ADVANCES IN MATHEMATICS **46**, 15–70 (1982)

One Fixed Point Actions on Spheres, II

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0. INTRODUCTION

The object of this paper is to treat the following old question of Montgomery and Samelson [M–S] and some of its consequences:

Which groups act smoothly on a closed homotopy sphere with exactly one fixed point and what are the isotropy representations of G which occur on the tangent space at the fixed point?

The first and only previously existing example of such an action was given by E. Stein for the group $SL(2, \mathbb{Z}_5)$ [St]. A related question was solved by Oliver [O_2]: Which groups act smoothly on a disk without fixed points? A group which acts on a sphere with one fixed point acts on a disk without fixed points. In [P_1] the author announced:

THEOREM A. *These groups act smoothly on a homotopy sphere with exactly one fixed point*:

(i) S^3, SO_3,

(ii) $SL(2, F)$, $PSL(2, F)$ *with characteristic F odd*,

(iii) *any odd order abelian group having at least three non-cyclic Sylow subgroups.*

This paper provides the proof of this theorem in case (i) and identifies some of the isotropy representations which occur. These are the representations occurring in the set $\mathscr{R}_0(G)$ defined in (6.9). Let G be S^3 or SO_3.

THEOREM B. *Given any integer $n \geqslant 0$ and $R \in \mathscr{R}_0$, there is a closed smooth homotopy G sphere Σ such that Σ^G consists of n points and the isotropy representation at each is R.*

THEOREM C. *If M is any smooth G manifold and $p \in M^G$ is a point whose isotropy representation A lies in $\mathscr{R}_0(G)$, there is a smooth G manifold M' having the same homotopy type as M, $M'^G = M^G - p \cup \{p_1 \cdots p_n\}$ and the isotropy representation at each p_i is A.*

0001-8708/82/100015–56$05.00/0

To the author's knowledge, it is not known whether there is a group which acts smoothly on a disk without fixed points yet cannot act smoothly on a sphere with one fixed point. Many groups which are not known to act smoothly act topologically on a spere with one fixed point. Take any smooth fixed point free action of G on a disk D [O_1, O_2]. Then G acts topologically on the sphere $D/\partial D$ with one fixed point. Here is a challenging question which directs attention to the central issues: *Is there any connected simple Lie group besides SO_3 which acts smoothly on a sphere with one fixed point?*

Aside from the main applications—Theorems A–C—this paper offers these general tools for constructing smooth actions on a manifold whose underlying homotopy type is fixed: (i) an equivariant transversality lemma (6.1), (ii) an equivariant surgery theory (Sections 3 and 4), (iii) an equivariant surgery induction theorem (5.10). These topics have independent interest in their own right. The Equivariant Transversality Lemma (6.1) is used in conjunction with the equivariant stable cohomotopy theory $\omega_G^*(Y)$ associated to the G space Y when G is a compact Lie group. An element of $\omega_G^0(Y)$ is represented by a proper G map ω: $Y \times M \to M$, where M is some representation of G. When Y is a smooth G manifold and the hypothesis of (6.1) is satisfied, ω is properly G homotopic to a map h transverse to 0. The manifold $X = h^{-1}(0)$ inherits some additional structure which serves as input for the equivariant surgery theory. In particular the composition $f: X \subset Y \times M \to Y$ is a G map whose degree is 1. There is a stable G vector bundle isomorphism $b: sTX \to f^*sTY$, where sTX is the stable G tangent bundle of X. Additionally there is a λ bundle isomorphism $d: \lambda(TX) \to \lambda(f^*TY)$ (Section 3). The datum $\mathscr{W} = (X, f, b, d)$ is called a G prenormal map. It is what is required for equivariant surgery theory. This theory describes a method of converting (via G cobordism) $\mathscr{W}$ to $\mathscr{W}' = (X', f', b', d')$ with f' a pseudoequivalence, i.e., $f': X' \to Y$ is a G map which is a homotopy equivalence. The Equivariant Induction Theorem describes conditions where pseudoequivalence is achievable. In general, an equivariant surgery induction theorem identifies a family $\mathscr{H}$ of proper subgroups of G such that if for each $H \in \mathscr{H}$ the restriction to H of $\mathscr{W}$ is the boundary of an H prenormal map $\mathscr{W}_H$, then $\mathscr{W}$ is G cobordant to $\mathscr{W}'$ with f' a pseudoequivalence. For the problem at hand we require that this cobordism is rel X^G so $X^G = X'^G$.

Briefly the program for applying (i)–(iii) to the problem of constructing smooth homotopy spheres with exactly one fixed point is this. Begin with $Y = S(A \oplus \mathbb{R})$— the unit sphere of the representation $A \oplus \mathbb{R}$ where A is a representation in $\mathscr{R}_0(G)$ with $A^G = 0$ and G acts trivially on $\mathbb{R}$. Thus Y^G has two fixed points p and q. A suitable $x \in \omega_G^0(Y)$ is produced with: (iv) requirements on $i^*x \in \omega_G^0(Y^G)$, i: $Y^G \to Y$ and (v) $\mathrm{res}_H x = 1 \in \omega_H^0(Y)$ for $H \in \mathscr{H}$. The Equivariant Transversality Lemma is used to produce a G prenormal map $\mathscr{W} = \mathscr{W}(x) = (X, f, b, d)$, $f: X \to Y$ whose properties implicitly depend on $x \in \omega_G^0(Y)$. Condition (iv) on i^*x is used to arrange that X^G = one point while (v) is used to satisfy the hypothesis of the Equivariant Induction Theorem, i.e., $\mathrm{res}_H \mathscr{W} = \partial \mathscr{W}_H$ for $H \in \mathscr{H}$. This theorem implies that $\mathscr{W}$ is G cobordant (rel X^G) to $\mathscr{W}'$ with f' a pseudoequivalence. Thus X' is a homotopy sphere and X'^G is one point.

These methods have been used to study other invariants of actions on homotopy spheres. We mention two explicitly. Let Y be a smooth G homotopy sphere. Suppose for each subgroup H of G, each component of Y^H has the same dimension—dim Y^H. Define an integral valued function on the subgroups of G by

$$\mathrm{Dim}\, Y(H) = \dim Y^H + 1.$$

The definition is so arranged that $\mathrm{Dim}\, S(A \oplus \mathbb{R})(H) = \dim A^H$ when A is a representation of G. It is an old theorem of Artin [Ar] that the values of $\mathrm{Dim}\, S(A \oplus \mathbb{R})$ are not independent and $\mathrm{Dim}\, S(A \oplus \mathbb{R})(G)$ is a function of $\{\mathrm{Dim}\, S(A \oplus \mathbb{R})(C) \mid C \text{ cyclic}\}$. Using the methods here, this functional dependence is seen to fail for general smooth G homotopy spheres.

THEOREM [D–P$_2$]. *There is a function* ψ_G *such that* $\mathrm{Dim}\, Y(G) = \psi_G\{\mathrm{Dim}\, Y(H) \mid H \neq G\}$ *for every smooth G homotopy sphere Y iff G is a non-cyclic group of prime power order. Compare also* [tD–P].

Another interesting invariant of a smooth action of G on a homotopy sphere Y is the set of isotropy representations $\{T_pY \mid p \in Y^G\}$ of G on the tangent space of Y at fixed points. In fact an old question of P. A. Smith asks if necessarily the representations of G at two isolated fixed points of an action of G on a homotopy sphere must be equal [Sm]. This question has a negative answer. The first examples [P$_4$] (following strong positive evidence by Atiyah and Bott [A–B] and Milnor [Mi]) of distinct isotropy representations were produced using a modification of the program outlined above for producing one fixed point actions. See also [P$_5$]. For other applications of the methods here, see also [P$_3$, P$_6$].

The author has expended considerable effort to make this paper as self-contained as possible. *In particular there is a self-contained account of the relevant equivariant surgery theory*. The only outside references to surgery required are to some of the less technical results in [W$_1$]. In spite of this most of the results in this paper appear for the first time in print here. At a few points the author had to sacrifice generality for clarity. In particular some of the hypothesis of (5.10) can be weakened; however, that is not relevant to our main applications. Sections 1 and 2 introduce the basic notation and background.

Here is a brief description of the setting for equivariant surgery and induction theorem (Sections 3–5) whose goal is to convert an equivariant map $f: X \to Y$ between smooth G manifolds into a pseudoequivalence. The basic datum for equivariant surgery is a $K - G$ prenormal map of triads $\mathscr{W} = (W, F, B, D)$ ((3.9) and (3.9′)). Here K is a subgroup of G. Briefly W is a K manifold whose boundary ∂W is the union of two invariant G manifolds X_0 and X_1 (which might be empty), $F: W \to Z$ is a map of $K - G$ manifold triads while B and D are certain equivariant bundle isomorphisms. In addition there are G prenormal maps $\mathscr{W}_i = (X_i, f_i, b_i, d_i)$ for $i = 0, 1$ with $f: X_i \to Y_i$, $\partial Z = Y_0 \cup Y_1$ such that restricting $\mathscr{W}_i$ to a K prenormal map gives the K prenormal map $\mathscr{W}|_{X_i}$. For $i = 0$ abbreviate $\mathscr{W}_0$ as (X, f, b, d), where $f: X \to Y$. Let Q be a subgroup of G and $\mu \in \pi_{k+1}(f^Q)$. Definition (3.17) (see (3.30)) tells what it means to do surgery on $(\mathscr{W}, \mu)$. It is a process which creates a new $K - G$ prenormal map $\mathscr{W}'$ with $\mathscr{W}'_1 = \mathscr{W}_1$. The class $\partial\mu \in \pi_k(X^Q)$ which lies in the kernel of $\pi_k(X^Q) \to \pi_k(Y^Q)$ is killed in X' where $\mathscr{W}'_0 = (X', f', b', d')$. Theorems (3.18), (3.31) and Corollary (3.19) tell when surgery on a class $\mu \in \pi_{k+1}(f^Q)$ is possible. For example, the condition in (3.19) is $k < \frac{1}{2} \dim X^Q$.

The role of equivariant vector bundle data (B and D appearing in the definition of the prenormal map $\mathscr{W}$ in (3.9) and the splitting (1.1)) in the process of equvariant surgery and transversality is much more prominent than in the case of equivariant surgery and transversality dealing with free actions. See [W$_1$] for equivariant surgery for free actions. Some reasons for this are made aparent in the motivational remarks in Section 3. Another reason is the relation between bundle data (3.9) in equivariant surgery and its relation to the subtle process of equivariant transversality. Indeed the Transversality Lemma (6.1) naturally provides the required bundle data (B, D) for a prenormal map. See Section 6 and also [D–P$_1$, Sect. 8]. Many of the results which deal with bundle data in Sections 3, 6 and 7 have no counterpart or are trivial for equivariant surgery involving free actions. Sections 3 and 7 dealing with the equivariant vector bundle aspects of surgery and transversality are the most demanding. To help motivate the material there we include some motivational discussion now and more in Section 3. Here we mention the unexpected role of the splitting (1.1) and the notion of stability in equivariant transversality and surgery.

In (1.1) we introduce the splitting of an equvariant vector bundle depending on a representation C. In particular if N is a smooth G manifold and K is a subgroup of G, we have a splitting of the normal bundle $\nu = \nu(N^K, N)$ of N^K in N as

$$\nu = \Lambda(\nu, C) \oplus \Lambda'(\nu, C).$$

This is a splitting of $N(K)$ (normalizer of K) vector bundles and C is a representation of G which contains the Lie algebra of G. The splitting is arranged so that each fiber $\Lambda'(\nu_x)$, $x \in N^K$, viewed as a representation of K

contains all the irreducible representations in ν_x which occur in C (viewed as a representation of K). From the splitting of ν we obtain a splitting

$$TN|_{N^K} = TN^K \oplus \Lambda(\nu) \oplus \Lambda'(\nu), \tag{0.1}$$

where C has been omitted, whose role in the Equivariant Transversality Lemma (6.1) we now explain.

Let M be a representation of G, Y a smooth G manifold, $N = Y \times M$ and let $\omega: N \to M$ be a proper G map. Lemma (6.1) gives a condition—Y is C stable (3.6)—under which ω is properly G homotopic to a map f transverse to $0 \in M$. Two general concepts are involved in the proof of (6.1). Compare [P_3, Chap. II]. The first is that the problem of equivariant transversality is involved with global phenomena in contrast to the non-equivariant situation where everything is local and trivial. The second is that Schur's lemma applied to the equivariant bundles involved with transversality gives a decomposition of the problem into two stages and provides the basis for the inductive proof (6.1).

To amplify the second point suppose $f: N \to M$ is transverse to 0 with $X = f^{-1}(0)$. Let $K \subset G$ and let $x \in X$ be a point with isotropy group K. Then the differential df_x of f at x gives a surjective K map

$$df_x: T_xN \to T_0M$$

between tangent spaces at x and 0 which splits according to (0.1) and Shur's lemma as

$$df_x = df_x^K \oplus \Lambda(df_x) \oplus \Lambda'(df_x). \tag{0.2}$$

The transversality condition and the splitting (0.2) imply each factor is surjective. Let $G(x)$ be the orbit of x in X and g_x be its tangent space at x. *Note* $g_x \in \operatorname{Ker}(df_x)$ because $f(x) = 0$ and f is a G map. This means $\Lambda'(g_x) = g_x \cap \Lambda'(\nu_x)$ is in $\operatorname{Ker} \Lambda'(df_x)$; so $\Lambda'(df_x)|_L$ *is already surjective on the complement* L *of* $\Lambda'(g_x)$ *in* $\Lambda'(\nu_x)$. On the other hand $g_x \cap \Lambda(\nu_x) = 0$; so there is no interaction between the Lie algebra of G and $\Lambda(\nu_x)$.

The idea of the proof of (6.1) is to reverse these observations to make df_x surjective. This is done inductively on the partial order in the set of isotropy groups of N. Let K be such an isotropy group and $x \in X^K$. We suppose df_y is surjective whenever $y \in X^H$ for $H > K$. We must make each factor of df_x in (0.2) surjective. (In the process X is altered rel $\bigcup_{H>K} X^H$.) To achieve surjectivity for the first factor traditional methods of non-equvariant transversality are used. This is the first stage and is easy. The second stage requires making the second and third factors in (0.2) surjective. The second requires work but no further hypothesis. Surjectivity for the third requires the stability assumption (3.6). In (6.4) we show how to use the transversality lemma to produce the data for equivariant surgery, i.e., a prenormal map (3.9). The stability condition (3.6) appears also in the process of equivariant surgery through Lemma (3.26).

As a point of interest we note that Dovermann and Rothenberg have extended the above outlined methods of equivariant transversality to topological actions of finite groups on manifolds. Madsen and Rothenberg have used the resulting topological transversality lemma in their work on topological equivalence of representations of finite groups.

Section 4 analyzes what happens when $k = \frac{1}{2} \dim W^Q$ and $\mu \in \pi_{k+1}(F^Q)$. Here homological considerations appear. Chiefly these are dictated by Smith Theory which asserts that if $F: W \to Z$ is a pseudoequivalence, then for each subgroup P whose connected component P_0 is a torus and P/P_0 is a p group (the set of all such groups is denoted by $\mathscr{P}$), F^P must be a mod p homology equivalence. This must hold for all primes p. This means that in the process of converting F to a pseudoequivalence we are led to the inductive situation where Q is one of the groups mentioned above, F^Q induces an isomorphism in homology up to the middle dimension of W^Q ((4.8) using Section 3) and for all P as above with $Q \lhd P$ and P/P_0 a p group not 1, F^P is a mod p homology equivalence. Then (4.6) $K_m(W^Q) = \operatorname{Ker}(H_m(W^Q) \to H_m(Z^Q))$, $m = \frac{1}{2} \dim W^Q$, is a projective $Z_{(p)}(\bar{Q}^0)$ module $\bar{Q} = N(Q)/Q$, $\bar{Q}^0 = \bar{Q}/\bar{Q}_0$. Above $Z_{(p)}$ (Z localized at p) coefficients are understood. We analyze when this is free (4.13). The main technical results (4.14) and (4.15) then tell when $\mathscr{W}$ is G prenormally cobordant (produced by equivariant surgery) to $\mathscr{W}'$, where F'^Q is a mod p homology equivalence and X^P is unaltered for $P > Q$.

Section 5 contains the induction theorem for $G = SO_3$ and S^3. As mentioned this asserts that if $\mathscr{W}$ is a G prenormal map and $\operatorname{res}_K \mathscr{W} = \partial \mathscr{W}_K$ for all K in a certain family of subgroups, then $\mathscr{W}$ is G prenormally cobordant to $\mathscr{W}'$ with $\mathscr{W}' = (X', f', b', d')$ and f' a pseudoequivalence. The condition $\operatorname{Res}_K \mathscr{W} = \partial \mathscr{W}_K$ is nothing more than $\mathscr{W}_K$ is a $K - G$ prenormal (3.9′) map of triads with $(\mathscr{W}_K)_0 = \mathscr{W}$ and $(\mathscr{W}_K)_1 = \varnothing$. Thus we can apply the results of Sections 3 and 4 to prove the induction theorem (5.10). The essential results in the proof of (5.10) are (4.13), (4.14) and (4.15).

Section 6 contains the proofs of Theorems A–C. Section 7 contains the proof of the Equivariant Transversality Lemma (6.1) and Section 8 is a technical check that the set $\mathscr{R}_0(G)$ is non-empty.

Because of the interval between announcement of the main results here and their publication, it is prudent to give some history. The Equivariant Transversality Lemma (6.1) has evolved since 1973 when the author first announced a version for finite groups in lectures at Heidelberg and Saarbrücken. It is used to establish the main results of this paper for finite groups. See also [P_3, P_3, P_8]. The equivariant surgery theory for finite groups was developed jointly with Dovermann [D–P_1]. Much of the material on equivariant surgery theory for general compact Lie groups appearing in

Section 4 was presented (in cruder form) in lectures at Aarhus in 1976. The article [P_9] from August 1975 on the projective class group and equivariant surgery (Section 4) was an outgrowth of the author's lectures at Mexico-Centro de Investigacion and the University of Chicago. The author acknowledges the hospitality and support of the above institutions. The Equivariant Surgery Induction Theorem (5.10) has an analog for finite groups which was jointly done with Dovermann [D–P_3]. The induction theorem there was used to treat the main results of this paper for finite groups [P_1]. The induction theorem of Dress [Dr] for the Wall surgery obstruction groups may be recast in the geometric terms mentioned above (except that all group actions must be free). The theorem of Dress is entirely algebraic and plays a role in the geometric proof of the induction theorem of [D–P_3] but has no role here.

The author expresses his gratitude for the comments of Dovermann and Oliver on the material in this article.

1. General Representation Theory

View a complex (real) representation of G both as a vector space A and a homomorphism A' of G into the complex (real) general linear group $GL(A)$ of A. This representation is denoted by A with A' understood. For $g \in G$, trace $\mathbf{A}(g)$ is denoted by $A(g)$. Let μ be normalized Haar measure on G and for two representations A and B of G set

$$\langle A, B\rangle = \int_G A(g)\,\bar{B}(g)\,d\mu(g).$$

When G is finite of order $|G|$, this becomes

$$\langle A, B\rangle = \frac{1}{|G|}\sum_{g\in G} A(g)\,\bar{B}(g).$$

Here an overbar denotes complex conjugation. Then A and B are said to be orthogonal if $\langle A, B\rangle = 0$ and A is irreducible if $\langle A, A\rangle = 1$.

When χ is an irreducible representation, $\chi \in A$ means χ occurs as a subrepresentation of A and A_χ is the maximal subspace of A orthogonal to all irreducible representations different from χ. The set of all real irreducible representations of G is denoted by $I(G)$. The one dimensional *real* trivial representation is denoted by 1. For $\chi \in I(G)$, the set of all real linear G equivariant endomorphisms of χ is a division algebra D_χ over the real numbers $\mathbb{R}$ whose dimension over $\mathbb{R}$ is denoted by d_χ. The invertible real linear G equivariant endomorphisms of a representation A of G are denoted by $\mathrm{Aut}_G A$. By Schur's lemma

$$(1.0)\quad \mathrm{Aut}_G A = \prod_{\chi \in I(G)} \mathrm{Aut}_G(A_\chi) \qquad \text{and} \qquad \mathrm{Aut}_G(A_\chi) = GL(\langle A, \chi\rangle, D_\chi);$$

so $\pi_k(\mathrm{Aut}_G A) \to \pi_k(\mathrm{Aut}_G A')$ is an isomorphism if $A \subset A'$ and $k < \min\{\langle A, \chi\rangle\, d_\chi - 1 \mid \langle A, \chi\rangle \neq 0\}$.

Let $H \subset N \subset G$ and suppose H is normal in N. Let E be an N vector bundle over an N space X with trivial H action and let C be a representation of G. We seek an orthogonal splitting

$$(1.1)\qquad E = E^H \oplus \Lambda(E, C) \oplus \Lambda'(E, C)$$

where for $x \in X$, the fibers over X are the H representations

$$\Lambda(E, C)_x = \bigoplus_{\chi \notin C, \chi \neq 1} E_{x\chi}; \qquad \Lambda'(E, C) = \bigoplus_{\chi \in C, \chi \neq 1} E_{x\chi}.$$

Here χ ranges over $I(H)$ and C is viewed as a representation of H. Note the splitting when it exists is functorial for N bundle maps because it is orthogonal. This also follows from Schur's lemma. It means that an N bundle map b splits as $b^H \oplus \Lambda(b, C) \oplus \Lambda'(b, C)$.

Here are two conditions which guarantee the existence of $\Lambda(E, C)$ and $\Lambda'(E, C)$:

(1.2) (a) Each $\chi \in I(H)$, $\chi \in C$ extends to $\tilde{\chi} \in I(NH)$.
(b) Each $\chi \in I(H)$, $\chi \notin C$ extends to $\tilde{\chi} \in I(NH)$.

Here NH is the normalizer of H. We say C *is good* if:

(1.2′) For each $H \subset G$, either (1.2a) or (1.2b) is valid.

More generally if $\mathscr{H}$ is a conjugation invariant family of subgroups of G which is closed under taking subgroups, we say C is $\mathscr{H}$ *good* if for each $H \subset G$, $H \notin \mathscr{H}$, either (1.2a) or (1.2b) is valid.

Let E' and E be N vector bundles over X and let $Hom_H(E', E)$ denote the vector bundle of H equivariant real linear vector bundle homomorphisms from E' to E. It is an N vector bundle with $(nb)(v) = nb(n^{-1}v)$ for $b \in \mathrm{Hom}_H(E', E)$, $v \in E'$ and $n \in N$. As a special case, let $E' = E = \tilde{\chi}$ be an irreducible representation of N which restricts to the irreducible representation χ of H. Then $\mathrm{Hom}_H(\tilde{\chi}, \tilde{\chi})$ is D_χ. It inherits an action of N as shown above and $\tilde{\chi}$ is a module over it. Directly from the definitions, we find

$$n(\lambda \cdot v) = (n\lambda) \cdot nv, \qquad \lambda \in D_\chi, \quad v \in \tilde{\chi};$$

so $\tilde{\boldsymbol{\chi}} \oplus_{D_\chi} \mathrm{Hom}_H(\tilde{\boldsymbol{\chi}}, E)$ is an N vector bundle over X with $n(w \otimes b) = nw \otimes nb$.

Here $\tilde{\chi}$ is $X \times \tilde{\chi}$. In case (1.2a) resp. (1.2b), define (Compare $[A_1]$ and $[A_2, \text{p. } 79]$)

$$(1.3) \quad \text{(a)} \quad A(E, C) = \bigoplus_{\chi \in \mathcal{S}_1} \tilde{\chi} \otimes_{D_\chi} \operatorname{Hom}(\tilde{\chi}, E),$$

$$\text{(b)} \quad A'(E, C) = \bigoplus_{\chi \in \mathcal{S}_2} \tilde{\chi} \otimes_{D_\chi} \operatorname{Hom}(\tilde{\chi}, E),$$

where $\mathcal{S}_1 = \{\chi \in I(H) \mid \chi \notin C, \chi \neq 1\}$, $\mathcal{S}_2 = \{\chi \in I(H) \mid \chi \in C, \chi \neq 1\}$ and $\tilde{\chi}$ is a fixed extension of χ. There are obvious monomorphisms of $A(E, C)$ and $A'(E, C)$ to E. Use (1.1) to define $A'(E, C)$ when (1.3a) is used to define $A(E, C)$. Similarly use (1.1) to define $A(E, C)$ when $A'(E, C)$ is defined by (1.3b). Note that $A(E, C)$ and $A'(E, C)$ are defined or any subgroup pair (H, N) with H normal in N whenever C is good. If C is $\mathcal{H}$ good, they are defined whenever $H \notin \mathcal{H}$.

Throughout this paper we deal with objects with G action. If $H \subset G$, Res_H denotes restriction of the action to H. Often from context it is clear that an object with G action should be viewed as one with H action. Then Res_H is omitted. E.g., if $\chi \in I(H)$ and A is a representation of G, $\chi \in A$ means $\chi \in \operatorname{Res}_H A$.

For any integer k, let t^k denote the complex one dimensional representation of the circle S^1 with $t(z) = t^k \cdot z$ for $z \in \mathbb{C}$ and $t \in S^1 \subset \mathbb{C}$. As real representations t^k and t^l are equivalent iff $k = \pm l$ and t^k is real irreducible unless $k = 0$. Since S^1 is the maximal torus of S^3 and SO_3, the representations of these two groups are determined by their restrictions to S^1.

Let G be S^3 viewed as the unit sphere in the quaternions $\mathbb{H}$. The quaternions are viewed as a right complex vector space. Then left multiplication by elements in S^3 makes $\mathbb{H}$ a complex two dimensional representation of S^3. The ring of complex polynomials in the two complex coordinates of $\mathbb{H}$ inherits the structure of an infinite dimensional representation of G. The action of G is defined by $gp(v) = p(g^{-1}v)$ for $g \in G$, $v \in \mathbb{H}$ and p a polynomial in the coordinates of v. The subspace of polynomials of degree k is a finite dimensional invariant subspace; so defines a complex representation S_k of G. From the immediate calculation

$$\operatorname{Res}_{S^1} \mathbb{H} = t + t^{-1},$$

we find that for k odd, resp. even,

$$(1.4) \quad \begin{aligned} \operatorname{Res}_{S^1} S_k &= t^k + t^{-k} + t^{k-2} + t^{-k+2} + \cdots + t + t^{-1}, \\ \operatorname{Res}_{S^1} S_k &= t^k + t^{-k} + t^{k-2} + t^{-k+2} + \cdots + t^2 + t^{-2} + 1. \end{aligned}$$

SO_3 is the quotient of S^3 by its center $Z_2 = \{1, -1\}$. Let $p: S^3 \to SO_3$ be the quotient map. Since $-1 \in Z_2$ acts trivially on polynomials of even degree, S_{2k} in fact comes from a representation T_k of SO_3. Formally

$p^*T_k = S_{2k}$. Since the circle in S^3 double covers the circle in SO_3, (1.4) implies that

$$\text{(1.5)} \qquad \operatorname{Res}_{S^1} T_k = t^k + t^{-k} + t^{k-1} + t^{-k+1} + \cdots + t + t^{-1} + 1.$$

Denote the realifications of T_k and S_k by $t_k\ s_k$.

Let g denote the Lie algebra of G. It is a real representation of G. Then

$$\text{(1.6)} \qquad \begin{array}{ll} g \otimes \mathbb{C} = T_1, & G = SO_3, \\ g \otimes \mathbb{C} = p^*T_1 = S_2, & G = S^3. \end{array}$$

This easy check is left to the reader.

LEMMA (1.7). *If $G = SO_3$, g is good. If G is S^3, g is $\mathscr{K}$ good, where $K \in \mathscr{K}$ iff K is cyclic of order* 4, 2 *or* 1.

The proof of this is postponed until the representations of the subgroups of these groups are discussed in Section 8.

COROLLARY (1.8). *Let G be SO_3 (S^3), $H \subset N \subset G$ with H normal in N (and $H \notin \mathscr{K}$). Then any N vector bundle E over an N/H space has a splitting $E = E^H \oplus \Lambda(E, g) \oplus \Lambda'(E, g)$.*

2. SUBGROUPS OF S^3 AND SO_3 AND EQUIVARIANT COHOMOTOPY

The subgroups of S^3 and SO_3 are well known. See [Wo]. The subgroups of SO_3 up to conjugacy are: O_2, the icosahedral group I, the octahedral group O, the tetrahedral group T, S^1, the cyclic group of order n, Z_n, the dihedral group D_n of order $2n$ and the trivial group 1. The maximal proper subgroups are O_2, I and O. They are their own normalizers. O_2 is the normalizer of S^1 and Z_n. The normalizer of D_n is D_{2n} unless $n = 1$. The normalizer of D_2 is O. The normalizer of T is O.

The subgroups of S^3 up to conjugacy are the groups $H' = p^{-1}H$ for $H \subset SO_3$ and the cyclic groups of odd order. Their normalizers are given by $NL = (Np(L))'$ for $L \subset S^3$.

The notation $H \subset G$ means H is a subgroup of G while $H < G$ means H is a proper subgroup and $H \lhd G$ means H is a normal subgroup. The set of all subgroups of G is denoted $\mathscr{S}(G)$. The counterimage of D_l in S^3 is the generalized quaternion group Q_l. Use J to denote either $I \subset SO_3$ or its counterimage $I' \subset S^3$. The context will determine the usage.

Let M be a real representation of G, $F = F(M)$ the space of proper self maps of M and $[Y, F]^G$ the G homotopy classes of maps of Y to F. For $G = S^3$ resp. SO_3, and $H \subset G$, we define a set $\mathscr{S}_H$ of real representations of

H by declaring $M \in \mathcal{S}_H$ iff $\chi \in M$ for $\chi \in I(H)$ implies $\chi \in s_8$ or $\chi = 1$ resp. $\chi \in t_4$ or $\chi = 1$. Then for $H \subseteq G$

$$\omega_H^0(\cdot) = \varinjlim_{M \in \mathcal{S}_H} [\cdot, F(M)]^H$$

is the zeroth term in an equivariant cohomology theory $\omega_H^*(\cdot)$ [Se]. Caution! $\omega_H^0(\cdot)$ usually refers to the case where $\mathcal{S}_H$ is the set of all representations of H.

The G vector bundle $Y \times M$ over Y is denoted by $\mathbf{M}$. As the base space Y is omitted from the notation, it must be determined by context. Any proper self G map ω of $\mathbf{M}$ which is properly G homotopic to a fiber preserving map determines a class $[\omega] \in \omega_G^0(Y)$ and any element of this group is represented this way for some M in $\mathcal{S}_G$. (Equivalently elements are represented by proper G maps $\omega\colon \mathbf{M} \to M$.) Actually $\omega_G^0(Y)$ is a ring with unit 1 represented by the identity map of $\mathbf{M}$. When Y^H is connected, there is a homomorphism

$$\deg_H : \omega_G^0(Y) \to Z$$

obtained by setting $\deg_H[\omega]$ equal to the fiber degree of $\omega^H\colon \mathbf{M}^H \to \mathbf{M}^H$. In the special case of a point q, we abbreviate $\omega_G^0(q)$ by ω_G^0 and note that $\omega_G^0(Y)$ is an ω_G^0 module.

When X is a G space and $x \in X$, G_x denotes the isotropy group of x and $\operatorname{Iso}(X) = \{G_x \mid x \in X\}$. This set is closed under conjugation by G. Let $\operatorname{Iso}(X)/G$ be denoted by $\mathbf{Iso}(X)$. It is the set of conjugacy classes of isotropy groups of X. The conjugacy class of a subgroup K is denoted by (K). For $G = S^3$ or SO_3 set

$$\mathcal{L} = \{G_x \mid x \in M, M \in \mathcal{S}_G\} \tag{2.0}$$

and let $\mathcal{S}'$ denote the set of compact G manifolds defined by $X \in \mathcal{S}'$ iff $\operatorname{Iso}(X) \subset \mathcal{L}$. Write $X \sim Y$ if the Euler characteristics $\chi(X^H)$ and $\chi(Y^H)$ are equal for all $H \subseteq G$. The Grothendieck group of the equivalence classes in $\mathcal{S}'$ with addition defined by disjoint union is denoted by $A(G)$. It is a ring with multiplication defined by Cartesian product. The class of $X \in \mathcal{S}$ in $A(G)$ is denoted by $[X]$. The unit 1 is the class of a point q.

The function $[X] \to \chi(X^H)$ for $H \subseteq G$ defines a homomorphism of $A(G)$ to Z. The collection of these homomorphisms as H ranges over subgroups of G gives an injective homomorphism of $A(G)$ into the ring of functions from subgroups of G to Z. From this one finds that

$$E' = [G/O_2] + [G/O] - [G/D_4] - [G/D_3] \tag{2.1}$$

is an idempotent (i.e., $E'^2 = E'$) in $A(G)$ for $G = SO_3$. (Compare T. tom Dieck, Idempotent elements in the Burnside ring, preprint.) This element is defined as an element of $A(G)$ because O_2, O, D_4 and D_3 are in $\mathrm{Iso}(t_4)$. This will become apparent later. One finds

$$(2.2)\qquad \begin{aligned} \chi(E'^H) &= 0, \qquad H = G, I, \\ &= 1, \qquad (H) \neq (G), (I). \end{aligned}$$

Set

$$(2.3)\qquad \begin{aligned} E &= 1 - E' \in A(G), & G &= SO_3, \\ E &= p^*(1 - E') \in A(G), & G &= S^3. \end{aligned}$$

Here $p^*: A(SO_3) \to A(S^3)$ is defined by viewing an SO_3 manifold as an S^3 manifold via the homomorphism p.

There is a ring homomorphism $\Phi: A(G) \to \omega_G^0$ with the property that

$$(2.4)\qquad \deg_H \Phi([X]) = \chi(X^H)$$

for $H \subset G$ and $[X] \in A(G)$. See [M–P, P_3]. Then

$$(2.5)\qquad e = \Phi(E) \in \omega_G^0$$

is an idempotent and $\mathrm{Res}_H(e) \in \omega_H^0$ is zero unless (H) is (G) or (I) for $G = SO_3$ or (H) is (G) or (I') for $G = S^3$. This follows from (2.2)–(2.5) and the fact that $[\omega] \in \omega_H^0$ is determined by the integers $\deg_K \omega$ for $K \subset H$.

LEMMA (2.6). *Let G be SO_3 or S^3 and Y a G space with $Y^J = Y^G$. Then the inclusion $i: Y^G \to Y$ induces an isomorphism $e \cdot \omega_G^0(Y) \to e \cdot \omega_G^0(Y^G)$.*

Compare [A–S, 1.1].

Proof. $e \cdot \omega_G^*(Y, Y^G) = 0$ because $\mathrm{Res}_H e = 0$ unless (H) is (G) or (J) and Y is obtained from Y^G up to G homotopy equivalence by adding G cells of type $D^i \times G/H$, $H \neq G$ or J. Note $\omega_G^*((D^i, S^{i-1}) \times G/H) \cong \omega_H^*(D^i, S^{i-1})$.

If $G = SO_3$, set $\mathscr{H} = \{O_2, O\}$. If $G = S^3$, set $\mathscr{H} = \{O_2', O'\}$. Let G be one of these two groups and $x \in \omega_G^0(Y)$. Then $\mathrm{Res}_{\mathscr{H}} x = 1_{\mathscr{H}}$ means by definition $\mathrm{Res}_H x$ is the identity $1_H \in \omega_H^0(Y)$ for $H \in \mathscr{H}$.

Let A be a representation of G and $S(A \oplus 1) = Y$ be the unit sphere of $A \oplus 1$. We may suppose Y is a G invariant subspace of $A \oplus 1$. If $A^G = 0$, Y^G consists of two points so $\omega_G^0(Y^G) = \omega_G^0 \oplus \omega_G^0$. Suppose $A^J = 0$; so $Y^J = Y^G$. Let $i: Y^G \to Y$ be the inclusion.

COROLLARY (2.7). *Let G be SO_3 or S^3 and $\alpha = \pm 1$. There is an $x(\alpha) \in \omega_G^0(Y)$ such that $i^* x(\alpha) = (1, 1 - \alpha e)$ and* $\mathrm{Res}_{\mathscr{H}} x(\alpha) = 1_{\mathscr{H}}$.

Proof. i^* maps $e\omega_G^0(Y)$ isomorphically to $e\omega_G^0(Y^G)$. Since $i^* 1 = (1, 1)$ and $i^* z = (0, \alpha e)$ for some $z \in e \cdot \omega_G^0(Y)$, $1 - z$ serves for $x(\alpha)$.

3. The H Vector Bundle Aspects of Surgery

In this section we treat the geometric aspects of equivariant surgery. In contrast to the process of equivariant surgery in the category of free actions, the role of equivariant bundle isomorphisms is a major consideration. Because of this we expend some effort to motivate the assumptions and results by first describing in an abbreviated way the role of bundle data in equivariant surgery. First we introduce some general notation. The main results of this section are (3.18), (3.19) and (3.31).

For any G space X, G acts on the partially ordered set

$$\Pi(X) = \coprod_{H \subset G} \pi_0(X^H). \tag{3.1}$$

An element α of $\Pi(X)$ lables a component X_α of X^H for some H. This is expressed by $\rho(\alpha) = H$. It gives a function ρ from $\Pi(X)$ to the set of subgroups of G. Set $\alpha \leqslant \beta$ if $X_\alpha \subset X_\beta$ and $\rho(\beta) \subset \rho(\alpha)$. Set $G_\alpha = \{ g \in G \mid g\alpha = \alpha \}$ and note $\rho(\alpha)$ is a normal subgroup of G_α. For example, when X^H is connected and $G_x = H$ for some $x \in X^H$, then $X_\alpha = X^H$ when $\rho(\alpha) = H$ and $G_\alpha = N(H)$.

Let ξ be a G vector bundle over X and C a good representation of G (1.2′). For $\alpha \in \Pi(X)$, define G_α vector bundles over X_α by

$$\begin{aligned} \lambda_\alpha(\xi) &= \lambda_\alpha(\xi, C) = \Lambda(\xi|_{X_\alpha}, C), \\ \lambda'_\alpha(\xi) &= \lambda'_\alpha(\xi, C) = \Lambda'(\xi|_{X_\alpha}, C). \end{aligned} \tag{3.2}$$

Note that whenever $\alpha \leqslant \beta$

$$\mathrm{Res}_{G_\alpha \cap G_\beta} \lambda_\alpha(\xi) = \lambda_\beta(\xi)|_{X_\alpha} \oplus V_{\alpha\beta}, \tag{3.3}$$

where $V_{\alpha\beta}$ is orthogonal to the first summand. This means a G_α vector bundle map $d_\alpha : \lambda_\alpha(\xi) \to \lambda_\alpha(\xi')$ defines a $G_\alpha \cap G_\beta$ bundle map $d_{\alpha\beta} : \lambda_\beta(\xi)|_{X_\alpha} \to \lambda_\beta(\xi')|_{X_\alpha}$. We abbreviate the collection $\{\lambda_\alpha(\xi) \mid \alpha \in \Pi(X)\}$ by $\lambda(\xi)$.

(3.4) A λ map $d: \lambda(\xi) \to \lambda(\xi')$ is by definition a collection $\{d_\alpha \mid \alpha \in \Pi(X)\}$, where each $d_\alpha : \lambda_\alpha(\xi) \to \lambda_\alpha(\xi')$ is a G_α vector bundle map satisfying $d_\beta|_{X_\alpha} = d_{\alpha\beta}$ whenever $\alpha \leqslant \beta$.

For example, if $b: \xi \to \xi'$ is a G vector bundle map, $\lambda(b) = \{\lambda_\alpha(b): \lambda_\alpha(\xi) \to \lambda_\alpha(\xi')\}$ is the collection of induced maps provided by the functoriality of the Λ construction. It is a λ map. A λ map d for which each d_α is a vector bundle isomorphism is called a λ isomorphism. Set

(3.5) $\lambda(X)=\lambda(TX, C)$ and $\lambda'(X)=\lambda'(TX, C)$, X a G manifold and C a good representation of G containing g.

Remark. (3.5) only makes sense when C is good. If C is $\mathscr{H}$ good, we define $\lambda(\xi, C)$ to be the collection $\{\lambda_\alpha(\xi, C) \mid \rho(\alpha) \notin \mathscr{H}\}$.

Let ξ be a G vector bundle over X. Define a G_α vector bundle $\pi_\alpha \xi$ over X_α by

$$\xi|_{X_\alpha} = \xi^H|_{X_\alpha} \oplus \pi_\alpha(\xi), \qquad \rho(\alpha)=H.$$

Then

$$\pi_\alpha(\xi)=\lambda_\alpha(\xi) \oplus \lambda'_\alpha(\xi).$$

The collection $\{\pi_\alpha(\xi) \mid \alpha \in \Pi(X)\}$ is denoted by $\pi(\xi)$. Let $b: \xi \to \xi'$ be a G vector bundle map. Then

$$b|_{X_\alpha} = b_\alpha \oplus \pi_\alpha(b),$$

where

$$b_\alpha: \xi^H|_{X_\alpha} \to \xi'^H|_{X_\alpha}, \qquad \pi_\alpha(b): \pi_\alpha(\xi) \to \pi_\alpha(\xi').$$

When $\xi = TX$ is the G tangent bundle of X, $\pi_\alpha(TX)$ is the normal bundle $\nu(X_\alpha, X)$ of X_α in X.

Let $\varepsilon(\xi)$ denote one of ξ, $\pi(\xi)$ or $\lambda(\xi)$. If b: $\xi \to \xi'$ is a G bundle map, $\varepsilon(b)$: $\varepsilon(\xi) \to \varepsilon(\xi')$ is the induced map. The stabilization of $\varepsilon(\xi)$ is defined to be

$$s(\varepsilon\xi) = \varepsilon(\xi \oplus \mathbf{M})$$

for an arbitrary G module M. Here $\mathbf{M}$ is G vector bundle $X \times M$ if X is the base space of ξ. If b: $\varepsilon(\xi) \to \varepsilon(\xi')$, $s(b)$ is $b \oplus \varepsilon(1_{\mathbf{M}})$, where $1_{\mathbf{M}}$ is the identity of M. When η and η' are G vector bundles and $b_t: \eta \to \eta'$, $0 \leqslant t \leqslant 1$, is a G homotopy of G vector bundle isomorphisms, we say b_0 and b_1 are *regularly G homotopic.*

Here in abbreviated form is a description of the role of bundle data in the process of equivariant surgery. Roughly the setting for surgery is this: F: $(W, X) \to (Z, Y)$ is an equivariant map between smooth G manifolds of the same dimension, $X=\partial W$, $Y=\partial Z$, ξ is a G vector bundle over Z, $H \subset G$, X_α is a component of X^H, Y_β is the component of Y^H into which X_α is mapped by F, B: $sTW \to F^*s\xi$ is a G bundle isomorphism and D_γ: $\lambda_\gamma W \to \lambda_\gamma F^*\xi$ is an $N=G_\alpha$ vector bundle isomorphism. There is a relation between B and D_γ (3.9), namely, $\lambda_\gamma(B)=s(D_\gamma)$.

Suppose $x \in X_\alpha$ is a point whose isotropy group G_x is H. Then the orbit of x $G(x)$ is G/H. The inclusion of $G(x)$ in X gives an injection of $T_x G(x) = g/h \to T_x X$, where $T_x X$ is the isotropy representation of H on the tangent space of X at x. The Lie algebras of G, N, H are g, n, h. Let $\Omega = \nu(X_\alpha, X)_x$ (fiber over x) and let Γ be the orthogonal complement in Ω of g/n.

Note that as a representation of H, g/h splits as $n/h \oplus g/n$ and $(g/h)^H = n/h$; so $g/n \subset \Omega$. Suppose $\dim W_\gamma = n+1$. Let $\iota' : S^k \to X_\alpha$ represent an element of $\ker(\pi_k(X_\alpha) \to \pi_k(Y_\beta))$, let $S = S^k \times D^{n-k}$, $i: S \to X_\alpha$ extend ι' and let $\mathbb{D}_0 = \operatorname{ind}_H^G S \times D(\Gamma)$. For any H space A, $\operatorname{ind}_H^G A$ is the G space $Gx_H A$. If B is a G space and $f: A \to B$ is an H map, there is a unique G map $\operatorname{ind}_H^G f: \operatorname{ind}_H^G A \to B$.

The aim is to produce a G imbedding $\iota: \mathbb{D}_0 \to X$ such that $\iota|_{S^k}$ is homotopic to ι', F extends to $F' : (W', X') \to (Z, Y)$ and (B, D_γ) extend to (B', D'_γ), where $W' = W \cup_\iota \mathbb{D}$. We use the bundle isomorphisms (B, D_γ) to produce the imbedding ι. The differential of this imbedding is related to these bundle isomorphisms in such a way that the extensions (B', D'_γ) exist ((3.27), (3.28) and the proof of (3.18)). We amplify this. The isomorphism $B_\gamma : sTW_\gamma \to (F^* s\xi)^H|_{W_\gamma}$ gives rise to an isomorphism $l(B_\gamma): sTS \oplus n/\mathbf{h} \to i^* sTX_\alpha$ (3.25i). By (3.27) there is an imbedding $i_0 : S \to X_\alpha$. The differential of $\operatorname{ind}_H^N i_0$ at S is stably regularly homotopic to $l(B_\gamma)$; so we can suppose i is an imbedding. We suppose $\operatorname{ind}_H^G i$ is also an imbedding. To extend this to an imbedding of $\mathbb{D}_0$ in X it is necessary and sufficient (by the G Tubular Neighborhood Theorem) to have an H vector bundle isomorphism $c: \mathbf{\Gamma} = \nu(\operatorname{ind}_H^G S, \mathbb{D}_0)|_S \to^{\cong} \nu(\operatorname{ind}_H^G S, X)|_S$. Lemma (3.26) produces c such that $\Lambda(c) = l(D_\gamma): \Lambda(\mathbf{\Gamma}) \to i^* \lambda_\alpha X$ and $\Lambda'(sc) = l(\lambda'_\gamma B): \Lambda'(s\mathbf{\Omega}) \to i^* \lambda'_\alpha X$. (Remark $\Lambda(\mathbf{\Gamma}) = \Lambda(\mathbf{\Omega})$.) See (3.23) for the definition of $l(E)$. Here E is D_γ or $\lambda'_\gamma(B)$. There is then an equivariant imbedding $\iota: (\mathbb{D}_0, \operatorname{ind}_H^G S) \to (X, \operatorname{ind}_H^G S)$ whose normal differential (see just before (3.27)) is $\operatorname{ind}_H^G c$.

By construction the differential of ι at S stably is the sum of three terms: $l(B_\gamma)$, $l(sD_\gamma) = l(\lambda_\gamma(B))$ (because of (3.9)) and $l(\lambda'_\gamma(B))$. Lemma (3.24) maintains then that B_γ, $\lambda_\gamma(B)$ and $\lambda'_\gamma(B)$ each extends to an isomorphism of bundles over W'_γ and Lemma (3.29) identifies the sources of these bundles as sTW'_γ, $\lambda_\gamma W'$ and $\lambda'_\gamma W'$. The sum of these extensions gives an extension over $sTW'|_{W'_\gamma} = sTW'_\gamma \oplus s\lambda\gamma(W') \oplus s\lambda'_\gamma(W)$. This is easily extended over sTW' giving B' which together with the extension D'_γ over $\lambda_\gamma(W')$ completes the process as far as the bundle isomorphisms are concerned.

This process of equivariant surgery is used on all components of fixed sets of all isotropy groups in a family $\mathscr{F}$ (4.2) of subgroups of G. This means D_γ must be defined whenever W_γ is such a component; so we must deal with a collection $D = \{D_\gamma\}$ of them; moreover, if $\gamma \leqslant \gamma'$ then the process of equivariant surgery applied to W_γ affects $W_{\gamma'}$. This leads to the relation between D_γ and $D_{\gamma'}$ incorporated in (3.4) and means that D is a λ map. See also (3.9).

We now begin the description of equivariant surgery. Suppose that X is a smooth G manifold and C is a good representation of G containing g. By definition X is C *stable* if

(3.6) $$\langle T_xX, 1\rangle \leqslant \langle T_xX, \chi\rangle - \langle g, \chi\rangle$$

(real inner product of real representations)

for all $\chi \in I(G_x)$ with $\langle T_xX, \chi\rangle \neq 0$, $\chi \in C$, $\chi \neq 1$. Here T_xX is the isotropy representation of G_x on the tangent space at x. When C is $\mathscr{H}$ good, (3.6) must be modified when $G_x \in \mathscr{H}$ by requiring the inequality to hold for all $\chi \in I(G_x)$, $\chi \neq 1$. A representation A of G is C stable if it is C stable as a G manifold. It is said to be stable if it is C stable for $C = g$. Since $T_xA = A$ for all $x \in A$, stability for A becomes:

(3.7) For all $H \in \operatorname{Iso}(A)$ and all $\chi \in I(H)$, $\chi \in g$, $\chi \neq 1$, $\langle A, 1\rangle \leqslant \langle A, \chi\rangle - \langle g, \chi\rangle$ whenever $\langle A, \chi\rangle \neq 0$. (Note $\langle A, \chi\rangle$ is the multiplicity of χ in A.)

Of course (3.7) is appropriately modified when g is $\mathscr{H}$ good.

Remark. If C is $\mathscr{H}$ good, condition (3.2) in the definition of C stability must be modified when $G_x \in \mathscr{H}$ by requiring the inequality to hold for all $\chi \in I(G_x)$, $\chi \notin 1$. Then (3.7) is modified accordingly.

Stability is applied through use of the following lemma: Suppose X is C stable, $\alpha \in \Pi(X)$, $\rho(\alpha) = H$ and $x \in X_\alpha$ with $G_x = H$. Set $G_\alpha = N$. Its Lie algebra is n and the Lie algebra of H is h. Let Ω be the H module $\nu(X_\alpha, X)_x$. Since the tangent space to the orbit of x is $g/h = g/n \oplus n/h$ as an H module and since n/h is the H fixed set of g/h, it follows that $g/n \subset \Omega$. Let Γ be its complement. In the next lemma $\Lambda'(\ )$ means $\Lambda'(\ , C)$. See (1.3b).

LEMMA (3.8). *Let $k + 1 < \dim X_\alpha$ and let ν be an H vector bundle over S^k. (H acts trivially on S^k.) For any H vector bundle isomorphism $b: \Lambda'(\mathbf{\Gamma} \oplus g/\mathbf{n}) \to \Lambda'(\nu \oplus g/\mathbf{n})$, there is an H vector bundle isomorphism $b': \Lambda'(\mathbf{\Gamma}) \to \Lambda'(\nu)$ such that $s(b')$ is regularly H homotopic to b.*

Proof. Since H acts trivially on S^k, $\nu = \oplus \nu_\chi$, $\nu_\chi = \boldsymbol{\chi} \otimes_{D_\chi} \bar{\nu}_\chi$, where $\bar{\nu}_\chi = \operatorname{Hom}_H(\boldsymbol{\chi}, \nu)$; see $[\mathrm{A}_1]$ and (1.3). Note that b_χ provides an isomorphism between $(\mathbf{\Gamma} \oplus g/\mathbf{n})_\chi$ and $(\nu \oplus g/\mathbf{n})_\chi$ for $\chi \in I(H)$, $\chi \in C$, $\chi \neq 1$. This means $\dim_{\mathbb{R}} \bar{\nu}_\chi = \langle \Gamma, \chi\rangle$; so $\dim \bar{\nu}_\chi = \langle \Gamma, \chi\rangle = \langle \Omega, \chi\rangle - \langle g/\mathbf{n}, \chi\rangle \geqslant \langle T_xX, \chi\rangle - \langle g, \chi\rangle \geqslant \langle T_xX, 1\rangle = \dim X_\alpha > k$. Since $\bar{\nu}_\chi$ is a stably trivial bundle over S^k, this implies $\bar{\nu}_\chi$ is trivial. Since in addition $\nu_{\chi:x} = \Gamma_\chi$, there is an H isomorphism $b'_\chi: \mathbf{\Gamma}_\chi \to \nu_\chi$. Let $b' = \oplus_{\chi \in C} b'_\chi: \Lambda'(\mathbf{\Gamma}) \to \Lambda'(\nu)$. Since $\langle \Gamma, \chi\rangle > k + 1$ for all $\chi \in C$, $\pi_k(\operatorname{Aut}_H \Lambda'(\Gamma)) \to \pi_k(\operatorname{Aut}_H \Lambda'(\Gamma \oplus g/n))$ is surjective (1.0). By composing b' with an element of the first homotopy group if necessary, we may suppose $s(b')$ is regularly H homotopic to b.

Remark. The statement and proof of (3.8) can be modified when X satisfies the stronger stability assumption in the remark following (3.7). The modified statement asserts the existence of $b'\colon \mathbf{\Gamma} \to \nu$ with $s(b')$ regularly H homotopic to $b\colon s(\mathbf{\Gamma} \oplus g/\mathbf{n}) \to s(\nu \oplus g/\mathbf{n})$ which is a given H vector bundle isomorphism. Compare the remark after (3.26).

A *G prenormal map* $\mathscr{W} = (W, F, B, D)$ consists of (often simply called a prenormal map):

(3.9) (i) A G map $F\colon W \to Z$ of degree 1 between smooth G manifolds.

(ii) A G vector bundle ξ over Z with $\dim \xi = \dim TZ$, a G vector bundle isomorphism $B\colon sTW \to F^*s\xi$ and a λ isomorphism $D\colon \lambda(W, C) \to \lambda(F^*\xi, C)$ such that $\lambda(B, C) = s(D)$ for some good (or $\mathscr{H}$ good) representation C of G.

(iii) W and ∂W are C stable.

If we wish to emphasize the group G, we say G prenormal map. When $\mathscr{W}$ is a G prenormal map and $H \subset G$ $\mathrm{Res}_H \mathscr{W}$ is the H prenormal map obtained by restricting data to H.

Let $X = \partial W$, $Y = \partial Z$ and suppose $f = F|_X\colon X \to Y$, $b = B|_X$ and $d = D|_X$. Then by definition

$$\partial\mathscr{W} = (X, f, b, d)$$

is again a prenormal map. Now let $K \subset G$ and let $\mathscr{W} = (W, F, B, D)$. Then

(3.9′) $\mathscr{W}$ is a $K - G$ prenormal map if $\mathscr{W}$ is a K prenormal map and $\partial\mathscr{W} = \mathrm{Res}_K \mathscr{W}_0$ for some G prenormal map $\mathscr{W}_0$.

Remarks. (3.9′) asserts the natural K data of $\partial\mathscr{W}$ extend to G data; so in particular the K representation C in (3.9i) is in fact the restriction of a G representation which in applications is taken to be g.

A map between G spaces is a *pseudoequivalence* if it is a G map which is a homotopy equivalence. The prenormal map $\mathscr{W}$ is a pseudoequivalence if F is a pseudoequivalence.

Throughout this paper *G manifold* will always mean *compact smooth oriented G manifold* where oriented means: for each subgroup H of G each component of the H fixed set comes with an orientation class.

A *manifold triad* (W, W_0, W_1) is a triple of manifolds such that $\partial W = W_0 \cup W_1$ and $W_0 \cap W_1 = \partial W_0 = \partial W_1$. We make the usual assumptions about compatibility of orientation classes. A *G manifold triad* is a manifold triad such that G respects the triad structure and so does a map of triads. A *prenormal map of G manifold triads* is a prenormal map which is also a map of triads of G manifolds.

Convention. If $\mathscr{W} = (W, F, B, D)$ is a prenormal map of triads, then we have induced prenormal maps $\mathscr{W}_i$, $i = 0, 1$, with $\mathscr{W}_i = (W_i, F_i, B|_{W_i}, D|_{W_i})$, where $F_i\colon W_i \to Z_i$ is the restriction of F. We also write (X, f, b, d) for $\mathscr{W}_0$ so $f\colon X \to Y = Z_0$.

DEFINITION (3.10). A prenormal map $\bar{\mathscr{W}}$ is equivalent to zero (~ 0) if there exists a prenormal map $\mathscr{W}$ of manifold triads such that $\mathscr{W}_0 = \bar{\mathscr{W}}$ and $\mathscr{W}_1$ and $\partial\mathscr{W}_0$ are pseudoequivalences.

If X is a G manifold, $T|X|$ denotes the triad $(X \times I,\ X \times 0,\ X \times 1 \cup \partial X \times I)$. Let $f_i\colon X_i \to Y$ be two G maps for $i = 0, 1$. If there is a G manifold pair (W, P) and a G map $F\colon (W, P) \to (Y \times I, \partial Y \times I)$ such that $\partial W = X_0 \cup P \cup X_1$, $\partial X_0 \cup \partial X_1 = \partial P = P \cap (X_0 \cup X_1)$, $F|_{X_i} = f_i$, where $F|_{X_i}\colon X_i \to Y \times i$, we say (W, P, F) is a G *cobordism* between (X_0, f_0) and (X_1, f_1). If $P = \partial X_0 \times I = \partial X_1 \times I$ and $F(x, t) = (f_0(x), t)$ for $x \in \partial X_0$, we say the cobordism is relative boundary (rel ∂). If $\mathscr{H}$ is a subset of $\mathscr{S}(G)$, $W^{\mathscr{H}} = X_0^{\mathscr{H}} \times I = X_1^{\mathscr{H}} \times I$ and $(p_1 F(x, 0), t) = F(x, t)$ for $x \in X_0^{\mathscr{H}}$ and p_1 is projection on the first factor, we say that the *cobordism is* rel $\mathscr{H}$. ($X^{\mathscr{H}}$ is the union of X^H for $H \in \mathscr{H}$.)

In analogy with the definition (3.9′) of a $K - G$ prenormal map we can define a $K - G$ prenormal triad $\mathscr{W} = (W, F, B, C)$. This is a $K - G$ prenormal map which also is a triad—similarly for the definition of $K - G$ prenormal cobordism. We shall sometimes abbreviate the phrase G prenormal or $K - G$ prenormal by prenormal.

(3.11) Hypothesis H: Let X be a smooth G manifold and $H \subset G$. Then X satisfies hypothesis H if $\dim X_\alpha = \dim X^H$ is independent of α for all α with $\rho(\alpha) = H$ and if each class in $\pi_k(X^H)$ for $k \leqslant \frac{1}{2}\dim X^H/N(H)$ can be homotoped into $X^{H^*} = \{x \mid G_x = H\}$.

Note that (3.11) implies that each component of X^H has a point whose isotropy group is H. For G finite (3.11) is guaranteed by supposing $2\dim X^L < \dim X^H$ whenever $L > H$. For non-finite groups the criterion is more complicated. For $G = SO_3$ or S^3, we give a simple criterion in (5.2).

(3.12) For any smooth G manifold X and $H \subset G$, $\hat{H} = \hat{H}_X$ denoted the unique minimal isotropy group of X containing H. This need not always exist. It does when X^H is connected. Note $X^{\hat{H}} = X^H$.

Suppose $\mathscr{W}_0 = (X, f, b, d)$ is a G prenormal map where $f\colon X \to Y$. Let X_α be a component of X^H and Y_β the component of Y^H into which f maps X_α so

$$f|_{X_\alpha} = f_\alpha : X_\alpha \to Y_\beta$$

and $\alpha \in \Pi X$, $\beta \in \Pi Y$. Let $\mu \in \pi_{k+1}(f_\alpha)$ (by definition $\pi_{k+1}(f_\alpha) =$

$\pi_{k+1}(Z_{f_\alpha}, X_\alpha)$, where Z_{f_α} is the mapping cylinder of f_α) be represented by the diagram

(3.13)
$$\mu:\ \begin{array}{ccc} S^k & \xrightarrow{\iota'} & X_\alpha \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle f_\alpha} \\ D^{k+1} & \xrightarrow{\kappa'} & Y_\beta \end{array}$$

$\partial\mu \in \pi_k(X_\alpha)$ is represented by ι'.

We *assume* the existence of a point $x \in \iota' S$ with $G_x = H$ and define the H representations Ω, Γ:

(3.14) (i) $\Omega = \nu(X_\alpha, X)_x$,

(ii) $\Omega = \Gamma \oplus g/n$. See discussion before (3.8).

Here is the Lie algebra of $N = G_\alpha$. Note that the tangent space to the orbit of $x \in X_\alpha$ is $g/h = n/h \oplus g/n$. For fixed H, k, $n - 1 = \dim X_\alpha / G_\alpha$, we define $(\mathbb{D}, \mathbb{D}_0)$ by

(3.15)
$$\mathbb{D} = \operatorname{ind}_H^G D \times D(\Gamma), \qquad D = D^{k+1} \times D^{n-k-1},$$
$$\mathbb{D}_0 = \operatorname{ind}_H^G S \times D(\Gamma), \qquad S = S^k \times D^{n-k-1}.$$

An extension $\mathscr{U} = \mathscr{U}(\mathscr{W}_0)$ of $\mu \in \pi_{k+1}(f_\alpha)$ is a commutative diagram of G which maps

(3.16)
$$\mathscr{U}:\ \begin{array}{ccc} \mathbb{D}_0 & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{D} & \xrightarrow{\kappa} & Y \end{array}$$

such that the restriction to (D^{k+1}, S^k) is μ.

Suppose $\mathscr{W}$ and $\mathscr{W}'$ are $K - G$ prenormal maps of triads, $\partial\mathscr{W} = \mathscr{W}_0 \cup \mathscr{W}_1$ and $\partial\mathscr{W}' = \mathscr{W}'_0 \cup \mathscr{W}_1$ with $\mathscr{W}_0 = (X, f, b, d)$ as above.

Definition (3.17). We say $\mathscr{W}'$ arises from $\mathscr{W}$ by surgery on $\mu \in \pi_{k+1}(f_\alpha)$ if there is an extension $\mathscr{U}$ of μ such that ι is an imbedding $W' = W \cup_\iota \mathbb{D}$, $F'|_{\mathbb{D}} = \kappa$ and the data of $\mathscr{W}'$ extend those of $\mathscr{W}$. (Recall $\mathscr{W} = (W, F, B, D)$ and $\mathscr{W}' = (W', F', B', D')$.)

If there exists a $\mathscr{W}'$ which arises from $\mathscr{W}$ by surgery on μ, we say *surgery on μ is possible*. The process of constructing $\mathscr{W}'$ from $\mathscr{W}$ is called surgery on $(\mathscr{W}, \mu)$, $\mu \in \pi_*(f_\alpha)$, or briefly surgery on μ. Let $\partial\mathscr{W}' = \mathscr{W}'_0 \cup \mathscr{W}_1$. There is an obvious $K - G$ prenormal cobordism between $\mathscr{W}_0$ and $\mathscr{W}'_0$ called the *trace* of the surgery.

We have just described surgery on $(\mathscr{W}, \mu)$ where $\mathscr{W}$ is a triad with $\partial\mathscr{W} = \mathscr{W}_0 \cup \mathscr{W}_1$, $\mathscr{W}_0 = (X, f, b, d)$ and $\mu \in \pi_*(f_\alpha)$. As a special case we can begin with any prenormal map $\mathscr{W} = (W, F, B, D)$ not necessarily a triad and form the triad

$$T[\mathscr{W}] = (T[W], F \times I, B \times I, D \times I),$$

where $T[W]$ is defined after (3.10). Then $T[\mathscr{W}]_0 = \mathscr{W}$; so for any $\mu \in \pi_*(F_\gamma)$ we have defined surgery on $(T[\mathscr{W}], \mu)$. This gives some new triad $\bar{\mathscr{W}}$. Set $\bar{\mathscr{W}}_0 = \mathscr{W}'$. Then $\mathscr{W} \mapsto \mathscr{W}'$ is called surgery on $(\mathscr{W}, \mu)$, $\mu \in \pi_*(F_\gamma)$. In particular if $\mathscr{W}$ itself is a triad, we can consider surgery on $(\mathscr{W}, \mu)$ with $\mu \in \pi_*(f_\alpha)$ as previously described or $\mu \in \pi_*(F_\gamma)$ as just described. We must emphasize that when $\mathscr{W}$ is a $K-G$ triad, surgery on $\mu \in \pi_*(f_\alpha)$ is G surgery in the sense that $W' = W \cup \mathbb{D}$, where $\mathbb{D} = \operatorname{ind}_H^G D \times D(\Gamma)$. However, surgery on $\mu \in \pi_*(F_\gamma)$ is K surgery in the sense that W' is defined by $\partial(W \times I \cup \mathbb{D}') = W' \cup W$, where $\mathbb{D}' = \operatorname{ind}_H^K D \times D(\Gamma)$. In particular $H \subset K$.

Let $i\colon S \to X_\alpha$ extend ι' (3.13). We shall show that the data of a prenormal map $\mathscr{W}$ with $\partial\mathscr{W} = \mathscr{W}_0 \cup \mathscr{W}_1$ together with μ determine a particular regular homotopy class of immersions of S in X_α ((3.27) and (3.28)) within the homotopy class of i when $k \leqslant n-3$. Specifically there is an immersion i_0 of S in X_α homotopic to i. The differential of $\operatorname{ind}_H^N i_0$ at S (see before (3.27)) viewed as an isomorphism of $TS \oplus n/\mathbf{h}$ and $i_0^* TX_\alpha$ is stably regularly homotopic to $l(B_\gamma)\colon sTS \oplus n/\mathbf{h} \to i_0^* sTX_\alpha$. (See (3.23) and (3.25i).) The following are the two main geometric steps in equivariant surgery:

THEOREM (3.18). *Suppose $\mathscr{W}$ is a $K-G$ prenormal map of triads and $\partial\mathscr{W} = \mathscr{W}_0 \cup \mathscr{W}_1$ with $\mathscr{W}_0 = (X, f, b, d)$. Let H be a subgroup of G, $k \leqslant n-3$ and $\mu \in \pi_{k+1}(f_\alpha)$ with $\rho(\alpha) = H$. Then surgery on μ is possible if there is a representative i_0 of the chosen regular homotopy class of i such that* $\operatorname{ind}_N^H \iota_0'$ *($\iota_0' = i_0|_S k$) is an embedding.*

We call a class $\mu \in \pi_{k+1}(f_\alpha)$ represented by (3.13) *trivial* if $\iota' S^k$ is contained in a disk in X_α. For such a class $\operatorname{ind}_H^N \iota'$ can always be assumed to be an imbedding. Alternatively by general position $\operatorname{ind}_H^N \iota'$ can always be assumed to be an imbedding if X satisfies hypothesis H (3.11) and $k < \frac{1}{2} \dim X_\alpha / G_\alpha$.

COROLLARY (3.19). *Let $\alpha \in \Pi(X)$ with $\rho(\alpha) = H$ and $\mu \in \pi_{k+1}(f_\alpha)$. If X is stable, satisfies hypothesis H (3.11), $\dim X_\alpha / G_\alpha \geqslant 6$ and if $k < \frac{1}{2} \dim X_\alpha / G_\alpha$ or μ is trivial, surgery on μ is possible.*

Remark. The effect on X_α of surgery on $\mu \in \pi_{k+1}(f_\alpha)$ is to kill $\partial\mu \in \pi_k(X_\alpha)$. Similarly the effect on W_γ of surgery on $\mu \in \pi_{k+1}(F_\gamma)$ is to kill $\partial\mu \in \pi_k(W_\gamma)$. The submanifolds $X_{\alpha'}$ for $\alpha' < \alpha$ are unaltered.

We begin the proofs of (3.18) and (3.19) with some preliminaries: Let N be a compact Lie group with closed subgroup H. Let $F: A \to B$ be an N equivariant map and $A_0 \subset A^H$, $B_0 \subset B^H$ with $f = F|_{A_0}: A_0 \to B_0$. Let $\mu \in \pi_{k+1}(f)$, n be an integer larger than k; $S = S^k \times D^{n-k}$, $D = D^{k+1} \times D^{n-k}$ and

$$\text{(3.20)} \qquad \begin{array}{ccc} S & \xrightarrow{i} & A_0 \\ {\scriptstyle \mu'}\big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle f} \\ D & \xrightarrow{h} & B_0 \end{array}$$

be a diagram which gives μ by restriction to $(D^{k+1}, S^k) \subset (D, S)$. Define

$$A' = A \bigcup_{\operatorname{ind} i} \operatorname{ind} D, \qquad \operatorname{ind} = \operatorname{ind}_H^N.$$

Here H acts trivially on D. Let $F'_\mu = F': A' \to B'$ be the unique N equivariant map extending F with $F'|_D = h$.

Let θ and ξ be N vector bundles over A respectively B and $E: \theta \to F^*\xi$ an N vector bundle isomorphism. Denote by V the H module θ_x for any $x \in iS$. Set

$$\text{(3.21)} \qquad \omega = \omega(\theta) = D \times V.$$

It is an H vector bundle over D. Choose any H vector bundle isomorphism

$$L(E): \omega \to h^*\xi_0, \qquad \xi_0 = \xi|_{B_0}.$$

Let E_0 be the H vector bundle map which covers j defined by the composition

$$\text{(3.22)} \qquad i^*\theta_0 \xrightarrow{i^*E} i^*f^*\xi_0 = j^*h^*\xi_0 \to h^*\xi_0,$$

where $\theta_0 = \theta|_{A_0}$. Let $l = l(E): \omega|_S \to i^*\theta_0$ be the unique H vector bundle isomorphsm which gives this commutative diagram

$$\text{(3.23)} \qquad \begin{array}{ccc} \omega|_S & \xrightarrow{l(E)} & i^*\theta_0 \\ \big\downarrow & & \big\downarrow{\scriptstyle E_0} \\ \omega & \xrightarrow{L(E)} & h^*\xi_0 . \end{array}$$

Define an N vector bundle θ' over A' by

$$\theta' = \theta \bigcup_{\text{ind } l(E)} \text{ind } \omega, \qquad \text{ind} = \text{ind}_H^N.$$

Extend E to $E'\colon \theta' \to F'^*\xi$ with $E' = L(E)$ on ω. This uniquely defines E' as we insist on N equivariance. Set

$$\Gamma(\theta, \mu, E) = \theta'$$

so $E'\colon \Gamma(\theta, \mu, E) \xrightarrow{\cong} F'^*\xi$. We record this construction as a lemma.

LEMMA (3.24). *Let* $F\colon A \to B$, $E\colon \theta \to F^*\xi$ *and* $\mu \in \pi_{k+1}(f)$, $f\colon A_0 \to B_0$, *be as above. Then* $F'_\mu = F'\colon A' = A \cup \text{ind } D \to B$ *is an* N *map extending* F *and* $E'\colon \Gamma(\theta, \mu, E) \to F'^*\xi$ *is an* N *vector bundle isomorphism extending* E.

Lemma (3.24) is used several times in the proof of Theorem (3.18). In the applications μ is given in (3.18) and (A, A_0) is either (W, X_α) or (W_γ, X_α), (B, B_0) is either (Z, Y_β) or (Z_δ, Y_β) and N is either G or G_α. Here β is the component of Y^H with $f_\alpha X_\alpha \subset Y_\beta$ and γ resp. δ is the component W^H resp. Z^H with $X_\alpha \subset W_\gamma$ resp. $Y_\beta \subset Z_\delta$. For each choice of bundle θ over W or W_γ (constructed from TW) and bundle isomorphism E, (3.23) provides an equivariant bundle isomorphism $l(E)\colon \omega(\theta)|_S \to i^*\theta|_{X_\alpha}$. The next lemma identifies the H bundles $\omega(\theta)|_S$. Recall from Theorem (3.18) that $\mathscr{W} = (W, F, B, D)$.

LEMMA (3.25). *The* H *vector bundles* $\omega(\theta)|_S$ (3.21) *for* θ *respectively* sTW_γ, $\lambda_\gamma W$, $s\lambda'_\gamma W$ *and* sTW *are*: $sTS \oplus n/\mathbf{h}$, $\Lambda(\mathbf{\Omega})$, $\Lambda'(s\mathbf{\Omega})$ *and* $sT\mathbb{D}_0|_S$; *so*

(i) $l(B_\gamma)\colon sTS \oplus n/\mathbf{h} \to i^*sTX_\alpha$,

(ii) $l(D_\gamma)\colon \Lambda(\mathbf{\Omega}) \to i^*\lambda_\alpha X$,

(iii) $l(\lambda'_\gamma(B))\colon \Lambda'(s\mathbf{\Omega}) \to i^*s\lambda'_\alpha X$,

(iv) $l(B)\, sT\mathbb{D}_0|_S \to i^*sTX|_{X_\alpha}$.

Proof. Since $\omega(\theta)$ is completely determined by the H module θ_x for any $x \in iS$, it suffices to check that the H vector bundles over S listed in (i)–(iv) all have the form $S \times V$, where $V = \theta_x$. (i) For $\theta = sTW_\gamma$, H acts trivially on θ_x and on the fibers of $sTS \oplus n/\mathbf{h}$. (ii) For $\theta = \lambda_\gamma W$, $\theta|_{X_\alpha} = \lambda_\alpha(X) = \Lambda(\nu(X_\alpha, X))$; so $\theta_x = \Lambda(\Omega)$ (3.14i); similarly for (iii): (iv) For $\theta = sTW$, $i^*\theta = sTX$; so $\theta_x = sT_xX_\alpha \oplus \nu(X_\alpha, X)_x = sT_xS \oplus n/h \oplus \Omega$ (3.14). Since $sT\mathbb{D}_0|_S = sTS \oplus n/\mathbf{h} \oplus \mathbf{\Omega}$ by (3.14i) and (3.15), $\omega(\theta)|_S = sT\mathbb{D}_0|_S$.

Remark. From (3.14ii) and (3.15), we see that $sT\mathbb{D}_0|_D$ is $(sTD \oplus n/\mathbf{h}) \oplus s\mathbf{\Omega}$. Since $\Omega = \Lambda(\Omega) \oplus \Lambda'(\Omega)$, $sT\mathbb{D}|_D$ splits functorially as a sum of three terms. Because of the assumption $\lambda(B)=s(D)$, we can and do choose $L(B)=(L(B_\gamma)\oplus sL(D_\gamma)\oplus L(\lambda'_\gamma B))$. This means $l(B)$ splits similarly.

LEMMA (3.26). *Suppose $i\colon S\to X_\alpha$ extends to a G imbedding of* $\operatorname{ind}_H^G S$ *in X. Then there is an H vector bundle isomorphism $c\colon \mathbf{\Gamma}\to \nu(\operatorname{ind}_H^G S, X)|_S=\nu'$ such that $\Lambda(c)=l(D_\gamma)\colon \Lambda(\mathbf{\Gamma})\to i^*\lambda_\alpha X$ and $\Lambda'(sc)=l(\lambda'_\gamma(B))$ up to regular H homotopy.*

Proof. First observe that $\Lambda(g/n)=0$ because as an H module $g/n\subset g$. See (1.1) and recall $g\subset C$. This means $\Lambda(\mathbf{\Gamma})=\Lambda(g/\mathbf{n}\oplus\mathbf{\Gamma})$ and $\Lambda(g/\mathbf{n}\oplus\nu')=\Lambda(\nu')$. Note $i^*\nu(X_\alpha, X)=\nu(\operatorname{ind}_H^N S, X)|_S=\nu(\operatorname{ind}_H^N S, \operatorname{ind}_H^G S)|_S\oplus\nu'=g/\mathbf{n}\oplus\nu'$; so $l(D_\gamma)\colon\Lambda(\mathbf{\Gamma})=\Lambda(\mathbf{\Omega})\to i^*\lambda_\alpha X=\Lambda(g/\mathbf{n}\oplus\nu')=\Lambda(\nu')$. Note also that $l(\lambda'_\gamma(B))\colon \Lambda'(g/\mathbf{n}\oplus s\mathbf{\Gamma})=\Lambda'(s\mathbf{\Omega})\to si^*\lambda'_\alpha X=\Lambda'(g/\mathbf{n}\oplus s\nu')$ (3.25iii). Since X is stable and $k\leqslant\frac{1}{2}\dim X_\alpha$, (1.0) and (1.1) and (3.8) imply there is an H vector bundle isomorphism $b_2\colon\Lambda'(\Gamma)\to\Lambda'(\nu')$ such that $s(b_2)$ is regularly H homotopic to $l(\lambda'_\gamma(B))$. Let $c=l(D_\gamma)\oplus b_2$: $\mathbf{\Gamma}\to\mathbf{v}'$.

Remark. The proof completed applies to the case C is good or is $\mathscr{H}$ good and $H\notin\mathscr{H}$. See (3.9ii). When $H\in\mathscr{H}$ the statement and proof of (3.26) are slightly altered. The statement asserts the existence of c with $sc=l(\pi_\gamma(B))$. The production of c uses the stronger stability assumption in the remark following (3.8).

We use the H vector bundle isomorphisms provided in (3.25) to produce an extension $\mathscr{U}(\mathscr{W}_0)$ of $\mu\in\pi_{k+1}(f_\alpha)$ with ι an imbedding. See (3.16). First we need some definitions. Let S' and X' be smooth manifolds of the same dimension and S resp. X a submanifold of S' resp. X' with $\dim S=\dim X$ and let $\iota\colon(S',S)\to(X',X)$ be an immersion of pairs. The composition $\nu(S,S')\to TS'|_S\xrightarrow{d\iota}TX'|_X\to\nu(X,X')$ induces an isomorphism between $\nu(S,S')$ and $(\iota|_S)^*\nu(X,X')$ called the *normal differential of ι at S*. The differential $d\iota$ of ι induces an isomorphism $d\iota\colon TS'|_S\to\iota^*|_S TX'$ also called the *differential of ι at S*.

LEMMA (3.27). *Suppose G acts freely on the smooth manifold X of dimension $l+\dim G$. Let $S=S^k\times D^{l-k}$ and $i\colon S\to X$ be a map. If $k<l-2$, any vector bundle isomorphism $b\colon sTS\oplus\mathbf{g}\to i^*sTX$ determines a G immersion of* $\operatorname{ind}_1^G S$ *in X G homotopic to* $\operatorname{ind}_1^G i$ *whose differential at S is stably regularly homotopic to b. If $k<\frac{1}{2}l$ the immersion may be taken to be an imbedding.*

Proof. If $p\colon X\to X/G$ is the orbit map, $TX=p^*T(X/G)\oplus\mathbf{g}$; so $b\colon sTS\oplus\mathbf{g}\to i^*p^*sT(X/G)\oplus\mathbf{g}$. By [H] (compare [$W_1$, D–$P_1$]), b determines an immersion of S in X/G which is homotopic to $p\circ i$ whose differential is stably regularly homotopic to b. This lifts to an immersion of S in X which uniquely extends to a G map of $\operatorname{ind}_1^G S$ in X. Its differential restricted to S is stably regularly G homotopic to $b \oplus 1_{\mathbf{g}}: (sTX \oplus \mathbf{g}) \oplus \mathbf{g} \to (i^*p^*sT(X/G) + \mathbf{g}) \oplus \mathbf{g}$.

LEMMA (3.28). *Let $\mathscr{W} = (W, F, B, D)$ be a $K - G$ prenormal map of triads, $\mathscr{W}_0 = (X, f, b, d)$ and $\mu \in \pi_{k+1}(f_\alpha)$. Then there is an extension $\mathscr{U}$* of μ (3.16) *such that ι is an immersion (imbedding if* $\operatorname{ind}_H^N \iota'$ *is an imbedding in particular if $k < \frac{1}{2} \dim X_\alpha / G_\alpha$). The differential of ι^H at S is stably regularly homotopic to $l(B_\gamma): sTS \oplus n/\mathbf{h} \to i^*sTX_\alpha$* (3.25i) and the normal differential of ι: $(\mathbb{D}_0, \operatorname{ind}_H^G S) \to (X, \iota \operatorname{ind}_H^G S)$ *at* $\operatorname{ind}_H^G S$ *is* $\operatorname{ind}_H^G c$, $c: \mathbf{\Gamma} \to \nu(\iota \operatorname{ind}_H^G S, X)|_S$, *where $\Lambda(c) = l(D_\gamma)$, $\Lambda'(sc) = l(\lambda'_\gamma(B))$ and $\iota^H|_S = i$.*

Proof. Let μ' be the diagram of (3.20) which gives $\mu \in \pi_{k+1}(f_\alpha)$ by restriction to (D^{k+1}, S^k). We may suppose $iS \subset X_\alpha^* = \{x \in X_\alpha \mid G_x = H\}$ because of (3.11). Since N/H acts freely on X_α^*, Lemma (3.27) applied to i and the group N/H gives an N immersion of $\operatorname{ind}_H^N S$ into X_α^* whose differential at S is stably regularly homotopic to $l(B_\gamma): sTS \oplus n/\mathbf{h} \to i^*sTX_\alpha$ (3.25i). Note that $\operatorname{ind}_1^{N/H} S$ is the same as $\operatorname{ind}_H^N S$ as an N manifold. (If $k < \frac{1}{2} \dim X_\alpha / G_\alpha$ the immersion may be taken to be an imbedding.) Thus we may suppose $\operatorname{ind}_H^N i$ is an immersion. Then $\operatorname{ind}_H^G i = \operatorname{ind}_N^G \circ \operatorname{ind}_H^G i$ is an immersion. The G Tubular Neighborhood Theorem [Br] provides a G immersion ι of $\mathbb{D}_0 = \operatorname{ind}_H^G S \times D(\Gamma)$ into X extending $\operatorname{ind}_H^G i$ whose normal differential at S is $c: \mathbf{\Gamma} \to i^*\nu(\iota \operatorname{ind}_H^G S, X)$. Note $\nu(\operatorname{ind}_H^G S, \mathbb{D}_0)|_S = \mathbf{\Gamma}$; so $\nu(\operatorname{ind}_H^G S, \mathbb{D}_0) = \operatorname{ind}_H^G \mathbf{\Gamma}$.

Since $(\mathbb{D}, \mathbb{D}_0)$ retracts equivariantly to $\operatorname{ind}_H^G(D, S)$, there is an extension of $h: D \to Z$ to $\kappa: \mathbb{D} \to Z$ giving a diagram $\mathscr{U}$ extending μ as in (3.16).

Lemma (3.28) is half way toward the proof of Theorem (3.18). What remains is to extend $B: sTW \to F^*\xi$ to $B': sTW' \to (F'^*\xi)$. Here $W' = W \cup_\iota \mathbb{D}$ and F' extends F with $F'|_{\mathbb{D}} = \kappa$. As $W \cup \operatorname{ind}_H^G D =^{\text{def}} O$ is a G deformation retract of W', it suffices to define the extensions B' and D' restricted to $sTW'|_O$ and $\lambda(W')|_O$. Because $k < n - 1$, ΠW and ΠO are the same set; moreover, for $\gamma' \in \Pi W$, $W_{\gamma'} = O_{\gamma'}$ unless $\gamma' \geqslant g\gamma$ for some $g \in G$ and then $y \in O_{\gamma'} - W_{\gamma'}$ is of the form $y = gx$ for $x \in D$. Note that $W'_\gamma = O_\gamma$. Suppose D'_γ has been defined extending D_γ. For $g \in G$ set $D'_{g\gamma} = gD'_\gamma g^{-1}$. For $\gamma' \geqslant g\gamma$ and $y = gx$, $x \in D$, $D'_{\gamma'}$ on the fiber over y is $D'_{g\gamma', \gamma'}$ over y. See (3.4). Compare [D–P$_1$]. Of course $D'_{\gamma'} = D_{\gamma'}$ on fibers over points of $W_{\gamma'}$. We emphasize this in the following remark.

Remark. The extension $D': \lambda(W') \to \lambda(F'^*\xi)$ exists if the extension $D'_\gamma: \lambda_\gamma(W') \to \lambda_\gamma(F'^*\xi)$ exists and the extension $B': sT'W' \to sF'^*\xi$ exists if the extension to $sTW'|_O$ exists.

Let $\Gamma_1 = \Gamma(sTW_\gamma, \mu, \beta_\gamma)$, $\Gamma_2 = \Gamma(\lambda_\gamma(W), \mu, D_\gamma)$ and $\Gamma_3 = \Gamma(sTW, \mu, B)$.

LEMMA (3.29). $\Gamma_1 = sTW'_\gamma$, $\Gamma_2 = \lambda_\gamma(W')$ and $\Gamma_3 = sTW'|_O$.

Proof. By definition, $\Gamma_1 = sTW_\gamma \cup \mathrm{ind}_H^N \omega$, where $\mathrm{ind}_H^N \omega$ is attached to sTW_γ along $\mathrm{ind}_H^N \omega|_S$ using $\mathrm{ind}_H^N l(B_\gamma)$. Since $l(B_\gamma)$ is stably regularly homotopic to the differential of the imbedding of S in X_α (by construction—Lemma (3.28)) used to form $W'_\gamma = W_\gamma \cup \mathrm{ind}_H^N D$, $\Gamma_1 = sTW'_\gamma$.

The bundle Γ_2 is $\lambda_\gamma(W) \cup \mathrm{ind}_H^N \omega$, where $\mathrm{ind}_H^N \omega$ is attached via $l(D_\gamma)$.The bundle $\lambda_\gamma(W')$ is $\Lambda(\nu(W'_\gamma, W'))$ by definition and $\nu(W'_\gamma, W') = \nu(W_\gamma, W) \cup \mathrm{ind}_H^N \omega$. Here $\mathrm{ind}_H^N \omega$ is attached via $\mathrm{ind}_H^N b$, where $b\colon \omega|_S \to \nu(X_\alpha, X)|_S$ is the restriction to S of the normal differential of the imbedding $\iota\colon (\mathbb{D}_0, \mathrm{ind}_H^N S) \to (X, X_\alpha)$. By construction, the normal differential of $\iota\colon (\mathbb{D}_0, \mathrm{ind}_H^G S) \to (X, \iota\, \mathrm{ind}_H^G S)$ at $\mathrm{ind}_H^G S$ is $\mathrm{ind}_H^G c$. Thus $\Lambda(b) = \Lambda(c) = l(D_\gamma)$ ((Lemma (3.26)); so $\Gamma_2 = \lambda_\gamma(W')$. (Note $\nu(\mathrm{ind}_H^N S, X)|_S = g/\mathbf{n} \oplus \nu(\mathrm{ind}_H^G S, X)|_S$; so $\Lambda(\nu(\mathrm{ind}_H^N S, H)|_S) = \Lambda(\nu(\mathrm{ind}_H^G S, X)|_S)$.)

The bundle Γ_3 is $sTW \cup \mathrm{ind}_H^G \omega$, where $\mathrm{ind}_H^G \omega$ is attached along $\mathrm{ind}_H^G \omega|_S$ via $\mathrm{ind}_H^G l(B)$. By the remark following (3.25) $l(B) = l(B_\gamma) \oplus sl(D_\gamma) \oplus sl(\lambda'_\gamma(B)) = l(B_\gamma) \oplus s(c)$ (3.26). But this is stably regularly homotopic to the differential of ι at S $d\iota$ by Lemma (3.28). Since $sTW'|_O$ is obtained from sTW by attaching $\mathrm{ind}_H^G \omega$ using the stabilization of $\mathrm{ind}_H^G d\iota$, Γ_3 is $sTW'|_O$ as asserted.

Proof of Theorem (3.18). By Lemma (3.28) there is an extension $\mathscr{U}$ of μ such that ι (3.16) is an imbedding. The differential of ι^H has the properties specified in (3.28). Form $W' = W \cup_\iota \mathbb{D}$. $O = W \cup_\iota \mathrm{ind}_H^G D$ and extend F to $F'\colon W' \to Z$ with $F'|_{\mathbb{D}} = \kappa$. By Lemmas (3.29) and (3.24) and the remark prior to (3.29), the extensions $B'\colon sTW' \to F'^*\xi$ and $D'\colon \lambda(W') \to \lambda(F'^*\xi)$ of B and D exist. They satisfy $\lambda(B') = s(D')$ because $\lambda(B) = s(D)$ and $\lambda_\gamma(B') = s(D'_\gamma)$. The latter is a consequence of the construction and the remark preceding Lemma (3.26). Thus $\mathscr{W}'' = (W', F', B', C')$ arises from $\mathscr{W}'$ by surgery on μ. This completes the proof of (3.18).

Remark (3.30). We modify the data of (3.13) when Y^H is connected. We then replace X_α and Y_β in (3.13) by X^H and Y^H. Then $\mu \in \pi_{k+1}(f^H)$; so (3.18) and (3.19) are correspondingly altered by changing f_α to f^H and G_α to $N(H)$.

Finally we consolidate some of the results of this section incorporating (3.30).

THEOREM (3.31). *Let* $\mathscr{W}' = W, F, B, D)$, $F\colon W \to Z$ *be a* $K-G$ *prenormal map of triads with* $\partial\mathscr{W}' = \mathscr{W}'_0 \cup \mathscr{W}'_1$, $\mathscr{W}'_0 = (X, f, b, d)$, $f\colon X \to Y$. *Suppose* $H \subset K$, Z^H *and* Y^H *are connected and there is a point* $x \in X$ *with* $G_x = H$. *Let* $\mu \in \pi_{k+1}(f^H)$ *resp.* $\mu \in \pi_{k+1}(F^H)$ *and* $i\colon S \to X^H$ *resp.* $i\colon S \to$ *interior* W^H *be a map such that* $\iota' = i|_{S^k}$ *represents* $\partial\mu$. *Then surgery on* μ *is possible if there is a representation* i_0 *of the regular homotopy class* i *such that* $\mathrm{ind}_H^N \iota'_0$ *is an imbedding for* $N = N(H)$ *resp.* $N = K \cap N(H)$.

Proof. This is a restatement of (3.18) when $\mu \in \pi_{k+1}(f^H)$. When $\mu \in \pi_{k+1}(F^H)$, it is a consequence of (3.18) applied to the triad $T(\mathscr{W})$ and the definition of surgery on μ.

4. Homological Aspects of Equivariant Surgery

In this section we treat the basic homolgical steps in equivariant surgery. The broad format is similar to $[W_1]$ which treats the case of free finite group actions. Input from transformation groups appears here throughout but in particular in Theorem (4.6). The following notation and assumptions hold:

(4.1) $\mathscr{W} = (W, F, B, D)$ is a $K-G$ prenormal map, $F: W \to Z$, $\partial\mathscr{W} = (X, f, b, d)$, $f: X \to Y$, Q is subgroup of K; X and W satisfy hypothesis Q (3.11), $\dim X^Q \geqslant 6$, $\dim Z^H$ and $\dim W^H$ are equal for all $H \subset K$, Z^Q and Y^Q are 1 connected.

(4.2) $\bar{Q} = N(Q)/Q$, $\bar{Q}_K = N_K(Q)/Q$, $N_K(Q) = K \cap N(Q)$, R is the integers Z or $Z_{(p)}$, p prime, G_0 is the connected component of G, $G^0 = G/G_0$, $\Lambda_{Q,K} = R(\bar{Q}_K^0)$, $\Lambda_Q = R(\bar{Q}^0)$, $\Gamma_{Q,K} = H_*((\bar{Q}_K)_0, R)$, $\Gamma_Q = H_*(\bar{Q}_0, R)$, $n = \dim W^Q/\bar{Q}_K$, $K_*(W^Q, R) = \operatorname{Ker}(H_*(W^Q, R) \to H_*(Z^Q, R))$. Sometimes Λ abbreviates $\Lambda_{Q,K}$ or Λ_Q and Γ abbreviates $\Gamma_{Q,K}$ or Γ_Q. Set

$$\mathscr{P} = \{P \mid |P^0| = p^n \text{ for some prime } p,\ n \geqslant 0 \text{ and } P_0 \text{ is a torus.}\}$$

We mention that often we have fixed a particular group G in the discussion. Then $P \in \mathscr{P}$ means $P \subset G$ and $P \in \mathscr{P}$. We also emphasize that (4.1) is to hold in this section except for a modification in (4.15). In particular Q is an isotropy group of X. The properties of the kernel groups $K_*(W, R)$ are reviewed in $[W_1$, Sect. 2$]$. In particular $K^*(W^Q, R)$ is the cokernel of $H^*(Z^Q, R) \to H^*(W^Q, R)$, $K_i(W^Q, R) \cong K^{l-i}(W^Q, X^Q, R)$, $l = \dim W^Q$, and there is a long exact sequence of $K_*(K^*)$ groups for the pair (W^Q, X^Q). This requires the assumption degree F^Q is a unit in R. Note $K_*(W^Q, R)$ is a module over $\Gamma_{Q,K}$ and $\Lambda_{Q,K}$.

For the remainder of this section all homology, cohomology and $K_*(K^*)$ groups will have coefficients in R and these will be suppressed from notation except for special emphasis. The integers are denoted by Z.

Definition (4.3). $F^Q \sim 0$ means:

(a) If $n = \dim W^Q/\bar{Q}_K = 2m$, $K_k(W^Q, Z) = 0$ for $k < m$ and $K_k(X^Q, Z) = 0$ for $k < m-1$. If $n = 2m+1$, $K_k(W^Q, Z) = 0$ for $k < m$, $K_k(X^Q, Z) = 0$ for $k < m$ and $K_m(W^Q, X^Q, Z) = 0$.

(b) W^Q, Z^Q, X^Q, Y^Q are 1 connected.

DEFINITION (4.4). $F^Q \equiv O(R)$ means $F^Q \sim 0$ and $K_*(W^Q, R) = 0$.

DEFINITION (4.5). $F^Q \approx O(R)$ means:

(i) $F^Q \sim 0$,

(ii) degree F^Q is a unit of R and

(iii) $F^P \equiv O(R_{(p)})$ whenever $Q \lhd P$, $P \neq Q$, $P/Q \in \mathscr{P}$, and $|(P/Q)^0| = p^l$, p prime in R, $l \geqslant 0$.

We remark that (4.5) implies that $0 \to K_*(W^Q) \to H_*(W^Q) \to H_*(Z^Q) \to 0$ is exact; so $\tilde{H}_{m+1}(M_{FQ}) = K_m(W^Q)$ if M_{FQ} is the mapping cone of F^Q. We also note that if degree $F = \pm 1$ and $P \in \mathscr{P}$, degree F^P is a unit in $Z_{(p)}$, if $|P^0| = p^l$, p prime, and is a unit in Z if $l = 0$. Compare [D–P$_1$, Theorem 1.26].

THEOREM (4.6). *Suppose $F^Q \approx O(R)$. Then the following are projective $\Lambda_{Q,K}$ modules: $K_m(W^Q)$, $K_{n-m}(W^Q, X^Q)$ (and $K_m(X^Q)$ when $n = 2m+1$). Moreover $K_m(W^Q)$ and $K_{n-m}(W^Q, X^Q)$ are $\Lambda_{Q,K}$ duals. The K_* groups of W^Q, (W^Q, X^Q) (and X^Q when $n = 2m+1$) are free $\Gamma_{Q,K}$ modules. They are obtained from $K_m(W^Q)$, $K_{n-m}(W^Q, X^Q)$ (and $K_m(X^Q)$ when $n = 2m+1$) by tensoring over R with $\Gamma_{Q,K}$.*

Proof. This is essentially [P$_9$, 6.1]. It suffices to treat the case $Q = 1$ which we do. Let M be the mapping cone of F, $q \in M^G$ be the canonical basepoint and d be the the dimension of G. Then $K_k(W, X) \cong K^{n+d-k}(W) = H^{n+d-k+1}(M, q)$. Since $K_k(W, X) = 0$ for $k < m$ when $n = 2m$ or for $k \leqslant m$ when $n = 2m+1$, $H^s(M, q) = 0$ for $s > m+d+1$. Since $K_k(W) = 0$ for $k < m$, $H_s(M, q) = 0$ for $k < m+1$; so $H^s(M, q) = 0$ for $s < m+1$ by the Universal Coefficient Theorem.

Let E be an acyclic space on which G acts freely. Define $H^*_G(M, q) = H^*(M \times_G E, q \times_G E)$ and similarly define $H^G_*(M, q)$. There is a spectral sequence $H^*_{G_0}(M, q, H^*(G_0)) \Rightarrow H^*(M, q)$ and a similar one in homology. Since $H^s(M, q)$ is non–zero only if $m+1 \leqslant s \leqslant m+d+1$, it follows that the spectral sequence collapses and (i) $H^s_{G_0}(M, q) = 0$ for $s \neq m+1$ and $H^{m+1}_{G_0}(M, q)$ is R torsion free. (Note $H^{G_0}_m(M, q) = 0$ because $H_s(M, q) = 0$, $s < m+1$.) Since also (ii) $H^*(M^P, q, R_p) = 0$, $R_p = R/p^R$ whenever $P \in \mathscr{P}$, $|P^0| = p^n \neq 1$, p prime in R, it follows from [P$_9$, 5.2] that $H^{m+1}_{G_0}(M, q) = H^{m+1}(M, q) = K^m(W)$ is a projective Λ module. By the Universal Coeffiecient Theorem so too is $K_m(W)$.

Now observe that $H^{G_0}_s(M, q) = 0$ for $s \neq m+1$ and $K_m(W) = H_{m+1}(M, q) = H^{G_0}_{m+1}(M, q)$ is R torsion free; so the homology spectral sequence $H^{G_0}_*(M, q, \Gamma) \Rightarrow H_*(M, q) = K_{*-1}(W)$ collapses and the associated graded group E_0 is $\Gamma \otimes H^{G_0}_*(M, q)$. Since $K_m(W) = H_{m+1}(M, q)$ is

R torsion free, E_0 is a free Γ module; hence, $H_*(M, q)$ is a free Γ module and $K_*(W) = H_*(M, q) = \Gamma \otimes H_{m+1}(M, q) = \Gamma \otimes K_m(W)$.

By considering the mapping cones of $\bar{F}: W/X \to Z/Y$ and $f: X \to Y$ and using the above argument, the remaining statements of the theorem are verified with the exception of the duality statement. For that note $K_{n-m}(W, X) \cong K^{d+m}(W) = H^d(G_0) \otimes K^m(W) = K^m(W)$. By the Universal Coefficient Theorem and $K_{m-1}(W) = 0$, $K^m(W)$ is the dual of $K_m(W)$; so then is $K_{n-m}(W, X)$.

Remark. The identification of $K_{n-m}(W^Q, X^Q)$ with the dual $\mathrm{Hom}_\Lambda(K_m(W^Q), \Lambda) = \mathrm{Hom}_R(K_m(W^Q), R)$ of $K_m(W^Q)$ arises from the R valued bilinear form $\lambda': K_m(W^Q) \times K_{n-m}(W^Q, X^Q) \to R$ defined by $\lambda'(x, y) = \langle \mu \cdot x, y \rangle$, where $\mu \cdot x$ denotes the Pontrjagin product of the fundamental class μ of $(\bar{Q}_K)_0$ and x. The intersection pairing between $K_i(W^Q)$ and $K_{n+d-i}(W^Q, X^Q)$ defined by Poincaré duality and cup product is denoted by $\langle\ ,\ \rangle$. There is a straightforward way to make λ' into a Λ valued bilinear form $\lambda: K_m(W^Q) \times K_{n-m}(W^Q, X^Q) \to \Lambda$. In the case $n = 2m$ and $f^Q \equiv O(R)$, $K_m(W^Q) = K_m(W^Q, X^Q)$ (because $K_n(X^Q) = 0$); so λ' and λ become non-singular forms on $K_m(W^Q)$. There is a more geometric definition of λ in this situation. Elements of $K_m(W^Q)$ can be represented by immersions of $S^m \times D^m$ in W^{Q^*} which project to an immersion in $W^{Q^*}/\bar{Q}_K$ by an appropriate application of (3.28). Apply either [W$_2$, Sect. 3] or [W$_1$, Sect. 5] to produce the intersection form λ and self intersection form μ giving a special Hermitian form $(K_m(W^Q), \lambda, \mu)$ over Λ in the sense of [W$_1$, Sect. 5].

The remainder of this section makes repeated use of the results of surgery in the preceding section. Since Z^Q and Y^Q are connected we use the setting of remark (3.30) and the result of (3.31). In addition we note that surgery on $(\mathscr{W}, \mu)$ for $\mu \in \pi_*(f^Q)$ or $\mu \in \pi_*(F^Q)$ does not alter W^H for $H \not\subset Q$ so the trace of such a surgery is a prenormal cobordism rel$\{H \not\subset Q\}$.

We begin with a description of the process of *handle subtraction.* (Compare [W$_1$, 1.3].) Let $\mathscr{W} = (W, F, B, D)$ be a $K - G$ prenormal map of triads with $F: (W, X) \to (Z, Y)$. We suppose a map $i: (D, S) \to (W^Q, X^Q)$ is given such that $F \circ i$ maps into Y^Q. This gives rise to an element $\omega \in K_{k+1}(W^Q, X^Q)$. Here $D = D^{k+1} \times D^{n-k-1}$, $S = S^k \times D^{n-k-1}$, $n = \dim W^Q/\bar{Q}$. The subset of $K_{k+1}(W^Q, X^Q)$ represented in this way is denoted by $\pi_{k+1}(F^Q, f^Q)$. (Compare [D–P$_1$, Sect. 4].) We suppose now that i is an imbedding whose image lies in $W^{Q^*} = \{w \mid G_w = Q\}$ and that the composition $D \to W^{Q^*} \to W^{Q^*}/\bar{Q}$ is an imbedding. Then $\mathrm{ind}_Q^N i$ is also an imbedding. Since $F^Q \circ i$ maps to Y^Q, the above data give a class $\mu \in \pi_{k+1}(f^Q)$ written $\mu = \partial\omega$. The assumptions imply (3.31) surgery on μ is possible giving $\mathscr{W}'$ with $W' = W \cup \mathbb{D}$ ((3.15)–(3.17)). There is a $k+1$ sphere in the interior of W' represented by $iD^{k+1} \cup D^{k+1} \subset W \cup D^{k+1}$. This represents a class x in $\pi_{k+1}(W'^Q)$. It is easy to see there is a class

$\mu' \in \pi_{k+1}(F'^Q)$ with $\partial\mu' = x$ (3.13). Perform surgery on $(\mathscr{W}', \mu')$. The result is a $K-G$ prenormal map $\mathscr{W}'' = (W'', F'', B'', D'')$. This process $\mathscr{W}' \mapsto \mathscr{W}''$ is called *handle subtraction* on $\omega \in K_{k+1}(W^Q, X^Q)$. When $N(Q) \subset K$, the reader may check that the effect on W^Q as a $\bar{Q}$ manifold is to convert W^Q to W''^Q with

$$W''^Q = \text{closure}(W^Q - \text{ind}_Q^{N(Q)} D). \tag{4.7}$$

This construction is discussed un $[W_1, 1.3]$ where the operation is done in the orbit space.

THEOREM (4.8). *Let $\mathscr{W}$ be a $K-G$ prenormal map satisfying* (4.1). (i) *If $N(Q) \subset K$, $\mathscr{W}$ is $K-G$ prenormally cobordant to $\mathscr{W}'$* $\text{rel}\{H \not\subset Q\}$ *with $F'^Q \sim 0$.* (ii) *$\mathscr{W}$ is $K-G$ prenormally cobordant* rel boundary *and* $\text{rel}\{H \not\subset Q\}$ *to $\mathscr{W}'$ with $K_k(W'^Q, Z) = 0$ for $k < m$; so if $f^Q \equiv O(R)$, $F'^Q \sim 0$.*

Proof of Theorem (4.8). The proof is by repeated surgery on $\mathscr{W}$. Each step produces a $\mathscr{W}'$ with certain properties. These properties are then assumed for the original $\mathscr{W}$; so the prime is dropped before the next stage. Each surgery step produces a cobordism $\text{rel}\{H \not\subset Q\}$.

First we achieve (4.3b). Since Y^Q is connected, there are classes $\mu_1, \ldots, \mu_l \in \pi_1(f^Q)$ such that $\partial\mu_1, \ldots, \partial\mu_l$ generate $\pi_0(X^Q)$. Use Theorem (3.31) to do surgery on these classes. This kills $\pi_0(X^Q)$. Since Y^Q is one connected, the same procedure applied to classes in $\pi_2(f^Q)$ kills $\pi_1(X^Q)$. Likewise W^Q is made one connected.

Now we treat (4.3a). The Hurewicz theorem relates $\pi_{k+1}(f^Q)$ and $K_{k+1}(X^Q, Z)$. If $K_j(X^Q, Z) = 0$ for $j < k$, then $K_k(X^Q, Z) \cong \pi_{k+1}(f^Q)$. If $k < \frac{1}{2}n$ (4.2), we can do surgery on classes in the latter group to kill it and hence kill $K_k(X^Q, Z)$. Note X^H is untouched if $H > Q$. Do this for all $k < m-1$ if $n = 2m$ and for $k < m$ if $n = 2m+1$. Thus we may suppose the properties required for X in (4.3) hold. By doing surgery on classes in $\pi_{k+1}(F^Q)$, we kill $K_k(W^Q, Z)$ for $k < m$, where $m = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$.

If $n = 2m$ the proof of (4.3a) is complete. If $n = 2m+1$, it remains to kill $K_m(W^Q, X^Q, Z)$. Since this group is $\pi_{m+1}(F^Q, f^Q)$, each class is represented by a map $\psi\colon (D^{m+1}, S^n) \to (W^Q, X^Q)$. Since W^Q and X^Q satisfy hypothesis Q, the range of this map may be supposed (W^{Q^*}, X^{Q^*}). In $[W_1, 1.4]$ it is shown how to make each ψ into an imbedding which projects to an imbedding in $(W^{Q^*}/\bar{Q}, X^{Q^*}/\bar{Q})$. This uses $N(Q) \subset K$. In fact we may suppose ψ extends to an imbedding $\hat{\psi}\colon (D, S) \to (W^{Q^*}, X^{Q^*})$ with these properties. Here $D = D^{m+1} \times D^m$, $S = S^m \times D^m$. These are the data required for handle subtraction. As in $[W_1, 1.4]$ handle subtraction on generators of $K_m(W^Q, X^Q, Z)$ kills this group. This uses (4.7) and in particular requires $N(Q) \subset K$.

THEOREM (4.9). *Let $\mathscr{W}$ be a $K-G$ prenormal map satisfying* (4.1). *Suppose $NQ \subset K$, $F^Q \approx O(R)$ and $K_m(W^Q)$ is a stably free Λ_Q module. Then $\mathscr{W}$ is prenormally cobordant to $\mathscr{W}'$* rel$\{H \not\subset Q\}$ *such that $F^Q \equiv O(R)$.*

Proof. Since $F^Q \approx O(R)$, we need only kill $K_m(W^Q)$ to achieve $F^Q \equiv O(R)$ by Theorem (4.6). There are two cases (compare [W_1, Sect. 4]):

Case $n = 2m$. By Theorem (4.6) and the hypotheses, $K_m(W^Q, X^Q)$ is a stably free Λ modules which may be assumed free. (Just do surgery on trivial classes in $\pi_m(f^Q)$.) As $K_k(W^Q, X^Q, Z)$ is zero for $k < m$, $\pi_{m+1}(F^Q, f^Q) = K_m(W^Q, X^Q, Z)$ and $K_m(W^Q, X^Q) = K_m(W^Q, X^Q, Z) \otimes R$. To kill (W^Q, X^Q) we represent the elements of a Λ base by maps $d_j: (D_j^m, S_j^m) \to (W^Q, X^Q)$. Each D_j^m is a copy of D^m. Denote

$$U = \coprod \mathrm{ind}_Q^{NQ} D_j, \qquad D_j = D_j^m \times D^m,$$

$$U_0 = \coprod \mathrm{ind}_Q^{NQ} S_j, \qquad S_j = S_j^{m-1} \times D^m.$$

The d_j give rise (piping argument) to a $\bar{Q}$ imbedding $\Phi: (U, U_0) \to (W^Q, X^Q)$ with $\Phi|_{D_j^m} = d_j$. This uses $N(Q) \subset K$. Set $W_0^Q = \text{closure}(W^Q - U)$. For the rest of the argument R coefficients are understood. We have a commutative diagram of $H_*(Q_0')$ homomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & H_*(U \cup X^Q, X^Q) \arrow[r] \arrow[d, "A"', "\cong"] & H_*(W^Q, X^Q) \arrow[r] \arrow[d, "Id"] & H_*(W^Q, U \cup X^Q) \arrow[r] \arrow[d, "C"] & 0 \\
0 \arrow[r] & K_*(W^Q, X^Q) \arrow[r] & H_*(W^Q, X^Q) \arrow[r] & H_*(Z^Q, Y^Q) \arrow[r] & 0.
\end{tikzcd}$$

By construction, A is an isomorphism. To see this note $H = H_*(U \cup X^Q, X^Q)$ and $K_*(W^Q, X^Q) = K$ are free $H_*(\bar{Q}_0) = \Gamma$ modules. The first by inspection, the second by Theorem (4.6). Moreover $H \otimes_\Gamma R$ and $K \otimes_\Gamma R$ are free Λ modules with bases $\{(D_j, S_j)\}$ in the first and their images in the second by construction; hence, $A \otimes_\Gamma 1_R$ is an isomorphism; so A is an isomorphism. It follows that C is an isomorphism. The composition of excision and C gives an isomorphism

$$H_*(W_0^Q, X_0^Q) \to H_*(W^Q, U \cup X^Q) \to H_*(Z^Q, Y^Q);$$

so $K_*(W_0^Q, X_0^Q)$ is zero. Then $K_*(W_0^Q)$ and $K_*(X_0^Q)$ are also. Here $X_0^Q = \partial W_0^Q$.

The imbeddings d_j represent classes in $\pi_{m+1}(F^Q, f^Q)$. Perform handle subtraction on these classes. This converts $\mathscr{W}$ to $\mathscr{W}' = (W', F', B', C')$, where $W'^Q = W_0^Q$ by (4.7). This uses $N(Q) \subset K$; thus $K_m(W'^Q) = 0$ as required.

Case $n = 2m+1$. Since $F^Q \sim 0$, $K_*(W^Q, X^Q)$, $K_*(W^Q)$ and $K_*(X^Q)$ are free Γ modules by Theorem (4.6). It follows from that theorem and $F^Q \approx O(R)$ that the sequence

$$0 \to K_{m+1}(W^Q, X^Q) \to K_m(X^Q) \to K_m(W^Q) \to 0$$

is exact and consists of free Λ modules. (The homomorphism $\psi: K_*(W^Q) \to K_*(W^Q, X^Q)$ is one of Γ modules and $K_*(W^Q)$ is generated as a Γ module by $K_m(W^Q)$; so ψ is zero.)

As $K_k(W^Q, Z^Q, Z) = 0$ for $k < m$, $\pi_{m+2}(F^Q, f^Q) = K_{m+1}(W^Q, X^Q, Z)$ and $K_{m+1}(W^Q, X^Q) = K_{m+1}(W^Q, X^Q) \otimes R$. Represent a Λ base $\{\omega_j\}$ for $K_{m+1}(W^Q, X^Q)$ by elements

$$d_j: (D_j^{m+1}, S_j^m) \to (W^Q, X^Q).$$

The techniques of $[W_1, \text{Sect. } 4]$ show that the map $\Phi: \coprod \operatorname{ind}_Q^{NQ} S_j^m \to X^Q$ induced by the d_j can be assumed to be an imbedding. This uses hypothesis Q and again involves mapping S_j^m into X^{Q^*} so the projection in $(X^Q)^*/Q_0'$ is an imbedding. Let $\mathscr{W}'$ be the result of surgery on the classes $\partial\omega_j \in \pi_m(f^Q)$. See (3.18) and (3.31). Then $W' = W \cup \operatorname{ind}_{NQ}^G U$, where $U = \coprod \operatorname{ind}_Q^{NQ} D_j$, $D_j = D_j^{m+1} \times D^m$. The sequence of kernel groups for the triple $X' \subset X' \cup U \subset W'$ is $0 \to K_{m+1}(W'^Q, X'^Q) \to K_{m+1}(W^Q, X^Q) \to K_m(U, U \cap X'^Q) \to K_m(W'^Q, X'^Q) \to 0$ because $K_*(U, U \cap X'^Q)$ is a free Γ module generated by $K_m(U, U \cap X'^Q)$.

The Λ module $K_m(U, U \cap X'^Q)$ is free with one basis element corresponding to each handle (represented by the core of the dual handle). The map from $K_{m+1}(W^Q, X^Q)$ to $K_m(U, U \cap X'^Q)$ is dual to the map $K_{m+1}(U, U \cap X^Q) \to K_m(W^Q)$ representing the attaching maps so it is zero. (See (4.6).) Thus $K_{m+1}(W^Q, X^Q)$ is unchanged. We acquire a free module $K_m(W'^Q, X'^Q)$ dual to $K_{m+1}(W')$; thus $K_m(X'^Q) = 0$.

Choose a base for $K_m(W')$. Represent those elements by classes in $\pi_{m+1}(F'^Q)$. Do surgery with respect to these classes. Denote the new map again by $\mathscr{W}'$. Then it follows as in $[W_1, \text{Sect. } 4]$ that $K_m(W'^Q) = 0$; so $K_*(W'^Q) = 0$ by Theorem (4.6).

In order to use (4.9) we have to decide when $K_m(W^Q)$ is stably free. This leads to the Grothendieck groups $K_0(\Lambda)$ and $G_0(\Lambda)$ of projective Λ modules and all Λ modules. The quotient of $K_0(\Lambda)$ resp. $G_0(\Lambda)$ by the subgroup generated by free modules is denoted by $\tilde{K}_0(\Lambda)$ resp. $\tilde{G}_0(\Lambda)$. The following lemma was pointed out to me by Oliver:

LEMMA (4.10). *$\tilde{K}_0(Z_{(p)}(G)) \to \prod_{(|C|, p)=1, C \subset G \text{ cyclic}} \tilde{G}_0(Z_p(C))$ is a monomorphism when G is finite. Each component homomorphism is induced by tensoring with Z_p and viewing the result as a $Z_p(C)$ module.*

Proof. The basic facts which may be found in [Ba] are: $K_0(A(G)) \to K_0(A_p(G))$ is injective if A is a local ring with maximal ideal p and $A_p = A/pA$. This homomorphism is induced by tensoring over A with A_p. See [Ba, p. 449]. $K_0(Z_pG) \to G_0(Z_pG)$ is injective [Ba, p. 532]. $G_0(Z_pG) \to \prod_{C\,\text{cyclic}} G_0(Z_p(C))$ is injective. $Z_p(G)$ is a local ring if $|G| = p^n$ and is semisimple if $(p, |G|) = 1$. Projective modules over a local ring are free.

Lemma (4.10) is exploited like this: Let X be a G space. Set $\Gamma_p = \Gamma \otimes Z_p$ (4.2) and define

$$\delta(X) = \sum (-1)^i [H_*(X, Z_p) \otimes_{\Gamma_p} Z_p]_i \in \tilde{G}_0(Z_p G^0).$$

(Observe—$\otimes_{\Gamma_p} Z_p$ is an exact functor.) The tensor product in this definition is graded. The subscript i denotes the term in degree i. From (4.6), $(K_*(W^Q) \otimes_{\Gamma_p} Z_p)_i$ is zero if $i \neq m$ and is $K_m(W^Q) \otimes Z_p$ if $i = m$. Thus from the sentence after (4.5)

$$[K_m(W^Q) \otimes Z_p] = (-1)^m (\delta(W^Q) - \delta(Z^Q))$$

in $\tilde{G}_0(Z_p \tilde{Q}_K^0)$. (Note Z_p is Γ_p projective so tensoring with Z_p is an exact functor.) Define

$$\mathscr{F}_G = \{H \subset G \mid H \not\subset G_0\}. \tag{4.11}$$

Let Σ be a set of subgroups of G and $\Omega \subset \Sigma$. Then Ω is said to be *closed* if (i) whenever $K \in \Sigma$, $H \in \Omega$ and $K \geqslant H$, then $K \in \Omega$ and (ii) if $H' \in \Sigma$ is conjugate to $H \in \Omega$, then $H' \in \Omega$. For any $\Omega \subset \Sigma$, Ω^* is the smallest closed subset of Σ containing Ω. Observe that $\mathscr{F}_L$ is closed in $\mathscr{S}(L)$.

LEMMA (4.12). *Let $X = \bigcup_{H \subset G, i} G/H \times D^i$ be a G c.w. complex. Then*

$$\delta(X) = \sum_{i, H \in \mathscr{F}_G} (-1)^i \, \delta(G/H) \in \tilde{G}_0(Z_p G^0).$$

Proof. If $H \subset G_0$, then $G/H/G_0 = G/G_0$; so $H_*(G/H) \otimes_{\Gamma_p} Z_p = Z_p(G^0) \otimes_{Z_p} (H_*(G/G_0 \cap H) \otimes_{\Gamma_p} Z_p)$ is a free $Z_p(G^0)$ module. Thus $\delta(G/H) = 0$ and so G/H contributes nothing to $\delta(X)$.

COROLLARY (4.12′). *Let X and Y bee G c.w. complexes $G = O_2$ or O_2'. If $\chi(X^H) = \chi(Y^H)$ whenever $H \in \mathscr{F}_G$, then $\delta(X) = \delta(Y)$.*

Proof. By Lemma (4.12) we may suppose Iso(X) and Iso(Y) lie in $\mathscr{F}_G = \mathscr{F}$. Let $\Sigma \subset \mathscr{F}$ be closed and suppose the lemma is true if Iso(X) and Iso(Y) are in $\bigcup$. Let $K \in \mathscr{F} - \Sigma$ be a maximal element and $\Sigma' = (\Sigma \cup K)^*$. Suppose Iso($X$) and Iso($Y$) lie in Σ'. Set $X' = \bigcup_{H \in \Sigma} X^H$ and likewise define

Y'. Then $\chi(X'^H) = \chi(Y'^H)$ for all $H \in \mathscr{F}$ (because for $H \in \mathscr{F}$ there are only finitely many isotropy groups containing H); so $\delta(X') = \delta(Y')$. Thus $\delta(X) - \delta(Y) = |\bar{K}|^{-1}(\chi(X/X'^K) - \chi(Y/Y'^K))\,\delta(G/K) = 0$. The proof is completed by induction.

LEMMA (4.13). *Let $\mathscr{W} = (W, F, B, D)$ be a $K-G$ prenormal map satisfying* (4.1). *Suppose $F^Q \approx O(R)$* (4.5) *and either* (i) *$\bar{Q}_K$ is finite and $\chi(W^H) = \chi(Z^H)$ for all $Q \lhd H \subset K$ with H/Q cyclic $\neq 1$ of order prime to p or* (ii) *$\bar{Q}_K$ is O_2 (or $O_2' \subset S^3$) and $\chi(W^H) = \chi(Z^H)$ whenever $Q \lhd H \subset K$ and $H/Q \notin (\bar{Q})_0$. Then $K_m = K_m(W^Q, R)$ is a stably free $\Lambda_{Q,K}$ module.*

Proof. Let $G = \bar{Q}_K$. By Lemma (4.10) it it suffices to show $(-1)^m[K_m \otimes Z_p] = \delta(W^Q) - \delta(Z^Q)$ is zero in $\tilde{G}_0(Z_p(L))$ whenever $L \in G^0$ is a non-trivial cyclic group of order prime to p. In case (i) this is implied by the hypothesis and [O$_1$, Lemma 4]. In case (ii) $G^0 = Z_2$; so there is only one L, namely, $L = G^0$. Now apply Corollary (4.12′) with $X = W^Q$ and $Y = Z^Q$.

THEOREM (4.14). *Let $\mathscr{W} = (W, F, B, D)$* be a *$K-G$ prenormal map satisfying* (4.1). *Suppose $N(Q) \subset K$ and $F^Q \approx O(R)$* (4.5). *Suppose either R is Z and $\tilde{K}_0(\Lambda_Q) = 0$ or R iz $Z_{(p)}$ for some prime p and either* (4.13i) *or* (4.13ii) *is satisfied. Then $\mathscr{W}$ is $K-G$ prenormally cobordant to $\mathscr{W}'$* rel$\{H \not\subset Q\}$ *with $F'^Q \cong O(R)$.*

Proof. Since $F^Q \approx O(R)$, $K_m = K_m(W^Q, R)$ is a projective Λ_Q module by (4.6). If this is stably free, we apply (4.9) to complete the proof. If $R = Z$, K_m is stably free because $\tilde{K}_0(\Lambda_Q) = 0$. If $R = Z_{(p))}$, then K_m is stably free by (4.13).

THEOREM (4.15). *Let $\mathscr{W} = (W, F, B, D)$ be a $K-G$ prenormal map satisfying* (4.1) *with the hypotheses*: (i) *X satisfies hypothesis Q and* (ii) $\dim X^Q \geqslant 6$ *deleted. In particular $X = \partial W$ may be empty. Require* $\dim W^Q \geqslant 6$. *Suppose $f^Q \equiv O(R)$ and $F^Q \approx O(R)$. Suppose either R is Z and $\tilde{K}_0(\Lambda_{Q,K})$ is zero or R is $Z_{(p)}$ for some prime p and either* (4.13i) *or* (4.13ii) *holds. Then there is an obstruction $\sigma_Q(\mathscr{W}) \in L_n(\Lambda_{Q,K})$* ($n = \dim W^Q/\bar{Q}_K$) *whose vanishing implies $\mathscr{W}$ is $K-G$ prenormally cobordant to $\mathscr{W}'$* rel$\{H \not\subset Q\}$ *and* rel boundary *such that $F'^Q \cong O(R)$.*

Before proving Theorem (4.15) we briefly review the definition of the L groups. (In Wall's notation these are the L^h groups.) See [W$_1$, W$_3$]. The definition of $L_n(\Lambda)$ depends on he parity of n.

$n = 2k$. The group in ths case consists of equivalence classes of pairs (M, ϕ), where M is a stably free Λ module and ϕ is a $(-1)^k = u$ quadratic form on M, i.e., in the notation in [W$_3$, p. 3] $(M, \phi) \in Q_{(\alpha, u)}(M)$. (The antiautomorphism α of Λ arises from the orientation homomorphism $e^Q: \pi_0(\bar{Q}_K) \to \{\pm 1\}$ defined by $g_*[W^Q] = e^Q(g)[W^Q]$, $g \in \pi_0(\bar{Q}_K)$ and $[W^Q]$ a chosen generator of the top dimensional homology of W^Q. Then $\alpha(g) = e^Q(g) \cdot g^{-1}$ for $g \in \pi_0(\bar{Q}_K) \in \Lambda$.) The bilinearization $\lambda = (\phi + T_u\phi)$ of ϕ is assumed non-singular. Here $T_u\phi$ is the transposed form [W_3, p. 3]. In [W_1, Sect. 5] there is an alternative equivalent definition based on triples (M, λ, μ) (called in [W_1, Sect. 5] a $(-1)^k$ Hermitian form) with M as above; λ is a Λ valued form (intersection form) satisfying $\lambda(x, y) = (-1)^k \lambda(y, x)$ with associated "quadratic form" μ (self intersection form). Note that we do *not* assume a preferred stable base for M. A triple M of this kind is obtained from a quadratic module (M, ϕ) by setting $\lambda = (1 + T_u)\phi$ with $\mu(x) = \phi(x, x)$. Let $M^* = \mathrm{Hom}_\Lambda(M, \Lambda)$. It is a Λ module and $M \oplus M^*$ supports a $(-1)^k$ quadratic form called te hyperbolic form and denoted by $\mathbb{H}(M)$. A quadratic module is by definition equivalent to zero if it is isomorphic to $\mathbb{H}(M)$ for some M.

$n = 2k + 1$. The group in this case consists of equivalence classes of formations over Λ. To define the data of a formation recall that a Lagrangian (or subkernel [W_1, 5.3]) L of a quadratic module (Q, ϕ) is a direct summand of Q on which $\lambda = (1 + T_u)\phi$ and $\phi(x, x)$ vanish and $L = \{x \in Q \mid \lambda(x, y) = 0 \text{ for all } y \in L\}$. Then a formation (Q, ϕ, Q_0, Q_1) over Λ consists of:

(i) A quadratic module (Q, ϕ) which is equivalent to zero.

(ii) Lagrangians Q_0 and Q_1 of (Q, ϕ). Here Q_0 and Q_1 are to be free Λ modules.

The further details of definition, addition and equivalence may be found in [R, Sects. 1 and 2].

Proof of (4.15). The proof depends on the parity of n. First we define $\sigma_Q(\mathscr{Y})$.

$n = 2m$. Since $F^Q \equiv O(R)$, $K_m(W^Q) \equiv K_m(W^Q, X^Q)$. By (4.6), $K_m(W^Q)$ is a projective $\Lambda = \Lambda_{Q,K}$ module. It is stably free by (4.13). By (4.6), $K_m(W^Q, X^Q) \cong K_m(W^Q)^*$. This isomorphism is induced by the intersection pairing λ. Since $K_m(W^Q) = K_m(W^Q, X^Q)$, λ is non-singular bilinear form over Λ. Let μ be the associated self intersection form. Then $\sigma_Q(\mathscr{Y})$ is the class of $(K_m(W^Q), \lambda, \mu)$ in $L_n(\Lambda)$.

$n = 2m + 1$. We follow [D–P_3] which turn uses the ideas in [W_2, Sect. 2] and construct a diagram as on p. 56 of [W_2]; this will give rise to a formation defining an element in $L_n(\Lambda)$.

Since $F^Q \approx O(R)$, $K_k(W, Z) = 0$ for $k < m$ so $K_m(W^Q, Z) = \pi_{m+1}(F^P)$. Define $K_m = K_m(W^Q, R) \cong K_m(W^Q, Z) \otimes R$. Pick classes $\{\mu_i\}$ in $\pi_{k+1}(F^Q)$

which generate K_m as a Λ module. Associated to the μ_i are imbeddings $\partial\mu_i: S^m \times D^{m+1} \to W^{Q'}$. We can suppose these induce an imbedding of $\coprod \operatorname{ind}_Q^{N_K(Q)} S_i \to W^{Q'}$, $S_i = S^m \times D^{m+1}$ by a general position argument.

Pick $z \in Z$ with $G_z = Q$. This exists because $Q \in \operatorname{Iso}(Z)$. We can suppose F^Q is transverse to z. This is easily seen using a (non-equivariant) transversality argument for F^Q and the equivariant homotopy extension theorem for F^Q. Hence, $(F^Q)^{-1}(z) = \{w_1, ..., w_r\}$, where r can be assumed to be the absolute value of the degree of F^Q. Construct a submanifold U_1 of W^Q. It consists of disks D_i around points w_i, the submanifolds $\partial\mu_i(S^m \times D^{m+1})$, tubes connecting $\partial\mu_i(S^m \times D^{m+1})$ with D_1 and tubes connecting D_1 with the other D_i's. We can suppose that the tubes and the $\partial\mu: (S^m \times D^{m+1})$ are mapped to a point $\bar{z}$ in ∂D_z (D_z a disk around z) while the disks D_i (up to a small deformation at the boundary) are mapped by a linear isomorphism. Also $F^Q(W^Q - \operatorname{int} U_1) \subset Z^Q - \operatorname{int} D_z$. Finally we suppose $U = \operatorname{ind}_Q^{N_K Q} U_1$ is imbedded in W^Q equivariantly extending the inclusion of U_1 in W^Q. Similarly $\tilde{D}$ denotes the equivariant image of $\operatorname{ind}_Q^{N_K Q} D_z$ in Z^Q. Define $W_0^Q = W^Q - \operatorname{int} U$ and $Z_0^Q = Z^Q - \operatorname{int} \tilde{D}$. So finally we can suppose

$$F^Q: (W^Q, W_0^Q, U) \to (Z^Q, Z_0^Q, \tilde{D})$$

and $\deg F^Q = \deg F^Q \mid W_0^Q$.

The exact sequence of K_i groups with R coefficients is

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] \arrow[dr] & & K_{m+1}(W^Q, W_0^Q) & = & K_{m+1}(U, \partial U) \arrow[rr] \arrow[dr] & & K_m(W_0^Q) \arrow[rr] \arrow[dr] & & 0 \\
& K_{m+1}(W^Q) \arrow[ur] \arrow[dr] & & & & K_m(\partial U) \arrow[ur] & & K_m(W^Q) \arrow[ur] \arrow[dr] & \\
0 \arrow[rr] \arrow[ur] & & K_{m+1}(W^Q, U) & = & K_{m+1}(W_0, \partial U) \arrow[rr, "j"] \arrow[ur] & & K_m(U) \arrow[rr] \arrow[ur, "i"] & & 0.
\end{tikzcd}$$

We justify the zeros on the left of this diagram. Let $F_0^Q = F^Q \mid W_0^Q$. Observe that $K_m(W_0, \partial U, Z)$, $K_m(U, \partial U, Z)$, $K_k(\partial U, Z)$ and $K_k(W_0, Z)$, $k < m$, all vanish; and degree $F_0^Q =$ degree F^Q is a unit of R. Also $F_0^P = F^P \equiv O(R_{(p)})$ whenever $Q \lhd P$, $Q \neq P$, $P/Q \in \mathscr{P}$, $|(P/Q)|^0 = p^n$, p prime in R; so $F_0^Q \approx O(R)$. Thus Theorem (4.6) implies $K_m(W_0^Q)$ is a projective Λ module and $K_*(W_0^Q)$ is a free Γ module generated by $K_m(W_0^Q)$. Similarly for $K_m(U)$ and $K_*(U)$. Then since $K_m(W_0, \partial U)$ and $K_m(U, \partial U)$ vanish, $K_{m+1}(W_0) \to K_{m+1}(W_0, \partial U)$ and $K_{m+1}(U) \to K_{m+1}(U, \partial U)$ are zero. This accounts for the zeros and shows $K_{m+1}(U, \partial U)$ and $K_{m+1}(W_0, \partial U)$ are Λ submodules of $K_m(\partial U)$. We claim they are free. If $R = Z$, there is nothing to show. Note that $W_0^K = W^K$ and $Z_0^K = Z^K$ whenever $Q < K$, so (4.13i) or (4.13ii) is satisfied with W_0^Q and Z_0^Q replacing W^Q and Z^Q. Thus Lemma (4.13) implies

$K_m(W_0) \cong K_{m+1}(W_0, \partial U)^*$ is a free Λ module when $R = Z_{(p)}$. Note $K_{m+1}(U, \partial U)$ is free by inspection.

We now show that $\tau = (K_m(\partial U), \phi, K_{m+1}(U, \partial U), K_{m+1}(W_0^Q, \partial U))$ is a formation. It represents the class $\sigma_Q(\mathscr{W}) \in L_n(\Lambda)$. Here ϕ is the quadratic form determined by the intersection form λ and self intersection form μ. Certain facts are clear: $(K_m(\partial U), \phi)$ is equivalent to zero; $K_{m+1}(U, \partial U)$ are Lagrangian; λ and μ vanish on $K_{m+1}(W_0^Q, \partial U)$ (same argument as in the proof of [W$_1$, 5.7]); so it too is a Lagrangian. Thus τ is a formation.

To complete the proof of (4.15) we must show that if $\sigma_Q(\mathscr{W})$ vanishes, then $\mathscr{W}$ is prenormally cobordant to $\mathscr{W}'$ with $F'^Q \equiv O(R)$. If $n = 2m$, $\sigma_Q(\mathscr{W}) = 0$ means the quadratic module determined by $(K_m(W^Q), \lambda, \mu)$ is hyperbolic say $\mathbb{H}(M)$ for some free Λ module M. As in [W$_1$, Sect. 4] surgery on generators of M kills $K_m(W^Q)$. Let $n = 2m + 1$ and $\sigma_Q(\mathscr{W}) = (K_m(\partial U), \phi, K_{m+1}(U, \partial U), K_{m+1}(W_0^Q, \partial U)) = 0$. We write this formation as $(\mathbb{H}(F), F, G)$, where $\mathbb{H}(F) = (K_m(\partial U), \phi)$. Let $\gamma \oplus \mu: G \to F \oplus F^* = \mathbb{H}(F)$ be the homomorphism which exhibits G as a submodule of $\mathbb{H}(F)$. Since this formation is equivalent to zero, there is a free Λ module L and a homomorphism $j: L \to F^*$ such that $\psi: F \oplus L \to G^* \oplus L^*$ is an isomorphism where

$$\psi = \begin{pmatrix} \mu^* & \gamma^* j \\ j^* & 0 \end{pmatrix}.$$

Identify F^* with $K_m(U)$. Let $x_1, \ldots, x_l$ generate L. Then surgery on the classes $\{\partial_j(x_i), i = 1 \cdots l\}$ kills $K_m(W^Q)$. See Butterfly diagram above. See [W, Sect. 5; R]. This is Ranicki's formulation of [W$_1$, Sect. 5].

5. An Induction Theorem for S^3 and SO_3

In this section G is SO_3 or S^3 and $\mathscr{H}$ is the set of two subgroups $\{O, O_2\}$ when G is SO_3 or their double covers when G is S^3. The Induction Theorem (5.10) asserts that if $\mathscr{W}_0$ is a G prenormal map and $\mathrm{Res}_K \mathscr{W}_0 = \partial \mathscr{W}_K$ for $K \in \mathscr{H}$, then $\mathscr{W}_0$ is G prenormally cobordant rel$\{G\}$ to a pseudoequivalence. The precise assumptions for this to be true are spelled out in (5.8) and (5.10). We note that much of the notation in this section is contained in (4.2).

The proof of (5.10) makes repeated use of (4.14) and (4.15) as Q ranges over a subset of $\mathscr{S}(G)$ which lies in $\mathscr{P}$. Both (4.14) and (4.15) involve a ring R which is a function of Q and single out a family of subgroups of $\mathscr{P}$ depending on Q and $K \subset G$. We specify this now.

(5.0) For $Q \in \mathscr{P}$, R_Q is Z if Q is connected and R_Q is $Z_{(q)}$ if $|Q^0| = q^s > 1$, q prime.

$$\mathscr{P}_{Q,K} = \left\{ P \in \mathscr{P} \;\middle|\; \begin{array}{l} Q \lhd P \subset K, P \neq Q, |/P^0| = p^s \\ \text{for some prime } p \text{ prime in } R_Q \end{array} \right\}.$$

Determination of the sets $\mathscr{P}_{Q,K}$ depends on knowledge of the subgroup structure of G and of the normalizers of groups in $\mathscr{P}$. This information for $G = S^3$ and SO_3 is recorded in Section 2. In particular we mention that a subgroup P of SO_3 in $\mathscr{P}$ is conjugate to a subgroup Q of O_2 and $N(Q) \subset O_2$ unless $Q = D_2$, then $N(Q) = 0$.

Before applying the results of Section 4 we need to have an effective means of verifying the hypotheses of (4.1), in particular hypothesis Q for X and W. Because in applications X and W are rarely explicitly given, we need to be able to verify (4.1) from conditions on Z and Y. The next few results show how this is done.

For certain G manifolds we can define an integral valued function $\mathrm{Dim}\, X$ on the set of conjugacy classes of subgroups of G by

$$\mathrm{Dim}\, X(H) = \dim X^H, \qquad H \subset G.$$

For this to make sense each component of X^H must have the same dimension. We say $\mathrm{Dim}\, X$ is even or odd if all values are even or odd.

In order to relate $\mathrm{Dim}\, X$ and (3.11), define $X_s^Q = \{x \in X \mid G_x > Q\}$. There is a well defined map of $G/L \times X^L$ to X which sends (gL, x) to gx. This induces a surjective map of $\bigcup (G/L)^Q \times_{L^*} X^L$ onto X_s^Q. The union is over the set of conjugacy classes of isotropy groups L of X strictly containing Q and $L^* = NL \cap NQ/L \cap NQ$. This set is finite so

$$\dim X_s^Q \leqslant \max_{(L)} \left(\dim_{L \in \mathrm{Iso}\, X} X^L + \dim_{L > Q} G/L^Q - \dim \frac{NL \cap NQ}{L \cap NQ} \right).$$

Lemma (5.1). *Suppose X is a smooth G manifold for which* $\mathrm{Dim}\, X$ *is defined. If for all $L > Q$*

$$\mathrm{Dim}\, X(L) + \mathrm{Dim}\, G/L(Q) - \dim \frac{NL \cap NQ}{L \cap NQ} < \frac{1}{2} (\mathrm{Dim}\, X(Q) + \dim \bar{Q})$$

(see (4.2) for $\bar{Q}$) then X satisfies hypothesis Q.

Proof. The hypothesis implies any sphere S^k with $k \leqslant \frac{1}{2} \dim X^Q/\bar{Q}$ can be homotoped into $X^{Q^*} = X^Q - X_s^Q$ because $\dim X_s^Q < \frac{1}{2}(\dim X^Q + \dim \bar{Q})$.

Definition (5.2). X satisfies the Gap-hypothesis if:

(i) $\hat{P}_X$ is defined for all $P \subset G$ with $P \in \mathscr{P}$. See (4.2) and (3.12).

(ii) The inequality in (5.1) is satisfied whenever $Q = \hat{P}_Y$ (3.12) for $P \in \mathscr{P}$.

(5.3) From now on a prenormal map $\mathscr{W} = (X, f, b, d), f: X \to Y$ (3.9) will satisfy these additional conditions: Dim $X =$ Dim Y and $\mathrm{Iso}(X) = \mathrm{Iso}(Y)$.

LEMMA (5.4). *Let $\mathscr{W}$ be a prenormal map as in* (5.3). *If Y satisfies the Gap-hypothesis* (5.2), *then X satisfies the Gap-hypothesis.*

Proof. As Iso $X =$ Iso Y, $\hat{P}_X$ is defined and equal to $\hat{P}_Y$ for $P \in \mathscr{P}$. The inequalities required for Dim X are implied by those for Dim Y.

A weaker inequality than (5.1) is useful for determining whether $Q \in \mathrm{Iso}(X)$. Consider these properties for X and $Q \subset G$:

(5.5-Q) For all $L > Q$, $\mathrm{Dim}\, X(L) \neq \mathrm{Dim}\, X(Q)$ implies $\mathrm{Dim}\, X(L) + \dim G < \mathrm{Dim}\, X(Q)$.

(5.6) (5.5-Q) holds for all $Q \subset G$.

LEMMA (5.7). *Suppose* Dim X *is defined and X satisfies* (5.5-Q). *If for all $L > Q$,* $\mathrm{Dim}\, X(L) \neq \mathrm{Dim}\, X(Q)$, *each component of X^Q contains a point whose isotropy group is Q. If X^Q is connected this is necessary and sufficient.*

Proof. The hypothesis implies $\dim X^Q_s < \dim X^Q$ so no component of X^Q is contained in X^Q_s. Finally observe that if X^Q is connected and $Q \in \mathrm{Iso}(X)$, $\dim X^L < \dim X^Q$ because X^L is a proper submanifold of X^Q.

The center of S^3 is denoted by C; so $SO_3 = S^3/C$. When $\mathscr{W} = (X, f, b, d)$ is an S^3 prenormal map, $\mathscr{W}^C = (X^C, f^C, b^C, d^C)$ will denote the resulting SO_3 prenormal map.

Let G be SO_3 or S^3 and $\mathscr{W}_0 = (X, f, b, d)$ be a G prenormal map which satisfies these conditions:

Set $\mathscr{F} = \mathscr{F}_{O_2} \cup \{O, T\}$ if $G = SO_3$ and $\mathscr{F} = \mathscr{F}_{O_2'} \cup \{O', T'\}$ if $G = S^3$. Recall $\mathscr{F}_G$ is defined in (4.11).

(5.8) (i) degree $f = 1$ and in addition degree $f^C = 1$ when $G = S^3$.

(ii) Dim $Y(H)$ is even for all $H \subset G$ and Dim $Y(H) \geqslant 6$ for $H \in \mathscr{H}$.

(iii) Y^H is connected for $H \in \mathscr{P} \cup \mathscr{F}$.

(iv) When $G = SO_3$, $\mathrm{Iso}(Y)$ contains $\mathscr{S} = \{(S^1, O_2, D_4, D_2, Z_2, 1)\}$ and $\hat{P}_Y \in \mathscr{P}$ for all $P \in \mathscr{P}$. When $G = S^3$, $\mathrm{Iso}(Y^C)$ contains $\mathscr{S}$, $\mathrm{Iso}(Y) = p^{-1}\,\mathrm{Iso}(Y^C) \cup 1$ and $\hat{P}_{Y^C} \in \mathscr{P}$ for all $P \in \mathscr{P}$. Here Y^C is viewed as an SO_3 manifold.

By an $\mathscr{H}$ prenormal map $\mathscr{W}_{\mathscr{H}}$, we sall mean $\mathscr{W}_{\mathscr{H}} = \{\mathscr{W}_K \mid K \in \mathscr{H}\}$, where

$\mathcal{W}'_K = (W_K, F_K, B_K, D_K)$ is a $K-G$ prenormal map (3.9′) with $F_K: W_K \to Z$, X a G manifold independent of K and $\partial\mathcal{W}'_K = (X, f, b, d) = \mathcal{W}'_0$ a G prenormal map which is independent of K. We write $\mathcal{W}'_0 = \partial\mathcal{W}'_{\mathcal{F}}$ and call Z the *target* of $\mathcal{W}'_{\mathcal{F}}$. If each $\mathcal{W}'_K$ is $K-G$ prenormally cobordant rel$\{H \not\subset Q\}$ to $\mathcal{W}'_K$ with common G cobordism between $\partial\mathcal{W}'_K$ and $\partial\mathcal{W}''_K$, we say $\mathcal{W}'_{\mathcal{F}}$ is prenormally cobordant to $\mathcal{W}''_{\mathcal{F}}$ rel$\{H \not\subset Q\}$.

LEMMA (5.9). *Let $\mathcal{W}'_{\mathcal{F}} = \{\mathcal{W}'_K\}$ be an $\mathcal{K}$ prenormal map. Suppose $\partial\mathcal{W}'_{\mathcal{F}} = \mathcal{W}'_0$ satisfies (5.8). Let $\Omega \subset \mathcal{F}$ be a closed set (Section 4) such that $\chi(W_K^H) = \chi(Z^H)$ whenever $H \in \Omega$, $H \subset K \in \mathcal{K}$.* Let *$Q \in \mathcal{F} - \Omega$ be a maximal element. Then $\mathcal{W}'_{\mathcal{F}}$ is prenormally cobordant* rel$\{H \not\subset Q\}$ *to $\mathcal{W}''_{\mathcal{F}}$ with $\chi(W_K'^H) = \chi(Z^H)$ whenever $H \in (\Omega \cup Q)^*$—the smallest closed set containing Ω and Q. In addition $\partial\mathcal{W}''_{\mathcal{F}}$ satisfies* (5.8).

Proof. One of the groups in $\mathcal{K}$ say J contains $N(Q)$. Let J' be the other group in $\mathcal{K}$. Suppose first $Q \notin \mathrm{Iso}(X)$. Since Y^Q is connected and $\mathrm{Iso}(X) = \mathrm{Iso}(Y)$, $\hat{Q}_X$ and $\hat{Q}_Y$ are defined and equal say equal to $\hat{Q}$. Then $X^Q = X^{\hat{Q}}$ and $Y^Q = Y^{\hat{Q}}$ (3.12). Since $X^J \subset X^Q$ and J is an isotropy group, $Q \lhd \hat{Q} \subset J$ so $\hat{Q} \in \mathcal{F}$. But then $\hat{Q} \in \Omega$ because Ω is closed; thus $\chi(X^Q) = \chi(X^{\hat{Q}}) = 2\chi(W^{\hat{Q}}) = 2\chi(Z^{\hat{Q}}) = \chi(Y^{\hat{Q}}) = \chi(Y^Q)$. Here the inductive hypothesis and the fact that $\dim X^{\hat{Q}}$ is even and is the boundary of $W^{\hat{Q}}$ are used. Since X^Q and Y^Q have the same Euler characteristics, W^Q and Z^Q do also for the same reason as above.

Thus we may suppose $Q \in \mathrm{Iso}(X)$. Since $\dim X^Q \geqslant 6$ (5.8ii), we can perform G surgeries on $(\mathcal{W}'_K, \mu)$ for $K = J, J'$ using trivial classes $\mu \in \pi_i(f^Q)$ (3.19) for $i = 1$ or 2 to achieve $\chi(W_J'^Q) = \chi(Z^Q)$. To justify this note that surgery on such a class alters W_J^Q up to G homotopy by adding $\bar{Q} \times D^{i+1}$. This alters the Euler characteristic by $(-1)^{i+1}\chi(\bar{Q})$. For $Q \in \mathcal{F}$, $\bar{Q}$ is finite so $\chi(\bar{Q}) = |\bar{Q}|$. Since W_J^H and Z_J^H have the same Euler characteristic for $H > Q$, their Euler characteristics for $H = Q$ have a difference divisible by $|\bar{Q}|$; so the above surgeries will kill this difference.

Having achieved equal Euler characteristics for $W_J'^Q$ and Z^Q means the same is true for X'^H and Y^H if H is any conjugate of Q. This implies equal Euler characteristics for $W_K'^H$ and Z^H for K either J of J' and H any conjugate of Q in K. This completes the proof.

THEOREM (5.10). *Let G be SO_3 or S^3 and $\mathcal{W}'_0 = (X, f, b, d)$, $f: X \to Y$ be a G prenormal map* (5.3) *satisfying* (5.8). *Suppose $\mathcal{W}'_0 = \partial\mathcal{W}'_{\mathcal{F}}$. Let Z be the target of $\mathcal{W}'_{\mathcal{F}}$. Suppose Z^P and Y^P are one connected for all $P \in \mathcal{P}$ and Y and* $\mathrm{Res}_K Z$ *for $K \in \mathcal{K}$ satisfy the Gap-hypothesis* (5.2). *Then $\mathcal{W}'_{\mathcal{F}}$ is prenormally cobordant* rel$\{G\}$ (*Section* 3) *to $\mathcal{W}''_{\mathcal{F}}$ with $\partial\mathcal{W}''_{\mathcal{F}} = (X', f', b', d')$, f' a pseudoequivalence* (*Introduction*).

Remarks. (R0) Because of (5.8iii) $\hat{P}_Y$ (3.11) is defined whenever $P \in \mathcal{P}$.

By (5.3), $\hat{P}_X = \hat{P}_Y$. From now on we drop the subscripts; *so $\hat{P}$ means the minimal isotropy groups in X or Y which contain P*. Observe that $X^P = X^{\hat{P}}$ and $Y^P = Y^{\hat{P}}$.

(R1) The proof of (5.10) uses (5.9) and the results of the preceding section. All those results require the blanket assumption (4.1) which is used for $Q = \hat{P}$ (3.11) for each $P \in \mathscr{P}$. These conditions are met because Y and $\operatorname{Res}_K Z$ for $K \in \mathscr{K}$ satisfy the Gap-hypothesis (5.2) by assumption. From (5.3) and (5.4) it follows that X and W_K satisfy the Gap-hypothesis. *Then X and W_K satisfy hypothesis Q whenever $Q = \hat{P}$ for $P \in \mathscr{P}$*. (Of course for W_K we need $Q \subset K$.) Note in particular that $Q = \hat{Q}$ whenever $Q \in \operatorname{Iso}(Y) \cap \mathscr{P}$.

(R2) Note that the assumption degree $f = 1$ (5.8i) implies degree f^P is a unit of R_P whenever $P \in \mathscr{P}$, $|P^0| = p^l$, p prime. See the remark following (4.5); moreover, when P is such a group and $P \subset K \in \mathscr{K}$, then degree F_K^P is a unit of R_p because $X^P = \partial W_K^P$.

(R3) If $H \subset L$, then $\chi(W_K^H) = \frac{1}{2}\chi(X^H)$ because $\dim X^H = \dim Y^H = \operatorname{Dim} Y(H)$ is even ((5.8ii) and (5.3)). This means $\chi(W_K^H) = \chi(Z^H)$ iff $\chi(X^H) = \chi(Y^H)$. We also note that $F_K^Q \equiv O(R_Q)$ (4.4) implies that $\chi(W_K^Q) = \chi(Z^Q)$ provided of course $Q \subset K$.

(R4) Theorems (4.14) and (4.15) require statements about $\tilde{K}_0(\Lambda)$ and $L_n(\Lambda)$ which are valid when these groups vanish. They vanish in these cases: $\tilde{K}_0(\Lambda) = 0$, $\Lambda = Z(\pi)$, $|\pi| = 1$ or 2; $L_n(\Lambda)$ vanishes if n is odd and $\Lambda = Z_{(2)}(H)$ for a 2 group H [Pa] or for $\Lambda = Z$.

There are several steps in the proof of (5.10). Each is an application of (4.14) or (4.15) to $\mathscr{W}_K$, Q, K and R with

$$Q = \hat{Q} \in \mathscr{P}, \qquad Q \subset K, \qquad R = R_Q.$$

(Note (5.8iv) which implies $\hat{P} \in \mathscr{P}$ whenever $P \in \mathscr{P}$.) The requirements for (4.14) and (4.15) then simplify. For example, X and W_K satisfy hypothesis Q as noted in R1. We restate these simplified conditions for reference in proof:

(4.14′) (i) $N(Q) \subset K$. (ii) $F_K^Q \approx O(R_Q)$ (this is (4.5′) below). (iii) If Q is connected, $\tilde{K}_0(Z(\bar{Q}^0)) = 0$; otherwise either (4.13i) or (4.13ii) is satisfied.

(4.15′) (i) $f^Q \equiv O(R_Q)$. (ii) $F_K^Q \approx O(R_Q)$ (this is (4.5′) below. (iii) If Q is connected, $\tilde{K}_0(Z(\bar{Q}^0)) = 0$; otherwise either (4.13i) or (4.13ii) is satisfied. (iv) $L_n(\Lambda) = 0$ when $n = \dim W^Q/\bar{Q}_K$ and $R_Q(\bar{Q}_K^0) = \Lambda$.

The condition (4.5) that $F^Q \approx O(R)$ for $F = F_K$ and $R = R_Q$ also simplifies in the presence of the assumption degree $f = 1$ (5.8). Then (4.5ii) is automatically satisfied by R2; *thus $F_K^Q \approx O(R_Q)$ is equivalent with*:

(4.5′) (i) $F_K^Q \sim 0$. (ii) $F_K^P \equiv O(R_P)$ for $P \in \mathscr{P}_{Q,K}$ (5.0).

Finally for reference we simply restate (4.13) which is required for (4.14′) and (4.15′). Since it is applied to $R = R_Q$, we must have $|Q^0| = q^s > 1$ for some prime q.

(4.13) $F_K^Q \approx O(R_Q)$, i.e., (4.5′) is satisfied and one of the following two conditions holds: (i) If $\bar{Q}_K$ is finite, then $\chi(W_K^Q) = \chi(Z^H)$ for all $Q \lhd K \subset H$ with H/Q cyclic $\neq 1$ of order prime to $|Q^0|$. (ii) If $\bar{Q}_K$ is O_2 (or $O_2' \subset S^3$), then $\chi(W_K^H) = \chi(Z^H)$ whenever $Q \lhd H \subset K$ and $H/Q \in (\bar{Q})_0$.

Each step in the proof of (5.10) involves an application of (4.14) or (4.15) to $\mathscr{W}_K^\cdot$ (K one of the two subgroups of $\mathscr{K}$) to achieve either $F_K^Q \equiv O(R_Q)$ or $\chi(W_K^Q) = \chi(Z^Q)$ for some Q and K. Each application produces a $K-G$ prenormal cobordism rel$\{G\}$ between $\mathscr{W}_K^\cdot$ and $\mathscr{W}_K'^\cdot$ to achieve the required condition. (We emphasize that the G fixed set of X is unchanged.) This produces a G prenormal cobordism $\mathscr{W}^\cdot$ between $\partial\mathscr{W}_K^\cdot$ and $\partial\mathscr{W}_K'^\cdot$ which is glued to $\mathscr{W}_L^\cdot$ (L the other group in $\mathscr{K}$) to produce $\mathscr{W}_L'^\cdot$ with $\partial\mathscr{W}_L'^\cdot = \partial\mathscr{W}_K'^\cdot$. This yields $\mathscr{W}_{\mathscr{K}}'^\cdot = \{\mathscr{W}_K'^\cdot, \mathscr{W}_L'^\cdot\}$. By construction $\mathscr{W}_K'^\cdot$ satisfies the required condition. The condition on $\mathscr{W}_L'^\cdot$ is implied by either that for K or by the preceding steps. For example, if Q is contained in both K and L, then $F_K^Q \approx O(R_Q)$ implies $f^Q \equiv O(R_Q)$ implies $\chi(X^Q) = \chi(Y^Q)$ implies $\chi(W_K^Q) = \chi(Z^Q)$. The first implication is a consequence of the definitions in (4.4) and Poincaré Duality. The second and third are consequences of R3. Since the cobordisms used to construct $\mathscr{W}_{\mathscr{K}}'^\cdot$ from $\mathscr{W}_{\mathscr{K}}^\cdot$ are rel$\{H \not\subset Q\}$, $\mathscr{W}_{\mathscr{K}}'^\cdot$ not only satisfies the condition specific to the particular step being discussed but also satisfies the conditions of the preceding steps. At this point the prime is dropped and $\mathscr{W}_{\mathscr{K}}^\cdot$ is assumed to have the properties of $\mathscr{W}_{\mathscr{K}}'^\cdot$.

In order to apply (4.14) or (4.15), we verify the relevant conditions (4.14′) and (4.15′) (and (4.5′)) at each step. *Remember this requires* $Q = \hat{Q}$ *and* $Q \in \mathscr{P}$ *or equivalently* $Q \in \mathrm{Iso}(Y)$ and $Q \in \mathscr{P}$. In steps 2–6 below Q has certain specific values. Each such Q must then be in Iso(Y) because $Q = \hat{Q}$. This accounts partially for (5.8iv).

We begin the proof of (5.10). *First we prove* (5.10) *for* $G = SO_3$. Let

$$\Sigma_0 = \{P \in O_2 \mid P \in \mathscr{P}, (P) \not\subset (D_2)\},$$

$$\mathscr{F}_0 = \{Q \in \mathscr{F} \mid (Q) \not\subset (D_2)\}.$$

($\mathscr{F}$ is defined just before (5.8).) Each $P \in \Sigma_0$ is unique up to conjugacy in O_2. Let $\Omega \subset \Sigma_0 \cup \mathscr{F}_0 = \Sigma$ be a closed set. We make this inductive assumption:

$A(\Omega)$: (i) For all $H \in \Omega$, $\chi(W_K^H) = \chi(Z^H)$ whenever $H, K \in \mathscr{K}$.

(ii) For all $P \in \Omega$ with $P \in \mathscr{P}$, $F_K^P \equiv O(R_P)$ for $K = O_2$.

Step 1. *Prove* $A(\Sigma)$. This is done inductively. By assumption $A(\Omega)$ holds. Let $Q \in \Sigma - \Omega$ be a maximal element in this set partially ordered by inclusion. We show how to achieve $A(\Omega')$ for $\Omega' = (\Omega \cup Q)^*$—the smallest closed set in Σ containing Ω and Q. There are two cases: (i) $Q \notin \mathscr{P}$ and (ii) $Q \in \mathscr{P}$. For case (i) use Lemma (5.9) to achieve $A(\Omega')$. In case (ii) there are two further cases (iia) $\hat{Q} \neq Q$ and (iib) $\hat{Q} = Q$.

Case (iia). Observe that $\hat{Q} \subset O_2 = K$. To see this note that $\hat{Q} \in \mathscr{P}$ by (5.8iv); so $\hat{Q}$ is conjugate to a subgroup of O_2 and this conjugate L of $\hat{Q}$ is an isotropy group because $\hat{Q}$ is. This conjugate L contains the corresponding conjugate of Q. Since subgroups of O_2 in Σ are unique up to conjugacy in O_2, L is conjugate in O_2 to an isotropy group L' containing Q. Then $L' = \hat{Q}$ by definition of $\hat{Q}$ (R0). Now observe $\hat{Q} > Q$ and $\hat{Q} \in \Sigma_0$. Since Ω is closed, $\hat{Q} \in \Omega$. Since $\hat{Q} \in \mathrm{Iso}(X)$ by definition of $\hat{Q}$ and $\mathrm{Res}_K X = W_K$, Q is an isotropy group of W_K (by the Equivariant Collar Neighborhood Theorem); so $W_K^{\hat{Q}} = W_K^Q$ and $F_K^{\hat{Q}} = F_K^Q$.

We assert that Q is a finite q group for some prime q and $\hat{Q}^0$ is also a q group possibly the trivial group. Grant this for the moment. Then R_Q is $Z_{(q)}$ and $R_{\hat{Q}}$ is either $Z_{(q)}$ or Z. In either case $F_K^{\hat{Q}} = F_K^Q \equiv O(R_{\hat{Q}})$ implies (α) $F_K^Q \equiv O(R_Q)$. Now we establish the assertion. Note $Q \in \mathscr{P}$ and $\hat{Q} \neq Q$ rules out $Q = S^1$ and $Q = O_2$ because these are isotropy groups of Y by (5.8). This means Q is cyclic of prime power order or dihedral of 2 power order. (These are the only other groups in $\mathscr{P}$ which are in O_2.) Since S^1 and O_2 are isotropy groups, $\hat{Q}$ is cyclic or S^1 in the first case while $\hat{Q}$ is dihedral or O_2 in the second. In either case this implies $\hat{Q}^0$ is a q group because $\hat{Q} > Q$ and $\hat{Q} \in \mathscr{P}$ by (5.8). Now suppose $Q \subset 0$. Then (β) $\chi(W_0^Q) = \frac{1}{2}\chi(X^Q) = \frac{1}{2}\chi(X^{\hat{Q}}) = \frac{1}{2}\chi(Y^{\hat{Q}}) = \chi(Z^Q)$. The first and fourth equalities follow from R3). The second follows from (R0) and the third follows from (R3) and $\chi(W_K^{\hat{Q}}) = \chi(Z^{\hat{Q}})$ because $\hat{Q} \in \Omega$ and $A(\Omega)$ holds. Put (α) and (β) together with $A(\Omega)$ to prove $A(\Omega')$. This completes case (iia).

Case (iib). $Q = \hat{Q}$, $Q \in \mathscr{P}$, $K = O_2$. We verify the requirements of (4.14′). (i) $N(Q) \subset K$ because the normalizer in SO_3 of any subgroup of O_2 in $\mathscr{P}$ other than D_2 lies in O_2 (Section 2). (ii) Use (4.8i) to achieve $F_K^Q \sim O$ (4.5′i). Then ($A(\Omega)$ii) implies (4.5′ii); so (4.5′) is satisfied, i.e., $F_K^Q \approx O(R_Q)$. (iii) If Q is connected, $Q = S^1$ and $\bar{Q}^0 = Z_2$; so $\tilde{K}_0(Z(\bar{Q}^0)) = 0$ by (R4). If Q is not conected, ($A(\Omega)$i) implies whichever of (4.13i) or (4.13ii) is relevant. The hypothesis of (4.14′) is satisfied; so we apply (4.14) to achieve $F_K^Q \equiv O(R_Q)$. If Q is also a subgroup of O, this implies $\chi(W_0^Q) = \chi(Z^Q)$ as in case (iia). As $A(\Omega)$ has not been disturbed in applying (4.14), we have achieved $A(\Omega')$. This completes case (iib).

Step 2. Achieve $F_K^Q \equiv O(R_Q)$ for $Q = D_4$, $K = 0$. We verify (4.15′). (i) $f^Q \equiv O(R_Q)$ because $F_{O_2}^Q \equiv O(R_Q)$ by ($A(\Sigma)$ii). (ii) Use (i) and (4.8ii) to achieve $F_K^Q \sim 0$ (4.5′i). Then (4.5′ii) is vacuously satisfied because $\mathscr{P}_{Q,K}$ is empty as D_4 is the 2 Sylow subgroup of O. This means (4.5′) is satisfied.

(iii) This is implied by $(A(\Sigma)\mathrm{i})$. (iv) $L_n(A)=0$: Since $|Q^0|=8$, $R_Q=Z_{(2)}$. Since $N_Q(D_4)=D_4$, $\bar{Q}_K=1$; so $A=Z_{(2)}$. Now observe that $n=\dim W_K^Q/\bar{Q}_K$ is odd as $\dim X^Q=\dim Y^Q=\mathrm{Dim}\, Y(Q)$ is even by (5.8). Then $L_n(A)=0$ is implied by (R4). This completes verification of (4.15′). Step 2 is now completed using (4.15).

Steps 3 and 5 proceed exactly like step 1 case (iib) while steps 4 and 6 proceeds exactly like step 2.

Step 3. Achieve $F_K^Q\equiv O(R_Q)$ for $Q=D_2$, $K=0$. Verify (4.14′). (i) $N(Q)\subset K$ as $N(Q)=0$. (ii) Use (4.8i) to achieve $F_K^Q\sim 0$ (4.5i). Note that $\mathscr{S}_{Q,K}$ (5.0) is $\{D_4\}$; so $F_K^P=O(R_P)$ for $P\in\mathscr{S}_{Q,K}$ (4.5′ii) by step 2. (iii) $(A(\Sigma)\mathrm{i})$ implies (4.13i). Having verified (4.14′), apply (4.14) to achieve what is required.

Step 4. Achieve $F_K^Q\equiv O(R_Q)$ for $Q=D_2$, $K=O_2$. Verify (4.15′). (i) $f^Q\equiv O(R_Q)$ because $F_0^Q\equiv O(R_Q)$ by step 3. (ii) Use (i) and (4.8ii) to achieve (4.5′i). Observe that $\mathscr{S}_{Q,K}=\{D_4\}$ because $N_{O_2}(D_2)=D_4$. Since $D_4\in\Sigma$, $(A(\Sigma)\mathrm{ii})$ implies (4.5′ii); so (4.5′) is satisfied. (iii) This is implied by $(A(\Sigma)\mathrm{ii})$. (iv) $L_n(A)=0$; $|Q^0|=4$; so R_Q is $Z_{(2)}$. Since $\bar{Q}_K$ is Z_2, $A=Z_{(2)}(Z_2)$. As in step 2(iv) n is odd. Then $L_n(A)=0$ is implied by (R4). Use (4.15) to achieve the required goal.

Step 5. Achieve $F_K^Q\equiv O(R_Q)$ for $Q=Z_2$, $K=O_2$. Here we take Z_2 to be the unique cyclic group of order 2 in $S^1\subset O_2$. We verify (4.14′). (i) $N(Q)=K$. This follows from our choice of Z_2 in K. (ii) Use (4.8i) to achieve (4.5i). If $P\in\mathscr{S}_{Q,K}$, then $P\in\Sigma_0$ or $P=D_2$; so (4.5ii) is implied by either $(A(\Sigma)\mathrm{ii})$ or by step 4. (iii) $(A(\Sigma)\mathrm{i})$ implies (4.13ii). Note $\bar{Q}_K=O_2$. Now apply (4.14) to achieve the required condition.

Step 6. *Final Step*: $f^Q\equiv O(R_Q)$ for $Q=1$; K is G. This is the final step because this implies by definition f is a pseudoequivalence as R_Q is Z. (See (4.4) and (5.0)). We apply (4.15) to $\mathscr{W}=(W, f, b, d)$. Care with notation is necessary at this point. In particular X and f play the role of W and F in (4.15) and X there is ∂W which is vacuous now. We verify (4.15′i). (i) This is automatic as $\partial X=\varnothing$. (ii) Use (i) and (4.8ii) to achieve $f^Q\sim 0$ (4.5′i). Note that each P in $\mathscr{S}_{Q,K}$ is conjugate to a subgroup in $\Sigma_0\cup\{D_2, Z_2\}$. But for P in this set, $F_{O_2}^P\equiv O(R_P)$ by preceding steps; so $f^P\equiv O(R_P)$ and this holds for any conjugate of P. Thus (4.5′) is satisfied and $f^Q\approx O(R_Q)$. (iii) As Q is 1 and K is G, $Q_K^0=1$ and $\tilde{K}_0(Z(Q_K^0))=0$ by (R4). (iv) $L_n(A)=0$: Note $n=\dim X/G$ is odd and A is Z. Apply $(R4)$. Thus (4.15′) is verified. Use (4.15) to achieve $f^Q\equiv O(R_Q)$.

Now we treat $G=S^3$. Observe that $\mathscr{W}_0^C$ is a $G/C=SO_3$ prenormal map which satisfies all the hypotheses of the case just completed; hence, we can suppose $\mathscr{W}_0^C$ is a pseudoequivalence. But then $f^P\equiv O(Z_{(p)})$ whenever

$G \supset P \in \mathscr{P}$, $|P^0| = p^n$, $P \neq 1$, because $\hat{P}_Y C \in \mathscr{P}$ whenever $P \in \mathscr{P}$. Now apply Theorem (4.15) to $\mathscr{W}_0$ with $Q = 1$. This completes the proof.

6. Proofs of the Main Results

One tool for constructing G prenormal maps is an equivariant transversality construction used in conjunction with the equivariant cohomology theory $\omega_G^*(\cdot)$. This we now explain. In this section G is SO_3 or S^3.

Let M be a representation of G and let Y be a smooth G manifold. Set $N = Y \times M$ and $\pi: N \to Y$ the projection on Y. The G vector bundle map (covering the point map q) which is defined by the composition $TN \to \pi^* N = \pi^* q^* M \to M$ is called s. Here M is viewed as a G vector bundle over a point. Let C be a G module. Suppose C is good (or $\mathscr{H}$ good) and $C \supset g$. The differential of a G map $\omega: N \to M$ is denoted by $D\omega: TN \to TM$ and $d\omega: TN \to M$ is $p_2 \circ D\omega$ where $p_2: TM = M \times M \to M$ is projection on the second factor. Set $TY > M$ if $sTY = s\mathbf{A}$ for some representation A of G and $\langle A, \chi \rangle = 0$ implies $\langle M, \chi \rangle = 0$ for each irreducible representation $\chi \neq 1$ of G.

Lemma (6.1). *Let $\omega: N \to M$ be a proper G map. Suppose $TY > M$,* $\mathrm{Iso}(N) = \mathrm{Iso}(Y)$, *$Y$ is C stable, $\Sigma \subset Y$ is a closed invariant set and ω is transverse to zero on $N|_\Sigma$ with $\lambda(d\omega, C) = \lambda(s, C)$ on $\lambda(TN|_\Sigma, C)$. Then ω is properly G homotopic* $\mathrm{rel}\, N|_\Sigma$ *to a map h transverse to zero with $\lambda(dh, C) = \lambda(s, C)$.*

The proof of this theorem is postponed until Section 7. Compare [P_3; D–P_1, Sect. 8]. In the meantime we show how to use it to construct prenormal maps which serve as input for (5.10). *Observe that ω in* (6.1) *gives rise to a class $[\omega]$ in $\omega_G^0(Y)$ which determines some of the properties of $X = h^{-1}(0)$ produced from ω by* (3.1). We make these blanket assumptions for the remainder of the section:

(6.2) Z and $\partial Z = Y$ are smooth G manifolds which satisfy: $\mathrm{Dim}\, Z$ and $\mathrm{Dim}\, Y$ (Section 5) are defined, both are C stable (3.6) for $C = g$, Y and $\mathrm{Res}_K Z$, $K \in \mathscr{H}$ (Section 5) satisfy the Gap-hypothesis (5.2), Y and Z satisfy (5.6) and

(a) $\mathrm{Iso}(Y) = \{H \in G \mid \mathrm{Dim}\, Y(L) \neq \mathrm{Dim}\, Y(H), \forall L > H\}$,

(b) $\mathrm{Iso}(\mathrm{Res}_K Z) = \{H \in K \mid \mathrm{Dim}\, Z'(L) \neq \mathrm{Dim}\, Z'(H), \forall L > H, L \subset K\}$, $Z' = \mathrm{Res}_K Z$.

Theorem (6.3). *Let Y satisfy* (6.2). *Suppose the hypotheses of* (6.1) *are satisfied (with $A = \varnothing$). Let $X = h^{-1}(0)$ and suppose $X^G \neq \varnothing$. Then* $\mathrm{Dim}\, X$ *is defined,* $\mathrm{Dim}\, X = \mathrm{Dim}\, Y$, $\mathrm{Iso}(X) = \mathrm{Iso}(Y)$, *and X is C stable.*

Proof. By transversality the G normal bundle of X in N is $X \times M$. Let $f: X \to Y$ be the composition $X \subset N \to^{\pi} Y$. Since $TN = \pi^*(TY \oplus Y \times M)$, $f^*TY \oplus X \times M = TN|_X = TX \oplus X \times M$; so $sTX \cong f^*sTY$. This implies $\dim X^H = \dim TX^H = \dim TY^H = \dim Y^H$ whenever $X^H \neq \varnothing$, but $X^G \neq \varnothing$ guarantees this. This shows $\operatorname{Dim} X = \operatorname{Dim} Y$. Let $H \in \operatorname{Iso}(Y)$. Then $\operatorname{Dim} Y(H) \neq \operatorname{Dim} Y(L)$ for $L > H$ (6.2); so $\operatorname{Dim} Y(L) + \dim G < \operatorname{Dim} Y(H)$ by (5.6). This inequality then holds if Y is replaced by X. By (5.7), $H \in \operatorname{Iso}(X)$; thus $\operatorname{Iso}(X) \supset \operatorname{Iso}(Y)$. As $X \subset N$, $\operatorname{Iso}(X) \subset \operatorname{Iso}(N) = \operatorname{Iso}(Y)$; so $\operatorname{Iso}(X) = \operatorname{Iso}(Y)$. Use this and $sTX \cong f^*sTY$ together with C stability for Y to conclude C stability for X.

THEOREM (6.4). *In addition to the hypothesis of* (6.3) *suppose* $\deg_1|\omega| = 1$ (*Section* 2). *Then there is a G prenormal map* $\tau(\omega) = (X, f, b, d) = \mathscr{W}_0$, $f: X \to Y$ *with* $d = \lambda(s, C)$ *and the conclusions of* (6.3) *hold for* X.

Proof. Let X be produced by G transversality as in (6.3). As noted in the proof of (6.3) there is a G vector bundle isomorphism $b: sTX \cong f^*sTY$. The λ isomorphism $d = \lambda(s, C)$ satisfies $s(d) = \lambda(b, C)$. For G finite and $C = g$ (so $\lambda(TX, g) = \pi TX = \nu X$) this was explained in [D–P$_1$, Sect. 8]. The procedure here is the same. The essential point is that $\lambda(dh, C) = \lambda(s, C)$ (6.1). It follows from (ordinary) transversality that degree $f = \deg_1|\omega| = 1$. The conditions (5.3) and (3.9) for $\tau(\omega) = (X, f, b, d)$ to be a G prenormal map are now verified using (6.3).

THEOREM (6.5). *Suppose Y and Z satisfy* (6.2) *and* $TZ > M$. *Let* $\omega: Y \times M \to M$ *be a proper G map which extends to a proper K map* $\omega_K: Z \times M \to M$ *for given* $K \subset G$. *Suppose* $\operatorname{Iso}(Y \times M) = \operatorname{Iso}(Y)$, $\operatorname{Iso}(\operatorname{Res}_K Z \times M) = \operatorname{Iso}(\operatorname{Res}_K Z)$, $\deg_1|\omega| = 1$ *and ω is transverse to* 0 *with* $X = \omega^{-1}(0)$, $X^G \neq \varnothing$. *Then there is a $G - K$ prenormal map* $\tau(\omega_K) = \mathscr{W}_K = (W_K, F_K, B_K, D_K)$, $F_K: W_K \to \operatorname{Res}_K Z$ *with* $\partial \mathscr{W}_K = \operatorname{Res}_K \mathscr{W}_0$, $\mathscr{W}_0 = \tau(\omega)$.

Proof. As ω is transverse to 0 as a G map, it is transverse as a K map. Use (6.1) with $\Sigma = Y$ and $N = Z \times M$ to construct a proper K homotopy rel $N|_\Sigma$ between ω_K and a map h_K which is transverse to 0. Set $\mathscr{W}_K = (W_K, F_K, B_K, D_K) = \tau(\omega_K)$. Apply (6.3) to $\operatorname{Res}_K Z$ to see W_K satisfies $\operatorname{Dim} W_K = \operatorname{Dim} \operatorname{Res}_K Z$, $\operatorname{Iso}(W_K) = \operatorname{Iso}(\operatorname{Res}_K Z)$, and W_K is $\operatorname{res}_K C$ stable. The conditions (5.3) and (3.9) required for $\mathscr{W}_K$ to be a K prenormal map are now verified using (6.3) and (6.4).

Manifolds Y and Z which satisfy (6.2) are $S(A \oplus \mathbb{R})$ and $D(A \oplus \mathbb{R})$ for certain representations A of G. We proceed to spell out the properties required for the representations.

LEMMA (6.6). *Let A be a complex representation of G satisfying* (5.6). *Then $K \in \operatorname{Iso}(A)$ iff for all $L > K$, $\operatorname{Dim} A(L) \neq \operatorname{Dim} A(K)$; moreover, $\operatorname{Iso}(A)$ is closed under intersection.*

Proof. The first statement is a consequence of (5.7) and the fact that A^K is connected for all K. To see the second note: $A \oplus A$ satisfies (5.6), $\operatorname{Iso}(A \oplus A) = \{H \cap K \mid H, K \in \operatorname{Iso}(A)\}$ and $\operatorname{Dim}(A \oplus A)(L) = 2 \operatorname{Dim} A(L)$ for all L; so $\operatorname{Dim}(A \oplus A)(L) \neq \operatorname{Dim}(A \oplus A)(K)$ iff $\operatorname{Dim} A(L) \neq \operatorname{Dim} A(K)$.

LEMMA (6.7). *Let A be a complex representation of G which satisfies* (5.6) *and* $\operatorname{Iso}(A) \supset \mathscr{S}$ (2.0). *Then each element of $\omega_G^0(Y)$, $Y = S(A \oplus \mathbb{R})$ (or $D(A \oplus \mathbb{R})$), can be realized as $|\omega|$ where ω: $Y \times M \to M$ for some $M \in \mathscr{S}_G$ with $\operatorname{Iso}(Y \times M) = \operatorname{Iso}(Y)$.*

Proof. Each element of $\omega_G^0(Y)$ is represented this way for some $M \in \mathscr{S}_G$; so $\operatorname{Iso}(M) \subset \mathscr{S}$. By (6.6), $\operatorname{Iso}(A) = \operatorname{Iso}(Y)$ is closed under intersections; so $\operatorname{Iso}(Y \times M) = \{H \cap K \mid K \in \operatorname{Iso}(Y), K \in \operatorname{Iso}(M)\} = \operatorname{Iso}(Y)$.

THEOREM (6.8). *Let A be a complex representation of G which satisfies* (5.6) *$A \supset T_4$ resp. S_8 for $G = SO_3$ resp. S^3 (Section* 1) *and* $\operatorname{Iso}(A) \supset \mathscr{S}$ (2.0). *Suppose $Y = S(A \oplus R)$ and $Z = D(A \oplus \mathbb{R})$ satisfy* (6.2) *and Y satisfies* (5.8ii–iv). *Let $x \in \omega_G^0(Y)$ with $\operatorname{res}_{\mathscr{F}} x = 1_{\mathscr{F}}$ and $\deg_G x_q \neq 0$ for some $q \in Y^G$. Suppose $\dim Y^G = 0$. Then there is a smooth G manifold X with $\operatorname{Dim} X = \operatorname{Dim} Y$, the cardinality of X^G is $\sum_{p \in Y^G} |\deg_G x_p|$ and there is a pseudoequivalence f: $X \to Y$.*

Proof. Apply (6.4) to Y and ω: $Y \times M \to M$ with $|\omega| = x \in \omega_G^0(Y)$ and $\operatorname{Iso}(Y \times M) = \operatorname{Iso}(Y)$ (6.7). (Note: $M \in \mathscr{S}_G$; $A \supset T_4$ resp. S_8 implies $TY > M$.) This produces $\tau(\omega) = \mathscr{W}_0 = (X, f, b, c)$. We must verify that $X^G = \sum_{p \in Y^G} |\deg_G x_p|$. If so $X^G \neq \varnothing$ as $\deg_G x_p \neq 0$ for some p by hypothesis. A look at the proof of (6.1) (Section 7) shows that the first step in making ω equivariantly transverse to zero is to make ω^G transverse to $0 \in M^G$. Here there is no group acting and this can be done in an arbitrary manner. Now $Y^G \times M^G = \coprod_{p \in Y^G} p \times M^G$. The restriction ω_p^G of ω^G to $p \times M^G$ has degree equal to $\deg_G x_p$ by definition. Make each ω_p^G transverse to 0 with $|\deg_G x_p|$ points in the inverse image of 0. Then ω^G is transverse to 0. Now complete the equivariant transversality construction on ω by producing h equivariantly homotopic (rel $Y^G \times M^G$) to ω and h transverse to zero. Since $X = h^{-1}(0)$ and $X^G = (h^G)^{-1}(0) = (\omega^G)^{-1}(0)$, the cardinality of X^G is $\sum_{p \in Y^G} |\deg_G x_p|$.

Now apply (6.5) to $Y = S(A \oplus \mathbb{R})$, $Z = D(A \oplus \mathbb{R})$ and ω: $Y \times M \to M$ with $|\omega| = x$ for each $K \in \mathscr{H}$. Note the assumption $\operatorname{res}_{\mathscr{F}} x = 1_{\mathscr{F}}$ means $\operatorname{res}_K x = 1_K \in \omega_K^0(Y)$ for each $K \in \mathscr{H}$; so ω does extend to ω_K: $Z \times H \to M$. Since $TZ > M$, (6.5) produces $\tau(\omega_K) = \mathscr{W}_K$ with the properties specified and

$\mathscr{H}'_0 = \partial\mathscr{H}'_{\mathscr{K}}$, where $\mathscr{H}'_{\mathscr{K}} = \{\mathscr{H}'_K \mid K \in \mathscr{K}\}$. Thus the hypothesis of (5.10) is satisfied. (Note for $G = S^3$, $\deg_C x = 1$ because $\operatorname{res}_{\mathscr{K}} x = 1_{\mathscr{K}}$. Thus degree $f^C = 1$. This is required in (5.8i).) By (5.10) we may suppose f is a pseudoequivalence (modulo a G prenormal cobordism rel$\{H \neq G\}$). This completes the proof.

We now describe a set $\mathscr{R}(G)$ of complex representations of G when $G = SO_3$ or S^3 which satisfy the hypothesis of (6.8). The set of their realifications is denoted by $\mathscr{R}_0(G)$.

(6.9) *For* $G = SO_3$, $A \in \mathscr{R}(G)$ iff A satisfies (5.6), $A^I = 0$, $A \oplus \mathbb{R}$ is stable, $A \supset T_4$ resp. S_8 for $G = SO_3$ resp. S^3 and:

(i) Whenever $Q < L \subset G$, $Q \in \mathscr{S}$ and $\dim_C A^L = \dim_C A^Q$, $2\dim_C A^L \leqslant \dim_C A^Q$. The inequality must be strict when Q is Z_l, l a power of 2, or (L, Q) is (D_l, Z_l), l any prime power. For S^3 the inequality must be strict when Q is $p^{-1}D_l$, l a power of 2.

(ii) $\dim_C A^H \geqslant 2$ for $H \neq G, I$ and $\dim_C A^P \geqslant 3$ for $P \in \mathscr{S}$.

(iii) $\hat{P}_A \in \mathscr{S}$ for all $P \in \mathscr{S}$ and $\operatorname{Iso}(A) \supset \mathscr{S}'$.

For $G = S^3$, $A \in \mathscr{R}(G)$ iff $A = B \oplus n\mathbb{H}$, where $\mathbb{H}$ is the quaternion field with standard action of S^3, $B \in \mathscr{R}(SO_3)$ and $4n > \dim_C B$.

Remarks (6.10). (a) If for all $H \in \operatorname{Iso}(A)$, $\langle A, 1\rangle$ is less than $\langle A, \chi\rangle$ for all $\chi \in I(H)$ with $\langle A, \chi\rangle \neq 0$, $\chi \in g$, $\chi \neq 1$, $3A \oplus \mathbb{R}$ is stable. (b) In addition $3A$ always satisfies (5.6). (c) It is left to the reader to check that if $A \in \mathscr{R}(G)$, then $Y = S(A \oplus R)$ and $Z = D(A \oplus \mathbb{R})$ satisfy (5.8), (5.10) and (6.2). The essential check (implied for $G = SO_3$ by (6.9i)) is the Gap-hypothesis (5.2) for Y and $\operatorname{Res}_K Z$ for $K \in \mathscr{K}$.

THEOREM (6.11). *Let G be SO_3 or S^3, $A \in \mathscr{R}_0(G)$ and $\alpha = \pm 1$. Then there is a smooth action of G on a closed homotopy sphere Σ such that Σ^G consists of $2 - \alpha$ points and the isotropy representation at each points is A.*

Proof. Let $Y = S(A \oplus \mathbb{R})$, $Z = D(A \oplus \mathbb{R})$ and $x \in \omega^0_G(Y)$ with $i^*(x) = (1, 1 - \alpha e)$ (2.7) and $\operatorname{Res}_{\mathscr{K}} X = 1_{\mathscr{K}}$. (Observe that A satisfies the hypothesis of (2.7) as $A^J = 0$.) Then Y, Z and x satisfy the hypothesis of (6.8). From (6.8) we obtain a homotopy sphere X with X^G having cardinality $1 + |1 - \alpha \deg_G e| = 2 - \alpha$ as $\deg_G e = 1$ by (2.2)–(2.5). Take $\Sigma = X$.

For (6.11) to be non-vacuous we need the following lemma whose proof is postponed until Section 8.

LEMMA (6.12). *$\mathscr{R}(G)$ and hence $\mathscr{R}_0(G)$ is non-empty.*

Proof of Theorems A–C. These are all corollaries of (6.11) and the fact that $\mathscr{R}_0(G) \neq \varnothing$. First note that (6.11) produces homotopy spheres Σ_i with Σ_i^G consisting of i points for $i = 1$ or 3 and the isotropy representation at each point is A. If M is any G manifold and $p \in M^G$ has isotropy representation A, then $M \# \Sigma_3 = M'$ has $M'^G = M^G - p \cup (p_1 \cup p_2)$ and the isotropy representation at p_1 and p_2 is A. Of course this connected sum is taken at the point of M whose isotropy representation is A. Repeat this process to complete the proof of Theorem C. Theorems A and B are immediate from Theorem C.

7. Proof of the Transversality Lemma

This section is devoted to the proof of the Transversality Lemma (6.1). Compare [P_3; D–P_3, Sect. 8]. We prove it under the assumption C is good. The proof in the more general situation where C is $\mathscr{H}$ good is a minor modification of this proof. We fix notation. Let C be a good representation of G with $g \subset C$. Abbreviate $\lambda(E, C)$ and $\lambda(b, C)$ by $\lambda(E)$ and $\lambda(b)$ when E is a G vector bundle and b is a G vector bundle map. View the representation M of G as a G vector bundle over $0 \in M$. Then $\lambda(M)$ and $\lambda'(M)$ are defined and $\lambda(M) = \{\lambda_\omega(M) \mid \omega \in \Pi(0)\}$. If N is any G manifold and $\alpha \in \Pi(N)$, set $\lambda_\alpha(M) = \lambda_\omega(M)$, where $\omega \in \Pi(0)$ is the unique component with $\rho(\omega) = \rho(\alpha)$, and set $M_\alpha = M^{\rho(\alpha)}$. The tangent space of M TM is $M \times M$ and $y \times M \subset M \times M$ is regarded as the tangent space $T_y M$ of M at y. The differential $Df\colon TN \to TM$ of a G map $f\colon N \to M$ is a G bundle map. Set $df = p_2 \circ Df\colon TN \to M$, where $p_2\colon M \times M \to M$ is projection on the second factor. Then f is transverse to $0 \in M$ iff for each $x \in f^{-1}(0)$, $df_x\colon T_x N \to M$ is surjective. For $\alpha \in \Pi(N)$ and $x \in N_\alpha$

$$df_x = df_{\alpha x} \oplus \lambda_\alpha(df_x) \oplus \lambda'_\alpha(df_x), \tag{7.1}$$

where $f_\alpha\colon N_\alpha \to M_\alpha$ is the restriction of f to N_α. This means df_x is surjective iff each term in (7.1) is surjective. We emphasize $df\colon TN \to M$ is a G bundle map covering the point map of N to 0; so $df_\alpha\colon TN_\alpha \to M_\alpha$, $\lambda_\alpha(df)\colon \lambda_\alpha(TN) \to \lambda_\alpha(M)$ and $\lambda'_\alpha(df)\colon \lambda'_\alpha(TN) \to \lambda'_\alpha(M)$.

Let N be a smooth G manifold provided with a G invariant inner product on its tangent space. For $x \in N$, let g_x be the tangent space to the orbit $Gx \subset M$; so $g_x \subset T_x N$. Set

$$\tilde{g} = \bigcup_{x \in N} g_x,$$

$$TN - \tilde{g} = \bigcup_{x \in N} (T_x N - g_x).$$

Here $T_xN - g_x$ is the orthogonal complement of g_x in T_xN. These are not G vector bundles as the dimension of g_x varies with x. With care they can be treated as G vector bundles. In particular the definition of $\lambda(E)$ and $\lambda'(E)$ for E either $\tilde{g}$ or $TN - \tilde{g}$ is formally the same as if E were a G vector bundle. We observe that $\lambda(\tilde{g}, C) = 0$. This follows from the fact that g_x is a sub G_x representation of g and $g \subset C$. Set

$$\lambda''(TN) = \lambda'(TN - \tilde{g}) \subset \lambda'(TN)$$

and denote the restriction of $\lambda'(df)$ to $\lambda''(TN)$ by $\lambda''(df)$. Its target remains $\lambda'(M)$. Now observe this *key point*. If $x \in N_\alpha$ and $f(x) = 0$, then $\lambda'_\alpha(g_x) \in \operatorname{Ker}(\lambda'_\alpha(df_x))$; so if $\lambda'(df_x)$ is surjective, then $\lambda''_\alpha(df_x)$ is surjective and conversely.

Here is one special situation where an equivariant transversality result follows immediately from the classical case where there is no group acting. Its proof is left to the reader.

LEMMA (7.2). *Let a group L act freely on a manifold S, $f: S \to T$ be a proper L map, $T' \subset T$ be an invariant L submanifold and $A \subset S$ be a closed invariant set such that f is transverse to T' on A. There is a proper L homotopy of f* rel *A to a map transverse to T'.*

Let W and Z be L/H spaces with L/H acting freely on W. Let E be an L bundle over W, E' an L bundle over Z, $h: W \to Z$ an L/H map and C an L representation. Set

$$\mu = \min\{d_\chi(\langle\chi, E_w\rangle - \langle\chi, E'_{h(w)}\rangle)\},$$

$$\chi \in I(H), \qquad \chi \in C, \qquad \chi \neq 1, \qquad \langle\chi, E_w\rangle \neq 0, \qquad w \in W.$$

LEMMA (7.3). *Suppose* $\dim W \leqslant \mu$, *$D \subset W$ is an L/H invariant subspace and $t: A'(E, C)|_D \to A'(F, C)|_D$ is a surjective L bundle map covering $h|_D$. Then t extends to a surjective L bundle map covering h. Here $F = h^*E'$.*

Proof. Sought is a $K = L/H$ section s in the space Γ of surjective H bundle maps from $A'(E, C)$ to $A'(F, C)$ which extends the section t of $\Gamma|_D$. Observe that Γ is the total space of a K fiber bundle over W whose fiber at $x \in W$ is Ω/Ω', where Ω is $\operatorname{Aut}_H(A'(E_x, C))$ and Ω' is $\operatorname{Aut}_H(\operatorname{Ker} \tau_x, C))$. By (1.0) and (1.1), $\pi_i(\Omega/\Omega')$ is zero if $i \leqslant \mu$. Now K sections of Γ extending t are in 1–1 correspondence with sections of the fiber bundle $\Gamma/K \to W/K$ which extend t/K. The fiber of this bundle is again Ω/Ω' as K acts freely on W. The existence of s now follows from obstruction theory.

Proof of (6.1). We refer to (6.1) for notation and hypothesis. We replace ω there by f. Set

$$\pi(N) = \{\alpha \in \Pi(N) \mid G_x = \rho(\alpha) \text{ for some } x \in N_\alpha\}.$$

Let $\theta \subset \pi(N)$ be a G invariant subset with the property that if $\beta \in \theta$ and $\gamma \leqslant \beta$ then $\gamma \in \theta$. Let $\alpha \in \pi(N) - \theta$ be a minimal element.

(7.4) Inductive hypothesis: U is an open invariant set in N containing $N|_\Sigma$ and N_β for $\beta \in \theta$ such that:

(a) $\lambda(df) = \lambda(s)$ on U,

(b) f is transverse to 0 on U.

In addition suppose

(c) f_α is transverse to $0 \subset M_\alpha$.

(7.5) Set $X = f^{-1}(0)$ and $H = \rho(\alpha)$.

View the G_α normal bundle ν_α of N_α in N as a G invariant subspace of N using the Equivariant Tubular Neighborhood Theorem [4]. Let B_0 and B_1 be closed $\underline{G_\alpha}$ invariant subsets of N_α with $N_\alpha \cap \overline{N|_\Sigma \cup \bigcup_{\beta \in \Sigma} N_\beta} \subset B_0 \subset \overline{N_\alpha - B_1} = B \subset U$. Here $\bar{S}$ denotes the closure of S. Set $F = \overline{N_\alpha - B_0}$ and $\nu' = \nu_\alpha - \tilde{g}|_F$. Observe that $\tilde{g}|_F$ and hence ν' is a G_α vector bundle over F. Let (D, S) resp. (D', S') be the unit disk; unit sphere bundle of ν_α resp. ν' and D_ε the vectors in D of norm not exceeding ε.

Choose ε so small that $D_{\varepsilon|B} \subset U$ and $0 < \varepsilon < 1$. Choose a G_α invariant function $\phi: D \to [0, 1]$ so that $\phi = 1$ on $D|_{B_0} \cup S$ and $\phi = \gamma \circ \rho_\alpha$ on D_ε [Wass]. Here $\rho_\alpha : \nu_\alpha \to N_\alpha$ is bundle projection and $\gamma: N_\alpha \to [0, 1]$ is a G_α map with $\gamma = 1$ on B_0 and $\gamma = 0$ on B_1.

Since f is transverse to 0 on B, df_x is surjective for $x \in B \cap X_\alpha$. This means each factor in (7.1) is surjective and this implies $\lambda''_\alpha(df_x)$ is surjective for $x \in B \cap X_\alpha$. Let $B_2 = B \cap F$ and $t_\alpha : \lambda''_\alpha(TN)|_F \to \lambda'_\alpha(M)$ extend $\lambda''_\alpha(df)|_{B_2 \cap X_\alpha}$ with $t_\alpha|_{F \cap X_\alpha}$ surjective. First produce $t_\alpha|_{F \cap X_\alpha}$ using Lemma (7.3). The extension of this to F is always possible. See [1; 1.4.1]. We apply (7.3) with $W = X_\alpha \cap F$, $D = X_\alpha \cap B_2$, $Z = 0$, h the point map, $L = G_\alpha$, $H = \rho(\alpha)$, $E = (TN - \tilde{g})|_W$ and $E' = M$ viewed as an L vector bundle over 0.

We verify the hypothesis of (7.3). Since $\alpha \in \pi(N)$, there is a point $x \in N_\alpha$ with $G_x = H$. If y is any point of N_α, its isotropy group contains H. If it strictly contains H, it is contained in U by definition of U. This means $L/H = K$ acts freely on F and so on $F \cap X_\alpha$. Since f_α is transverse to $0 \subset M_\alpha$ with $X_\alpha = f_\alpha^{-1}(0)$ and since $N = Y \times M$, it follows that $\dim X_\alpha = \dim Y^H$. By hypothesis $sTY = \mathrm{A}$ for some representation A of G. Let $\chi \in I(H)$, $\chi \in C$, $\chi \neq 1$ and $\langle T_pN - \tilde{g}_p, \chi \rangle \neq 0$ for some $p \in X_\alpha \cap F \subset N$. Since $T_pN = T_yT \oplus M$, $\pi(p) = y$ and $TY > M$, it follows that $\langle A, \chi \rangle \neq 0$. Since $H \in \mathrm{Iso}(N) = \mathrm{Iso}(Y)$ and Y is C stable ((3.6) and (3.7)),

$$\dim Y^H = \langle A, 1 \rangle \leqslant \langle A, \chi \rangle - \langle g, \chi \rangle; \tag{7.6}$$

so

$$\dim W = \dim X_\alpha = \dim Y^H \leqslant \langle A, \chi\rangle - \langle g, \chi\rangle \leqslant \langle T_pN - \tilde{g}_p, \chi\rangle - \langle M, \chi\rangle.$$

This means $\dim W \leqslant \mu$ (7.3) and the conditions of (7.3) are verified.

Note that v' is $\lambda_\alpha(TN) \oplus \lambda''_\alpha(TN)$ restricted to F. Define a G_α map $L: D' \to M = M_\alpha \oplus \lambda_\alpha(M) \oplus \lambda'_\alpha(M)$ by

$$L = (f_\alpha \circ \rho_\alpha, \lambda_\alpha(s) \oplus t_\alpha).$$

Define $h: D' \to M$ by

$$h = \phi f + (1-\phi) L.$$

$$dh = \phi\, df + (1-\phi)\, dL + \Delta, \tag{7.7}$$

where in terms of local coordinates

$$\Delta = ((f_j - L_j)\, \partial\phi/\partial x_i).$$

Assertions. $\lambda_\delta(dh_x) = \lambda_\delta(s_x)$ for $\delta \in \pi(N)$ and $x \in D'_\varepsilon \cap N_\delta$, $\lambda'_\alpha(dh_x) = t_{\alpha x}$ whenever $x \in X_\alpha \cap F$, $h = f$ on $D'|_{F \cap B_0} \cup S'|_F$ and $f_\alpha = h_\alpha$. These are evident from the definitions and this observation: Whenever $x \in D'_\varepsilon \cap N_\delta$, $\lambda_\delta(\Delta_x)$ and $\lambda'_\delta(\Delta_x)$ are zero ($\phi = \gamma \circ \rho_\alpha$ there) and $\lambda_\delta(dL_x) = \lambda_\delta(s_x)$.

For points $x \in F \cap X_\alpha$, dh_x is $df_{\alpha x} \oplus \lambda_\alpha(s_x) \oplus t_{\alpha x}$. Each map is surjective; so h is transverse to zero on $F \cap X_\alpha$. Let f' be f on $N - G \times_{G_\alpha} D'$ and be the unique G extension of h to $G \times_{G_\alpha} D'$. Then f' is transverse to zero on a neighborhood V of N_α. Let U' be $U \cup (V \cap G \times_{G_\alpha} D')$. Then: (1) U' contains N_β for $\beta \in \theta \cup G \cdot \alpha = \theta'$. (2) $\lambda(df') = \lambda(s)$ on U'. (3) f' is transverse to 0 on U'. Let $\delta \in \pi(N) - \theta'$ be a minimal element. Then f'_δ is transverse to $0 \in M_\delta$ on $N_\delta \cap U'$. Replace U' if necessary by a smaller G invariant set again called U' so these properties hold for $\bar{U}'$ the cosure of U'. Use (7.2) and the equivariant homotopy extension theorem to produce a proper G homotopy of f' rel $\bar{U}'$ to a map f'' such that f''_δ is transverse to $0 \in M_\delta$. Note that $G_\delta/\rho(\delta)$ acts freely on $N_\delta - \bar{R}$, where R is an open invariant set satisfying (1)–(3) and $\bar{R} \subset U$. (7.2) is aplied to this space and f_δ restricted to it.

This provides the inductive step for the proof of (6.1). It also takes care of the initial step in the induction where $\theta = \varnothing$ and $U = \varnothing$. The induction is completed when $\theta = \pi(N)$.

8. Representations of Subgroups of SO_3 and S^3

The aim of this section is to prove g is good when $G = SO_3$ (1.7) and to show $\mathscr{R}(G)$ non–empty (6.12). These are statements which involve information about representations of G and its subgroups. The proofs of (1.7) and (6.12) involve easy computations from representation theory once a few specific facts about the representations of subgroups of SO_3 are collected. The principal computations in (1.7) and (6.12) are involved with determining which $\chi \in I(H)$ are contained in g for each subgroup $H \leqslant G$ and in computing $\langle A, \chi \rangle$ for a representation A of G when $\chi \in I(H)$ and $\chi \in g$ or $\chi = 1$. In particular the first is equivalent to describing $\mathrm{Res}_H\, g$ for each $H \subset G$, so our first task is to do this.

In order to avoid confusion, we write $\langle\ \rangle_H$ for the inner product defined on representations of H. When necessary to distinguish between real and complex inner products, we use $\langle\ \rangle'$ to denote the latter.

For each integer $k \neq 0$, n_k denotes the real two dimensional representation of O_2 defined as follows: View $\mathbb{R}^2$ as the complex numbers $\mathbb{C}$. For $t \in S^1 \subset O_2$ and $z \in \mathbb{C}$, set $t \circ z = t^k \cdot z$ and $\tau \circ\ = \bar{z}$ the complex conjugate of z. Since S^1 and τ generate O_2 this defines the representation. Then n_k is irreducible, n_k is equivalent to n_{-k} and is the unique representation of O_2 whose restriction to S^1 is t^k. Let d denote the real one dimensional representation of O_2 with $tx = x$ for $t \in S^1$ and $x \in \mathbb{R}$ while $\tau x = -x$.

Using the definition of T_k (Section 1) as the set of complex polynomials in the coordinates of $\mathbb{H}$ of degree $2k$, it is easy to verify that

$$\mathrm{Res}_{O_2} T_s = (N_s \oplus N_{s-1} \oplus \cdots \oplus N_1 \oplus d^S \otimes \mathbb{C}), \tag{8.1}$$

where $N_l = n_l \otimes \mathbb{C}$. In fact $N_l \subset T_s$ is the O_2 invariant subspace generated by $z^{2s-l}w^l$ and $z^l w^{2s-l}$ if the complex coordinates of $\mathbb{H}$ are z and w while $d^s \otimes \mathbb{C}$ is generated by $z^s w^s$. It follows that

$$\begin{aligned} \mathrm{Res}_{O_2} t_s &= 2(n_s \oplus n_{s-1} \oplus \cdots \oplus n_1 \oplus d^s), \\ \mathrm{Res}_{O_2}\, g &= n_1 \oplus d. \end{aligned} \tag{8.2}$$

Note $T_1 = g \otimes \mathbb{C}$ (Section 1); so $t_1 = 2g$.

One easily checks that $\mathrm{Res}_H n_r = \mathrm{Res}_H n_s$ when $H = D_k$ iff $r \equiv \pm s \bmod k$ and $< n_r, 1\rangle_H \neq 0$ iff $k \mid r$ and then $\mathrm{Res}_H n_r = \mathrm{Res}_H (1 \oplus d)$. Let C_{sk} be the number of integers l, $1 \leqslant l \leqslant s$, which are $\pm 1 \bmod k$ and let D_{sk} be the number of these l which are $0 \bmod k$. Then

$$\begin{aligned} \langle t_s, n_1 \rangle_H &= 2 \cdot C_{sk}, \\ \langle t_s, d \rangle_H &= 2D_{sk} + \begin{cases} 2, & s \equiv 1(2), \\ 0, & s \equiv 0(2), \end{cases} \\ \langle t_s, 1 \rangle_H &= 2D_{sk} + \begin{cases} 0, & s \equiv 1(2), \\ 2, & s \equiv 0(2). \end{cases} \end{aligned} \tag{8.3}$$

We remark that (8.3) is a consequence of (8.2), the fact that the restrictions to H of d and 1 are irreducible, the restriction to H of n_1 is irreducible if $k \neq 2$ and the interpretation of $\langle t_s, \chi \rangle$ as the multiplicity of χ in t_s when χ is irreducible.

For the cyclic group $K = Z_k$, the numbers $\langle t_s, \chi \rangle_K$ for $\chi \in I(K)$, $\chi \in g$ can be determined from either (8.4) or preferably from the character formula (1.5) and (1.6). Unless $K = 1$, there are two $\chi \in I(K)$ with $\chi \in g$. One is $\chi = 1$. The other is called ψ. We find

$$(8.4) \qquad \begin{gathered} \langle t_s, \psi \rangle = 2C_{sk} k \neq 2 \qquad \text{and} \qquad 4C_{s2} k = 2, \\ \langle t_s, 1 \rangle = 4D_{sk} + 2. \end{gathered}$$

With the aid of a character table for $I = A_5$, $O = S_4$ and $T = A_4$, the fact that each element of G is conjugate to an element of $S^1 \subset G$ and the character formula for T_s (1.5), the following formulas are verified:

(8.5) $\langle T_\lambda, T_\omega \rangle'_H$ for $H = O, T$ is

$$\begin{aligned} &\tfrac{1}{24}\{(2\lambda + 1)(2\omega + 1) + 9(-1)^{\lambda+\omega} + 6a_\lambda a_\omega\}, \qquad && H = O \\ &\tfrac{1}{12}\{(2\lambda + 1)(2\omega + 1) + 3(-1)^{\lambda+\omega} + 8b_\lambda b_\omega\}, \qquad && H = T. \end{aligned}$$

Here a_λ is 1 for λ congruent to 0 or 1 mod 4 and is -1 if λ is 2 or 3 mod 4; b_λ is 0, 1, -1 as λ is 1, 0, -1 mod 3.

(8.6) $\langle T_\lambda, T_0 \rangle'$ is

$$\tfrac{1}{60}\{2\lambda + 1 + (-1)^\lambda\, 15 + 20b_\lambda + 12c_\lambda\}.$$

Here c_λ is 2, 1, 0, -1, -2 as λ is 0, 1, 2, 3, 4 mod 5. Observe that $T_0 = 1 \otimes \mathbb{C}$.

LEMMA (8.7). *$\mathrm{Res}_H\, g$ is irreducible for $H = 0, T, I$.*

Proof. $g \otimes \mathbb{C} = T_1$ (Section 1). So $\mathrm{Res}_H\, g$ is real irreducible if $\mathrm{Res}_H T_1$ is complex irreducible. It suffices to take $H = T$ because T is a common subgroup. The result follows from $\langle T_1, T_1 \rangle'_T = 1$ by (8.5).

Proof of (1.7). In the case $G = SO_3$, we must verify either (1.2a) or (1.2b) where $C = g$ for each subgroup H of G. For $G = S^3$ we need to verify this unless H is cyclic of order 4, 2 or 1. First SO_3. If $NH = H$ or $\mathrm{Res}_H\, g$ is irreducible, there is nothing to show. This occurs for $H = I$, O, O_2 and T. If H is not one of these, it is a subgroup of O_2. Now $\mathrm{Res}_{O_2}\, g = n_1 \oplus d$ and $\mathrm{Res}_H d$ is irreducible for all H while $\mathrm{Res}_H n_1$ is irreducible unless H is D_2, Z_2 or 1. So if H is not one of these groups and χ is one of $\mathrm{Res}_H\, n_1$ or $\mathrm{Res}_H\, d$, then $\tilde{\chi}$ is $\mathrm{Res}_{NH}\, \psi$ for $\psi = n_1$ or d. Note in these cases $NH \subset O_2$ (Section 2).

For $H = D_2$, $\mathrm{Res}_H\, g$ contains the three non-trivial representations of D_2 different from 1; so $\chi \notin g$ implies $\chi = 1$ and this lifts to NH. For $H = Z_2$, $\mathrm{Res}_H\, g$ contains both irreducible representations of H; so there is nothing to show.

Now take $G = S^3$. For any subgroup H of G, $NH = p^{-1}NpH$. For $H \neq p^{-1}K$, where $K = D_2$ or Z_2, (1.2a) for K implies (1.2a) for H. Note $p^{-1}D_2 = Q_8 = H$ is the quaternion group of order 8 and $I(H) = \mathrm{Res}_H \mathbb{H} \cup p^*I(D_2)$. Since 1 and $\mathrm{Res}_H \mathbb{H}$ lift to G, the condition (1.2b) is verified for $H = Q_8$.

Proof of (6.12). It suffices to prove $\mathscr{R}(G) \neq \varnothing$ for $G = SO_3$. Let $A = 3(T_3 \oplus T_4) \oplus 6T_7$. We claim $3A \in \mathscr{R}(G)$, so $\mathscr{R}_0(G) \neq 0$. The conditions (6.9) must be verified for $3A$. This requires the determination of $\dim_{\mathbb{C}} A^L$ for all $L \subset G$. For some L these are listed in these two tables:

L:	G	O_2	S^1	O	T
$\dim_{\mathbb{C}} A^L$:	0	3	12	3	12

L:	D_7	D_6	D_5	D_4	D_3	D_2	Z_7	Z_6	Z_5	Z_4	Z_3	Z_2
$\dim_{\mathbb{C}} A^L$:	9	9	9	12	21	30	24	24	24	30	48	66

The dimensions of fixed point sets of the groups not in this table can be determined from the table via: $A^I = A^G$, $A^{D_l} = A^{O_2}$ and $A^{Z_l} = A^{S^1}$ for $l > 7$. The tables and subsequence equalities use (8.2)–(8.6) and $\dim_{\mathbb{C}} A^L = \langle A, 1 \otimes \mathbb{C}\rangle'_L = \frac{1}{2}\langle A, 1\rangle_L$. By inspection (6.9i–ii) and $A^I = 0$ are now verified for A and hence $3A$. In view of (5.7) and (6.10), the conjugacy classes of isotropy groups of $3A$ are those listed in the table; thus (6.9iii) holds by inspection of these tables. Note (2.0) that $\mathscr{L} = \mathrm{Iso}(3t_4)$. Since $3T_4 \subset A$, $\mathscr{L} \subset \mathrm{Iso}(3A)$.

To verify $3A \oplus 1$ is stable, we use (6.10a). For the subgroups $H \in \mathrm{Iso}(A)$ and $H \subset O_2$, (6.10a) is verified by using (8.2)–(8.4). For $H = O$ or T, use (8.5) and these hints: $\mathrm{Res}_H\, g$ is irreducible. Thus the condition for $H = O$ or T in (6.10a) is $\langle A, 1\rangle_H < \langle A, g\rangle_H$. We verify this inequality

$$\langle A, g\rangle_H \geqslant 2\langle A, g \otimes \mathbb{C}\rangle'_H = 2\langle A, T_1\rangle'_H \geqslant 2\langle A, T_0\rangle'_H = \langle A, 1\rangle_H.$$

The next to last inequality requires (8.5). Condition (6.10a) is vacuous for $H = G$ because g is irreducible and $\langle A, g\rangle = 0$.

Finally note that (6.10b) implies condition $3A$ satisfies (5.6).

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