \prod_i C_{S_i}(P^{(i)})$ and that diagram (1) is a pullback square so the fibres of the vertical arrows are isomorphic.

Finally, axiom (C) for $(S, \mathcal{F}, \mathcal{L})$ follows by applying axiom (C) to each component of a morphism $f \in \mathcal{L}(P, Q)$ and each $g \in P \leq \prod_i P^{(i)}$. \square

2.16. Remark. Using the notation of Definition 2.15, if $\{\delta_{P,Q}^i\}$ are systems of lifts in \mathcal{L}_i , there results a system of lifts in $\prod_i \mathcal{L}_i$ as follows. If $P, Q \leq S$ are \mathcal{F} -centric, then $\delta_{P,Q}$ is $\prod_i \delta_{P^{(i)}, Q^{(i)}} : \prod_i N_{S_i}(P^{(i)}, Q^{(i)}) \rightarrow \prod_i \mathcal{L}_i(P^{(i)}, Q^{(i)})$.

2.17. Proposition. Given p -local finite groups $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ for $i = 1, \dots, n$, the category $\prod_i \mathcal{L}_i$ is a full subcategory of $\times_i \mathcal{L}_i$ and the inclusion $j : \prod_i \mathcal{L}_i \rightarrow \times_i \mathcal{L}_i$ induces a homotopy equivalence on nerves. In particular, $\prod_{i=1}^n |\mathcal{L}_i| \simeq |\times_{i=1}^n \mathcal{L}_i|$.

Proof. Set $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$. The category $\prod_i \mathcal{L}_i$ is a full subcategory of \mathcal{L} by Definition 2.15 and the fact that $\prod_i \mathcal{F}_i$ is a full subcategory of $\times_i \mathcal{F}_i$. The assignment $P \mapsto \prod_i P^{(i)}$ and the inclusion $\mathcal{L}(P, Q) \subseteq \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)})$ give rise to a functor $r_{\mathcal{L}} : \mathcal{L} \rightarrow \prod_{i=1}^n \mathcal{L}_i$ (see the pullback diagram in Definition 2.15) which is a retract to the inclusion j by Lemma 2.14. Also there is a natural transformation $\text{Id} \rightarrow j \circ r$ which is defined on an object $P \in \mathcal{L}$ by $\iota_P^{r(P)} : P \rightarrow r(P) = \prod_{i=1}^n P^{(i)}$ (see Remark 2.16 and Definition 2.5). This shows that $|r|$ is a homotopy inverse to $|j| : \prod_i |\mathcal{L}_i| \rightarrow |\mathcal{L}|$. \square

2.18. Remark. Given a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, Definition 2.15 allows us to consider its n -fold product with itself denoted $(S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})$. By construction, the action of the symmetric group Σ_n on $S^{\times n}$ extends to an action on the fusion system $\mathcal{F}^{\times n}$ and the linking system $\mathcal{L}^{\times n}$ by permuting the factors. Moreover, the functor $\pi : \mathcal{L}^{\times n} \rightarrow \mathcal{F}^{\times n}$ and the distinguished monomorphisms $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}^{\times n}}(P)$ for every $\mathcal{F}^{\times n}$ -centric $P \leq S^{\times n}$ are Σ_n -equivariant from the construction in Definition 2.15. Therefore, also the inclusion $\mathcal{B}S^{\times n} \xrightarrow{\delta_{S^{\times n}}} \mathcal{B}\text{Aut}_{\mathcal{L}^{\times n}}(S^{\times n}) \rightarrow \mathcal{L}^{\times n}$ is Σ_n -equivariant and so is the induced map $\Theta : \mathcal{B}S^{\times n} \rightarrow |\mathcal{L}^{\times n}| \simeq |\mathcal{L}|^{\times n}$.

The choice of $\delta_{P,Q}$ in $\mathcal{L}^{\times n}$ made in Remark 2.16 is easily seen to be equivariant with respect to the action of Σ_n as well.

Finally, the functor j and the homotopy equivalence in Proposition 2.17 are also equivariant with respect to the action of Σ_n by permuting coordinates.

3. The wreath product of spaces

Let G be a finite group and X a G -space. The Borel construction X_{hG} is the orbit space of $EG \times X$ where EG is a contractible space on which G acts freely on the right. Recall from 2.9 that $\mathcal{B}G$ is the small category with one object and G as a morphism set. Then X can be viewed as a functor $X : \mathcal{B}G \rightarrow \mathbf{Top}$ and the Borel construction is a model for $\text{hocolim}_{\mathcal{B}G} X$. There is a natural map $X_{hG} \rightarrow X/G$ to the orbit space of X induced by the map $EG \rightarrow *$.

A standard model for EG is the geometric realization of the simplicial set $\mathcal{E}G$ whose set of n -simplices is the $(n + 1)$ -fold product $G \times \dots \times G$ with face and degeneracy maps defined using deletion and duplication and where G acts diagonally via right translations. The identity element of G equips EG with a natural choice of a basepoint (which is not invariant under G). This basepoint provides an augmentation map $\kappa(X) : X \rightarrow X_{hG}$ which is an inclusion map and it fits into the fibration sequence

$$X \xrightarrow{\kappa(X)} X_{hG} \rightarrow BG. \tag{3.1}$$

We will tend to simply write κ instead of $\kappa(X)$ whenever X is understood from the context. A fixed point $x \in X$ corresponds to a G -map $* \rightarrow X$ and gives rise to a section $s : BG \rightarrow X_{hG}$ for this fibration.

Now assume that G is a semidirect product $H \rtimes N$. Consider spaces, namely, simplicial sets X, Y and Z such that X has a left action of G and Z has a right action of H . Assume further that Y has a left action of H and a right action of N such that $h \cdot (y \cdot n) = (h \cdot y) \cdot (hnh^{-1})$ for all $y \in Y, h \in H$ and $n \in N$. Note that the actions of N and H on Y do not commute. Then $Z \times Y$ admits a right G -action defined by $(z, y) \cdot (h, n) = (z \cdot h, (h \cdot y) \cdot n)$ where $g = hn \in H \rtimes N$ and $(z, y) \in Z \times Y$. Moreover, by inspection, there is a homeomorphism

$$(Z \times Y) \times_G X \cong Z \times_H (Y \times_N X).$$

Taking $Z = EH$ and $Y = EN$ where the left H -action on EN is via conjugation, we obtain a homeomorphism

$$(EH \times EN) \times_G X \xrightarrow{\cong} EH \times_H (EN \times_N X). \tag{3.2}$$

Moreover there is an obvious isomorphism of simplicial sets

$$\mathcal{E}H \times \mathcal{E}N \xrightarrow{\cong} \mathcal{E}G$$

which in turn induces a homeomorphism $EH \times EN \approx EG$ of G -spaces. It now becomes clear that

$$X \xrightarrow{\kappa} X_{hN} \xrightarrow{\kappa} (X_{hN})_{hH} \xrightarrow{\cong} X_{hG} \text{ is equal to } X \xrightarrow{\kappa} X_{hG}. \tag{3.3}$$

3.4. Definition. The wreath product of a space X with a subgroup G of Σ_k is the space

$$X \wr G := (X^{\times k})_{hG}$$

where G acts by permuting the factors of $X^{\times k}$. The diagonal map $\Delta_X : X \rightarrow X^{\times k}$ together with $\kappa : X^{\times k} \rightarrow X \wr G$ give rise to a natural map

$$\Delta(X) : X \rightarrow X \wr G.$$

We shall use a left normed notation for iteration of the wreath product construction. That is, by convention, $X \wr G_1 \wr G_2 \wr \dots \wr G_n$ denotes $(\dots((X \wr G_1) \wr G_2) \wr \dots) \wr G_n$. Applying (3.2) and (3.3) iteratively it is left as an easy exercise to prove

3.5. Proposition. *Given permutation groups $G_i \leq \Sigma_{k_i}$ where $i = 1, \dots, n$, there is a homeomorphism*

$$\alpha_n : X \wr G_1 \wr G_2 \wr \dots \wr G_n \xrightarrow{\cong} X \wr (G_1 \wr G_2 \wr \dots \wr G_n)$$

which is natural in X . Moreover, the composite

$$X \xrightarrow{\Delta} X \wr G_1 \xrightarrow{\Delta} (X \wr G_1) \wr G_2 \xrightarrow{\Delta} \dots \xrightarrow{\Delta} X \wr G_1 \wr G_2 \wr \dots \wr G_n \xrightarrow{\alpha_n} X \wr (G_1 \wr G_2 \wr \dots \wr G_n)$$

is equal to $\Delta : X \rightarrow X \wr (G_1 \wr G_2 \wr \dots \wr G_n)$ via the above homeomorphism.

3.6. Remark. Clearly Σ_k fixes all the points in the image of the diagonal map $X \rightarrow X^k$. If $X \neq \emptyset$, then the fibre sequence (3.1) $X^k \rightarrow X \wr G \rightarrow BG$ splits for any $G \leq \Sigma_k$ and the long exact sequence in homotopy groups gives rise to isomorphisms

$$\begin{aligned} \pi_1(X \wr G) &\cong (\pi_1 X) \wr G \quad \text{and} \\ \pi_i(X \wr G) &\cong (\pi_i X)^k \quad \text{for all } i \geq 2. \end{aligned}$$

Moreover, $\kappa : X^k \rightarrow X \wr G$ induces inclusions $\prod_k \pi_* X \leq \pi_*(X \wr G)$ on which $G \leq \pi_1(X \wr G)$ acts on higher homotopy groups by permuting the factors.

In particular, if $X = BH$ for a discrete group H , there is a homotopy equivalence $(BH) \wr G \simeq B(H \wr G)$ and $\Delta: BH \rightarrow (BH) \wr G \simeq B(H \wr G)$ is homotopic to the map induced by the diagonal inclusion $H \leq H \wr G$.

Let Y be a G -space. For any space X , $\text{map}(X, Y)$ becomes a G -space, and the evaluation map $X \times \text{map}(X, Y) \xrightarrow{\text{ev}} Y$ is clearly G -equivariant. Therefore, applying the Borel construction, it gives rise to a map $\text{ev}_{hG}: X \times \text{map}(X, Y)_{hG} \rightarrow Y_{hG}$ whose adjoint is denoted

$$(\text{ev}_{hG})^\# : \text{map}(X, Y)_{hG} \rightarrow \text{map}(X, Y_{hG}).$$

If the component $\text{map}^f(X, Y)$ of some $f: X \rightarrow Y$ is invariant under the G -action then inspection of the adjunction shows that $(\text{ev}_{hG})^\#$ restricts to

$$(\text{ev}_{hG})^\# : \text{map}^f(X, Y)_{hG} \rightarrow \text{map}^{\kappa(Y) \circ f}(X, Y_{hG}).$$

Moreover, the composite

$$\text{map}^f(X, Y) \xrightarrow{\kappa} \text{map}^f(X, Y)_{hG} \xrightarrow{(\text{ev}_{hG})^\#} \text{map}^{\kappa \circ f}(X, Y_{hG}) \tag{3.7}$$

coincides with the natural map induced by $Y \xrightarrow{\kappa(Y)} Y_{hG}$.

3.8. Proposition. Fix a map $f: A \rightarrow X$ and $G \leq \Sigma_k$. Denote the adjoint of

$$A \times (\text{map}^f(A, X) \wr G) = A \times \text{map}^{\Delta_X \circ f}(A, X^k)_{hG} \xrightarrow{\text{ev}_{hG}} (X^k)_{hG} = X \wr G$$

by $\gamma: \text{map}^f(A, X) \wr G \rightarrow \text{map}^{\Delta(X) \circ f}(A, X \wr G)$. Then:

(a) The triangle below is commutative:

$$\begin{array}{ccc} \text{map}^f(A, X) & & \\ \Delta \downarrow & \searrow \text{map}(A, \Delta(X)) & \\ \text{map}^f(A, X) \wr G & \xrightarrow{\gamma} & \text{map}^{\Delta(X) \circ f}(A, X \wr G) \end{array}$$

(b) If A is a non-empty path connected CW-complex then γ is a homotopy equivalence.

Proof. (a) Note that $\prod_k \text{map}^f(A, X) = \text{map}^{\Delta_X \circ f}(A, X^k)$ and that this component is invariant under the action of $G \leq \Sigma_k$. The commutativity of the triangle follows from (3.7) and Definition 3.4.

(b) First, we check that the evaluation $\text{ev}: \text{map}^c(A, BG) \rightarrow BG$ at some $a \in A$ is a homotopy equivalence where the domain is the path component of the null-homotopic maps. Since this map between connected spaces has a section $\text{const}: BG \rightarrow \text{map}^c(A, BG)$, its homotopy fibre $\text{map}_*^c(A, BG)$ is connected. But it is in fact contractible because $\Omega \text{map}_*(A, BG) \simeq \text{map}_*(A, G) \simeq *$. Then the section is also a homotopy equivalence.

Now consider the following ladder in which the rows are fibre sequences and π_* is induced by $X \rightarrow *$:

$$(1) \quad \begin{array}{ccccc} \text{map}^f(A, X)^k & \longrightarrow & \text{map}^f(A, X) \wr G & \longrightarrow & BG \\ \text{incl} \downarrow & & \gamma \downarrow & & \simeq \downarrow \text{const} \\ F & \longrightarrow & \text{map}^{\Delta(X) \circ f}(A, X \wr G) & \xrightarrow{\pi_*} & \text{map}^c(A, BG) \end{array}$$

It commutes because the right-hand square commutes as a consequence of the commutativity of the following square and adjunction

$$\begin{array}{ccc} A \times \text{map}^{\Delta X \circ f}(A, X^k)_{hG} & \longrightarrow & A \times \text{map}(A, *)_{hG} \\ \text{ev}_{hG} \downarrow & & \downarrow \text{proj} = \text{ev}_{hG} \\ (X^{\times k})_{hG} & \xrightarrow{\pi} & *_hG = BG \end{array}$$

Now, F is a union of path components of $\text{map}(A, X^k)$ because it is the fibre of the fibration $\text{map}(A, X \wr G) \rightarrow \text{map}(A, BG)$ over the component of the constant map. Moreover, F clearly contains the component $\text{map}^{\Delta X \circ f}(A, X^k)$ and inspection of γ shows that the map between the fibres is simply the inclusion. Comparison of the long exact sequences in homotopy of the fibre sequences in (1) shows that F is connected, whence $F = \text{map}^f(A, X)^{\times k}$. Application of the five lemma to the exact sequences in homotopy now yields the result. \square

4. Killing homotopy groups

The aim of this section is to study the effect on homotopy groups of the composite map $X \xrightarrow{\Delta(X)} X \wr \Sigma_k \xrightarrow{\eta} (X \wr \Sigma_k)_p^\wedge$ where $\Delta(X)$ was defined in the last section and η is the p -completion map.

4.1. Proposition. *Let X be a pointed space. Then the kernel of $\pi_* X \rightarrow \pi_*(X_p^\wedge)$ contains all the elements whose order is prime to p .*

Proof. Let $[\Theta] \in \pi_*(X)$ be an element of order k prime to p . Then the map $\Theta : S^n \rightarrow X$ factors through the Moore space $M(\mathbb{Z}/k, n)$, which is a nilpotent space with the mod p homology of a point. It follows that $\eta \circ \Theta : S^n \rightarrow X_p^\wedge$ factors through $M(\mathbb{Z}/k, n)_p^\wedge \simeq *$ (see [2, Chapter VI.5]), and therefore is null-homotopic. \square

An element of exponent n in a group G is an element whose order divides n . For the proof of the next result, recall that for any space, $\pi_1(X)$ acts on the groups $\pi_* X$, see, e.g. [23, Corollary 7.3.4] or [25, Chapter III]. We write α^ω for the image of the action of $\omega \in \pi_1 X$ on $\alpha \in \pi_n X$.

4.2. Lemma. *Fix an integer $n \geq 3$ and a pointed space X . Then the kernel of*

$$\pi_* X \xrightarrow{\Delta(X)_*} \pi_*(X \wr \Sigma_n) \xrightarrow{\eta_*} \pi_*((X \wr \Sigma_n)_p^\wedge)$$

contains all the elements of exponent n in $\pi_ X$.*

Proof. We recall from Remark 3.6 that

$$\begin{aligned} \pi_1(X \wr \Sigma_n) &= (\pi_1 X) \wr \Sigma_n, \\ \pi_i(X \wr \Sigma_n) &= \bigoplus_n \pi_i X \quad \text{for } i \geq 2. \end{aligned}$$

Furthermore, $\kappa: \prod_n X \rightarrow X \wr \Sigma_n$ induces the inclusion $\prod_n \pi_* X \leq \pi_*(X \wr \Sigma_n)$. The section $s: B\Sigma_n \rightarrow X \wr \Sigma_n$ defined by the fixed point $(*, \dots, *) \in X^n$ induces the group inclusion $\Sigma_n \leq \pi_1(X \wr \Sigma_n)$ which acts by permuting the factors of $\pi_*(X^n) \leq \pi_*(X \wr \Sigma_n)$.

We can choose elements $\omega_k \in \Sigma_n$ whose order is prime to p and $\omega_k(1) = k$ for all $k = 1, \dots, n$. Indeed, if $p > 2$ we can choose the involutions $\omega_k = (1, k)$. If $p = 2$ we can choose ω_k to be 3-cycles (note that $n \geq 3$). In both cases we choose $\omega_1 = \text{id}$.

For every $k = 1, \dots, n$ let $j_k: X \rightarrow \prod_n X$ denote the inclusion into the k th factor with respect to the basepoint of X . Note that $\Delta_X: X \rightarrow X^n$ induces $(\Delta_X)_*(\theta) = (\theta, \dots, \theta) \in \prod_n \pi_* X$. By inspection of the action of $\omega_k \in \pi_1(X \wr \Sigma_n)$, it follows that for any $\theta \in \pi_i X$, $(\kappa \circ j_k)_*(\theta) = ((\kappa \circ j_1)_*(\theta))^{\omega_k} \in \pi_i(X \wr \Sigma_n)$. Now fix some $\theta \in \pi_i X$ of exponent n . Since $\Delta(X)$ is defined as the composite $X \xrightarrow{\Delta_X} \prod_n X \xrightarrow{\kappa} X \wr \Sigma_n$, we have

$$\Delta(X)_*(\theta) = \prod_{k=1}^n (\kappa \circ j_k)_*(\theta) = \prod_{k=1}^n ((\kappa \circ j_1)_*(\theta))^{\omega_k}.$$

Now consider the p -completion map $X \wr \Sigma_n \xrightarrow{\eta} (X \wr \Sigma_n)_p^\wedge$ and note that it maps ω_k to the trivial element by Proposition 4.1. By applying η_* and using the naturality of the action of the fundamental group we see that

$$\begin{aligned} (\eta \circ \Delta(X))_*(\theta) &= \prod_{k=1}^n \eta_*((\kappa \circ j_1)_*(\theta))^{\omega_k} = \prod_{k=1}^n \eta_*((\kappa \circ j_1)_*(\theta))^{\eta_*(\omega_k)} \\ &= (\eta_*((\kappa \circ j_1)_*(\theta)))^n = \eta_*((\kappa \circ j_1)_*(\theta^n)) = 0. \quad \square \end{aligned}$$

4.3. Lemma. Fix some $k \geq 3$ and consider a map $f: X \rightarrow Y$. Assume that every element of $\pi_i \text{map}^f(X, Y)$ has exponent k and that $\text{map}^{\eta \circ \Delta(Y) \circ f}(X, (Y \wr \Sigma_k)_p^\wedge)$ is p -complete. Then the homomorphism

$$\pi_i \text{map}^f(X, Y) \xrightarrow{\text{map}(X, \eta \circ \Delta(Y))_*} \pi_i \text{map}^{\eta \circ \Delta(Y) \circ f}(X, (Y \wr \Sigma_k)_p^\wedge)$$

is trivial.

Proof. According to Proposition 3.8(a) the triangle in the diagram below commutes up to homotopy:

$$\begin{array}{ccccc} \text{map}^f(X, Y) & \xrightarrow{\Delta(Y)_*} & \text{map}^{\Delta \circ f}(X, Y \wr \Sigma_k) & \xrightarrow{\eta_*} & \text{map}^{\eta \circ \Delta \circ f}(X, (Y \wr \Sigma_k)_p^\wedge) \\ & \searrow \Delta & \uparrow \gamma & & \uparrow \text{dotted} \\ & & \text{map}^f(X, Y) \wr \Sigma_k & \xrightarrow{\eta} & (\text{map}^f(X, Y) \wr \Sigma_k)_p^\wedge \end{array}$$

Since $\text{map}^{\eta \circ \Delta(Y) \circ f}(X, (Y \wr \Sigma_k)_p^\wedge)$ is p -complete, the map $(\eta_* \circ \gamma)_p^\wedge$ gives rise to a choice of a map for the dotted arrow so that the square is homotopy commutative. We can now apply Lemma 4.2 to the diagonal arrow Δ and the bottom arrow η . \square

5. The wreath product of p -local finite groups

Given a finite group G , the space $(BG) \wr \Sigma_k$ is the classifying space of the group $G \wr \Sigma_k$ (see 3.6). In this section we prove an analogous result for p -local finite groups.

Recall that a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ admits an S -system of lifts $\{\delta_{P,Q}\}$, see Definition 2.5 and the remarks below it. Thus, an element $s \in S$ permutes the set of all morphisms \mathcal{L} , by either pre-composition with $\widehat{s^{-1}}$ (i.e. $\varphi \mapsto \varphi \circ \widehat{s^{-1}}$) or by post-composition with \widehat{s} (i.e. $\varphi \mapsto \widehat{s} \circ \varphi$) where $s \in N_S(Q, sQs^{-1})$. These assignments form a left and right action of S on \mathcal{L} and we obtain an action of S on \mathcal{L} by conjugation of the subgroups $P \leq S$ and by conjugation of morphisms $\varphi \mapsto \widehat{s} \circ \varphi \circ \widehat{s^{-1}}$.

5.1. Definition. The action of a group G on S is called fusion preserving if the image of $G \xrightarrow{\tau} \text{Aut}(S)$ consists of fusion preserving automorphisms, that is, for every $\varphi \in \mathcal{F}(P, Q)$ and every $g \in G$ the composite $\tau_g \circ \varphi \circ \tau_g^{-1}$ belongs to $\mathcal{F}(\tau_g(P), \tau_g(Q))$.

In this section we prove Theorem 5.2 which is a variant of [3, Theorem 4.6]. While condition (2) of Theorem 5.2 offers some simplifications, we relax the assumption imposed in [3] that G is a finite p -group. The main idea of the proof remains the same but some new arguments were needed. We also felt that some details are missing in [3] and we therefore decided to present a complete proof of Theorem 5.2.

5.2. Theorem. *Let G be a finite group which acts on the centric linking system \mathcal{L}_0 of a p -local finite group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$. The action of $g \in G$ on $\varphi \in \text{Mor}(\mathcal{L}_0)$ is denoted by $\varphi \mapsto g \cdot \varphi \cdot g^{-1}$. Assume that $S_0 \triangleleft G$ and let S be a Sylow p -subgroup of G . Assume further that:*

- (1) *Each $g \in G$ acts on $\text{Ob}(\mathcal{L})$ by sending P to gPg^{-1} . For each $g \in G$ and each $\varphi \in \mathcal{L}_0(P, Q)$, $\pi_0(g\varphi g^{-1}) = c_g \circ \pi_0(\varphi) \circ c_{g^{-1}} \in \mathcal{F}_0(gPg^{-1}, gQg^{-1})$.*
- (2) *If $P_0 \leq S_0$ is \mathcal{F}_0 -centric and if a homomorphism $c_g : P_0 \rightarrow S_0$ for some $g \in G$ belongs to \mathcal{F}_0 , then $g \in S_0$.*
- (3) *The action of G on \mathcal{L}_0 extends the conjugation action of S_0 on \mathcal{L}_0 .*
- (4) *There is a G -equivariant system of lifts in \mathcal{L}_0 , that is, $g \cdot \widehat{s} \cdot g^{-1} = \widehat{gsg^{-1}}$ for any $g \in G$ and any $s \in N_{S_0}(P, Q)$.*
- (5) *If $Q \leq S_0$ is not \mathcal{F}_0 -centric but $\bar{Q} := N_{S_0}(Q)$ is \mathcal{F}_0 -centric, then there exists $\bar{\varphi} \in \mathcal{L}_0(\bar{Q}, S_0)$ such that $\pi_0(\bar{\varphi})(Q)$ does not contain its S_0 -centralizer and moreover, for any $x \in N_S(Q)$ there exists some $s \in S_0$ such that $x\bar{\varphi}x^{-1} = \widehat{s} \circ \bar{\varphi}$.*

Then, there exists a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ with the following properties:

- (a) *There are inclusions $\mathcal{F}_0 \subseteq \mathcal{F}$, $\mathcal{F}_0^c \subseteq \mathcal{F}^c$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ in such a way that the distinguished monomorphisms δ_P in \mathcal{L} extend the ones in \mathcal{L}_0 . The map $i : |\mathcal{L}_0| \rightarrow |\mathcal{L}|$ induced by the inclusion fits in a homotopy fibre sequence*

$$|\mathcal{L}_0| \xrightarrow{i} |\mathcal{L}| \rightarrow B(G/S_0).$$

Moreover, if S_0 has a complement K in G , that is $G = S_0 \rtimes K$, then:

- (b) There is a homotopy equivalence $|\mathcal{L}_0|_{hK} \xrightarrow{\cong} |\mathcal{L}|$ such that the composite $|\mathcal{L}_0| \rightarrow |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$ is homotopic to $|\mathcal{L}_0| \xrightarrow{1} |\mathcal{L}|$ and such that $\Theta : BS \rightarrow |\mathcal{L}|$ is homotopic to the composite

$$BS \xrightarrow{B \text{ incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|.$$

- (c) Up to isomorphism $(S, \mathcal{F}, \mathcal{L})$ is the unique p -local finite group with the properties in (b).

As a corollary we obtain the proof of Theorem A in Section 1.

Proof of Theorem A. By Remark 2.18 there is an action of Σ_n on the n -fold product $(S_0, \mathcal{F}_0, \mathcal{L}_0) = (S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})$ by permuting the factors.

The action of S_0 on \mathcal{L}_0 by conjugation clearly extends to an action of $S_0 \rtimes \Sigma_n$ because $S_0 = S^{\times n}$ acts on every coordinate of $\mathcal{L}_0 = \mathcal{L}^{\times n}$ and Σ_n acts by permuting the factors of \mathcal{L}_0 and the factors of $S_0 = S^{\times n}$. Set $G = S \wr K = S_0 \rtimes K$. We shall now show that the action of G on \mathcal{L}_0 satisfies hypotheses (1)–(5) of Theorem 5.2.

Hypothesis (1) is clearly satisfied because K acts on S_0 by permuting the factors which is an automorphism of $\mathcal{F}_0 = \mathcal{F}^{\times n}$. Note that $\pi : \mathcal{L}_0 \rightarrow \mathcal{F}_0$ is Σ_n -equivariant and it is also S_0 -equivariant since $\pi(\hat{s}) = c_s$ for any $s \in S$. Hypothesis (3) holds by the definition of the action of $G = S_0 \rtimes K$ on \mathcal{L}_0 . For hypothesis (4) choose a system of lifts $\{\delta_{P,Q}\}$ in \mathcal{L} (see Remark 2.6) and use Remarks 2.16 and 2.18 together with the obvious fact that the system $\{\delta_{P,Q}\}$ is S_0 -equivariant.

We now check hypothesis (2). Fix an \mathcal{F}_0 -centric subgroup $P_0 \leq S_0$ and let $P_0^{(i)}$ be defined as in 2.13. Since $P_0^{(i)}$ are \mathcal{F} -centric for $i = 1, \dots, n$ by Lemma 2.14 and $S \neq 1$, it follows that $P_0^{(i)} \neq 1$ whence $Z(P_0^{(i)}) \neq 1$ for all $i = 1, \dots, n$. Also note that $\prod_i Z(P_0^{(i)}) = \prod_i C_S(P_0^{(i)}) = C_{S_0}(P_0) \leq P_0$ because P_0 is \mathcal{F}_0 -centric. Fix some $g = (s_1, \dots, s_n; \sigma) \in G = S \wr K$ and assume that $g \notin S_0$, namely $\sigma \neq 1$. Without loss of generality we can assume that $\sigma(1) = 2$. Choose $1 \neq z_1 \in C_S(P_0^{(1)})$ and consider $(z_1, 1, \dots, 1; \text{id}) \in \prod_{i=1}^n Z(P_0^{(i)}) \leq P_0$. Then

$$\begin{aligned} c_g((z_1, 1, \dots, 1; \text{id})) &= (s_1, \dots, s_n; \sigma)(z_1, 1, \dots, 1; \text{id})(s_{\sigma^{-1}(1)}^{-1}, \dots, s_{\sigma^{-1}(n)}^{-1}; \sigma^{-1}) \\ &= (1, s_2 z_1 s_2^{-1}, 1, \dots, 1; \text{id}). \end{aligned}$$

Therefore $c_g \notin \mathcal{F}_0(P_0, S_0)$ because it cannot be a restriction of a morphism in $\prod_n \mathcal{F}$.

Finally we prove that hypothesis (5) is satisfied. Assume that $Q \leq S_0$ is not \mathcal{F}_0 -centric but $\bar{Q} := N_{S_0}(Q)$ is \mathcal{F}_0 -centric. Observe that $N_S(Q^{(i)})$ are all \mathcal{F} -centric because $N_{S_0}(Q)^{(i)}$ are all \mathcal{F} -centric by Lemma 2.14 and $N_{S_0}(Q)^{(i)} \leq N_S(Q^{(i)})$.

For every i we choose a morphism $\varphi_i \in \mathcal{L}(N_S(Q^{(i)}), S)$ such that $\pi(\varphi)(Q^{(i)})$ is fully \mathcal{F} -centralized (see [6, A.2(b)]), and define a morphism $(\varphi_1, \dots, \varphi_n) \in \mathcal{L}_0(\prod_i N_S(Q^{(i)}), S_0)$. Let $\tilde{\varphi} \in \mathcal{L}_0(\bar{Q}, S_0)$ be its restriction to \bar{Q} . Then $\pi(\tilde{\varphi})(Q)$ is fully centralized since $\pi_0(\tilde{\varphi})(Q)^{(i)} = \pi(\varphi_i)(Q^{(i)})$ are fully centralized for all i . By assumption Q is not \mathcal{F}_0 -centric, hence $\pi_0(\tilde{\varphi})(Q)$ does not contain its S_0 -centralizer.

It remains to show that for any $g \in N_G(Q)$ there exists some $s \in S_0$ such that $g\tilde{\varphi}g^{-1} = \hat{s} \circ \tilde{\varphi}$. Set $W = N_G(Q)/N_{S_0}(Q) \leq \Sigma_n$. Choose $u\sigma \in N_G(Q)$ where $u \in \prod_i N_S(Q)^{(i)}$ and $\sigma \in W$ and assume that $\sigma(i) = j$. Given $x \in Q^{(i)}$, choose $\underline{x} \in Q \leq \prod_i Q^{(i)}$ with $x_i = x$. Note that

$$u\sigma \cdot \underline{x} \cdot \sigma^{-1}u^{-1} = u \cdot (x_{\sigma(i)}) \cdot u^{-1}.$$

Thus, $u_j x u_j^{-1} \in Q^{(j)}$. It follows then that $u_j Q^{(i)} u_j^{-1} \subseteq Q^{(j)}$, that is, $Q^{(i)}$ is S -conjugate to a subgroup of $Q^{(j)}$. By symmetry $Q^{(i)}$ and $Q^{(j)}$ are S -conjugate. Thus, after conjugating by an appropriate element in S_0 we may assume that $Q^{(i)} = Q^{(j)}$ whenever $\sigma(i) = j$ for some $\sigma \in W$. Note that this does not change W . Moreover, in the definition of $\tilde{\varphi}$, we can take $\varphi_i = \varphi_j$ if $\sigma(i) = j$ for some $\sigma \in W$. Finally, for any $g \in N_G(Q)$ we can write $g = y\sigma$ for some $\sigma \in W$ and $y \in N_{S_0}(Q) \subseteq \prod_i N_S(Q)^{(i)}$. By the choice of the morphisms φ_i , it is clear that $\sigma\tilde{\varphi}\sigma^{-1} = \tilde{\varphi}$, hence

$$g\tilde{\varphi}g^{-1} = y\sigma\tilde{\varphi}\sigma^{-1}y^{-1} = y\tilde{\varphi}y^{-1} = \hat{y} \circ \tilde{\varphi} \circ \widehat{y^{-1}} = \hat{y} \circ \tilde{\varphi}(y^{-1}) \circ \tilde{\varphi} = \hat{s} \circ \tilde{\varphi}$$

where $s \in S_0$.

Now we apply Theorem 5.2(b) to conclude that there exists a p -local finite group $(S', \mathcal{F}', \mathcal{L}')$ with $(|\mathcal{L}_0|)_{hK} \simeq |\mathcal{L}'|$ such that

$$(1) \quad BS' \xrightarrow{B \text{ incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}'|$$

is homotopic to $\Theta' : BS' \rightarrow |\mathcal{L}'|$. Also observe that the horizontal arrows in

$$\begin{array}{ccc} (BS)^{\times n} & \xlongequal{\quad} & BS_0 \\ \Theta^{\times n} \downarrow & & \downarrow \Theta_0 \\ |\mathcal{L}|^{\times n} & \xrightarrow{\simeq} & |\mathcal{L}_0| \end{array}$$

form a Σ_n -equivariant map of the vertical arrows. It follows that the composite in (1) is homotopic to the map

$$BS' \xrightarrow{B \text{ incl}} BG \simeq (BS) \wr K \xrightarrow{\Theta \wr K} |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$$

which is therefore homotopic to $\Theta' : BS' \rightarrow |\mathcal{L}'|$. The uniqueness of $(S', \mathcal{F}', \mathcal{L}')$ with this property is guaranteed by part (c) of Theorem 5.2. \square

5.3. Remark. If the p -local finite group in Theorem A is associated with a finite group G then $(S', \mathcal{F}', \mathcal{L}')$ satisfies $|\mathcal{L}'|_p^\wedge \simeq (|\mathcal{L}'|_p^\wedge \wr K)_p^\wedge \simeq (BG_p^\wedge \wr K)_p^\wedge \simeq B(G \wr K)_p^\wedge$. Those equivalences follow from the Serre spectral sequence associated to $|\mathcal{L}'|^n \times_K EK$ and [2, Lemma I.5.5] since the spaces involved are p -good [6, Proposition 1.12]. Thus, \mathcal{L}' is the linking system associated to $G \wr K$.

In the remainder of this section we will prove Theorem 5.2. From now on, the hypotheses and notation set up in Theorem 5.2 are in force. Its proof, namely the construction of $(S, \mathcal{F}, \mathcal{L})$, is obtained in a sequence of Definitions and Lemmas 5.4–5.16. Their proofs are given after 5.16.

5.4. Definition. Let \mathcal{H}_0 denote the set of all the \mathcal{F}_0 -centric subgroups of S_0 . Fix once and for all a Sylow p -subgroup S of G and for every $P \leq S$ let P_0 denote $P \cap S_0$.

The action of G on the set of all subgroups of S_0 by conjugation restricts to an action on the set \mathcal{H}_0 of all the \mathcal{F}_0 -centric subgroups of S_0 because G acts via fusion preserving automorphisms of S_0 by hypothesis (1).

5.5. Definition. Let \mathcal{F}_1 be the fusion system on S_0 generated by \mathcal{F}_0 and $\text{Aut}_G(S_0)$. Define a category \mathcal{L}_1 whose object set is \mathcal{H}_0 and

$$\text{Mor}(\mathcal{L}_1) = (G \times \text{Mor}(\mathcal{L}_0)) / (gs, \varphi) \sim (g, \hat{s} \circ \varphi) \quad (s \in S_0).$$

The morphism set $\mathcal{L}_1(P_0, Q_0)$ where $P_0, Q_0 \in \mathcal{H}_0$ consists of the equivalence classes $[g : \varphi]$ such that $g \in G$ and $\varphi \in \mathcal{L}_0(P_0, Q_0^s)$. Composition is given by the formula

$$[g : \varphi] \circ [h : \psi] = [gh : (h^{-1}\varphi h) \circ \psi],$$

and identities are the elements of the form $[1 : \text{id}_{P_0}]$. We check later that composition is well defined.

Define a functor $\pi_1 : \mathcal{L}_1 \rightarrow \mathcal{F}_1$ which is the identity on the set of objects and

$$\pi_1([g : \varphi]) = c_g \circ \pi_0(\varphi).$$

We also define functions $\hat{\delta}_{P_0, Q_0} : N_G(P_0, Q_0) \rightarrow \mathcal{L}_1(P_0, Q_0)$ by $g \mapsto [g : \iota_{P_0}^{Q_0^g}]$ and denote the image of g by \hat{g} .

We will prove the following properties relating \mathcal{L}_1 and \mathcal{L}_0 .

5.6. Lemma. *The category \mathcal{L}_1 satisfies the following properties:*

- (a) *There is an inclusion functor $j : \mathcal{L}_0 \rightarrow \mathcal{L}_1$ which is the identity on objects and $\varphi \mapsto [1 : \varphi]$ on morphisms.*
- (b) *Every morphism in \mathcal{L}_1 has the form $\hat{g} \circ \varphi$ where φ is a morphism in $\mathcal{L}_0 \subseteq \mathcal{L}_1$. If $\varphi \in \mathcal{L}_0(P_0, Q_0)$ and $x \in N_G(P_0)$, then $\varphi \circ \hat{x} = \hat{x} \circ (x^{-1}\varphi x)$.*
- (c) *There is a homotopy fibre sequence*

$$|\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \rightarrow B(G/S_0).$$

If S_0 admits a complement K in G then there is a homotopy equivalence $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ such that the composite $|\mathcal{L}_0| \rightarrow |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ is homotopic to the map induced by the inclusion j . Moreover, the composite

$$BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$$

is homotopic to the map $BG \rightarrow |\mathcal{L}_1|$ induced by the functor $k : \mathcal{B}G \rightarrow \mathcal{L}_1$ with $k(\bullet_G) = S_0$ and $k(g) = [g : 1_{S_0}]$.

The next step in our construction is to define the following category.

5.7. Definition. Define a category \mathcal{L}_2 whose object set is

$$\mathcal{H} = \{P \leq S: P_0 = P \cap S_0 \in \mathcal{H}_0\}$$

and whose morphism sets are defined by

$$\mathcal{L}_2(P, Q) = \{\psi \in \mathcal{L}_1(P_0, Q_0): \forall x \in P \exists y \in Q \text{ s.t. } (\psi \circ \hat{x} = \hat{y} \circ \psi)\}.$$

By construction $\mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$ and composition of morphisms is obtained by composing them in \mathcal{L}_1 . Identities id_P have the form $[1 : \text{id}_{P_0}]$. Also define the following maps $\hat{\delta}_{P,Q}: N_G(P, Q) \rightarrow \mathcal{L}_2(P, Q)$ by $g \mapsto [g : \iota_{P_0}^{Q_0}]$ and denote the image of g by \hat{g} .

The main properties of the category \mathcal{L}_2 and its relation to the previously defined \mathcal{L}_1 are contained in next two lemmas.

5.8. Lemma. *The category \mathcal{L}_1 is the full subcategory of \mathcal{L}_2 on the objects \mathcal{H}_0 and the inclusion $j: \mathcal{L}_1 \rightarrow \mathcal{L}_2$ induces a homotopy equivalence on nerves.*

5.9. Lemma. *Let $P, Q \leq S$. The category \mathcal{L}_2 satisfies the following properties:*

- (a) *For every morphism $\psi \in \mathcal{L}_2(P, Q)$ there exists a unique monomorphism $\pi_2(\psi): P \rightarrow Q$ which satisfies $\psi \circ \hat{x} = \pi_2(\psi)(x) \circ \psi$ in \mathcal{L}_2 for all $x \in P$. Moreover, $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$.*
- (b) *$\pi_2(\text{id}_{P_0}) = \text{id}_P$ and $\pi_2(\lambda) \circ \pi_2(\psi) = \pi_2(\lambda \circ \psi)$ for every $P \xrightarrow{\psi} Q \xrightarrow{\lambda} R$ in \mathcal{L}_2 .*
- (c) *For every $\hat{g} \in \mathcal{L}_2(P, Q)$ with $g \in N_G(P, Q)$, we have $\pi_2(\hat{g}) = c_g$.*
- (d) *Given $\psi \in \mathcal{L}_2(P, Q)$, if $\pi_2(\psi)$ is an isomorphism of groups then ψ is an isomorphism in \mathcal{L}_2 .*

Lemma 5.9 justifies the following definition.

5.10. Definition. Let \mathcal{F}_2 be the category whose object set is \mathcal{H} , see Definition 5.7, and whose morphism sets $\mathcal{F}_2(P, Q)$ are the set of group monomorphisms $\pi_2(\mathcal{L}_2(P, Q))$ defined by Lemma 5.9. By the properties shown in this lemma, there results a projection functor $\pi_2: \mathcal{L}_2 \rightarrow \mathcal{F}_2$ which is the identity on objects.

5.11. Lemma. *The category \mathcal{F}_2 satisfies the following properties:*

- (a) *For every $P, Q \in \mathcal{H}$, $\text{Hom}_G(P, Q) \subseteq \mathcal{F}_2(P, Q)$. In particular, \mathcal{F}_2 contains all the inclusions $P \leq Q$ of groups in \mathcal{H} .*
- (b) *Every morphism in \mathcal{F}_2 factors as an isomorphism in \mathcal{F}_2 followed by an inclusion. In particular, every isomorphism of groups $f: P \rightarrow Q$ in \mathcal{F}_2 is an isomorphism in \mathcal{F}_2 .*

Thus, \mathcal{F}_2 falls short of being a fusion system on S only because its set of objects \mathcal{H} need not contain all the subgroups of S .

5.12. Definition. Let \mathcal{F} denote the fusion system on S generated by \mathcal{F}_2 .

5.13. Lemma. *The fusion system \mathcal{F} satisfies the following properties:*

- (a) \mathcal{F}_2 is the full subcategory of \mathcal{F} generated by the objects in \mathcal{H} .
- (b) Every $P \in \mathcal{H}$ is \mathcal{F} -centric. In particular, $\mathcal{H}_0 \subseteq \mathcal{F}^c$.
- (c) Every morphism $f \in \mathcal{F}(P, Q)$ restricts to a morphism $f|_{P_0} \in \mathcal{F}(P_0, Q_0)$.

5.14. Lemma. *The functor $\pi_2 : \mathcal{L}_2 \rightarrow \mathcal{F}$ satisfies all the axioms of a centric linking system on the object set \mathcal{H} .*

Finally, the last step in the proof is to show that the fusion system (S, \mathcal{F}) defined in 5.12 is saturated and that \mathcal{L}_2 can be extended to a unique centric linking system \mathcal{L} associated to \mathcal{F} .

5.15. Lemma. *\mathcal{F} is a saturated fusion system on S .*

5.16. Lemma. *There exists a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ such that \mathcal{L}_2 is a full subcategory of \mathcal{L} and $\pi_2 : \mathcal{L}_2 \rightarrow \mathcal{F}$ is the restriction of $\pi : \mathcal{L} \rightarrow \mathcal{F}$. Moreover, $\hat{\delta}_P : P \rightarrow \text{Aut}_{\mathcal{L}_2}(P)$ are the distinguished monomorphisms of $(S, \mathcal{F}, \mathcal{L})$ for all $P \in \mathcal{H}$, and the inclusion $\mathcal{L}_2 \subseteq \mathcal{L}$ induces a homotopy equivalence on nerves.*

Assuming Definitions and Lemmas 5.4–5.16, we can now prove Theorem 5.2.

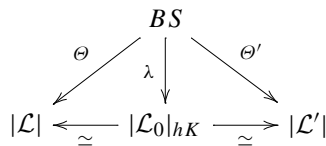
Proof of Theorem 5.2. The p -local finite group $(S, \mathcal{F}, \mathcal{L})$ is constructed in Lemma 5.16. Together with Lemma 5.8 we obtain inclusions of full subcategories $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$ which induce homotopy equivalences on nerves. By Lemma 5.6(c), there results the homotopy fibre sequence of part (a).

Now assume that S_0 has a complement K in G and we prove points (b) and (c). Lemma 5.6(c) shows that there are homotopy equivalences $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1| \simeq |\mathcal{L}|$ such that $|\mathcal{L}_0| \rightarrow |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$ is homotopic to the map induced by the inclusion $\mathcal{L}_0 \subseteq_j \mathcal{L}_1 \subseteq \mathcal{L}$. Moreover the map

$$BS \xrightarrow{B \text{ incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$$

is induced by the functor $\Lambda_0 : BS \rightarrow \mathcal{L}$ which sends \bullet_S to S_0 and defined on morphisms by $s \mapsto [s : 1_{S_0}] = \hat{s} \in \text{Aut}_{\mathcal{L}}(S_0)$ (see Lemmas 5.16, 5.6 and Definition 5.7). The map $\Theta : BS \rightarrow |\mathcal{L}|$ is the realization of the functor $\Lambda_1 : BS \rightarrow \mathcal{B} \text{Aut}_{\mathcal{L}}(S) \rightarrow \mathcal{L}$ where $s \mapsto \hat{s} \in \text{Aut}_{\mathcal{L}}(S)$, then the lift of the inclusion $\iota_{S_0}^S \in \mathcal{L}(S_0, S)$ provides a natural transformation $\Lambda_0 \rightarrow \Lambda_1$ because $\hat{s} \circ \iota_{S_0}^S = \iota_{S_0}^S \circ \hat{s}$, see Definition 2.5. Therefore $|\Lambda_0|$ and $|\Lambda_1|$ are homotopic and the proof of point (b) is complete.

Now assume that $(S, \mathcal{F}', \mathcal{L}')$ is another p -local finite group which satisfies the properties in point (b). Let λ denote the composite $BS \rightarrow BG = (BS_0)_{hK} \rightarrow |\mathcal{L}_0|_{hK}$. By assumption there is a homotopy commutative diagram



The isomorphism of $(S, \mathcal{F}, \mathcal{L})$ and $(S, \mathcal{F}', \mathcal{L}')$ follows from [6, Theorem 7.7]. \square

In the rest of the section we fill in the details needed for the construction in 5.5–5.16.

Proof that Definition 5.5 makes \mathcal{L}_1 a small category and makes $\pi_1 : \mathcal{L}_1 \rightarrow \mathcal{F}_1$ a functor. The verification that composition of morphisms is well defined is similar to the one in [3, Theorem 4.6]. Specifically, for any $g_0, h_0 \in S_0$

$$\begin{aligned}
 [gg_0 : \varphi] \circ [hh_0 : \psi] &= [gg_0hh_0 : (h_0^{-1}h^{-1}\varphi hh_0) \circ \psi] = \text{by hypothesis (3),} \\
 [gg_0h : (h^{-1}\varphi h) \circ \widehat{h_0} \circ \psi] &= [gh : \widehat{h^{-1}g_0h} \circ (h^{-1}\varphi h) \circ \widehat{h_0} \circ \psi] = \text{by hypothesis (4),} \\
 [gh : h^{-1}(\widehat{g_0} \circ \varphi)h \circ \widehat{h_0} \circ \psi] &= [g : \widehat{g_0} \circ \varphi] \circ [h : \widehat{h_0} \circ \psi].
 \end{aligned}$$

Associativity is straightforward as well as checking that the morphisms $[1 : 1_{P_0}]$ are identity morphisms $P_0 \rightarrow P_0$.

It is evident from the definition that π_1 maps identity morphisms in \mathcal{L}_1 to identities in \mathcal{F}_1 . It also respects compositions by the following calculation which uses hypothesis (1) in the third equality

$$\begin{aligned}
 \pi_1([g : \varphi]) \circ \pi_1([h : \psi]) &= c_g \circ \pi_0(\varphi) \circ c_h \circ \pi_0(\psi) = c_{gh} \circ (c_{h^{-1}} \circ \pi_0(\varphi) \circ c_h) \circ \pi_0(\psi) \\
 &= c_{gh} \circ \pi_0(h^{-1}\varphi h) \circ \pi_0(\psi) = c_{gh} \circ \pi_0(h^{-1}\varphi h \circ \psi) \\
 &= \pi_1([gh : h^{-1}\varphi h \circ \psi]) = \pi_1([g : \varphi] \circ [h : \psi]). \quad \square
 \end{aligned}$$

Proof of Lemma 5.6. (a) By Definition 5.5 we have $[1 : \varphi] \circ [1 : \varphi'] = [1 : \varphi \circ \varphi']$ so j is clearly associative and unital. It is an inclusion functor because $[1 : \varphi] = [1 : \varphi']$ if and only if $\varphi = \varphi'$ by the definition of morphisms in \mathcal{L}_1 .

(b) Clearly, every morphism ψ in \mathcal{L}_1 has the form $[g : \varphi] = [g : 1] \circ [1 : \varphi] = \widehat{g} \circ \varphi$. Given φ and x as in the statement, by Definition 5.5

$$\varphi \circ \widehat{x} = [1 : \varphi] \circ [x : 1] = [x : x^{-1}\varphi x] = [x : 1_{Q_0^x}] \circ [1 : x^{-1}\varphi x] = \widehat{x} \circ x^{-1}\varphi x.$$

(c) Set $\bar{G} = G/S_0$ and denote its elements by $\bar{g} = gS_0$. There is a functor $\Pi : \mathcal{L}_1 \rightarrow \mathcal{B}(\bar{G})$ which sends every object of \mathcal{L}_1 to $\bullet_{\bar{G}}$ and maps $[g : \varphi] \mapsto \bar{g}$.

Now, consider the comma category $(\bullet_{\bar{G}} \downarrow \Pi)$. Its objects are pairs (\bar{g}, P_0) and morphisms $(\bar{g}, P_0) \rightarrow (\bar{h}, Q_0)$ are morphisms $[x : \lambda] \in \mathcal{L}_1(P_0, Q_0)$ such that $\bar{x} = \bar{h}\bar{g}^{-1}$. We can easily check that $\widehat{g} : P_0^g \rightarrow P_0$ provides an isomorphism $(\bar{e}, P_0^g) \rightarrow (\bar{g}, P_0)$ in $(\bullet_{\bar{G}} \downarrow \Pi)$. Therefore, the set of objects of the form (\bar{e}, P_0) form a skeletal full subcategory of $(\bullet_{\bar{G}} \downarrow \Pi)$, that is, it contains an element from every isomorphism class of objects. This subcategory is clearly isomorphic to \mathcal{L}_0 and moreover the composite $\mathcal{L}_0 \subseteq (\bullet_{\bar{G}} \downarrow \Pi) \rightarrow \mathcal{L}_1$ is the inclusion j in part (a).

Moreover, any morphism $\bar{g} \in \mathcal{B}\bar{G}$ clearly induces an automorphism of the category $(\bullet_{\bar{G}} \downarrow \Pi)$. Therefore, Quillen’s theorem B [20] applies in this situation to show that the sequence $|(\bullet_{\bar{G}} \downarrow \Pi)| \rightarrow |\mathcal{L}_1| \rightarrow |\mathcal{B}(G/S_0)|$ is a homotopy fibre sequence. Finally, using the homotopy equivalence $|j|$ we obtain the homotopy fibre sequence $|\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \xrightarrow{|\Pi|} BG/S_0$.

Now suppose that S_0 has a complement K in G . Recall that G acts on the category \mathcal{L}_0 and we view the restriction of this action to K as a functor $\mathcal{BK} \rightarrow \mathbf{Cat}$. Let $\text{Tr}_K(\mathcal{L}_0)$ denote the transporter category (or Grothendieck construction) of this functor; see, e.g. [24]. The object set of $\text{Tr}_K(\mathcal{L}_0)$ is \mathcal{H}_0 , and the morphisms $P_0 \rightarrow Q_0$ are pairs (k, φ) where $\varphi \in \mathcal{L}_0(kP_0, Q_0)$.

Composition is given by the following formula: $(k_2, \varphi_2) \circ (k_1, \varphi_1) = (k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1})$. Define a functor $\Phi : \text{Tr}_K(\mathcal{L}_0) \rightarrow \mathcal{L}_1$ which is the identity on objects and

$$\Phi : \text{Tr}_K(\mathcal{L}_0)(P_0, Q_0) \rightarrow \mathcal{L}_1(P_0, Q_0) \text{ is defined by } (k, \varphi) \mapsto [k : k^{-1}\varphi k].$$

It is clear that $\Phi(1, \text{id}) = [1 : \text{id}]$ and for any pair of composable morphisms (k_2, φ_2) and (k_1, φ_1) in $\text{Tr}_K(\mathcal{L}_0)$,

$$\begin{aligned} \Phi(k_2, \varphi_2) \circ \Phi(k_1, \varphi_1) &= [k_2 : k_2^{-1}\varphi_2k_2] \circ [k_1 : k_1^{-1}\varphi_1k_1] \\ &= [k_2k_1 : k_1^{-1}k_2^{-1}\varphi_2k_2k_1 \circ k_1^{-1}\varphi_1k_1] = \Phi(k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1}). \end{aligned}$$

By definition Φ is bijective on the object set. It is bijective on morphism sets because $K \cap S_0 = 1$ so every morphism in $\mathcal{L}_1(P_0, Q_0)$ has a unique representative of the form $[k : \varphi]$ where $k \in K$ and $\varphi \in \mathcal{L}_0$.

Thomason [24] constructed a homotopy equivalence $|\mathcal{L}_0|_{hK} \xrightarrow{\beta} |\text{Tr}_K(\mathcal{L}_0)|$ such that the map $|\mathcal{L}_0| \rightarrow |\mathcal{L}_0|_{hK} \simeq |\text{Tr}_K(\mathcal{L}_0)|$ is homotopic to the map induced by the inclusion $\mathcal{L}_0 \subseteq \text{Tr}_K(\mathcal{L}_0)$ via $\varphi \mapsto [\bar{e} : \varphi]$. Furthermore, by inspection Φ carries the subcategory of \mathcal{L}_0 in $\text{Tr}_K(\mathcal{L}_0)$ onto $\mathcal{L}_0 \subseteq \mathcal{L}_1$ via the identity map. We deduce that $|\Phi| \circ \beta$ is a homotopy equivalence $|\mathcal{L}_0|_{hK} \rightarrow |\mathcal{L}_1|$ whose composition with $|\mathcal{L}_0| \rightarrow |\mathcal{L}_0|_{hK}$ is homotopic to the map induced by the inclusion $j : \mathcal{L}_0 \rightarrow \mathcal{L}_1$.

To complete the proof we now consider the subcategory \mathcal{BS}_0 of $\mathcal{BAut}_{\mathcal{L}_0}(S_0) \subseteq \mathcal{L}_0$ via the monomorphism $\delta_{S_0} : S_0 \rightarrow \text{Aut}_{\mathcal{L}_0}(S_0)$ and observe that it is invariant under the action of K by hypothesis (4). Thus, there is an inclusion of subcategories $\text{Tr}_K \mathcal{BS}_0 \subseteq \text{Tr}_K \mathcal{L}_0$ induced by $\text{Tr}_K(\delta_{S_0})$. By inspection there is an isomorphism of categories $\text{Tr}_K \mathcal{BS}_0 \cong \mathcal{BG}$ via the functor $(k, s) \mapsto sk$ such that the composite

$$\mathcal{BG} \cong \text{Tr}_K(\mathcal{BS}_0) \subseteq \text{Tr}_K(\mathcal{L}_0) \xrightarrow{\Phi} \mathcal{L}_1$$

is the functor which sends \bullet_G to S_0 and $g \mapsto [g : 1] \in \text{Aut}_{\mathcal{L}_1}(S_0)$. \square

Here are more properties of \mathcal{L}_1 that we will need later in order to study the properties of the category \mathcal{L}_2 .

5.17. Lemma. *Let $P_0, Q_0, R_0 \in \mathcal{H}_0$. Then*

- (a) *For every $g \in N_G(P_0, Q_0)$ and $h \in N_G(Q_0, R_0)$ the equality $\hat{h} \circ \hat{g} = \widehat{hg}$ holds in \mathcal{L}_1 .*
- (b) *Fix $\psi \in \mathcal{L}_1(P_0, Q_0)$ of the form $[g : \varphi]$. Then, for every $x \in N_G(P_0)$ there exists at most one $y \in N_G(Q_0)$ such that $\psi \circ \hat{x} = \hat{y} \circ \psi$. In this case $y = gxg^{-1}s_0$ for a unique $s_0 \in S_0$. Moreover, if $x \in P_0$ then $y = \pi_1(\psi)(x)$ satisfies $\psi \circ \hat{x} = \hat{y} \circ \psi$.*
- (c) *Every morphism in $\mathcal{L}_1(P_0, Q_0)$ is both a monomorphism and an epimorphism.*
- (d) *Fix $\psi \in \mathcal{L}_1(P_0, Q_0)$ such that $\pi_1(\psi)(P_0) \leq R_0$ for some $R_0 \leq Q_0$. Then there exists $\lambda \in \mathcal{L}_1(P_0, R_0)$ such that $\psi = \iota \circ \lambda$ where $\iota = \hat{e} \in \mathcal{L}_1(R_0, Q_0)$.*
- (e) *If $\pi_1(\psi) = \pi_1(\psi')$ where $\psi, \psi' \in \mathcal{L}_1(P_0, Q_0)$ then $\psi' = \psi \circ \hat{z}$ for a unique $z \in Z(P_0)$.*
- (f) *Fix $P_0 \in \mathcal{H}_0$ and set $H := \{g \in G \mid gP_0g^{-1} \text{ is } \mathcal{F}_0\text{-conjugate to } P_0\}$. Then H is a subgroup of G which contains S_0 and $|\text{Aut}_{\mathcal{L}_1}(P_0) : \text{Aut}_{\mathcal{L}_0}(P_0)| = |H : S_0|$.*

Proof. (a) $\hat{h} \circ \hat{g} = [h : \hat{e}] \circ [g : \hat{e}] = [hg : \hat{e}] = \widehat{hg}$ by Definition 5.5 and hypothesis (4).

(c) Every morphism $[g : \varphi]$ factors as $[g : 1] \circ [1 : \iota_{\varphi(P_0)}^{Q_0}] \circ [1 : \varphi']$ for some isomorphism $\varphi' : P_0 \rightarrow \varphi(P_0)$ in \mathcal{L}_0 . Since $[g : 1]$ and $[1 : \varphi']$ are isomorphisms it is enough to show that morphisms of the form $[1 : \iota_{P_0}^{Q_0}] \in \mathcal{L}_1(P_0, Q_0)$, which we denote by ι , are monomorphisms and epimorphisms.

Consider morphisms $[h : \varphi], [h' : \varphi'] \in \mathcal{L}_1(R_0, P_0)$ and assume that $\iota \circ [h : \varphi] = \iota \circ [h' : \varphi']$. Then $[h, \hat{e} \circ \varphi] = [h' : \hat{e} \circ \varphi']$ and therefore $h' = hs$ for some $s \in S_0$ and $\hat{e} \circ \varphi = \hat{e} \circ \hat{s} \circ \varphi'$ where $\hat{e} \in \mathcal{L}_0(\varphi(P_0), hQ_0h^{-1})$. Since \hat{e} is a monomorphism in \mathcal{L}_0 it follows that $\varphi = \hat{s} \circ \varphi'$ and therefore $[h : \varphi] = [h' : \varphi']$. This shows that ι is a monomorphism.

If $[h : \varphi] \circ \iota = [h' : \varphi'] \circ \iota$ then a direct calculation shows that $[h : \varphi \circ \hat{e}] = [h' : \varphi' \circ \hat{e}]$. A similar argument to the one above using the fact that \hat{e} is an epimorphism in \mathcal{L}_0 shows that $h' = hs$ and $\varphi = \hat{s} \circ \varphi'$, whence $[h : \varphi] = [h' : \varphi']$.

(b) If y exists then it is unique because by part (c), ψ is an epimorphism. Since we have $\psi \circ \hat{x} = [gx : x^{-1}\varphi x]$ and $\hat{y} \circ \psi = [yg : \varphi]$ we see that there is a unique $s \in S_0$ such that $gx = ygs^{-1}$, whence $y = gxg^{-1} \cdot gsg^{-1}$.

If $x \in P_0$ then axiom (C) satisfied by \mathcal{L}_0 , see Definition 2.4, implies that

$$\begin{aligned} \psi \circ \hat{x} &= [g : \varphi] \circ [x : 1] = [gx : \widehat{x^{-1} \circ \varphi \circ \hat{x}}] = [g : \varphi \circ \hat{x}] = [g : \widehat{\pi_0(\varphi)(x)} \circ \varphi] \\ &= [c_g(\pi_0(\varphi)(x)) \cdot g : \varphi] = c_g(\widehat{\pi_0(\varphi)(x)}) \cdot g \circ \psi = \pi_1(\psi)(x) \circ \psi. \end{aligned}$$

(d) Write $\psi = [g : \varphi]$ for some $\varphi \in \mathcal{L}_0(P_0, Q_0^S)$. Then $\varphi = \hat{e} \circ \bar{\varphi}$ for some $\bar{\varphi} \in \mathcal{L}_0(P_0, gR_0g^{-1})$ and $e \in N_{S_0}(P_0, gR_0g^{-1})$. By inspection $\psi = [1 : \hat{e}] \circ [g : \bar{\varphi}]$.

(e) Write $\psi = [g : \varphi]$ and $\psi' = [g' : \varphi']$ in $\mathcal{L}_1(P_0, Q_0)$. By assumption and Definition 5.5, $c_g \circ \pi_0(\varphi) = c_{g'} \circ \pi_0(\varphi')$, whence $\pi_0(\varphi) = c_{g^{-1}g'} \circ \pi_0(\varphi')$. Since $\pi_0(\varphi), \pi_0(\varphi') \in \mathcal{F}_0$, we obtain that $c_{g^{-1}g'} \in \mathcal{F}_0(P_0, P_0)$ so hypothesis (2) implies that $g^{-1}g' \in S_0$, namely $g' = gs$ for some $s \in S_0$. Then $\pi_0(\varphi) = c_s \circ \pi_0(\varphi')$ implies that $\hat{s} \circ \varphi' = \varphi \circ \hat{z}$ for some $z \in Z(P_0)$. Therefore, $\psi \circ \hat{z} = [g : \varphi \circ \hat{z}] = [g : \hat{s} \circ \varphi'] = [g' : \varphi'] = \psi'$.

(f) By hypothesis (1) in Theorem 5.2, if Q_0 is \mathcal{F}_0 -conjugate to Q'_0 then gQ_0g^{-1} is \mathcal{F}_0 -conjugate to gQ'_0g^{-1} for any $g \in G$. This implies that H is a subgroup of G and it contains S_0 because $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$.

Let g_1, \dots, g_n be representatives for the cosets of S_0 in H . By Definition 5.5 every element $\psi \in \text{Aut}_{\mathcal{L}_1}(P_0)$ can be written uniquely as $\psi = [g_i : \varphi]$ for some $i = 1, \dots, n$ where $\varphi \in \mathcal{L}_0(P_0, {}^{S_i}P_0)$. Also note that $|\mathcal{L}_0(P_0, {}^{S_i}P_0)| = |\text{Aut}_{\mathcal{L}_0}(P_0)|$ because ${}^{S_i}P_0$ is \mathcal{F}_0 -conjugate to P_0 . This shows that $|\text{Aut}_{\mathcal{L}_1}(P_0)| = n \cdot |\text{Aut}_{\mathcal{L}_0}(P_0)| = |H : S_0| \cdot |\text{Aut}_{\mathcal{L}_0}(P_0)|$. \square

We now turn to the study of the properties of the category \mathcal{L}_2 .

Proof that Definition 5.7 makes \mathcal{L}_2 a small category. Given morphisms $\psi \in \mathcal{L}_2(P, Q)$ and $\rho \in \mathcal{L}_2(Q, R)$, we leave it as an easy exercise to check that $\rho \circ \psi \in \mathcal{L}_1(P_0, R_0)$ belongs to $\mathcal{L}_2(P, R)$. Thus, composition of morphisms in \mathcal{L}_2 is well defined. It is easily seen to be unital and associative because this is the case in \mathcal{L}_1 .

Since $S_0 \triangleleft G$ it follows that $N_G(P, Q) \subseteq N_G(P_0, Q_0)$. Now fix some $g \in N_G(P, Q)$ and $x \in P$ and set $y = gxg^{-1} \in Q$. It follows from Lemma 5.17(a) that $\hat{g} \circ \hat{x} = \widehat{gx} = \widehat{yg} = \hat{y} \circ \hat{g}$. Therefore $\hat{g} \in \mathcal{L}_2(P, Q)$. \square

We are now ready to prove Lemmas 5.8 and 5.9.

Proof of Lemma 5.8. By construction $\mathcal{L}_2(P_0, Q_0) \subseteq \mathcal{L}_1(P_0, Q_0)$ for any $P_0, Q_0 \in \mathcal{H}_0$. For every $x \in P_0$ and every $\psi = [g : \varphi] \in \mathcal{L}_1(P_0, Q_0)$, it follows from Lemma 5.17(b) that there is an equality $\psi \circ \hat{x} = \hat{y} \circ \psi$ in \mathcal{L}_1 where $y = \pi_1(\psi)(x) \in Q_0$. Therefore $\psi \in \mathcal{L}_2(P_0, Q_0)$ and we conclude that $\mathcal{L}_1(P_0, Q_0) = \mathcal{L}_2(P_0, Q_0)$.

The inclusion functor $j : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ has a left inverse $r : \mathcal{L}_2 \rightarrow \mathcal{L}_1$ which maps an object P to P_0 and maps morphisms via the inclusions $\mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$. Observe that $r \circ j = \text{Id}_{\mathcal{L}_1}$ because $\mathcal{L}_2(P_0, Q_0) = \mathcal{L}_1(P_0, Q_0)$.

By Lemma 5.17(b) we see that $\mathcal{L}_2(P_0, P)$ contains $[e : 1_{P_0}] = \hat{e}$. These morphisms define a natural transformation $j \circ r \rightarrow \text{Id}$. This is because we recall that $[e : 1_{P_0}]$ and $[e : 1_{Q_0}]$ are the identities of P_0 and Q_0 in \mathcal{L}_1 and for any $\psi \in \mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$

$$\psi \circ [e : 1_{P_0}] = [e : 1_{Q_0}] \circ \psi.$$

Then it follows that j and r yield homotopy equivalences on nerves. \square

Proof of Lemma 5.9. (a) By Definition 5.7, for every $x \in P$ there exists some $y \in Q$ such that $\psi \circ \hat{x} = \hat{y} \circ \psi$. Since $P \leq N_G(P_0)$ and $Q \leq N_G(Q_0)$, Lemma 5.17(b) implies that y is unique. There results a well-defined function $\pi_2(\psi) : P \rightarrow Q$ defined by $\pi_2(\psi)(x) = y$. In addition, since \hat{x} and $\hat{y} = \pi_2(\psi)(x)$ are morphisms in \mathcal{L}_2 (see Definition 5.7) and $\mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$, we deduce that the equation $\psi \circ \hat{x} = \pi_2(\psi)(x) \circ \psi$ holds in \mathcal{L}_2 . Moreover, $\pi_2(\psi) : P \rightarrow Q$ is the unique function that satisfies this equality for all $x \in P$. The fact that $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$ follows from the last assertion in Lemma 5.17(b).

Given $x, x' \in P$, set $y = \pi_2(\psi)(x)$ and $y' = \pi_2(\psi)(x')$. Then, by Lemma 5.17(a)

$$\psi \circ \widehat{xx'} = \psi \circ \hat{x} \circ \hat{x}' = \hat{y} \circ \psi \circ \hat{x}' = \hat{y} \circ \hat{y}' \circ \psi = \widehat{yy'} \circ \psi.$$

This shows that $\pi_2(\psi)$ is a homomorphism. If $x \in \ker \pi_2(\psi)$ then $\psi \circ \hat{x} = \hat{1} \circ \psi = \psi$. Since ψ is a monomorphism by Lemma 5.17(c), we deduce that $\hat{x} = \text{id}$, hence $x = 1$. Therefore $\pi_2(\psi)$ is a monomorphism.

(b) Clearly $\pi_2([e : 1_{P_0}]) = \text{Id}_{P_0}$. Now given $P \xrightarrow{\psi} Q \xrightarrow{\lambda} R$ in \mathcal{L}_2 , set $y = \pi_2(\psi)(x)$ and $z = \pi_2(\lambda)(y)$. Then $\psi \circ \hat{x} = \hat{y} \circ \psi$ and $\lambda \circ \hat{y} = \hat{z} \circ \lambda$ so $\lambda \circ \psi \circ \hat{x} = \hat{z} \circ \lambda \circ \psi$ whence, by the uniqueness statement in Lemma 5.17(b), we conclude that $z = \pi_2(\lambda \circ \psi)(x)$.

(c) This follows from Lemma 5.17(a) because for any $x \in P$ we have $\hat{g} \circ \hat{x} = \widehat{gx} = \widehat{c_g(x)g} = \widehat{c_g(x)} \circ \hat{g}$ in \mathcal{L}_1 so $\pi_2(\hat{g}) = c_g$.

(d) Write $\psi = [g : \varphi]$. Observe that $\pi_2(\psi)(P_0) = \pi_1(\psi)(P_0) \leq Q_0$ by statement (a). Since $\pi_2(\psi) : P \rightarrow Q$ is an isomorphism, for every $y_0 \in Q_0 \leq Q$ there exists some $x \in P$ such that $\pi_2(\psi)(x) = y_0$, namely $\psi \circ \hat{x} = \hat{y}_0 \circ \psi$. By Lemma 5.17(b) we know that $y_0 = gxg^{-1}s_0$ for some $s_0 \in S_0$. We deduce then that $x \in S_0 \cap P = P_0$ because $S_0 \triangleleft G$. This shows that $\pi_2(\psi)(P_0) = Q_0$ and therefore $\pi_1(\psi)$ is an isomorphism of groups.

Since $\pi_1(\psi)$ is an isomorphism, φ is an isomorphism in \mathcal{L}_0 and therefore ψ is an isomorphism in \mathcal{L}_1 . Given any $y \in Q$ there is a unique $x \in P$ with $\psi \circ \widehat{x^{-1}} = \widehat{y^{-1}} \circ \psi$ because $\pi_2(\psi)$ is an isomorphism. By taking inverses one sees that ψ^{-1} belongs to \mathcal{L}_2 so ψ is an isomorphism in \mathcal{L}_2 . \square

For later use we also need the following technical lemma.

5.18. Lemma. Fix $P \in \mathcal{H}$ and consider $N_S(P_0)$ as a subgroup of $\text{Aut}_{\mathcal{L}_1}(P_0)$ via the monomorphism $\hat{\delta}_{P_0, P_0} : N_S(P_0) \rightarrow \text{Aut}_{\mathcal{L}_1}(P_0)$. Let $Q \leq N_S(P_0)$ and assume that $Q = \psi P \psi^{-1}$ for some $\psi \in \text{Aut}_{\mathcal{L}_1}(P_0)$. Then $P_0 = Q_0$ and ψ is an isomorphism in \mathcal{L}_2 from P to Q .

Proof. Recall from Lemma 5.8 that $\text{Aut}_{\mathcal{L}_1}(P_0) = \text{Aut}_{\mathcal{L}_2}(P_0)$. For $x \in P_0$ set $\hat{y} = \psi \hat{x} \psi^{-1} \in Q$. Thus $\psi \circ \hat{x} = \hat{y} \circ \psi$ and by Definition 5.10, $y = \pi_2(\psi)(x) \in P_0$. This shows that $P_0 = \psi P_0 \psi^{-1}$ and, in particular, $P_0 \leq Q_0$.

Since $P_0 \leq Q_0$, we may consider $\iota := \hat{e} \in \mathcal{L}_1(P_0, Q_0)$ where $e \in G$ is the identity element, and define $\lambda = \iota \circ \psi \in \mathcal{L}_1(P_0, Q_0)$. For every $x \in P$, set $\hat{y} = \psi \hat{x} \psi^{-1}$. By definition $y \in Q$. Note that $P_0 \triangleleft Q$ because $P_0 \triangleleft P$. So Lemma 5.17(a) implies

$$\lambda \circ \hat{x} = \iota \circ \psi \circ \hat{x} = \iota \circ \hat{y} \circ \psi = \hat{y} \circ \hat{e} \circ \psi = \hat{y} \circ \psi.$$

We conclude from Definition 5.7 that $\lambda \in \mathcal{L}_2(P, Q)$. Furthermore, $\pi_2(\lambda)$ is an isomorphism because it is a monomorphism by Lemma 5.9(a) and $|P| = |Q|$. Lemma 5.9(d) now shows that λ is an isomorphism in \mathcal{L}_2 and, in particular, it is an isomorphism of the objects P_0 and Q_0 in \mathcal{L}_1 . In particular $|P_0| = |Q_0|$ and therefore $\lambda = \psi$. \square

We now check the main properties of the category \mathcal{F}_2 .

Proof of Lemma 5.11. (a) This is immediate from Lemma 5.9(c). By taking $e \in N_G(P, Q)$ for any inclusion $P \leq Q$ in \mathcal{H} we obtain $\text{incl}_P^Q \in \mathcal{F}_2(P, Q)$.

(b) Fix a homomorphism $f : P \rightarrow Q$ in \mathcal{F}_2 and set $R = f(P)$. By definition, $f = \pi_2(\psi)$ for some $\psi \in \mathcal{L}_2(P, Q)$. Also note that every $y \in R$ must normalize $f(P_0)$ because f is an isomorphism and that by Lemma 5.9(a), $f(P_0) = \pi_1(\psi)(P_0)$.

Write $\psi = [g : \varphi]$. Then there is an isomorphism $\bar{\varphi}$ in \mathcal{L}_0 such that $\psi = [1 : \iota_{f(P_0)}^{Q_0}] \circ [g : \bar{\varphi}]$. Since $\psi \in \mathcal{L}_2$, for every $x \in P$ there exists $y \in R$ such that

$$[1 : \iota_{f(P_0)}^{Q_0}] \circ [g : \bar{\varphi}] \circ \hat{x} = \hat{y} \circ [1 : \iota_{f(P_0)}^{Q_0}] \circ [g : \bar{\varphi}] = [1 : \iota_{f(P_0)}^{Q_0}] \circ \hat{y} \circ [g : \bar{\varphi}].$$

By Lemma 5.17(c), $[1 : \iota_{f(P_0)}^{Q_0}]$ is a monomorphism and we deduce that $[g : \bar{\varphi}]$ is an isomorphism $P \rightarrow R$ in \mathcal{L}_2 . Also $f = \text{incl}_R^Q \pi_2([g : \bar{\varphi}])$. This completes the proof. \square

5.19. Lemma. Consider $P \leq S$ such that $P_0 \in \mathcal{H}_0$. Then $C_G(P) = C_{S_0}(P) = Z(P_0)^P$ where P acts on $Z(P_0)$ by conjugation.

Proof. If $g \in C_G(P)$ then $c_g|_{P_0} = \text{id}_{P_0} \in \text{Aut}_{\mathcal{F}_0}(P_0)$. By hypothesis (2), $g \in S_0$, and it follows that $C_G(P) = C_{S_0}(P)$. Now, $C_{S_0}(P) \leq C_{S_0}(P_0) = Z(P_0)$ because P_0 is \mathcal{F}_0 -centric. Therefore, $C_G(P) = C_{Z(P_0)}(P) = Z(P_0)^P$. \square

Lemmas 5.13 and 5.14 state the main properties of the fusion system \mathcal{F} .

Proof of Lemma 5.13. (a) Clearly \mathcal{H} is closed to taking supergroups because \mathcal{H}_0 is closed to taking supergroups in S_0 . Since \mathcal{F} is generated by inclusions and restriction of homomorphisms in \mathcal{F}_2 , Lemma 5.11 shows that for any $P, Q \in \mathcal{H}$ the inclusion $\mathcal{F}_2(P, Q) \subseteq \mathcal{F}(P, Q)$ is an equality.

(b) By definition, $P_0 \in \mathcal{H}_0$. By Lemma 5.19, $C_S(P) = Z(P_0)^P \leq P$. Assume that Q is \mathcal{F} -conjugated to P . By part (a) there exists some $\psi \in \mathcal{L}_2(P, Q)$ such that $\pi_2(\psi)(P) = Q$. Parts (a) and (d) of Lemma 5.9 imply that ψ is an isomorphism in \mathcal{L}_2 . From Definition 5.7 it is clear that ψ is an isomorphism in $\mathcal{L}_1(P_0, Q_0)$ and in particular $Q_0 \in \mathcal{H}_0$, namely Q_0 is \mathcal{F}_0 -centric. It follows from Lemma 5.19 that $C_S(Q) = Z(Q_0)^Q \cong Z(P_0)^P$, whence P is \mathcal{F} -centric.

(c) For any $f \in \mathcal{F}(P, Q)$ where $P, Q \in \mathcal{H}$, part (a) implies that $f = \pi_2(\psi)$ for some $\psi \in \mathcal{L}_2(P, Q) \subseteq \mathcal{L}_2(P_0, Q_0)$. The result follows from Lemma 5.9(a) which shows that $f|_{P_0} = \pi_1(\psi)$ whose image is contained in Q_0 by Definition 5.5. \square

Proof of Lemma 5.14. The monomorphisms $\delta_P : P \rightarrow \text{Aut}_{\mathcal{L}_2}(P)$ are the restrictions of the maps $\hat{\delta}_{P,Q} : N_G(P, Q) \rightarrow \mathcal{L}_2(P, Q)$, i.e. $\delta_P(g) = [g : 1_{P_0}]$.

To verify axiom (A) in [6, Definition 1.7], see also 2.4, we need to show that for any $P, Q \in \mathcal{H}$ the set $\pi_2^{-1}(f)$ where $f \in \mathcal{F}(P, Q)$ admit a transitive free action of $C_S(P)$ via $\delta_P : N_S(P) \rightarrow \text{Aut}_{\mathcal{L}_2}(P)$. Note that $\mathcal{F}(P, Q) = \mathcal{F}_2(P, Q)$ by Lemma 5.13. Consider $\psi, \psi' \in \mathcal{L}_2(P, Q) \subseteq \mathcal{L}_1(P_0, Q_0)$ such that $\pi_2(\psi) = \pi_2(\psi')$. By restriction to P_0 , Lemma 5.9(a) shows that $\pi_1(\psi) = \pi_1(\psi')$. Lemma 5.17(f) shows that there exists $z \in Z(P_0)$ such that $\psi' = \psi \circ \hat{z}$ in \mathcal{L}_1 . Note that $\hat{z} \in \text{Aut}_{\mathcal{L}_2}(P_0)$ by Definition 5.5 so the equality $\psi' = \psi \circ \hat{z}$ also holds in \mathcal{L}_2 . Furthermore, Lemma 5.17(c) implies that

$$\pi_2(\psi) = \pi_2(\psi') = \pi_2(\psi \circ \hat{z}) = \pi_2(\psi) \circ c_z.$$

As a consequence $z \in C_S(P)$ and we conclude that $C_S(P)$ acts transitively on the fibres of $\pi_2 : \mathcal{L}_2(P, Q) \rightarrow \mathcal{F}(P, Q)$. The action is free by Lemma 5.19 and the uniqueness assertion in Lemma 5.17(e).

Axiom (B) holds by Lemma 5.9(c). To verify axiom (C) we fix a morphism $\psi \in \mathcal{L}_2(P, Q)$ and an element $g \in P$. Set $f = \pi_2(\psi) \in \mathcal{F}(P, Q)$. By the definition of the morphisms in \mathcal{L}_2 , see Lemma 5.9(a) we have $\psi \circ \hat{g} = \widehat{f(g)} \circ \psi$, which is what we need. \square

Notation. We shall write $P \simeq_{\mathcal{F}} Q$ for the statement that $P, Q \leq S$ are \mathcal{F} -conjugate.

The next step is to prove Lemma 5.15 which shows that \mathcal{F} is a saturated fusion system.

Clearly S_0 acts on \mathcal{H}_0 by conjugation and $[P_0]_{S_0}$ denotes the orbit of P_0 , i.e. the conjugacy class of P_0 in S_0 . Since G acts via fusion preserving automorphisms, it acts on the set $\mathcal{H}_0/\mathcal{F}_0$ of the \mathcal{F}_0 -conjugacy classes of the subgroups $P_0 \in \mathcal{H}_0$ which we denote $[P_0]_{\mathcal{F}_0}$. The stabilizer of $[P_0]_{\mathcal{F}_0}$ under this action of G is denoted, as usual, by $G_{[P_0]_{\mathcal{F}_0}}$. Now, $G_{[P_0]_{\mathcal{F}_0}}$ acts on the set $[P_0]_{\mathcal{F}_0} \subseteq \mathcal{H}_0$. Clearly, $S_0 \leq G_{[P_0]_{\mathcal{F}_0}}$ because $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$. Moreover, since $S_0 \triangleleft G$, this action induces an action of $G_{[P_0]_{\mathcal{F}_0}}$ on the set \mathcal{P} of all the S_0 -conjugacy classes of the subgroups of S_0 that are \mathcal{F}_0 -conjugate to P_0 .

5.20. Lemma. *For every $P \in \mathcal{H}$ there exist $\bar{P}, P' \in \mathcal{H}$ such that:*

- (a) $\bar{P} = {}^aP$ for some $a \in G$ and $\bar{P} \simeq_{\mathcal{F}} P'$, whence $P \simeq_{\mathcal{F}} P'$, and
- (b) P'_0 is fully \mathcal{F}_0 -normalized and $P'_0 \simeq_{\mathcal{F}_0} \bar{P}_0$.

In addition, $\bar{S} := N_S(P'_0)S_0$ is a Sylow p -subgroup of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ and \bar{S}/S_0 fixes the S_0 -conjugacy class $[P'_0]_{S_0}$.

Proof. The argument follows the one in the proof of step 3 in [3, Theorem 4.6].

Clearly $S_0 \cdot P \leq G_{[P_0]_{\mathcal{F}_0}}$ because $P \leq N_G(P_0)$ and $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$. Choose $S' \in \text{Syl}_p(G_{[P_0]_{\mathcal{F}_0}})$ which contains $S_0 \cdot P$. By Sylow's theorems, there exists some $a \in G$ such that $S' = G_{[P_0]_{\mathcal{F}_0}} \cap S^a$. Set $\bar{P} = {}^aP$ and observe that

$$\bar{P} = {}^aP \leq {}^a(G_{[P_0]_{\mathcal{F}_0}} \cap S^a) \leq S.$$

Also $\bar{P}_0 = {}^aP_0 \in \mathcal{H}_0$, so $\bar{P} \in \mathcal{H}$. In addition, $G_{[\bar{P}_0]_{\mathcal{F}_0}} = {}^a(G_{[P_0]_{\mathcal{F}_0}})$. It follows that

$$\bar{S} := S \cap G_{[\bar{P}_0]_{\mathcal{F}_0}} = {}^a(S') \in \text{Syl}_p(G_{[\bar{P}_0]_{\mathcal{F}_0}}).$$

Consider now the set \mathcal{P}_{f_n} of all the S_0 -conjugacy classes of the fully \mathcal{F}_0 -normalized subgroups $R \leq S_0$ which are \mathcal{F}_0 -conjugate to \bar{P}_0 . Since G normalizes S_0 and it is fusion preserving, it carries fully \mathcal{F}_0 -normalized subgroups of S_0 to ones, and therefore $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ acts on \mathcal{P}_{f_n} .

We now restrict the action of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ on \mathcal{P}_{f_n} to \bar{S} . By [3, Proposition 1.16] we know that $|\mathcal{P}_{f_n}| \not\equiv 0 \pmod p$. Therefore \bar{S}/S_0 must have some fixed point $[R_0]_{S_0}$. Thus, R_0 is fully \mathcal{F}_0 -normalized and is \mathcal{F}_0 -conjugate to \bar{P}_0 . For every $g \in \bar{S} \leq S$ we have $gR_0g^{-1} \simeq_{S_0} R_0$ so $\bar{S} \leq N_S(R_0)S_0$. On the other hand $S_0N_S(R_0) \leq G_{[R_0]_{\mathcal{F}_0}} = G_{[\bar{P}_0]_{\mathcal{F}_0}}$ and \bar{S} is a Sylow p -subgroup of the latter group, hence

$$\bar{S} = S_0 \cdot N_S(R_0).$$

It remains to find some $P' \in \mathcal{H}$ such that $P' \simeq_{\mathcal{F}} \bar{P}$ and such that $P'_0 = R_0$. Now, since $\bar{P} \leq \bar{S}$, it must stabilize $[R_0]_{S_0}$. We conclude that \bar{P}/\bar{P}_0 acts on

$$X := \{[f] \in \text{Rep}_{\mathcal{F}_0}(\bar{P}_0, S_0) : \text{Im } f \text{ is } S_0\text{-conjugate to } R_0\}$$

via $[f_0] \mapsto [c_g \circ f_0 \circ c_g^{-1}]$. Clearly X is not empty because by construction $\bar{P}_0 \simeq_{\mathcal{F}_0} R_0$. Choose some $f \in \mathcal{F}_0(\bar{P}_0, R_0)$. Then every element of X has the form $[\alpha \circ f]$ for some $\alpha \in \text{Aut}_{\mathcal{F}_0}(R_0)$. Moreover $[\alpha \circ f] = [\beta \circ f]$ if and only if $\alpha^{-1}\beta \in \text{Aut}_{S_0}(R_0)$. Therefore

$$|X| = \frac{|\text{Aut}_{\mathcal{F}_0}(R_0)|}{|\text{Aut}_{S_0}(R_0)|} \not\equiv 0 \pmod p$$

because R_0 is fully \mathcal{F}_0 -normalized. Since \bar{P} is a finite p -group, there is some $[f_0] \in X^{\bar{P}}$ where $f_0 \in \mathcal{F}_0(\bar{P}_0, S_0)$ and $\text{Im } f_0 = R_0$. Let $\psi_0 \in \mathcal{L}_0(\bar{P}_0, S_0)$ be a lift of f_0 .

Recall from Lemma 5.6(a) that we may consider ψ_0 as a morphism in $\mathcal{L}_1(\bar{P}_0, S_0)$ via an inclusion $\mathcal{L}_0 \subseteq \mathcal{L}_1$. Fix some $x \in \bar{P}$. Since \bar{P} fixes $[f_0]$, there exists some $s \in S_0$ such that

$$c_x^{-1} \circ f_0 \circ c_x = c_s \circ f_0.$$

Lifting to \mathcal{L}_0 and using hypothesis (1), we see that there exists a unique $z \in C_{S_0}(\bar{P}_0) = Z(\bar{P}_0)$ such that

$$(1) \quad x^{-1}\psi_0x = \hat{s} \circ \psi_0 \circ \hat{z} = \widehat{sf_0(z)} \circ \psi_0 \quad \text{in } \mathcal{L}_0.$$

Set $y := xsf_0(z)$ and note that $y \in \bar{P} \cdot S_0 \cdot Z(R_0) \leq S$. Lemma 5.6(c), Eq. (1) and the properties of S -systems of lifts (see Definition 2.5) imply that

$$\psi_0 \circ \hat{x} = \hat{x} \circ (x^{-1} \psi_0 x) = \hat{x} \circ \widehat{sf_0(z)} \circ \psi_0 = \hat{y} \circ \psi_0.$$

Therefore, by definition, $\psi_0 \in \mathcal{L}_2(\bar{P}, S)$. Consider $f = \pi_2(\psi_0) \in \mathcal{F}(\bar{P}, S)$ and set $P' = f(\bar{P})$. By Lemmas 5.13(a) and 5.11(b), f restricts to an isomorphism $f : \bar{P} \rightarrow P'$ in \mathcal{F} . By Lemmas 5.9(a) and 5.6(a) we see that $f|_{\bar{P}_0} = \pi_0(\psi_0) = f_0 \in \mathcal{F}_0(\bar{P}_0, R_0)$. Since $f \in \mathcal{F}(\bar{P}, P')$ is an isomorphism we deduce from Lemma 5.13(c) that $P'_0 = f(\bar{P}_0) = R_0$. This completes the proof since f is an \mathcal{F} -isomorphism between \bar{P} and P' which restricts to an \mathcal{F}_0 -isomorphism f_0 between \bar{P}_0 and $R_0 = P'_0$. \square

5.21. Lemma. *If $P \leq S$ is \mathcal{F} -centric but $P \notin \mathcal{H}$, then there exists $P' \leq S$ which is \mathcal{F} -conjugate to P such that*

$$\text{Out}_S(P') \cap O_p(\text{Out}_{\mathcal{F}}(P')) \neq 1.$$

Proof. The argument is almost repeated from step 4 in the proof of [3, Theorem 4.6] if we find a subgroup $\hat{P} \leq S$ which is \mathcal{F} -conjugate to P and such that \hat{P}_0 does not contain its S_0 -centralizer.

Assume to the contrary that there is some P which is \mathcal{F} -centric, P_0 is not \mathcal{F}_0 -centric, and for which there does not exist \hat{P} as above. Choose $P \leq S$ so that P_0 has the maximal possible order. If $N_{S_0}(P_0)$ is \mathcal{F}_0 -centric we choose $\tilde{\varphi} \in \mathcal{L}_0(N_{S_0}(P_0), S_0)$ as in hypothesis (5) of Theorem 5.2. Then $\tilde{\varphi}$ represents a morphism $N_S(P_0) \rightarrow S$ in \mathcal{L}_2 because for any $x \in N_S(P_0)$

$$\tilde{\varphi} \circ \hat{x} = [1 : \tilde{\varphi}] \circ [x : 1] = [x : x^{-1} \tilde{\varphi} x] = [x : \hat{s} \circ \hat{\varphi}] = [xs : \tilde{\varphi}] = [xs : 1] \circ [1 : \tilde{\varphi}] = \widehat{xs} \circ \tilde{\varphi}.$$

Thus, $\pi_2(\tilde{\varphi})$ is a morphism $f : N_S(P_0) \rightarrow S$ in \mathcal{F} whose restriction to P gives rise to an \mathcal{F} -conjugate \hat{P} and by Lemmas 5.13(c) and 5.9(a), $\hat{P}_0 = f(P_0)$ does not contain its S_0 -centralizer. This is a contradiction, hence $N_{S_0}(P_0)$ cannot be \mathcal{F}_0 -centric.

Now, P must normalize $N_{S_0}(P_0)$ and we consider $Q := PN_{S_0}(P_0)$. Clearly $Q_0 = N_{S_0}(P_0)$. By the maximality of $|P_0|$ we deduce that Q is \mathcal{F} -conjugate to some \hat{Q} such that \hat{Q}_0 does not contain its S_0 -centralizer. By restriction to $P \leq Q$ we see that P is \mathcal{F} -conjugate to some $\hat{P} \leq \hat{Q}$ and \hat{P}_0 cannot contain its S_0 -centralizer because $\hat{Q}_0 \geq \hat{P}_0$ does not contain its S_0 -centralizer. This is again a contradiction.

Finally, our notation was chosen in such a way that the argument in step 4 in [3, proof of Theorem 4.6] can be now read verbatim to complete the proof. \square

Proof of 5.15. By [4, Theorem 2.2] and Lemma 5.21, \mathcal{F} is saturated if the saturation axioms of Definition 2.2 hold for all subgroups in \mathcal{H} . To show this, we slightly modify the argument in [3, Theorem 4.6].

Condition I. Fix $P \in \mathcal{H}$ which is fully \mathcal{F} -normalized. We have to show that it is fully \mathcal{F} -centralized and that $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$. By Lemma 5.13(b) we know that P is \mathcal{F} -centric and in particular fully \mathcal{F} -centralized.

Consider \bar{P} and P' as in Lemma 5.20. Recall that $\bar{S} = N_S(P'_0)S_0$ is a Sylow p -subgroup of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$. Lemma 5.6(a) shows that $\text{Aut}_{\mathcal{L}_0}(\bar{P}_0) \leq \text{Aut}_{\mathcal{L}_1}(\bar{P}_0)$ and by Lemmas 5.17(f), 5.11 and 5.13

$$|\text{Aut}_{\mathcal{L}_1}(P'_0) : \text{Aut}_{\mathcal{L}_0}(P'_0)| = |G_{[\bar{P}_0]_{\mathcal{F}_0}} : S_0|. \tag{5.22}$$

By definition $N_{S_0}(P'_0) = S_0 \cap N_S(P'_0)$ so

$$|N_S(P'_0)/N_{S_0}(P'_0)| = |N_S(P'_0)S_0/S_0| = |\bar{S}/S_0|. \tag{5.23}$$

Now, P'_0 is fully \mathcal{F}_0 -normalized and is \mathcal{F}_0 -centric so

$$|\text{Aut}_{\mathcal{L}_0}(P'_0) : N_{S_0}(P'_0)| \neq 0 \pmod p. \tag{5.24}$$

Since $|G_{[P'_0]_{\mathcal{F}_0}} : \bar{S}| \neq 0 \pmod p$, we deduce from (5.22), (5.23) and (5.24) that

$$|\text{Aut}_{\mathcal{L}_1}(P'_0) : N_S(P'_0)| = \frac{|\text{Aut}_{\mathcal{L}_1}(P'_0)|}{|\text{Aut}_{\mathcal{L}_0}(P'_0)|} \cdot \frac{|\text{Aut}_{\mathcal{L}_0}(P'_0)|}{|N_{S_0}(P'_0)|} \cdot \frac{|N_{S_0}(P'_0)|}{|N_S(P'_0)|} \neq 0 \pmod p,$$

namely $N_S(P'_0) \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_1}(P'_0))$.

Fix $\psi \in \text{Aut}_{\mathcal{L}_1}(P'_0)$ such that

$$\psi^{-1}N_S(P'_0)\psi \supseteq R \in \text{Syl}_p(N_{\text{Aut}_{\mathcal{L}_1}(P'_0)}(P'_0)) \tag{5.25}$$

and set

$$P'' = \psi P' \psi^{-1} \leq N_S(P'_0).$$

Lemma 5.18 shows that $P'_0 = P''_0$ and that $\psi \in \mathcal{L}_2(P', P'')$ is an isomorphism. In particular, P'' is \mathcal{F} -conjugate to P' , hence also to P because $P' = {}^aP$ for some $a \in G$ and $\hat{a} \in \mathcal{L}_2(P, P')$ is an isomorphism. We now claim that

$$(i) \quad \text{Aut}_{\mathcal{L}_2}(P'') = N_{\text{Aut}_{\mathcal{L}_1}(P'_0)}(P'') \quad \text{and} \quad (ii) \quad N_S(P'') = N_{N_S(P'_0)}(P'').$$

Clearly (i) follows from the definition of the morphisms in \mathcal{L}_2 because

$$\begin{aligned} \lambda \in \text{Aut}_{\mathcal{L}_2}(P'') &\iff \forall x \in P'' \exists y \in P'' (\lambda \circ \hat{x} \circ \lambda^{-1} = \hat{y}) \\ &\iff \lambda \in N_{\text{Aut}_{\mathcal{L}_1}(P'_0)}(P''). \end{aligned}$$

For (ii), note that $P'' \subseteq N_S(P'_0) \subseteq \text{Aut}_{\mathcal{L}_1}(P'_0)$ so by the choice of ψ in Eq. (5.25),

$$N_{N_S(P'_0)}(P'') = N_S(P'_0) \cap N_{\text{Aut}_{\mathcal{L}_1}(P'_0)}(P'') \in \text{Syl}_p(N_{\text{Aut}_{\mathcal{L}_1}(P'_0)}(P'')).$$

On the other hand,

$$N_{N_S(P'_0)}(P'') \leq N_S(P'') \leq N_{\text{Aut}_{\mathcal{L}_1}(P'_0)}(P''),$$

hence $N_S(P'') = N_{N_S(P_0)}(P'')$. We deduce that $N_S(P'') \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_2}(P''))$. Finally, $\text{Aut}_{\mathcal{L}_2}(P) \cong \text{Aut}_{\mathcal{L}_2}(P'')$ because P'' and P are isomorphic in \mathcal{L}_2 (via $\psi \circ \hat{a}$). Also, $|N_S(P)| \geq |N_S(P'')|$ because P is fully \mathcal{F} -normalized. Therefore $N_S(P) \in \text{Syl}_p(\text{Aut}_{\mathcal{L}_2}(P))$ and Lemma 5.14 implies that $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$.

Condition II. Fix $P \in \mathcal{H}$ and $f \in \mathcal{F}(P, S)$. Parts (a) and (b) of Lemma 5.13 show that $f(P) \in \mathcal{H}$ and that $f(P)$ is \mathcal{F} -centric and in particular it is fully \mathcal{F} -centralized. We have to prove that f extends to some morphism $N_f \rightarrow S$ in \mathcal{F} where

$$N_f = \{g \in N_S(P) : f \circ c_g = c_s \circ f \text{ for some } s \in S\}.$$

Note that s in the definition of N_f belongs to $N_S(\text{Im } f)$. Set $Q = N_f$. We observe that

$$Q \leq N_S(Q_0) \quad \text{and} \quad Q \leq N_S(P) \leq N_S(P_0). \tag{5.26}$$

By construction of \mathcal{F}_2 , there exists $\varphi \in \mathcal{L}_2(P, S)$ such that $f = \pi_2(\varphi)$. Now φ in a morphism in $\mathcal{L}_1(P_0, S_0)$ and we write $\varphi = [g : \varphi_0]$. By definition of $Q = N_f$, for any $q \in Q$ there exists $t \in S$ such that $f \circ c_q = c_t \circ f$. Lemma 5.13(c) and (5.26) imply that $f_0 \circ c_q = c_t \circ f_0$ where $f_0 : P_0 \rightarrow S_0$ is the restriction of f . By Lemma 5.17(b), $f_0 = \pi_1(\varphi)$ so part (e) of that lemma implies that $\varphi \circ \hat{q} = \hat{t} \circ \varphi \circ \hat{z}$ for some $z \in Z(P_0)$. Part (b) of that lemma applies again to show that $\varphi \circ \hat{q} = \widehat{s_q} \circ \varphi$ for some $s_q \in S$.

Now, if $q \in Q_0$ then $\varphi \circ \hat{q} = [gq : q^{-1}\varphi_0q]$ and $\widehat{s_q} \circ \varphi = [s_qg : \varphi_0]$. Therefore there is $s \in S_0$ such that

$$gqs = s_qg \quad \text{and} \quad \widehat{s} \circ \varphi_0 = \widehat{q^{-1}} \circ \varphi_0 \circ \hat{q}.$$

In particular $s_q \in S_0$ and $\pi_0(\varphi_0) \circ c_q = c_{q_s} \circ \pi_0(\varphi_0)$. This shows that $Q_0 \subseteq N_{\pi_0(\varphi_0)}$ and we may extend φ_0 to some $\psi_0 \in \mathcal{L}_0(Q_0, S_0)$ because $Q_0 \supseteq P_0$ which is \mathcal{F}_0 -centric. Define $\psi = [g : \psi_0]$ and note that $\varphi = \psi \circ [1 : \iota_{P_0}^{Q_0}]$. From (5.26), for any $q \in Q$

$$\psi \circ \hat{q} \circ [1 : \iota_{P_0}^{Q_0}] = \psi \circ [1 : \iota_{P_0}^{Q_0}] \circ \hat{q} = \varphi \circ \hat{q} = \widehat{s_q} \circ \varphi = \widehat{s_q} \circ \psi \circ [1 : \iota_{P_0}^{Q_0}].$$

Since $[1 : \iota_{P_0}^{Q_0}]$ is an epimorphism in \mathcal{L}_1 by Lemma 5.17(c), we deduce that $\psi \in \mathcal{L}_2(Q, S)$. Finally, $f = \pi_2(\varphi) = \pi_2(\psi) \circ \text{incl}_P^Q$. This completes the proof. \square

Proof of Lemma 5.16. Our notation was chosen in such a way that the argument in [3, Theorem 4.6, step 7] can be read verbatim and we shall therefore avoid reproducing it. \square

6. Maps from a homotopy colimit

Let \mathcal{C} be a small category, and $X : \mathcal{C} \rightarrow \mathbf{Top}$ be a diagram of spaces over \mathcal{C} . The values taken by the functor will be denoted by $X(c)$ and $X(\varphi)$ where $c \in \mathcal{C}$, $\varphi \in \text{Mor}_{\mathcal{C}}(c, c')$. The homotopy colimit of the diagram X is the space

$$\text{hocolim}_{\mathcal{C}} X = \left(\coprod_{n \geq 0} \coprod_{c_0 \rightarrow \dots \rightarrow c_n} X(c_0) \times \Delta^n \right) / \sim$$

where we divide by the usual face and degeneracy identifications [2, Chapter XII].

There is a filtration of $\text{hocolim}_{\mathcal{C}} X$ by spaces $F_n X$ where $F_n X$ is the image of the union of $X(c) \times \Delta^m$ in $\text{hocolim}_{\mathcal{C}} X$ for all $m \leq n$. Notice that $F_0 X$ is just $\coprod_{c \in \mathcal{C}} X(c)$ and $F_1 X$ is the union of the mapping cylinders of all $\varphi \in \text{Mor}(\mathcal{C})$. Observe that a map $f_1 : F_1 X \rightarrow Y$ is the same as a set of maps $f_1(c) : X(c) \rightarrow Y$ together with homotopies $f_1(c') \circ X(\varphi) \simeq f_1(c)$ for every $\varphi \in \mathcal{C}(c, c')$. Equivalently, these are paths $f(c) \rightsquigarrow f_1(c') \circ X(\varphi)$ in $\text{map}^{f(c)}(X(c), Y)$. A set of maps $X(-) \xrightarrow{f(-)} Y$ which admits such homotopies is called a *system of homotopy compatible maps* and it gives rise to an element in the set $\varprojlim_{\mathcal{C}} [X(c), Y]$.

Fix a system of homotopy compatible maps $X(-) \xrightarrow{f(-)} Y$. By the remark above it gives rise to a map $f_1 : F_1 X \rightarrow Y$ where $f_1|_{X(c)} = f(c)$. Wojtkowiak [26] addressed the question whether f_1 can be extended, up to homotopy, to a map $\tilde{f} : \text{hocolim}_{\mathcal{C}} X \rightarrow Y$. The method is to extend f_1 by induction on the spaces $F_n X$.

Given a map $\tilde{f}_n : F_n X \rightarrow Y$ whose restriction to $X(c)$ is homotopic to $f(c)$, Wojtkowiak developed an obstruction theory for extending it to $F_{n+1} X$ without changing it on $F_{n-1} X$. The existence of such an extension depends on the vanishing of a certain obstruction class in $\varprojlim_{\mathcal{C}}^{n+1} \pi_n(\text{map}^{f(c)}(X(c), Y))$. The extension from $F_1 X$ to $F_2 X$ involves in general a functor into the category of groups and representations, whose \varprojlim^2 term is described in Wojtkowiak’s work. Fortunately, if these groups are abelian then Wojtkowiak’s definition of \varprojlim^2 coincides with the usual one from homological algebra. Once the map has been extended to $F_2 X$, there are homotopies between the paths $f_1(c) \rightsquigarrow f_1(c'') \circ X(\psi \circ \varphi)$ and $f(c) \rightsquigarrow f_1(c') \circ X(\varphi) \rightsquigarrow f_1(c'') \circ X(\psi) \circ X(\varphi)$ for all $c \xrightarrow{\varphi} c' \xrightarrow{\psi} c''$. Thanks to these homotopies there are functors $c \mapsto \pi_n(\text{map}^{f(c)}(X(c), Y))$ into \mathbf{Ab} for all $n > 1$.

Given two maps $\tilde{f}_1, \tilde{f}_2 : \text{hocolim}_{\mathcal{C}} X \rightarrow Y$ whose restrictions to $X(c)$ are homotopic to $f(c)$ for all $c \in \mathcal{C}$, Wojtkowiak also studies an obstruction theory for the construction of a homotopy $\tilde{f}_1 \simeq \tilde{f}_2$. Clearly, \tilde{f}_1 and \tilde{f}_2 give rise to a homotopy $\tilde{f}_1|_{F_0 X} \xrightarrow{H_0} \tilde{f}_2|_{F_0 X}$. The idea is to extend the homotopy H_0 inductively to $I \times F_n X$. Given a homotopy $\tilde{f}_1|_{F_{n-1} X} \xrightarrow{H_{n-1}} \tilde{f}_2|_{F_{n-1} X}$, the possibility of extending it to a homotopy $\tilde{f}_1|_{F_n X} \xrightarrow{H_n} \tilde{f}_2|_{F_n X}$ without changing it on $F_{n-2} X$, depends on the vanishing of an obstruction class in $\varprojlim_{\mathcal{C}}^n \pi_n(\text{map}^{f(c)}(X(c), Y))$.

6.1. Definition. (See [6, Definition 3.3].) Fix a prime p . A small category \mathcal{C} has *p -height d* if for every functor $F : \mathcal{C} \rightarrow \mathbb{Z}_{(p)}\text{-mod}$ the groups $\varprojlim_{\mathcal{C}}^i F$ vanish for all $i > d$. The p -height of \mathcal{C} is infinite if no such d exists and it is finite otherwise.

6.2. Theorem. Let \mathcal{C} be a finite category of p -height $d < \infty$. Consider a sequence of maps $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \dots \xrightarrow{g_d} Y_{d+1}$ and let $y_i = g_i \circ \dots \circ g_0 : Y_0 \rightarrow Y_{i+1}$. Given a functor $X : \mathcal{C} \rightarrow \mathbf{Top}$ and a system of homotopy compatible maps $f(-) : X(-) \rightarrow Y_0$, define new systems of homotopy compatible maps $f_i(-) = y_i \circ f(-) : X(-) \rightarrow Y_{i+1}$ for all $i = 0, \dots, d$. Assume that

(i) For every $c \in \mathcal{C}$ and every $i = 1, \dots, d$ the induced map

$$\pi_i \text{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \pi_i \text{map}^{f_i(c)}(X(c), Y_{i+1})$$

is the trivial homomorphism between abelian groups.

(ii) The groups $\pi_{* > 0} \text{map}^{f_i(c)}(X(c), Y_i)$ are $\mathbb{Z}_{(p)}$ -modules for all $c \in \mathcal{C}$ and all i .

Then

- (a) There exists a map $\tilde{f} : \text{hocolim}_{\mathcal{C}} X \rightarrow Y_d$ which renders the following square homotopy commutative for all $c \in \mathcal{C}$,

$$\begin{array}{ccc}
 X(c) & \xrightarrow{f(c)} & Y_0 \\
 \iota(c) \downarrow & & \downarrow y_{d-1} \\
 \text{hocolim}_{\mathcal{C}} X & \xrightarrow{\tilde{f}} & Y_d
 \end{array}$$

- (b) If $\tilde{f}_1, \tilde{f}_2 : \text{hocolim}_{\mathcal{C}} X \rightarrow Y_0$ satisfy $\tilde{f}_1|_{X(c)} \simeq \tilde{f}_2|_{X(c)} \simeq f(c)$ for all $c \in \mathcal{C}$ then the composites $\text{hocolim}_{\mathcal{C}} X \xrightarrow{\tilde{f}_1, \tilde{f}_2} Y_0 \xrightarrow{y_d} Y_{d+1}$ are homotopic.

Proof. (a) We will define by induction maps $\tilde{f}_i : F_i X \rightarrow Y_i$ for all $i = 1, \dots, d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{i-1}(c)$ for all $c \in \mathcal{C}$.

Note that, by definition of a system of homotopy compatible maps, we can construct a map $\tilde{f}_0 : F_1 X \rightarrow Y_0$. Let $\tilde{f}_1 = g_0 \circ \tilde{f}_0$. Assume by induction that $\tilde{f}_i : F_i X \rightarrow Y_i$ with $\tilde{f}_i|_{X(c)} \simeq f_{i-1}$ has been constructed for some $1 \leq i < d$. The obstruction class Θ'_{i+1} for the extension of \tilde{f}_i to $F_{i+1} X$ is mapped by the homomorphism

$$\varinjlim_{\mathcal{C}^{\text{op}}}^{i+1} \pi_i \text{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \varinjlim_{\mathcal{C}^{\text{op}}}^{i+1} \pi_i \text{map}^{f_i(c)}(X(c), Y_{i+1})$$

to the obstruction class Θ_{i+1} for the extension of $g_i \circ \tilde{f}_i$ to $F_{i+1} X$. When $i \geq 1$, by hypothesis (i) the groups are abelian and this homomorphism is trivial, whence $\Theta_{i+1} = 0$. Wojtkowiak’s obstruction theory guarantees the existence of a map $\tilde{f}_{i+1} : F_{i+1} X \rightarrow Y_{i+1}$ which agrees with $g_i \circ \tilde{f}_i$ on $F_{i-1} X$ and such that $\tilde{f}_{i+1}|_{X(c)} \simeq g_i \circ f_{i-1}(c) = f_i(c)$. This completes the induction step.

Hypothesis (ii) and the assumption on \mathcal{C} imply that the groups

$$\varinjlim_{\mathcal{C}^{\text{op}}}^i \pi_{i-1} \text{map}^{f_{d-1}}(X(c), Y_d)$$

are trivial for all $i \geq d + 1$. Thus, the obstructions to the extension of \tilde{f}_d to $F_i X$ where $i > d$ must all vanish. We can therefore construct by induction on $i \geq d + 1$ maps $\tilde{f}_i : F_i X \rightarrow Y_d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{d-1}(c)$ for all $c \in \mathcal{C}$ and such that \tilde{f}_{i+1} agrees with \tilde{f}_i on $F_{i-1} X$. We can finally define $\tilde{f} : \text{hocolim}_{\mathcal{C}} X = \bigcup_i F_i X \rightarrow Y_d$ with the required properties. In fact, $\tilde{f}|_{F_n X} = \tilde{f}_{n+1}|_{F_n X}$ for all $n > d$.

(b) First, we construct by induction homotopies $y_i \circ \tilde{f}_1|_{F_i X} \xrightarrow{H_i} y_i \circ \tilde{f}_2|_{F_i X}$ for all $i = 0, \dots, d$. Recall that $F_0 X = \coprod_{c \in \mathcal{C}} X(c)$ and we define H_0 as the sum of the homotopies $y_0 \circ \tilde{f}_1|_{X(c)} \simeq y_0 \circ \tilde{f}_2|_{X(c)}$.

Assume by induction that $H_i : y_i \circ \tilde{f}_1|_{F_i X} \simeq y_i \circ \tilde{f}_2|_{F_i X}$ has been constructed where $0 \leq i < d$. The obstruction \mathcal{T}'_i for the extension of H_i to a homotopy $y_i \circ \tilde{f}_1|_{F_{i+1} X} \simeq y_i \circ \tilde{f}_2|_{F_{i+1} X}$ is mapped by the homomorphism

$$\varprojlim_{\mathcal{C}^{op}}^{i+1} \pi_{i+1} \text{map}^{f_i(c)}(X(c), Y_{i+1}) \xrightarrow{(g_{i+1})^*} \varprojlim_{\mathcal{C}^{op}}^{i+1} \pi_{i+1} \text{map}^{f_{i+1}(c)}(X(c), Y_{i+2})$$

to the obstruction class \mathcal{T}_i for the extension of $g_{i+1} \circ H_i : I \times F_i X \rightarrow Y_{i+2}$ to $I \times F_{i+1} X$. This homomorphism is trivial by hypothesis (i). Therefore $\mathcal{T}_i = 0$, and by Wojtkowiak’s theory there is a homotopy $y_{i+1} \circ \tilde{f}_1|_{F_{i+1} X} \xrightarrow{H_{i+1}} y_{i+1} \circ \tilde{f}_2|_{F_{i+1} X}$. This completes the induction step.

Now, the hypothesis on \mathcal{C} together with (ii) imply that the groups

$$\varprojlim_{\mathcal{C}^{op}}^i \pi_i \text{map}^{f_d(c)}(X(c), Y_{d+1})$$

are trivial for all $i \geq d + 1$. We can therefore construct by induction on $i \geq d + 1$ homotopies $y_d \circ \tilde{f}_1|_{F_i X} \xrightarrow{H_i} y_d \circ \tilde{f}_2|_{F_i X}$ such that H_{i+1} and H_i agree on $I \times F_{i-1} X$. There results a homotopy $y_d \circ \tilde{f}_1 \simeq y_d \circ \tilde{f}_2$. \square

7. Maps between p -local finite groups

7.1. Definition. Let (S, \mathcal{F}) be a fusion system. A map $f : BS \rightarrow X$ is called \mathcal{F} -invariant, if for every $\varphi \in \mathcal{F}(P, S)$ the composite $BP \xrightarrow{B\varphi} BS \xrightarrow{f} X$ is homotopic to $f|_{BP} = f \circ B \text{incl}_P^S$.

7.2. Example. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. The map $\Theta : BS \rightarrow |\mathcal{L}|$ of 2.9 is \mathcal{F} -invariant by Proposition 2.10.

Given a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, the question we address in this section is when an \mathcal{F} -invariant map $f : BS \rightarrow X$ can be extended to a map $|\mathcal{L}| \rightarrow X$. Here is the main result of this section which uses the constructions in Section 3.

7.3. Theorem. Let $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ be p -local finite groups and consider an \mathcal{F} -invariant map $f : BS \rightarrow |\mathcal{L}'|_p^\wedge$. Then:

- (a) There exists $m \geq 0$ and a map $\tilde{f} : |\mathcal{L}| \rightarrow (|\mathcal{L}'| \wr \Sigma_{p^m})_p^\wedge$ which renders the following square homotopy commutative

$$\begin{array}{ccc} BS & \xrightarrow{f} & |\mathcal{L}'|_p^\wedge \\ \Theta \downarrow & & \downarrow \Delta_p^\wedge \\ |\mathcal{L}| & \xrightarrow{\tilde{f}} & (|\mathcal{L}'| \wr \Sigma_{p^m})_p^\wedge \end{array}$$

- (b) There exists $e > 0$ such that for any two maps $\tilde{f}_1, \tilde{f}_2 : |\mathcal{L}| \rightarrow |\mathcal{L}'|_p^\wedge$ with $\tilde{f}_1 \circ \Theta \simeq \tilde{f}_2 \circ \Theta \simeq f$, the composites $|\mathcal{L}| \xrightarrow{\tilde{f}_1, \tilde{f}_2} |\mathcal{L}'|_p^\wedge \xrightarrow{\Delta_p^\wedge} (|\mathcal{L}'| \wr \Sigma_{p^e})_p^\wedge$ are homotopic.

7.4. Example. If $f = \Theta : BS \rightarrow |\mathcal{L}|$ then \tilde{f} can be chosen as the identity on $|\mathcal{L}|_p^\wedge$.

The main tool for proving Theorem 7.3 is Theorem 6.2 but we will need some preliminary facts about the homotopy groups of mapping spaces and p -completion of wreath products of spaces.

Consider a group G . Its abelianization is denoted by G_{ab} . Its maximal p -perfect subgroup [2, Chapter VII.3] is denoted by $O^p(G)$. This is the maximal subgroup of K such that $H_1(K; \mathbb{F}_p) = 0$. It is clearly characteristic in G . It is also clear that $O^p(G)$ contains every element of G with finite order prime to p . In particular, if G is finite then $G/O^p(G)$ is a finite p -group. For a finite abelian group A , set $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$; this is the set of p -power order elements in A .

7.5. Proposition. *Set $H = G \wr \Sigma_k$ for some group G . If either $p > 2$ and $k \geq 2$ or if $p = 2$ and $k \geq 3$ then $H/O^p(H)$ is an abelian p -group. In particular, if G is finite then $H/O^p(H)$ is a finite abelian p -group.*

Proof. Note that H contains $G^{\times k}$ as a normal subgroup and we will write $g_{(i)}$ for the element $(1, \dots, 1, g, 1, \dots, 1) \in H$ with $g \in G$ in the i th position. To prove the result it suffices to show that $O^p(H)$ contains all the elements of the form $g_{(1)}g_{(i)}^{-1}$ for all $i > 1$, all the elements of the form $g_{(1)}$ where $g \in [G, G]$ is a commutator in G and that it contains $A_k \leq \Sigma_k$. Indeed, the quotient of H by the normal subgroup generated by the elements of the first and second type is $G_{\text{ab}} \times \Sigma_k$, so throwing in A_k would guarantee that the quotient $H/O^p(H)$ is abelian.

If $p > 2$ then clearly the involutions $\tau = (1, i) \in \Sigma_k$ belong to $O^p(H)$. Therefore, for any $g \in G$ also $g_{(1)}\tau g_{(1)}^{-1}$ belongs to $O^p(H)$. As a consequence we also have $g_{(1)}\tau g_{(1)}^{-1}\tau^{-1} = g_{(1)}g_{(i)}^{-1} \in O^p(H)$. These are the elements of the first type.

Now, given $a, b \in G$ we observe that

$$\{a_{(1)}^{-1}a_{(2)}\} \cdot \{b_{(1)}^{-1}b_{(2)}\} \cdot \{(ab)_{(1)}(ab)_{(2)}^{-1}\} = (a^{-1}b^{-1}ab)_{(1)},$$

so $O^p(H)$ contains all the elements of the form $g_{(1)}$ with $g \in [G, G]$. Finally, since Σ_k is generated by involutions, Σ_k is also contained in $O^p(H)$. This completes the proof in this case.

If $p = 2$ and $k \geq 3$, consider some $g \in G$ and a 3-cycle $(1, i, j)$. By inspection $[g_{(1)}, \tau] = g_{(1)}g_{(i)}^{-1}$ so $O^p(H)$ contains all the elements of the first type. Also $O^p(H)$ contains all the elements of the second type, namely $g_{(1)} \in H$ where $g \in [G, G]$ by the same argument we used for odd p . Finally, $O^p(H)$ contains all the 3-cycles in Σ_k , whence it contains A_k . \square

7.6. Corollary. *Let X be a p -good space, then $\pi_1((X \wr \Sigma_k)_p^\wedge)$ is abelian if $k \geq 3$. It is a finite abelian p -group if $\pi_1(X_p^\wedge)$ is finite.*

Proof. We may replace X with X_p^\wedge by next Lemma 7.7. Set $Y = X \wr \Sigma_k$ and $\pi = \pi_1 Y$. By Remark 3.6 and Proposition 7.5 we see that $\pi/O^p(\pi)$ is an abelian group and that it is a finite abelian p -group if $\pi_1 X$ is finite. Let $E \rightarrow Y$ be the principal fibration obtained by pulling back the covering map $B(O^p(\pi)) \rightarrow B\pi$ along the first Postnikov section $Y \rightarrow B\pi$. Clearly, $\pi_1 E = O^p(\pi)$ and we obtain a fibre sequence $E \rightarrow Y \rightarrow B(\pi/O^p(\pi))$. By [2, Chapter VII.3.2], E is p -good and E_p^\wedge is simply connected. By fibrewise p -completion [2, Chapter I.8.3], there

is a fibre sequence $E_p^\wedge \rightarrow \dot{Y}_p^\wedge \rightarrow B(\pi/O^p(\pi))$ where $Y \rightarrow \dot{Y}_p^\wedge$ is a mod- p equivalence because E is p -good. We deduce that $\pi_1(\dot{Y}_p^\wedge) = \pi/O^p(\pi)$ which is abelian. The description in [1, Propositions 5.5, 5.6] of the fundamental group of the p -completion of a space implies that $\pi_1(Y_p^\wedge) = \pi_1((\dot{Y}_p^\wedge)_p^\wedge)$ is the p -adic completion of $\pi/O^p(\pi)$ which is also an abelian group. It is an abelian finite p -group if $\pi_1(X)$ is finite (see [1, 5.7(vi)]). \square

7.7. Lemma. *Let X be a p -good space. Then, for any $G \leq \Sigma_n$ the diagram*

$$\begin{array}{ccc}
 X_p^\wedge & \xrightarrow{\Delta(X_p^\wedge)} & X_p^\wedge \wr G \\
 \Delta(X)_p^\wedge \downarrow & & \downarrow \eta \\
 (X \wr G)_p^\wedge & \xrightarrow[\cong]{(\eta \wr G)_p^\wedge} & (X_p^\wedge \wr G)_p^\wedge
 \end{array}$$

is homotopy commutative where $(\eta \wr G)_p^\wedge$ is a homotopy equivalence.

Proof. The first statement follows from the naturality of η and of $-\wr G$. The map $\eta \wr G$ is a mod p homology equivalence by a Serre spectral sequence argument, hence $(\eta \wr G)_p^\wedge$ is a homotopy equivalence by [2, Lemma 1.5.5]. \square

7.8. Proposition. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group, let P be a finite p -group and consider a map $f : BP \rightarrow |\mathcal{L}|_p^\wedge$. Then*

- (a) $\pi_i(|\mathcal{L}|_p^\wedge)$ are finite p -groups for all $i \geq 1$.
- (b) $\pi_i(\text{map}^{\eta \circ \Delta \circ f}(BP, (|\mathcal{L}|_p^\wedge \wr \Sigma_k)_p^\wedge))$ are finite p -groups for all $i \geq 1$ and $k \geq 0$. Moreover, if $k \geq 3$, $\pi_1(\text{map}^{\eta \circ \Delta \circ f}(BP, (|\mathcal{L}|_p^\wedge \wr \Sigma_k)_p^\wedge))$ is abelian.

Proof. (a) The fundamental group $\pi_1(|\mathcal{L}|_p^\wedge)$ is a finite p -group by [3, Theorem B]. Using a Serre class argument (see [23, Chapter 9.6, Theorem 15]), we only need to show that the integral homology is finite at each degree. In [21], it is proven that the suspension spectrum $\Sigma^\infty |\mathcal{L}|_p^\wedge$ is a retract of $\Sigma^\infty BS$, hence its integral homology groups are finite abelian p -groups.

(b) If $S = 1$ then $|\mathcal{L}| = *$ hence $(|\mathcal{L}|_p^\wedge \wr \Sigma_k)_p^\wedge \simeq (B\Sigma_k)_p^\wedge$ and $\eta \circ \Delta \circ f$ is null-homotopic. Dwyer–Zabrodsky’s result [10] shows that the space under study is homotopy equivalent to $(B\Sigma_k)_p^\wedge$ and the result follows from Proposition 7.5 together with [5, Proposition A.2] and part (a).

We now assume that $S \neq 1$. By [6, Theorem 4.4(a)] f is homotopic to

$$BP \xrightarrow{B\rho} BS \xrightarrow{\Theta} |\mathcal{L}| \xrightarrow{\eta} |\mathcal{L}|_p^\wedge$$

for some $\rho : P \rightarrow S$. Denote $f' = \Theta \circ B\rho$. We may, and will, assume that $f = \eta \circ f'$. By Theorem 1 there exists a p -local finite group $(S', \mathcal{F}', \mathcal{L}')$ together with a homotopy equivalence

$\omega : |\mathcal{L}| \wr \Sigma_k \xrightarrow{\cong} |\mathcal{L}'|$. Since $|\mathcal{L}|$ is p -good by [6, Proposition 1.12], Lemma 7.7 now implies that $(|\mathcal{L}| \wr \Sigma_k)_p^\wedge \simeq (|\mathcal{L}'| \wr \Sigma_k)_p^\wedge$. The following diagram

$$\begin{array}{ccc} \text{map}^{\Delta \circ f'}(BP, |\mathcal{L}| \wr \Sigma_k) & \xrightarrow{(\eta_*) \circ (\eta \wr \Sigma_k)_*} & \text{map}^{\eta \circ \Delta \circ f}(BP, (|\mathcal{L}'|_p^\wedge \wr \Sigma_k)_p^\wedge) \\ \omega_* \downarrow \simeq & & \simeq \downarrow (\omega_p^\wedge)_* \\ \text{map}^{\omega \circ \Delta \circ f'}(BP, |\mathcal{L}'|) & \xrightarrow{\eta_*} & \text{map}^{\eta \circ \omega \circ \Delta \circ f'}(BP, |\mathcal{L}'|_p^\wedge) \end{array}$$

is homotopy commutative. By Proposition 2.11(b) both horizontal maps induce, after p -completion, split surjections on all homotopy groups. Moreover note that the spaces at the right-hand side of the diagram are p -complete by Proposition 2.11(a). Therefore it suffices to prove that the homotopy groups of

$$(\text{map}^{\Delta \circ f}(BP, |\mathcal{L}'|_p^\wedge \wr \Sigma_k))_p^\wedge \tag{7.9}$$

are finite p -groups. It follows from Proposition 3.8(b) and Remark 3.6 that

$$\begin{aligned} \pi_1 \text{map}^{\Delta \circ f}(BP, |\mathcal{L}'|_p^\wedge \wr \Sigma_k) &\cong \pi_1(\text{map}^f(BP, |\mathcal{L}'|_p^\wedge) \wr \Sigma_k) \quad \text{and} \\ \pi_i \text{map}^{\Delta \circ f}(BP, |\mathcal{L}'|_p^\wedge \wr \Sigma_k) &\cong \bigoplus_k \pi_i(\text{map}^f(BP, |\mathcal{L}'|_p^\wedge)) \quad \text{for } i > 1. \end{aligned}$$

Since $\text{map}^f(BP, |\mathcal{L}'|_p^\wedge)$ is the p -completed classifying space of a p -local finite group by Proposition 2.11(a), its homotopy groups are finite p -groups by (a). Now [2, Proposition VII.4.3] shows that the homotopy groups of (7.9) are finite p -groups. Finally, if $k \geq 3$, Proposition 3.8(b) and Corollary 7.6 show that the fundamental group is abelian. \square

Proof of Theorem 7.3. First, we assume that $S \neq 1$, or else the result is a triviality. We begin by constructing a sequence of spaces and maps $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \dots$ where $Y_0 = |\mathcal{L}'|_p^\wedge$ with the following properties:

- (i) For every $i \geq 0$ there exists some $m_{i+1} \geq 2$ such that $Y_{i+1} = (Y_i \wr \Sigma_{p^{m_{i+1}}})_p^\wedge$ and such that g_i is the composite $Y_i \xrightarrow{\Delta} Y_i \wr \Sigma_{p^{m_i}} \xrightarrow{\eta} Y_{i+1}$.
- (ii) $\pi_{* > 0} \text{map}^{g_{i-1} \circ \dots \circ g_0 \circ f|_{BP}}(BP, Y_i)$ are finite abelian p -groups for all $i \geq 1$.
- (iii) The homomorphism

$$\pi_i \text{map}^{g_{i-1} \circ \dots \circ g_0 \circ f|_{BP}}(BP, Y_i) \xrightarrow{(g_i)_*} \pi_i \text{map}^{g_i \circ \dots \circ g_0 \circ f|_{BP}}(BP, Y_{i+1})$$

is trivial for all $i \geq 1$ and all $P \leq S$ in \mathcal{F}^c .

Property (i) states explicitly how to construct the sequence from Y_0 and the m_i 's. Since $Y_0 = |\mathcal{L}'|_p^\wedge$, property (i) with Lemma 7.7 and Theorem A shows that for every $i \geq 0$ there is a homotopy equivalence $Y_i \simeq |\mathcal{L}_i|_p^\wedge$ for some p -local finite group $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ with $S_i \neq 1$. We assume this fact from now on.

To begin with, set $\mathcal{L}_0 = \mathcal{L}'$ and $Y_0 = |\mathcal{L}_0|_p^\wedge$ and $m_1 = 2$. Let $g_0: Y_0 \rightarrow Y_1$ be the composite $Y_0 \xrightarrow{\eta \circ \Delta} (Y_0 \wr \Sigma_{p^2})_p^\wedge$. Proposition 7.8 guarantees that (ii) holds for Y_1 since $p^{m_1} = p^2 > 3$.

Assume by induction that we have constructed $Y_0 \xrightarrow{g_0} \dots \xrightarrow{g_{k-1}} Y_k$, where $k \geq 1$, for which conditions (i)–(iii) hold. By hypothesis (ii) on Y_k we can choose $m_{k+1} \geq 2$ such that $p^{m_{k+1}}$ annihilates every element in the abelian group

$$\bigoplus_{P \in \mathcal{F}^c} \pi_k \operatorname{map}^{g_{k-1} \circ \dots \circ g_0 \circ f|_{BP}}(BP, Y_k).$$

Define $Y_{k+1} = (Y_k \wr \Sigma_{p^{m_{k+1}}})_p^\wedge$ and $g_k = \eta \circ \Delta(Y_k)$. Thus, condition (i) holds for $Y_k \xrightarrow{g_k} Y_{k+1}$. Proposition 7.8 implies that condition (ii) holds for $i = k + 1$ since $p^{m_{k+1}} \geq p^2 > 3$. It now follows from Proposition 2.11 that the mapping space $\operatorname{map}^{g_k \circ \dots \circ g_0 \circ f|_{BP}}(BP, Y_{k+1})$ is p -complete and we are in position to apply Lemma 4.3 (with $Y = Y_k$ and $X = BP$) to deduce that condition (iii) holds for g_k . This completes the inductive step of the construction.

We now prove inductively that for every $k \geq 0$ there is a homotopy equivalence $Y_k \simeq (|\mathcal{L}'| \wr G_k)_p^\wedge$, where $G_k \leq \Sigma_{p^{m_1 + \dots + m_k}}$, such that

$$(1) \quad |\mathcal{L}'|_p^\wedge \xrightarrow{g_{k-1} \circ \dots \circ g_0} Y_k \simeq (|\mathcal{L}'| \wr G_k)_p^\wedge \text{ is homotopic to } |\mathcal{L}'|_p^\wedge \xrightarrow{\Delta(|\mathcal{L}'|)_p^\wedge} (|\mathcal{L}'| \wr G_k)_p^\wedge.$$

This is a triviality when $k = 0$. We assume inductively for $k \geq 1$ that the left triangle in the following diagram is homotopy commutative:

$$\begin{array}{ccccc} & & Y_k & \xrightarrow{\Delta(Y_k)} & Y_k \wr \Sigma_{p^{m_{k+1}}} & \xrightarrow{\eta} & Y_{k+1} \\ & \nearrow^{g_{k-1} \circ \dots \circ g_0} & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ |\mathcal{L}'|_p^\wedge & \xrightarrow{\Delta(|\mathcal{L}'|)_p^\wedge} & (|\mathcal{L}'| \wr G_k)_p^\wedge & \xrightarrow{\Delta} & (|\mathcal{L}'| \wr G_k)_p^\wedge \wr \Sigma_{p^{m_{k+1}}} & \xrightarrow{\eta} & ((|\mathcal{L}'| \wr G_k)_p^\wedge \wr \Sigma_{p^{m_{k+1}}})_p^\wedge \end{array}$$

The composite at the top is $g_k \circ \dots \circ g_0$. By Theorem A and [6, Proposition 1.12], $|\mathcal{L}'| \wr G_k$ is p -good. The induction step now follows from Lemma 7.7 and Proposition 3.5.

Now consider the category $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)^{\text{op}}$ and the functor $\tilde{B}: \mathcal{C} \rightarrow \mathbf{Top}$ recalled in 2.8. Clearly $f: BS \rightarrow |\mathcal{L}'|_p^\wedge$ gives rise to a system of homotopy compatible maps $f_0: \tilde{B}(-) \rightarrow |\mathcal{L}'|_p^\wedge$ in the sense described in Section 6. Recall from [6, Corollary 3.4] that \mathcal{C} is a finite category with p -height $d < \infty$ (see Definition 6.1). Theorem 6.2 applied to f_0 and $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \dots$ shows that there is a map $\tilde{f}_0: |\mathcal{L}| \rightarrow Y_d$ such that $\tilde{f}_0 \circ \Theta \simeq g_{d-1} \circ \dots \circ g_0 \circ f$. Part (a) of the theorem now follows because the following diagram is homotopy commutative by (1):

$$\begin{array}{ccc} BS & \xrightarrow{f} & |\mathcal{L}'|_p^\wedge \\ \Theta \downarrow & & \downarrow \Delta(|\mathcal{L}'|)_p^\wedge \\ |\mathcal{L}| & \xrightarrow{\tilde{f}_0} & Y_d \simeq (|\mathcal{L}'| \wr G_d)_p^\wedge \longrightarrow (|\mathcal{L}'| \wr \Sigma_{p^{m_1 + \dots + m_d}})_p^\wedge \end{array}$$

Part (b) of the theorem follows similarly: Given $\tilde{f}_1, \tilde{f}_2 : |\mathcal{L}| \rightarrow Y_0$ such that $\tilde{f}_1 \circ \Theta \simeq \tilde{f}_2 \circ \Theta \simeq f$, we have $g_d \circ \dots \circ g_0 \circ \tilde{f}_1 \simeq g_d \circ \dots \circ g_0 \circ \tilde{f}_2$ which implies that the composites $|\mathcal{L}| \xrightarrow{\tilde{f}_1, \tilde{f}_2} |\mathcal{L}'|_p \xrightarrow{\Delta_p} Y_{d+1} \simeq (|\mathcal{L}'| \wr \Sigma_{p^{m_1+\dots+m_{d+1}}})_p^\wedge$ are homotopic. \square

Proof of Theorem B. The induced map $BS \xrightarrow{B\rho} BS' \xrightarrow{\eta \circ \Theta'} |\mathcal{L}'|_p^\wedge$ is clearly \mathcal{F} -invariant because $BS' \rightarrow |\mathcal{L}'|_p^\wedge$ is \mathcal{F}' -invariant by 7.2 and ρ is fusion preserving. The result is now a direct consequence of Theorem 7.3 and Theorem A. \square

We say that $\rho : S \rightarrow \Sigma_n$ is \mathcal{F} -invariant if $\rho|_P$ and $\rho \circ \varphi$ are equivalent representations for every $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$.

7.10. Proposition. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and let $\rho : S \rightarrow \Sigma_n$ be a homomorphism. Then the following statements are equivalent:*

- (1) ρ is \mathcal{F} -invariant.
- (2) $B\rho : BS \rightarrow B\Sigma_n$ is an \mathcal{F} -invariant map.
- (3) $\eta \circ B\rho : BS \rightarrow (B\Sigma_n)_p^\wedge$ is an \mathcal{F} -invariant map.

Proof. It follows immediately from a result of Mislin [15, proof of the main theorem] which gives rise to bijections $\text{Rep}(P, \Sigma_n) \approx [BP, B\Sigma_n] \xrightarrow{\eta_*} [BP, (B\Sigma_n)_p^\wedge]$ for all $P \leq S$. \square

7.11. Proposition. *The regular permutation representation of a finite p -group S induces an \mathcal{F} -invariant map $B\text{reg}_S : BS \rightarrow B\Sigma_{|S|}$ for any fusion system \mathcal{F} on S .*

Proof. By Proposition 7.10, it is enough to check that $\text{reg}_S : S \rightarrow \Sigma_{|S|}$ is \mathcal{F} -invariant. Note that S acts freely on S via $\text{reg}_S : S \rightarrow \Sigma_{|S|}$, that is all the isotropy subgroups are trivial. In particular, any group monomorphism $\varphi : P \rightarrow S$ where $P \leq S$ renders S a free P -set via $\text{reg}_S \circ \varphi$. Since any two free P -sets of the same cardinality are equivalent, it follows that $\text{reg}_S|_P$ and $\text{reg}_S \circ \varphi$ are conjugate in Σ_n . \square

By Example 7.2 and Proposition 7.10, every map $f : |\mathcal{L}| \rightarrow (B\Sigma_n)_p^\wedge$ gives rise to an \mathcal{F} -invariant representation ρ of S of rank n where $B\rho \simeq f|_{BS}$. Not every \mathcal{F} -invariant representation of S arises necessarily in this way. However, the next proposition gives a partial answer to that question.

7.12. Proposition. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group.*

- (a) *Given $\rho \in \text{Rep}_n(\mathcal{F})$, there exists some $k \geq 0$ and an element $\tilde{f} \in \text{Rep}_{p^k n}(\mathcal{L})$ such that $\tilde{f}|_{BS}$ is homotopic to $BS \xrightarrow{B(p^k \cdot \rho)} B\Sigma_{p^k n} \xrightarrow{\eta} (B\Sigma_{p^k n})_p^\wedge$.*
- (b) *Consider $f_1, f_2 \in \text{Rep}_n(\mathcal{L})$ such that $f_1|_{BS} \simeq f_2|_{BS}$. Then there exists some $e \geq 0$ such that $p^e \cdot f_1 = p^e \cdot f_2$ in $\text{Rep}_{p^e n}(\mathcal{L})$.*

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be the p -local finite group associated with Σ_n . Since $|\mathcal{L}|$ is p -good by [6, Proposition 1.12], a standard Serre spectral sequence argument shows that

$$(1) \quad (B\Sigma_n)_p^\wedge \simeq |\mathcal{L}|_p^\wedge \xrightarrow{\Delta_p^\wedge} (|\mathcal{L}|_p^\wedge \wr \Sigma_k)_p^\wedge \simeq ((B\Sigma_n)_p^\wedge \wr \Sigma_k)_p^\wedge \xrightarrow{B \text{ incl}_p^\wedge} (B\Sigma_{nk})_p^\wedge \quad \text{and}$$

$$(B\Sigma_n)_p^\wedge \xrightarrow{(B\Delta)_p^\wedge} (B\Sigma_{nk})_p^\wedge$$

where $\Delta: \Sigma_n \hookrightarrow \Sigma_{nk}$ is the diagonal inclusion, are homotopic. Both (a) and (b) follow directly from Proposition 7.10, Theorem 7.3 and (1) taking into account the definition of the operation $+$ in $\coprod_{n \geq 0} \text{Rep}_n(\mathcal{F})$ and $\coprod_{n \geq 0} \text{Rep}_n(\mathcal{L})$. \square

7.13. Proposition. Every S -regular permutation representation $|\mathcal{L}| \xrightarrow{f} (B\Sigma_n)_p^\wedge$ is a homotopy monomorphism at p .

Proof. By [5, Lemma 2.3], $H^*(S; \mathbb{F}_p)$ is a finitely generated module over the Noetherian \mathbb{F}_p -algebra $H^*(B\Sigma_{m \cdot |S|}; \mathbb{F}_p)$ via the homomorphism $(m \cdot \text{reg}_S)^*$. Finally, $H^*(|\mathcal{L}|; \mathbb{F}_p)$ is a submodule of $H^*(S; \mathbb{F}_p)$ by [6, Theorem B], whence it is finitely generated. \square

Proof of Theorem C. Apply Propositions 7.11 and 7.12(a) to obtain $f \in \text{Rep}_{p^k \cdot |S|}(\mathcal{L})$ such that $f|_{BS}$ is homotopic to $\eta \circ B(p^k \cdot \text{reg}_S)$, that is, $\Phi(f) = p^k \cdot \text{reg}_S$. By Proposition 7.13, f is a homotopy monomorphism at p . \square

8. The p -local index of the Sylow subgroup

Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group and let $f: |\mathcal{L}| \rightarrow (B\Sigma_n)_p^\wedge$ be a map. The restriction $f|_{BS} = f \circ \Theta$ is \mathcal{F} -invariant by Example 7.2 and is homotopic to $(B\rho)_p^\wedge$ for a unique $\rho \in \text{Rep}(S, \Sigma_n)$ which is \mathcal{F} -invariant by Proposition 7.10 and [10]. These results maps $\text{Rep}_n(\mathcal{L}) \rightarrow \text{Rep}_n(\mathcal{F})$ which are compatible with the operations $+$ and \times defined in Section 1. They give rise to a ring homomorphism

$$\Phi: \text{Rep}(\mathcal{L}) \rightarrow \text{Rep}(\mathcal{F}).$$

8.1. Proposition. Additively, $\ker(\Phi)$ and $\text{coker}(\Phi)$ are p -torsion.

Proof. An element in $\ker(\Phi)$ has the form $f_1 - f_2$ where $f_1, f_2 \in \text{Rep}_n(\mathcal{L})$ for some n and $f_1|_{BS} \simeq f_2|_{BS}$. Proposition 7.12 implies that $p^e \cdot (f_1 - f_2) = 0$ in $\text{Rep}(\mathcal{L})$ and it follows that $\ker(\Phi)$ is p -torsion.

An element of $\text{Rep}(\mathcal{F})$ has the form $\rho_1 - \rho_2$ for some $\rho_1 \in \text{Rep}_{n_1}(\mathcal{F})$ and $\rho_2 \in \text{Rep}_{n_2}(\mathcal{F})$. By Proposition 7.12, the definition of Φ and the definition of the operations $+$ in $\text{Rep}(\mathcal{F})$ and $\text{Rep}(\mathcal{L})$, we see that there exist integers $k_1, k_2 \geq 0$ and representations $f_1 \in \text{Rep}_{p^{k_1}n_1}(\mathcal{L})$ and $f_2 \in \text{Rep}_{p^{k_2}n_2}(\mathcal{L})$ such that $\Phi(f_1) = p^{k_1} \cdot \rho_1$ and $\Phi(f_2) = p^{k_2} \cdot \rho_2$. Then $\omega = p^{k_2} \cdot f_1 - p^{k_1} \cdot f_2$ is an element of $\text{Rep}(\mathcal{L})$ such that $\Phi(\omega) = p^{k_1+k_2}(\rho_1 - \rho_2)$. It follows that $\text{coker}(\Phi)$ is p -torsion. \square

By Proposition 7.11 the ring $\text{Rep}(\mathcal{F})$ contains $\text{reg}_S: S \rightarrow \Sigma_{|S|}$ which generates an (additive) infinite cyclic group $\text{Rep}^{\text{reg}}(\mathcal{F}) := \{n \cdot \text{reg}_S\}_{n \in \mathbb{Z}}$ in $\text{Rep}(\mathcal{F})$. Similarly let $\text{Rep}^{\text{reg}}(\mathcal{L})$ denote the additive subgroup of the ring $\text{Rep}(\mathcal{L})$ generated by all the S -regular representations of $(S, \mathcal{F}, \mathcal{L})$ (see Definition 1.2).

It follows directly from the definitions that Φ restricts to a group homomorphism

$$\Phi^{\text{reg}} : \text{Rep}^{\text{reg}}(\mathcal{L}) \rightarrow \text{Rep}^{\text{reg}}(\mathcal{F}).$$

8.2. Corollary. *The cokernel of Φ^{reg} is a cyclic p -group. The kernel of Φ^{reg} is an abelian torsion p -group and $\text{Rep}^{\text{reg}}(\mathcal{L})$ is isomorphic to the direct sum of \mathbb{Z} with an abelian p -torsion group.*

Proof. This follows from Proposition 8.1 which in particular implies that the image of Φ^{reg} is isomorphic to \mathbb{Z} , whence it splits off from $\text{Rep}^{\text{reg}}(\mathcal{L})$. \square

Given a finite group G there is a natural one-to-one correspondence between equivalence classes of permutation representations $G \rightarrow \Sigma_n$ and equivalence classes of G -sets of cardinality n . Sum and products of representations (as described in Section 1) correspond to disjoint unions and products of the associated G -sets. Note that reg_G corresponds to a free G -set with one orbit.

Let us return to discuss $\text{Rep}(\mathcal{F})$. Since the product of a free S -set with any other S -set is again a free set, it follows that $\text{Rep}^{\text{reg}}(\mathcal{F})$ and $\text{Rep}^{\text{reg}}(\mathcal{L})$ are in fact ideals in $\text{Rep}(\mathcal{F})$ and $\text{Rep}(\mathcal{L})$ and that Φ^{reg} is a ring homomorphism.

8.3. Example. Let $(S, \mathcal{F}, \mathcal{L})$ be the p -local finite group of a finite group G . The restriction of $(B \text{reg}_G)_p^\wedge : |\mathcal{L}|_p^\wedge \rightarrow (B \Sigma_{|G|})_p^\wedge$ to BS is homotopic to $n \cdot (B \text{reg}_S)_p^\wedge$ where $n = |G : S|$ because $\text{reg}_G|_S = n \cdot \text{reg}_S$. In particular $(B \text{reg}_G)_p^\wedge$ is an element in $\text{Rep}^{\text{reg}}(\mathcal{L})$ which is mapped by Φ to $n \cdot \text{reg}_S$. It follows that $|G : S| \in \text{Im}(\Phi^{\text{reg}})$, whence $|\text{coker}(\Phi^{\text{reg}})|$ divides $|G : S|$.

8.4. Definition. Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Define the upper and lower p -local index of S in \mathcal{L} by

$$\begin{aligned} \text{Uind}_p(\mathcal{L} : S) &= |\text{coker}(\Phi^{\text{reg}})|, \\ \text{Lind}_p(\mathcal{L} : S) &= |\text{Rep}^{\text{reg}}(\mathcal{F}) : \text{Rep}^{\text{reg}}(\mathcal{F}) \cap \text{Im}(\Phi)|. \end{aligned}$$

Clearly $\text{Lind}_p(\mathcal{L} : S)$ divides $\text{Uind}_p(\mathcal{L} : S)$ because $\text{Im}(\Phi^{\text{reg}}) \leq \text{Im}(\Phi) \cap \text{Rep}^{\text{reg}}(\mathcal{F})$.

8.5. Lemma. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Then $\text{Uind}_p(\mathcal{L} : S)$ is a p -power. If there exists a permutation representation $\rho : |\mathcal{L}| \rightarrow (B \Sigma_n)_p^\wedge$ such that $\rho|_{BS} \simeq B(n \cdot \text{reg}_S)$ with $n \geq 1$ prime to p , then $\text{Uind}_p(\mathcal{L} : S) = 1$, and in particular also $\text{Lind}_p(\mathcal{L} : S) = 1$.*

Proof. The first statement follows from Corollary 8.2. The existence of ρ shows that $n \in \text{Im}(\Phi^{\text{reg}})$ hence, $\text{Uind}_p(\mathcal{L} : S) = 1$. \square

The depth of a fusion system \mathcal{F} on S is the largest number of elements in a chain of proper inclusions of \mathcal{F} -centric \mathcal{F} -radical subgroups of S . This includes chains ending in S . Following [22] we call these subgroups “ \mathcal{F} -Alperin.” Thus, if the depth of \mathcal{F} is n then there exists no chain $P_1 \leq \dots \leq P_{n+1}$ of proper inclusions of \mathcal{F} -Alperin subgroups.

8.6. Proposition. *If the depth of a p -local finite group $(S, \mathcal{F}, \mathcal{L})$ is equal to 2 then $\text{Uind}_p(\mathcal{L} : S) = 1$.*

Proof. Let \mathcal{R} denote the collection of the \mathcal{F} -Alperin subgroups of S . Fix representatives S, P_1, \dots, P_n for the \mathcal{F} -conjugacy classes in \mathcal{R} where P_i are fully \mathcal{F} -normalized. Consider the poset $\text{sd } \mathcal{R}$ defined in [12, Definition 1.3]. Its objects are the \mathcal{F} -conjugacy classes $[P]$ of elements $P \in \mathcal{R}$ and the \mathcal{F} -conjugacy classes $[P \lesssim S]$ of proper inclusion $P \lesssim S$ in \mathcal{R} . Here we use the fact that \mathcal{F} has depth 2. The only relations in $\text{sd } \mathcal{R}$ are $[P \lesssim S] < [P]$ and $[P \lesssim S] < [S]$. By Alperin’s fusion theorem [6, Theorem A.10], if $[Q] = [P_i]$ then $[Q \lesssim S] = [P_i \lesssim S]$. It follows that $\text{sd } \mathcal{R}$ is isomorphic to the poset \mathcal{C}_n whose objects are $\{c_0, c_1^i, c_2^i \mid i = 1, \dots, n\}$ and whose only relations are $c_1^i < c_0$ and $c_1^i < c_2^i$ for all $i = 1, \dots, n$. Specifically, $c_0 = [S], c_2^i = [P_i]$ and $c_1^i = [P_i \lesssim S]$. We view \mathcal{C}_n as a small category with an arrow $x \rightarrow y$ if $x < y$.

In [12, Theorem A] a functor $F : \mathcal{C}_n \rightarrow \mathbf{Top}$ with the following properties is constructed. The values of F are the classifying spaces of finite groups G_0, G_1^i and G_2^i for $i = 1, \dots, n$ and the maps $F(c_1^i) \rightarrow F(c_0)$ and $F(c_1^i) \rightarrow F(c_2^i)$ are induced by inclusion of groups $G_1^i \leq G_0$ and $G_1^i \leq G_2^i$. In addition S is a subgroup of $G_0 = \text{Aut}_{\mathcal{L}}(S)$ of index prime to p . Also, the numbers $k_i = |G_2^i : G_1^i|$ are prime to p by [12, Theorem A] and the fact that P_i is fully \mathcal{F} -normalized, whence $N_S(P_i)$ is a Sylow p -subgroup of $G_1^i = \text{Aut}_{\mathcal{L}}(P_i \leq S)$ which is a subgroup of $G_1^i = \text{Aut}_{\mathcal{L}}(P_i)$ by [12, Proposition 1.5]. Finally, the map $\Theta : BS \rightarrow |\mathcal{L}|$ factors up to homotopy through $BG_0 \simeq F(c_0) \rightarrow \text{hocolim}_{\mathcal{C}_n} F \simeq |\mathcal{L}|$.

Set $k = \prod_1^n k_i$ and $k_0 = |G_0| \cdot k$. Note that k_0 is divisible by $|G_1^i|$ and $|G_2^i|$ for all i because $k_0 = k \cdot |G_0| = k \cdot |G_1^i| \cdot |G_0 : G_1^i|$ and k_i divides k . Set $\ell_i = k_0 / |G_1^i|$ and $m_i = k_0 / |G_2^i|$. Consider the following permutation representations for $i = 1, \dots, n$

$$k \cdot \text{reg}_{G_0} : G_0 \rightarrow \Sigma_{k_0}, \quad \ell_i \cdot \text{reg}_{G_1^i} : G_1^i \rightarrow \Sigma_{k_0}, \quad m_i \cdot \text{reg}_{G_2^i} : G_2^i \rightarrow \Sigma_{k_0}.$$

Note that $(k \cdot \text{reg}_{G_0})|_{G_1^i}$ and $(m_i \cdot \text{reg}_{G_2^i})|_{G_1^i}$ are equivalent to $\ell_i \cdot \text{reg}_{G_1^i}$ because all of them give the set $\{1, \dots, k_0\}$ the structure of a free G_1^i -set with ℓ_i orbits. By taking classifying spaces there results a system of homotopy compatible maps $F \rightarrow B\Sigma_{k_0}$. It can be replaced by a system of compatible maps $F \rightarrow B\Sigma_{k_0}$ as follows. First, set the maps $F(c_1^i) \rightarrow B\Sigma_{k_0}$ to be the composite $F(c_1^i) \rightarrow F(c_0) \rightarrow B\Sigma_{k_0}$. Next, replace the maps $F(c_1^i) \rightarrow F(c_2^i)$ by cofibrations and change the maps $F(c_2^i) \rightarrow B\Sigma_n$ up to homotopy to obtain a system of compatible maps $F \rightarrow B\Sigma_{k_0}$.

There results a map $f : |\mathcal{L}| \simeq \text{hocolim } F \rightarrow B\Sigma_{k_0}$ such that the restriction to the Sylow p -subgroup $f|_{BS} = f \circ B\iota_S^{G_0} \simeq k \cdot |G_0 : S| \cdot B\text{reg}_S$ where $k \cdot |G_0 : S|$ is prime to p . Now Lemma 8.5 applies. \square

We shall now prove Theorem D. In fact we prove the following stronger result.

8.7. Theorem. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p -local finite group. Then $\text{Uind}_p(\mathcal{L} : S) = 1$ if*

- (1) $(S, \mathcal{F}, \mathcal{L})$ is associated with a finite group, or
- (2) $(S, \mathcal{F}, \mathcal{L})$ is one of the exotic examples in [6, Examples 9.3 and 9.4] or in [22] or in [9] or in [7, Example 5.3].

Proof. (1) This follows from Lemma 8.5 and Example 8.3.

(2) We will apply Proposition 8.6. The p -local finite groups in [6, Examples 9.3, 9.4] as well as the ones in [22] and in [9] were shown to have depth 2 in Examples 7.6, 7.7, 7.3 and 7.4 of [13], respectively. The information on the structure of the exotic p -local finite groups in

[7, Example 5.3] implies quite directly that these fusion systems have depth 2. We leave the straightforward details to the reader. \square

8.8. Conjecture. *For all p -local finite groups $\text{Uind}_p(\mathcal{L} : S) = 1$.*

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