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Fault-tolerant embedding of paths in crossed cubes

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ABSTRACT

The crossed cube CQ_n is an important variant of the hypercube Q_n and possesses many desirable properties for interconnection networks. This paper shows that in CQ_n with f_v faulty vertices and f_e faulty edges there exists a fault-free path of length ℓ between any two distinct fault-free vertices for each ℓ satisfying $2^{n-1} - 1 \leq \ell \leq 2^n - f_v - 1$ provided that $f_v + f_e \leq n - 3$, where the lower bound of ℓ and the upper bound of $f_v + f_e$ are tight for some n. Moreover, this result improves the known result that CQ_n is $(n-3)$-Hamiltonian connected.

1. Introduction

It is well-known that the underlying topology of an interconnection network can be represented by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network. In this paper, we use graphs and interconnection networks interchangeably.

The embedding problem, which maps a guest graph into a host graph, is an important topic in recent years. Many graph embeddings take paths, cycles, trees, and meshes as guest graphs [6,8–12,15,18,19,21], because they are the architectures widely used in parallel computing systems. In particular, paths are probably the most common structure of graph embedding in parallel computing since paths are often used to model linear arrays [3].

It is well known that the hypercube network Q_n is one of the most popular interconnection networks since it has a simple structure and is easy to implement. As an important variant of Q_n, the crossed cube CQ_n, proposed first by Efe [4,5], has many properties superior to Q_n. For example, CQ_n has a diameter $\lceil (n+1)/2 \rceil$ about half of the diameter of Q_n, and a $(2^n - 1)$-vertex complete binary tree can be embedded into CQ_n but not into Q_n [15]. The variously desirable properties of CQ_n have been extensively investigated in the literature (see, for example, [1,6,7,13–17,19,21,22]).

Failures are inevitable when a network is put in use. Therefore, it is practically meaningful to consider faulty networks. In this paper, we study embedding of paths of different lengths between any two vertices in faulty crossed cubes. Use f_v and f_e to denote the numbers of faulty vertices and faulty edges, respectively, in CQ_n. We prove that there exists a fault-free path of length ℓ between any two distinct fault-free vertices in CQ_n, for each ℓ satisfying $2^{n-1} - 1 \leq \ell \leq 2^n - f_v - 1$, provided that $f_v + f_e \leq n - 3$. The lower bound of ℓ and the upper bound of $f_v + f_e$ for a successful embedding are tight for some n. In other words, the result does not hold if $\ell \leq 2^{n-1} - 2$ or $f_v + f_e \geq n - 2$. Moreover, our result improves the result that CQ_n is $(n-3)$-Hamiltonian connected obtained by Huang et al. [13] and Chen et al. [2].

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The rest of the paper is organized as follows. Section 2 gives some definitions and notations. Section 3 discusses path embedding in faulty crossed cubes. Finally, we conclude the paper in Section 4.

2. Preliminaries

Let $G = (V, E)$ be a connected graph, where $V = V(G)$ and $E = E(G)$ denote the vertex set and the edge set of G, respectively. Two vertices u and v of G are *adjacent* if $(u, v) \in E$. A *path* is a sequence of adjacent vertices, written as $P = \langle u, u_1, \ldots, u_k, v\rangle$, in which all vertices are distinct. We use uv-path to denote a path between vertices u and v. For a uv-path P, $P + (u, v)$ is called a *cycle*. The length of a path (or a cycle) is the number of edges contained in it. A cycle which contains each vertex in G exactly once is called a *Hamiltonian cycle*. A graph G is *Hamiltonian* if there is a Hamiltonian cycle in G. A path P can also be denoted by $P = \langle u, P_1, u_i, u_{i+1} \ldots, u_j, P_2, u_t, u_{t+1}, \ldots, u_k, v\rangle$, where P_1 is the path $\langle u, u_1, \ldots, u_i\rangle$ and P_2 is the path $\langle u_j, u_{j+1}, \ldots, u_t\rangle$. If (x, u) is an edge not on a path $P = \langle u, u_1, \ldots, v\rangle$, then the path between x and v obtained by adding edge (x, u) to P is denoted by $P + (x, u)$.

Let $F \subset V(G) \cup E(G)$ denote a faulty set in G. The graph $G - F$ is the subgraph obtained from G by deleting all elements in F. A subgraph of G is fault-free if it contains no faulty elements in F. A graph G is k-Hamiltonian if $G - F$ remains Hamiltonian for any $F \subset E(G) \cup V(G)$ with $|F| \le k$.

We now recall the definition of the crossed cube proposed by Efe in [5]. An n-bit string x is written as $x = x_{n-1}x_{n-2} \ldots x_1x_0$, where $x_i \in \{0, 1\}$ for $i = 0, 1, \ldots, n-1$. The complement of x_i is denoted by $\bar{x}_i$ (i.e., $\bar{0} = 1$ and $\bar{1} = 0$). Two binary strings $x = x_1x_0$ and $y = y_1y_0$ are pair-related, denoted as $x \sim y$, if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. The n-dimensional crossed cube CQ_n has 2^n vertices labelled by n-bit strings and can be recursively defined as follows.

CQ_1 is a complete graph with two vertices labelled by 0 and 1. For $n \ge 2$, CQ_n is obtained by taking two copies of CQ_{n-1}, denoted by CQ^0_{n-1} with vertex-set

$$V(CQ^0_{n-1}) = \{0x_{n-2} \ldots x_1x_0 : x_i = 0 \text{ or } 1, i = 0, 1, \ldots, n-2\}$$

and CQ^1_{n-1} with vertex-set

$$V(CQ^1_{n-1}) = \{1y_{n-2} \ldots y_1y_0 : y_i = 0 \text{ or } 1, i = 0, 1, \ldots, n-2\},$$

respectively, and adding an edge joining $0x_{n-2} \ldots x_1x_0 \in V(CQ^0_{n-1})$ and $1y_{n-2} \ldots y_1y_0 \in V(CQ^1_{n-1})$ if and only if

(1) $x_{n-2} = y_{n-2}$ if n is even, and

(2) $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \le i < \lfloor \frac{n-1}{2} \rfloor$.

According to the definition, we can denote $CQ_n = L \oplus R$, where $L = CQ^0_{n-1}$ and $R = CQ^1_{n-1}$. A subgraph of CQ_n, which is also a crossed cube of smaller dimension, is called a subcube of CQ_n. Thus L and R are two $(n-1)$-dimensional subcubes of CQ_n.

We recall some structural properties of CQ_n presented in [5], which are to be used in our construction of paths of different lengths. For $\ell < n$, the ℓ-prefix of $x = x_{n-1}x_{n-2} \ldots x_1x_0$, denoted by $p_\ell(x)$, is the substring $x_{n-1}x_{n-2} \ldots x_{n-\ell}$. For an ℓ-bit string x with $\ell \le n$, denote by $P_x(CQ_n)$ the subgraph of CQ_n induced by the set of all vertices with the ℓ-prefix x. For two distinct ℓ-bit strings x and y with $\ell < n$, $P_x(CQ_n)$ and $P_y(CQ_n)$ are adjacent subgraphs if $P_x(CQ_n)$ and $P_y(CQ_n)$ can be linked by an edge in CQ_n. Let $P_{x,y}(CQ_n)$ denote the subgraph of CQ_n induced by $P_x(CQ_n) \cup P_y(CQ_n)$. Then $P_x(CQ_n)$ is isomorphic to $CQ_{n-\ell}$ and $P_{x,y}(CQ_n)$ is isomorphic to $CQ_{n-\ell+1}$ if $P_x(CQ_n)$ and $P_y(CQ_n)$ are adjacent. In particular, if x and y are both 2-bit strings and (x, y) is an edge of CQ_2, then $P_x(CQ_{2k})$ and $P_y(CQ_{2k})$ are adjacent subgraphs isomorphic to CQ_{2k-2}, and $P_{x,y}(CQ_{2k})$ is isomorphic to CQ_{2k-1}.

For even n, we can contract those vertices in CQ_n having the same prefix of length two into one vertex and obtain a graph with four vertices. It is shown in Fig. 1(a) that this four-vertex graph is isomorphic to CQ_2.

Similarly, if n is odd, we can contract those vertices in CQ_n with the same prefix of length three into one vertex and obtain a graph with eight vertices, as shown in Fig. 1(b), which is isomorphic to CQ_3.

3. Fault-tolerant embedding of paths in crossed cubes

In this section, we prove our main result. For all the terminology and notation not defined here, we follow [20]. The following two lemmas will be used in the proof of our theorem.

Lemma 1 (*Fan et al. [6]*). *If $n \ge 3$, then for any two different vertices u and v in CQ_n, there exists a uv-path of every length from $\lceil (n+1)/2 \rceil + 1$ to $2^n - 1$.*

Lemma 2 (*Huang et al. [13], Chen et al. [2]*). *CQ_n is $(n-2)$-Hamiltonian for $n \ge 3$.*

Theorem 3. *For $n \ge 3$ and any $F \subset V(CQ_n) \cup E(CQ_n)$ with $|F| \le n-3$, there exists a path of length ℓ between any two distinct vertices in $CQ_n - F$ for each ℓ satisfying $2^{n-1} - 1 \le \ell \le 2^n - f_v - 1$, where f_v is the number of vertices in F.*

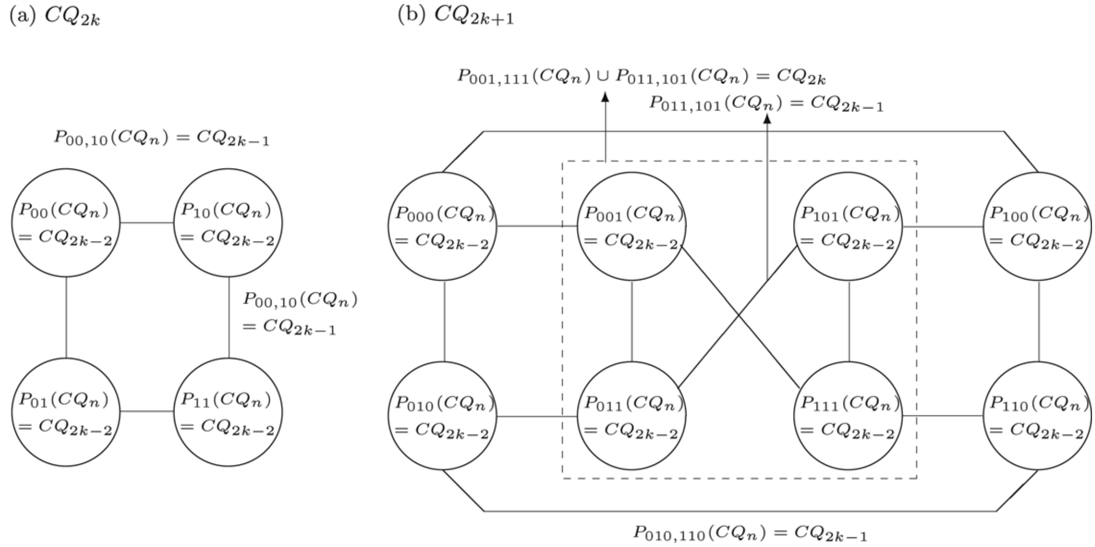

Fig. 1. Subgraphs of CQ_{2k} and CQ_{2k+1}.

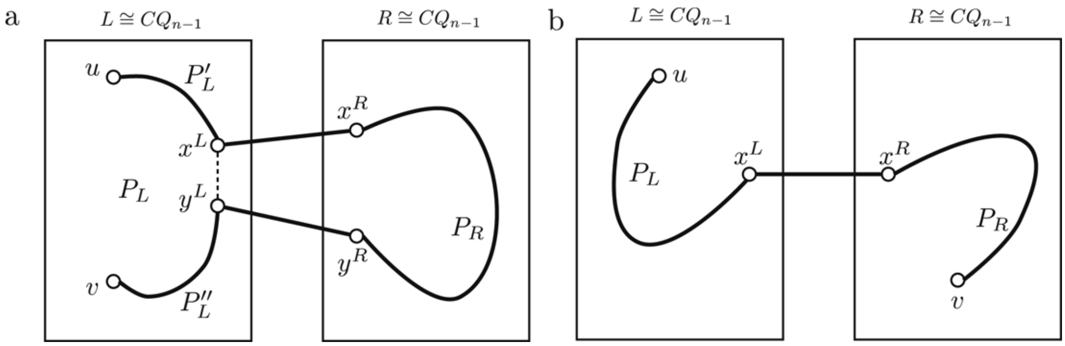

Fig. 2. Illustration for the proof of Case 1. (A straight line represents an edge, a curved line represents a path between two vertices and a dashed line represents a removed edge.)

Proof. We prove the theorem by induction on $n \geq 3$. By Lemma 1, the conclusion is true for CQ_3. Assume that the theorem is true for CQ_{n-1} with $n \geq 4$. We now consider CQ_n. Let $F \subset V(CQ_n) \cup E(CQ_n)$ be a set of faulty elements in $CQ_n = L \oplus R$ with $|F| \leq n-3$. We denote $F^L = F \cap L$, $F^R = F \cap R$, $F_v = F \cap V(CQ_n)$, $f_v = |F_v|$, $f_v^L = |F_v \cap V(L)|$, $f_v^R = |F_v \cap V(R)|$. Without loss of generality, we may assume $|F^L| \geq |F^R|$. Let u and v be any two fault-free vertices in CQ_n. From the structure of CQ_n (See Fig. 1), there are many choices of L and R such that $CQ_n = L \oplus R$ (L and R are two $(n-1)$-dimensional subcubes of CQ_n). We choose L and R such that L contains as few as possible elements in F. We will construct the desired paths according to the following two cases.

Case 1. $|F^L| \leq n-4$.

Case 1.1. Both u and v are in L or R. Without loss of generality, we may assume that both u and v are in L.

For $2^{n-1}-1 \leq \ell \leq 2^n - f_v - 1$, we can write $\ell = \ell_0 + \ell_1 + 1$ where $2^{n-2}-1 \leq \ell_0 \leq 2^{n-1} - f_v^L - 1$ and $2^{n-2}-1 \leq \ell_1 \leq 2^{n-1} - f_v^R - 1$. By the induction hypothesis, there exists a fault-free uv-path P_L of length ℓ_0 in L. Since $\ell_0 \geq 2^{n-2}-1$, there must exist an edge (x^L, y^L) on the path $P_L = \langle u, P'_L, x^L, y^L, P''_L, v\rangle$ such that the two crossed edges (x^L, x^R) and (y^L, y^R) are fault-free. Suppose, to the contrary, that there does not exist such an edge, then there are at least $\lceil (2^{n-2}-1)/2 \rceil = 2^{n-3}$ faults outside L. However, $n-4 \geq |F^L| \geq 2^{n-3} > n-3$ for $n \geq 4$, a contradiction. Since $|F^R| \leq |F^L| \leq n-4$ and $2^{n-2}-1 \leq \ell_1 \leq 2^{n-1} - f_v^R - 1$, by the induction hypothesis, there exists a fault-free $x^R y^R$-path P_R of length ℓ_1 in R. Then $\langle u, P'_L, x^L, x^R, P_R, y^R, y^L, P''_L, v\rangle$ is a uv-path of length ℓ in $CQ_n - F$ (See Fig. 2(a)).

Case 1.2. $u \in L$ and $v \in R$.

Since $|F| \leq n-3$ and there are 2^{n-1} crossed edges between L and R, there exists a fault-free crossed edge (x^L, x^R) in CQ_n where $x^L \neq u$ and $x^R \neq v$. For $2^{n-1}-1 \leq \ell \leq 2^n - f_v - 1$, we can write $\ell = \ell_0 + \ell_1 + 1$ where $2^{n-2}-1 \leq \ell_0 \leq 2^{n-1} - f_v^L - 1$ and $2^{n-2}-1 \leq \ell_1 \leq 2^{n-1} - f_v^R - 1$. By the induction hypothesis, there exists a fault-free ux^L-path P_L of length ℓ_0 in L and there exists a fault-free $x^R v$-path P_R of length ℓ_1 in R. Then $\langle u, P_L, x^L, x^R, P_R, v\rangle$ is a uv-path of length ℓ in $CQ_n - F$ (See Fig. 2(b)).

Case 2. $|F^L| = n-3$ for any choice of L and R.

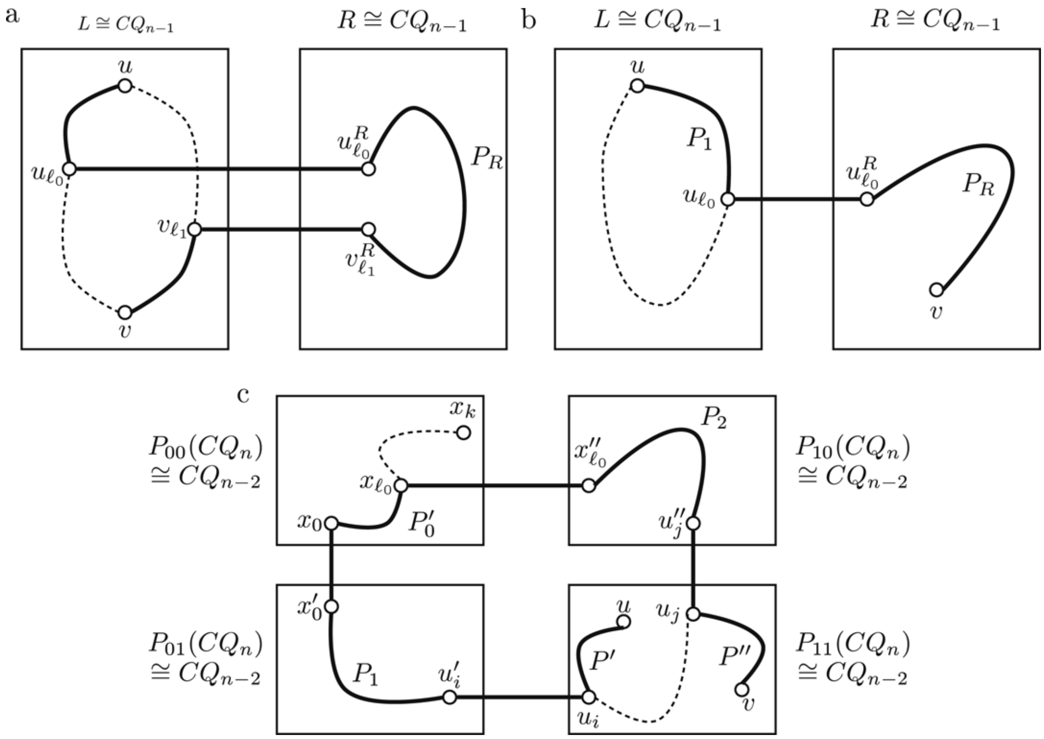

Fig. 3. Illustration for the proof of Case 2. (A straight line represents an edge, a curved line represents a path between two vertices.)

Case 2.1. Both u and v are in L. By Lemma 2, there is a Hamiltonian cycle $\langle u, u_1, \ldots, u_m, v, v_1, \ldots, v_n, u\rangle$ in $L - F$ where $m + n = 2^{n-1} - f^L_v - 2$. (Notice that m or n may be equal to 0.)

For $2^{n-1} - 1 \le \ell \le 2^n - f_v - 1$, we can write $\ell = \ell_0 + \ell_1 + \ell_2 + 2$ where $0 \le \ell_0 \le m$, $0 \le \ell_1 \le n$ and $2^{n-2} - 1 \le \ell_2 \le 2^{n-1} - 1$. We use u^R_i and v^R_j to denote the neighbors of u_i and v_j in R, respectively. For any two vertices u^R_i and v^R_j in R and any integer ℓ_2 with $2^{n-2} - 1 \le \ell_2 \le 2^{n-1} - 1$, by the induction hypothesis, there exists a $u^R_i v^R_j$-path P_R of length ℓ_2 in R. Then $\langle u, u_1, \ldots, u_{\ell_0}, u^R_{\ell_0}, P_R, v^R_{\ell_1}, v_{\ell_1}, \ldots, v_1, v\rangle$ is a uv-path of length ℓ in $CQ_n - F$ (See Fig. 3(a)).

Case 2.2. $u \in L$ and $v \in R$. By Lemma 2, there is a Hamiltonian cycle $C = \langle u, u_1, \ldots, u_m, u\rangle$ in $L-F$ where $m = 2^{n-1} - f^L_v - 1$.

For $2^{n-1} - 1 \le \ell \le 2^n - f_v - 1$, we can write $\ell = \ell_0 + \ell_1 + 1$ where $1 \le \ell_0 \le m$ and $\ell_0 \ne (m+1)/2$ if m is an odd integer, $2^{n-2} \le \ell_1 \le 2^{n-1} - 1$. There are two different paths $P_1 = \langle u, u_1, \ldots, u_{\ell_0}\rangle$, $P_2 = \langle u, u_m, \ldots, u_{m-\ell_0+1}\rangle$ of length ℓ_0 on the cycle C and $u_{\ell_0} \ne u_{m-\ell_0+1}$ since $\ell_0 \ne (m+1)/2$. We use $u^R_{\ell_0}$ and $u^R_{m-\ell_0+1}$ to denote the neighbors of u_{ℓ_0} and $u_{m-\ell_0+1}$ in R, respectively. Then at least one of $u^R_{\ell_0}$ and $u^R_{m-\ell_0+1}$ is different from v. Without loss of generality, assume $u^R_{\ell_0} \ne v$. By the induction hypothesis, there exists a $u^R_{\ell_0} v$-path P_R of length ℓ_1 in R. Then $\langle u, u_1, \ldots, u_{\ell_0}, u^R_{\ell_0}, P_R, v\rangle$ is a uv-path of length ℓ in $CQ_n - F$ (See Fig. 3(b)).

Case 2.3. Both u and v are in R for any choice of L and R.

Case 2.3.1. n is an even integer. We can split CQ_n into four $(n-2)$-dimensional subcubes $P_{00}(CQ_n)$, $P_{01}(CQ_n)$, $P_{10}(CQ_n)$, $P_{11}(CQ_n)$ (See Fig. 1(a)). Without loss of generality, we may assume $L = P_{00,01}(CQ_n)$ and $R = P_{10,11}(CQ_n)$.

If $F \cap P_{00}(CQ_n) \ne \emptyset$ and $F \cap P_{01}(CQ_n) \ne \emptyset$, then we can express $CQ_n = L' \oplus R'$, where $L' = P_{00,10}(CQ_n) \cong CQ_{n-1}$ and $R' = P_{01,11}(CQ_n) \cong CQ_{n-1}$. Clearly, $|F^{L'}| \le n - 4$ and $|F^{R'}| \le n - 4$, which contradicts the choices of L and R. Thus, $F \cap P_{00}(CQ_n) = \emptyset$ or $F \cap P_{01}(CQ_n) = \emptyset$. Without loss of generality, assume $F \cap P_{01}(CQ_n) = \emptyset$. In other words, the faulty elements are all in $P_{00}(CQ_n)$.

If both u and v are in $P_{10}(CQ_n)$, then we can express $CQ_n = L' \oplus R'$, where $L' = P_{00,10}(CQ_n) \cong CQ_{n-1}$ and $R' = P_{01,11}(CQ_n) \cong CQ_{n-1}$. Then $|F^{L'}| \le n - 3$ and $|F^{R'}| = 0$, u and v are in L', which contradicts the choices of u and v.

If $u \in P_{10}(CQ_n)$ and $v \in P_{11}(CQ_n)$, we can express $CQ_n = L' \oplus R'$, where $L' = P_{00,10}(CQ_n) \cong CQ_{n-1}$ and $R' = P_{01,11}(CQ_n) \cong CQ_{n-1}$. Then $|F^{L'}| \le n - 3$ and $|F^{R'}| = 0$, $u \in L'$ and $v \in R'$, which contradicts the choices of u and v.

Thus, both u and v are in $P_{11}(CQ_n)$. Note that the faulty elements are all in $P_{00}(CQ_n)$. Then $P_{01}(CQ_n)$, $P_{10}(CQ_n)$ and $P_{11}(CQ_n)$ are all fault-free. $P_{00}(CQ_n)$ is an $(n-2)$-dimensional subcube of CQ_n, by Lemma 2, it is $(n-4)$-Hamiltonian for $n \ge 5$. For $n = 4$, we leave this particular case to the Appendix. Since $|F| = n - 3$ and $F \subset P_{00}(CQ_n)$, there is a fault-free Hamiltonian path $P_0 = \langle x_0, x_1, \ldots, x_k\rangle$ of length $2^{n-2} - f_v - 1$ in the faulty $P_{00}(CQ_n)$. By Lemma 1, there exists a uv-path $P_3 = \langle u, u_1, \ldots, u_m, v\rangle$ of length $2^{n-2} - 1$ in $P_{11}(CQ_n)$. We use u'_i and u''_i to denote the neighbors of u_i in $P_{01}(CQ_n)$ and $P_{10}(CQ_n)$, respectively; x'_i and x''_i to denote the neighbors of x_i in $P_{01}(CQ_n)$ and $P_{10}(CQ_n)$, respectively.

For $2^{n-1}-1 \le \ell \le 2^n - f_v - 1$ and $n > 4$, we can write $\ell = \ell_0 + \ell_1 + \ell_2 + \ell_3 + 4$ where $0 \le \ell_0 \le 2^{n-2} - f_v^L - 1$, $\lceil (n-1)/2 \rceil + 1 \le \ell_1 \le 2^{n-2} - 1$, $\lceil (n-1)/2 \rceil + 1 \le \ell_2 \le 2^{n-2} - 1$ and $4 \le \ell_3 \le 2^{n-2} - 2$. For $0 \le \ell_0 \le 2^{n-2} - f_v^L - 1$, there is a subpath $P_0' = \langle x_0, x_1, \ldots, x_{\ell_0} \rangle$ of length ℓ_0 on the path P_0. For $\ell_3 \ge 4$, we can write $\ell_3 = \ell' + \ell''$ where $1 \le \ell' \le \ell_3 - 1$ and $1 \le \ell'' \le \ell_3 - 1$. We can select two subpaths $P' = \langle u, u_1, \ldots, u_i \rangle$ and $P'' = \langle u_j, \ldots, u_m, v \rangle$ on the path P_3 such that u_i'(the neighbor of u_i in $P_{01}(CQ_n)$) is different from x_0' and u_j''(the neighbor of u_j in $P_{10}(CQ_n)$) is different from x_{ℓ_0}'' and the lengths of P' and P'' are ℓ' and ℓ'', respectively. By Lemma 1, there is an $x_0'u_i'$-path P_1 of length ℓ_1 in $P_{01}(CQ_n)$ and there is an $x_{\ell_0}''u_j''$-path P_2 of length ℓ_2 in $P_{10}(CQ_n)$. Then $\langle u, P', u_i, u_i', P_1, x_0', x_0, P_0', x_{\ell_0}, x_{\ell_0}'', P_2, u_j'', u_j, P'', v \rangle$ is a uv-path of length ℓ in $CQ_n - F$ (see Fig. 3(c)).

Case 2.3.2. n is an odd integer. We can also split CQ_n into four $(n-2)$-dimensional subcubes $P_{000,010}(CQ_n)$, $P_{001,011}(CQ_n)$, $P_{101,111}(CQ_n)$, $P_{100,110}(CQ_n)$ (See Fig. 1(b)). Without loss of generality, we may assume $L = P_{000,010}(CQ_n) \cup P_{001,011}(CQ_n)$ and $R = P_{101,111}(CQ_n) \cup P_{100,110}(CQ_n)$.

If $F \cap P_{000,010}(CQ_n) \neq \emptyset$ and $F \cap P_{001,011}(CQ_n) \neq \emptyset$, then we can express $CQ_n = L' \oplus R'$, where $L' = P_{000,010}(CQ_n) \cup P_{100,110}(CQ_n) \cong CQ_{n-1}$ and $R' = P_{001,011}(CQ_n) \cup P_{101,111}(CQ_n) \cong CQ_{n-1}$. Clearly, $|F^{L'}| \le n-4$ and $|F^{R'}| \le n-4$, which contradicts the choices of L and R.

Thus, $F \cap P_{000,010}(CQ_n) = \emptyset$ or $F \cap P_{001,011}(CQ_n) = \emptyset$. Without loss of generality, assume $F \cap P_{001,011}(CQ_n) = \emptyset$, that is, the faulty elements are all in $P_{000,010}(CQ_n)$.

If both u and v are in $P_{100,110}(CQ_n)$, then we can express $CQ_n = L' \oplus R'$, where $L' = P_{000,010}(CQ_n) \cup P_{100,110}(CQ_n) \cong CQ_{n-1}$ and $R' = P_{001,011}(CQ_n) \cup P_{101,111}(CQ_n) \cong CQ_{n-1}$. Then $|F^{L'}| \le n-3$ and $|F^{R'}| = 0$, u and v are in L', which contradicts the choices of u and v.

If $u \in P_{100,110}(CQ_n)$ and $v \in P_{101,111}(CQ_n)$, we can express $CQ_n = L' \oplus R'$, where $L' = P_{000,010}(CQ_n) \cup P_{100,110}(CQ_n) \cong CQ_{n-1}$ and $R' = P_{001,011}(CQ_n) \cup P_{101,111}(CQ_n) \cong CQ_{n-1}$. Then $|F^{L'}| \le n-3$ and $|F^{R'}| = 0$, $u \in L'$ and $v \in R'$, which contradicts the choices of u and v.

Thus, both u and v are in $P_{101,111}(CQ_n)$. Note that the faulty elements are all in $P_{000,010}(CQ_n)$. Then the three $n-2$-dimensional subcubes $P_{001,011}(CQ_n)$, $P_{101,111}(CQ_n)$ and $P_{100,110}(CQ_n)$ are all fault-free.

In this subcase, the faulty elements are all in $(n-2)$-dimensional subcube $P_{000,010}(CQ_n)$, both u and v are in $P_{101,111}(CQ_n)$. Then the desired uv-path can be constructed using the method similar with that in Case 2.3.1, the details are omitted.

Remarks. The lower bound on path length ℓ and the upper bound on $|F|$ for a successful embedding are tight in the following sense:

(1) For $n \ge 3$, if $\ell \le 2^{n-1} - 2$, then the theorem does not hold. For example, there is no path of length 2 between any two adjacent vertices in CQ_3.

(2) For $n \ge 3$, if $|F| \ge n-2$, then the theorem does not hold. For example, let $F = \{001\}$, then there is no path of length 6 between the two vertices 010 and 100 in $CQ_3 - F$.

4. Conclusion

In this paper, we prove that there exists a fault-free path of length ℓ between any two distinct fault-free vertices in CQ_n with $|F| \le n-3$ for each ℓ satisfying $2^{n-1} - 1 \le \ell \le 2^n - f_v - 1$. The lower bound on path length ℓ and the upper bound on $|F|$ for a successful embedding are tight for some n.

Since every component in the network may have different reliability, it is important to consider properties of a network with some conditional faults. An interesting question is whether we can do better in some conditional faulty crossed cubes. That is, even when the number of faulty elements is larger than $n-3$, the conclusion is also true in some conditional faulty cases.

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Appendix

In the following, we will construct uv-paths of lengths from 7 to $2^n - f_v - 1$ in CQ_4 where the only faulty element is in $P_{00}(CQ_4)$ and both u and v are in $P_{11}(CQ_4)$ (see Fig. 4). Since $P_{10,11}(CQ_4) \cong CQ_3$ is fault-free, by Lemma 1, there is a fault-free uv-path of length 7 in $P_{10,11}(CQ_4)$, also in the faulty CQ_4.

If the faulty element is a vertex, since CQ_4 is vertex-transitive [14], without loss of generality, we may assume it is 0000. The fault-free paths of lengths from 6 to 14 between 1101 and 1110 are listed as follows:

$P_6 = \langle 1101, 1011, 0001, 0011, 0010, 0110, 1110 \rangle$

$P_7 = \langle 1101, 1011, 0001, 0011, 0101, 0100, 0110, 1110 \rangle$

$P_8 = \langle 1101, 1011, 1010, 0010, 0011, 0101, 0100, 0110, 1110 \rangle$

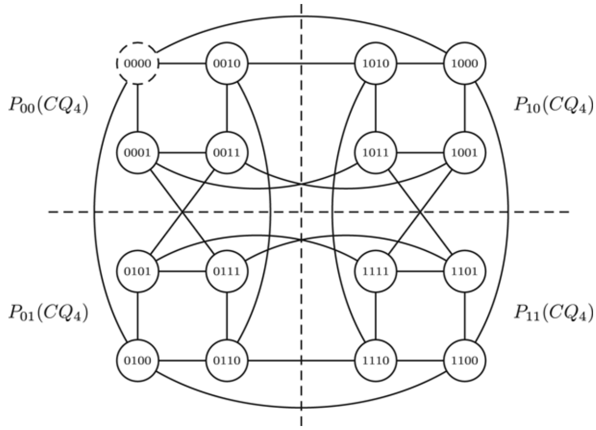

Fig. 4. The crossed cube CQ_4.

$P_9 = \langle 1101, 1011, 1001, 0011, 0001, 0111, 0101, 0100, 0110, 1110\rangle$
$P_{10} = \langle 1101, 1011, 1001, 1000, 1010, 0010, 0011, 0101, 0100, 0110, 1110\rangle$
$P_{11} = \langle 1101, 1011, 1010, 1000, 1001, 0011, 0001, 0111, 0101, 0100, 0110, 1110\rangle$
$P_{12} = \langle 1101, 1011, 1001, 1000, 1010, 0010, 0011, 0001, 0111, 0101, 0100, 0110, 1110\rangle$
$P_{13} = \langle 1101, 1100, 1000, 1001, 1011, 1010, 0010, 0011, 0001, 0111, 0101, 0100, 0110, 1110\rangle$
$P_{14} = \langle 1101, 1111, 1001, 1000, 1010, 1011, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1110\rangle$.

It is clear that the above paths of lengths from 6 to 12 between 1101 and 1110 do not contain any edges in $P_{11}(CQ_4)$.

Then $(1111, 1101) + P_i + (1110, 1100)$ $(i = 6, 7, \ldots, 12)$ are fault-free paths of lengths from 8 to 14 between 1111 and 1100. $(1111, 1101) + P_i$ $(i = 7, 8, \ldots, 13)$ are fault-free paths of lengths from 8 to 14 between 1111 and 1110. $P_i + (1110, 1111)$ $(i = 7, 8, \ldots, 13)$ are fault-free paths of lengths from 8 to 14 between 1101 and 1111. $P_i + (1110, 1100)$ $(i = 7, 8, \ldots, 12)$ are fault-free paths of lengths from 8 to 13 between 1101 and 1100. $P = \langle 1101, 1111, 1110, 1010, 1000, 1001, 1011, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100\rangle$ is a fault-free path of length 14 between 1101 and 1100. $(1100, 1101) + P_i$ $(i = 7, 8, \ldots, 12)$ are fault-free paths of lengths from 8 to 13 between 1100 and 1110. $P = \langle 1110, 1111, 1001, 1000, 1010, 1011, 0001, 0011, 0010, 0110, 0100, 0101, 0111, 1101, 1100\rangle$ is a fault-free path of length 14 between 1100 and 1110. Hence, the conclusion is true if the faulty element is the vertex 0000.

If the faulty element is an edge, without loss of generality, we may assume it is incident with 0000, then the above fault-free uv-paths of lengths from 8 to 14 is also fault-free. The fault-free uv-paths of length 15 are listed as follows.

The faulty edge is (0000, 0010).
$\langle 1101, 1100, 1000, 1001, 1011, 1010, 0010, 0011, 0001, 0000, 0100, 0101, 0111, 0101, 1111, 1110\rangle$
$\langle 1101, 1100, 1000, 1001, 1011, 1010, 0010, 0011, 0001, 0000, 0100, 0101, 0111, 0110, 1110, 1111\rangle$
$\langle 1101, 1111, 1110, 1010, 1011, 1001, 1000, 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100\rangle$
$\langle 1111, 1101, 1100, 1000, 1001, 1011, 1010, 0010, 0011, 0001, 0000, 0100, 0101, 0111, 0110, 1110\rangle$
$\langle 1111, 1101, 1011, 1001, 1000, 1010, 0010, 0011, 0001, 0000, 0111, 0101, 0100, 0110, 1110, 1100\rangle$
$\langle 1110, 1010, 1011, 1001, 1000, 0000, 0001, 0011, 0010, 0110, 0111, 1101, 1111, 0101, 0100, 1100\rangle$.
The faulty edge is (0000, 0001).
$\langle 1101, 1111, 1001, 1000, 1010, 1011, 0001, 0011, 0010, 0000, 0100, 0101, 0111, 0110, 1110, 1100\rangle$
$\langle 1101, 1100, 1110, 1010, 1011, 1001, 1000, 0000, 0010, 0011, 0001, 0111, 0110, 0100, 0101, 1111\rangle$
$\langle 1101, 1111, 1001, 1011, 1010, 1000, 0000, 0010, 0110, 0111, 0001, 0011, 0101, 0100, 1100, 1110\rangle$
$\langle 1110, 1010, 1011, 1001, 1000, 0000, 0010, 0011, 0001, 0111, 0110, 0100, 0101, 1111, 1101, 1100\rangle$
$\langle 1111, 1101, 1100, 1000, 1001, 1011, 1010, 0010, 0000, 0100, 0101, 0011, 0001, 0111, 0110, 1110\rangle$
$\langle 1111, 1101, 1011, 1010, 1000, 1001, 0011, 0001, 0111, 0101, 0100, 0000, 0010, 0110, 1110, 1100\rangle$.
Hence, the conclusion is true.

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