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Note

On the complexity of deciding bimatrix games similarity

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ABSTRACT

In this paper, we show that it is NP-complete to decide whether two bimatrix games share a common Nash equilibrium. Furthermore, it is co-NP-hard to decide whether two bimatrix games have exactly the same set of Nash equilibria.

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1. Introduction

Game theory is a mathematical model to study and predict behaviors of rational decision makers. A noncooperative finite strategic form game [1] is guaranteed to have a Nash equilibrium. However, Nash's proof of existence is nonconstructive. Since the 1960's, mathematicians and computer scientists have been studying computational aspects of finding Nash equilibria. On the algorithmic side, people focus on bimatrix games, which are the simplest cases of normal form games. It is well known [2] that Nash equilibria of a zero-sum bimatrix game can be solved by linear programming. Lemke and Howson [3] designed an algorithm, named after them as the LH algorithm, to compute Nash equilibria of bimatrix games based on a special pivoting method. However, it is showed [4] that the LH algorithm may need to take exponential steps to find a Nash equilibrium in the worst case. On the complexity side, Papadimitriou [5] first showed that computing a Nash equilibrium is in PPAD. In 2005, there was a big breakthrough in this area. Daskalakis, Goldberg and Papadimitriou [7] showed that computing a Nash equilibrium is indeed PPAD-complete for 4-player games. Chen and Deng [8] later showed that computing a Nash equilibrium even for 2-player games is also PPAD-complete, which is quite surprising. Furthermore, computing an approximate Nash equilibrium in the order of $\frac{1}{n^{\Theta(1)}}$ [6] still remains to be PPAD-complete. Besides the set of results concerning about computing an arbitrary Nash equilibrium, people also studied complexities of determining certain properties of Nash equilibria. Conitzer et al. [9] and Gilboa et al. [11] showed that it is NP-hard to decide whether a game has more than one Nash equilibrium as well as decide whether there is a Nash equilibrium that gives players a certain payoff. Codenotti et al. [10] further extended these results to $(0, 1)$ bimatrix games. Recently, people begin to study the relationship between two different games. Gabarró et al. [12] defined *Game Isomorphism* between two games, which preserves certain properties such as payoffs, preferences, partial structures etc. of Nash equilibria of two games under permutations. They showed that it is in general computational hard to decide whether two games are isomorphism or not.

In this paper, we study the relationship between two games in a similar spirit of [12]. In particular, we would like to address a specific question: whether two games share a common Nash equilibrium, which is called *weakly similar*, or whether they have exactly the same set of Nash equilibria, which is called *strongly similar*. We will study the computational aspects of these two questions. It is motivated from several folds. First, we can consider the set of Nash equilibria as the spectrum of a game, which is analogous to the set of eigenvalues of a matrix. Two matrices that share the same set of eigenvalues are

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(strongly) similar. In game theory, it would be natural and mathematically interesting to develop such a *similarity* concept. Second, when we want to compute a Nash equilibrium of a game that could be computational hard, we may transform it to a relatively easier game that shares some of the Nash equilibria with the original game. Thus, it is quite natural to ask the general similarity question as we propose. However, we do not consider permutations in our definition of similarity, which is different from [12]. W.L.O.G, we will focus on bimatrix games in this paper. We show that it is NP-complete to decide whether two bimatrix games share a common Nash Equilibrium while it is co-NP-hard to decide whether two bimatrix games have exactly the same set of Nash equilibria.

2. Preliminaries

Bimatrix games are the simplest cases of normal form games. In a bimatrix game, there are two players called the *row player* and the *column player*, respectively. Let $\Sigma_R = \{r_1, \ldots, r_m\}$ be the pure strategy space of the row player while $\Sigma_R = \{c_1, \ldots, c_n\}$ be the pure strategy space of the column player. We can use two matrices A and B to represent the payoffs to the row player and the column player when they play different combinations of pure strategies. Specifically, when the row player plays r_i and the column player plays c_j, the payoff to the row player is $A(r_i, c_j)$ while the payoff to the column player is $B(r_i, c_j)$. A strategy profile is a probability distribution on the strategy space. Next we would like to define Nash equilibrium following the definition in [6]:

Definition 1. Given a bimatrix game (A, B), a strategy profile (x^*, y^*) is called Nash equilibrium (NE) if $\forall x'$, $x^{*\mathrm{T}}Ay^* \geq x'^{\mathrm{T}}Ay^*$ and $\forall y'$, $x^{*\mathrm{T}}By^* \geq x^{*\mathrm{T}}By'$.

Conitzer et al. [9] showed a bunch of complexity results based on a single reduction from the SAT problem to Nash equilibria of a bimatrix game. We will make use of their reduction in our paper. Therefore, we formally state their construction in the following. Let ϕ be a boolean formula, V be its variables, $L = \{\pm v : v \in V\}$ be the set of corresponding literals, and C be the set of clauses. Contitzer et al. [9] constructed a symmetric bimatrix game $G(\phi)$ from ϕ such that the strategy space Σ of $G(\phi)$ is $L \bigcup V \bigcup C \bigcup \{f\}$ where f is a special pure strategy. The payoff matrices A and B are defined to be:

(i) $A(l^1, l^2) = B(l^2, l^1) = 1$ for all $l^1, l^2 \in L$ with $l^1 \neq -l^2$;
(ii) $A(l, -l) = B(-l, l) = -2$ for all $l \in L$;
(iii) $A(l, x) = B(x, l) = -2$ for all $l \in L, x \in \Sigma - L$;
(iv) $A(v, l) = B(l, v) = 2$ for all $v \in V, l \in L$ with $v(l) \neq v$;
(v) $A(v, l) = B(l, v) = 2 - n$ for all $v \in V, l \in L$ with $v(l) = v$;
(vi) $A(v, x) = B(x, v) = -2$ for all $v \in V, x \in \Sigma - L$;
(vii) $A(c, l) = B(l, c) = 2$ for all $c \in C, l \in L$ with $l \notin c$;
(viii) $A(c, l) = B(l, c) = 2 - n$ for all $c \in C, l \in L$ with $l \in c$;
(ix) $A(c, x) = B(x, c) = -2$ for all $c \in C, x \in \Sigma - L$;
(x) $A(f, f) = B(f, f) = 0$;
(xi) $A(f, x) = B(x, f) = 1$ for all $x \in \Sigma - \{f\}$.

Based on the above construction, Contitzer et al. showed the following nice property of G:

Theorem 1 ([9]). *If $(l_1, l_2, \ldots, l_n)$ satisfies ϕ, then there is a Nash equilibrium of $G(\phi)$ where both players play l_i with probability $\frac{1}{n}$. The only other Nash equilibrium is the one where both players play f.*

Basically the theorem says that there is a one-to-one correspondence between mixed strategy NEs of $G(\phi)$ and satisfying assignments of ϕ. The next corollary, although it is not explicitly stated in [9], follows trivially from Theorem 1.

Corollary 1. *It is NP-complete to decide whether a bimatrix game has a mixed strategy Nash equilibrium.*

3. Bimatrix game similarity

Now we are ready to study the complexity of deciding bimatrix games similarity. First, we formally define the concept of similarity in below.

Definition 2. Given two bimatrix games $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, if there exists a strategy profile (x, y) such that it is a Nash equilibrium for both G_1 and G_2, G_1 and G_2 are weakly similar. Furthermore, if G_1 and G_2 have exactly the same set of Nash equilibria, they are strongly similar.

In the next, we show a simple result about deciding whether two bimatrix games share a common mixed strategy NE.

Proposition 1. *Given two bimatrix games $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, it is NP-complete to decide whether they share a common mixed strategy NE.*

Proof. Given a strategy profile (x, y), it only takes polynomial time to check whether it is a mixed strategy NE for both G_1 and G_2. Therefore, it is trivial that the problem is in NP.

In order to prove the NP-hardness of this problem, we first prove the following claim: for two boolean formulas ϕ_1 and ϕ_2 that share the same set of variables, it is NP-complete to decide whether they share a common satisfying assignment. We would like to call this problem COMMON-ASSIGNMENT. We reduce the SAT problem, which is well known to be NP-complete, to it. Given a formula $\phi_1 = \bigwedge_i C_i$ where C_i is a clause and V is the set of variables of ϕ_1, we construct another formula $\phi_2 = \bigwedge_{v_i \in V}(v_i \bigvee -v_i)$. It is obvious that ϕ_2 is always satisfied and ϕ_1, ϕ_2 have the same set of variables. If ϕ_1 and ϕ_2 have a common satisfying assignment, ϕ_1 is satisfiable. The reverse direction is also true. Thus, the COMMON-ASSIGNMENT problem is NP-complete. Now, we reduce the COMMON-ASSIGNMENT problem to our problem. Given two formulas ϕ_1 and ϕ_2, they can be encoded by two bimatrix games $G(\phi_1)$ and $G(\phi_2)$, respectively, using the method in [9]. However, the two games $G(\phi_1)$ and $G(\phi_2)$ may have different size of strategy spaces after the straightforward encodings. In order to conquer this, we introduce the following padding technique. For a variable $v \in V$, we can add clauses $(v \bigvee -v)$ to ϕ_1 and ϕ_2 in sense of conjunction. As we know, this operation does not change satisfying properties of ϕ_1 and ϕ_2. In this way, we can make sure that the clauses sets of ϕ_1 and ϕ_2 have the same size. Thus, $G(\phi_1)$ and $G(\phi_2)$ have the same size of strategy spaces after the encodings and paddings. As shown in [9], the mixed strategy Nash equilibria of $G(\phi_1)$(as well as $G(\phi_2)$) have a one-to-one correspondence with the satisfying assignments of formula ϕ_1(as well as ϕ_2). Therefore, G_1 and G_2 have a common mixed strategy NE iff the answer to the COMMON-ASSIGNMENT problem is "yes". The proposition is proved. ■

Actually, there is a more straightforward proof of Proposition 1 by setting $G_2 = (E, E)$ where E is an all one matrix, and making use of Corollary 1. However, in the proof of the next theorem, we need the NP-hardness result of the COMMON-ASSIGNMENT problem. Therefore, we prove Proposition 1 in a little bit complicated way to make the following proofs more clean. Please note that when deciding the COMMON-ASSIGNMENT problem and the similarity property later, we do not consider the effect of permutations.

Our target is to show the complexity of deciding the weak similarity of bimatrix games. It can not be harder than the problem of deciding whether two bimatrix games share a common mixed strategy NE. The reason is that it only takes polynomial time to check whether two bimatrix games share a common pure strategy NE or not. Combining with this fact, an algorithm to solve the problem in Proposition 1 can decide the weak similarity problem. Moreover, computing a NE of a bimatrix game is known to be PPAD-complete [8]. Thus, we may conjecture that deciding weak similarity of bimatrix games is not exactly NP-hard. The next theorem shows it is indeed NP-complete. However the reduction used in [9] does not work anymore since any two games constructed there always share a common pure strategy NE, which is the one where both the row and the column player play f. We need to bypass that. The main idea of the proof in the following theorem is to modify the matrices $G(\phi_1)$ and $G(\phi_2)$ in different ways such that they can not have a common pure strategy NE. Moreover, we should maintain the set of mixed strategy NEs of $G(\phi_1)$ (respectively, $G(\phi_2)$) in the modification.

Theorem 2. *Given two bimatrix games $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$, it is NP-complete to decide whether they share a common NE, i.e., it is NP-complete to decide whether two bimatrix games are weakly similar.*

Proof. Given a strategy profile (x, y), it only takes polynomial time to check whether it is a NE for both G_1 and G_2. Therefore, it is trivial that the problem is in NP.

In order to prove the NP-hardness result, we still reduce the COMMON-ASSIGNMENT problem to it. Given two formulas ϕ_1 and ϕ_2, if they have different sets of variables, we just reject them since ϕ_1 and ϕ_2 can not have a common satisfying assignment. Otherwise, let V be the same set of variables shared by ϕ_1 and ϕ_2. We can transform ϕ_1 and ϕ_2 to two bimatrix games $G(\phi_1) = (A_1, B_1)$ and $G(\phi_2) = (A_2, B_2)$, respectively, using the encoding method in [9] and the padding technique introduced in Proposition 1. By the padding technique, the two games $G(\phi_1)$ and $G(\phi_2)$ have the same size of strategy spaces. In the next, we would like to modify the matrices in the following way. Please note that the last row of A_1 (and the last column of B_1 because of symmetry) is $(1, 1, \ldots 1, 0)^{\mathrm{T}}$ while the last column of A_1 (and the last row of B_1) is $(-2, -2, \ldots, -2, 0)$. This special property will play an important role in our reduction. For A_1 and B_1, we add a vector $(0, \ldots, -1)^{\mathrm{T}}$ to both of them as the last column. Thus, we get

$$A_1' = \begin{pmatrix} \cdots & -2 & 0 \\ \cdots & -2 & 0 \\ \cdots & . & . \\ 11..1 & 0 & -1 \end{pmatrix}_{n\times(n+1)} \quad \text{and} \quad B_1' = \begin{pmatrix} \cdots & 1 & 0 \\ \cdots & 1 & 0 \\ \cdots & . & . \\ -2-2..-2 & 0 & -1 \end{pmatrix}_{n\times(n+1)}.$$

By this modification, it is obvious that the strategy profile $((0, \ldots, 0, 1), (0, \ldots 0, 1, 0))$ is the only pure NE for (A_1', B_1'). Furthermore, for any NE (x, y) of (A_1', B_1'), the column player does not play the $(n + 1)$th pure strategy. That is because for every probability vector $\widetilde{x}^{\mathrm{T}}$, the $(n + 1)$th element of $\widetilde{x}^{\mathrm{T}} B_1'$ is strictly less than the nth element. Therefore, $((x_1, \ldots x_n), (y_1, \ldots, y_n, 0))$ is a NE of (A_1', B_1') iff $((x_1, \ldots x_n), (y_1, \ldots, y_n))$ is a NE of (A_1, B_1). Thus, mixed strategy NEs of (A_1', B_1') have a one-to-one correspondence with satisfying assignments of ϕ_1. For A_2 and B_2, we add a vector $(0, \ldots, 2)^{\mathrm{T}}$ to both of them as the last column. Thus, we get

$$A'_2 = \begin{pmatrix} \dots & -2 & 0 \\ \dots & -2 & 0 \\ \dots & . & . \\ 11..1 & 0 & 2 \end{pmatrix}_{n\times(n+1)} \quad \text{and} \quad B'_2 = \begin{pmatrix} \dots & 1 & 0 \\ \dots & 1 & 0 \\ \dots & . & . \\ -2-2..-2 & 0 & 2 \end{pmatrix}_{n\times(n+1)}.$$

By this modification, it is obvious that the strategy profile $((0, \dots, 0, 1), (0, \dots 0, 0, 1))$ is the only pure NE for (A'_2, B'_2). We claim that if $((x_1, \dots x_n), (y_1, \dots, y_n))$ is a mixed strategy NE of (A_2, B_2), $((x_1, \dots x_n), (y_1, \dots, y_n, 0))$ is a NE of (A'_2, B'_2). To see this, please note that according to Theorem 1, for every mixed strategy NE of (A_2, B_2), the nth element x_n must be 0. Thus, the last element of $x^T B'_2$ should be 0. Then the $(n+1)$th strategy of the column player can not be a best response given x. Therefore, $(y_1, \dots, y_n, 0)$ gives the column player the maximum payoff when the row player plays x. On the other hand, given that the column player plays $(y_1, \dots, y_n, 0)$, $(x_1, \dots x_n)$ gives the row player the maximum payoff.

Now we would like to show that ϕ_1 and ϕ_2 share a common satisfying assignment iff (A'_1, B'_1) and (A'_2, B'_2) share a common NE. If ϕ_1 and ϕ_2 share a common satisfying assignment, (A_1, B_1) and (A_2, B_2) will have a common mixed strategy NE $((x_1, \dots x_n), (y_1, \dots, y_n))$. According to our analysis above, (A'_1, B'_1) and (A'_2, B'_2) should have a common NE $((x_1, \dots x_n), (y_1, \dots, y_n, 0))$. For the other direction, if (A'_1, B'_1) and (A'_2, B'_2) share a common NE (x', y'), it must be a mixed strategy NE since (A'_1, B'_1) and (A'_2, B'_2) can not have the same pure strategy NE. Moreover, the NEs of (A'_1, B'_1) have a one-to-one correspondence with NEs of (A_1, B_1). We know that (x', y') only takes positive values on literals of ϕ_1 that give a satisfying assignment of ϕ_1. Therefore, (x', y') also only takes positive values on the same set of literals of ϕ_2. Thus, y'_{n+1} must be 0. Ignoring the last element of y', we get the strategy profile (x, y) and it is a mixed strategy NE of (A_2, B_2). Thus, the supports of (x, y)(i.e., the supports of (x', y')) give a satisfying assignment of ϕ_2. Equivalently, the same set of literals that correspond to the supports of (x', y') give a satisfying assignment to both formulas ϕ_1 and ϕ_2. Therefore, if (A'_1, B'_1) and (A'_2, B'_2) share a common NE, ϕ_1 and ϕ_2 share a common satisfying assignment. The theorem is proved. ■

The next natural question is what is the complexity to decide whether two bimatrix games are strongly similar or not. It is well known [13] that given two boolean formulas ϕ_1 and ϕ_2, it is co-NP-hard to determine whether the two formulas are equivalent. Given the reduction in [9], there is a one-to-one correspondence between mixed NEs of $G(\phi_1)$ (respectively, $G(\phi_2)$) and satisfying assignments of formula ϕ_1 (respectively, ϕ_2). Therefore, $G(\phi_1)$ and $G(\phi_2)$ have the same set of NEs iff ϕ_1 and ϕ_2 are equivalent. Thus,

Theorem 3. *It is co-NP-hard to decide whether two bimatrix games are strongly similar.*

4. Conclusion

Computing a Nash equilibrium of a bimatrix game is PPAD-complete [8] while deciding whether a bimatrix game [9, 11] has more than one Nash equilibrium is NP-hard. In this paper, we introduce the concept of bimatrix games similarity. We show that it is NP-complete to determine whether two bimatrix games are weakly similar while it is co-NP-hard to determine whether they are strongly similar. In general, the complexity class PPAD only concerns about the complexity of computing an arbitrary Nash equilibrium. However, complexities are moved to NP/co-NP when we ask questions related to multiplicity of Nash equilibria. We regard this work as the first step to understanding the relationship between PPAD and NP/co-NP. Moreover, it is an interesting topic to further explore computational aspects of studying relationships between two different games.

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