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One Fixed Point Actions on Spheres, I

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0. THE PITCH

The object of this paper is to treat the following old question of Montgomery and Samelson [8] and some of its consequences:

Which groups act smoothly on a closed homotopy sphere with exactly one fixed point and what are the isotropy representations of G which occur on the tangent space at the fixed point?

The first and only previously existing example of such an action was given by Stein for the group $SL(2, \mathbb{Z}_5)$ [14]. A related question was solved by Oliver [6]: Which groups act smoothly on a disk without fixed points? A group which acts on a sphere with one fixed point acts on a disk without fixed points. In [11] the author announced:

THEOREM A. *These groups act smoothly on a homotopy sphere with exactly one fixed point:*

(i) S^3, SO_3,

(ii) $SL(2, F)$, $PSL(2, F)$ *with characteristic F odd,*

(iii) *any odd order abelian group having at least three non-cyclic Sylow subgroups.*

This paper provides the proof of this theorem in case (iii) and identifies some of the isotropy representations which occur. These are the representations occurring in the set $\mathscr{R}_0$ defined in (1.1). Let G satisfy (iii).

THEOREM B. *Given any integer $n \geqslant 0$ and $R \in \mathscr{R}_0$, there is a closed smooth homotopy G sphere Σ such that Σ^G consists of n points and the isotropy representation at each is R.*

THEOREM C. *If M is any smooth G manifold and $p \in M^G$ is a point whose isotropy representation R lies in $\mathscr{R}_0$, there is a smooth G manifold M′*

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having the same homotopy type as M, $M'^G = M^G - p \cup \{p_1, \dots, p_n\}$ *and the isotropy representation at each* p_i *is* R.

In [10] we use Theorem B to construct smooth actions of G on a homotopy sphere Σ such that Σ^G is two points with distinct isotropy representations. Compare [1, 7]. In [9] we treat Theorem A(i). Of course isotropy representations are just one aspect of the equivariant tangent bundle. The relation between the equivariant tangent bundle and the fixed set of an action on a sphere is an interesting one. In addition to the above references, see also [3, 12] regarding this relation.

The main tools used in A and B are Theorems (2.2) and (2.8). The latter uses equivariant transversality to construct a manifold and map satisfying the hypothesis of (2.2). The transversality construction is used in conjunction with the equivariant cohomology theory $\omega_G^*(\cdot)$. Starting from any representation $R \in \mathscr{R}_0$ and an $x \in \omega_G^0(Y)$, where Y is the unit sphere of $R \oplus \mathbb{R}$, we use (2.8) to construct a G map $f: X \to Y$. Here $\mathbb{R}$ is the trivial one dimensional representation of G. The properties of (X, f) depend on x and Y. All the following lemmas are involved with establishing the relation among (X, f), x and Y and showing the existence of an x such that X^G is one point and (X, f) satisfies the hypothesis of (2.2). Then (2.2) gives A(iii). The Burnside ring of G plays a role because $\omega_G^*(\cdot)$ is a module over the Burnside ring and it is used to express one of the conditions of (2.2).

1. The Setup

Throughout this paper G is an odd order abelian group which has at least three non-cyclic Sylow subgroups and $\mathscr{F}$ is a family of subgroups of G. These examples of $\mathscr{F}$ are used: $\mathscr{P}$—the family of groups of prime power order, $\tilde{\mathscr{P}}$—the family of hyperelementary groups which here are the product of a cyclic group and a p group of relatively prime order and

$$\mathscr{H} = \{H \subset G \mid G/H \notin \mathscr{P}\}.$$

Observe that $\mathscr{H} \supset \tilde{\mathscr{P}} \supset \mathscr{P}$.

Let X be a G space and G_x the isotropy group of $x \in X$. Then $\mathrm{Iso}(X)$ is the set of G_x for $x \in X$. We say X satisfies the Gap hypothesis if $\dim X^K < \frac{1}{2} \dim X^H$ whenever $K \supset H$ and $X^K \neq X^H$ provided $\dim X^H \neq 0$. Towards identifying the isotropy representations of one fixed point actions on spheres, we define a set $\mathscr{R}$ of complex G modules (alias complex representations of G). *The set of realifications of representations in* $\mathscr{R}$ *is denoted by* $\mathscr{R}_0$.

Definition (1.1). A complex G module V is in $\mathscr{R}$ if:

(a) $V^G = 0$.
(b) V satisfies the Gap hypothesis.
(c) $\mathrm{Iso}(V - 0) = \mathscr{H}$.
(d) $\dim_C V^H \geqslant 2$ for $H \in \mathscr{H}$.
(e) $\dim_C V^P \geqslant 3$ for $P \in \mathscr{P}$.

LEMMA (1.2). *$\mathscr{R}$ is not empty.*

Proof. Let $\mathbb{C}(G)$ be the complex regular representation of G, 1 the trivial representation and $\rho(G) = \mathbb{C}(G) - 1$. As G is abelian, $G = \prod P_j$ is the product of its Sylow subgroups P_j. The projection of G on P_j makes any complex P_j module a complex G module. Set $V = \rho(G) - \sum \rho(P_j)$. We claim $V \in \mathscr{R}$. The verification is left to the reader. Note: (i) $H \subset G$ is in $\mathrm{Iso}(V)$ iff $\dim V^H > \dim V^{H'}$ whenever H' strictly contains H. (ii) The complex dimension of $\mathbb{C}(G)^H$ is $|G|\,|H|^{-1}$. Here $|G|$ denotes the order of G. (iii) If $H \subset G$, $H = \prod H_j$, $H_j \subset P_j$. (iv) $\rho(P_j)^H = \rho(P_j^{H_j})$.

LEMMA (1.3). *If V_1 and V_2 are in $\mathscr{R}$, $V_1 \oplus V_2$ is in $\mathscr{R}$. If S is a complex G module and $\mathrm{Iso}(S - 0) \subset \mathscr{H}$, there is a $W \in \mathscr{R}$ such that $W \oplus S \in \mathscr{R}$.*

Proof. For any W and S, $\mathrm{Iso}(W \oplus S) = \{H \cap K \mid H \in \mathrm{Iso}(W),\ K \in \mathrm{Iso}(S)\}$. Thus if $\mathrm{Iso}(S - 0) \subset \mathscr{H}$ and $\mathrm{Iso}(W - 0) = \mathscr{H}$, $\mathrm{Iso}(W \oplus S - 0) = \mathscr{H}$. This remark is the essential observation in the first statement. The reader may then verify that $nV \oplus S \in \mathscr{R}$ for n large and V as in (1.2).

The Burnside ring of G, $\Omega(G)$, is the Grothendieck group of equivalence classes of compact G spaces with addition defined by disjoint union and X equivalent to Y if the Euler characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal for all $H \subset G$. See, e.g., [6]. The class of a space X in $\Omega(G)$ is written $[X]$. The multiplication is defined by $[X] \cdot [Y] = [X \times Y]$. The identity 1 is $[p]$ for any point p. $\Omega(G)$ is the free abelian group generated by the classes $[G/H]$ as H runs over conjugacy classes of subgroups of G. The subgroup generated by $\{[G/H] \mid H \in \mathscr{F}\}$ is denoted by $\Omega(G, \mathscr{F})$. It is an ideal when $\mathscr{F}$ is closed under taking subgroups. There is a homomorphism

$$\chi_H \colon \Omega(G) \to Z$$

defined by $\chi_H[X] = \chi(X^H)$.

Formally extend the definition of $\Omega(G)$ to $\Omega(U, G) = \oplus_U \Omega(G)$ for any finite set U with trivial G action. Define a function

$$\mu \colon \Omega(G) \to \Omega(G)$$

by $\mu(\sum a_H|G/H|) = \sum |a_H|[G/H]$, where $|a_H|$ is the absolute value of the integer a_H. Extend μ to $\Omega(U, G)$ coordinate wise. Define $\sigma: \Omega(U, G) \to \Omega(G)$ by $\sigma(\oplus_{x\in U} E_x) = \sum_{x\in U} \mu(E_x)$.

Let $H \subset G$. Then Res_H denotes restriction of G data to H data. This is used in several contexts, e.g., to define the ideal

$$\Delta(\mathscr{F}) = \bigcap_{H\subset\mathscr{F}} \mathrm{Ker}(\mathrm{Res}_H: \Omega(G) \to \Omega(H))$$

in $\Omega(G)$ associated to the family of subgroups $\mathscr{F}$ of G. To be more specific, Res_H here is the homomorphism defined by $\mathrm{Res}_H|X| = [\mathrm{Res}_H X]$, where $\mathrm{Res}_H X$ means to view the G space X as an H space. Set

$$\Delta(G) = \Delta(\tilde{\mathscr{P}})$$

and for $\varepsilon = \pm 1$ set

$$\Delta^\varepsilon = \{x \in \Delta(\mathscr{H}) \mid \chi_G(x) = \varepsilon\}.$$

The former is an ideal in $\Omega(G)$.

LEMMA (1.4). *Δ^1 is not empty.*

Proof. Since $\mathrm{Res}_C S$ and $\chi_C(S)$ are multiplicative as a function of S in $\Omega(G)$, it suffices to exhibit for each cyclic group C which is not of prime power order an element $X = X(C) \in \Omega(C)$ with $\chi_C(X) = 1$ and $\chi_1(X) = 0$. Then $S = \prod_{H\in\mathscr{F}} i_H^* X(C_H) \in \Delta^1$, where $i_H: G \to C_H$ is a surjective homomorphism onto a cyclic group C_H not of prime power order such that the kernel of i_H contains H.

If C is cyclic of order pq, $(p, q) = 1$, take $x = (1 + a|C/H| + b|C/K| + d|C|)$, where H and K are cyclic subgroups of C of orders p and q, respectively, and a, b and d are integers. Since $(p, q) = 1$, the equations $\chi_C(X) = 1$ and $\chi_1(X) = 0$ are solvable for a, b and d.

LEMMA (1.5). *For any $S \in \Delta^1$, there is a U in $\Omega(G)$ such that $E = U \cdot S$ is in Δ^ε and $\mu(1 - E) \equiv (1 - E) \bmod \Omega(G, \mathscr{H})$.*

Proof. First we find a U in $\Omega(G)$ such that $\chi_G(U) = 1$ and the coefficients of $|G/K|$ in $U \cdot S$ have prescribed signs whenever $G \neq K \notin \mathscr{H}$. This is done by induction.

Let $\mathscr{K}$ be the set of subgroups not in $\mathscr{H}$; so $K \in \mathscr{K}$ iff $G/K \in \mathscr{P}$. Say $\mathscr{F}$ is closed if $K \subset K'$ and $K \in \mathscr{F}$ implies $K' \in \mathscr{F}$. E.g., $\mathscr{K}$ is closed. Express S as $1 + X + Y$, where $X = \sum_{K\in\mathscr{K}-\{G\}} a_K[G/K]$ and $Y \in \Omega(G, \mathscr{H})$. Let $\mathscr{K}' \subset \mathscr{K}$ be closed. Suppose a_K has given sign whenever $K \in \mathscr{K}'$. Let $H \in \mathscr{K} - \mathscr{K}'$ be a maximal group in this set. Set $\mathscr{K}'' = \mathscr{K}' \cup H$. Then $\mathscr{K}''$

is closed; so $\Omega(G,\mathscr{H}'')$ is an ideal for $\mathscr{H}''=\{H\notin\mathscr{K}''\}$. We compute modulo $\Omega(G,\mathscr{H}'')$.

Let b_H be an integer to be determined. Then $T=(1+b_H[G/H])$. S is congruent to $1+\sum_{K\neq G,H} a_K[G/K]+(b_H+a_H+b_H\cdot\sum_{K\supset H} a_K\,|G/K|)[G/H]$. This uses the fact that $[G/H]\cdot[G/K]=a_{H,K}[G/H\cap K]$; so $H\cap K\in\mathscr{H}''$ unless $H=H\cap K$ and if K contains H, $a_{H,K}=|G|\,|K|^{-1}$. Since $H\notin\mathscr{H}$, $G/H\in\mathscr{P}$; so the order of G/H is a power of p for some p. Then p divides each $|G/K|$ whenever $K\supset H$ and $K\neq G$. This means $\sum_{G\neq K\supset H} a_K\,|G/K|$ is never -1. Now it is evident that b_H can be chosen so that the coefficient of $[G/H]$ in T has a prescribed sign. The coefficients of $[G/K]$ in T and S are the same for $K\in\mathscr{K}'$. Since χ_G is a ring homomorphism and $\chi_G(U)=1$ for $U=1+b_H[G/H]$ whenever $H\neq G$, it follows that $\chi_G(T)=1$; so $T\in\Delta^1$.

Thus there is a U' so that $U'\cdot S=E'\in\Delta^1$ and the coefficients of $[G/K]$ in E are negative for K in $\mathscr{K}-\{G\}$. There is a U'' so that $U''\cdot S\in\Delta^1$ and these coefficients are positive. Then $E''=-U''\cdot S\in\Delta^{-1}$ and these coefficients are negative. The lemma is satisfied with $E=E'$ when $\varepsilon=1$ and $U=E''$ when $\varepsilon=-1$.

Let M be a real G module, $F=F(M)$ the space of proper self maps of M and $[Y,F]^G$ the G homotopy classes of maps of Y to F. Then

$$\omega_G^0(\cdot)=\varinjlim_{M}\,[\cdot,F(M)]^G$$

is the zeroth term of an equivariant cohomology theory $\omega_G^*(\cdot)$ [13]. Let $\mathbf{M}$ denote the G vector bundle $Y\times M$ over Y. As the base space Y of this vector bundle is omitted from the notation, it must be determined by context. Any proper self G map ω of $\mathbf{M}$ which is properly G homotopic to a fiber preserving map determines a class $[\omega]\in\omega_G^0(Y)$ and any element of this group is represented this way for some M. Actually $\omega_G^0(Y)$ is a ring with unit 1 represented by the identity map of $\mathbf{M}$. When Y^H is connected, there is a homomorphism

$$\deg_H\colon \omega_G^0(Y)\to Z$$

obtained by setting $\deg_H[\omega]$ equal to the fiber degree of $\omega^H\colon\mathbf{M}^H\to\mathbf{M}^H$. In the special case of a point p, we abbreviate $\omega_G^0(p)$ by ω_G^0 and note that $\omega_G^0(Y)$ is an ω_G^0 module.

Reference [13] asserts ω_G^0 and $\Omega(G)$ are naturally isomorphic. In fact there are inverse isomorphisms Φ and ψ giving a commutative diagram

$$\begin{array}{ccc} \Omega(G) & \underset{\psi}{\overset{\Phi}{\rightleftarrows}} & \omega_G^0 \\ & {\scriptstyle\chi_H}\searrow \quad \swarrow{\scriptstyle\deg_H} & \\ & Z & \end{array}$$

See [11, 6.2 and 6.8]. These extend to isomorphisms again called Φ and ψ between $\omega_G^0(U)$ and $\Omega(G, U)$ for any finite set U with trivial G action. Define a function

$$\theta\colon \omega_G^0(U) \to \Omega(G)$$

by $\theta = \sigma \circ \psi$.

Localization provides an important relation among $\Delta(\mathscr{F})$, $\omega_G^0(Y)$ and $\omega_G^0(Y^G)$ when $\mathrm{Iso}(Y - Y^G) \subset \mathscr{F}$. Let $S \in \Delta(\mathscr{F})$, $s = \Phi(S)$ and t be the set of powers of s. This is a multiplicative set in ω_G^0 so $t^{-1}\omega_G^0(Y)$ is defined. Let i be the inclusion of Y^G in Y. As in [2]:

LEMMA (1.6). *If* $\mathrm{Iso}(Y - Y^G) \subset \mathscr{F}$, $t^{-1}i^*\colon t^{-1}\omega_G^0(Y) \to t^{-1}\omega_G^0(Y^G)$ *is an isomorphism.*

Proof. $s \cdot \omega_G^*(Y, Y^G) = 0$ because $\mathrm{Res}_H\, s = 0$ for all $H \in \mathscr{F}$ and Y is obtained from Y^G up to G homotopy equivalence by adding G cells of type $D^i \times G/H$, $H \in \mathscr{F}$. Note $\omega_G^*(G/H \times (D^i, S^{i-1})) \simeq \omega_H^*(D^i, S^{i-1})$.

Let $x \in \omega_G^0(Y)$. Then $\mathrm{Res}_{\mathscr{H}}\, x = 1_{\mathscr{H}}$ means $\mathrm{Res}_H\, x$ is the identity of $\omega_H^0(Y)$ for all $H \in \mathscr{H}$.

LEMMA (1.7). *Suppose* Y^G *consists of two points* p *and* q *and* $\mathrm{Iso}(Y - Y^G) \subset \mathscr{H}$. *Then there is an* $x \in \omega_G^0(Y)$ *and an* $E \in \Delta^\varepsilon$ *with* $\mu(1 - E) \equiv (1 - E) \bmod \Omega(G, \mathscr{H})$ *such that* $i^*x = \Phi(1, 1 - E)$, $\mathrm{Res}_{\mathscr{H}}\, x = 1_{\mathscr{H}}$ *and* $\chi_G\,\theta(i^*x) = 2 - \varepsilon$.

Proof. $\omega_G^0(Y^G) = \omega_G^0(p) \oplus \omega_G^0(q)$. Let $s = \Phi(S)$, where $S \in \Delta^1$. Lemma (1.6) supplies a z in the ideal $s\omega_G^0(Y)$ with $i^*z = (0, s^\lambda)$ for some λ. Since $S^\lambda \in \Delta^1$, there is a U in $\Omega(G)$ such that $E = U \cdot S^\lambda \in \Delta^\varepsilon$ and $\mu(1 - E) \equiv (1 - E) \bmod \Omega(G, \mathscr{H})$ (1.5). Let $u = \Phi(U) \in \omega_G^0$ and $x = 1 - uz$. Then $i^*x = \Phi(1, 1 - E)$ and $\mathrm{Res}_{\mathscr{H}}\, x = 1_{\mathscr{H}}$. The last statement follows from $\theta(i^*x) = 1 + \mu(1 - E)$ and $\chi_G(E) = \varepsilon$.

2. THE STING

For any G space X, G acts on the partially ordered set

$$\Pi(X) = \coprod_{H \subset G} \pi_0(X^H).$$

An element α of $\Pi(X)$ labels a component X_α of X^H for some H. This is expressed by $\rho(\alpha) = H$. It gives a function ρ from $\Pi(X)$ to the set of subgroups of G. Set $\alpha \leqslant \beta$ if $X_\alpha \subset X_\beta$ and $\rho(\beta) \subset \rho(\alpha)$. Let $G_\alpha =$

$\{g \in G \mid ga = \alpha\}$. When ξ is a G vector bundle over X, define a G_α vector bundle $\pi_\alpha \xi$ over X_α by

$$\xi|_{X_\alpha} = \xi^H|_{X_\alpha} \oplus \pi_\alpha \xi, \qquad \rho(\alpha) = H.$$

The collection $\pi\xi = \{\pi_\alpha \xi \mid \alpha \in \Pi(X)\}$ is called a Π bundle over X. The most important example comes from the G tangent bundle TX of a G manifold X. In that case πTX is denoted by νX. Let $\varepsilon\xi$ denote either ξ or $\pi\xi$. The stabilization of $\varepsilon\xi$ is defined to be

$$s(\varepsilon\xi) = \varepsilon(\xi \oplus \mathbf{M})$$

for some G module M.

Note that any H vector bundle isomorphism $a: \xi|_{X_\alpha} \to \xi'|_{X_\alpha}$, $\rho(\alpha) = H$, splits as $a = a^H \oplus a_H$. A Π vector bundle isomorphism $b: \pi\xi \to \pi\xi'$ is a collection of G_α vector bundle isomorphisms $b_\alpha: \pi_\alpha \xi \to \pi_\alpha \xi'$ such that $b_\alpha|_{X_\beta} = (b_\beta)_H$ whenever $\beta \leqslant \alpha$. In particular a G vector bundle isomorphism $a: \xi \to \xi'$ defines a Π bundle isomorphism $\pi(a): \pi\xi \to \pi\xi'$. The stabilization of a G or Π vector bundle isomorphism $b: \varepsilon\xi \to \varepsilon\xi'$ is defined to be

$$s(b) = b \oplus \varepsilon(1_{\mathbf{M}}): s(\varepsilon\xi) \to s(\varepsilon\xi').$$

Let X be a smooth G manifold and R a real G module whose dimension is $\dim X$.

(2.1) An *R structure on X* consists of a G vector bundle isomorphism $b: sTX \to s\mathbf{R}$ and a Π vector bundle isomorphism $c: \nu X \to \pi\mathbf{R}$ such that $\pi(b) = s(c)$.

When X has an R structure for some R, we say X *is framed*. If X is the boundary of W and W has an $R \oplus \mathbb{R}$ structure which restricts to the R structure on X, we say X is the framed boundary of W. Here $\mathbb{R}$ is the trivial one dimensional G module. If in addition each component of W^H has a non-empty boundary for all $H \subset G$, we write $X = \partial_p W$. We say X and X' are *framed cobordant* if $X \cup X'$ is the framed boundary of some manifold W and $\mathrm{Iso}(X) = \mathrm{Iso}(X') = \mathrm{Iso}(W)$. They are framed cobordant rel boundary if $\partial X = X \cap X' = \partial X'$. The framed cobordism is rel $\mathscr{F}$ if W^K is $X^K \times I$, $I = [0, 1]$ for $K \in \mathscr{F}$. These definitions extend in an obvious way to pairs (X, f), where $f: X \to Y$. For example, $(X, f) = \partial_p(W, F)$ if $X = \partial_p W$, $Y = \partial Z$ and $F: W \to Z$ extends f. Let $Y = \partial Z$, $f: X \to Y$ and $(W_{\mathscr{F}}, F_{\mathscr{F}}) = \{(W_H, F_H) \mid H \in \mathscr{F}\}$. Each W_H is an H manifold and $F_H: W_H \to Z$ is an H map. Then $\partial_p(W_{\mathscr{F}}, F_{\mathscr{F}}) = \mathrm{Res}_{\mathscr{F}}(X, f)$ means $\partial_p(W_H, F_H) = \mathrm{Res}_H(X, f)$ for all $H \in \mathscr{F}$.

Let $\mathscr{F}$ be a family of subgroups of G with the property that if $H \in \mathscr{F}$ and $K \subseteq H$, then $K \in \mathscr{F}$. Suppose $\mathscr{F}$ contains all hyperelementary subgroups of

G and that G is an odd order abelian group. The following theorem is the key tool used in Theorems A and B.

THEOREM (2.2) [5, 2.6]. *Let X and Y be smooth closed even dimensional G manifolds and $f: X \to Y$ a G map. Suppose*

(i) *degree $f = 1$, X has an R structure and satisfies the Gap hypothesis.*

(ii) *$Y = \partial Z$, Z^H and Y^H are 1 connected whenever $H \in \mathscr{P}$.*

(iii) *For all $H \in \mathscr{P}$, each component of X^H has a point whose isotropy group is H and* $\dim X^H = \dim Y^H \geqslant 6$.

(iv) $\operatorname{Res}_{\mathscr{F}}(X,f) = \partial_p(W_{\mathscr{F}}, F_{\mathscr{F}})$.

(v) *For all $H \in \mathscr{F}$,* $\dim X^H \geqslant 3$ *and* $\operatorname{Iso}(X) \supset \mathscr{F}$.

(vi) $[Y] - [X] \in \Delta(G) + 2\Omega(G, \mathscr{F})$.

Then (X,f) is framed cobordant to (X',f') $\operatorname{rel}\{K \subseteq G \mid K \notin \mathscr{F}\}$ *with f' a homotopy equivalence. In particular $X^G = X'^G$ and $T_pX = R$ for all $p \in X^G$.*

A smooth G manifold Y is a unique fixed point dimension manifold (abbreviated UFD) if the dimension of each component of Y^H only depends on H for all $H \subseteq G$.

LEMMA (2.3). *If Y is a UFD, $K \in \operatorname{Iso}(Y)$ if* $\dim Y^H < \dim Y^K$ *whenever H strictly contains K. The converse is true if Y^K is connected.*

Proof. $K \in \operatorname{Iso}(Y)$ iff Y^K strictly contains Y^H when $H > K$, e.g., when $\dim Y^H < \dim Y^K$. If Y^K is connected, strict containment implies inequality. This happens for a smooth G manifold iff $\dim Y^H < \dim Y^K$.

LEMMA (2.4). *If Y is a UFD, each component of Y^K has a point whose isotropy group is K whenever* $\dim Y^K > \dim Y^H$ *for all $H > K$.*

Proof. Let Y_α be a component of Y^K and H the intersection of the isotropy groups of points of Y_α. If there is no point in Y_α whose isotropy group is K, $H > K$ and $Y_\alpha = Y_\beta$ for some component Y_β of Y^H. Then $\dim Y_\alpha = \dim Y^K > \dim Y^H = \dim Y_\beta$.

LEMMA (2.5). *If Y has an R structure, Y is a UFD.*

Proof. For any component Y_α of Y^H $\dim Y_\alpha = \dim R^H$.

LEMMA (2.6). *Suppose X and Y have R structures and $f: X \to Y$ is a G map. Then* $\operatorname{Iso}(X) \supset \{H \in \operatorname{Iso}(Y) \mid Y^H$ *is connected and* $X^H \neq \varnothing\} = \mathscr{F}$. *If $I \in \mathscr{F}$, each component of X^I has a point whose isotropy group is I.*

Proof. If $I \in \mathscr{F}$, $\dim Y^K < \dim Y^I$ if $K > I$, by (2.3) and (2.5). Since X^I is not empty $\dim X^I = \dim Y^I > \dim Y^K \geqslant \dim X^H$; so $I \in \mathrm{Iso}(X)$ by (2.3) and each component of X^I has a point whose isotropy group is 1 by (2.4).

LEMMA (2.7). *Suppose $F: (W, \partial W) \to (Z, \partial Z)$ is a G map, W and Z have R structures and* $\mathrm{Iso}(Z)$ *contains I. If Z^I is connected and* $\dim Z^I > 0$, *(W, F) is framed cobordant* rel ∂W *and* rel$\{K$ *not contained in* $I\}$ *to (W', F'), where W'^I is connected.*

Proof. If W^I is empty, there is nothing to prove; so suppose $W^I \neq \varnothing$. By (2.6) each component of W^I has a point whose isotropy group is I; moreover, $\dim W^I = \dim Z^I > 0$. The conclusion follows from [4, Sect. 9].

We review the G transversality construction of [4, 11] in the special case dealing with $\omega_G^0(Y)$. Let $\omega: \mathbf{M} \to \mathbf{M}$ be a proper G map representing an element $[\omega]$ of $\omega_G^0(Y)$. Define M_H via $\mathrm{Res}_H M = M^H \oplus M_H$. The normal derivative of ω at $x \in \omega^{-1}(Y)$ is the $H = G_x$ endomorphism of M_H defined as $(\pi\, d\omega_x\, i)_H: M_H \to M_H$, where $\pi\, d\omega_x\, i$ is the composition

$$M \overset{i}{\subset} T\mathbf{M}_x \xrightarrow{d\omega_x} T\mathbf{M}\omega(x) \xrightarrow{\pi} \nu(Y, \mathbf{M})_{f(x)} = M.$$

THEOREM (2.8) [4, 11]. *Let $A \subset Y$ be a closed invariant set. Suppose ω is transverse to $Y \subset \mathbf{M}$ on $\mathbf{M}|_A$ such that for each $x \in \mathbf{M}|_A \cap \omega^{-1}(Y)$, the normal derivative at x is the identity. Then ω is properly G homotopic* rel $\mathbf{M}|_A$ *to a map ω' transverse to Y such that the normal derivative of ω' is the identity for all $x \in \omega'^{-1}(Y)$.*

Given $x \in \omega_G^0(Y)$, represent x as $[\omega]$ for some ω. Use (2.8) to produce ω' transverse to Y and set

$$\tau(x) = (X, f),$$

where $X = \omega'^{-1}(Y)$ and $f = \omega'|_X: X \to Y$. When Y has an R structure, X has an R structure and the framed cobordism class of $\tau(x)$ is well defined.

Suppose Y and Z are UFDs and $Y = \partial Z$. We record the properties of the transversality construction $\tau(x)$ when $x \in \omega_G^0(Y)$ and its relation to $\tau(w)$ when $w \in \omega_G^0(Z)$ extends x. Let then $\mathbf{M}$ be $Y \times M$ and η be a proper self map of $\mathbf{M}$ representing x in $\omega_G^0(Y)$ or let $\mathbf{M}$ be $Z \times M$ and $\tilde{\eta}$ a proper G self map of $\mathbf{M}$ which extends η. Set $(X, f) = \tau(x)$ and $(W, F) = \tau(w)$. We emphasize that first η is made transverse to Y to produce $\tau(x)$. Then $\tilde{\eta}$ is made transverse to Z rel $\mathbf{M}|_Y$ to produce $\tau(w)$. In particular $\partial(W, F) = (X, f)$. These further properties hold:

(P.1) *If Y has an R structure, X has an R structure.*

As pointed out in [4, Sect. 8] the transversality construction $\tau(x)$ provides a G vector bundle isomorphism $b: sTX \to f^*sTY$ and a Π vector bundle isomorphism $c: \nu X \to f^*\nu Y$ which satisfy $\pi(b) = s(c)$. These combine with an R structure on Y to produce an R structure on X.

(P.2) *If the R structure on Y extends to an $R \oplus \mathbb{R}$ structure on Z, the R structure on X extends to an $R \oplus \mathbb{R}$ structure on W.*

The isomorphisms b and c extend to $B: sTW \to F^*sTZ$ and $C: \nu W \to F^*\nu Z$.

(P.3) $\operatorname{degree} f^H = \deg_H x$.

Let η' be properly G homotopic to η and be transverse to Y with $X = \eta'^{-1}(Y)$. Then η'^H is transverse to Y^H for all H and $X^H = (\eta'^H)^{-1} Y^H$. Since η'^H is properly G homotopic to a fiber preserving self map of $\mathbf{M}^H$ whose fiber degree is $\deg_H x$ by definition and since $f^H = \eta'^H|_{X^H}$, $\operatorname{degree} f^H = \deg_H x$.

(P.4) *X is a UFD. Either* $\dim X^H = \dim Y^H$ *or X^H is empty. The former surely occurs when* $\deg_H x \neq 0$. *Likewise W is a UFD and* $\dim W^H$ *and* $\dim Z^H$ *are so related.*

Since X has an R structure, Lemma (2.5) implies that X is a UFD. If $\deg_H x \neq 0$, $\operatorname{degree} f^H \neq 0$ by (P.3) so $X^H \neq \varnothing$. If $X^H \neq \varnothing$, $\dim X^H = \dim R^H = \dim Y^H$.

LEMMA (2.9). *Suppose Y has an R structure, $Y = \partial_p Z$, Z^H is connected and* $\dim Z^H > 0$ *for all $H \in \mathscr{H}$ and* $\operatorname{Iso}(Y) \supset \mathscr{H}$. *If $x \in \omega_G^0(Y)$ and* $\operatorname{Res}_{\mathscr{H}} x = 1_{\mathscr{H}}$, *then $\tau(x) = (X, f)$ has these properties: X has an R structure,* $\operatorname{Iso}(X) \supset \mathscr{H}$ *and* $\operatorname{Res}_{\mathscr{H}}(X, f) = \partial_p(W_{\mathscr{H}}, F_{\mathscr{H}})$.

Proof. When $H \in \mathscr{H}$ let $(W_H, F_H) = \tau(1_H)$, where 1_H is the identity in $\omega_H^0(Z)$. Since $\operatorname{Res}_H x$ extends to 1_H, $\partial(W_H, F_H) = \operatorname{Res}_H(X, f)$. By (P.1), X has an R structure. By (P.2), W_H has a $\operatorname{Res}_H(R \oplus \mathbb{R})$ structure extending the $\operatorname{Res}_H R$ structure on X. Since $\deg_H x = 1$ for all $H \in \mathscr{H}$, (P.3), (P.4) and (2.4) imply that $\operatorname{Iso}(X)$ contains $\mathscr{H}$.

Apply (2.7) with G replaced by H to produce a framed H cobordism rel X of (W_H, F_H) to (W'_H, F'_H) with $W_H'^I$ connected for all $I \subseteq H$. Do this for each $H \in \mathscr{H}$. Set $(W_{\mathscr{H}}, F_{\mathscr{H}}) = \{(W'_H, F'_H) \mid H \in \mathscr{H}\}$ to complete the lemma.

LEMMA (2.10). *Suppose Y has an R structure, $\partial_p Z = Y$, Z^H is connected for $H \in \mathscr{H}$,* $\operatorname{Iso}(Y) \supset \mathscr{H}$, $\dim Y^G = 0$ *and $Y^K = Y^G$ whenever $K \notin \mathscr{H}$. Let $x \in \omega_G^0(Y)$ with* $\operatorname{Res}_{\mathscr{H}} x = 1_{\mathscr{H}}$. *Then $\tau(x) = (X, f)$ has these properties:*

(a) *X has an R structure and* $\operatorname{degree} f = 1$,

(b) $\dim X^H = \dim Y^H$, $H \in \mathscr{H}$, *and* $\dim X^K \leqslant 0$ *for $K \notin \mathscr{H}$,*

(c) $\operatorname{Iso}(X) \supset \mathscr{H}$,

(d) $\mathrm{Res}_{\mathscr{F}}(X, f) = \partial_p(W_{\mathscr{F}}, F_{\mathscr{F}})$,

(e) *the cardinality of* X^G *is* $\chi_G \theta(i^*x)$, $i\colon Y^G \to Y$,

(f) $[X] \equiv \theta(i^*x) \bmod \Omega(G, \mathscr{H})$.

Proof. Let $x = [\omega]$, $\omega\colon \mathbf{M} \to \mathbf{M}$. By the G homotopy extension theorem and Theorem (2.8) applied to $i^*\omega\colon i^*\mathbf{M} \to i^*\mathbf{M}$, we may suppose that $i^*\omega$ is transverse to Y^G as a submanifold of $i^*\mathbf{M}$; moreover by [11, (6.9)] we may suppose $X' = (i^*\omega)^{-1}(Y^G)$ represents $\theta(i^*x)$ in $\Omega(G)$. Note that ω is automatically transverse to Y on $i^*\mathbf{M}$.

Use Theorem (2.8) again with $A = Y^G$ to make ω transverse to Y rel $i^*\mathbf{M}$. Call the resulting map again ω and set $X = \omega^{-1}Y$ and $f = \omega|_X$ so $(X, f) = \tau(\omega)$. By Lemma (2.9), (c) and (d) are satisfied. Since $\deg_1 \omega = 1$, degree $f = 1$ by (P.3) and X has an R structure by (P.1); so (a) holds while (b) is a consequence of (P.4) and the hypothesis on Y. Since $Y^K = Y^G$ for $K \notin \mathscr{H}$, it follows that $(i^*\mathbf{M})^K = \mathbf{M}^K$ for $K \notin \mathscr{H}$ and this implies $X'^K = X^K$ for $K \notin \mathscr{H}$. Property (f) follows from this while property (e) follows from (f) and the fact that $\dim X^G$ is zero; so $\chi(X^G)$ is the cardinality of X^G.

LEMMA (2.11). *Suppose in addition to the hypothesis of Lemma* (2.10), *that* Y^G *consists of two points,* $i^*x = \Phi(1, 1 - E)$ *for* $E \in \Delta(\mathscr{H})$, *where* $\mu(1 - E) \equiv (1 - E) \bmod \Omega(G, \mathscr{H})$, *and* $|Y| = |Y^G| = 2$. *Then* $|Y| - [X] \in \Delta(G) + 2\Omega(G, \mathscr{H})$.

Proof. By definition of θ, $\theta(i^*x) = 1 + \mu(1 - E)$; so the hypothesis and (2.10(f)) easily imply that $[Y] - [X] = E + \Omega$ for some $\Omega \in \Omega(G, \mathscr{H})$. We claim $\Omega \in 2\Omega(G, \mathscr{H})$. Since the order of G is odd, this statement is true iff $\chi_H(\Omega) \equiv 0(2)$ for all $H \in \mathscr{H}$. Since $E \in \Delta(\mathscr{H})$, $\chi_H(E) = 0$ for all $H \in \mathscr{H}$. Since X^H and Y^H are boundaries for all $H \in \mathscr{H}$, their Euler characteristics are zero modulo 2; so $\chi_H(\Omega) = \chi_H[X] - \chi_H[Y]$ is zero mod 2. Since $\mathscr{F} \subset \mathscr{H}$, $\Delta(\mathscr{H}) \subset \Delta(G)$ so $E \in \Delta(G)$.

THEOREM (2.12). *For any* $R \in \mathscr{R}_0$ *and* $\varepsilon = \pm 1$, *there is a smooth closed homotopy sphere* Σ *with* Σ^G *consisting of* $2 - \varepsilon$ *points. In addition* Σ *has an* R *structure and* $\mathrm{Res}_{\mathscr{F}} \Sigma = \partial_p W_{\mathscr{F}}$ *for some family* $W_{\mathscr{F}}$.

Proof. Let Y, resp. Z, be the unit sphere, resp. disk, of $R \oplus \mathbb{R}$. Then Y satisfies the hypothesis of (1.7). Let $x \in \omega_G^0(Y)$ be given by that lemma. Set $\tau(x) = (X, f)$. Then X^G consists of $2 - \varepsilon$ points by (2.10) and (1.7). We verify the hypothesis of (2.2) with $\mathscr{F} = \mathscr{H}$. By (1.1), Y satisfies the Gap hypothesis. The hypotheses of (2.10) are satisfied. By (2.10(b)), X satisfies the Gap hypothesis. Then (2.10(a)) implies (2.2(i)) while (2.2(ii)) is obvious; (2.10(b,c)) plus (1.1) gives (2.2(iii,v)) while (2.10(d)) is a restatement of (2.2(iv)). Condition (2.2(vi)) follows from (2.11). Theorem (2.2) finishes the proof.

Proof of Theorems A, B *and* C. These are all corollaries of (2.12) and the fact that $\mathscr{R}_0 \neq \varnothing$ (1.2). First note that (2.12) produces homotopy spheres Σ_i with Σ_i^G consisting of i points for $i = 1$ or 3 and the isotropy representation at each is R. If M is any G manifold and $p \in M^G$ has isotropy representation R, then $M \# \Sigma_3 = M'$ has $M'^G = M^G - p \cup \{p_1 \cup p_2\}$ and the isotropy representations at p_1 and p_2 are R. Of course this connected sum is taken at the point of M with isotropy representation R. Repeat this process to complete the proof of C. Theorems A and B are consequences of Theorem C.

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