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Heegaard–Floer homology of broken fibrations over the circle

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Abstract

We extend Perutz's Lagrangian matching invariants to 3-manifolds which are not necessarily fibered using the technology of holomorphic quilts. We prove an isomorphism of these invariants with Ozsváth–Szabó's Heegaard–Floer invariants for certain extremal spin^c structures. As applications, we give new calculations of Heegaard–Floer homology of certain classes of 3-manifolds, and a characterization of Juhász's sutured Floer homology.

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1. Introduction

In this paper, we study two seemingly different Floer theoretical invariants of three- and four-manifolds. These are Perutz's Lagrangian matching invariants and Ozsváth and Szabó's Heegaard–Floer theoretical invariants. The main result of this paper is an isomorphism between the 3-manifold invariants of these theories for certain spin^c structures, namely *quilted Floer homology* and *Heegaard–Floer homology*. We also outline how the techniques here can be generalized to obtain an identification of 4-manifold invariants.

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Before giving a review of both of the above mentioned theories, we give the definition of a broken fibration over S^1, which will be an important part of the topological setting that we will be working with.

Definition 1. A map $f : Y \to S^1$ from a closed oriented smooth 3-manifold Y to S^1 is called a broken fibration if f is a circle-valued Morse function with all of the critical points having index 1 or 2.

The terminology is inspired from the terminology of broken Lefschetz fibrations on 4-manifolds, to which we will return later in this paper in Section 5. We remark that a 3-manifold admits a broken fibration if and only if $b_1(Y) > 0$, and if it admits one, it admits a broken fibration with connected fibers.

We will restrict ourselves to broken fibrations with connected fibers and we will denote by $\Sigma_{\max}$ and $\Sigma_{\min}$ two fibers with maximal and minimal genus respectively. We denote by $\mathcal{S}(Y|\Sigma_{\min})$, the spin^c structures $\mathfrak{s}$ on Y such that $\langle c_1(\mathfrak{s}), [\Sigma_{\min}]\rangle = \chi(\Sigma_{\min})$ (those spin^c structures which satisfy the adjunction *equality* with respect to the fiber with minimal genus).

Definition 2. The universal Novikov ring Λ over $\mathbb{Z}_2$ is the ring of formal power series $\Lambda = \Sigma_{r\in\mathbb{R}} a_r t^r$ with $a_r \in \mathbb{Z}_2$ such that $\#\{r|a_r \neq 0, r < N\} < \infty$ for any $N \in \mathbb{R}$.

We first give a definition of a new invariant $QFH'(Y, f, \mathfrak{s}; \Lambda)$ for all spin^c structures in $\mathcal{S}(Y|\Sigma_{\min})$ and prove an isomorphism between this variant of quilted Floer homology of a broken fibration $f : Y \to S^1$ (with coefficients in the universal Novikov ring) and the Heegaard–Floer homology of Y perturbed by a closed 2-form η that pairs positively with the fibers of f.

Theorem 3. *$QFH'(Y, f, \mathfrak{s}; \Lambda) \simeq HF^{\pm}(Y, \eta, \mathfrak{s})$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$.*

When $g(\Sigma_{\min})$ is at least 2, the coefficients can be taken to be in $\mathbb{Z}_2$ (in this case admissibility of our diagrams are automatic, therefore we do not need to use perturbations).

Corollary 4. *Suppose that $g(\Sigma_{\min}) > 1$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ we have*

$$QFH'(Y, f, \mathfrak{s}; \mathbb{Z}_2) \simeq HF^+(Y, \mathfrak{s}; \mathbb{Z}_2).$$

As corollaries of this result, we give new calculations of Heegaard Floer homology groups for certain manifolds for which $QFH'(Y, f, \mathfrak{s})$ is easy to calculate. We give several such calculations among which the following is particularly interesting.

Corollary 5. *Suppose f has only two critical points, and let $\alpha, \beta \subset \Sigma_{\max}$ be the vanishing cycles of these critical points. Then, $\oplus_{\mathfrak{s}\in\mathcal{S}(Y|\Sigma_{\min})} HF^+(Y, \eta, \mathfrak{s})$ is free of rank $\iota(\alpha, \beta)$, the geometric intersection number between α and β. Furthermore, if $g(\Sigma_{\min}) > 1$ then the result holds over $\mathbb{Z}_2$, i.e.*

$$\oplus_{\mathfrak{s}\in\mathcal{S}(Y|\Sigma_{\min})} HF^+(Y, \mathfrak{s}) = \mathbb{Z}_2^{\iota(\alpha,\beta)}.$$

The second main theorem proves that the invariants $QFH'(Y, f, \mathfrak{s}; \Lambda)$ that we defined are isomorphic to the quilted Floer homology groups coming from Perutz's theory of Lagrangian matching invariants. Unlike $QFH'(Y, f, \mathfrak{s}; \Lambda)$, for technical reasons these latter invariants are only defined in the case $g(\Sigma_{\max}) < 2g(\Sigma_{\min})$. Thus, we have the following theorem.

Theorem 6. *Suppose that Y admits a broken fibration with $g(\Sigma_{\max}) < 2g(\Sigma_{\min})$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$, $QFH(Y, f, \mathfrak{s}; \Lambda)$ is well-defined and*

$$QFH'(Y, f, \mathfrak{s}; \Lambda) \simeq QFH(Y, f, \mathfrak{s}; \Lambda).$$

As before, we have the same result over $\mathbb{Z}_2$ when $g(\Sigma_{\min})$ is at least 2.

In Section 2, we construct a Heegaard diagram associated with a broken fibration and investigate the properties of this diagram. We also give a calculation of perturbed Heegaard–Floer homology of fibered 3-manifolds for $\mathfrak{s} \in \mathcal{S}(Y|F)$. In Section 3, we give a definition of quilted Floer homology in the language of Heegaard Floer theory and prove that it is isomorphic to the Heegaard–Floer homology for the spin^c structures under consideration. Here, we give several corollaries of our first main result, including new calculations of Heegaard–Floer homology groups and a characterization of Juhász's sutured Floer homology. In Section 4, we give a complete definition of $QFH(Y, f, \mathfrak{s})$ and we prove our second main theorem, namely that the group defined in Section 3 is isomorphic to the original definition of quilted Floer homology in terms of holomorphic quilts. Finally in Section 5 we discuss the extension of this isomorphism to four-manifold invariants.

We now proceed to review the theories and the notation that are involved in our theorem.

1.1. (Perturbed) Heegaard–Floer homology

In this section, we review the construction of Heegaard–Floer homology, introduced by Ozsváth and Szabó [16]. The usual construction involves certain admissibility conditions, however there is a variant of Heegaard–Floer homology where Novikov rings and perturbations by closed 2-forms are introduced in order to make the Heegaard–Floer homology group well-defined without any admissibility condition. Our account will be brief since this theory has been well developed in the literature. The reader is encouraged to turn to [4] for a more detailed account of perturbed Heegaard Floer theory. Furthermore, we will mostly find it convenient to work in the setup of Lipshitz's cylindrical reformulation of the Heegaard–Floer homology [11].

Let $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a pointed Heegaard diagram of a 3-manifold Y. This gives rise to a pair of Lagrangian tori $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ in $\mathrm{Sym}^g(\Sigma_g)$, together with a holomorphic hypersurface $Z = z \times \mathrm{Sym}^{g-1}(\Sigma_g)$. The Heegaard–Floer homology of Y is the Lagrangian Floer homology of these tori, where one uses the orbifold symplectic form pushed down from $\Sigma_g^{\times g}$, though one can also use honest symplectic forms (see [21]). The differential is twisted by keeping track of the intersection number n_z of holomorphic disks contributing to the differential with Z. More precisely, the Heegaard–Floer chain complex $CF^+(Y)$ is freely generated over $\mathbb{Z}$ by $[\mathbf{x}, i]$ where $\mathbf{x}$ is an intersection point of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ and $i \in \mathbb{Z}_{\geq 0}$, and the differential is given by

$$\partial^+([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\varphi \in \pi_2(\mathbf{x},\mathbf{y}), n_z(\varphi) \leq i} \#\widehat{\mathcal{M}}(\varphi)[y, i - n_z(\varphi)]$$

where $\#\widehat{\mathcal{M}}(\varphi)$, as usual in Lagrangian–Floer homology, refers to a count of holomorphic disks with boundary on $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ connecting $\mathbf{x}$ and $\mathbf{y}$. The above definition only makes sense under certain admissibility conditions so that the sum on the right hand side of the differential is finite. In general, one can consider a twisted version of the above chain complex by a closed 2-form in $\Omega^2(Y)$. This is called the *perturbed Heegaard–Floer homology*. The chain complex $CF^+(Y, \eta)$ is freely generated over Λ (see Definition 2) by $[\mathbf{x}, i]$ where $\mathbf{x}$ is an intersection point and i is a nonnegative integer as before, and the differential is twisted by the area $\int_{[\varphi]} \eta$ of the holomorphic disks that contribute to the differential. More precisely, the differential of the perturbed theory is given by

$$\partial^+([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\varphi \in \pi_2(\mathbf{x},\mathbf{y}), n_z(\varphi) \leq i} \#\widehat{\mathcal{M}}(\varphi) t^{\eta(\varphi)} [y, i - n_z(\varphi)].$$

Note that if φ_1, φ_2 are two holomorphic disks that connect an intersection point $\mathbf{x}$ to $\mathbf{y}$, then their difference is a periodic domain P and we have the equality $\eta(\varphi_1) - \eta(\varphi_2) = \eta([P])$, where the latter only depends on the cohomology class of η. We remark that although the differential depends on the choice of a representative of the class $[\eta]$, the isomorphism class of the homology groups is determined by $\mathrm{Ker}(\eta) \cap H_2(Y; \mathbb{Z})$.

Recall that a 2-form is said to be generic when $\mathrm{Ker}(\eta) \cap H_2(Y; \mathbb{Z}) = \{0\}$. For a generic form coming form an area form on the Heegaard surface, $HF^+(Y, \eta)$ is defined without any admissibility conditions on the Heegaard diagram.

1.2. Quilted Floer homology of a 3-manifold

In this section, we review the definition of quilted Floer homology of a 3-manifold Y equipped with a broken fibration $f : Y \to S^1$. The general theory of holomorphic quilts is under systematically developed by Wehrheim and Woodward [28], though the case we consider also appears in the work of Perutz [20]. The relevant part of the theory in the setting of 3-manifolds is obtained from Perutz's construction of Lagrangian matching conditions associated with critical values of broken fibrations, which we now review from [22].

Given a Riemann surface (Σ, j) and an embedded circle $L \subset \Sigma$, denote by Σ_L the surface obtained from Σ by surgery along L, i.e., by removing a tubular neighborhood of L and gluing in a pair of disks. To such data, Perutz associates a distinguished Hamiltonian isotopy-class of Lagrangian correspondences $V_L \subset \mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\Sigma_L)$ (where the symmetric products are equipped with Kähler forms in suitable cohomology classes, see [22]). These are described in terms of a symplectic degeneration of $\mathrm{Sym}^n(\Sigma)$. More precisely, one considers an elementary Lefschetz fibration over D^2 with regular fiber Σ and a unique vanishing cycle L which collapses at the origin. Then one passes to the relative Hilbert scheme, $\mathrm{Hilb}^n_{D^2}(\Sigma)$, of this fibration (the resolution of the singular variety obtained by taking fiber-wise symmetric products). The regular fibers of the induced map from $\mathrm{Hilb}^n_{D^2}(\Sigma)$ are identified with $\mathrm{Sym}^n(\Sigma)$, and the fiber above the origin has a codimension 2 singular locus which can be identified with $\mathrm{Sym}^{n-1}(\Sigma_L)$. V_L then arises as the vanishing cycle of this fibration.

Given a 3-manifold Y and a broken fibration $f : Y \to S^1$, the *quilted Floer homology* of Y, $QFH(Y, f)$, is a Lagrangian intersection theory graded by spin^c structures on Y. Let $p_1, p_2, \ldots, p_k$ be the set of critical values of f. Pick points $p_i^{\pm}$ in a small neighborhood of each p_i so that the fiber genus increases from p_i^- to p_i^+. For $\mathfrak{s} \in \mathrm{spin}^c(Y)$, let $\nu : S^1 \setminus \mathrm{crit}(f) \to \mathbb{Z}_{\geq 0}$ be the locally constant function defined by $\langle c_1(\mathfrak{s}), [F_s] \rangle = 2\nu(s) + \chi(F_s)$, where $F_s = f^{-1}(s)$. Then the construction in the previous paragraph gives Lagrangian correspondences $L_{p_i} \subset \mathrm{Sym}^{\nu(p_i^+)}(F_{p_i^+}) \times \mathrm{Sym}^{\nu(p_i^-)}(F_{p_i^-})$. The quilted Floer homology of Y, $QFH(L_{p_1}, \ldots, L_{p_k})$, is then generated by horizontal (with respect to the gradient flow of f) multi-sections of f which match along the Lagrangians $L_{p_1}, \ldots, L_{p_k}$ at the critical values of f, and the differential counts rigid holomorphic "quilted cylinders" connecting the generators, [20,28] (see Section 4.1 for a detailed definition).

There are various technical difficulties involved in the definition of $QFH(Y, f, \mathfrak{s})$ due to bubbling of holomorphic curves. These are addressed by different means depending on the value of

$\langle c_1(\mathfrak{s}), [\Sigma_{\max}]\rangle$. The easiest case is the (positively) monotone case, that is when $\langle c_1(\mathfrak{s}), [\Sigma_{\max}]\rangle > 0$, where holomorphic bubbles are a priori excluded. However, for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ we will almost never be in the monotone case. In the strongly negative case, that is when $\langle c_1(\mathfrak{s}), [\Sigma_{\max}]\rangle \leq \chi(\Sigma_{\max})/2$, one can still eliminate bubbles a priori by standard means. For the rest of the cases, bubbles might and will occur in general, therefore complications arise. One then tries to establish a proper combinatorial rule for handling bubbled configurations. One could also try to use the more technical machinery of [13] or [3] in order to tackle this case. Another related issue is showing that quilted Floer homology is an invariant of a three manifold. The isomorphism constructed in this paper shows this in an indirect way for the spin^c structures under consideration. We will return to this question and various well-definedness questions in [10].

In this paper, we will deal with the spin^c structures $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$. In this case, quilted Floer homology has been defined only in the strongly negative case, which is equivalent to requiring $g(\Sigma_{\max}) < 2g(\Sigma_{\min})$ (see Section 4 for details). However, we will define a variant of quilted Floer homology, which we will denote by $QFH'(Y, f, \mathfrak{s})$ that suits our purposes and avoids these technical issues, hence is well-defined in all cases; see Section 3.1 for the definition. We will prove that in the case when $QFH(Y, f, \mathfrak{s})$ is defined, it is isomorphic to $QFH'(Y, f, \mathfrak{s})$ (this is the content of our Theorem 6). Then, Theorem 3, which establishes an isomorphism between $QFH'(Y, f, \mathfrak{s})$ and $HF^+(Y, \mathfrak{s})$, will show that $QFH(Y, f, \mathfrak{s})$, when defined, is isomorphic to $HF^+(Y, \mathfrak{s})$.

Finally, we remark that in the case when $f : Y \to S^1$ is a fibration, $QFH(Y, f)$ is given as a fixed point Floer homology theory on the moduli space of vortices and was first introduced by Salamon in [25]. In this case, the spin^c structures $\mathfrak{s} \in \mathcal{S}(Y|\Sigma)$ correspond to taking the zeroth symmetric product of the fibers. In this case, it is natural to set $QFH(Y, f) = \Lambda$ if $\mathfrak{s}$ is the canonical tangent spin^c structure, and $QFH(Y, f) = 0$ for other $\mathfrak{s} \in \mathcal{S}(Y|\Sigma)$.

2. Heegaard diagram for a broken fibration on Y

2.1. A standard Heegaard diagram

We start with a 3-manifold Y with $b_1 > 0$. Then Y admits a broken fibration over S^1. Consider such a Morse function $f : Y \to S^1$ with the following additional properties.

- $F_{-1} = \Sigma_{\max}$ has the maximal genus $g_{\max} = g$ and $F_1 = \Sigma_{\min}$ has the minimal genus $g_{\min} = k$ among fibers of f.
- The fibers are connected.
- The genera of the fibers are in decreasing order as one travels clockwise and counter-clockwise from -1 to 1.

It is easy to see that a broken fibration with these properties exists if and only if $b_1 > 0$. In fact, any broken fibration with connected fibers can be deformed into one with these properties by an isotopy that changes the order of the critical values.

We will now construct a Heegaard diagram for Y adapted to f. Roughly speaking, the Heegaard surface Σ will be obtained by connecting $\Sigma_{\max}$ and $\Sigma_{\min}$ by two "tubes" traveling clockwise and counter-clockwise from $\Sigma_{\max}$ to $\Sigma_{\min}$. More precisely, start with a section γ of f over S^1. Then we can pick a metric for which γ is a gradient flow line of f, and since γ is disjoint from the critical points of f, it also avoids the stable/unstable manifolds of the critical points. Now pick two distinct points p and q on $\Sigma_{\max}$ sufficiently close to the point where γ intersects $\Sigma_{\max}$, connect p to $\Sigma_{\min}$ by the gradient flow line above the northern semi-circle in the base S^1

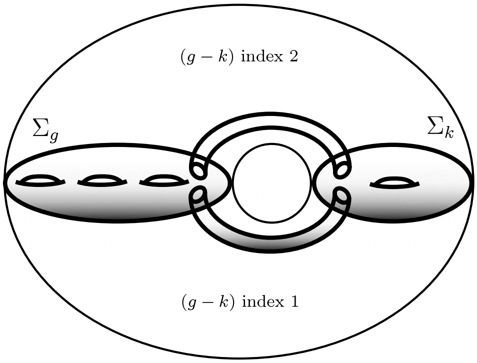

Fig. 1. Heegaard surface for a broken fibration.

which connects -1 to 1 in the clockwise direction and connect q to $\Sigma_{\min}$ by the gradient flow line above the southern semi-circle, avoiding the critical points of f in both cases. Denote these flow lines by γ_p and γ_q and their end points in $\Sigma_{\min}$ by $\bar{p}$ and $\bar{q}$. Then the Heegaard surface that we are interested in is obtained by removing disks around $p, q, \bar{p}$ and $\bar{q}$ and connecting $\Sigma_{\max}$ to $\Sigma_{\min}$ along γ_p and γ_q (see Fig. 1). We denote the resulting surface by

$$\Sigma = \Sigma_{\max} \cup_{\partial N(\gamma_p) \cup \partial N(\gamma_q)} \Sigma_{\min}$$

where $N(\gamma_p)$ and $N(\gamma_q)$ stand for normal neighborhoods of γ_p and γ_q.

Note that $g(\Sigma) = g+k+1$. Denote the point where γ intersects $\Sigma_{\max}$ by w and the point where γ intersects $\Sigma_{\min}$ by z. Next, we will describe α and β curves on Σ in order to get a Heegaard decomposition of Y. First, set α_0 to be $\partial N(\gamma_p) \cap f^{-1}(-i)$ and set β_0 to be $\partial N(\gamma_p) \cap f^{-1}(i)$. The preimage of the northern semi-circle is a cobordism from $\Sigma_{\max}$ to $\Sigma_{\min}$ which can be realized by attaching $(g-k)$ 2-handles to $\Sigma_{\max} \times I$, and hence can be described by the data of $g-k$ disjoint attaching circles on $\Sigma_{\max}$. These we declare to be $\alpha_1, \ldots, \alpha_{g-k}$. Similarly the preimage of the southern semi-circle is a cobordism from $\Sigma_{\max}$ to $\Sigma_{\min}$, encoded by $g-k$ disjoint attaching circles $\beta_1, \ldots, \beta_{g-k}$ on $\Sigma_{\max}$. Alternatively, these two sets correspond to the stable and unstable manifolds of the critical points of f. More precisely, orienting the base S^1 in the clockwise direction, $\alpha_1, \ldots, \alpha_{g-k}$ are the intersections of the stable manifolds of the critical points above the northern semi-circle with $\Sigma_{\max}$, similarly $\beta_1, \ldots, \beta_{g-k}$ are the intersections of the unstable manifolds of the critical points above the southern semi-circle with $\Sigma_{\max}$. Note that by choosing p and q sufficiently close to w we can ensure that they lie in the connected component as w in the complement of $\alpha_1, \ldots, \alpha_{g-k}$ and $\beta_1, \ldots, \beta_{g-k}$.

Next, we describe the remaining curves, $(\alpha_{g-k+1}, \ldots, \alpha_{g+k}, \beta_{g-k+1}, \ldots, \beta_{g+k})$. Let F be the part of Σ which consists of $\Sigma_{\max}$ (except the two disks removed around p and q) together with halves of the connecting tubes up to α_0 and β_0. Thus F is a genus g surface with 2 boundary components α_0 and β_0. Also, denote by $\bar{F}$ the complement of $\mathrm{Int}(F)$ in Σ. Thus $\bar{F}$ is a genus k surface with boundary consisting of α_0 and β_0 and $\Sigma = F \cup_{\alpha_0 \cup \beta_0} \bar{F}$. Let us also pick p^+ and q^+ on the boundary of the disks deleted around p and q, and $\bar{p}^+$ and $\bar{q}^+$ their images under the gradient flow (so that they lie on the boundary of the disks deleted around $\bar{p}$ and $\bar{q}$). Now we can find two $2k$-tuples of "standard" pairwise disjoint arcs in $\bar{F}$, $(\bar{\xi}_1, \ldots, \bar{\xi}_{2k})$, $(\bar{\eta}_1, \ldots, \bar{\eta}_{2k})$ such that $\bar{\xi}_i$ intersect $\bar{\eta}_j$ only if $i = j$, in which case the intersection is transverse at one point. Furthermore, we can arrange that the points z, $\bar{p}^+$ and $\bar{q}^+$ lie in the same connected component in the complement of these arcs in $\bar{F}$. A nice visualization of these curves on $\bar{F}$ can be obtained

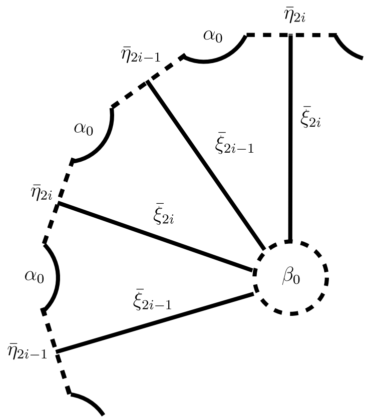

Fig. 2. The curves $(\bar{\xi}_{2i-1}, \bar{\xi}_{2i})$, $(\bar{\eta}_{2i-1}, \bar{\eta}_{2i})$.

by considering a representation of $\bar{F}$ by a $4k$-sided polygon. First, represent a genus k surface by gluing the sides of $4k$-gon in the way prescribed by the labeling $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_k b_k a_k^{-1} b_k^{-1}$ of the sides starting from a vertex and labeling in the clockwise direction. Now remove a neighborhood of each vertex of the polygon and a neighborhood of a point in its interior. This now represents a genus k surface with two boundary components. Let us put β_0 at the boundary of the interior puncture and α_0 at the boundary near the vertices then the curves $(\bar{\xi}_{2i-1}, \bar{\xi}_{2i})$ coincide with the portions of the edges labeled (a_i, b_i) left after removing a neighborhood of each vertex and the curves $(\bar{\eta}_{2i-1}, \bar{\eta}_{2i})$ connect the midpoints of $(\bar{\xi}_{2i-1}, \bar{\xi}_{2i})$ radially to β_0; see Fig. 2.

Now, using the gradient flow of f we can flow the arcs $(\bar{\xi}_1, \ldots, \bar{\xi}_{2k})$ above the northern semi-circle to obtain disjoint arcs $(\xi_1, \ldots, \xi_{2k})$ in F which do not intersect with $\alpha_1, \ldots, \alpha_{g-k}$. (Generic choices ensure that the gradient flow does not hit any critical points.) The flow sweeps out disks in Y which bound $(\alpha_{g-k+1}, \ldots, \alpha_{g+k}) = (\xi_1 \cup \bar{\xi}_1, \ldots, \xi_{2k} \cup \bar{\xi}_{2k})$. Similarly, we define $(\beta_{g-k+1}, \ldots, \beta_{g+k})$ by flowing the arcs $(\bar{\eta}_1, \ldots, \bar{\eta}_{2k})$ above the southern semi-circle. To complete the Heegaard decomposition of (Y, f) we set the base point on $\bar{F}$ to be z which lies in the same region as $\bar{p}^+$ and $\bar{q}^+$. Therefore, we constructed a Heegaard decomposition of (Y, f). We will also make use of a filtration associated with the base point w which we can ensure to be located in the same region as p^+ and q^+ by picking p and q sufficiently close to w, which is the image of z under the gradient flow above the northern and southern semi-circles. Roughly speaking, this point will be used to keep track of the domains passing through the connecting "tubes".

Note that the Heegaard diagram constructed above might be highly inadmissible. An obvious periodic domain with nonnegative coefficients is given by F, which represents the fiber class. However, the standard winding techniques will give us a Heegaard diagram where F (or its multiples) is the *only* potential periodic domain which might prevent our Heegaard diagram from being admissible (which happens if and only if $k = 1$). In fact, we can achieve this by only changing the diagram in the interior of F, so that the standard configuration of curves on $\Sigma_{\min}$ is preserved. Furthermore, we will make sure that, in the new Heegaard diagram, the points w, p^+ and q^+ remain in the same connected component. To get started, fix an arc δ in F, disjoint from all the α and β curves and arcs in $\mathrm{Int}(F)$, that connects the two boundary components of F and passes through p^+ and q^+. We claim that there are $g + k$ simple closed curves $\{\gamma_1, \ldots, \gamma_{g+k}\}$ in F such that γ_i do not intersect δ and the algebraic intersection of γ_i with α_j is 1 if $i = j$ and 0 otherwise. (Note that we do not require the curves $\gamma_1, \ldots, \gamma_{g+k}$ to be disjoint.) For that, we will show that the curves $\alpha_1, \ldots, \alpha_{g-k}, \xi_1, \ldots, \xi_{2k}, \delta$ are linearly independent in $H_1(F, \partial F)$. Then the Poincaré–Lefschetz duality implies the existence of the desired simple closed curves in F which do not intersect δ.

Lemma 7. *The curves $\alpha_1, \ldots, \alpha_{g-k}, \xi_1, \ldots, \xi_{2k}, \delta$ are linearly independent in $H_1(F, \partial F)$.*

Proof. It suffices to show that the complement of $\alpha_1, \ldots, \alpha_{g-k}, \xi_1, \ldots, \xi_{2k}, \delta$ in F is connected. Take any two points a, b in the complement. Now use the gradient flow along the northern semi-circle to obtain $\bar{a}$ and $\bar{b}$. Also let $\bar{\delta}$ be the image of δ under the flow. Connect $\bar{a}$ and $\bar{b}$ in the complement of $\bar{\xi}_1, \ldots, \bar{\xi}_{2k}$ in $\bar{F}$ with a path that is disjoint from $\bar{\delta}$. (This is easy because of the standard configuration of curves in $\bar{F}$.) Now flow the connecting path back to obtain a path that connects a and b in the complement of $\alpha_1, \ldots, \alpha_{g-k}, \xi_1, \ldots, \xi_{2k}$. □

Lemma 8. *Given a basis of the abelian group of periodic domains in the form $F, P_1, \ldots, P_n$, after winding the α curves sufficiently many times along the curves $\{\gamma_1, \ldots, \gamma_{g+k}\}$, we can arrange that any periodic domain which is given as a linear combination of P_i has both positive and negative regions on the Heegaard surface. Furthermore, for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$, the resulting diagram is weakly admissible if $k > 1$.*

Proof. This follows by winding ([16] Section 5) successively along the curves $\{\gamma_1, \ldots, \gamma_{g+k}\}$ in F, first wind along γ_1 all the α curves that intersect γ_1, then wind the resulting curves around γ_2, etc. In this way the α curves stay disjoint (each winding is actually a diffeomorphism of F supported near γ_i, and maps disjoint curves/arcs to disjoint curves/arcs). Furthermore, because winding along γ_i is a diffeomorphism of F isotopic to identity, it preserves the property that α_j and γ_k have algebraic intersection numbers 1 if $j = k$, 0 otherwise. If we had a periodic domain with a nontrivial boundary along α_i, then after winding sufficiently along γ_i, the multiplicity of some region of the periodic domain with boundary in α_i becomes negative. The argument for that relies on the observation that, since the total boundary of the periodic domain has algebraic intersection number 0 with γ_i, and since all the other α curves have algebraic intersection number 0, while α_i has nonzero algebraic intersection, the boundary of the periodic domain must also include a β curve which has nonzero algebraic intersection number with γ_i. Thus after each winding along γ_i, the domain of the periodic domain which has boundary on α_i has a region where the multiplicity is decreased. Hence after sufficiently many windings, we can ensure that any periodic domain with boundary in one of $\alpha_1, \ldots, \alpha_{g+k}$ has at least one negative region.

Furthermore, note that a periodic domain is uniquely determined by the part of its boundary which is spanned by $\{[\alpha_0], \ldots, [\alpha_{g+k}]\}$. Therefore, given a basis $F, P_1, \ldots, P_n$, after winding sufficiently many times, we can make sure that each P_i has sufficiently large multiplicities both positive and negative in certain regions of the Heegaard diagram where all other P_j's have small multiplicities. Thus for a periodic domain to have only positive multiplicities, it must be of the form $mF + m_1 P_1 + \cdots + m_n P_n$ such that m is much larger than $|m_i|$. Then $\langle c_1(\mathfrak{s}), mF + m_1 P_1 + \cdots + m_n P_n \rangle = m\langle c_1(\mathfrak{s}), F\rangle + \sum_{i=1}^{n} m_i \langle c_1(\mathfrak{s}), P_i\rangle$ must be non-zero when $k \neq 1$ since $m\langle c_1(\mathfrak{s}), F\rangle$ dominates the sum and $\langle c_1(\mathfrak{s}), F\rangle = 2 - 2k$ is non-zero. Thus the diagram can be made weakly admissible when $k > 1$. □

We remark that the configuration of the curves on $\bar{F}$ is left intact. Also, the curve δ in F has not been changed. Therefore, after winding we still have the points p and q lying in the same region of the Heegaard diagram. From now on, we will use the notation $(\Sigma, \alpha_0, \ldots, \alpha_{g+k}, \beta_0, \ldots, \beta_{g+k}, z, w)$ for this diagram, which is weakly admissible if $k > 1$. We will refer to this kind of diagrams as *almost admissible*. In order to make sense of Heegaard–Floer homology groups for our special Heegaard diagram in the case when the lowest genus fiber is a torus (i.e. $k = 1$), we will need to work in the perturbed setting since the periodic domain F prevents the diagram from being weakly admissible. However, because we have an "almost admissible" diagram, it suffices to perturb only in the "direction of the fiber class".

Lemma 9. *Given a basis of the abelian group of periodic domains in the form $F, P_1, \ldots, P_n$, we can find an area form A on the Heegaard surface such that $A([F]) > 0$ and $A(span\{P_1, \ldots, P_n\}) = 0$.*

Proof. By the previous lemma, we can arrange that any periodic domain in the linear span of $\{P_1, \ldots, P_n\}$ has both positive and negative regions on the Heegaard surface. The rest of the proof now follows from Farkas' lemma in the theory of convex sets. See [12] Lemmas 4.17–4.18. □

Now an area form A on the Heegaard surface gives a real cohomology class $[\eta] \in H^2(Y; \mathbb{R})$ via the bijection between periodic domains and $H_2(Y; \mathbb{Z})$. Namely, set $[\eta](P) = A(P)$. Choosing a representative $\eta \in [A]$ we can consider the perturbed Heegaard Floer homology $HF^+(Y, f, \eta)$. Since F is the only periodic domain which prevents weak admissibility (only in the case $k = 1$) and $\eta([F]) > 0$, we have a well-defined group $HF^+(Y, f, \eta)$ by the following lemma.

Lemma 10. *Given $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i, j \in \mathbb{Z}_{\geq 0}$ and $r, s \in \mathbb{R}$ there are only finitely many homology classes $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$, with $n_z(\varphi) = i - j$ and $\eta(\varphi) = r - s$ which have positive domains.*

Proof. Let φ and ψ be in $\pi_2(\mathbf{x}, \mathbf{y})$, then $\varphi - \psi \in \pi_2(\mathbf{x}, \mathbf{x})$. We can write $\varphi - \psi = mF + m_1 P_1 + \cdots + m_n P_n + n\Sigma$. Since $n_z(\varphi) = n_z(\psi)$, we have $n = 0$. Also since $\eta(\varphi) = \eta(\psi)$ and $\eta(F) \neq 0$ while $\eta(P_i) = 0$, we conclude that $m = 0$. Finally, since $A(P_i) = 0$, we have $A(\varphi) = A(\psi)$ but then there are only finitely many nonnegative domains which have a fixed area. □

Now, as explained in the introduction $HF^+(Y, f, \eta)$ is an invariant of $(Y, [\eta])$, in fact it only depends on $\mathrm{Ker}(\eta) \cap H_2(Y; Z)$, hence is independent of the value of $\eta([F])$.

The usual invariance arguments of Heegaard–Floer theory, as in [16], imply that $HF^+(Y, f, \eta)$ is independent of the choice of f within its smooth isotopy class. Also note that a geometric way of choosing η is by choosing a section γ of f (a section of f always exists) and letting $[\eta]$ be the Poincaré dual of $[\gamma]$. In that case, we will write $HF^+(Y, f, \gamma)$ for this perturbed Heegaard–Floer homology group. In fact, the choice of the base points w and z as above gives a section of f. Namely, note that we have arranged so that the image of z under the flow above both the northern and the southern semi-circles lies in the same region as w. The union of these two gradient flow lines can therefore be perturbed into a section of f, which we will denote by γ_w. The group $HF^+(Y, f, \gamma_w)$ will be one of the main protagonists in this paper. The differential of this group can be made more explicit as follows: choose a basis of the group of periodic domains in the form $F, P_1, \ldots, P_n$ such that F is the fiber of f and P_i are periodic domains so that the boundary of P_i does not include α_0 or β_0. (This can be arranged by subtracting a multiple of F.) Then if we choose $\eta \in PD[\gamma_w]$ we have $\eta(\mathrm{span}(P_1, \ldots, P_m)) = 0$ and $\eta(F) = n_w(F) = 1$. Therefore for any periodic domain P, we have $\eta(P) = n_w(P)$. Thus there exists a function $\lambda : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{R}$ such that for any $\varphi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have $\eta(\varphi) - n_w(\varphi) = \lambda(\mathbf{x}) - \lambda(\mathbf{y})$. Hence, we can define the differential for $HF^+(Y, f, \gamma_w)$ as follows:

$$\partial^+([\mathbf{x}, i]) = \sum_{\mathbf{y}} \sum_{\varphi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\varphi) \leq i} \#\widehat{\mathcal{M}}(\varphi) t^{n_w(\varphi)} [y, i - n_z(\varphi)].$$

This yields the same homology groups as the original definition where the differential is weighted by $t^{\eta(\varphi)}$: namely, the two chain complexes are related by rescaling each generator $[\mathbf{x}, i]$ to $t^{\lambda(\mathbf{x})}[\mathbf{x}, i]$. When we consider $HF^+(Y, f, \gamma_w)$, we will always consider the differential above.

2.2. Splitting the Heegaard diagram

As explained in the introduction, we will only consider the spin^c structures on Y that satisfy the adjunction equality with respect to $\Sigma_{\min}$; the set of isomorphism classes of such spin^c structures was denoted by $\mathcal{S}(Y|\Sigma_{\min})$. In this section we observe that for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$, we obtain a nice splitting of the generators of the Heegaard–Floer complex into intersections in F and $\bar{F}$. Furthermore, we prove a key lemma en route to understanding the holomorphic curves contributing to the differential.

Let us denote by I_{left} the intersection of $\alpha_1 \times \cdots \times \alpha_{g-k}$ and $\beta_1 \times \cdots \times \beta_{g-k}$ in $\text{Sym}^{g-k}(\Sigma)$, and by I_{right} the set of intersection points of $\alpha_0 \times \alpha_{g-k+1} \times \cdots \times \alpha_{g+k}$ and $\beta_0 \times \beta_{g-k+1} \times \cdots \times \beta_{g+k}$ in $\text{Sym}^{2k+1}(\Sigma)$ such that each intersection point lies in $\bar{F}$. Thus, each element of I_{right} consists of one point from the set of $4k$ intersection points of α_0 with $\eta_1, \ldots, \eta_{2k}$, another point from the set of $4k$ intersection points of β_0 with $\xi_1, \ldots, \xi_{2k}$ and finally $2k - 1$ points from the set of $2k$ points consisting of the intersections of $\bar{\xi}_i$ with $\bar{\eta}_i$ for $i = 1, \ldots, 2l$.

We have $I_{\text{left}} \otimes I_{\text{right}} \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, where $\mathbb{T}_\alpha = \alpha_0 \times \cdots \times \alpha_{g+k}$ and $\mathbb{T}_\beta = \beta_0 \times \cdots \times \beta_{g+k}$ are the Heegaard tori in $\text{Sym}^{g+k+1}(\Sigma)$. Denote by C_{left} and C_{right} the free Λ-modules generated by I_{left} and I_{right} respectively.

Lemma 11. *An intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ induces a* spin^c *structure $\mathfrak{s}_z(\mathbf{x}) \in \mathcal{S}(Y|\Sigma_{\min})$ if and only if $\mathbf{x} \in C_{\text{left}} \otimes C_{\text{right}}$.*

Proof. This follows easily from the following index formula from [17] (cf. Lemma 4.11 in [11]):

$$\langle c_1(\mathfrak{s}_z(\mathbf{x})), F\rangle = e(F) + 2n_{\mathbf{x}}(F)$$

where $n_{\mathbf{x}}(F)$ is the number of components of the tuple $\mathbf{x}$ which lie in F. Since $\mathfrak{s}_z(\mathbf{x}) \in \mathcal{S}(Y|\Sigma_{\min})$, we have $\langle c_1(\mathfrak{s}_z(\mathbf{x})), F\rangle = \langle c_1(\mathfrak{s}_z(\mathbf{x})), \Sigma_{\min}\rangle = 2 - 2k$. Also $e(F) = -2g$, hence the above formula gives

$$n_{\mathbf{x}}(F) = 1 + g - k$$

which is satisfied if and only if $\mathbf{x} \in C_{\text{left}} \otimes C_{\text{right}}$. □

Next, we prove an important lemma about the behavior of holomorphic disks on the tubular regions to the left of α_0 and β_0. This lemma lies at the heart of most of the arguments about the behavior of holomorphic curves that we are going to consider subsequently. For the purpose of the next lemma, let a and b be parallel pushoffs of α_0 and β_0 to the left into the interior of F. Let us label the connected components of the domains in the cylindrical region between a and α_0 by $a_1, \ldots, a_{4k}$ and the cylindrical region between b and β_0 by $b_1, \ldots, b_{4k}$. Choose the labeling so that a_1 and b_1 are in the same region as the arc δ, hence $n_{a_1} = n_{b_1} = n_w$. We will adapt the set-up of Lipshitz's cylindrical reformulation of Heegaard–Floer homology [11]. Let us also call an almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ admissible if it satisfies the axioms (J1-5) of [11], and the differential is obtained via a count of J-holomorphic curves $u : S \to \Sigma \times [0, 1] \times \mathbb{R}$ for an admissible J which satisfy the axioms (M1-6) of [11].

Lemma 12. *Let $\mathbf{x} = \mathbf{x}_{\text{left}} \otimes \mathbf{x}_{\text{right}}$ and $\mathbf{y} = \mathbf{y}_{\text{left}} \otimes \mathbf{y}_{\text{right}}$ be in $C_{\text{left}} \otimes C_{\text{right}}$ and $A \in \pi_2(\mathbf{x}, \mathbf{y})$ and let u be a Maslov index* 1 *holomorphic curve in the homology class A. Assume moreover that the* *contribution of curves in the class A to the differential is non-zero. Then,*

$$n_w(u) = n_{a_1}(u) = \cdots = n_{a_{4k}}(u) = n_{b_1}(u) = \cdots = n_{b_{4k}}(u).$$

Furthermore, if the projection of the image of u to the Heegaard surface $\Sigma = F \cup_{\alpha_0 \cup \beta_0} \bar{F}$ *lies entirely in F, one can find almost complex structures* j_0 *and* j_1 *on* Σ *and an admissible almost complex structure J on* $\Sigma \times [0,1] \times \mathbb{R}$ *such that* $J|_{\Sigma \times \{0\} \times \mathbb{R}} = j_0$ *and* $J_{\Sigma \times \{1\} \times \mathbb{R}} = j_1$ *with the property that u restricted to the boundary does not hit* $\{a, b\} \times \{0, 1\} \times \mathbb{R}$. *(In other words, u converges to Reeb orbits around a and b upon neck stretching.)*

Proof. The first part of the proof will be obtained by "stretching the neck" along the curves a and b. Suppose that there is an i (mod $4k$) such that $n_{a_i}(u) \neq n_{a_{i+1}}(u)$ (one can argue in the same way for b_i's). Thus the source S of u has a piece of boundary which maps to the β arc that separates a_i and a_{i+1}. Let β_j be the curve containing that arc. The crucial observation that we will make use of is the fact that the disk u has no corners in $\beta_j \cap F$, since $\mathbf{x}$ and $\mathbf{y}$ have no components in $\beta_j \cap F$.

We now degenerate Σ along the curves a and b. Specifically, this means that one takes small cylindrical neighborhoods of the curves a and b, and changes the complex structure in those neighborhoods so that the modulus of the cylindrical neighborhoods gets larger and larger. Topologically this degeneration can be understood as follows: after degenerating along a and b, Σ degenerates into Σ_{max} and Σ_{min} and the homology class A splits into A_{left} and A_{right} corresponding to the induced domains on Σ_{max} and Σ_{min} from the domain of A on Σ. (The definition of homology classes $\pi_2(\mathbf{x}, \mathbf{y})$ in this degenerated setting is given in Definition 4.14 of [12]. It is the homology classes of maps to $\Sigma_{\text{max}} \times [0,1] \times \mathbb{R}$ (and to $\Sigma_{\text{min}} \times [0,1] \times \mathbb{R}$) which have strip-like ends converging to $\mathbf{x}$ and $\mathbf{y}$, and to Reeb chords at points of degeneration.)

Next we analyze the holomorphic degeneration of u. Suppose that the moduli space of holomorphic curves representing A is non-empty for all large values of the stretching parameter. Then we conclude by Gromov compactness that there is a subsequence converging to a pair of holomorphic combs of height 1 (in the sense of [12] Section 5.4; see Proposition 5.23 for the proof of Gromov compactness in this setting) u_0 representing A_{left} and u_1 representing A_{right}. (The limiting curves have height 1 because otherwise one of the stages would have index ≤ 0, contradicting transversality—see Proposition 5.6 of [12]). By assumption, the degeneration of u involves breaking along a Reeb chord ρ contained in a with one of the ends of ρ on $a \cap \beta_j$. Hence some component S_0 of the domain of u_0 has a boundary component Γ, consisting of arc components separated by boundary marked points, such that one of the arcs is mapping to β_j and, at one end of that arc, u_0 has a strip-like end converging to the Reeb chord ρ. Now, since there are no corner points on any of the β-arcs in Σ_{max}, the marked points on Γ are all labeled by Reeb chords on a (corresponding to arcs connecting intersection points of β curves with a), and any two consecutive punctures on Γ are connected by an arc which is mapped to part of a β arc which lies on the left half of the Heegaard diagram. Thus, in particular there are no arcs in Γ which map to α curves. Now, we can extend u_0 at the punctures on Γ by sending the marked points to the point of Σ_{max} to which a collapses upon neck-stretching. (This is possible since, after collapsing a, $u_0|_{S_0}$ viewed as a map to Σ_{max} admits a continuous extension at these points. Note that the projection to $[0,1] \times \mathbb{R}$ also extends continuously at the punctures by the definition of holomorphic combs; see the proof of Proposition 5.23 of [12] for more details regarding this.) Therefore, the image of the boundary component Γ under the projection to $[0,1] \times \mathbb{R}$ remains bounded and is entirely contained in $0 \times \mathbb{R}$. Moreover, since the projection is holomorphic, the projection of Γ to $0 \times \mathbb{R}$ is a non-increasing function. Hence we conclude that Γ maps to a constant. Now, the maximum principle implies that the entire component S_0 has to be mapped to a constant value in $0 \times \mathbb{R}$. Therefore, S_0 has all of its boundary components mapped to β curves. Now, u_0 maps all of its boundary to β curves in $\Sigma_{\max}$ which remain linearly independent in homology even after degeneration. However, $u_0(S_0)$ gives a homological relation between those curves. The only way this could be is if the relation is trivial, that is, the boundary of $u_0(S_0)$ traces each β curve algebraically zero times, but that contradicts the assumption that $n_{a_i}(u_{0|S_0}) \neq n_{a_{i+1}}(u_{0|S_0})$ and thus proves the first part of the lemma.

To prove the second part, suppose that u contributes to the differential between $\mathbf{x} = \mathbf{x}_{\text{left}} \otimes \mathbf{x}_{\text{right}}$ and $\mathbf{y} = \mathbf{y}_{\text{left}} \otimes \mathbf{y}_{\text{right}}$ and the image of u lies entirely in F (the left side of the Heegaard diagram). This implies that $\mathbf{x}_{\text{right}} = \mathbf{y}_{\text{right}}$ by using the first part of the lemma which we have already established. Let us describe how we choose the complex structure j_0 (j_1 is constructed in a completely analogous way). Let β_j be the curve such that $\beta_j \cap \alpha_0$ is an intersection point that appears in $\mathbf{x}_{\text{right}} = \mathbf{y}_{\text{right}}$. Note that u cannot have any boundary component that maps to any other β curves that intersect with α_0. Recall also that we have the closed β curves $\beta_1, \ldots, \beta_{g-k}$ in F. After stretching the neck around a and b sufficiently, suppose that u restricted to the boundary still intersects a. As before, in the limit u degenerates and we restrict our attention to the component u_0 which has a boundary component that maps to β_j. We identify the left side of the degenerated Heegaard surface with $\Sigma_{\max}$. From now on, we also think of β_j as a closed curve since after the degeneration along a, the two end points of β_j come together. By exactly the same argument as in the first part, we conclude that u_0 maps all of its boundary components to $\beta_1 \cup \cdots \cup \beta_{g-k} \cup \beta_j$ and its projection to $[0, 1] \times \mathbb{R}$ is constant and lies on $\{0\} \times \mathbb{R}$. To arrive at a contradiction, we would like to ensure that no such u_0 exists by choosing the almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ appropriately such that the restriction of the induced complex structure on $\Sigma_{\max} \times [0, 1] \times \mathbb{R}$ to $\Sigma_{\max} \times \{0\} \times \mathbb{R}$ does not allow such a curve. To that end, we adopt the idea used in Lemma 8.2 of [11] (which in turn is adapted from the idea in Proposition 3.16 of [16]). Namely, since all of $\beta_1, \ldots, \beta_{g-k}, \beta_j$ are linearly independent, we can find disjoint curves on $\Sigma_{\max}$ not intersecting these β curves such that their complement in $\Sigma_{\max}$ is a disjoint union of $(g - k + 1)$ punctured surfaces with each surface having genus at least 1 and such that each β curve is contained in one and only one of these surfaces. We further degenerate the complex structure by stretching along these curves. Note that, crucially, these curves are also disjoint from the original curves a and b along which we degenerate. Therefore we can do the degeneration simultaneously. Let us denote by j_n a sequence of complex structures on $\Sigma_{\max}$ where we stretch along the specified curves as n tends to infinity. Following the argument in Lemma 8.2 of [11], suppose we have a degeneration J_n of admissible almost complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$, corresponding to stretching along a and b and when restricted to a neighborhood of $\Sigma \times \{0\} \times \mathbb{R}$, it has a further degeneration of the form $J_n = j_n \times j_{[0,1]\times\mathbb{R}}$ corresponding to stretching along the other curves that we discussed (which we said are disjoint from a and b). Suppose now that there is a sequence of J_n holomorphic curves that converge to a curve whose projection to $[0, 1] \times \mathbb{R}$ is constant to a point $p \in \{0\} \times \mathbb{R}$. By composing u_n with projection to $[0, 1] \times \mathbb{R}$, and rescaling near p, we get a sequence of maps $u_n : (S_n, \partial S_n) \to (\mathbb{H}, \mathbb{R})$, where S_n is obtained from $\Sigma_{\max}$ by cutting along the β curves. On the other hand, by hypothesis the projection of u_n to Σ is supported in F, which in the limit degenerates to $g - k + 1$ disjoint surfaces of genus at least 1 such that each β curve is in a separate surface. Therefore, by passing to a subsequence and restricting to the complement of the β curves, in the limit we obtain a $(g - k + 1)$-fold covering map from a disjoint union of $g - k + 1$ punctured surfaces of genus at least 1 to $(\mathbb{H}, \mathbb{R})$, which cannot exist. Hence, for a sufficiently large n, if we set $j_0 = j_n$, we can conclude that the map u_0 cannot exist. Hence, by

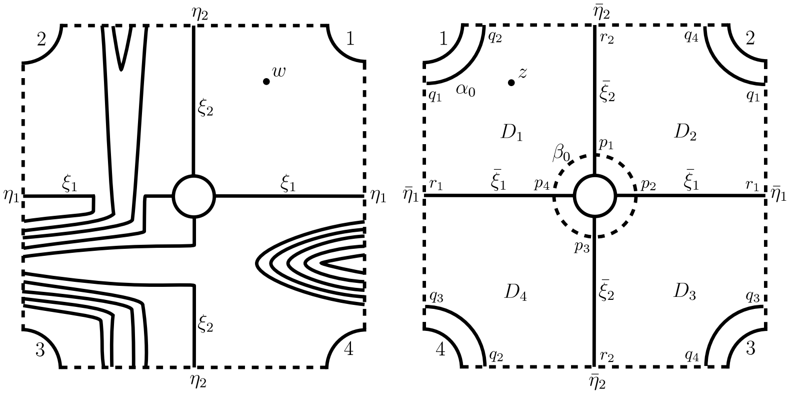

Fig. 3. Torus bundles.

stretching as described, we can ensure that u does not map any of its boundary to a for some J_n where n is sufficiently large. One can similarly arrange by a stretching supported near $\Sigma \times \{1\} \times \mathbb{R}$ that u restricted to its boundary does not intersect b. □

2.3. *Calculations for fibered 3-manifolds and* C_{right}

Before delving into a general study of Heegaard–Floer homology for broken fibrations, here we will calculate $HF^+(Y, \eta)$ in the case of fibered 3-manifolds. Some of these calculations were done independently by Wu in [31], where perturbed Heegaard–Floer homology for $\Sigma_g \times S^1$ is calculated for all spin^c structures. We take the liberty to reconstruct some of the arguments presented there in this section since these calculations will play a role for the calculations we do for general fibered 3-manifolds. Even though we will do calculations in general for any fibered 3-manifold, we will restrict ourselves to spin^c structures in $\mathcal{S}(Y|F)$, which will simplify the calculations. Our conclusion is that $\oplus_{\mathfrak{s}\in\mathcal{S}(Y|F)} HF^+(Y, \eta)$ has rank 1. See also [1] for a different approach in the case of torus bundles.

For fibered 3-manifolds, we have $g = k$, thus the Heegaard diagram has the curves α_0, β_0, and the rest of the diagram is constructed from the standard configuration of curves $\bar{\xi}_1, \ldots, \bar{\xi}_{2k}$, $\bar{\eta}_1, \ldots, \bar{\eta}_{2k}$ as in Fig. 2. Also we will see below that, for the spin^c structures in $\mathcal{S}(Y|F)$, the generators of our chain complex are given by the intersection points in C_{right}.

We first discuss the case of torus bundles. It will then be clear that the general case is just a matter of notational complication. Also note that, in the case of torus bundles, we have to use a perturbation η with $\eta([F]) > 0$ as explained in the previous section since our diagram is not weakly admissible. For higher genus fibrations, the diagram is weakly admissible hence our calculation also determines the unperturbed Heegaard–Floer homology $HF^+(Y)$. When doing explicit calculations we will always consider the case of $HF^+(Y, f, \gamma_w)$ but clearly all arguments go through for any perturbation with η satisfying $\eta([F]) > 0$, or for the unperturbed case whenever the diagram is weakly admissible.

Fig. 3 shows the Heegaard diagram for T^3. Both the left and the right figure are twice punctured tori, and are identified along the two boundaries (the one in the middle and the one formed by the four corners) where the gluing of the left and right figures is made precise by the labels at the four corners. On the right side the standard set of arcs $\bar{\xi}_1, \bar{\xi}_2, \bar{\eta}_1, \bar{\eta}_2$ are depicted; the left side is constructed by taking the images of these arcs under the horizontal flow (which is the identity map for T^3), and winding ξ_1 and ξ_2 along transverse circles so that the diagram becomes almost admissible. (Note that the winding process avoids the region where w is placed, as required: first ξ_2 is wound once along a horizontal circle, then ξ_1 is wound twice along a vertical circle.) For general torus bundles, the same construction will give a Heegaard diagram, where ξ_1 and ξ_2 are replaced by their images under the monodromy of the torus bundle. The important observation here is that the right side of the diagram is always standard. We will show that all the calculations that we need can be done on the right side of the diagram for the spin^c structures we have in mind. The calculation for T^3 is essentially the same as in [31]. However, we will see that Lemma 12 plays a crucial role in the calculation for general torus bundles. We first do the calculation for T^3.

Proposition 13. *$HF^+(T^3, f, \gamma_w, \mathfrak{s}_0) = \Lambda$ where $\mathfrak{s}_0 \in \mathcal{S}(T^3|T^2)$ is the unique torsion* spin^c *structure on T^3.*

Proof. As in Lemma 11, $\mathfrak{s}_{\mathbf{x}}(z) \in \mathcal{S}(T^3|T^2)$ if and only if $\mathbf{x} \in C_{\text{right}}$, hence $\mathbf{x}$ can be one of the following tuples of intersections depicted in Fig. 3:

$$\begin{array}{llll} \mathbf{x}_1 = p_1q_2r_1 & \mathbf{x}_2 = p_2q_1r_2 & \mathbf{x}_3 = p_3q_4r_1 & \mathbf{x}_4 = p_4q_3r_2 \\ \mathbf{y}_1 = p_4q_1r_2 & \mathbf{y}_2 = p_1q_4r_1 & \mathbf{y}_3 = p_2q_3r_2 & \mathbf{y}_4 = p_3q_2r_1. \end{array}$$

Next, we apply the adjunction inequality for the other T^2 components, this implies that the Heegaard–Floer groups vanish except for the unique torsion spin^c structure, $\mathfrak{s}_0$ which has $c_1(\mathfrak{s}_0) = 0$. The two other torus components are realized by periodic domains in Fig. 3, one of them is the domain P_1 including $D_2 \cup D_3$ and bounded by α_2 and β_1, the other one is the domain P_2 including $D_3 \cup D_4$ and bounded by α_1 and β_2. Then the formula $\langle c_1(\mathfrak{s}_z(\mathbf{x})), P_i\rangle = e(P_i) + 2n_{\mathbf{x}}(P_i)$, implies that the only intersection points for which $\mathfrak{s}_z(\mathbf{x}) = \mathfrak{s}_0$ are $\mathbf{x}_1$ and $\mathbf{y}_1$. Furthermore, note that D_1 is a hexagonal region connecting $\mathbf{x}_1$ to $\mathbf{y}_1$, hence it is represented by a holomorphic disk $\varphi_1 \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$, and the algebraic number of holomorphic disks in the corresponding moduli space of disks in the homology class of φ_1 is given by $\#\widehat{\mathcal{M}}(\varphi_1) = \pm 1$. (See Appendix in [24].)

Now, given any other Maslov index 1 homology class $A \in \pi_2(\mathbf{x}_1, \mathbf{y}_1)$, we have $A = D_1 + mF + m_1P_1 + m_2P_2$. In particular, note that $n_z(A) = 1$. Furthermore, if we restrict to those with $n_w = 0$ (that is $m = 0$), since $m_1P_1 + m_2P_2$ has both positive and negative domains by almost admissibility, unless $m_1 = m_2 = 0$ there is no holomorphic representative of A.

We conclude that $\partial^+[\mathbf{x}_1, i] = f(t)[\mathbf{y}_1, i-1]$, where $f(t) = \pm 1 + \mathrm{O}(t)$ is invertible in the Novikov ring. This implies that $[\mathbf{y}_1, i]$ is in the image of ∂^+. Thus in particular we have $\partial^+[\mathbf{y}_1, i] = 0$ for all i. Finally, there is no Maslov index 1 disk with $n_w = 0$ which connects $\mathbf{x}_1$ to itself or $\mathbf{y}_1$ to itself. Thus we conclude that in $CF^+(T^3, f, \gamma_w, \mathfrak{s}_0)$:

$$\begin{array}{ll} \partial^+[\mathbf{x}_1, 0] = 0 & \partial^+[\mathbf{y}_1, i] = 0 \\ \partial^+[\mathbf{x}_1, i] = (\pm 1 + \mathrm{O}(t))[\mathbf{y}_1, i-1] & \text{for } i > 0. \end{array}$$

Hence the homology is generated by $[\mathbf{x}_1, 0]$, in other words $HF^+(T^3, f, \gamma_w, \mathfrak{s}_0) = \Lambda$ as required. □

From now on, we will simply write $\mathbf{x}_1$ for $[\mathbf{x}_1, 0]$. The next theorem generalizes this calculation to any torus bundle.

Theorem 14. *Let (Y, f) be a torus bundle and let $\mathfrak{s}$ be in $\mathcal{S}(Y|T^2)$. Then, $HF^+(Y, f, \gamma_w, \mathfrak{s}) = \Lambda$ if $\mathfrak{s} = \mathfrak{s}_0$ where $\mathfrak{s}_0$ is the* spin^c *structure corresponding to the vertical tangent bundle and $HF^+(Y, f, \gamma_w, \mathfrak{s}) = 0$ otherwise.*

Proof. The main difficulty for the general torus bundle case that makes the calculation different from the calculation for T^3 is that we cannot a priori eliminate the generators $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ and $\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$. In fact, if the first Betti number of the torus bundle is equal to 1, these generators are in the same spin^c class as $\mathbf{x}_1$ and $\mathbf{y}_1$.

Now, the domains D_i are homology classes in $\pi_2(\mathbf{x}_i, \mathbf{y}_i)$, which have holomorphic representatives φ_i with $\#\mathcal{M}(\varphi_i) = \pm 1$. Since any non-trivial periodic domain has to pass through some region to the left of α_0 or β_0, any other homology class in $\pi_2(\mathbf{x}_i, \mathbf{y}_i)$ which contributes to the differential has to have $n_w \neq 0$ by Lemma 12. For the same reason, any homology class in $\pi_2(\mathbf{x}_i, \mathbf{y}_j)$ for some $i \neq j$ which contributes to the differential has to have $n_w \neq 0$ since there is no homology class in $\pi_2(\mathbf{x}_i, \mathbf{y}_j)$ that lies in the right side of the diagram (this can be verified either by inspection, or referring to the case of T^3, where $\mathbf{x}_i$ and $\mathbf{y}_j$ represent different spin^c classes for $i \neq j$). Moreover, the classes in $\pi_2(\mathbf{x}_i, \mathbf{x}_j)$ all have even Maslov index, hence do not contribute to the differential. Therefore, we have

$$\partial^+[\mathbf{x}_1, i] = [\mathbf{y}_1, i-1] \pmod t \quad \text{for } i > 0$$
$$\partial^+[\mathbf{x}_1, 0] = 0 \pmod t \qquad \partial^+[\mathbf{x}_2, i] = [\mathbf{y}_2, i] \pmod t$$
$$\partial^+[\mathbf{x}_3, i] = [\mathbf{y}_3, i] \pmod t \qquad \partial^+[\mathbf{x}_4, i] = [\mathbf{y}_4, i] \pmod t$$

where the higher order terms do not involve the $\mathbf{x}_j$'s. As before, because we are working over a Novikov ring of power series, we conclude that $[\mathbf{y}_1, i]$, $[\mathbf{y}_2, i]$, $[\mathbf{y}_3, i]$ and $[\mathbf{y}_4, i]$ are all in the image of ∂^+. Furthermore, the only possible generator which might be in the kernel of ∂^+ is $[\mathbf{x}_1, 0]$. Finally Lemma 15 shows that there is no holomorphic disk starting at $\mathbf{x}_1$ with $n_z = 0$ and $n_w \neq 0$. Hence we have $\partial^+[\mathbf{x}_1, 0] = 0$ and the homology group $\oplus_{\mathfrak{s} \in \mathcal{S}(Y|T^2)} HF^+(Y, f, \gamma_w, \mathfrak{s})$ is generated by $[\mathbf{x}_1, 0]$. Furthermore, $\mathfrak{s}_z(\mathbf{x}_1) = \mathfrak{s}_0$ so the theorem is proved. □

Note also that the adjunction inequality implies that $HF^+(Y, f, \gamma_w, s)$ vanishes for $\mathfrak{s} \notin \mathcal{S}(Y|T^2)$. Therefore the above calculation is in fact a complete calculation of perturbed Heegaard–Floer homology for torus bundles.

The following lemma which we used in the above calculation holds in general (not only in the fibered case). Let Y be any 3-manifold with $b_1 > 0$, and $f : Y \to S^1$ a broken fibration with connected fibers. Construct the almost admissible Heegaard diagram for f as before and let $\mathbf{x}_1 \in C_{\text{right}}$ be given by the union of the intersection points in $\alpha_0 \cap \bar{\eta}_2$, $\bar{\xi}_2 \cap \beta_0$, and $\bar{\xi}_i \cap \bar{\eta}_i$ for $i \neq 2$, where the intersection points in $\alpha_0 \cap \beta_2$ and $\alpha_2 \cap \beta_0$ are chosen so that the region containing z includes them as corners. (In the case of the torus bundle this is the generator $[\mathbf{x}_1, 0]$.) Note that the generators of C_{right} can always be described from the standard diagram since the right hand sides of our Heegaard diagrams are always the same.

Lemma 15. *Let $\varphi \in \pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{y}_{\text{right}})$ be a holomorphic disk in a class that contributes non-trivially to the differential for given $\mathbf{x}_{\text{left}}, \mathbf{y}_{\text{left}}, \mathbf{y}_{\text{right}}$. If $n_z(\varphi) = 0$, then $\mathbf{y}_{\text{right}} = \mathbf{x}_1$ and the domain of φ is contained on the left side of the Heegaard diagram (i.e. it is contained in F).*

Proof. Consider the component of $\mathbf{x}_1$ which is an intersection point on β_0, say p_1. Now, among the four regions which have p_1 as one of their corners, one includes z, namely D_1, and two of them lie in the left half of the diagram, hence by Lemma 12, they must have the same multiplicity. Denote these regions by L_1 and L_2, so that L_1 and D_1 share an edge on β_0. If the component of φ which is asymptotic to p_1 is constant, then p_1 is also part of $\mathbf{y}_{\text{right}}$. Otherwise, since φ has a corner which leaves p_1 and $n_z(\varphi) = 0$, we must have a non-zero multiplicity at L_2, but since L_1 and L_2 must have the same multiplicity, this implies that p_1 has to be a member in $\mathbf{y}_{\text{right}}$.

The same conclusion applies for the point of $\mathbf{x}_1$ which lies on α_0. But then there is a unique way to complete these two intersection points to a generator in C_{right}, hence we conclude that $\mathbf{y}_{\text{right}} = \mathbf{x}_1$. Thus φ is in $\pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{x}_1)$.

Furthermore, since φ fixes $\mathbf{x}_1$, the intersection of the domain of φ with $\bar{F}$ must coincide with the intersection of some periodic domain for $S^1 \times \Sigma_k$ with $\bar{F}$ (since any domain that has no corners on the right side, can be completed to a periodic domain on the Heegaard diagram of $S^1 \times \Sigma_k$ by reflecting). However, it is easy to identify all the periodic domains of $S^1 \times \Sigma_k$ and observe that no non-trivial combination of periodic domains for $S^1 \times \Sigma_k$ (if we leave out F and its multiples), can have the same multiplicity in the regions immediately to the left of α_0 and β_0. However, by Lemma 12 this property has to hold. This proves the lemma. □

Theorem 16. *Let (Y, f) be a fiber bundle with fiber a genus g surface and let $\mathfrak{s}$ be in $\mathcal{S}(Y|\Sigma_g)$. Then, $HF^+(Y, f, \gamma_w, \mathfrak{s}) = \Lambda$ if $\mathfrak{s} = \mathfrak{s}_0$ where $\mathfrak{s}_0$ is the* spin^c *structure corresponding to a vertical tangent bundle and $HF^+(Y, f, \gamma_w, \mathfrak{s}) = 0$ otherwise.*

Proof. The proof is essentially the same as the proof of the corresponding theorem for the torus bundles. The only difference is the number of generators which are canceled out: there are now $8g$ generators $\mathbf{x}_1, \ldots, \mathbf{x}_{4g}$, $\mathbf{y}_1, \ldots, \mathbf{y}_{4g}$, and the $4g$ hexagonal regions of $\bar{F}$ (see Fig. 2) give $\partial^+[\mathbf{x}_1, i] = [\mathbf{y}_1, i-1] \pmod t$ and $\partial^+[\mathbf{x}_j, i] = [\mathbf{y}_j, i] \pmod t$ for $j \geq 2$. Arguing as before, the only generator left is again $\mathbf{x}_1$ which gives $\mathfrak{s}_z(\mathbf{x}_1) = \mathfrak{s}_0$. □

Note that this gives a new way of obtaining the results of the original calculation of Ozsváth and Szabó in [16] for fibered 3-manifolds.

Corollary 17. *Let (Y, f) be a fiber bundle with fiber a genus $g > 1$ surface and let $\mathfrak{s}$ be in $\mathcal{S}(Y|\Sigma_g)$. Then, $HF^+(Y, \mathfrak{s}) = \mathbb{Z}$ if $\mathfrak{s} = \mathfrak{s}_0$ where $\mathfrak{s}_0$ is the* spin^c *structure corresponding to vertical tangent bundle and $HF^+(Y, \mathfrak{s}) = 0$ otherwise.*

Proof. Since the diagram is weakly admissible, we can let $t = 1$ and the result follows from the previous theorem. □

In general, let $\partial^+_{\text{right}}$ be the contribution to the Heegaard–Floer differential from the holomorphic disks whose domain lies in $\bar{F}$ (i.e. the disks which lie on the right half of our almost admissible Heegaard diagrams), also we write $CF^+_{\text{right}} = C_{\text{right}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$ for the chain complex associated with the right side of the diagram for the purpose of constructing HF^+ theory, that is, the chain complex freely generated over Λ by $[\mathbf{x}, i]$ where $\mathbf{x} \in C_{\text{right}}$ and $i \in \mathbb{Z}_{\geq 0}$.

Corollary 18. *$(CF^+_{\text{right}}, \partial^+_{\text{right}})$ is a chain complex with rank 1 homology generated by $\mathbf{x}_1$.*

Proof. This is only a reformulation of the above results. □

3. The isomorphism

In this section, we prove the main theorem of this paper. Namely, we prove that the perturbed Heegaard–Floer homology group $HF^+(Y, f, \gamma_w)$ is isomorphic to the Floer homology of the chain complex $(C_{\text{left}}, \partial_{\text{left}}; \Lambda)$. Before stating our theorem let us digress to give a rigorous definition of the latter chain complex.

3.1. A variant of Heegaard–Floer homology for broken fibrations over the circle

Let Y be a 3-manifold with $b^1 > 0$, and let $f : Y \to S^1$ be a broken fibration with connected fibers, and satisfying the conditions at the beginning of Section 2.1. As before consider the highest genus fiber Σ_g and let $\alpha_1, \ldots, \alpha_{g-k}$ and $\beta_1, \ldots, \beta_{g-k}$ be tuples of $g-k$ disjoint linearly independent simple closed curves on Σ_g obtained from the attaching circles corresponding to the critical values of f, and let w be a base point that is in the complement of α and β curves. As in Lemma 8, we can arrange weak admissibility for $k > 1$ by winding if necessary. For $k = 1$, we need to keep track of the intersection with the point w and have to work over Λ. We define the Floer homology of such a configuration in a manner similar to the usual Heegaard–Floer theory by defining the chain complex to be the Λ-module freely generated by intersection points of $\mathbb{T}_\alpha^{g-k} = \alpha_1 \times \cdots \alpha_{g-k}$ and $\mathbb{T}_\beta^{g-k} = \beta_1 \times \cdots \beta_{g-k}$ in $\mathrm{Sym}^{g-k}(\Sigma_g)$, equipped with a differential given as follows:

$$\partial \mathbf{x} = \sum_{\varphi \in \pi_2(\mathbf{x},\mathbf{y}), \mu(\varphi)=1} \#\widehat{\mathcal{M}}(\varphi) t^{n_w(\varphi)} \mathbf{y}.$$

For reasons that will be clarified in Section 4, we will denote the homology group that we expect to get from this construction $QFH'(Y, f; \Lambda)$. This stands for *quilted Floer homology* of the broken fibration (Y, f) with coefficients in Λ. There are at least two obvious issues that we need to address in order to make sure that $QFH'(Y, f; \Lambda)$ is well-defined. The first issue is the compactness of the moduli space $\mathcal{M}(\varphi)$. The second issue is proving that $\partial^2 = 0$. The setup here is more delicate than the usual setup of Heegaard–Floer homology due to the fact that $\mathrm{Sym}^{g-k}(\Sigma_g)$ is not a (positively) monotone symplectic manifold when $k > 0$ (it has $\langle c_1, [\Sigma_g]\rangle = 2-2k$). Therefore, one expects the existence of configurations with negative Chern number bubbles. However, we will adopt the cylindrical setting of Lipshitz [11], whereby one considers pseudo-holomorphic curves in $\Sigma_g \times [0, 1] \times \mathbb{R}$ instead of disks in $\mathrm{Sym}^{g-k}(\Sigma_g)$, and choose our almost complex structures from a sufficiently general class. Namely, one chooses a translation-invariant almost-complex structure J on $\Sigma_g \times [0, 1] \times \mathbb{R}$ such that J preserves a 2-plane distribution ξ on $\Sigma_g \times [0, 1]$ which is tangent to Σ_g near $(\boldsymbol{\alpha} \cup \boldsymbol{\beta}) \times [0, 1]$ and near $\Sigma_g \times \partial[0, 1]$ (see [11], axiom J5'). Now we can show that transversality can be achieved for holomorphic curves in the homology class of the fiber of the projection $\Sigma_g \times [0, 1] \times \mathbb{R} \to [0, 1] \times \mathbb{R}$. However the expected dimension of these curves is negative, hence bubbling at interior points can be ruled out a priori (see [11] Lemma 8.2). Furthermore, since we assumed that all the fibers are connected, the $(g-k)$-tuples of curves are linearly independent in homology; this implies that any boundary bubble lifts to a spherical class in $\pi_2(\mathrm{Sym}^{g-k}(\Sigma_g))$. By choosing almost complex structures such that $J_{|\Sigma \times \{0\} \times \mathbb{R}}$ and $J_{|\Sigma \times \{1\} \times \mathbb{R}}$ are of special type as described in Lemma 8.2 of [11] (cf. Lemma 12), we can also avoid boundary bubbles. Hence, the compactness of $\mathcal{M}(\varphi)$ is ensured. We will once and for all restrict our choice of almost complex structures to this class of almost complex structures.

The drawback of this approach is that it does not correspond in a straightforward way to the original setting in $(\mathrm{Sym}^{g-k} \Sigma_g, \mathbb{T}_\alpha^{g-k}, \mathbb{T}_\beta^{g-k})$ since such general almost complex structures prevent the fibers of the projection to $[0, 1] \times \mathbb{R}$ from being complex. In this case, in order to be able to work in $\mathrm{Sym}^{g-k}(\Sigma_g)$ one needs to establish a proper combinatorial rule for handling bubbled configurations (for example by applying the general machinery of virtual fundamental cycles [13]). It is reasonable to expect that one would then get the same differential as above, but the argument would be technically very involved. However, there is an exception to this, namely when we are in the strongly negative case, that is when $g < 2k$. We show in Section 4 that in this case we can indeed use integrable complex structures of the form $\mathrm{Sym}^{g-k}(j_s)$ for a path j_s of complex structures on Σ and still avoid bubbling since such complex structures are sufficient to achieve transversality and the assumption $g < 2k$ ensures that the expected dimension of bubbles is negative.

The proof of $\partial^2 = 0$ for $QFH'(Y, f; \Lambda)$ will be part of the proof of the isomorphism that we will construct between $QFH'(Y, f; \Lambda)$ and $HF^+(Y, f, \gamma_w)$. Namely, this follows from an identification between the Maslov index 1 moduli spaces in both theories. Furthermore, we will also see in this section that $QFH'(Y, f; \Lambda)$ is an invariant of $(Y, [f])$, that is it only depends on Y and the homotopy class of f through the homology class $[\Sigma_{\min}] \in H_2(Y)$.

As usual in Floer homology theories, the groups $QFH'(Y, f; \Lambda)$ are graded by equivalence classes of spin^c structures. Given an intersection point in $\mathbf{x} \in \mathbb{T}_\alpha^{g-k} \cap \mathbb{T}_\beta^{g-k}$ one gets a spin^c structure $\mathfrak{s}(\mathbf{x}) \in \mathcal{S}(Y|\Sigma_{\min})$, as in Heegaard–Floer theory, except we do not need to consider any additional base point since the intersection point $\mathbf{x}$ gives a matching of index 1 and 2 critical points of f, which in turn determines a spin^c structure by taking the gradient vector field of f outside of tubular neighborhoods of these matching flow lines and extending it in a non-vanishing way to the tubular neighborhoods. We remark that in our setup of the Heegaard diagram for (Y, f), we have the equality $\mathfrak{s}(\mathbf{x}_{\text{left}}) = \mathfrak{s}_z(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1)$ (where $\mathbf{x}_1$ is as in Lemma 15).

Remark. Note that if we restrict to the case where we only count $n_w = 0$ curves we obtain Juhász's sutured Floer homology groups associated with the diagram $(F, \alpha_1, \ldots, \alpha_{g-k}, \beta_1, \ldots, \beta_{g-k})$ (see [5]). We will return to this below.

3.2. Isomorphism between $QFH'(Y, f; \Lambda)$ and $HF^+(Y, f, \gamma_w)$

We now proceed to prove an isomorphism between $QFH'(Y, f; \Lambda)$ and $HF^+(Y, f, \gamma_w)$. As a first step, we make use of the calculations of the previous section. Let $CF^+_{\text{left}} = C_{\text{left}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$ and $CF^+_{\text{right}} = C_{\text{right}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$, using the splitting of generators of $HF^+(Y, f, \gamma_w)$ as discussed in Section 2.2, so that we have $CF^+(Y, f, \gamma_w) = CF^+_{\text{left}} \otimes CF^+_{\text{right}}$. We denote by ∂_F and $\partial_{\bar{F}} = \mathbb{1} \otimes \partial_{\text{right}}$ the contributions to the Heegaard–Floer differential from holomorphic curves whose domains lie in F and $\bar{F}$ respectively. Furthermore, we denote by $\partial_{\text{left}} \otimes \mathbb{1}$, the contribution of those holomorphic curves whose domain lies in F and which act by identity on C_{right} with respect to the splitting $C_{\text{left}} \otimes C_{\text{right}}$. Since the boundary of F includes points of intersections occurring in C_{right}, this is a priori more restrictive than ∂_F. However, Lemma 15 implies that $\partial_{\text{left}} \otimes \mathbb{1}$ is a differential on $C_{\text{left}} \otimes \mathbf{x}_1$ (and the argument given in the proof of Lemma 15 more generally shows that $\partial_F = \partial_{\text{left}} \otimes \mathbb{1}$). The next proposition says that the homology of this differential is isomorphic to $HF^+(Y, f, \gamma_w)$.

Proposition 19. *$HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq H(C_{\text{left}} \otimes \mathbf{x}_1, \partial_{\text{left}} \otimes \mathbb{1}, \gamma_w, \mathfrak{s})$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$.*

Proof. Both homology groups are filtered by n_w. Therefore, there are spectral sequences converging to both sides induced by the n_w filtration. Furthermore, we claim that there is a chain map:

$$F : C_{\text{left}} \otimes \mathbf{x}_1 \to CF^+_{\text{left}} \otimes CF^+_{\text{right}}$$

given by

$$F(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1) = [\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, 0]$$

which induces an isomorphism of E^1-pages of the spectral sequences associated with both chain complexes. The fact that F is a chain map, is a consequence of Lemma 15. More precisely, Lemma 15 gives that if a holomorphic map contributing to the differential originates at $[\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, 0]$ then it has to converge to a generator of the form $[\mathbf{y}_{\text{left}} \otimes \mathbf{x}_1, 0]$, and the domain of the map has to lie on the left half of the Heegaard diagram; these are exactly the contributions to the differential captured by $\partial_{\text{left}} \otimes \mathbb{1}$.

Furthermore, showing that F induces an isomorphism on the E^1-pages of the spectral sequences on both sides amounts to checking that

$$F' : (C_{\text{left}} \otimes \mathbf{x}_1, \partial^0_{\text{left}} \otimes \mathbb{1}) \to (CF^+_{\text{left}} \otimes CF^+_{\text{right}}, \partial^0_{\text{left}} \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}})$$

is an isomorphism in homology, where $\partial^0_{\text{left}} \otimes \mathbb{1}$ denotes those holomorphic maps contributing to the differential ∂_F with $n_w = 0$. (Here we have used Lemma 12 to identify $n_w = 0$ part of ∂^+ with $\partial^0_{\text{left}} \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$.) The injectivity of F' in homology follows from the fact that, by Corollary 18 (see also the proof of Theorem 16), $\mathbf{x}_1$ does not lie in the image of ∂_{right}. Thus, we only need to check that F' is surjective in homology. Suppose that $\mathbf{a}_1\mathbf{x}_1 + \cdots + \mathbf{a}_{4k}\mathbf{x}_{4k} + \mathbf{b}_1\mathbf{y}_1 + \cdots + \mathbf{b}_{4k}\mathbf{y}_{4k} \in CF^+_{\text{left}} \otimes CF^+_{\text{right}}$ is in the kernel of $\partial^0_{\text{left}} \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$, where we have chosen the notation so that $\mathbf{a}_i$ and $\mathbf{b}_i$ are elements in $CF^+_{\text{left}} = C_{\text{left}} \otimes \Lambda[\mathbb{Z}_{\geq 0}]$, and $\mathbf{x}_i$ and $\mathbf{y}_i$ are the generators of C_{right} as in Theorem 16. Now, because this element is in the kernel of $\partial^0_{\text{left}} \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}}$, we have

$$\partial^0_{\text{left}}\mathbf{a}_1 = 0 \quad \text{and} \quad U\mathbf{a}_1 + \partial^0_{\text{left}}\mathbf{b}_1 = 0$$
$$\partial^0_{\text{left}}\mathbf{a}_i = 0 \quad \text{and} \quad \mathbf{a}_i + \partial^0_{\text{left}}\mathbf{b}_i = 0 \quad \text{for } i \neq 1$$

where $U : CF^+_{\text{left}} \to CF^+_{\text{left}}$ is the usual map in Heegaard–Floer theory which maps $[\mathbf{a}, i] \to [\mathbf{a}, i-1]$. It appears in the above equation because the disk D_1 connecting $\mathbf{x}_1$ to $\mathbf{y}_1$ intersects the base point z with multiplicity 1. (Here we also chose an orientation system so that $\partial_{\text{right}}\mathbf{x}_i = \mathbf{y}_i$, one can also do the same calculation if $\partial_{\text{right}}\mathbf{x}_i = -\mathbf{y}_i$.)

Now, observe that the above equations give

$$(\partial^0_{\text{left}} \otimes \mathbb{1} + \mathbb{1} \otimes \partial_{\text{right}})(\mathbf{b}_i\mathbf{x}_i) = -\mathbf{a}_i\mathbf{x}_i + \mathbf{b}_i\mathbf{y}_i \quad \text{for } i \neq 1.$$

Therefore, $-\mathbf{a}_i\mathbf{x}_i + \mathbf{b}_i\mathbf{y}_i$ is in the kernel, but by assumption we also had $\mathbf{a}_i\mathbf{x}_i + \mathbf{b}_i\mathbf{y}_i$ in the kernel. This gives us that $2\mathbf{b}_i\mathbf{y}_i$ is in the kernel, which in turn, implies that $\partial^0_{\text{left}}\mathbf{b}_i = 0$. (This holds unless we work over a field of characteristic 2, see below for that case.) Thus, $\mathbf{a}_i = 0$ and $\mathbf{b}_i\mathbf{y}_i$ is in the image of $\partial^0_{\text{left}} \otimes 1 + 1 \otimes \partial_{\text{right}}$. (In characteristic 2, we directly conclude that $\mathbf{a}_i\mathbf{x}_i + \mathbf{b}_i\mathbf{y}_i$ is in the image.) Therefore, in either case we can ignore all the terms other than $\mathbf{a}_1\mathbf{x}_1 + \mathbf{b}_1\mathbf{y}_1$. Furthermore, note that

$$(\partial^0_{\text{left}} \otimes 1 + 1 \otimes \partial_{\text{right}})(U^{-1}\mathbf{b}_1\mathbf{x}_1) = U^{-1}\partial^0_{\text{left}}\mathbf{b}_1\mathbf{x}_1 + \mathbf{b}_1\mathbf{y}_1 = -\mathbf{a}_1\mathbf{x}_1 + \mathbf{b}_1\mathbf{y}_1.$$

Thus, we conclude that $2\mathbf{b}_1\mathbf{y}_1$ is in the kernel, which implies that $\partial^0_{\text{left}}\mathbf{b}_1 = 0$ hence, $U\mathbf{a}_1 = 0$ and $(\partial^0_{\text{left}} \otimes 1 + 1 \otimes \partial_{\text{right}})(U^{-1}\mathbf{b}_1\mathbf{x}_1) = \mathbf{b}_1\mathbf{y}_1$ hence we can ignore the term $\mathbf{b}_1\mathbf{y}_1$ and the fact that $U\mathbf{a}_1 = 0$ implies that $\mathbf{a}_1\mathbf{x}_1$ is in the image of F as desired.

This concludes the proof of Proposition 19 since a filtered chain map that induces an isomorphism of E^1-pages induces an isomorphism at all pages of the spectral sequences (see e.g. Theorem 3.5 of [15]), in particular the E^∞-pages are the groups that we have written in the statement of Proposition 19. □

An immediate corollary that follows from the proof of Proposition 19 is that the U-action on $HF^+(Y, f, \gamma_w, \mathfrak{s})$ is trivial for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$. In fact, we have a splitting of the long exact sequence induced by the U-action, which implies the following relation with the hat-version of the Heegaard–Floer homology where the differential counts the holomorphic curves with $n_z = 0$ (see [16]).

Corollary 20. *For* $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$, $\widehat{HF}(Y, f, \gamma_w, \mathfrak{s}) \simeq HF^+(Y, f, \gamma_w, \mathfrak{s}) \oplus HF^+(Y, f, \gamma_w, \mathfrak{s})$ [1]. □

Note that in the case that $g(\Sigma_{\min}) = k > 1$, there is no perturbation required, thus the above equality holds for the homology groups with $\mathbb{Z}_2$ coefficients. In particular, this implies that $HF^+(Y, \mathfrak{s})$ is algorithmically computable for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ since there are known algorithms for computing $\widehat{HF}(Y, \mathfrak{s})$.

Finally, we are ready to state and prove our main result. Over the course of the proof of the following theorem, we will see why the variant of the Heegaard–Floer homology that we denoted by $QFH'(Y, f, \mathfrak{s}; \Lambda)$ is well-defined. More precisely, we will see that the differential that we defined for $QFH'(Y, f, \mathfrak{s}; \Lambda)$ squares to zero.

Theorem 21. $HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq QFH'(Y, f, \mathfrak{s}; \Lambda)$ *for* $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$.

Proof. Because of Proposition 19, it suffices to prove that

$$H(C_{\text{left}} \otimes \mathbf{x}_1, \partial_{\text{left}} \otimes \mathbb{1}, \gamma_w, \mathfrak{s}) \simeq H(C_{\text{left}}, \partial, \mathfrak{s})$$

where the latter group is what we previously called $QFH(Y, f, s)$. Clearly, we have a one-to-one correspondence between the generators. Next, we will show that there is an isomorphism of chain complexes. In fact, we will show that the signed counts of Maslov index 1 holomorphic curves in $\pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{x}_1)$ which contribute to $\partial_{\text{left}} \otimes 1$ and Maslov index 1 holomorphic curves in $\pi_2(\mathbf{x}_{\text{left}}, \mathbf{y}_{\text{left}})$ that contribute to the differential ∂ are equal. First observe that for curves which stay away from the necks at α_0 and β_0, which are precisely those with $n_w = 0$, this one to one correspondence is clear. (These are the curves that contribute to the differential $\partial^0_{\text{left}} \otimes 1$ in Proposition 19.)

Next, we discuss the curves which have $n_w \neq 0$. The correspondence in this case will be obtained by stretching the necks along a and b, which are respectively parallel pushoffs of α_0 and β_0 to the left of the Heegaard diagram (into the region F).

Let us first describe the holomorphic curves that contribute to $\partial_{\text{left}} \otimes \mathbb{1}$ with $n_w \neq 0$, more precisely. Remember that by definition $\partial_{\text{left}} \otimes \mathbb{1}$ counts those holomorphic curves whose domain lies in F, hence they have $n_z = 0$. Now, recall that Lemma 12 says that by choosing the almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ appropriately, one can arrange that the projection to the Heegaard surface is an unbranched cover around the necks a and b (i.e. the holomorphic curve converges to Reeb orbits around a and b). Let $A \in \pi_2(\mathbf{x}_{\text{left}} \otimes \mathbf{x}_1, \mathbf{y}_{\text{left}} \otimes \mathbf{x}_1)$ be a Maslov index 1 homology class which is contributing to $\partial_{\text{left}} \otimes \mathbb{1}$. By degenerating the almost complex structure around a and b on Σ, we get two homology classes $A_{\text{left}} \in \pi_2(\mathbf{x}_{\text{left}}, \mathbf{y}_{\text{left}})$ and $A_{\text{right}} \in \pi_2(\mathbf{x}_1, \mathbf{x}_1)$. The domain of A_{left} lies on $\Sigma_{\max}$ and it determines a homology class for the type of holomorphic curves contributing to the differential ∂. The domain of A_{right} has two components A^a_{right} and A^b_{right}, both supported in disks which are the domains between α_0 and a, with a collapsed to a point, and between β_0 and b with b collapsed to a point. We claim that the Maslov index of A_{left} is equal to 1, and the Maslov indices of each of the components in A_{right} are equal to $2n_w$. Since the degeneration is along Reeb orbits, we have the formula

$$\mathrm{ind}(A) = \mathrm{ind}(A_{\mathrm{left}}) + \mathrm{ind}(A^a_{\mathrm{right}}) + \mathrm{ind}(A^b_{\mathrm{right}}) - 2(N_a + N_b)$$

where N_a and N_b are the numbers of connected components of the unramified covering in the necks at a and b (clearly $N_a, N_b \in [1, n_w]$). Therefore, it suffices to see that $\mathrm{ind}(A^a_{\mathrm{right}}) = \mathrm{ind}(A^b_{\mathrm{right}}) = 2n_w$. This follows from the usual formula $\mathrm{ind}(A^a_{\mathrm{right}}) = \langle c_1(\mathfrak{s}), A^a_{\mathrm{right}}\rangle = e(A^a_{\mathrm{right}}) + 2n_{\mathbf{x}}(A^a_{\mathrm{right}}) = 2n_w$ (since the homology class A^a_{right} is n_w times the disk with boundary on α_0, $e(A^a_{\mathrm{right}}) = n_w$ and $n_{\mathbf{x}}(A^a_{\mathrm{right}}) = n_w/2$); similarly for A^b_{right}. We deduce that $\mathrm{ind}(A_{\mathrm{left}}) = 1 + 2(N_a + N_b) - 4n_w$, which implies that $\mathrm{ind}(A_{\mathrm{left}}) = 1$ and the coverings in the cylindrical necks near a and b are both trivial (in other terms, after neck-stretching we have n_w distinct cylinders passing through each neck).

Furthermore, we have the evaluation maps:

$$ev^a_{\mathrm{left}} : \mathcal{M}(A_{\mathrm{left}}) \to \mathrm{Sym}^{n_w}(\mathbb{D})$$
$$ev^a_{\mathrm{right}} : \mathcal{M}(A^a_{\mathrm{right}}) \to \mathrm{Sym}^{n_w}(\mathbb{D})$$
$$ev^b_{\mathrm{left}} : \mathcal{M}(A_{\mathrm{left}}) \to \mathrm{Sym}^{n_w}(\mathbb{D})$$
$$ev^b_{\mathrm{right}} : \mathcal{M}(A^b_{\mathrm{right}}) \to \mathrm{Sym}^{n_w}(\mathbb{D})$$

given by taking the preimages of the degeneration points of a and b and projecting to $\mathbb{D} = [0, 1] \times \mathbb{R}$. We claim that the moduli space $\mathcal{M}(A)$ can be identified with the fiber product of moduli spaces $\mathcal{M}(A_{\mathrm{left}}) \times_B \mathcal{M}(A_{\mathrm{right}})$, where $B = \mathrm{Sym}^{n_w}(\mathbb{D}) \times \mathrm{Sym}^{n_w}(\mathbb{D})$ and the fiber product is taken with respect to the above evaluation maps. This is a consequence of a gluing theorem (see [18] Theorem 5.1 for the proof in a very closely related situation).

Finally, we will prove that $(ev^a_{\mathrm{right}}, ev^b_{\mathrm{right}}) : \mathcal{M}(A^a_{\mathrm{right}}) \times \mathcal{M}(A^b_{\mathrm{right}}) \to \mathrm{Sym}^{n_w}(\mathbb{D}) \times \mathrm{Sym}^{n_w}(\mathbb{D})$ has degree 1. This implies that, for the purpose of counting pseudoholomorphic curves, the fiber product of moduli spaces $\mathcal{M}(A_{\mathrm{left}}) \times_B \mathcal{M}(A_{\mathrm{right}})$ can be identified with $\mathcal{M}(A_{\mathrm{left}})$. Therefore, we can identify the moduli spaces $\mathcal{M}(A)$ and $\mathcal{M}(A_{\mathrm{left}})$, as required.

To see that the evaluation maps have degree 1, we argue as follows. First, we represent the domain of the strips in $\mathcal{M}(A^a_{\mathrm{right}})$ by the upper half of the unit disk so that the upper half circle maps to α_0 and the interval $[-1, 1]$ maps to the β curve. Also, represent the target disk by the unit disk, so that α_0 corresponds to the unit circle and the β arc is represented by the real positive axis; furthermore the degeneration point of a as used to define the map ev^a_{right} is mapped to the origin in this representation. Thus, the moduli space $\mathcal{M}(A^a_{\mathrm{right}})$ consists of holomorphic maps from the upper half disk to the unit disk and ev^a_{right} records the positions of the n_w zeros of these maps. Now, any holomorphic map from the upper half disk to the unit disk can be reflected ($u(1/\bar{z}) := 1/\overline{u(z)}$) to get a holomorphic map from the upper half-plane to $\mathbb{P}^1$, mapping the real axis to the real positive axis. This can then be further reflected about the real axis to get holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ which are hence rational fractions of degree $2n_w$, with real coefficients (forced by the invariance under conjugation) and with equivariance under $z \to 1/\bar{z}$. Now, such holomorphic maps are classified by their zeros (the poles are the reflections of the zeros). In our case, there are $2n_w$ zeros and none of these are real, so they are n_w pairs of complex conjugate points. Finally, we note that ev^a_{right} maps any such holomorphic map to the positions of its n_w zeros which lie inside the upper half-disk. Therefore, $ev^a_{\mathrm{right}} : \mathcal{M}(A^a_{\mathrm{right}}) \to \mathrm{Sym}^{n_w}(\mathbb{D})$ is in fact a diffeomorphism. In particular, it has degree 1. □

Note that when the minimal genus fiber has genus greater than 1, there is no perturbation required since the diagrams that we consider are weakly admissible in that case. Hence, we get the above isomorphism for the homology groups with $\mathbb{Z}_2$ coefficients.

Corollary 22. *Suppose that* $g(\Sigma_{\min}) = k > 1$, *then for* $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$ *we have*

$$QFH'(Y, f, \mathfrak{s}; \mathbb{Z}_2) \simeq HF^+(Y, \mathfrak{s}).$$

Proof. This follows from the above result by letting $t = 1$. □

In some cases, the quilted Floer homology groups can be calculated easily, the following special case is an example of this. Given two simple closed curves α and β on a surface of genus greater than 1, let $\iota(\alpha, \beta)$ denote the geometric intersection number of α and β, i.e. the number of transverse intersections of their geodesic representatives for a hyperbolic metric.

Corollary 23. *Suppose that f has only two critical points, and let $\alpha, \beta \subset \Sigma_{\max}$ be the vanishing cycles for these critical points. Then $\oplus_{\mathfrak{s}\in(\mathcal{S}|\Sigma_{min})} HF^+(Y, f, \gamma_w, \mathfrak{s})$ is free of rank $\iota(\alpha, \beta)$.*

Proof. When f has only two critical points, $QFH'(Y, f)$ reduces to the Lagrangian Floer homology of the simple closed curves α and β on the surface $\Sigma_{\max}$. This is easily calculated by representing the free homotopy classes of simple closed curves α and β by geodesics, which ensures that there are no non-constant holomorphic disks contributing to the differential. In fact, any holomorphic disk would lift to a holomorphic disk in the universal cover $\mathbf{H}^2$, which would contradict the fact that there is a unique geodesic between any two points in $\mathbf{H}^2$. Therefore, the quilted Floer homology is freely generated by the number of intersection points of geodesic representatives of α and β. □

We remark that if $\iota(\alpha, \beta) = 1$, then the critical values can be canceled. Thus for non-fibered manifolds which admit a broken fibration with only 2 critical points the rank of quilted Floer homology is greater than 1.

3.3. An application to the sutured Floer homology

Juhász introduced an extension of the Heegaard–Floer homology to "balanced sutured 3-manifolds". (See [5] for the definition.) A connected balanced sutured manifold Y is a compact oriented 3-manifold with boundary Y and it can be equipped with a broken fibration $f : Y \to [0, 1]$ whose fibers are surfaces with non-empty boundary and $f^{-1}(0)$ and $f^{-1}(1)$ are homeomorphic surfaces such that each connected component has exactly one boundary component (balanced condition). We can always arrange that $f^{-1}(1/2) = \Sigma$ is the highest genus fiber which is connected and as one travels from $1/2$ to 0 one attaches two handles along $\beta_1, \ldots, \beta_m$ and as one travels from $1/2$ to 1 one attaches two handles along $\alpha_1, \ldots, \alpha_m$ which are realized as vanishing cycles of f on Σ. The balanced condition translates to the condition that the set of α curves and respectively the set of β curves are linearly independent in $H_1(\Sigma)$. The sutures $s(\gamma)$ of Y correspond to the boundary components of $\partial\Sigma$ and the annular neighborhoods $A(\gamma)$ of Y are obtained from $s(\gamma)$ by flowing using the gradient flow of f along [0, 1] with respect to a metric such that the gradient vector field of f preserves the boundary of Y.

In [5], Juhász constructs a variant of the Heegaard–Floer homology for sutured 3-manifolds. This is simply, the Lagrangian–Floer homology group $HF(\mathrm{Sym}^m(\Sigma), \alpha_1 \times \cdots \times \alpha_m, \beta_1 \times \cdots \times \beta_m)$ where the projections of the holomorphic curves contributing to the differential on Σ_m are required to stay away from the boundary of Σ_m.

In [6], Kronheimer and Mrowka construct an invariant of sutured manifolds using the monopole (resp. instanton) Floer homology, by constructing a closed 3-manifold Y_n and setting the sutured Floer homology of Y by defining it to be a summand of the monopole (resp. instanton) Floer homology of Y_n. The construction of (Y_n, f_n) is by first gluing $T \times [0, 1]$ where T is an oriented connected genus $n \geq 1$ surface with non-empty boundary, so that $\partial T \times [0, 1]$ is glued to the union of annuli $A(\gamma)$, and then identifying the fibers over 0 and 1 by choosing a homeomorphism between them. Note that the balanced condition implies that f_n has connected fibers. In the monopole (resp. instanton) setting, Kronheimer and Mrowka define the sutured monopole (resp. instanton) Floer homology of Y to be $\bigoplus_{\mathfrak{s}\in\mathcal{S}(Y|\Sigma_{\min})} HM(Y_n, \mathfrak{s})$ and prove that this is an invariant of the sutured manifold Y (in particular, it is also independent of the genus n of T and the homeomorphism chosen in identifying fibers over 0 and 1). It was raised in [6] as a question, whether one can recover Juhász's definition of the sutured Floer homology from the construction given above applied in the setting of the Heegaard–Floer homology. In the next theorem, we give an affirmative answer to this.

Theorem 24. *For $n \geq 1$,*

$$SFH(Y) \simeq \bigoplus_{\mathfrak{s}\in\mathcal{S}(Y_n|\Sigma_{\min})} HF^+(Y_n, \mathfrak{s}).$$

Note that this theorem in particular implies that the group on the right hand side is independent of n and the chosen surface homeomorphism in the construction of Y_n. As usual, in the case that the lowest genus fiber of f_1 has genus 1, one needs to use coefficients in Λ.

Proof. Theorem 21 applied to (Y_n, f_n) yields that $\bigoplus_{\mathfrak{s}\in\mathcal{S}(Y_n|\Sigma_{\min})} HF^+(Y_n, \mathfrak{s}) = QFH'(Y_n, f_n)$. Therefore, the proof will follow once we establish that $SFH(Y, f) \simeq QFH'(Y_n, f_n)$. This in turn relies on a simple observation about the Heegaard diagrams used in the definition of these groups, namely let us denote the maximal genus fiber of f by Σ, and the maximal genus fiber of f_n by $\Sigma \cup T$. Now, if an admissible sutured Heegaard diagram of (Y, f) is given by $(\Sigma, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$, then the Heegaard diagram for calculating $QFH'(Y_n, f_n)$ is given by $(\Sigma \cup T, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$. Note that there is no α or β curve entering T. Thus, the proof will be complete once we prove that holomorphic curves contributing to the differential of $QFH'(Y_n, f_n)$ do not enter to the region including T. Note that because of the admissibility condition of the sutured Heegaard diagram of (Y, f) we can use an almost complex structure which is vertical in a neighborhood of $\Sigma \times [0, 1] \times \mathbb{R}$ (by vertical, we mean that the fibers of the projection $\Sigma \times [0, 1] \times \mathbb{R} \to [0, 1] \times \mathbb{R}$ are holomorphic) so that the holomorphic curves contributing to the differential of the sutured Floer homology appear as holomorphic curves contributing to the differential of $QFH'(Y_n, f_n)$. On the other hand, we use a non-vertical almost complex structure as in Section 3.1, along $T \times [0, 1] \times \mathbb{R}$ away from the boundary of T. Now, let $u : (S, \partial S) \to (\Sigma \cup T) \times [0, 1] \times \mathbb{R}$ be a holomorphic map contributing to the differential of $QFH'(Y_n, f_n)$. We would like to show that the image of the projection of u to the Heegaard surface does not hit T. This follows from a degeneration argument. Namely, suppose that the image of the projection of u does hit T, then we can degenerate along Reeb orbits corresponding to the attaching region of T to Σ, this would on one side give a holomorphic map $u_T : \tilde{S} \to \tilde{T} \times [0, 1] \times \mathbb{R}$ where $\tilde{T}$ is the closed surface obtained by shrinking each boundary component of T to a point and $\tilde{S}$ is a part of the domain of the degenerated map. It follows for example from Corollary 4.3 in [11] that the index of u_T is equal to the Euler measure of the projection onto $\tilde{T}$. (Strictly speaking Corollary 4.3 in [11] is proved to hold for holomorphic curves with corners. Here we apply it in the degenerate case where there are no corners. This can be justified as follows: degeneration of u along the attaching region results in u degenerating into two pieces u_Σ and u_T. Then Corollary 4.3 in [11] can be applied to both u and u_Σ and a short calculation using the additivity of the index then implies that index of u_T is the Euler measure of $\tilde{T}$.) Furthermore, in this case (since there are no corners) the Euler measure is simply $\chi(\tilde{T})$ times the multiplicity of the domain supported in the whole of $\tilde{T}$. The multiplicity is positive by holomorphicity of u_T and $\chi(\tilde{T})$ is negative by assumption. So, we conclude that the index is negative. Furthermore, we have restricted to the class of almost complex structures so that the fiber of the projection $(\Sigma \cup T) \times [0, 1] \times \mathbb{R} \to (\Sigma \cup T) \times [0, 1]$ is not a holomorphic surface, this ensures that one can choose an almost complex structure giving transversality as in Proposition 3.7 of [11]. This yields the desired contradiction (note that in the case that T has genus 1, we still obtain a contradiction since we get a negative dimension for the transversely cut moduli space after quotienting by the $\mathbb{R}$ action). □

4. Isomorphism between $QFH(Y, f; \Lambda)$ and $QFH'(Y, f; \Lambda)$

In this section, we relate $QFH'(Y, f)$ defined as a variant of Heegaard–Floer homology as in Section 3.1 with the original definition in terms of holomorphic quilts given in the introduction, which we called $QFH(Y, f)$ (see below for a detailed definition). More precisely, we show that these two groups are isomorphic whenever they are defined.

There are two main ingredients in this isomorphism. The first one is a general result in the theory of holomorphic quilts which proves an isomorphism between quilted Floer homology groups under transverse and embedded compositions of Lagrangians (see below for definitions). This result is originally due to Wehrheim and Woodward [28], which was proved in the positively monotone setting. However, here we are situated in the (strongly) negative setting in which case the arguments of Wehrheim and Woodward are no longer valid. To resolve this issue, in [8] we gave a new proof of Wehrheim and Woodward's theorem which applies in the current situation. The new proof applies under both positive and (strongly) negative monotonicity assumptions.

The second main ingredient in the proof of the isomorphism is a detailed study of the Lagrangian correspondences that are involved in the definition of $QFH(Y, f)$. In the next section, we give a detailed definition of $QFH(Y, f, \mathfrak{s})$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$ and $g < 2k$. In particular, we give a detailed description of monotonicity which is required to have a rigorous definition over $\mathbb{Z}_2$ when $k > 1$. Proving the isomorphism involves showing that various compositions of these Lagrangians correspondences are Hamiltonian isotopic to product tori, that appear in the definition of $QFH'(Y, f)$. This part of the proof has appeared in author's thesis [9], and it can also be found in the upcoming work [10] where a more general set-up is developed.

Finally, we remark that all the theorems are stated for Floer homology groups over the universal Novikov ring Λ, but as before, in the case where the lowest genus fiber has genus greater than 1, one can take coefficients to be in $\mathbb{Z}_2$.

4.1. Definition of $QFH(Y, f, \mathfrak{s})$

We now give a detailed definition of $QFH(Y, f, \mathfrak{s})$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\max})$ when $g(\Sigma_{\max}) < 2g(\Sigma_{\min})$. Recall that we start with a broken fibration $f : Y \to S^1$ with connected fibers and with a distinguished maximal genus fiber $\Sigma_g = f^{-1}(-1)$ and a minimal genus fiber $\Sigma_k = f^{-1}(1)$. There are $g - k$ critical values $p_1, \ldots, p_{g-k}$ on the northern semi-circle in clockwise order and $g - k$ critical values $q_1, \ldots, q_{g-k}$ on the southern semi-circle in counter-clockwise order. For a critical value p fix two nearby points p^+, p^- on S^1 such that the genus $f^{-1}(p^+)$ is greater than the genus of $f^{-1}(p^-)$, that is, p^+ is to the left of p^-. Furthermore, we arrange that $p_i^+ = p_{i+1}^-$ and that $p_1^+ = q_1^+ = -1$ and $p_{g-k}^- \neq q_{g-k}^-$.

Next choose a Riemannian metric g on Y, we then have embedded curves $\alpha_i \subset f^{-1}(p_i^+)$ and $\beta_i \subset f^{-1}(q_i^+)$ cut out by the unstable manifolds of p_i and q_i. By abuse of notation, we also denote by $\alpha_i, \beta_i \subset \Sigma_g = f^{-1}(-1)$ the embedded curves cut out by the intersection of the unstable manifolds of p_i and q_i with Σ_g.

Finally, choose an area form ξ_i and a compatible complex structure j_i on each $f^{-1}(p_i^+)$, $f^{-1}(q_i^+)$ and $f^{-1}(1)$. Note that the gradient flow gives an identification of $f^{-1}(p_i^+)$ and $f^{-1}(p_i^-)$ (resp. for $f^{-1}(q_i^+)$ and $f^{-1}(q_i^-)$) outside of their intersection with the stable and unstable manifolds associated with p_i (resp. q_i) and we ask that this identification is a complex isomorphism when the fibers are equipped with the complex structures. Note that $f^{-1}(p_{g-k}^-)$ and $f^{-1}(q_{g-k}^-)$ are diffeomorphic surfaces and we can and do in fact arrange them to be symplectomorphic, however we cannot in general demand that they are isomorphic complex surfaces as this will put severe restrictions on the monodromy.

Let $B = S^1 - \left(\left(\bigcup_{i=1}^{g-k} (p_i^+, p_i^-) \cup (q_i^+, q_i^-) \right) \cup (p_{g-k}^-, q_{g-k}^-) \right)$ be the finite set of $2(g-k)+1$ points, one between every consecutive pair of critical points with the exception of p_{g-k}^- and q_{g-k}^- which lie between the consecutive critical points p_{g-k} and q_{g-k}. Except p_{g-k}^- and q_{g-k}^-, any two consecutive points (as a subset of S^1) gives a quintuple $(\Sigma, j, C, \overline{\Sigma}, \overline{j})$ where (Σ, j) and $(\overline{\Sigma}, \overline{j})$ are connected Riemann surfaces, C is an embedded non-separating curve on Σ and there is a canonical diffeomorphism from Σ_C (the result of surgery on C) to $\overline{\Sigma}$.

For $b \in B$, let $F_b = f^{-1}(b)$ be the fiber of f equipped with a complex structure and a compatible area form ξ_b as above. We can then consider $\mathrm{Sym}^n(F_b)$ as a complex manifold for any n. There are two distinguished classes in $H^2(\mathrm{Sym}^n(F_b))$ which span the invariant subspace of the second cohomology group under the action of the mapping class group of F_b. These are η, Poincaré dual to $\{pt\} \times \mathrm{Sym}^{n-1}(F_b)$ and θ, which can be concisely described using the fact that $c_1(T\mathrm{Sym}^n(F_b)) = (n+1-g_b)\eta - \theta$, where g_b is the genus of F_b.

In [21], Perutz constructs Kähler forms ω_{F_b} on $\mathrm{Sym}^n(F_b)$ with the property that ω_{F_b} agrees with $\mathrm{Sym}^n \xi_b$ outside of a neighborhood of the diagonal and $[\omega_{F_b}] = \eta + \lambda\theta$ for any sufficiently small fixed real parameter $\lambda > 0$, and which tames $\mathrm{Sym}^n(j)$ on all of $\mathrm{Sym}^n(F_b)$.

We are now ready to state the fundamental construction of Perutz.

Theorem 25 (*Perutz [22]*)**.** *Starting from a quintuple $(\Sigma, j, C, \overline{\Sigma}, \overline{j})$ as above, one can construct a Lagrangian correspondence $L_C \subset (Sym^{n-1}(\overline{\Sigma}) \times Sym^n(\Sigma), -\omega_{\overline{\Sigma}} \oplus \omega_{\Sigma})$ canonically up to Hamiltonian isotopy.*

We note here two topological properties of L_C from [22]. First, there are maps

$$\mathrm{Sym}^n(\Sigma) \xleftarrow{i} L_C \xrightarrow{\pi} \mathrm{Sym}^{n-1}(\overline{\Sigma})$$

such that i is a codimension 1 embedding and π is a trivial S^1 fibration. Second, note that for $n > 1 \pi_1(\mathrm{Sym}^n(\Sigma)) = H_1(\Sigma)$ and the homology class of L_C in $\mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\overline{\Sigma})$ is given by $C \times \mathrm{Sym}^{n-1}(\overline{\Sigma})$.

For $\mathfrak{s}$ a spin^c structure on Y, let us define the integer n_b by the formula $\langle c_1(\mathfrak{s}), F_b \rangle = 2n_b + \chi(F_b)$.

Perutz's construction applied to our set-up gives a sequence of Lagrangian correspondences between $\mathrm{Sym}^{n_{p^-_{g-k}}}(F_{p^-_{g-k}})$ and $\mathrm{Sym}^{n_{q^-_{g-k}}}(F_{q^-_{g-k}})$. Furthermore, these latter two symplectic manifolds are canonically identified by a symplectomorphism induced by the symplectomorphism of $F_{p^-_{g-k}}$ and $F_{q^-_{g-k}}$. (The fact that the complex structures on $F_{p^-_{g-k}}$ and $F_{q^-_{g-k}}$ are compatible with the same symplectic structure provides a path of complex structures interpolating between the given two complex structures on the underlying surface, this in turn gives a tautological Kähler isomorphism between $\mathrm{Sym}^{n_{p^-_{g-k}}}(F_{p^-_{g-k}})$ and $\mathrm{Sym}^{n_{q^-_{g-k}}}(F_{q^-_{g-k}})$. See for example [27] for further details of this identification where a definition of the Floer homology group that we are discussing here was given for the special of fibered 3-manifolds.) In this paper, we are only concerned with the spin^c structures $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$. Thus, for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$, we have $2n_b = 2 - 2k - \chi(F_b)$. Hence, we have $n_{p^-_{g-k}} = n_{q^-_{g-k}} = 0$ and so the above identification is trivially the identity map.

In any case, with the above identification in mind, we obtain a cyclic set of Lagrangian correspondences $L_{\alpha_1}, \ldots, L_{\alpha_{g-k}}$ and $L_{\beta_1}, \ldots, L_{\beta_{g-k}}$ between the cyclic set of symplectic manifolds $\{\mathrm{Sym}^{n_b}(F_b)\}_{b\in B}$.

Starting from such data, we define the quilted Floer homology of (Y, f) as the quilted Floer homology of this cyclic set of Lagrangians as developed by [28] (see also [20]):

$$QFH(Y, f) := HF(L_{\alpha_1}, \ldots, L_{\alpha_{g-k}}, L_{\beta_{g-k}}, \ldots, L_{\beta_1}).$$

Let us recall the basic definition of quilted Floer homology of a cyclic set of Lagrangians, as we will need some of this notation in order to prove that our quilted Floer homology is well-defined. Let us choose a cyclic ordering of the set B, write b_i for the ith element (where the indices are always considered in mod $2(g-k)$) and write $\underline{L} = (L_1, L_2, \ldots, L_{2(g-k)}) = (L_{\alpha_1}, \ldots, L_{\alpha_{g-k}}, L_{\beta_{g-k}}, \ldots, L_{\beta_1})$ so that $L_i \subset \mathrm{Sym}^{n_{b_i}}(F_{b_i}) \times (\mathrm{Sym}^{n_{b_{i+1}}}(F_{b_{i+1}}))$.

The quilted Floer chain complex $CF(\underline{L})$ is freely generated over the base ring by the generalized intersection points:

$$I(\underline{L}) = \{\underline{x} = (x_1, \ldots, x_{2(g-k)}) | (x_{2(g-k)}, x_1) \in L_1,$$
$$(x_1, x_2) \in L_2, \ldots, (x_{2(g-k)-1}, x_{2(g-k)}) \in L_{2(g-k)}\}.$$

This set can be arranged to be finite by requiring the curves α_i and β_j intersect at finitely many points, for all i, j. (This will be made clear below.) Next consider the path space

$$\mathcal{P}(\underline{L}) = \{(\gamma_1, \ldots, \gamma_{2(g-k)}) | \gamma_i : [0,1] \to \mathrm{Sym}^{n_{b_i}}(F_{b_i}), (\gamma_i(1), \gamma_{i+1}(0)) \in L_i\}.$$

Note that all the symplectic manifolds $\mathrm{Sym}^{n_{b_i}}(F_{b_i})$ are equipped with symplectic forms ω_i in the class $\eta + \lambda\theta$ and we choose compatible almost complex structures J_i. The Floer differential is then obtained in the usual way by counting moduli space of finite energy quilted holomorphic strips connecting generalized intersection points $\underline{x}$ and $\underline{y}$ as follows:

$$\mathcal{M}(\underline{x}, \underline{y}) = \Big\{ u_i : \mathbb{R} \times [0,1] \to \mathrm{Sym}^{n_{b_i}}(F_{b_i}) |\ \bar{\partial}_{J_i} u_i = 0,$$
$$E(u_i) = \int u_i^* \omega_i < \infty$$
$$\lim_{s\to-\infty} u_i(s, \cdot) = x_i,\ \lim_{s\to+\infty} u_i(s, \cdot) = y_i$$
$$(u_i(s,1), u_{i+1}(s,0)) \in L_i \text{ for all } i = 1, \ldots, 2(g-k) \Big\} /\mathbb{R}.$$

For people shy of holomorphic quilts, one can alternatively think of the latter group as a Floer homology group of two Lagrangians $\mathbb{L}_1 = L_1 \times L_3 \times \cdots \times L_{2(g-k)-1}$ and $\mathbb{L}_2 = L_2 \times L_4 \times \cdots \times L_{2(g-k)}$ in the product symplectic manifold $\mathbb{M} = \prod_{b \in B} \mathrm{Sym}^{n_b}(F_b)$.

The quilted Floer homology group is defined under monotonicity assumptions and we have to show that our set-up falls into (strongly) negative monotone case (compare [20], Definition 1.8). We address these technicalities now.

Transversality and avoiding bubbles: This follows from standard arguments in Floer theory; see for example, [26] Lemma 2.4. In the strongly negative monotone case, in addition to transversality for moduli space of Floer trajectories, transversality for the moduli space of bubbles can be achieved by an identical argument (see [20] Lemma 3.5). We insist on using a path of almost complex structures which are of the form $\mathrm{Sym}^{n_b}(j_s)$ near the diagonal as in [16] on each component $\mathrm{Sym}^{n_b}(F_b)$ of $\mathbb{M}$. Though, we warn the reader that one cannot necessarily achieve transversality by considering complex structures J_s on $\mathbb{M}$ which is a product of generic complex structures on each factor (see [30] for a discussion of this issue). Therefore, outside of the neighborhood of the diagonal one uses a generic complex structure on $\mathbb{M}$ (i.e., not of split-type).

To avoid disk or sphere bubbles, we pick a generic complex structure from our class of almost complex structure described above, which achieves transversality for Floer trajectories as well as disk and sphere bubbles. Then under the assumption $g < 2k$ (this is the strong negativity assumption), one can calculate the dimension of the moduli space of disk and sphere bubbles and get a negative number. In view of transversality, this proves that there are no non-constant bubbles. The calculation of the dimension of the moduli space of disk and sphere bubbles follows from Section 4 of [23]. We spell out this calculation here for completeness:

Note that the symplectic manifolds that we are dealing with are $M = \mathrm{Sym}^n(\Sigma)$ where $n = g(\Sigma) - k$ and $g(\Sigma)$ takes values between $g(\Sigma_{\max}) = g$ and $g(\Sigma_{\min}) = k$. We equipped M with a symplectic form in the class $\eta + \lambda\theta$ where $\lambda > 0$ is a fixed parameter that is determined by the monotonicity condition as follows.

The monotonicity constant τ is determined by the equation:

$$[\eta + \lambda\theta] = \tau[c_1(\mathrm{Sym}^n(\Sigma))] = \tau[(n + 1 - g)\eta - \theta].$$

Therefore, $\tau = \frac{1}{n+1-g} < 0$ is the fixed monotonicity constant which is the same for any of the symplectic manifolds we consider since $n - g = -g(\Sigma_{\min})$.

Now, Perutz calculates in Section 4 of [23] that for $n > 1$ the Hurewicz map $\pi_2(\mathrm{Sym}^n(\Sigma)) \to H_2(\mathrm{Sym}^n(\Sigma))$ has rank 1 and generated by a class h which satisfies $\eta(h) = 1$ and $\theta(h) = 0$. On the other hand, $c_1(\mathrm{Sym}^n(\Sigma)) = (n+1-g(\Sigma))\eta - \theta$. Therefore, any simple holomorphic sphere would have $[u] = h$ and its index would be:

$$2(\langle c_1(\mathrm{Sym}^n(\Sigma)), h\rangle + n - 3) = 4n - 2g(\Sigma) - 4 = 2g(\Sigma) - 4k - 4.$$

The assumption $g < 2k$ now implies that this quantity is strictly less than -4, which suffices for our purpose. (For $n = 1$, we cannot have any holomorphic spheres since $\pi_2(\Sigma) = 0$.)

Similarly, for a disk bubble we need to verify the assumptions for our Lagrangian $L_C \subset \mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\overline{\Sigma})$, where as before $n = g(\Sigma) - k = g(\overline{\Sigma}) + 1 - k$. In light of the fact that Perutz proves in Lemma 3.18 of [22] that any disk in $\pi_2(\mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\overline{\Sigma}), L_C)$ lifts to a sphere it follows that

$$\mu_{L_C}([u]) = 2(\langle c_1(\mathrm{Sym}^n(\Sigma) \times \mathrm{Sym}^{n-1}(\overline{\Sigma})), [u]\rangle).$$

Now, the positive area disks u for which the value $\mu_{L_C}([u])$ is maximal, have index given by

$$2(n + 1 - g(\Sigma)) + (2n - 1) - 3 = 2g(\Sigma) - 4k - 2.$$

Again, the assumption $g < 2k$ ensures that this value is strictly less than -2, which guarantees the non-existence of disk bubbles in the relevant moduli space of indices 0, 1 and 2.

Monotonicity (admissibility): We have seen that the symplectic manifolds $\mathrm{Sym}^b(F_b)$ are negatively monotone with the same monotonicity constant $\tau = \frac{1}{2-2k} < 0$ as $2c_1(T\mathrm{Sym}^b(F_b)) = 2(n_b - g_b + 1)\eta - 2\theta$ and $\omega_{F_b} = \eta + \lambda\theta$, and that each Lagrangian L_{α_i} and L_{β_i}, hence their products $\mathbb{L}_1$ and $\mathbb{L}_2$ are monotone with the same constant τ. Note that the value of $\lambda > 0$ is irrelevant since θ vanishes on spherical classes and any disk with boundary on $\mathbb{L}_i$ comes from a spherical class ([23], Section 4).

The only missing ingredient is that monotonicity for the pair $(\mathbb{L}_1, \mathbb{L}_2)$. That is to say, we need to show that index and the area functions on $\pi_1(\mathcal{P}(\underline{L}))$ are proportional with the monotonicity constant τ as above. This is needed in two places, first we need to have an a priori energy bound for low index moduli spaces in order to appeal to the Gromov compactness theorem to say that at the boundary of moduli spaces we either get broken trajectories or bubbled configurations. The bubbled configurations are then eliminated using the strong negativity assumptions. The second place where monotonicity is needed is to show that there are only finitely many homotopy classes of disks between given two intersection points, hence the Floer differential can be defined.

In fact, monotonicity for the pair does not always hold and depends on the relative position of the curves α_i and β_i and our task is to show that it can be ensured whenever the diagram $(\Sigma_g, \alpha_1, \ldots, \alpha_{g-k}, \beta_1, \ldots, \beta_{g-k})$ is an admissible diagram in the sense defined below (see also Section 2).

Our strategy will be to show that admissibility implies monotonicity for the Heegaard tori $(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ in $\mathrm{Sym}^{g-k}(\Sigma_g)$ for a symplectic form ω_ξ in the class η which tames the same complex structure as ω and deduce from that the required monotonicity properties for the pair $(\mathbb{L}_1, \mathbb{L}_2)$.

We will postpone this until we put our Lagrangians L_{α_i} and L_{β_i} in a nice position (by a Hamiltonian isotopy) so as to relate them to Heegaard tori $\alpha_1 \times \cdots \times \alpha_{g-k}$ and $\beta_1 \times \cdots \times \beta_{g-k}$ in $\mathrm{Sym}^{g-k}(\Sigma_g)$.

4.2. *Heegaard tori as a composition of Lagrangian correspondences*

Recall that given two Lagrangian correspondences, $L_1 \subset X \times Y$ and $L_2 \subset Y \times Z$ such that $L_1 \times L_2$ is transverse to the diagonal in Y, the composition $L_1 \circ L_2$ is a Lagrangian correspondence in $X \times Z$ given by the union of tuples (x, z) such that there exists a $y \in Y$ with the property that $(x, y) \in L_1$ and $(y, z) \in L_2$.

Now, for the class of almost complex structures j that are sufficiently stretched along the vanishing cycles of f near its critical points, we have the following important technical lemma about these correspondences which was conjectured by Perutz in [22].

Lemma 26. *For $g > k$, $L_{\alpha_1} \circ \cdots \circ L_{\alpha_{g-k}}$ and $L_{\beta_1} \circ \cdots \circ L_{\beta_{g-k}}$ are respectively Hamiltonian isotopic to $\alpha_1 \times \cdots \times \alpha_{g-k}$ and $\beta_1 \times \cdots \times \beta_{g-k}$ in $\mathrm{Sym}^{g-k}(\Sigma)$ equipped with a Kähler form ω which lies in the cohomology class $\eta + \lambda\theta$ with $\lambda > 0$.* □

The proof of this lemma can be found in [10,9]. The proof is obtained by carrying out the construction of Lagrangian correspondences as a family of degenerations. As the required technical set-up is developed extensively in [10], for the sake of brevity we choose to omit it from here.

Back to periods and admissibility: Lemma 26 is accomplished by carrying out the construction of L_{α_i} simultaneously which enables us to show that for a careful choice of degeneration one in fact has the exact equality:

$$L_{\alpha_1} \circ \cdots \circ L_{\alpha_{g-k}} = \alpha_1 \times \cdots \times \alpha_{g-k}$$
$$L_{\beta_1} \circ \cdots \circ L_{\beta_{g-k}} = \beta_1 \times \cdots \beta_{g-k}.$$

From now on, we will work with this situation and prove that the quilted Floer homology is well-defined in an admissible situation. We can then appeal to Hamiltonian isotopy provided by Lemma 26 to conclude that quilted Floer homology will be well-defined even in the cases where the above equalities might not hold. (Monotonicity still holds after Hamiltonian isotopy.)

Let us denote the Heegaard tori in $\mathrm{Sym}^{g-k}(\Sigma_g)$ by $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ and the path space connecting these by $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. Note that the above equalities imply that there is bijection between the $CF(L_{\alpha_1}, \ldots, L_{\alpha_{g-k}}, L_{\beta_{g-k}}, \ldots, L_{\beta_1})$ (generated by $\mathbb{L}_1 \cap \mathbb{L}_2$) and $CF(L_{\alpha_1} \circ \cdots \circ L_{\alpha_{g-k}}, L_{\beta_1} \circ \ldots \circ L_{\beta_{g-k}})$ (generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$). Recall that we denoted the path space that is used in the definition of quilted Floer homology by $\mathcal{P}(\underline{L})$. This is canonically identified with the path space $\Omega(\mathbb{L}_1, \mathbb{L}_2)$ for the Lagrangians $\mathbb{L}_1$ and $\mathbb{L}_2$. The main topological lemma about these path spaces is the following.

Lemma 27. *There exists an inclusion map* $\iota : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \Omega(\mathbb{L}_1, \mathbb{L}_2) = \mathcal{P}(\underline{L})$, *which induces a bijection between* $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *and* $\mathbb{L}_1 \cap \mathbb{L}_2$ *and an isomorphism* $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbf{x}) \to \pi_1(\Omega(\mathbb{L}_1, \mathbb{L}_2), \iota(\mathbf{x}))$ *for any* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Proof. Let $\gamma : [0, 1] \to \mathrm{Sym}^{g-k}(\Sigma_g)$ be a path in $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$. In particular, $\gamma(0) \in \mathbb{T}_\alpha = L_{\alpha_1} \circ \cdots \circ L_{\alpha_{g-k}}$. As our Lagrangian correspondences are circle bundles $\pi_i : L_{\alpha_i} \to \mathrm{Sym}^{g-k-i}(F_{b_i})$, $\gamma(0)$ determines a tuple $(\pi_1(\gamma(0)), \pi_2 \circ \pi_1(\gamma(0)), \ldots, \pi_{g-k} \circ \ldots \circ \pi_1(\gamma(0)))$, call these $(x_2, \ldots, x_{g-k})$. Similarly, $\gamma(1) \in \mathbb{T}_\beta$ determines a tuple $(x_{2(g-k)}, x_{2(g-k)-1}, \ldots, x_{g-k+1})$. The map $\iota : \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \mathcal{P}(\underline{L})$ is given by:

$$\gamma \to (\gamma, x_2, x_3, \ldots, x_{2(g-k)}).$$

In other words, all but the first component are constant paths which are in turn determined by the first component.

Now, the statement about the bijection between $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbb{L}_1 \cap \mathbb{L}_2$ follows from the definition of the map ι. Let us fix an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We also write $\mathbf{x}$ for the corresponding intersection point in $\mathbb{L}_1 \cap \mathbb{L}_2$.

Now, consider the path component of $\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ containing $\mathbf{x}$, one has evaluation maps on both sides which induces the Serre fibration:

$$\Omega_{\mathbf{x}}(\mathrm{Sym}^{g-k}(\Sigma_g)) \to \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \mathbb{T}_\alpha \times \mathbb{T}_\beta.$$

There is a similar Serre fibration for $\Omega(\mathbb{L}_1, \mathbb{L}_2)$. The map ι we defined above now gives a map between the two Serre fibrations:

$$\begin{array}{ccccc} \Omega_{\mathbf{x}}(\mathrm{Sym}^{g-k}(\Sigma_g)) & \longrightarrow & \Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta) & \longrightarrow & \mathbb{T}_\alpha \times \mathbb{T}_\beta \\ \downarrow \iota & & \downarrow \iota & & \downarrow \iota \\ \Omega_{\mathbf{x}}(\mathbb{M}) & \longrightarrow & \Omega(\mathbb{L}_1, \mathbb{L}_2) & \longrightarrow & \mathbb{L}_1 \times \mathbb{L}_2 \end{array}$$

Note that the leftmost arrow can also be seen as induced from the based inclusion of $\mathrm{Sym}^{g-k}(\Sigma_g)$ to $\mathbb{M}$. We next consider the long exact sequences induced by these Serre fibrations. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(\mathrm{Sym}^{g-k}(\Sigma_g)) & \longrightarrow & \pi_1\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta) & \longrightarrow & \pi_1(\mathbb{T}_\alpha\times\mathbb{T}_\beta) & \longrightarrow & \pi_1(\mathrm{Sym}^{g-k}(\Sigma)) \\
\downarrow & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} \\
\pi_2(\mathbb{L}_1\times\mathbb{L}_2) & \xrightarrow{i} & \pi_2(\mathbb{M}) & \longrightarrow & \pi_1\Omega(\mathbb{L}_1,\mathbb{L}_2) & \longrightarrow & \pi_1(\mathbb{L}_1\times\mathbb{L}_2) & \xrightarrow{p} & \pi_1(\mathbb{M})
\end{array}$$

From this, one can obtain the following set of short-exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta) & \longrightarrow & \frac{H_1(\Sigma_g)}{[\alpha_1],\dots,[\alpha_{g-k}],[\beta_1],\dots,[\beta_{g-k}]} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\iota} & & \downarrow \\
0 & \longrightarrow & \mathrm{coker}\, i & \longrightarrow & \pi_1\Omega(\mathbb{L}_1,\mathbb{L}_2) & \longrightarrow & \ker p & \longrightarrow & 0
\end{array}$$

From the topological properties of L_{α_i} and L_{β_j} mentioned after Theorem 25 it follows that all except the middle arrow is an isomorphism. We appeal to five-lemma to conclude that the middle arrow is also an isomorphism, as desired. □

Recall that our goal is to show that the index and the area are proportional on $\pi_1(\Omega(\mathbb{L}_1,\mathbb{L}_2),\mathbf{x})$ for the Lagrangian intersection problem $(\mathbb{M};\mathbb{L}_1,\mathbb{L}_2)$. We next show that in view of the above lemma, it suffices to show that the index and the area are proportional on $\pi_1(\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta),\mathbf{x})$ for the Lagrangian intersection problem $(\mathrm{Sym}^{g-k}(\Sigma_g);\mathbb{T}_\alpha,\mathbb{T}_\beta)$.

Lemma 28. *Let $P \in \pi_1(\Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta),\mathbf{x})$ and $\iota(P)$ be the image of P under the map defined in Lemma 27. Then we have the following equalities:*

$$Index_{(\mathbb{T}_\alpha,\mathbb{T}_\beta)}(P) = Index_{(\mathbb{L}_1,\mathbb{L}_2)}(\iota(P))$$
$$Area_{Sym^{g-k}(\Sigma_g)}(P) = Area_{\mathbb{M}}(\iota(P)).$$

Proof. Recall that the map $\iota : \Omega(\mathbb{T}_\alpha,\mathbb{T}_\beta) \to \Omega(\mathbb{L}_1,\mathbb{L}_2)$ sends a path $\gamma \to (\gamma, x_2, x_3, \dots, x_{2(g-k)})$ where $x_2, x_3, \dots, x_{2(g-k)}$ are constant paths. Thus, if P is a path of paths, written as $\gamma_s(t)$, the image of P under ι is given by $(\gamma_s(t), x_2(s), x_3(s), \dots, x_{2(g-k)}(s))$, hence, all but the first component has a vanishing t-derivative. Therefore, the areas are the same as claimed since the area of the image of P in $\mathbb{M}$ has contribution only from the component mapping to $\mathrm{Sym}^{g-k}(\Sigma_g)$. Similarly, the fact that the Maslov–Viterbo index is the same follows form the fact that $\mathbb{T}_\alpha\cap\mathbb{T}_\beta = \mathbb{L}_1\cap\mathbb{L}_2$ and crossing-forms used in calculation of the Maslov indices agree, this in turn follows from the fact that only the first component of $\iota(P)$ has a non-vanishing t-derivative (see proof of Lemma 3.1.6 in [29] for a similar argument). □

In view of Lemmas 27 and 28, to conclude monotonicity for the pair $(\mathbb{L}_1,\mathbb{L}_2)$, all we need to show is that index and area are proportional for the pair of Lagrangians $(\mathbb{T}_\alpha,\mathbb{T}_\beta)$ in $\mathrm{Sym}^{g-k}(\Sigma_g)$. Until now, we have been working with the symplectic forms ω on $\mathbb{M} = \prod_{b\in B}\mathrm{Sym}^{n_b}(F_b)$ that Perutz constructed which have the cohomology class in $\eta + \lambda\theta$ for sufficiently small $\lambda > 0$ on each component. The reason for not using symplectic forms in the arguably simpler class η is because Perutz's construction that we appeal to in Theorem 25 is only known to work for a symplectic form in class $\eta + \lambda\theta$ for non-zero λ (because of the fact that the relative Hilbert scheme used in the construction is not affine).

Now, we claim that if we isotope α and β curves in such a way to form an admissible diagram, then for a symplectic form ω_ξ (to be made explicit below) in the class η monotonicity is satisfied. By construction in Proposition 1.1 of [21] both ω and ω_ξ can be arranged to agree outside a neighborhood of the diagonal and tame the same regular almost complex structure $J_s = \mathrm{Sym}^{g-k}(j_s)$ everywhere. This suffices for the purpose of proving that the Floer homology is well-defined. (It also follows that $QFH(Y, f)$ does not depend on the precise value of λ as long as it is sufficiently small.)

Let us recall the essential properties of ω_ξ and ω from [21]. ω_ξ for ξ an area form on Σ_g is characterized by the following equation. Let P be a 2-dimensional region in $\mathrm{Sym}^{g-k}(\Sigma_g)$ and $D(P) \subset \Sigma_g$ be its projection to Σ_g defined by first taking the preimage of P by the map $\pi : \Sigma_g^{\times(g-k)} \to \mathrm{Sym}^{g-k}(\Sigma_g)$ and projecting to the first component (cf. [16] Definition 2.13). Then we have

$$\int_P \omega_\xi = \frac{1}{(g-k)!} \int_{D(P)} \xi.$$

Such symplectic forms ω_ξ were constructed by Perutz in [21] and they represent the cohomology class $\pi_*([\xi^{\times(g-k)}]) = s\eta$ where $s = \frac{1}{(g-k)!}\int_{\Sigma_g} \xi$ and coincides with $\pi_*(\xi^{\times(g-k)})$ outside a neighborhood of the diagonal Δ. We are free to scale ξ if necessary in order to adjust s.

The construction of the form ω in class $\eta + \lambda\theta$ in [21] is given by modifying ω_ξ near the diagonal. Indeed $\omega = \omega_\xi - \epsilon\delta$ for $\epsilon > 0$ small where δ denotes a closed $(1, 1)$ form representing the diagonal class $[\Delta] = (4g - 2k - 2)\eta - 2\theta$ [14] and supported in a small neighborhood of Δ. By taking ϵ sufficiently small, and adjusting s by scaling ξ, we can obtain a symplectic form ω in class $\eta + \lambda\theta$ for $\lambda > 0$ sufficiently small. Then, it is easy to see that ω and ω_ξ tame the same regular complex structures $\mathrm{Sym}^{g-k}(j_s)$ (since the modification is compactly supported and sufficiently small).

Next, we adopt the definition of strong admissibility from [16] to our setting. Note that as in [16], $\mathbf{x}$ determines a spin^c structure $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$. (In fact, again as in [16], one can see that $\pi_0(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta)) \simeq \mathcal{S}(Y|\Sigma_k)$.) Recall also that if $P \in \pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbf{x})$ then we denote the homology class $H(P) \in H_2(Y)$ determined by P obtained as follows: P can be projected onto the surface Σ_g such that the projection bounds various α_i and β_j, these boundary components then can be capped off inside Y by the corresponding cores of the handles attached to α_i and β_j (see [16]).

Definition 29. Let $(\Sigma_g, \alpha_1, \ldots, \alpha_{g-k}, \beta_1, \ldots, \beta_{g-k})$ be a diagram obtained from a broken fibration on Y and $k > 1$, we say that the diagram is strongly admissible if there exists an area form ξ on Σ_g such that for any $P \in \pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbf{x})$ the following holds:

$$(2-2k)\int_{D(P)} \xi = \langle c_1(\mathfrak{s}), H(P)\rangle \int_{\Sigma_g} \xi$$

where $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$ is the spin^c structure determined by $\mathbf{x}$ and $[P] \in H_2(Y)$ is the homology class determined by P.

Note that splicing a sphere representing $\pi_2(\mathrm{Sym}^{g-k}(\Sigma_g)) = \mathbb{Z}$, changes the index by

$$2\langle c_1(T\mathrm{Sym}^{g-k})(\Sigma_g), [\Sigma_{g-k}]\rangle = ((1-k)\eta - \theta)([\Sigma_g]) = (2-2k)$$

which is exactly equal to $\langle c_1(\mathfrak{s}), [\Sigma_g]\rangle$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$. This is why the normalization factor in our definition of strong admissibility differs form the one in [16].

Now, if we consider an arbitrary diagram $(\Sigma_g, \alpha_1, \ldots, \alpha_{g-k}, \beta_1, \ldots, \beta_{g-k})$, by winding (say α curves) as in [16] (cf. Lemma 8), we can make the diagram strongly admissible for $k > 1$ (for $k = 1$, we again need to use the Novikov ring Λ and keep track of the intersection with a basepoint w). The following lemma shows that for $k > 1$ strong admissibility implies monotonicity for a symplectic form ω_ξ in the class η (cf. [10]) and concludes the proof that $QFH(Y, f, \mathfrak{s})$ is well-defined for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$ and $g < 2k$.

Proposition 30. *For a strongly admissible diagram $(\Sigma_g, \alpha_1, \ldots, \alpha_{g-k}, \beta_1, \ldots, \beta_{g-k})$, the ω_ξ-area and the index maps from $\pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbf{x}) \to \mathbb{R}$ are proportional.*

Proof. If $P \in \pi_1(\Omega(\mathbb{T}_\alpha, \mathbb{T}_\beta), \mathbf{x})$, it follows from the same argument as in Theorem 4.9 of [16] that $\text{Index}(P) = \langle c_1(\mathfrak{s}), H(P)\rangle$, where $H(P) \in H_2(Y)$ is the homology class determined by P. Strong admissibility then implies that:

$$(2-2k)\int_{D(P)} \xi = \text{Index}(P)\int_{\Sigma_g} \xi. \quad \Box$$

Corollary 31. *When $k > 1$, $QFH(Y, f, \mathfrak{s})$ is well-defined over $\mathbb{Z}_2$ for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_k)$ and $g < 2k$.* $\Box$

4.3. The isomorphism

Recall that when defining $QFH'(Y, f, \Lambda)$ as a variant of the Heegaard–Floer homology we have used Lipshitz's cylindrical reformulation, by setting up the theory in $\Sigma_{\max} \times [0, 1] \times \mathbb{R}$. This was convenient because of the bubbling issues that may occur in the negatively monotone manifold $\text{Sym}^{g-k}(\Sigma_{\max})$. However, in the strongly negative case, when $g < 2k$, bubbling can be ruled out for a generic path J_s of almost complex structures on $\text{Sym}^{g-k}(\Sigma_{\max})$ on the grounds that the moduli space of bubbles in this case has negative virtual dimension. In fact, the proof of the lemma below shows that in this case, one can also use a path of integrable complex structures as in the case of the usual Heegaard–Floer homology to make sense of this group. Thus, the Floer homology groups can be formulated as a Lagrangian intersection theory in $\text{Sym}^{g-k}(\Sigma_{\max})$.

Lemma 32. *Suppose that Y admits a broken fibration with $g < 2k$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$,*

$$QFH'(Y, f, \mathfrak{s}; \Lambda) \simeq HF(Sym^{g-k}(\Sigma_{\max}); \alpha_1 \times \cdots \times \alpha_{g-k}, \beta_1 \times \cdots \times \beta_{g-k}, \mathfrak{s}; \Lambda).$$

Proof. We first argue that for a generic path of almost complex structures $\{j_s\}$ on $\Sigma_{\max}$ the induced integrable complex structures $\text{Sym}^{g-k}(j_s)$ achieve transversality for the holomorphic disks mapping to $\text{Sym}^{g-k}(\Sigma_{\max})$ which contribute to the differential and furthermore for these complex structures no bubbling can occur because of the strong negativity assumption $g < 2k$. The fact that these complex structures achieve transversality is standard and follows exactly as in the case of the usual Heegaard–Floer homology set-up; see for example Proposition A.5 of [11]. To avoid bubbling, we make use of the Abel–Jacobi map:

$$\text{AJ} : \text{Sym}^{g-k}(\Sigma_{\max}) \to \text{Jac}(\Sigma_{\max}).$$

The assumption $g < 2k$ ensures that the Abel–Jacobi map is injective for j chosen outside of a subset of complex codimension at least 1 (so that for a generic path j_s it is injective for all s). A generic choice of j_s therefore ensures that there cannot be any non-constant holomorphic spheres mapping to $\mathrm{Sym}^{g-k}(\Sigma_{\max})$ since $\pi_2(\mathrm{Jac}(\Sigma_{\max})) = 0$. One can also rule out disk bubbles in the same way: since the inclusions of $\alpha_1 \times \cdots \times \alpha_{g-k}$ and $\beta_1 \times \cdots \times \beta_{g-k}$ to $\mathrm{Sym}^{g-k}(\Sigma_{\max})$ are injective at the level of fundamental groups and since the Abel–Jacobi map is injective and induces an isomorphism on the first homology when $g < 2k$, the image of a holomorphic disk by the Abel–Jacobi map represents a trivial relative homology class, therefore it is trivial. Hence, there cannot be any non-constant holomorphic disk bubbles.

Now, applying the reformulation of Lipshitz, as in [11], allows us to translate the Lagrangian–Floer homology in $\mathrm{Sym}^{g-k}(\Sigma_{\max})$ "tautologically" to the cylindrical set-up in $\Sigma_{\max} \times [0, 1] \times \mathbb{R}$ (see Appendix A in [11]). □

Finally, we are ready to state our theorem that establishes the isomorphism between quilted Floer homology groups arising from Lagrangian correspondences with Heegaard–Floer homology.

Theorem 33. *Suppose that Y admits a broken fibration with $g < 2k$. Then for $\mathfrak{s} \in \mathcal{S}(Y|\Sigma_{\min})$,*

$$HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq QFH'(Y, f; \mathfrak{s}, \Lambda) \simeq QFH(Y, f; \mathfrak{s}, \Lambda).$$

Proof. The proof will be obtained by putting together the results obtained so far together with the composition theorem for correspondences [8,28] mentioned in the introduction to this section. This allows one to compose Lagrangian correspondences and obtain isomorphic Floer homology groups. More precisely, Theorem 21 and Lemma 32 give us that $HF^+(Y, f, \gamma_w, \mathfrak{s}) \simeq QFH'(Y, f; \mathfrak{s}, \Lambda) \simeq HF(\mathrm{Sym}^{g-k}(\Sigma_{\max}); \alpha_1 \times \cdots \times \alpha_{g-k}, \beta_1 \times \cdots \times \beta_{g-k}; \Lambda)$. Now, Lemma 26 expresses the Lagrangians $\alpha_1 \times \cdots \times \alpha_{g-k}$ and $\beta_1 \times \cdots \times \beta_{g-k}$ as transverse and embedded compositions of the Lagrangians L_{α_i} and L_{β_j}. Therefore, we are in a position to apply Wehrheim–Woodward's composition theorem which says that quilted Floer homology groups associated with a cyclic set of Lagrangian correspondences is invariant under transverse and embedded compositions of the Lagrangians. One important technicality that arises in the proof of Wehrheim–Woodward is the possibility of "figure-eight" bubbles. To avoid those, Wehrheim–Woodward originally proved their theorem only in the positively monotone case (and exact case), whereas we are in the strongly negative setting and it is not clear that the approach of Wehrheim and Woodward can be generalized to this situation.

We have addressed this problem in another place (see [8]) where we gave a new proof of the Wehrheim–Woodward result which applies equally well in the strongly negative setting.

Therefore, we obtain an isomorphism between the Floer homology of the Lagrangians $\alpha_1 \times \cdots \times \alpha_{g-k}, \beta_1 \times \cdots \times \beta_{g-k}$ and the quilted Floer homology of the Lagrangian correspondences $L_{\alpha_1}, \ldots, L_{\alpha_{g-k}}$ and $L_{\beta_1}, \ldots, L_{\beta_{g-k}}$. This completes the proof. □

5. Discussion: 4-manifold invariants

We first recall the definition of broken Lefschetz fibrations on smooth 4-manifolds.

Definition 34. A broken fibration on a closed 4-manifold X is a smooth map to a closed surface with singular set $A \cup B$, where A is a finite set of singularities of Lefschetz type near which a local model in oriented charts is the complex map $(w, z) \to w^2 + z^2$, and B is a 1-dimensional submanifold along which the fibration is locally modeled by the real map $(t, x, y, z) \to (t, x^2 + y^2 - z^2)$, B corresponding to $t = 0$.

It was proven in [7] that every closed oriented smooth 4-manifold admits an *equatorial* broken Lefschetz fibration to S^2 (see also [2] where the authors give a new proof of this result using handlebody calculus). Equatorial here means that the 1-dimensional part of the critical value set is a set of embedded parallel circles on S^2. Lagrangian matching invariants of a 4-manifold as defined by Perutz in [22] are obtained by counting quilted holomorphic sections of a broken Lefschetz fibration associated with the 4-manifold. These invariants, which are conjecturally equal to Seiberg–Witten invariants, have a TQFT-like structure where the three manifold invariants are the quilted Floer homology groups that we have discussed in this paper. Similarly, the Heegaard–Floer homology is the three manifold part of a TQFT-like structure, which underlies the construction of Ozsváth–Szabó 4-manifold invariants [19].

By cutting a broken Lefschetz fibration along a family of circles that are transverse to the equatorial circles of critical values, one can obtain a cobordism decomposition of the 4-manifold, such that each cobordism is an elementary cobordism, namely it is a cobordism obtained by either a one or two handle attachment. Therefore, because of Theorem 33, in order to equate the above mentioned four-manifold invariants for the spin^c structures which satisfy the adjunction equality with respect to the minimal genus fiber of the broken Lefschetz fibration, one needs to check only that the cobordism maps for one and two handle attachments in both theories coincide. This will be in turn obtained by extending the techniques developed in this paper to cobordism maps. We plan to investigate this latter claim in a sequel to this paper. This will in particular prove that for the spin^c structures considered, the Lagrangian matching invariants are independent of the broken Lefschetz fibration that is chosen on the 4-manifold.

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