Theoretical Computer Science 407 (2008) 448–457

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Computational complexity of quantified Boolean formulas with fixed maximal deficiency ☆

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ARTICLE INFO

Article history:
Received 18 March 2007
Received in revised form 3 March 2008
Accepted 23 July 2008
Communicated by X. Deng

Keywords:
Quantified boolean formula
Deficiency
Model
Minimal false formula
Complexity

ABSTRACT

The paper investigates the computational complexity of quantified Boolean formulas with fixed maximal deficiency. The satisfiability problem for quantified Boolean formulas with maximal deficiency 1 is shown to be solvable in polynomial time. For $k \geq 1$, it is shown that true formulas with fixed maximal deficiency k have models in which all Boolean functions can be represented as CNF formulas over at most $2^{4k/3}$ universal variables. As a consequence, the satisfiability problem for QCNF formulas with fixed maximal deficiency is in NP and for fixed deficiency the minimal falsity problem is in D^P. For two subclasses of quantified Boolean formulas with PSPACE-complete evaluation problem, QEHORN and QE2-CNF, we show that for fixed deficiency the minimal falsity problem can be decided in polynomial time.

1. Introduction

A propositional formula in conjunctive normal form (CNF) is called *minimal unsatisfiable* (MU) if the formula is unsatisfiable and any proper subformula is satisfiable. The problem of deciding whether a formula is minimal unsatisfiable is known to be D^P-complete [11]. D^P is the class of problems which can be described as the difference of two NP-problems, and the completeness is based on many-one polynomial time reductions. A measure for the structural complexity of formulas is the so-called *deficiency*. The deficiency is defined as the difference between the number of clauses and the number of variables. For any fixed k, the minimal unsatisfiability problem for formulas with deficiency k is solvable in polynomial time [3,8]. The maximal deficiency of a CNF formula φ is the maximum of deficiencies of all subformulas of φ. For formulas with fixed maximum deficiency the satisfiability can be decided in polynomial time [3,8].

The concept of minimal unsatisfiability for CNF has been extended to QCNF, the class of quantified Boolean formulas with matrix in CNF, for which the satisfiability problem is PSPACE-complete. A QCNF formula is said to be *minimal false* if the formula is false and deleting any clause results in a true formula. Clearly, minimal falsity is a generalization of minimal unsatisfiability. The minimal falsity problem is still PSPACE-complete [7]. The notion of deficiency can be extended to QCNF formulas. The deficiency for quantified Boolean formulas with CNF matrices is defined as the difference between the number of clauses and the number of *existential* variables. The set of minimal false QCNF formulas with deficiency k is denoted as MF(k). We will show that the minimal falsity problem MF(1) is solvable in polynomial time. In contrast to the propositional case, for $k > 1$ the computational complexity of the minimal falsity problem MF(k) remains open.

☆ A preliminary version has been published in the proceedings of the Conference SAT06, LNCS 4121, pp. 339–352.

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0304-3975/$ – see front matter
doi:10.1016/j.tcs.2008.07.022

For any QCNF formula Φ, the maximal deficiency is the maximum of deficiencies of all subformulas. For any minimal false formula, its deficiency coincides with its maximal deficiency. It will be shown that for any fixed $k > 1$, the satisfiability problem for formulas with maximal deficiency k is in NP, whereas the minimal falsity problem MF(k) is in D^P.

The proofs are based on a property on models of quantified Boolean formulas. Let $\Phi = \forall x_1 \exists y_1 \forall x_2 \exists x_2 \cdots \forall x_k \exists y_k \varphi$ be a closed quantified Boolean formula. The formula is true if and only if there are Boolean functions $f_i : \{x_1, \ldots, x_i\} \to \{0, 1\}$ such that $\forall x_1 \cdots \forall x_k \varphi[y_1/f_1(x_1), \ldots, y_k/f_k(x_1, \ldots, x_k)]$ is true. A sequence of Boolean functions $M = (f_1, \ldots, f_k)$ is called a *model* for Φ, if the formula is true for these functions. The Boolean functions f_i are represented as propositional formulas. We shall show that for any fixed k and for any true QCNF formula Φ with maximal deficiency k, there is a set U of at most $2^{4k/3}$ universal variables such that Φ has a model in which each Boolean function is a propositional formula over variables in U. That property can be used to prove that for the subclasses of quantified Boolean formulas QEHORN and QE2-CNF the minimal falsity problem MF(k) is solvable in polynomial time. QEHORN (resp. QE2-CNF) is the set of QNF formulas for which after the deletion of the universal variables the formulas are in QHORN (resp. Q2-CNF). That means, the existential part of the formulas are Horn formulas (resp. 2-CNF formulas). The satisfiability problem is PSPACE–complete for QEHORN and QE2-CNF [4].

Besides the theoretical insight to the complexity of the satisfiability problem, we think that the work of this paper may also have its practical value. Although there seems no practical problem which can be encoded to a formula with low deficiency, during the running of DPLL-like QSAT-solves, the current formula may have very low deficiency. Therefore, efficient algorithms solving formulas with lower deficiency might provide a possibility to improve the performance of QSAT-solvers by determining directly the truth value of the formulas with lower deficiency instead of splitting further.

The paper is organized as follows. After the notations and the definition of models, deficiency, and minimal falsity in Section 3, we prove in Section 4 the polynomial solvability of the minimal falsity problem for formulas with deficiency 1. In Section 5 we discuss the computational complexity of the minimal falsity problem for formulas with fixed (maximal) deficiency. The section contains the main theorem, that any quantified Boolean formula with maximal deficiency k, which is true, has model functions depending on at most $2^{4k/3}$ universal variables.

Here we would like to mention the related research which studies the ratio of the number of clauses to the number of variables. For k-CNF formulas, there is a critical ratio $c(k)$. Below $c(k)$, almost k-CNF formulas are satisfiable, whereas above $c(k)$ almost all are unsatisfiable. It is known that $c(2) = 1$ [5], but determining the exact location of $c(k)$ for $k \geq 3$ remains open. To understand the hardness level from 2SAT to 3SAT, authors in [10,12] investigated formulas with mixtures of 2-clauses and 3-clauses. Their work shows that the critical ratio and the complexity of satisfiability have close relationship with the ratio of the number of 3-clauses to the number of all clauses in the formulas.

2. Notations

A literal is a variable or a negated variable. Let X be a set of variables, then $lit(X)$ is the set of literals over the variables in X. Clauses are disjunctions of literals. Clauses are also considered as sets of literals. A propositional formula in conjunctive normal form (CNF) is a conjunction of clauses. We forbid tautological clauses in CNF formulas because deleing them does not change the satisfiability. CNF formulas will be considered as multi-sets of clauses. Thus, they may contain multiple occurrences of clauses. The set of all variables occurring in a formula φ is denoted as $var(\varphi)$.

QBF is the class of all closed quantified Boolean formulas. Any formula Φ in QCNF has the form $\Phi = Q_1 x_1 \cdots Q_n x_n \varphi$, where $Q \in \{\exists, \forall\}$ and φ is a CNF formula. $Q_1 x_1 \cdots Q_n x_n$ is the prefix of Φ, and φ is called the matrix of Φ. Sometimes we use an abbreviation and write $\Phi = Q\varphi$. The set of universal variables is denoted as $var_\forall(\Phi)$ and $var(\Phi)$ is the set of all variables.

Let $\Phi = Q_1 x_1 \cdots Q_n x_n \varphi, \Phi' = Q_1 x_1 \cdots Q_n x_n \varphi'$ be two QCNF formulas. We say Φ' is a subformula of Φ, denoted as $\Phi' \subseteq \Phi$, if φ' is a subformula of φ.

Suppose we have a formula $Q\,\varphi \in$ QCNF with $\exists$-variables $x_1, \ldots, x_n$. Then $\varphi_{|\exists}$ is the conjunction of clauses we obtain after the deletion of all occurrences of universal literals in φ. For example, if $\Phi = \forall y_1 \forall y_2 \exists x \varphi$ with $\varphi = (y_1 \vee x) \wedge (y_2 \vee \neg x) \wedge (\neg y_1 \vee x)$, then $\varphi_{|\exists} = x \wedge \neg x \wedge x$. Please note, that the propositional formula $\varphi_{|\exists}$ may contain multiple occurrence of clauses. A clause is termed universal if the clause contains only universal literals.

QEHORN is the set of QCNF formulas $\Phi = Q\varphi$, for which $\varphi_{|\exists}$ is a HORN formula. That means, the existential part of the matrix is a Horn formula. Analogously, QE2CNF is the class of QCNF formulas, whose existential part is a 2CNF formula, i.e. any existential part of the clauses contains at most two literals.

In our investigations we make use of substitutions of variables by formulas. For a quantified Boolean formula Φ, $\Phi[y_1/f_1, \ldots, y_n/f_n]$ denotes the formula obtained by simultaneously replacing in the matrix the occurrences of variables y_i by the formula f_i and eliminating the existential quantifiers from the prefix. For example, for $\Phi := \forall x \exists y (x \vee y) \wedge (\neg x \vee \neg y)$, $\Phi[y/\neg x]$ is the formula $\forall x (x \vee \neg x) \wedge (\neg x \vee x)$. For $\vec{y} = y_1, \ldots, y_n$, and $M = (f_1, \ldots, f_n)$, we sometimes write $\Phi[\vec{y}/M]$ instead of $\Phi[y_1/f_1, \ldots, y_n/f_n]$.

3. Models, deficiency, and minimal falsity

Let $\Phi = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_k \exists y_k \varphi$ be a closed quantified Boolean formula. The formula is true if and only if there are Boolean functions $f_i : \{x_1, \ldots, x_i\} \to \{0, 1\}$ such that $\forall x_1 \cdots \forall x_k \varphi[y_1/f_1(x_1), \ldots, y_k/f_k(x_1, \ldots, x_k)]$ is true. A sequence of

Boolean functions $M = (f_1, \dots, f_k)$ is called a *model* for Φ, if the formula is true for these functions. Subsequently, we assume that the Boolean functions f_i are represented as propositional formulas. If the Boolean functions f_i are given as CNF–formulas, then we call $M = (f_1, \dots, f_k)$ a *CNF–model*.

In our proofs we will make use of the length of models. We introduce the following definition.

Definition 1. Let $t : \mathbb{N} \to \mathbb{N}$ be a function and $\mathcal{B}$ a subclass of QBF.
Suppose, every true formula Φ in $\mathcal{B}$ has a model $M = (f_1, \dots, f_m)$, such that the size of each function given as propositional formula is smaller than or equal to $t(|\Phi|)$. Then we say $\mathcal{B}$ has $t(n)$-size models. If t is a polynomial, then we say $\mathcal{B}$ has polynomial size models.

For some subclasses of QCNF the structure and the length of models are known. For example, QHORN has linear size models and Q2-CNF has constant size models [6]. Note that the problem of determining whether a sequence of Boolean functions is a model of a quantified Boolean formula is in co-NP. Whereas deciding whether a quantified Boolean formula has a model is a PSPACE problem, since a formula is true if and only if it has a model. Therefore, assuming PSPACE $\neq \Sigma_2^P$, QCNF has no polynomial size propositional models.

Next we recall the definition of deficiency for CNF and QCNF. Let φ be a CNF formula over n variables with $n + k$ clauses, then we say k is the *deficiency* of φ. For the deficiency of a formula φ we write $d(\varphi)$.

Definition 2. For a formula $\Phi = Q\varphi \in$ QCNF, the deficiency of Φ, denoted by $d(\Phi)$, is the difference between the number of clauses and the number of *existential* variables.

The maximal deficiency of Φ is defined as $d^*(\Phi) := \max\{d(\Phi') \mid \Phi' \subseteq \Phi\} = \max\{d(\varphi'_{|\exists}) \mid \varphi' \subseteq \varphi\} := d^*(\varphi_{|\exists})$.

A formula Φ is termed *stable* if for any proper subformula $\Phi' \subset \Phi$, $d(\Phi') < d(\Phi)$.

The notion of stable QCNF formulas can be understood as a generalization of matching lean CNF formulas [9]. A formula $\varphi \in$ CNF is called matching lean if $d(\varphi') < d(\varphi)$ for any proper subformula $\varphi' \subset \varphi$. By Definition 2, Φ is stable if and only if $\varphi_{|\exists}$ is matching lean.

A formula $\phi_1 \wedge \cdots \wedge \phi_n$ in CNF is called minimal unsatisfiable, if the formula is unsatisfiable and for any clause ϕ_i the formula $\phi_1 \wedge \cdots \wedge \phi_{i-1} \wedge \phi_{i+1} \wedge \cdots \wedge \phi_n$ is satisfiable. The class of minimal unsatisfiable formulas is denoted by MU.

The definition can be extended to formulas in QCNF as follows:
A formula $Q(\phi_1 \wedge \cdots \wedge \phi_n)$ in QCNF is called minimal false, if the formula is false and for any clause ϕ_i the formula $Q(\phi_1 \wedge \cdots \wedge \phi_{i-1} \wedge \phi_{i+1} \wedge \cdots \wedge \phi_n)$ is true. The class of minimal false formulas is denoted by MF.

Definition 3. Let k be fixed. Then we define
MU(k) $:= \{\varphi : \varphi \in MU$ and $d(\varphi) = k\}$, and
MF(k) $:= \{\Phi : \Phi \in MF$ and $d(\Phi) = k\}$.

The class MU is D^P-complete [11], whereas MF is PSPACE-complete [7]. Any minimal unsatisfiable formula has deficiency greater than 0 [1]. Moreover, it has been shown that MU(k) is solvable in polynomial time [3].

Lemma 1. *Let $\Phi = Q\varphi$ be a formula in QCNF with matrix φ.*

(1) *If $\varphi_{|\exists}$ is satisfiable then Φ is true. Moreover, if $d^*(\Phi) = 0$ then Φ is true.*
(2) *If $d^*(\Phi) > 0$, then Φ can be divided into two formulas $\Phi' = Q\varphi_1$ and $\Phi'' = Q\varphi_2$, that is, $\Phi = Q(\varphi_1 \wedge \varphi_2)$, such that*
- *Φ' is stable,*
- *Φ is true iff Φ' is true, and*
- *there is a set E of existential variables occurring only in Φ'' and a truth assignment for E satisfying Φ''.*

(3) *If $\Phi \in MF$, then Φ is stable, i.e., $d^*(\Phi) = d(\Phi)$ and for any proper subformula Φ' it holds $d(\Phi') < d(\Phi)$.*
(4) *Any minimal false formula has deficiency greater than 0.*
(5) *Let $\Phi = Q\varphi$ be in MF(1), then we have $\varphi_{|\exists} \in$MU(1).*

Proof. (1) Obviously, any satisfying truth assignment of $\varphi_{|\exists}$ is a model of Φ. Thus, if $\varphi_{|\exists}$ is satisfiable, then Φ has a model, and hence it is true. Suppose $d^*(\Phi) = 0$. Then by definition any subformula of $\varphi_{|\exists}$ has deficiency less or equal than 0. Since any unsatisfiable formula in CNF contains a minimal unsatisfiable formula, and minimal unsatisfiable formulas have at least deficiency 1 [1], $\varphi_{|\exists}$ must be satisfiable.

(2) By Definition 2, $\Phi = Q\varphi$ is stable if and only if $\varphi_{|\exists}$ is matching lean. Since the union of two matching lean formulas is also matching lean [9], it follows that for two stable QCNF formulas $Q\varphi_1$ and $Q\varphi_2$, the formula $Q(\varphi_1 \cup \varphi_2)$ remains stable. Moreover, the union of all stable subformulas of Φ is the largest stable formula of Φ. Theorem 7.5 of [9] allows another characterization of the largest stable subformula. The largest stable subformula of Φ is the smallest subformula Φ' with $d^*(\Phi') = d^*(\Phi)$.

Suppose $\Phi' = Q\varphi_1$ is the largest stable subformula of Φ. Then $d(\Phi') = d^*(\Phi)$. Let $\Phi'' := Q(\varphi - \varphi_1)$, and let Ψ'' be obtained from Φ'' by the deletion of all positive and negative occurrences of existential variables occurring in the matrix of Φ'. Since Φ' has maximal deficiency, we can see that $d^*(\Psi'') = 0$ (otherwise, we would get a subformula with deficiency greater than $d^*(\Phi)$). Then Ψ'' is true, hence Φ'' is true. Since Φ' and Ψ'' have distinct existential variables and Φ'' is true for a truth assignment for E the existential variables not occurring in Φ', Φ'' is true independently to Φ'. Consequently, Φ is true if and only if Φ' is true.

(3) Let Φ be minimal false. Because of (1), we have $d^*(\Phi) > 0$. Then by (2) there is a stable subformula Φ' such that $d(\Phi') = d^*(\Phi)$ and Φ' is also false. However, since Φ is minimal false, it follows that $\Phi' = \Phi$. Therefore, Φ is stable, and we obtain our desired equation $d(\Phi) = d^*(\Phi)$.

(4) Let Φ be minimal false. Because of (1) and the falsity of Φ, we know $d^*(\Phi) > 0$. Together with (3) we have $d(\Phi) > 0$.

(5) Suppose $\Phi = Q\varphi \in \mathrm{MF}(1)$. Then $\varphi_{|\exists}$ is unsatisfiable and $d(\varphi_{|\exists}) = 1$. Using (3) we see that any proper subformula Φ' of Φ has deficiency $d(\Phi') < 1$. That means any proper subformula of $\varphi_{|\exists}$ has a deficiency less than 1. That implies the satisfiability of the proper subformulas of $\varphi_{|\exists}$, because any unsatisfiable propositional formula contains a minimal unsatisfiable formula with deficiency greater than 0. Altogether we have shown $\varphi_{|\exists} \in \mathrm{MU}(1)$. ■

The inverse of part (5) in Lemma 1 is invalid. For an example of a true formula $\Phi = Q\varphi$ with $\varphi_{|\exists} \in \mathrm{MU}(1)$ please see Example 1 in Section 4.

4. Maximal deficiency 1

In this chapter we will show: (1) The satisfiability problem for QCNF formulas with maximal deficiency 1 is solvable in polynomial time, and (2) for any true QCNF formula with $d^*(\Phi) = 1$, there exists one universal variable y and a model for Φ such that all model functions are either the constants 0 or 1, or the formulas $y, \neg y$.

Corollary 7.10 of [9] says that the largest matching lean subformula can be computed in polynomial time. This implies that the largest stable formula can also be computed in polynomial time. More precisely, given a QCNF formula $\Phi = Q\varphi$, we compute the largest matching lean subformula ψ of $\varphi_{|\exists}$, then we add to the clauses in ψ the removed occurrences of the universal literals, the resulting formula is the subformula $\varphi' \subseteq \varphi$ such that $(\varphi')_{|\exists}$ is ψ. Clearly, $Q\varphi'$ is the largest stable subformula of Φ. Since a QCNF formula has the same truth as its largest stable subformula, it is sufficient to consider stable formulas. Please note, that any false stable formula Φ with deficiency 1 must be minimal false. Thus, we only need to consider the complexity of MF(1).

In our polynomial time algorithm for deciding the minimal falsity problem MF(1) we make use of the *connectivity* of clauses. In order to motivate the definition at first we give some examples.

Example 1. Let $\Phi := \exists x \forall y \exists x_1 \exists x_2 \varphi$ be a quantified Boolean formula, where φ is the following formula (the columns are the clauses)

$$\varphi = \left\{ \begin{array}{cccc} & & x & \neg x \\ y & & \neg y & \\ \neg x_1 & x_1 & & \\ & \neg x_2 & x_2 & \end{array} \right\}.$$

That is, $\varphi = (y \vee \neg x_1) \wedge (x_1 \vee \neg x_2) \wedge (x \vee \neg y \vee x_2) \wedge (\neg x)$. Clearly, $\varphi_{|\exists}$ is in MU(1). The first and the third clauses contain the complementary pair of universal variables y and $\neg y$. Moreover, these clauses are connected by means of the complementary pairs of existential variables $(\neg x_1, x_1)$ and $(\neg x_2, x_2)$. None of these existential variables dominates the universal variable y in the prefix. We can say that the first and the third clauses are connected without the dominating existential variable x. The formula Φ is true, because for $y = 1$ we can choose $x_1 = x_2 = 1$. For $\neg y = 1$ we set $\neg x_1 = \neg x_2 = 1$.

Example 2. Let $\Phi := \exists x_1 \exists x_3 \forall y \exists x_2 \exists x_4\, \varphi$, where φ is the following formula

$$\varphi = \left\{ \begin{array}{ccccc} x_1 & \neg x_1 & \neg x_3 & x_3 & x_4 \\ \neg x_2 & y & x_2 & \neg y & \\ & \neg x_2 & & \neg x_4 & \end{array} \right\}.$$

As in Example 1, the existential part of this φ is also in MU(1). However, we can check that Φ is false. The difference here from Example 1 lies in the fact that the two clauses containing a pair of complementary universal literals y and $\neg y$, i.e. the second and the fourth clauses, are not connected without the dominating existential variables x_1 and x_3. Our task is to show that the unconnectedness is the reason of the falsity.

Definition 4. Suppose $\Phi = Q\varphi$ is a formula with $\varphi_{|\exists} \in MU(1)$, f, g are clauses in φ, and X is a subset of $var(\varphi_{|\exists})$.
We say f and g are *directly connected without* X in Φ if $f = g$, or if there is some existential literal $L \notin lit(X)$ such that $L \in f$ and $\neg L \in g$.
We say f and g are *connected without* X in Φ if there are in φ clauses $f = f_1, f_2, \ldots, f_n = g$ such that f_i and f_{i+1} are directly connected without X.

Notice, that if X is empty, then any two clauses in φ are connected without X, because $\varphi_{|\exists}$ is minimal unsatisfiable.

For convenience to state the main result, we introduce a natation: For a formula $\Phi = Q\varphi \in$ QCNF, and a universal variable y, we use $\varphi(y)$ to denote the formula obtained from φ by deleting of all universal literals from φ except y and $\neg y$. For example, if $\Phi = \exists x_1 \forall y_1 \forall y_2 \exists x_2 \varphi$ with $\varphi = (x_1 \vee y_1 \vee y_2 \vee \neg x_2) \wedge (\neg x_1 \vee y_1 \vee x_2) \wedge (x_1 \vee \neg y_2 \vee \neg x_2)$ then $\varphi(y_1) = (x_1 \vee y_1 \vee \neg x_2) \wedge (\neg x_1 \vee y_1 \vee x_2) \wedge (x_1 \vee \neg x_2)$.

Theorem 1. *Let $\Phi = \exists X_1 \forall Y_1 \cdots \exists X_m \forall Y_m \exists X_{m+1} \varphi$ be a formula with $\varphi_{|\exists}$ in $MU(1)$. Then we have:*

$$\begin{array}{lll} \Phi \text{ is false} & \Leftrightarrow & \text{for all variables } y \in Y_i,\ 1 \le i \le m, \text{ for all } f, g \in \varphi(y): \\ & & (y \in f, \neg y \in g) \Rightarrow f \text{ and } g \text{ are not connected without} \\ & & \qquad X_1 \cup \cdots \cup X_i \text{ in } \varphi(y). \end{array}$$

Corollary 1. *MF(1) can be solved in polynomial time.*

Proof. By Theorem 1, to see whether a formula $\Phi = \exists X_1 \forall Y_1 \cdots \exists X_m \forall Y_m \exists X_{m+1} \varphi$ is in MF(1), first check that $\varphi_{|\exists} \in \text{MU}(1)$. If this condition holds, check for each $y \in Y_i$, $i = 1, \ldots, m$, and for any two clauses f, g containing y and $\neg y$ respectively, that f and g are not connected without $X_1 \cup \cdots \cup X_i$.

Whether two clauses are connected without X can be decided in polynomial time as follows: For every clause we have a node labelled with the clause, two clauses are joined by an edge if there is some $x \notin X$ such that x occurs in one of the clauses and $\neg x$ in the other clause. Then two clauses are connected without X if and only if there is a path between the clauses. That can be decided in polynomial time.

Since there at most quadratic many pairs of clauses containing complementary universal literals, and the connectivity can be tested in polynomial time, MF(1) can be solved in polynomial time. ■

The proof of Theorem 1 can be divided into two parts (Lemmas 2 and 3). Lemma 2 reduces the number of alternations and universal variables in the prefix to one universal variable. Lemma 3 states the relationship between the truth of a formula and the property "connected without dominating variables".

Lemma 2. *Let $\Phi = \exists X_1 \forall Y_1 \cdots \exists X_m \forall Y_m \exists X_{m+1} \varphi$ be a formula with $\varphi_{|\exists}$ in $MU(1)$. Then,*

$$\begin{array}{lll} \Phi \text{ is false} & \Leftrightarrow & \text{for all variables } y \in Y_i,\ 1 \le i \le m, \text{ the formula} \\ & & \exists X_1 \cdots \exists X_i \forall y \exists X_{i+1} \cdots \exists X_{m+1} \varphi(y) \text{ is false.} \end{array}$$

Lemma 3. *Let $\Phi = \exists X \forall y \exists Z \varphi$ be a formula with $\varphi_{|\exists}$ in $MU(1)$. Then,*

$$\begin{array}{lll} \Phi \text{ is false} & \Leftrightarrow & \forall f, g \in \varphi : (y \in f, \neg y \in g) \Rightarrow \\ & & \qquad f \text{ and } g \text{ are not connected without } X \text{ in } \varphi. \end{array}$$

Our next task is to prove Lemmas 2 and 3. In our proofs we need some properties of MU(1) formulas, i.e. minimal unsatisfiable formulas with deficiency 1.

For any formula $\varphi \in \text{MU}(1)$ and $x \in var(\varphi)$, $\varphi[x/0]$ (resp. $\varphi[x/1]$) contains a unique minimal unsatisfiable subformula which is also in MU(1) [2], denoted by φ_x (reps. $\varphi_{\neg x}$). We call $(\varphi_x, \varphi_{\neg x})$ a splitting of φ over x. In general, the splitting formulas φ_x and $\varphi_{\neg x}$ may have common clauses. However, we have the following nice structural property.

Proposition 1 ([2]). *For any formula $\varphi \in MU(1)$, there is a variable $x \in var(\varphi)$ such that φ_x and $\varphi_{\neg x}$ have distinct variables and therefore no common clause.*

Let $\varphi, x, \varphi_{\neg x}, \varphi_x$ be as in Proposition 1, then we call $(\varphi_x, \varphi_{\neg x})$ a disjunctive splitting of φ over x.

Example 3. Let φ be the following formula:

$$\varphi = \begin{pmatrix} x_1 & \neg x_1 & & & & & \\ & x_2 & \neg x_2 & & & & \\ & & x_3 & \neg x_3 & & & \\ & & x & & \neg x & \neg x & \\ & & & & x_4 & \neg x_4 & \\ & & & & x_5 & x_5 & \neg x_5 \end{pmatrix}.$$

We can see

$$\varphi_x = \begin{pmatrix} x_1 & \neg x_1 & & \\ & x_2 & \neg x_2 & \\ & & x_3 & \neg x_3, \end{pmatrix}, \quad \varphi_{\neg x} = \begin{pmatrix} x_4 & \neg x_4 & \\ x_5 & x_5 & \neg x_5 \end{pmatrix}.$$

Suppose φ_L is a splitting formula, $y \in var(\varphi_L)$, then we can split φ_L again and get formulas $(\varphi_L)_K$ with $K \in \{y, \neg y\}$. For simplicity, we write $(\varphi_L)_K$ as φ_{LK}. Generally, we have $\varphi_{L_1 \cdots L_k}$ (which we still call a splitting formula) after several steps of splitting. Please notice, that when performing splitting, we remove the occurrences of splitting literals. During the proof of the polynomial-time solvability of MF(1) we have to recover some of the removed occurrences of literals. Suppose θ is a splitting formula, L a literal, θ^L denotes the formula obtained from θ by recovering the occurrences of L properly. Please notice, that if the original clauses from which θ is obtained (by deleting some splitting literals) does not contain L, then L will not occur in θ^L.

Example 4. Let φ be the formula in Example 3. We can see

$$\varphi_x = \begin{pmatrix} x_1 & \neg x_1 & & & & \\ & x_2 & \neg x_2 & & & \\ & & x_3 & \neg x_3 & & \end{pmatrix}, \qquad \varphi_{xx_2} = (x_1, \neg x_1), \qquad \varphi_{x\neg x_2} = (x_3, \neg x_3).$$

Then

$$\varphi_x^x = \begin{pmatrix} x_1 & \neg x_1 & & \\ & x_2 & \neg x_2 & \\ & & x_3 & \neg x_3 \\ & & x & \end{pmatrix}, \qquad \varphi^x_{xx_2} = \varphi_{xx_2}, \qquad \varphi^x_{x\neg x_2} = \begin{pmatrix} x_3 & \neg x_3 \\ x & \end{pmatrix}.$$

Proposition 2. *Suppose φ ∈MU(1), $x \in var(\varphi)$ with a disjunctive splitting $(\varphi_x, \varphi_{\neg x})$. Suppose further $y \in var(\varphi_L)$ for some $L \in \{x, \neg x\}$. Then for $K \in \{y, \neg y\}$, we have the following:*
If φ^L_{LK} does not contain L then $\varphi_K = \varphi_{LK}$.
If φ^L_{LK} contains L then $\varphi_K = \varphi^L_{LK} + \varphi^{\neg L}_{\neg L}$, moreover, $\varphi_{LK} = \varphi_{KL}$.

Proof. If φ^L_{LK} does contain L then $\varphi_{LK} = \varphi^L_{LK}$. That means $\varphi_{LK} \subseteq \varphi[K/0]$. Thus $\varphi_K = \varphi_{LK}$. Suppose φ^L_{LK} contains L then $\varphi^L_{LK} + \varphi^{\neg L}_{\neg L}$ is in MU(1), and a subformula of $\varphi[K/0]$, therefore, it must be φ_K. ■

In Examples 3 and 4, we can see that $\varphi_{x_2} = \varphi_{xx_2}$, while $\varphi_{\neg x_2} = \varphi^x_{x\neg x_2} + \varphi^{\neg x}_{\neg x}$.

Lemma 4. *Let φ be a MU(1) formula and $x \in var(\varphi)$. Then $\varphi[L/0] - \varphi_L$ can be satisfied by a partial truth assignment defined on $var(\varphi[L/0] - \varphi_L) - var(\varphi_L)$, for $L \in \{x, \neg x\}$.*

Proof. We prove the lemma by induction on the number of variables in φ. If φ has only one variable, the lemma clearly holds. Suppose φ has more than one variable. If $(\varphi_x, \varphi_{\neg x})$ is a disjunctive splitting, then the assertion follows. So, we assume $(\varphi_x, \varphi_{\neg x})$ is non-disjunctive. By Proposition 1, there is some $y \neq x$ such that $(\varphi_y, \varphi_{\neg y})$ is a disjunctive splitting. W.o.l.g., we assume that $x \in var(\varphi_y)$. There are two possibilities.

Case 1. y occurs in φ^y_{yx}. Then we have $\varphi_x = \varphi^y_{yx} + \varphi^{\neg y}_{\neg y}$. Hence

$$\varphi[x/0] - \varphi_x = (\varphi_y[x/0] - \varphi_{yx})^y.$$

Now the assertion follows from the induction hypothesis.

Case 2. y does not occur in φ^y_{yx}. Then $\varphi_x = \varphi_{yx}$. Thus

$$\varphi[x/0] - \varphi_x = (\varphi_y[x/0] - \varphi_{yx})^y + \varphi^{\neg y}_{\neg y}.$$

By the induction hypothesis, $\varphi_y[x/0] - \varphi_{yx}$ is satisfied by a truth assignment t defined on $var(\varphi_y[x/0] - \varphi_{yx}) - var(\varphi_{yx})$. Please note that $y \notin var(\varphi_x)$ and $var(\varphi_{\neg y}) \cap var(\varphi_y) = \emptyset$. Thus we can extend t to t' such that $t'(y) = 0$ and t' satisfies $\varphi^{\neg y}_{\neg y}$. Hence, the assertion is valid.

By the same argument we can show the assertion for $\varphi[x/1] - \varphi_{\neg x}$. ■

Suppose $\Phi = Q\varphi$ is a QCNF formula with $\varphi_{|\exists} \in$ MU(1), and $((\varphi_{|\exists})_x, (\varphi_{|\exists})_{\neg x})$ is a splitting of $\varphi_{|\exists}$ over x. Please note that $\varphi_{|\exists}$ is obtained from φ by removing all occurrences of universal literals. Now we add the occurrences of all universal literals according to their original occurring places to the clauses in $((\varphi_{|\exists})_x, (\varphi_{|\exists})_{\neg x})$, the result is denoted as $(\varphi_x, \varphi_{\neg x})$ which we still call a splitting of φ over x. In fact, φ_x (resp. $\varphi_{\neg x}$) is the subset of $\varphi[x/0]$ (resp. $\varphi[x/1]$) such that $(\varphi_x)_{|\exists} = (\varphi_{|\exists})_x$ (resp. $(\varphi_{\neg x})_{|\exists} = (\varphi_{|\exists})_{\neg x}$).

Corollary 2. *Let $\Phi = \exists x Q\varphi$ be a QCNF formula with $\varphi_{|\exists} \in MU(1)$. Then $\Phi[x/0]$ (resp. $\Phi[x/1]$) and $Q\varphi_x$ (resp. $Q\varphi_{\neg x}$) have the same truth. Moreover, Φ is false if and only if both $Q\varphi_x$ and $Q\varphi_{\neg x}$ are false.*

Proof of Lemma 2. We first show the direction from left to right. If Φ is false then $\exists X_1 \cdots \exists X_i \forall y \exists X_{i+1} \cdots X_{m+1}\varphi(y)$ is false, because the deletion of some universal literals preserves the falsity. Here, $\varphi(y)$ is the result of the deletion of all universal literals from φ except y and $\neg y$.

For the other direction, we suppose Φ is true. Pick $x \in X_1$, let $X_1' := X_1 - \{x\}$. W.l.o.g we assume $\Phi[x/0] = \exists X_1' \forall Y_1 \cdots \exists X_m \forall Y_m \exists X_{m+1}\varphi[x/0]$ is true. Then by Corollary 2, the formula $\Psi := \exists X_1' \forall Y_1 \cdots \exists X_m \forall Y_m \exists X_{m+1}\varphi_x$ is true.

If X_1' is empty and $m = 1$, then $\Psi = \forall Y_1 \exists X_2 \varphi_x$. Since Ψ is true and the existential part of φ_x is in MU(1), there must be some $y \in Y_1$ occurring positively and negatively. Then $\forall y \exists X_2(\varphi_x)(y)$ is true. By Lemma 4, then $\forall y \exists X_2(\varphi[x/0])(y)$ is true, hence $\exists X_1 \forall y \exists X_2 \varphi(y)$ is true.

Suppose X_1' is empty but $m > 1$. Then $\Psi = \forall Y_1 \exists X_2 \forall Y_2 \cdots \exists X_m \forall Y_m \exists X_{m+1}\varphi_x$. Suppose some $y \in Y_1$ occurs both negatively and positively in φ_x, then it is easy to see that $\forall y \exists X_2 \cdots \exists X_{m+1}\ (\varphi_x)(y)$ is true. By Lemma 4, $\forall y \exists X_2 \cdots \exists X_{m+1}(\varphi[x/0])(y)$ is true. Therefore, $\exists X_1 \forall y \exists X_2 \cdots \exists X_{m+1}\varphi(y)$ is true. Suppose no variable $y \in Y_1$ occurs both negatively and positively, then $\exists X_2 \forall Y_2 \cdots \exists X_m \forall Y_m \exists X_{m+1}(\varphi_x)'$ is also true, here $(\varphi_x)'$ is obtained from φ_x by the deletion of all occurrences of y or $\neg y$ for all $y \in Y_1$. Now Lemma 2 follows from the induction hypothesis and Lemma 4.

If X_1' is non-empty, Lemma 2 follows from the induction hypothesis and Lemma 4. ■

Lemma 5. *Let $\Phi = \exists X\forall y\exists Z\varphi$ be in QCNF with $\varphi_{|\exists} \in MU(1)$. Then, Φ is true if and only if there exist $L_1, \ldots, L_s \in lit(X)$ such that $var(\varphi_{L_1\cdots L_s}) \cap X = \emptyset$ and $\forall y\exists Z\varphi_{L_1\cdots L_s}$ is true.*

Proof. For $X = \{x_1, \ldots, x_n\}$, we proceed by an induction on n. For $n = 1$ the claim follows from Corollary 2. Suppose $n > 1$. Again by Corollary 2, Φ is true if and only if either $\exists x_2 \cdots \exists x_n Q\varphi_{x_1}$ is true or $\exists x_2 \cdots \exists x_n Q\varphi_{\neg x_1}$ is true. Now the lemma follows from the induction hypothesis. ■

Before proving Lemma 3 we will show some propositions on splitting.

Proposition 3. *Suppose $\varphi \in MU(1)$, $x \in var(\varphi)$ with $(\varphi_x, \varphi_{\neg x})$ a disjunctive splitting. We consider $\varphi_{LL_1\cdots L_s}$ for $L \in \{x, \neg x\}$.*

(1) *Suppose $\varphi^L_{LL_1\cdots L_s}$ does not contain L, then $\varphi_{L_1\cdots L_s} = \varphi_{LL_1\cdots L_s}$.*

(2) *Suppose $\varphi^L_{LL_1\cdots L_s}$ contains L, then $\varphi_{L_1\cdots L_s} = \varphi^L_{LL_1\cdots L_s} + \varphi^{\neg L}_{\neg L}$.*

(3) *Suppose $\varphi^x_{xL_1\cdots L_s}$ contains x, and $\varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$ contains $\neg x$, then $\varphi_{L_1\cdots L_m} = \varphi^x_{xL_1\cdots L_s} + \varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$.*

Proof. (1) This can be seen by induction on s. When $s = 1$ the assertion follows from Proposition 2. Suppose $s > 1$. If $\varphi^L_{LL_1}$ does not contain L then $\varphi_{L_1} = \varphi_{LL_1}$, and the assertion follows. So, we assume $\varphi^L_{LL_1}$ contains L. Then by Proposition 2, $\varphi_{LL_1} = \varphi_{L_1L}$. Thus, $\varphi_{LL_1\cdots L_s} = \varphi_{L_1LL_2\cdots L_s}$. Then by the induction hypothesis, $\varphi_{L_1L_2\cdots L_s} = \varphi_{L_1LL_2\cdots L_s}$, and the assertion follows.

(2) If $s = 1$, the claim follows from Proposition 2. Suppose $s > 1$. From Proposition 2, $\varphi_{LL_1} = \varphi_{L_1L}$, then $\varphi_{LL_1\cdots L_s} = \varphi_{L_1LL_2\cdots L_s}$. Thus by the induction hypothesis, $\varphi_{L_1L_2\cdots L_s} = \varphi^L_{L_1LL_2\cdots L_s} + \varphi^{\neg L}_{L_1\neg L}$. Since $\varphi_{L_1} = \varphi^L_{LL_1} + \varphi^{\neg L}_{\neg L}$, $\varphi_{L_1\neg L} = \varphi_{\neg L}$. The claim follows.

(3) Please notice, that for each $i = 1, \cdots s$, $\varphi^x_{L_1\cdots L_i}$ contains x. Thus from Proposition 2, $\varphi_{xL_1L_2\cdots L_s} = \varphi_{L_1xL_2\cdots L_s} = \cdots = \varphi_{L_1\cdots L_s x}$. Then by (2) we have

$$\varphi_{L_1\cdots L_s} = \varphi^x_{xL_1\cdots L_s} + \varphi^{\neg x}_{\neg x} = \varphi^x_{L_1\cdots L_s x} + \varphi^{\neg x}_{\neg x}.$$

Again by (2) we obtain $\varphi_{L_1\cdots L_m} = \varphi^x_{xL_1\cdots L_s} + \varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$. ■

Lemma 6. *Suppose $\varphi \in MU(1)$ and $X \subseteq var(\varphi)$, $f, g \in \varphi$. Then,*
f and g are connected without X if and only if there is a splitting formula $\varphi_{L_1\cdots L_m}$ such that $(f - \{L_1, \ldots, L_m\})$ and $(g - \{L_1, \ldots, L_m\}) \in \varphi_{L_1\cdots L_m}$, $L_i \in lit(X)$, $i = 1, \ldots, m$, and $\varphi_{L_1\cdots L_m}$ contains no variable in X.

Proof. ($\Leftarrow$) Suppose the splitting formula $\varphi_{L_1\cdots L_m}$ contains the two clauses $(f - \{L_1, \ldots, L_m\})$ and $(g - \{L_1, \ldots, L_m\})$, but contains no variable in X. Then, $(f - \{L_1, \ldots, L_m\})$ and $(g - \{L_1, \ldots, L_m\})$ are connected without X, hence f, g must be connected without X.

($\Rightarrow$) Suppose f and g are connected without X. We proceed by induction on the number of variables in φ. Suppose φ has only one variable, say x. If X is empty, clearly the assertion is true since φ itself contains no variable in X. Suppose $X = \{x\}$, then there is no variable outside X. That means, $f = g$, say the unit clause x, then the empty clause $f - \{x\}$ is in φ_x which contains no variable. Hence, the assertion is true.

Suppose, φ has more than one variable. Let x be a variable such that $(\varphi_x, \varphi_{\neg x})$ is a disjunctive splitting.

Case 1. $x \in X$. Since f and g are connected without X, f and g must be in the same part. Say for example $(f - \{x\}), (g - \{x\}) \in \varphi_x$. Obviously, $(f - \{x\}), (g - \{x\})$ are connected without $X \cap var(\varphi_x)$. Then by the induction hypothesis, there is a splitting formula $\varphi_{xL_1\cdots L_m}$ which do not contain any variable in X, but contains both $(f - \{x, L_1, \ldots, L_m\})$ and $(g - \{x, L_1, \ldots, L_m\})$. The lemma follows.

Case 2. $x \notin X$.

Subcase 2.1. f and g lie in the same part. W.l.o.g., we assume that $(f - \{x\}), (g - \{x\}) \in \varphi_y$. Now $(f - \{x\}), (g - \{x\})$ are still connected without $X \cap var(\varphi_y)$. Then by the induction hypothesis there is a splitting formula $\varphi_{xL_1,\cdots L_m}$ which do not contain any variable in X, but contains both $(f - \{x, L_1, \ldots, L_s\})$ and $(g - \{x, L_1, \ldots, L_s\})$. If $\varphi^x_{xL_1\cdots L_s}$ contains no x, then $\varphi_{L_1\cdots L_s} = \varphi_{xL_1\cdots L_s}$ by Proposition 3(1), and the lemma follows. So, suppose $\varphi^x_{xL_1\cdots L_s}$ contains x. Let $\neg x \vee h \in \varphi$ which is connected to itself. Then by the induction hypothesis there is a splitting formula $\varphi_{\neg xL_{s+1}\cdots L_m}$ in which no variable of X occurs but the clause $(h - \{L_{s+1} \cdots L_m\})$ appears. Now we can see that $\varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$ contains $\neg x$. Then

$$\varphi^x_{xL_1\cdots L_s} + \varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$$

is in MU(1), which is in fact $\varphi_{L_1\cdots L_m}$ by Proposition 3(3). The lemma follows.

Subcase 2.2. f and g lie in different part. Say, $(f - \{x\}) \in \varphi_x$ while $(g - \{\neg x\}) \in \varphi_{\neg x}$. Please notice that f and g are connected without X. Thus, f must be connected without X to a clause f' containing x, and g must be connected without X to clause g' containing $\neg x$. Then by the induction hypothesis, there are $\varphi_{xL_1\cdots L_s}$ and $\varphi_{\neg xL_{s+1}\cdots L_m}$ such that they do not contains variables in X, but $(f - \{x, L_1, \ldots, L_s\}), (f' - \{x, L_1, \ldots, L_s\}) \in \varphi_{xL_1\cdots L_s}$, and $(g - \{\neg x, L_{s+1}, \ldots, L_m\}), (g' - \{\neg x, L_{s+1}, \cdots L_m\}) \in \varphi_{\neg xL_{s+1}\cdots L_m}$. Then we can see that $\varphi^x_{xL_1\cdots L_s}$ contains x and $\varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$ contains $\neg x$. Thus $\varphi^x_{xL_1\cdots L_s} + \varphi^{\neg x}_{\neg xL_{s+1}\cdots L_m}$ is in MU(1), which is in fact $\varphi_{L_1\cdots L_m}$ by Proposition 3(3). The assertion follows. ■

Proof of Lemma 3. For $\Phi = \exists X \forall y \exists Z \varphi$ with $\varphi_{|\exists}$ in MU(1) we have

Φ is true (*Lemma* 5)
$\Leftrightarrow$ there exist $L_1, \ldots, L_s \in lit(X)$ such that in $\varphi_{L_1 \cdots L_s}$ no variable of X occurs and $\forall y \exists Z \varphi_{L_1 \cdots L_s}$ is true.
$\Leftrightarrow$ there exist $L_1, \ldots, L_s \in lit(X)$ such that in $\varphi_{L_1 \cdots L_s}$ no variable of X occurs and both y and $\neg y$ occur in $\varphi_{L_1 \cdots L_s}$.
$\Leftrightarrow$ there exist $L_1, \ldots, L_s \in lit(X)$ such that in $\varphi_{L_1 \cdots L_s}$ no variable of X occurs and there are $f, g \in \varphi$ such that $y \in f$, $\neg y \in g$ and $f - \{L_1, \ldots, L_s\},\ g - \{L_1, \ldots, L_m\} \in \varphi_{L_1 \cdots L_s}$. (*reordering*)
$\Leftrightarrow$ there are $f, g \in \varphi$ with $y \in f$, $\neg y \in g$ and there exist $L_1, \ldots, L_s \in lit(X)$ such that in $\varphi_{L_1 \cdots L_s}$ no variable of X occurs and $f - \{L_1, \ldots, L_s\},\ g - \{L_1, \ldots, L_s\} \in \varphi_{L_1 \cdots L_s}$. (*by Lemma* 6)
$\Leftrightarrow$ there are $f, g \in \varphi$ such that $y \in f$, $\neg y \in g$ and f, g are connected without X. ■

We shall conclude this section by showing that there is a polynomial-time algorithm which can find models for true QCNF formulas with maximal deficiency 1. First, we need the following proposition.

Proposition 4. *Suppose $\varphi \in MU(1)$, $f_1, \cdots f_n$ are some clauses in φ and $x_1, \ldots, x_{n-1}$ some variables of φ. If $x_i \in f_i$ and $\neg x_i \in f_{i+1}$ for each i, then $\varphi - \{f_1, \ldots, f_n\}$ can be satisfied by a truth assignment defined on $var(\varphi) - \{x_1, \ldots, x_{n-1}\}$.*

Proof. We shall prove the proposition by induction on the number of variables in φ.

Suppose $\varphi = \{x, \neg x\}$. Then there are two cases: $n = 1$ or $n = 2$. If $n = 1$ then the proposition is true since we need not remove any variable. When $n = 2$ the assertion is also true because after the deletion of the clauses f_1, f_2, the resulting formula is the empty formula.

Now suppose φ has more than one variables. Then there is a variable x such that $(\varphi_x, \varphi_{\neg x})$ is a disjunctive splitting. Please note that the proposition holds when $n = 1$. Thus we assume $n > 1$.

Case 1. $x \notin \{x_1, \ldots, x_n\}$. Then the clauses $f_1, \ldots, f_n$ must be in one part because of the disjunctive splitting. The assertion follows easily from the induction hypothesis.

Case 2. x is x_i for some $i = 1, \ldots, n$. Then $f_1, \ldots, f_i$ are in one part while $f_{i+1}, \ldots, f_n$ lie in the other part. Obviously, the proposition follows from the induction hypothesis. ■

Lemma 7. *There is a polynomial-time algorithm which, can decide the truth value of a input formula Φ with $d^*(\Phi) = 1$, and besides, can compute a model for Φ if it is true.*

Proof. Let Φ be the input formula with $d^*(\Phi) = 1$. First compute a stable sub-formula Φ' of Φ such that $d(\Phi') = 1$. Please note that Φ' can be computed in polynomial time [9]. Then the remaining part can be satisfied by a truth assignment t defined on existential variables not occurring in the matrix of Φ'. Again, such a t can be computed efficiently. Then any model of Φ' can be easily extended to a model of Φ by assigning the remaining existential variables the constant functions determined by t. Thus w.o.l.g. we assume that Φ itself is stable.

Then check the satisfiability of $\varphi_{|\exists}$. If it is satisfiable, then compute a $\{0, 1\}$-model for Φ. This can be done in polynomial time since $d^*(\Phi) = 1$. So, we suppose $\varphi_{|\exists}$ is unsatisfiable, that is, $\varphi_{|\exists} \in$ MU(1).

If for any universal variable y, there are no clauses $(y \vee f)$ and $(\neg y \vee g)$ such that they are connected without $X := \{x \mid$ "$\exists x$" procedes "$\forall y$" in the prefix of $\Phi\}$, then output false and stop. Else, find a universal variable y, clauses $f_1 := (y \vee f),\ f_2, \ldots, f_{n-1}, f_n := (\neg y \vee g)$, and literals $L_1, L_2, \ldots, L_{n-1}$ satisfying

(1) For each $1 \le i < n$, L_i is an existential literal not in $X \cup \{\neg x \mid x \in X\}$.
(2) For each $1 \le i < n$, $L_i \in f_i$, $\neg L_i \in f_{i+1}$.

This can also be done in polynomial time since a path from two nodes can be found efficiently if they are connected.

By Proposition 4 we know that $Q(\varphi - \{f_1, \ldots, f_n\})$ can be satisfied by a $\{0, 1\}$-model M defined on all existential variables except those of $L_1, \ldots, L_{n-1}$. From the proof of Proposition 4 it is not hard to see that M can be find in polynomial time.

Now we extend M to M' as follows. For each L_i $(1 \le i \le n)$, if L_i is positive, say x_j, then assign $\neg y$ to x_j; if L_i is $\neg x_j$, then assign y to x_j. That is, L_i always takes the same value as $\neg y$. Clearly, for all clauses among $f_1, \ldots, f_n$, after replacing each L_i by $\neg y$, the resulting clauses are tautological. Thus, M' is a model of Φ. This completes the proof. ■

5. Maximal deficiency *k*

In this chapter we prove the general theorem that the truth of formulas with maximal deficiency k depends, besides the existential variables, only on a fixed number of universal variables. As a consequence we can show that the satisfiability problem for formulas with fixed deficiency is in NP and that the minimal falsity problem MF(k) is in D^P. Moreover, we prove the polynomial–time solvability of MF(k) if we restrict ourselves to the subclasses QEHORN and QE2-CNF.

Theorem 2. *For any $k \ge 1$ and any true QCNF formula $\Phi = Q\varphi$ with $d^*(\Phi) = k$, there is a set U of universal variables with $|U| \le 2^{4k/3}$, and there is a CNF-model $M = (f_1, \ldots, f_m)$ for Φ such that for all i, $var(f_i) \subseteq U$ and the number of clauses of f_i is not more than 2^k.*

Proof. Suppose, Φ is not stable and has the form $Q\varphi$. Then the formula can be divided into two parts, $Q(\varphi_1 \wedge \varphi_2)$, where $\Phi_1 := Q\varphi_1$ is a stable subformula with $d(\Phi_1) = k$ (Lemma 1(2)). The formula φ_2 contains a non-empty set E of existential variables, such that none of the variables in E occurs in the clauses of φ_1, but any clause of φ_2 contains such a variable. The formula $Q\varphi_2$ can be made true independently of the stable subformula and the model formulas are either 0 or 1. Subsequently, we therefore assume that the formulas are stable.

We prove the theorem by an induction on $k \geq 1$.
Suppose $k = 1$. If $\varphi_{|\exists}$ is satisfiable, then clearly Φ has a $\{0, 1\}$-model, and the theorem follows. So, we suppose $\varphi_{|\exists}$ is unsatisfiable. Because Φ is stable, that means any proper subformula has a deficiency less than 1, any subformula of $\varphi_{|\exists}$ is satisfiable. Therefore, $\varphi_{|\exists}$ is in MU(1).

Notice that Φ is true. Theorem 1 implies, that the truth of the formula depends besides the existential variables only on at most one universal variable. Say y is that universal variable. Then we can delete all the other universal variables and obtain a formula $\exists x_1 \cdots \exists x_m \forall y \exists z_1 \cdots \exists z_n \phi$. Obviously, the model formulas for x_i are constants and the model formulas for z_j are constants or depend only on the universal variable y.

Suppose $k > 1$ and the assertion holds for formulas with maximal deficiency less than k. We proceed by an induction on the length of the prefix. In case of one quantifier, that means a formula of the form $\exists y \phi$, obviously the model is a constant 0 or 1. Now, assume that the prefix contains more than one quantifier. There are two cases according to the left-most quantifier.

Case 1: $\Phi = \exists y Q\varphi$. Then the model formula for y is either 0 or 1. We simplify the formula by means of this truth value. Then $Q\varphi[y/1]$ (resp. $Q\varphi[y/0]$) is true. Moreover, the maximal deficiency of $Q\varphi[y/1]$ (resp. $Q\varphi[y/0]$) is at most k, because Φ is stable. By the induction hypothesis the theorem holds.

Case 2. $\Phi = \forall x Q\varphi$. If x (resp $\neg x$) is a pure literal, then Φ is true if and only if $Q\varphi'$ is true. Here, φ' is obtained from φ by the deletion of all occurrences of x (resp. $\neg x$). Thus, any model of $Q\varphi'$ is a model of Φ, too. Now the assertion follows from the induction hypothesis.

We assume that x occurs positively and negatively in Φ. At first we consider $Q\varphi[x/1]$. Clearly, $(\varphi[x/1])_{|\exists}$ is a proper subset of $\varphi_{|\exists}$. Thus, $Q\varphi[x/1]$ has maximal deficiency at most $k - 1$. Then by the induction hypothesis, let U_1 be a set of universal variables with $|U_1| \leq 2^{4(k-1)/3}$, and $(f_1, f_2, \ldots, f_m)$ be a CNF-model of $Q\varphi[x/1]$ such that $var(f_i) \subseteq U_1$, and f_i has at most 2^{k-1} clauses for each $i = 1, \ldots, m$.

By the same argument, let U_0 be a set of universal variables with $|U_0| \leq 2^{4(k-1)/3}$, and $(g_1, g_2, \ldots, g_m)$ be a CNF-model of $Q\varphi[x/0]$ such that $var(g_i) \subseteq U_0$, and g_i has at most 2^{k-1} clauses for each $i = 1, \ldots, m$.

Now we define h_i for each $1 \leq i \leq m$ as follows: $h_i = (\neg x \vee f_i) \wedge (x \vee g_i)$. Since $h_i[x/1] = f_i$, $h_i[x/0] = g_i$, it follows that $(h_1, h_2, \ldots, h_m)$ is a model of Φ.

Note that h_i is not a CNF formula, however, by the distributive law we can distribute $\neg x$ (resp. x) to each clause of f_i (resp. g_i). Then we get a CNF-model $(h'_1, h'_2, \ldots, h'_m)$. The number of clauses of h'_i is at most $2^{k-1} + 2^{k-1} = 2^k$.

Let $U := U_0 \cup U_1 \cup \{x\}$. Then for each i, it holds $var(h_i) \subseteq U$. Because of $k > 1$, we have

$$|U| \leq |U_1| + |U_0| + 1 \leq 2^{4(k-1)/3} + 2^{4(k-1)/3} + 1 = 2^{4k/3-1/3} + 1 \leq 2^{4k/3}. \quad \blacksquare$$

The proof of Theorem 2 just tell us that there exists a small set U of universal variables, it does not provide a way to construct the set U. Because such a set U exists only for true formulas, there seems no algorithms which constructs the set U without testing the truth value of the formula.

Let Φ be in QCNF and U a set of universal variables, that means $U \subseteq var_\forall(\Phi)$. We delete all universal variables which are not in U. The resulting formula is written as Φ^U. For fixed $k > 0$, we introduce the sets QSAT(k) and SubQSAT(k). Roughly speaking, QSAT(k) is the class of QCNF formulas Φ such that Φ has the maximal deficiency k, and that for each clause ϕ of Φ, $\Phi - \{\phi\}$ is true. While subQSAT(k) is the class of true QCNF formulas with maximal deficiency $\leq k$. More technically,

$$\begin{aligned}\text{QSAT(k)} \;=\; &\{\Phi \in \text{QCNF} \mid d^*(\Phi) = k \text{ and for each clause } \phi \text{ there is some} \\ &\;U \subseteq var_\forall(\Phi) \text{with } |U| \leq 2^{4k/3} \text{ and } (\Phi - \{\phi\})^U \text{ is true}\}\end{aligned}$$

$$\text{SubQSAT}(k) = \{\Phi \in \text{QCNF} \mid d^*(\Phi) \leq k \text{ and there is some subset } U \subseteq var_\forall(\Phi) \text{ with } |U| \leq 2^{4k/3} \text{ and } \Phi^U \text{ is true}\}.$$

Theorem 3. *Let k be fixed.*

(1) *The satisfiability problem for QCNF with maximal deficiency k is in NP.*
(2) *The minimal falsity problem MF(k) for QCNF is in D^P.*

Proof. (1) Suppose a formula Φ has maximal deficiency k. If k is less than 1 then Φ is true, because of Lemma 1. Therefore, we can assume $k \geq 1$. Let $\Phi = \forall x_1 \exists y_1 \cdots \forall x_n \exists y_n \varphi$ be a QCNF formula. To determine whether the formula is true, we can guess a set U with at most $2^{4k/3}$ universal variables and a sequence $(f_1, f_2, \ldots, f_n)$ of CNF formulas such that $var(f_i) \subseteq U$ for all $i = 1, \ldots, n$. Then we check whether $\varphi^U[y_1/f_1, \ldots, y_n/f_n]$ is tautological. The correctness of the above procedure follows from Theorem 2. Please note that $\varphi^U[y_1/f_1, \ldots, y_n/f_n]$ has at most $2^{4k/3}$ variables. Hence, whether it is a tautology can be decided in linear time depending on the length of the formula Φ.

(2) Because for $d^*(\Phi) = k$ the formulas Φ^U and $(\Phi - \varphi)^U$ have maximal deficiency k, the classes QSAT(k) and SubQSAT(k) belong to NP. It remains to show that MF(k) = QSAT(k) − SubQSAT(k).

Suppose $\Phi \in \mathrm{MF}(k)$. We have $d^*(\Phi) = k$, because of Lemma 1. Since Φ is false and by means of Theorem 2, we see that Φ does not belong to SubQSAT(k). Because for each clause $\phi \in \Phi$, the formula $\Phi - \{\phi\}$ is true, Φ is in QSAT(k). That shows $\Phi \in \mathrm{QSAT}(k) - \mathrm{SubQSAT}(k)$.

Now suppose, we have $\Phi \in \mathrm{QSAT}(k) - \mathrm{SubQSAT}(k)$. By means of Theorem 2 and because of $\Phi \in \mathrm{QSAT}(k)$, Φ is true after the deletion of an arbitrary clause. Since $\Phi \notin \mathrm{SubQSAT}(k)$, the formula Φ is false. That proves $\Phi \in \mathrm{MF}(k)$. Because of Lemma 1(3) and $d^*(\Phi) = k$ we have $d(\Phi) = k$ and therefore $\Phi \in \mathrm{MF}(k)$. ■

The classes QEHORN and QE2-CNF are subclasses of QCNF, for which the existential part of the matrix is a Horn formula or a 2-CNF formula. The evaluation problem for these two classes remains PSPACE-complete. The unfolding is a procedure to transform a QCNF formula equivalently into an existentially quantified formula. More precisely, step by step as long as an universal variable exists we can apply the following unfolding step to the formula: Replace $Q\forall x\exists\vec{y}\varphi$ by $Q\exists\vec{y}\exists\vec{z}(\varphi[x/1] \wedge \varphi[x/0, \vec{y}/\vec{z}])$ for new variables $\vec{z}$. Clearly, if $Q\forall x\exists\vec{y}\varphi$ is in QEHORN (resp. QE2-CNF) then $Q\exists\vec{y}\exists\vec{z}(\varphi[x/1] \wedge \varphi[x/0, \vec{y}/\vec{z}])$ remains in QEHORN (resp. QE2-CNF). Thus, by unfolding formulas in QEHORN (resp. QE2-CNF) we get an existentially quantified QCNF with matrix in HORN (resp. 2-CNF). It is not hard to see that if we fix the number universal variables, then the unfolding procedure costs polynomial time. Since the satisfiability problem for HORN and 2-CNF is tractable, by Theorem 2 we have the following lemma.

Lemma 8. *Let k be fixed.*

(1) *The satisfiability problem for formulas in QEHORN and QE2-CNF with maximal deficiency k is solvable in polynomial time,*
(2) *The minimal falsity problem for formulas in QEHORN and QE2-CNF with deficiency k can be decided in polynomial time.*

Proof. We only prove (1) since (2) follows directly from (1). We apply the following polynomial-time algorithm to decide the truth value of QEHORN (resp. Q2CNF) formulas with fixed maximal deficiency. Given an arbitrary input formula Φ in QEHORN (resp. QE2CNF) with $d^*(\phi) = k$. For each subset U of universal variables with $|U| \le 2^{4k/3}$, unfolding Φ^U to an existentially quantified HORN (resp. 2CNF) formula, denoted by Ψ_U. If Ψ_U is true for some U, then output YES. Otherwise output NO. ■

6. Conclusion

Based on the observation that any true QCNF formula with maximal deficiency $k > 0$ has a model over at most $2^{4k/3}$ universal variables, we have shown that the evaluation problem for formulas with fixed maximal deficiency is in NP and that the minimal falsity problem for MF(k) is in D^P. Moreover, for two subclasses QEHORN and QE2-CNF of QCNF with PSPACE–complete satisfiability problem a polynomial-time algorithm MF(k) has been established. It is known that MF(1) is solvable in polynomial time, whereas for $k > 1$, whether MF(k) can be solved in polynomial time remains open. The model-size approach seems not sufficient to analysis the exact complexity of MF(k), because it does not provide further information about models (e.g., structure of model functions), and hence, to decide the truth value we have to non-deterministically guess a potential model and check.

Acknowledgements

The second author's research was partially supported by the NSFC project under grant number: 60573011, 10410638 and a MOE project under grant number: 05JJD72040122.

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