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On varieties of meet automata

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Eilenberg's variety theorem gives a bijective correspondence between varieties of languages and varieties of finite monoids. The second author gave a similar relation between conjunctive varieties of languages and varieties of semiring homomorphisms. In this paper, we add a third component to this result by considering varieties of meet automata. We consider three significant classes of languages, two of them consisting of reversible languages. We present conditions on meet automata and identities for semiring homomorphisms for their characterization.

1. Introduction

The core problem of the algebraic theory of regular languages is to decide the membership of a given language in certain significant classes of languages. Eilenberg's theorem establishes a one-to-one correspondence between varieties of languages and varieties of finite monoids (also called pseudovarieties). Reiterman's theorem presents the equational logic for the latter classes. Pin's chapter in the Handbook of Formal Languages [11] nicely surveys the extensive theory. Also the book [1] by Almeida is of high interest. The crucial item is to derive properties of languages from their (ordered) syntactic semigroups and monoids. In [13] the second author introduced the notion of syntactic semiring and proved a new Eilenberg-type theorem relating conjunctive varieties of languages and varieties of finite idempotent semirings.

Recently, new horizons were opened by Ésik and Ito [5] by considering literal varieties of languages and more generally by Straubing [18] with his C-varieties, where C is a category of finitely generated free monoids with certain monoid homomorphisms. The classical Eilenberg correspondence was modified to relate the literal varieties of languages and the literal varieties of homomorphisms from finitely generated free monoids onto finite monoids by Ésik and Larsen [6] and more generally by Straubing [18]. The equational logic for C-varieties of homomorphisms of monoids was created by Kunc in [8] and modified to D-varieties of homomorphisms of semirings in [16], where D is a category of finitely generated free idempotent semirings with certain semiring homomorphisms. The last paper also presents the most general variant of Eilenberg's theorem to date. The whole progress is surveyed in [17].

An Eilenberg-type theorem was decomposed using varieties of automata in [5] and by Chaubard, Pin and Straubing in [4] (under the name varieties of actions). The passage from languages to automata is done by taking the minimal complete deterministic automaton.

In the present paper we summarize results on syntactic structures and classes of languages in Sections 2 and 3 based on [17]. The next section introduces three basic examples. The first two are related to certain kinds of reversible automata. We learned about the first class from Golovkins and Pin [7] and the second class is mentioned also in Angluin [3]. The last example consists of languages in which all the words have equal lengths. None of these classes is a variety in the previous sense — we really need to consider conjunctive D-varieties. Section 5 studies classes of meet automata. We decompose

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the Eilenberg-type correspondence from [16] via varieties of meet automata. The passage from languages to automata is done by using canonical meet automata. In Section 6 we characterize those varieties for our three examples. Section 7 presents our classes by identities for the syntactic semiring homomorphisms. Clearly, each such identity is a property of the transformation semirings of meet automata. Notice that our properties from Section 6 are not of such a kind. We conclude with Final Remarks.

2. Syntactic structures

An *idempotent semiring* is a structure $(S, \cdot, \vee)$ where

(i) $(S, \cdot)$ is a monoid with the neutral element 1,
(ii) $(S, \vee)$ is a semilattice with the smallest element 0,
(iii) $(\forall\, a, b, c \in S)\ (a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc)$.
(iv) $(\forall\, a \in S)\ a0 = 0a = 0$.

Such a structure becomes an ordered set with respect to the relation $\leq$ defined by $a \leq b$ if and only if $a \vee b = b,\ a, b \in S$.

Homomorphisms are mappings between two semirings preserving the operations $\cdot$ and $\vee$ and sending 0 to 0 and 1 to 1.

Let A^+ be the free semigroup over an alphabet A and let $A^* = A^+ \cup \{\lambda\}$ be the free monoid over A. Let $|u|$ denote the length of the word $u \in A^*$. For a finite set A, let $A^\square$ be the set of all finite subsets of A^*. This set equipped with the multiplication $UV = \{uv \mid u \in U,\ v \in V\}$ and usual union form the free idempotent semiring over the set A; here $0 = \emptyset$, and $1 = \{\lambda\}$.

There are several natural categories whose objects are the finitely generated free idempotent semirings. Basic examples are the categories

$$\mathsf{D}_{\mathsf{all}},\ \mathsf{D}_{\mathsf{nk}},\ \mathsf{D}_{\mathsf{mi}},\ \mathsf{D}_{\mathsf{mlit}},\ \mathsf{D}_{\mathsf{mlm}},$$

where the morphisms are all, all non-killing, all monoid induced, all multiliteral, and all multi length-multiplying homomorphisms:

(1) $f \in \mathsf{D}_{\mathsf{nk}}(B^\square, A^\square)$ if and only if, for each $b \in B,\ f(\{b\}) \neq \emptyset$,
(2) $f \in \mathsf{D}_{\mathsf{mi}}(B^\square, A^\square)$ if and only if, for each $b \in B$, there exists $u \in A^*$ such that $f(\{b\}) = \{u\}$,
(3) $f \in \mathsf{D}_{\mathsf{mlit}}(B^\square, A^\square)$ if and only if, for each $b \in B,\ f(\{b\}) \subseteq A$, and
(4) $f \in \mathsf{D}_{\mathsf{mlm}}(B^\square, A^\square)$ if and only if there exists $k \geq 1$ such that, for each $b \in B,\ \emptyset \neq f(\{b\}) \subseteq A^k$.

Let D be a category whose objects are the finitely generated free idempotent semirings. A class $\mathfrak{X}$ of surjective homomorphisms from idempotent semirings freely generated by finite sets onto finite (idempotent) semirings is a D-*variety of semiring homomorphisms* if it satisfies:

(i) for each $(\varphi : A^\square \twoheadrightarrow S) \in \mathfrak{X}$ and surjective semiring homomorphism $\sigma : S \twoheadrightarrow T$, we have $\sigma\varphi \in \mathfrak{X}$,
(ii) for each $f \in \mathsf{D}(B^\square, A^\square)$ and $(\varphi : A^\square \twoheadrightarrow S) \in \mathfrak{X}$ we have $(\varphi f : B^\square \twoheadrightarrow \mathrm{im}(\varphi f)) \in \mathfrak{X}$ (for a mapping $g : C \to D$, we write $\mathrm{im} g = \{g(c) \mid c \in C\}$),
(iii) for each finite A, $(A^\square \twoheadrightarrow \{1\}) \in \mathfrak{X}$, and for each $(\varphi : A^\square \twoheadrightarrow S),\ (\psi : A^\square \twoheadrightarrow T) \in \mathfrak{X}$ we have $((\varphi, \psi) : A^\square \twoheadrightarrow \mathrm{im}(\varphi, \psi)) \in \mathfrak{X}$ (here $(\varphi, \psi)(U) = (\varphi(U), \psi(U)) \in S \times T,\ U \in A^\square$).

In (iii) we used products of zero factors and products of couples. It follows that $\mathfrak{X}$ is closed with respect to products of arbitrary finite families.

A language $L \subseteq A^*$ defines the *syntactic semiring congruence* $\sim_L$ on $(A^\square, \cdot, \cup)$ by $\{u_1, \ldots, u_k\} \sim_L \{v_1, \ldots, v_l\}$ if and only if

$$(\forall\, p, q \in A^*)\ (pu_1q, \ldots, pu_kq \in L \iff pv_1q, \ldots, pv_lq \in L).$$

The quotient structure $(A^\square, \cdot, \cup)/\sim_L$ is called the *syntactic semiring* of L; we denote it by $(\mathsf{S}(L), \cdot, \vee)$. The mapping $\varphi(L) : A^\square \twoheadrightarrow \mathsf{S}(L),\ U \mapsto U\sim_L$ is a surjective semiring homomorphism. We call it the *syntactic semiring homomorphism*.

3. Classes of languages

All the languages considered in this paper are assumed to be regular. For finite sets A and B, a semiring homomorphism $f : (B^\square, \cdot, \cup) \to (A^\square, \cdot, \cup)$ and $K \subseteq A^*$, we define

$$f^{[-1]}(K) = \{v \in B^* \mid f(\{v\}) \subseteq K\}.$$

The languages of the form $p^{-1}Kq^{-1} = \{u \in A^* \mid puq \in K\}$, where $p, q \in A^*$, are called *derivatives* of the language $K \subseteq A^*$.

A *class* of languages is an operator $\mathscr{L}$ assigning to every finite set A a set $\mathscr{L}(A)$ of (regular) languages over the alphabet A.

Such a class is a *conjunctive variety* if

(i) each $\mathscr{L}(A)$ is closed with respect to finite intersections (in particular $A^* \in \mathscr{L}(A)$) and derivatives, and
(ii) for each A and B and $f : B^\square \to A^\square,\ K \in \mathscr{L}(A)$ implies $f^{[-1]}(K) \in \mathscr{L}(B)$.

It is a *conjunctive* D-*variety* if (i) is true and (ii) is satisfied for all $f \in \mathsf{D}(B^\square, A^\square)$.

One gets the classical *positive* varieties, see [11], if (i) is strengthened to

(i′) each $\mathscr{L}(A)$ is closed with respect to finite unions, finite intersections and derivatives

and (ii) weakened to

(ii′) for each A and B and $f : B^* \to A^*$, $K \in \mathscr{L}(A)$ implies $f^{-1}(K) \in \mathscr{L}(B)$.

Finally, $\mathscr{L}$ is a *Boolean* variety if it satisfies (ii′) and

(i″) each $\mathscr{L}(A)$ is closed with respect to complements, finite unions, finite intersections and derivatives.

Fix a category D. We can assign to any class of languages $\mathscr{L}$ the D-variety

$$\mathsf{S}_{\mathsf{D}}(\mathscr{L}) = \langle \{\varphi(L) \mid A \text{ is a finite alphabet}, L \in \mathscr{L}(A)\}\rangle_{\mathsf{D}}$$

of semiring homomorphisms generated by the syntactic semiring homomorphisms of members of $\mathscr{L}$.

Conversely, for a class $\mathfrak{X}$ of homomorphisms of idempotent semirings and a finite set A, we put

$$(\mathsf{L}(\mathfrak{X}))(A) = \{L \subseteq A^* \mid \varphi(L) \in \mathfrak{X}\}.$$

Theorem 1 (*Polák [16]*). *The assignments* $\mathscr{L} \mapsto \mathsf{S}_{\mathsf{D}}(\mathscr{L})$ *and* $\mathfrak{X} \mapsto \mathsf{L}(\mathfrak{X})$ *are mutually inverse bijections between the conjunctive* D-*varieties of languages and* D-*varieties of homomorphisms of idempotent semirings.*

Recall that $(\pi_S)_{S\in\mathfrak{S}}$ is an *n-ary implicit operation* ($n \geq 0$) for the class $\mathfrak{S}$ of all finite idempotent semirings if $\pi_S : S^n \to S$, $S \in \mathfrak{S}$, is a mapping and for each semiring homomorphism $\sigma : S \to T$ and $s_1, \ldots, s_n \in S$, we have

$$\sigma(\pi_S(s_1, \ldots, s_n)) = \pi_T(\sigma(s_1), \ldots, \sigma(s_n)).$$

We denote the set of all n-ary implicit operations for $\mathfrak{S}$ by I_n. Widely used is $x^\omega \in \mathrm{I}_1$ where $x^\omega_S(a)$, for $a \in S$, is the only idempotent in the subsemigroup of S generated by a. We denote this element by a^ω and notice that, for each $S \in \mathfrak{S}$, there is $m \geq 1$ such that, for each $a \in S$, we have $a^\omega = a^m$. For the relevant background see Almeida [1].

An *n-ary pseudoidentity* is an ordered pair $\pi = \rho$ of n-ary implicit operations. Let $x_1, x_2, \ldots$ be a fixed sequence of pairwise different variables and let $X_n = \{x_1, \ldots, x_n\}$ for each $n \geq 0$. We may write $x, y, z, t, \ldots$ instead of $x_1, x_2, x_3, x_4, \ldots$. Let D be a category of homomorphisms of finitely generated free idempotent semirings. The n-ary pseudoidentity $\pi = \rho$ is D-*satisfied* in a semiring homomorphism $\varphi : A^\square \twoheadrightarrow S$ if for each $f \in \mathsf{D}(X_n^\square, A^\square)$ we have

$$\pi_S((\varphi f)(x_1), \ldots, (\varphi f)(x_n)) = \rho_S((\varphi f)(x_1), \ldots, (\varphi f)(x_n)).$$

We write $\varphi \models_{\mathsf{D}} \pi = \rho$ in such a case and for a set Π of pseudoidentities $\varphi \models_{\mathsf{D}} \Pi$ means $(\forall\, (\pi = \rho) \in \Pi)(\varphi \models_{\mathsf{D}} \pi = \rho)$. Let $\mathsf{Mod}_{\mathsf{D}}(\Pi) = \{\varphi \mid \varphi \models_{\mathsf{D}} \Pi\}$.

The following was discovered by Kunc for monoids and modified by the second author for idempotent semirings.

Theorem 2 (*Kunc [8], Polák [16]*). *Let the category* D *contains all isomorphisms. Then a class* $\mathfrak{X}$ *is a* D-*variety of semiring homomorphisms if and only if it is of the form* $\mathsf{Mod}_{\mathsf{D}}(\Pi)$ *for a set* Π *of pseudoidentities.*

4. Three basic examples of conjunctive D-varieties of languages

A state q of a complete deterministic automaton $\mathcal{D} = (D, A, \cdot, i, F)$ is *absorbing* if, for each $a \in A$, we have $q \cdot a = q$.

We recall the classical construction of the minimal automaton. For $L \subseteq A^*$ and $u \in A^*$, we write

$$u^{-1}L = \{w \in A^* \mid uw \in L\}, \qquad \mathsf{D}(L) = \{u^{-1}L \mid u \in A^*\}.$$

We assign to L its *canonical minimal automaton* $\mathcal{D}(L) = (\mathsf{D}(L), A, \cdot, L, F)$. A letter $a \in A$ acts on $u^{-1}L$ by $u^{-1}L \cdot a = a^{-1}(u^{-1}L)$. The state L is the initial state and $q \in \mathsf{D}(L)$ is a final state, i.e. an element of F, if and only if $\lambda \in q$. Note that if $\mathcal{D}(L)$ contains a final absorbing state then that state is A^* and if $\mathcal{D}(L)$ contains a nonfinal absorbing state then it is $\emptyset$.

4.1. *The conjunctive* D_{mi}-*variety* $\mathscr{P}$

For a language L over an alphabet A, we put $L \in \mathscr{P}(A)$ if and only if $L = A^*$ or L is recognized by a complete deterministic finite automaton $\mathcal{A}$ satisfying the following properties:

(i) $\mathcal{A}$ has at most one absorbing state and if there exists such state then it is a nonfinal state,
(ii) the action of each letter is injective on the set of all nonabsorbing states.

Such automata are called *injective* finite automata and denoted by IFA-R in Golovkins and Pin [7].

Note that all finite languages belong to the class $\mathscr{P}$. In the definition of the class $\mathscr{P}$ we can not concentrate only on the minimal automaton of a language. For example, the language $L = \{aa, ab, bb\}$ over the alphabet $A = \{a, b\}$ is finite and thus $L \in \mathscr{P}(A)$, but the minimal automaton of L does not satisfy the second condition in the definition of the class $\mathscr{P}$. Indeed, we have $a^{-1}L = \{a, b\}$, $b^{-1}L = \{b\}$ and $b^{-1}(a^{-1}L) = b^{-1}(b^{-1}L) = \{\lambda\}$.

The characterization of the class $\mathscr{P}$ using minimal automata follows.

Theorem 3 (*Ambainis and Freivalds [2]*). *A language over an alphabet A belongs to $\mathscr{P}(A)$ if and only if its minimal automaton $\mathcal{D}$ satisfies the following condition: for each word $u \in A^*$ and each state p in $\mathcal{D}$, the condition $p \neq p \cdot u = p \cdot u^2$ implies that $p \cdot u$ is a nonfinal absorbing state.*

The next proposition summarizes the closure properties of the class $\mathscr{P}$.

Proposition 4 (*Golovkins & Pin [7]*). *The class $\mathscr{P}$ is a conjunctive D_{mi}-variety of languages, but it is not closed under the following operations: finite union, complement and preimages of semiring homomorphisms.*

Proof. Almost all items are easy exercises and were mentioned in [7]. Here we only show an example of a language $L \in \mathscr{P}(A)$ and a semiring homomorphism $f \in \mathsf{D}_{\mathsf{all}}(B^\square, A^\square)$ such that $f^{[-1]}(L) \notin \mathscr{P}(B)$.

Put $A = \{a\}$, $B = \{a, b\}$ and let $L = \{a\}^* \setminus \{a^4\}^*$ be the language of all words over the alphabet $\{a\}$ whose length is not divisible by 4. Let $f : B^\square \to A^\square$ be defined by rules $f(\{a\}) = \{a\}, f(\{b\}) = \{\lambda, a^2\}$. Set $K = f^{[-1]}(L)$. For $u \in B^*$ we have $bu \in K$ if and only if the number of a's in u is odd and the same is true for $b^2u \in K$. This means that in the minimal automaton $\mathcal{D}(K)$ of the language K we have

(i) $K \neq b^{-1}K$ because $a^2 \in K$ and $ba^2 \notin K$,

(ii) $b^{-1}K = (b^2)^{-1}K$ because $bu \in K$ if and only if $b^2u \in K$,

(iii) the state $b^{-1}K$ is not an absorbing nonfinal state, i.e. $\emptyset$, because $ba \in K$.

The conditions (i), (ii) and (iii) mean that K does not satisfy the condition from Theorem 3, i.e. $K \notin \mathscr{P}(B)$. □

4.2. *The conjunctive* D_{nk}*-variety* $\mathscr{N}$

For each alphabet A, $\mathscr{N}(A)$ is the set of languages over A in which any two distinct left derivatives are disjoint.

These languages were studied by Angluin in [3] under the name of 0-*reversible* languages. The following easy observations were also given in [3] in a slightly different form. In particular, the membership of $L \subseteq A^*$ in the class $\mathscr{N}(A)$ can be seen from the minimal complete deterministic automaton of L.

Proposition 5 (*See also Angluin [3]*). *A language over an alphabet A belongs to $\mathscr{N}(A) \setminus \{A^*\}$ if and only if each action by a letter on its minimal automaton $\mathcal{D}$ is an injection on nonabsorbing states and $\mathcal{D}$ contains at most one final state and if it exists then this state is not absorbing.*

Proof. Note that the minimal automaton of the language L does not contain any final state if and only if $L = \emptyset$.

Let $L \notin \{A^*, \emptyset\}$ be a language from $\mathscr{N}(A)$. Assume that q_1 and q_2 are final states. Then $\lambda \in q_1$ and $\lambda \in q_2$ and hence $q_1 \cap q_2 \neq \emptyset$, which implies $q_1 = q_2$. Thus F consists of a single state. This final state is not absorbing, since otherwise $q = A^*$ and A^* is not disjoint with L. Let $u, v \in A^*, a \in A$ be such that $(u^{-1}L) \cdot a = (v^{-1}L) \cdot a \neq \emptyset$. Then there is a word w in $(u^{-1}L) \cdot a$ and hence $aw \in u^{-1}L$ and $aw \in v^{-1}L$. It follows that $u^{-1}L \cap v^{-1}L \neq \emptyset$ and thus finally $u^{-1}L = v^{-1}L$.

Let $L \notin \{A^*, \emptyset\}$ be a language over an alphabet A whose minimal automaton has a single final nonabsorbing state p and for which any action by a letter a is an injection on nonabsorbing states. Then from $w \in u^{-1}L \cap v^{-1}L$ it follows that $uw \in L$ and $vw \in L$. This implies that if we read w from both states $u^{-1}L$ and $v^{-1}L$ we reach the same final state. Because actions are injective, we obtain $u^{-1}L = v^{-1}L$. □

For $u = a_1 \ldots a_k,\ a_1, \ldots, a_k \in A,\ k \geq 1$, we write $u^{\mathsf{R}} = a_k \ldots a_1$. Moreover, for $L \subseteq A^*$, we put $L^{\mathsf{R}} = \{u^{\mathsf{R}} \mid u \in L\}$. Proposition 5 and the fact that $(u^{-1}L)^{\mathsf{R}} = L^{\mathsf{R}}(u^{\mathsf{R}})^{-1}$ gives the following.

Proposition 6. *Let A be an arbitrary alphabet. Then*

1. *The inclusion $\mathscr{N}(A) \subseteq \mathscr{P}(A)$ holds.*
2. *Each language of $\mathscr{P}(A)$ is a union of languages from $\mathscr{N}(A)$.*
3. *$L \in \mathscr{N}(A)$ if and only if every two different right derivatives of L are disjoint.*
4. *If $L \in \mathscr{N}(A)$ then the reverse language L^{R} belongs to $\mathscr{N}(A)$ too.* □

Now we are interested in closure properties of the class $\mathscr{N}$.

Proposition 7. *The class $\mathscr{N}$ is a conjunctive D_{nk}-variety of languages. The sets $\mathscr{N}(A)$ are not closed under finite unions nor complements.*

Proof. We explain here only that $\mathscr{N}$ is closed under semiring homomorphic preimages with respect to the category $\mathsf{D_{nk}}$, the rest are easy exercises. Let $L \in \mathscr{N}(A)$ and let $f \in \mathsf{D_{nk}}(B^\square, A^\square)$ be a semiring homomorphism. Then $f(\{u\}) \neq \emptyset$ for any $u \in B^*$. Let $K = f^{[-1]}(L)$. Assume that $M = u^{-1}K$ and $N = v^{-1}K$, where $u, v \in B^*$, are such that there is $w \in B^*$ which belongs to $M \cap N$. We want to show that $M = N$. Assume that $f(\{u\}) = \{u_1, \dots, u_k\} \subseteq A^*$ and $f(\{v\}) = \{v_1, \dots, v_l\} \subseteq A^*$. We have $uw \in K$, $vw \in K$, hence $f(\{uw\}) \subseteq L$, $f(\{vw\}) \subseteq L$. It follows that $f(\{u\})f(\{w\}) \subseteq L$ and $f(\{v\})f(\{w\}) \subseteq L$. Hence for any $i = 1, \dots, k$ we have $\emptyset \neq f(\{w\}) \subseteq u_i^{-1}L$ and similarly for any $j = 1, \dots, l$ we have $\emptyset \neq f(\{w\}) \subseteq v_j^{-1}L$. Because $L \in \mathscr{N}(A)$ we see that $u_1^{-1}L = \cdots = u_k^{-1}L = v_1^{-1}L = \cdots = v_l^{-1}L$. Now, let $t \in B^*$ be an arbitrary word from $M = u^{-1}K$. Then $ut \in K$, i.e. $f(\{u\})f(\{t\}) \subseteq L$ and $f(\{t\}) \subseteq u_1^{-1}L$ follows. So, we can deduce $f(\{t\}) \subseteq v_1^{-1}L = \cdots = v_l^{-1}L$ and we see that $f(\{v\})f(\{t\}) \subseteq L$. This implies $vt \in K$, i.e. $t \in N = v^{-1}K$. This shows the inclusion $M \subseteq N$ and the opposite inclusion can be proved in a similar way. □

4.3. *The conjunctive* $\mathsf{D_{mlm}}$*-variety* $\mathscr{C}$

For an alphabet A, we have $L \in \mathscr{C}(A)$ if and only if $L = A^*$ or there is a nonnegative integer k such that $L \subseteq A^k$. Note that all these languages (which are different from A^*) are finite. Hence $\mathscr{C}(A) \subseteq \mathscr{P}(A)$. The membership of a language to the class $\mathscr{C}(A)$ can be tested again from its minimal automaton.

Proposition 8. *A language L over an alphabet A with the minimal automaton $\mathcal{D}(L)$ belongs to $\mathscr{C}(A) \setminus \{A^*, \emptyset\}$ if and only if $\mathcal{D}(L)$ has one final state which is not absorbing and for any state p and a pair of words of different lengths u and v, $p \cdot u = p \cdot v$ implies that $p \cdot u$ is an absorbing state.*

Proof. The proof is clear. □

Proposition 9. *The class $\mathscr{C}$ is a conjunctive* $\mathsf{D_{mlm}}$*-variety of languages. The sets $\mathscr{C}(A)$ are not closed under finite unions nor complements. The class $\mathscr{C}$ is not closed under preimages of semiring homomorphisms.*

Proof. The statements are obvious. □

5. Classes of meet automata

Our automata have basically no input/final states. A structure $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ is a *meet automaton* over the finite *alphabet A* if we have

(i) Q is a nonempty finite set of *states*,
(ii) $\cdot : Q \times A \to Q$ is the *transition function*,
(iii) $\wedge : Q \times Q \to Q$ is a semilattice operation on Q,
(iv) $\top \in Q$ and $\top \wedge q = q$ for each $q \in Q$,
(v) $(p \wedge q) \cdot a = p \cdot a \wedge q \cdot a$ for each $p, q \in Q,\ a \in A$,
(vi) $\top$ is an absorbing state.

For $p, q \in Q$, we put $p \leq q$ if and only if $p \wedge q = p$. Clearly, $\leq$ is an order relation on Q with the smallest element $\bot = \bigwedge Q$ and the greatest element $\top$. If $\bot$ is an absorbing state we speak about the *hell*.

Given a meet automaton $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$, the mapping $\cdot$ naturally extends to $\cdot : Q \times A^* \to Q$. Moreover, for each finite set $\{u_1, \dots, u_k\}$ of words, we define

$$q \cdot \{u_1, \dots, u_k\} = q \cdot u_1 \wedge \cdots \wedge q \cdot u_k, \quad q \in Q$$

and

$$[\{u_1, \dots, u_k\}]_{\mathcal{Q}} : Q \to Q,\ q \mapsto q \cdot \{u_1, \dots, u_k\}.$$

For $k = 0$, we get the constant map on the state $\top$. Often we write $[u_1, \dots, u_k]_{\mathcal{Q}}$ instead of $[\{u_1, \dots, u_k\}]_{\mathcal{Q}}$. In the following definitions we fix the meet automata $\mathcal{P} = (P, A, \cdot, \wedge, \top)$ and $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$.

A mapping $\alpha : P \to Q$ is a *homomorphism* of $\mathcal{P}$ into $\mathcal{Q}$ if

(i) for each $p \in P,\ a \in A$, we have $\alpha(p \cdot a) = \alpha(p) \cdot a$,
(ii) for each $p, q \in P$, we have $\alpha(p \wedge q) = \alpha(p) \wedge \alpha(q)$,
(iii) $\alpha(\top) = \top$.

In case that α is surjective we say that $\mathcal{Q}$ is a *homomorphic image* of $\mathcal{P}$.

The *trivial* meet automaton over A is the structure $\mathcal{T} = (\{\top\}, A, \cdot, \wedge, \top)$.

We define the *product* $\mathcal{P} \times \mathcal{Q}$ of meet automata as $(P \times Q, A, \cdot, \wedge, (\top, \top))$ where, for each $p, r \in P,\ q, s \in Q,\ a \in A$, we have

$$(p, q) \cdot a = (p \cdot a, q \cdot a),\ (p, q) \wedge (r, s) = (p \wedge r, q \wedge s).$$

Remark. It is clear how to define $\mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, $n \geq 1$. From the categorical point of view our products are both categorical products and categorical sums of automata over the identical alphabets. In particular, the trivial meet automaton is both initial and terminal.

For a finite set B and a homomorphism $f : B^\square \to A^\square$, the set R *induces a subautomaton* of the meet automaton $\mathcal{Q}$ *with respect to* f if

(i) $\top \in R \subseteq Q$,
(ii) $r, s \in R$ implies $r \wedge s \in R$,
(iii) $r \in R,\ b \in B$ implies $r \circ b = r \cdot f(\{b\}) \in R$.

In this case we also say that $\mathcal{R} = (R, B, \circ, \wedge, \top)$ is a *subautomaton* of $\mathcal{Q}$ *with respect to* f.

For a category D, a D-*variety* $\mathbb{V}$ of meet automata consists of families $\mathbb{V}(A)$ of meet automata over A, for each finite alphabet A, if it satisfies the conditions:

(i) each $\mathbb{V}(A)$ is closed with respect to homomorphic images,
(ii) each $\mathbb{V}(A)$ contains the trivial meet automaton over A and it is closed with respect to products of couples,
(iii) for each $A, B,\ f \in \mathsf{D}(B^\square, A^\square),\ \mathcal{Q} \in \mathbb{V}(A),$ each subautomaton of $\mathcal{Q}$ with respect to f belongs to $\mathbb{V}(B)$.

Examples. 1. The class $\mathbb{V}_\mathsf{b}$ of all meet automata $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ in which the action of each letter $a \in A$ (i.e. the transformation of Q given by $q \mapsto q \cdot a$) is a bijection on Q, forms a D_{mi}-variety of meet automata.
2. The class $\mathbb{V}_\mathsf{i}$ of all meet automata $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ in which the action of each letter $a \in A$ is increasing (i.e. $p \leq p \cdot a$ for $p \in Q$) forms a $\mathsf{D}_{\mathsf{all}}$-variety of meet automata.
3. In contrast to 2, the class $\mathbb{V}_\mathsf{d}$ of all meet automata $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ in which the action of each letter $a \in A$ is decreasing (i.e. $p \cdot a \leq p$ for $p \in Q$) forms only a D_{nk}-variety of meet automata.
4. The class $\mathbb{V}_\mathsf{a}$ of all meet automata $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ with acyclic transitions, i.e. those satisfying condition

$$(\forall\, p \in Q,\ U, V \in A^\square)\ ((p \cdot U) \cdot V = p \Longrightarrow p \cdot U = p),$$

is a $\mathsf{D}_{\mathsf{all}}$-variety.

An equivalence relation $\sim$ on the set Q is a *congruence relation* on $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ if

(i) for each $p, q \in Q,\ a \in A$, if $p \sim q$ then $p \cdot a \sim q \cdot a$,
(ii) for each $p, q, r \in Q$, if $p \sim q$ then $p \wedge r \sim q \wedge r$.

We define the *quotient* automaton $\mathcal{Q}/\!\sim\, = (Q/\!\sim, A, \cdot_\sim, \wedge_\sim, \top\!\sim)$ by

$$Q/\!\sim\, = \{q\sim \mid q \in Q\} \quad \text{where } q\sim\, = \{p \in Q \mid p \sim q\},$$
$$q\sim \cdot_\sim a = (q \cdot a)\sim,\ \ p\sim \wedge_\sim q\sim\, = (p \wedge q)\sim, \quad \text{for all } p, q \in Q,\ a \in A.$$

We write more simply $\mathcal{Q}/\!\sim\, = (Q/\!\sim, A, \cdot, \wedge, \top)$. Notice that the assignment $\mathsf{nat}\sim : q \mapsto q\sim$ is a surjective homomorphism of $\mathcal{Q}$ onto $\mathcal{Q}/\!\sim$.

For a meet automaton $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ and $q, t \in Q$, we define the language

$$\mathsf{L}(\mathcal{Q}, q, t) = \{u \in A^* \mid t \leq q \cdot u\}.$$

We say that a language $L \subseteq A^*$ is *recognized* by a meet automaton $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ if there exist $q, t \in Q$ such that $L = \mathsf{L}(\mathcal{Q}, q, t)$.

We define the *canonical meet automaton* of a language $L \subseteq A^*$ as the structure $\mathcal{U}(L) = (\mathsf{U}(L), A, \cdot, \cap, A^*)$ where

$$\mathsf{U}(L) = \{u_1^{-1}L \cap \cdots \cap u_k^{-1}L \mid k \geq 0,\ u_1, \ldots, u_k \in A^*\} \quad (\text{we get } A^* \text{ for } k = 0)$$
$$\text{and for } q \in \mathsf{U}(L),\ a \in A, \quad \text{we have } q \cdot a = a^{-1}q.$$

Clearly, it is a meet automaton and $q \in \mathsf{U}(L)$ is the hell if and only if $q = \emptyset$.

Realize that

$$\mathcal{U}(\emptyset) = (\{\emptyset, A^*\}, A, \cdot, \cap, A^*) \quad \text{and} \quad \mathcal{U}(A^*) = (\{A^*\}, A, \cdot, \cap, A^*).$$

Lemma 10. *Let $L \subseteq A^*$ and let*

$$t = \bigcap \{q \in \mathsf{U}(L) \mid \lambda \in q\} \in \mathsf{U}(L).$$

Then $\mathsf{L}(\mathcal{U}(L), q, t) = q$ for each $q \in \mathsf{U}(L)$. In particular, the language L is recognized by $\mathcal{U}(L)$. Moreover, there exists an absorbing state in $\mathcal{U}(L)$ different from A^* if and only if $\bot = \emptyset$ (in this case there are exactly two absorbing states: A^* and $\bot$).

Proof. Let $u \in A^*$. Then $u \in \mathsf{L}(\mathcal{U}(L), q, t)$ is equivalent to $\lambda \in q \cdot u$ and this is equivalent to $u \in q$. The rest is now clear. $\square$

Lemma 11. *Let $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ be a meet automaton. Let $t \in Q$. Then the relation $\sim_t$ on Q defined by*

$$p \sim_t q \quad \text{if and only if } \mathsf{L}(\mathcal{Q}, p, t) = \mathsf{L}(\mathcal{Q}, q, t)$$

is a congruence relation on $\mathcal{Q}$.

Proof. The proof is straightforward. □

The quotient automata from the last proposition are called *minimalizations* of $\mathcal{Q}$.

The following lemma assures that the canonical meet automata play with respect to the class of all meet automata a similar role as minimal DFA's play with respect to all complete DFA's.

Lemma 12. *Let $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ be a meet automaton, let $q, t \in Q$ and $L = \mathsf{L}(\mathcal{Q}, q, t)$. Let $\mathcal{R}$ be a subautomaton of $\mathcal{Q}$ with respect to id_A induced by $R = \{q \cdot U \mid U \in A^\square\}$ and put $t' = \bigwedge \{q \cdot u \mid t \leq q \cdot u\}$. Then the mapping $r \sim_{t'} \mapsto \mathsf{L}(\mathcal{Q}, r, t)$ defines an isomorphism of $\mathcal{R}/\sim_{t'}$ onto $\mathcal{U}(L)$.*

Proof. Observe first that, for each $r \in R$, we have $\mathsf{L}(\mathcal{R}, r, t') = \mathsf{L}(\mathcal{Q}, r, t)$. Next we show that, for each $u_1, \dots, u_k \in A^*$,

$$\mathsf{L}(\mathcal{Q}, q \cdot \{u_1, \dots, u_k\}, t) = u_1^{-1}L \cap \cdots \cap u_k^{-1}L.$$

Indeed, w is an element of the left hand side if and only if $t \leq q \cdot u_1 w, \dots, q \cdot u_k w$, which is equivalent to $u_1 w, \dots, u_k w \in L$ and the claim follows.

Therefore the mapping from lemma is a bijection. It is an isomorphism since both $r \mapsto r \sim_{t'}$ and $q \cdot \{u_1, \dots, u_k\} \mapsto u_1^{-1}L \cap \cdots \cap u_k^{-1}L$ are surjective homomorphisms of $\mathcal{R}$ onto $\mathcal{R}/\sim_{t'}$ and $\mathcal{U}(L)$, respectively. □

Notice that the set $\mathsf{U}(L)$ is also the set of all states of the *universal* automaton of L. This automaton is nondeterministic, in fact $q \in p \cdot a$ if and only if $p \cdot a \supseteq q$, a state p is initial if $p \subseteq L$ and q is final if $\lambda \in q$. See, among others, [15] and Lombardy and Sakarovich [10].

Let $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ be a meet automaton. We put

$$\mathsf{T}(\mathcal{Q}) = \{[U]_\mathcal{Q} \mid U \in A^\square\}.$$

Clearly, for $q \in Q$ and $U, V \in A^\square$ we have

$$q \cdot (U \cup V) = q \cdot U \wedge q \cdot V \quad \text{and} \quad q \cdot (U \cdot V) = (q \cdot U) \cdot V.$$

Therefore the structure

$$\mathbf{T}(\mathcal{Q}) = (\mathsf{T}(\mathcal{Q}), \cdot, \vee),$$

where the operations are the composition and $\vee$ defined by

$$[U]_\mathcal{Q} \ \vee \ [V]_\mathcal{Q} = [U \cup V]_\mathcal{Q} \quad \text{for } U, V \in A^\square,$$

is an idempotent semiring.

Moreover, it follows that the mapping

$$\varphi(\mathcal{Q}) : A^\square \twoheadrightarrow \mathsf{T}(\mathcal{Q}), \ U \mapsto [U]_\mathcal{Q},$$

is a surjective semiring homomorphism of $(A^\square, \cdot, \cup)$ onto $\mathbf{T}(\mathcal{Q})$.

Notice that

$$\mathsf{T}(\mathcal{U}(\emptyset)) = \{\text{the identity, the constant map onto } A^*\} \quad \text{and} \quad \mathsf{T}(\mathcal{U}(A^*)) = \{\text{the identity}\}.$$

For a class $\mathfrak{X}$ of homomorphisms of idempotent semirings, we define the class $\mathsf{A}(\mathfrak{X})$ of meet automata as follows: for each A,

$$(\mathsf{A}(\mathfrak{X}))(A) = \{\mathcal{Q} \mid \mathcal{Q} \text{ is a meet automaton over } A \text{ and } \varphi(\mathcal{Q}) \in \mathfrak{X}\}.$$

The next proposition states first a result which is analogous to the well-known fact that the transformation monoid of the minimal complete deterministic automaton for L is isomorphic to the syntactic monoid of L. The second part relates classes of meet automata and classes of homomorphisms of semirings.

Proposition 13. 1. *The mapping $\kappa : [U]_{\mathcal{U}(L)} \mapsto U \sim_L, \ U \in A^\square$, defines an isomorphism of the transformation semiring $\mathbf{T}(\mathcal{U}(L))$ of the canonical meet automaton $\mathcal{U}(L)$ of $L \subseteq A^*$ onto the syntactic semiring $(\mathsf{S}(L), \cdot, \vee)$ of L. Thus $\kappa \circ \phi(\mathcal{U}(L)) = \phi(L)$.*
2. *Let $\mathfrak{X}$ be a* D*-variety of homomorphisms of idempotent semirings. Then $\mathsf{A}(\mathfrak{X})$ is a* D*-variety of meet automata.*

Proof. Item 1 comes from [14], Section 5.
2. (i) : Let α be a surjective homomorphism of $\mathcal{P} = (P, A, \cdot, \wedge, \top) \in (\mathsf{A}(\mathfrak{X}))(A)$ onto $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$. Then $\sigma : [U]_\mathcal{P} \mapsto [U]_\mathcal{Q}, \ U \in A^\square$, defines a surjective homomorphism of $\mathbf{T}(\mathcal{P})$ onto $\mathbf{T}(\mathcal{Q})$. Thus $\varphi(\mathcal{Q}) = \sigma(\varphi(\mathcal{P})) \in \mathfrak{X}$ and thus $\mathcal{Q} \in (\mathsf{A}(\mathfrak{X}))(A)$.
(ii) : The transformation semiring of the trivial meet automaton $\mathcal{T}$ is the trivial (= one element) semiring and thus $\varphi(\mathcal{T})$ is trivial.

Further,

$$\mathsf{im}(\varphi(\mathcal{P}), \varphi(\mathcal{Q})) = \{([U]_\mathcal{P}, [U]_\mathcal{Q}) \mid U \in A^\square\}$$

and $([U]_\mathcal{P}, [U]_\mathcal{Q}) \mapsto [U]_{\mathcal{P} \times \mathcal{Q}}$ is an isomorphism of $(\varphi(\mathcal{P}), \varphi(\mathcal{Q}))$ onto $\varphi(\mathcal{P} \times \mathcal{Q})$.
(iii) : Let $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ with $\varphi(\mathcal{Q}) \in \mathfrak{X}$. Let $\mathcal{P}$ be a subautomaton of $\mathcal{Q}$ with respect to $f \in \mathsf{D}(B^\square, A^\square)$. Then $\varphi(\mathcal{P}) = \varphi(\mathcal{Q}) \cdot f$, and consequently $\varphi(\mathcal{P}) \in \mathfrak{X}$. □

For a class $\mathbb{V}$ of meet automata, we define the class $\mathsf{J}(\mathbb{V})$ of languages as follows:

$$(\mathsf{J}(\mathbb{V}))(A) = \{L \subseteq A^* \mid \mathcal{U}(L) \in \mathbb{V}(A)\}, \quad \text{for each } A.$$

The following proposition relates classes of languages and classes of meet automata.

Proposition 14. 1. *Let $\mathbb{V}$ be a* D*-variety of meet automata. Then $\mathsf{J}(\mathbb{V})$ is a conjunctive* D*-variety of languages.*
2. *$L \in (\mathsf{J}(\mathbb{V}))(A)$ if and only if there exist $\mathcal{Q} = (Q, A, \cdot, \wedge, \top) \in \mathbb{V}(A)$, $q, t \in Q$ such that $L = \mathsf{L}(\mathcal{Q}, q, t)$.*
3. *The operator* J *maps different* D*-varieties of meet automata to different classes of languages.*
4. *The composition of* A *followed by* J *is exactly the operator* L.

Proof. 1. (i) : $A^* \in (\mathsf{J}(\mathbb{V}))(A)$ since $\mathcal{U}(A^*)$ is the trivial automaton. Further, let $K, L \in (\mathsf{J}(\mathbb{V}))(A)$. Notice first that

$$\{(u_1^{-1}(K) \cap \cdots \cap u_k^{-1}(K), u_1^{-1}(L) \cap \cdots \cap u_k^{-1}(L)) \mid k \geq 0,\ u_1, \ldots, u_k \in A^*\}$$

induces a D-subautomaton of $\mathcal{U}(K) \times \mathcal{U}(L)$ with respect to the identity of $A^\square$. Further, $(p, q) \mapsto p \cap q$ is a surjective homomorphism of this automaton onto the automaton $\mathcal{U}(K \cap L)$. Thus $K \cap L \in (\mathsf{J}(\mathbb{V}))(A)$.

Let $L \in (\mathsf{J}(\mathbb{V}))(A)$, $v, w \in A^*$. Notice first that $\mathcal{U}(v^{-1}L)$ with the obvious operations is a D-subautomaton of $\mathcal{U}(L)$ with respect to the identity of $A^\square$. Further, $q \mapsto qw^{-1}$ is a surjective homomorphism of the last automaton onto the automaton $\mathcal{U}(v^{-1}Lw^{-1})$. Thus $v^{-1}Lw^{-1} \in (\mathsf{J}(\mathbb{V}))(A)$.
(ii) : Let $f \in \mathsf{D}(B^\square, A^\square)$ and $K \in (\mathsf{J}(\mathbb{V}))(A)$. Then $\{K \cdot f(V) \mid V \in B^\square\}$ is a D-subautomaton of $\mathcal{U}(K)$ with respect to f. Since

$$w \in v^{-1}f^{[-1]}(K) \quad \text{if and only if } f(\{w\}) \subseteq K \cdot f(\{v\}),$$

the assignment $K \cdot f(V) \mapsto f^{[-1]}(K) \cdot V$ correctly defines a surjective homomorphisms of the last automaton onto $\mathcal{U}(f^{[-1]}(K))$. Thus $f^{[-1]}(K) \in (\mathsf{J}(\mathbb{V}))(B)$.
2. $\Rightarrow$: It follows from Lemma 10.
$\Leftarrow$: It follows from Lemma 12.
3. Let $\mathbb{V}$ and $\mathbb{W}$ be D-varieties of meet automata such that, for each A, $(\mathsf{J}(\mathbb{V}))(A) \subseteq (\mathsf{J}(\mathbb{W}))(A)$. Let $\mathcal{Q} = (Q, A, \cdot, \wedge, \top) \in \mathbb{V}(A)$.
Suppose first that

$$\text{there exists } i \in Q \text{ such that } Q = \{i \cdot U \mid U \in A^\square\}. \qquad (\diamond)$$

Let $Q = \{q_1, \ldots, q_m\}$. For each $j \in \{1, \ldots, m\}$, we have

$$L_j = \mathsf{L}(\mathcal{Q}, i, q_j) \in (\mathsf{J}(\mathbb{V}))(A) \subseteq (\mathsf{J}(\mathbb{W}))(A).$$

Therefore, there exist $\mathcal{Q}_j = (Q_j, A, \cdot, \wedge, \top) \in \mathbb{W}(A)$, $i_j, t_j \in Q_j$ such that $\mathsf{L}(\mathcal{Q}_j, i_j, t_j) = L_j$.

Now $\{(i_1 \cdot U, \ldots, i_m \cdot U) \mid U \in A^\square\}$ induces a subautomaton of the automaton $\mathcal{Q}_1 \times \cdots \times \mathcal{Q}_m \in \mathbb{W}(A)$ and the rule $(i_1 \cdot U, \ldots, i_m \cdot U) \mapsto i \cdot U$ correctly defines a homomorphism of this automaton onto $\mathcal{Q}$. Indeed, suppose that $(i_1 \cdot U, \ldots, i_m \cdot U) = (i_1 \cdot V, \ldots, i_m \cdot V)$ and $i \cdot U \neq i \cdot V$. Let $U = \{u_1, \ldots, u_k\}$, $V = \{v_1, \ldots, v_l\}$ and $i \cdot U \not\leq i \cdot V$ (the case $i \cdot V \not\leq i \cdot U$ would be treated in a similar way). There exists $g \in \{1, \ldots, l\}$ such that $i \cdot u_1 \wedge \cdots \wedge i \cdot u_k \not\leq i \cdot v_g$. Let $q_j = i \cdot U$. Then $v_g \notin \mathsf{L}(\mathcal{Q}, i, q_j)$, $u_1, \ldots, u_k \in \mathsf{L}(\mathcal{Q}, i, q_j) = \mathsf{L}(\mathcal{Q}_j, i_j, t_j)$ and $i_j \cdot V \not\geq t_j$, $i_j \cdot U \geq t_j$ leads to a contradiction.

If $(\diamond)$ is not satisfied and $Q = \{q_1, \ldots, q_n\}$ put $P_i = \{q_i \cdot U \mid U \in A^\square\}$ for $i = 1, \ldots, n$. Then P_i induces a subautomaton $\mathcal{P}_i$ of $\mathcal{Q}$ with respect to the identity ($i = 1, \ldots, n$) and

$$\alpha : P_1 \times \cdots \times P_n \to Q,\ (p_1, \ldots, p_n) \mapsto p_1 \wedge \cdots \wedge p_n$$

defines a surjective homomorphism of $\mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ onto $\mathcal{Q}$.

The Statement 4 follows from Proposition 13.1. $\square$

Theorem 15. (i) *The operator* A *is a bijection of the class of all* D*-varieties of homomorphisms of idempotent semirings onto the class of all* D*-varieties of meet automata.*
(ii) *The operator* J *is a bijection of the class of all* D*-varieties of meet automata onto the class of all conjunctive* D*-varieties of languages.*
(iii) *Both* A *and* J *and their inverses preserve inclusions.*

Proof. First, by Propositions 13.2 and 14.1 the operators A and J map into the above mentioned classes. By Theorem 1 the operator L is a bijection and by Proposition 14.4 it is of the form JA. Therefore J is onto and A is one-to-one. By Proposition 14.3 J is also one-to-one. Finally, A is surjective since $(\mathsf{AS_D J})(\mathbb{V}) = \mathbb{V}$. Indeed, an application of J to $(\mathsf{AS_D J})(\mathbb{V}) \neq \mathbb{V}$ would lead to a contradiction with Theorem 1.

Item (iii) is clear. $\square$

Examples. If we return to the four examples given after the definition of a D-variety of meet automata, we can observe that the corresponding classes of languages are the following.
1. The conjunctive D_{mi}-variety $\mathsf{J}(\mathbb{V}_{\mathsf{b}})$ is the well-known variety of group languages and $\mathsf{A}^{-1}(\mathbb{V}_{\mathsf{b}})$ is given by the identity $x^{\omega} = 1$. Note the important fact that $x^{\omega} = 1$ is only D_{mi}-satisfied. Indeed, the $\mathsf{D}_{\mathsf{all}}$-satisfiability of $x^{\omega} = 1$ would imply $(y \vee 1)^{\omega} = 1$ which has a consequence $1 \geq y \geq y^2 \geq \cdots \geq y^{\omega} = 1$ and the identity $1 = y$ would follow.
2. and 3. $\mathsf{A}^{-1}(\mathbb{V}_{\mathsf{i}})$ and $\mathsf{A}^{-1}(\mathbb{V}_{\mathsf{d}})$ are given by identities $1 \leq x$ and $x \leq 1$ respectively, which are $\mathsf{D}_{\mathsf{all}}$-satisfied and D_{nk}-satisfied.
4. One can show that, for each language L, the condition $\mathcal{U}(L) \in \mathbb{V}_{\mathsf{a}}$ follows from $\mathcal{D}(L) \in \mathbb{V}_{\mathsf{a}}$ and thus the class $\mathsf{J}(\mathbb{V}_{\mathsf{a}})$ corresponds tho the pseudovariety of all finite $\mathcal{R}$-trivial monoids (see e.g. [12]).

In fact none of the examples mentioned needs a characterization via meet automata because the corresponding classes form varieties or positive varieties of languages. For that reason we omit detailed arguments. Significant examples of conjunctive D-varieties are studied in the next section and other are mentioned in Final Remarks.

6. Varieties of meet automata for our basic examples

We consider the following conditions concerning a meet automaton $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$:

$$(\forall\, q \in Q,\ u \in A^*) \quad (q \neq q \cdot u = q \cdot u^2 \text{ implies } q \cdot u \text{ is the hell }), \tag{$*$}$$

$$(\forall\, q \in Q,\ u, v \in A^*) \quad (q \cdot u \neq q \cdot v \text{ implies } q \cdot u \wedge q \cdot v \text{ is the hell}), \tag{$\dagger$}$$

$$(\forall\, q \in Q \setminus \{\top\},\ u, v \in A^*) \quad (|u| \neq |v| \text{ implies } q \cdot u \wedge q \cdot v \text{ is the hell}). \tag{$\ddagger$}$$

Proposition 16. *Let $L \subseteq A^*$ be a regular language. Then :*
1. *$L \in \mathscr{P}(A)$ if and only if $\mathcal{U}(L)$ satisfies the condition ($*$).*
2. *$L \in \mathscr{N}(A)$ if and only if $\mathcal{U}(L)$ satisfies the condition ($\dagger$).*
3. *$L \in \mathscr{C}(A)$ if and only if $\mathcal{U}(L)$ satisfies the condition ($\ddagger$).*

Proof. 1. The statement is true for $L = A^*$.

Let $L \neq A^*$ belong to $\mathscr{P}(A)$. Let $k \geq 1$ be such that, for any $u \in A^*$, the transformation of the minimal automaton $\mathcal{D}(L)$ induced by the word u^k is idempotent, i.e. for any $q \in \mathsf{D}(L)$, we have $(q \cdot u^k) \cdot u^k = q \cdot u^k$.

Assume that $p \in \mathsf{U}(L)$ and $u \in A^*$ are such that $p \neq p \cdot u = p \cdot u^2$. We want to prove that $p \cdot u = \emptyset$. Because $p \cdot u^m = p \cdot u$ for any $m \geq 1$, we have also $p \neq p \cdot w = p \cdot w^2$ for $w = u^k$. Moreover, any state of the canonical meet automaton $\mathcal{U}(L)$ is an intersection of states of $\mathcal{D}(L)$. So, we can write $p = p_1 \cap p_2 \cap \cdots \cap p_n$, where $p_i \in \mathsf{D}(L)$. Because $p \cdot w \neq p$ there is some $i \in \{1, \ldots, n\}$ such that $p_i \cdot w \neq p_i$. We know that $p_i \cdot w^2 = p_i \cdot w \in \mathsf{D}(L)$ because $w = u^k$. The automaton $\mathcal{D}(L)$ of the language $L \in \mathscr{P}(A)$ satisfies the characterization given in Theorem 3. Hence $p_i w$ is the nonfinal absorbing state of $\mathcal{D}(L)$ and thus $p_i \cdot w = \emptyset$. Now $p \cdot w = p_1 \cdot w \cap \cdots \cap p_n \cdot w \subseteq p_i \cdot w = \emptyset$, i.e. $p \cdot u$ is the hell.

The reverse implication is clear, because $\mathcal{D}(L)$ is a subautomaton (in a classical sense) of the meet automaton $\mathcal{U}(L)$.

Statement 2 is clear.

3. Let $L \in \mathscr{C}(A)$. We know that $L = A^*$ if and only if $\mathcal{U}(L)$ is a trivial automaton for which the condition ($\ddagger$) is trivially satisfied.

Let $L \neq A^*$ be a language from $\mathscr{C}(A)$ with the canonical meet automaton $\mathcal{U}(L)$. It is easy to see that for any state $p \in \mathsf{U}(L) \setminus \{\top\}$ there is $k \geq 1$ such that $p \subseteq A^k$. Assume that $u, v \in A^*$ are words of different lengths, i.e. assume that $u \in A^m$, $v \in A^n$, $m \neq n$. If $k < m$ or $k < n$ then we see that $p \cdot u = \emptyset$ or $p \cdot v = \emptyset$. If $k \geq m$ and $k \geq n$ then $p \cdot u \subseteq A^{k-m}$ and $p \cdot v \subseteq A^{k-n}$, hence $p \cdot u \cap p \cdot v \subseteq A^{k-m} \cap A^{k-n} = \emptyset$. In any case $p \cdot u \cap p \cdot v = \emptyset$.

Now, assume that $L \notin \mathscr{C}(A)$. Then there are two words in L of different lengths, say $u = a_1 \ldots a_k$, $v = b_1 \ldots b_l$, where $k \neq l$, $k, l \geq 0$. Then in the canonical meet automaton of L we have $\lambda \in L \cdot u\ \cap\ L \cdot v$. Hence $L \cdot u\ \wedge\ L \cdot v \neq \emptyset$. $\square$

Note that a product of two meet automata satisfying ($*$) does not need to satisfy ($*$). Similarly for ($\dagger$) and ($\ddagger$).

We modify the above conditions to :

$$\text{each minimalization of } \mathcal{Q} \text{ satisfies } (*), \tag{$*'$}$$

$$\text{each minimalization of } \mathcal{Q} \text{ satisfies } (\dagger), \tag{$\dagger'$}$$

$$\text{each minimalization of } \mathcal{Q} \text{ satisfies } (\ddagger). \tag{$\ddagger'$}$$

Theorem 17. 1. *The class of all meet automata satisfying ($*'$) forms a D_{mi}-variety.*
2. *The class of all meet automata satisfying ($\dagger'$) forms a D_{nk}-variety.*
3. *The class of all meet automata satisfying ($\ddagger'$) forms a $\mathsf{D}_{\mathsf{mlm}}$-variety.*
4. *A canonical minimal automaton satisfies ($*$) if and only if it satisfies ($*'$).*
5. *A canonical minimal automaton satisfies ($\dagger$) if and only if it satisfies ($\dagger'$).*
6. *A canonical minimal automaton satisfies ($\ddagger$) if and only if it satisfies ($\ddagger'$).*

Before the proof we need several lemmas.

Lemma 18. *Let $\mathcal{P} = (P, A, \cdot, \wedge, \top)$ and $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ be meet automata. Then:*
1. *Let $\alpha : \mathcal{P} \twoheadrightarrow \mathcal{Q}$ be a surjective homomorphism and $t \in Q$. Let s be the smallest $p \in P$ with $\alpha(p) = t$. Then $\beta : p \sim_s \mapsto \alpha(p) \sim_t$ defines a surjective homomorphism of $\mathcal{P}/\sim_s$ onto $\mathcal{Q}/\sim_t$ with the property $\mathsf{nat} \sim_t \cdot\alpha = \beta \cdot \mathsf{nat} \sim_s$.*
2. *Let $s \in P,\ t \in Q$. Then the rule*

$$(p \sim_s, q \sim_t) \sim_{(s\sim_s, t\sim_t)} \mapsto (p, q) \sim_{(s,t)}$$

defines an isomorphism of $(\mathcal{P}/\sim_s \times \mathcal{Q}/\sim_t)/\sim_{(s\sim_s,t\sim_t)}$ onto $(\mathcal{P} \times \mathcal{Q})/\sim_{(s,t)}$.
3. *Let $\mathcal{R} = (R, B, \circ, \wedge, \top)$ be a subautomaton of $\mathcal{Q}$ with respect to $f : B^{\square} \to A^{\square}$ and let $t \in R$. We have to distinguish $\sim_t(\mathcal{R})$ and $\sim_t(\mathcal{Q})$. Moreover, let $\sim_t(\mathcal{Q}, f)$ be the congruence on $\mathcal{Q}$ defined by*

$$p \sim_t (\mathcal{Q}, f)\, q \text{ if and only if } \{v \in B^* \mid p \cdot f(\{v\}) \geq t\} = \{v \in B^* \mid q \cdot f(\{v\}) \geq t\}.$$

Then the rule $q \sim_t (\mathcal{Q}) \mapsto q \sim_t (\mathcal{Q}, f)$ defines a surjective homomorphism of $\mathcal{Q}/\sim_t(\mathcal{Q})$ onto $\mathcal{Q}/\sim_t(\mathcal{Q}, f)$ and the rule $r \sim_t(\mathcal{R}) \mapsto r \sim_t(\mathcal{Q}, f)$ defines an isomorphism of $R/\sim_t(\mathcal{R})$ onto a subautomaton of $\mathcal{Q}/\sim_t(\mathcal{Q}, f)$ with respect to f.

Proof. 1. Correctness : we have to show that $p \sim_s q$ implies $\alpha(p) \sim_t \alpha(q)$. Clearly $p \cdot u \geq s$ implies $\alpha(p) \cdot u \geq t$ and the opposite implication follows from the choice of s. Clearly, β is a homomorphism satisfying the equality above.
2. All follows from the fact that s (resp. t) is the smallest element in its $\sim_s$-class (resp. $\sim_t$-class).
Item 3 is clear. □

Lemma 19. *Let $\mathcal{P} = (P, A, \cdot, \wedge, \top)$ and $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ be meet automata. Then :*
1. *Let $\mathcal{P}$ satisfy the condition $(*)$ and let $\alpha : \mathcal{P} \twoheadrightarrow \mathcal{Q}$ be a surjective homomorphism. Then also $\mathcal{Q}$ satisfies $(*)$.*
2. *Let both of $\mathcal{P}$ and $\mathcal{Q}$ satisfy the condition $(*)$ and let $(s, t) \in P \times Q$. Then $(\mathcal{P} \times \mathcal{Q})/\sim_{(s,t)}$ satisfies $(*)$.*
3. *Let $\mathcal{Q}$ satisfy the condition $(*)$ and let $\mathcal{R} = (R, B, \circ, \wedge, \top)$ be a subautomaton of $\mathcal{Q}$ with respect to $f \in \mathrm{D_{mi}}(B^{\square}, A^{\square})$. Then also $\mathcal{R}$ satisfies $(*)$.*

Proof. 1. Let $q \in Q,\ u \in A^*$ be such that $q \neq q \cdot u = q \cdot u^2$. Take $p \in P$ with $\alpha(p) = q$. Then $q \cdot u = \alpha(p \cdot u) = \alpha(p \cdot u^2) = \cdots$ and there exist $k, d \geq 1$ such that $p \cdot u^{k+d} = p \cdot u^k$. Let $l \in \{k, \ldots, k + d - 1\}$ be divisible by d. Then $p \neq p \cdot u^l = p \cdot u^{2l}$. Thus $p \cdot u^l$ is the hell of $\mathcal{P}$ and $q \cdot u$ is the hell of $\mathcal{Q}$.
2. Let $(p, q) \in P \times Q,\ u \in A^*$ be such that $(p, q) \sim_{(s,t)} \neq (p, q) \sim_{(s,t)} \cdot u = (p, q) \sim_{(s,t)} \cdot u^2$. Similarly as above, there exists $l \geq 1$ such that $(p, q) \neq (p, q) \cdot u^l = (p, q) \cdot u^{2l}$. Let $p \neq p \cdot u^l$ (the case $q \neq q \cdot u^l$ would be treated in a similar way). Then $p = \perp$ is the hell.
If $s = \perp$ then $(\mathcal{P} \times \mathcal{Q})/\sim_{(\perp,t)}$ is isomorphic to $\mathcal{Q}/\sim_t$ via $(p, q) \sim_{(\perp,t)} \mapsto q \sim_t$. and we can use Part 1 of our Lemma.
If $s \neq \perp$ then $(\perp, q \cdot u^l) \sim_{(s,t)} (\perp, \perp) \sim_{(s,t)} (\perp, \perp) \cdot u$ for each $u \in A^*$.
Statement 3 is clear. □

Lemma 20. *Let $\mathcal{P} = (P, A, \cdot, \wedge, \top)$ and $\mathcal{Q} = (Q, A, \cdot, \wedge, \top)$ be meet automata. Then :*
1. *Let $\mathcal{P}$ satisfy the condition $(\dagger)$ and let $\alpha : \mathcal{P} \twoheadrightarrow \mathcal{Q}$ be a surjective homomorphism. Then also $\mathcal{Q}$ satisfies $(\dagger)$.*
2. *Let both of $\mathcal{P}$ and $\mathcal{Q}$ satisfy the condition $(\dagger)$ and let $(s, t) \in P \times Q$. Then $(\mathcal{P} \times \mathcal{Q})/\sim_{(s,t)}$ satisfies $(\dagger)$.*
3. *Let $\mathcal{Q}$ satisfy the condition $(\dagger)$ and let $\mathcal{R} = (R, B, \circ, \wedge, \top)$ be a subautomaton of $\mathcal{Q}$ with respect to $f \in \mathrm{D_{nk}}(B^{\square}, A^{\square})$. Then also $\mathcal{R}$ satisfies $(\dagger)$.*

Proof. Item 1 is clear.
2. Let $p \in P,\ q \in Q,\ u, v \in A^*$ and let $(p \cdot u, q \cdot u) \sim_{(s,t)} \neq (p \cdot v, q \cdot v) \sim_{(s,t)}$. Let $p \cdot u \neq p \cdot v$ (the case $q \cdot u \neq q \cdot v$ would be treated in a similar way). Then $p \cdot u \wedge p \cdot v$ is the hell.
For $s \neq \perp$ we have $(\perp, q \cdot u \wedge q \cdot v) \sim_{(s,t)} (\perp, \perp) \sim_{(s,t)} (\perp, \perp) \cdot w$ for each $w \in A^*$. For $s = \perp$ the structure $(\mathcal{P} \times \mathcal{Q})/\sim_{(s,t)}$ is isomorphic to $\mathcal{Q}/\sim_t$.
Item 3 is clear. □

Proof of Theorem 17. 1 (i) The closure with respect to homomorphic images follows from Lemmas 18.1 and 19.1.
(ii) Clearly, the trivial automaton satisfies $(*')$. For the product $\mathcal{P} \times \mathcal{Q}$, it follows from Lemmas 18.2, 19.2 and 19.1.
(iii) Let $\mathcal{R}$ be a subautomaton of $\mathcal{Q}$ with respect to $f \in \mathrm{D_{mi}}(B^{\square}, A^{\square})$. Then $r \neq r \cdot u = r \cdot u^2$ in $\mathcal{R}$ implies $r \neq r \cdot f(u) = r \cdot (f(u))^2$ in $\mathcal{Q}$. The statement follows from Lemmas 18.3, 19.1 and 19.3.
2 (i) : It follows from Lemmas 18.1 and 20.1.
(ii) : Clearly, the trivial automaton satisfies $(\dagger')$. Let both $\mathcal{P}$ and $\mathcal{Q}$ satisfy $(\dagger')$. Then also the product $\mathcal{P} \times \mathcal{Q}$ satisfies $(\dagger')$ by Lemmas 18.2, 20.2 and 20.1.
(iii) : It follows from Lemmas 18.3, 20.1 and 20.3.
3. Modify Lemmas 19 and 20 for the condition $(\ddagger)$ and follow the proofs of 1. and 2.
4–6 : From Lemma 10 it follows that a concrete minimalization of $\mathcal{U}(L)$ is isomorphic to $\mathcal{U}(L)$. The rest follows from Lemmas 19.1, 20.1. and a similar lemma for $(\ddagger)$. □

7. Varieties of semiring homomorphism for our basic examples

7.1. *The conjunctive* D_{mi}*-variety* $\mathscr{P}$

Proposition 21. *Let L be a language over an alphabet A. Then the language L belongs to $\mathscr{P}(A)$ if and only if the pseudoidentities $1 \le x^\omega$ and $x^\omega y \le y \vee x^\omega z$ are D_{mi}-satisfied in the syntactic semiring homomorphism $\varphi(L)$.*

Proof. Let L be a language over an alphabet A with the canonical minimal automaton $\mathcal{U}(L)$. Let k be a natural number such that u^k is an idempotent transformation of $\mathsf{U}(L)$ for each $u \in A^*$. Because the syntactic semiring is isomorphic to the transformation semiring of the canonical meet automaton (Proposition 13.1) we want to prove the following claim.
Claim: The canonical meet automaton $\mathcal{U}(L)$ satisfies the condition $(*)$ if and only if

$$(\forall\, p \in \mathsf{U}(L),\ u, v, w \in A^*) \quad (p \cdot u^k \subseteq p \text{ and } p \cdot (v \vee u^k w) \subseteq p \cdot (u^k v)). \qquad (**)$$

Assume first that $\mathcal{U}(L)$ satisfies the condition $(*)$. Then for each $p \in \mathsf{U}(L)$ and $u \in A^*$ we know that $p \cdot u^k = p$ or $p \cdot u^k = \emptyset$. The condition $p \cdot u^k \subseteq p$ follows.

If $p \cdot u^k = \emptyset$ then $p \cdot (v \vee u^k w) = p \cdot v \cap (p \cdot u^k) \cdot w = p \cdot v \cap \emptyset \cdot w = \emptyset \subseteq p \cdot (u^k v)$. If $p \cdot u^k = p$ then $p \cdot (v \vee u^k w) = p \cdot v \cap (p \cdot u^k) \cdot w = p \cdot v \cap p \cdot w \subseteq p \cdot v = (p \cdot u^k) \cdot v$. In both cases the condition $p \cdot (v \vee u^k w) \subseteq p \cdot (u^k v)$ holds and we proved the implication "$\Rightarrow$" of the claim.

Now, assume that $\mathcal{U}(L)$ satisfies the condition $(**)$ but does not satisfy the condition $(*)$. Then there are $p, q \in \mathsf{U}(L)$ and $u \in A^*$ such that $p \ne q \ne \emptyset$ and $p \cdot u = q = q \cdot u$. From the condition $(**)$ we have $q = p \cdot u^k \subseteq p$. Hence $\emptyset \ne q \subsetneq p$, i.e. there are words $v, w \in A^*$ such that $v \in p$, $v \notin q$ and $w \in q$. Then $p \cdot (v \vee u^k w) = p \cdot v \cap (p \cdot u^k) \cdot w = v^{-1} p \cap w^{-1} q$ which contains the empty word λ. Hence by condition $(**)$ $\lambda \in (p \cdot u^k) \cdot v = q \cdot v = v^{-1} q$ which is a contradiction with the assumption $v \notin q$. The proof of the claim is finished. □

Remark. Note that the pseudoidentity $x^\omega y \le y \vee x^\omega z$ is not $\mathsf{D}_{\mathsf{all}}$-satisfied in the syntactic semiring homomorphism of a language from $\mathscr{P}(A)$. For example, let $L = (ba + bb)^*(\lambda + b)$ be the language of all words which have at all odd positions the letter b.

Denote $p = L$ and let $q = b^{-1}L$ be the language of all words which have at all even positions the letter b. We can see $q \cdot a = q \cdot b = p$ and $p \cdot a = \emptyset$. For $U = \{\lambda, b\}$, we see $p \cdot U = p \cap p \cdot b = p \cap q = b^*$ and $(p \cap q) \cdot U = p \cap q$, hence $p \cdot U^k = p \cap q$. If we put $V = \{ba\}$, $W = \{\lambda\}$ then $p \cdot V = p$, $q \cdot V = \emptyset$ and we have $p \cdot (V \vee U^k W) = p \cdot V \cap (p \cdot U^k) \cdot W = p \cap (p \cap q) = p \cap q$. On the other hand, $p \cdot (U^k V) = (p \cap q) \cdot V = p \cdot V \cap q \cdot V = \emptyset$. In other words, the pseudoidentity $x^\omega y \le y \vee x^\omega z$ is not $\mathsf{D}_{\mathsf{all}}$-satisfied in $\varphi(L)$.

It is not hard to see that the pseudoidentity $1 \le x^\omega$ is $\mathsf{D}_{\mathsf{all}}$-satisfied in $\varphi(L)$ for $L \in \mathscr{P}(A)$.

Note that if the pseudoidentities $1 \le x^\omega$ and $x^\omega y \le y \vee x^\omega z$ are D_{mi}-satisfied in a semiring homomorphism then also the pseudoidentity $x^\omega y^\omega = x^\omega \vee y^\omega$ and consequently the pseudoidentity $x^\omega y^\omega = y^\omega x^\omega$ are D_{mi}-satisfied in the same semiring homomorphism. Notice that the pseudoidentities $1 \le x^\omega$ and $x^\omega y^\omega = y^\omega x^\omega$ characterize the closure of $\mathscr{P}$ to a positive variety (see [7]).

7.2. *The conjunctive* D_{nk}*-variety* $\mathscr{N}$

Proposition 22. *Let L be a language over an alphabet A. The language L belongs to $\mathscr{N}(A)$ if and only if the pseudoidentity $xy \le xt \vee zt \vee zy$ is D_{mi}-satisfied in the syntactic semiring homomorphism $\varphi(L)$.*

Proof. Let L be a language over an alphabet A with the canonical meet automaton $\mathcal{U}(L)$.

Assume that $L \in \mathscr{N}(A)$. We want to prove that for each state $p \in \mathsf{U}(L)$ and each $u, u', v, v' \in A^*$ we have $p \cdot (uv' \vee u'v' \vee u'v) \subseteq p \cdot (uv)$. Because $L \in \mathscr{N}(A)$, either a state $p \in \mathsf{U}(L)$ is $\emptyset$ or A^*, for which the previous inclusion is trivial, or p is a left derivative of L. Assume $w \in p \cdot uv' \cap p \cdot u'v' \cap p \cdot u'v$. Then $v'w \in u^{-1}p$ and $v'w \in u'^{-1}p$ which implies $p \cdot u = p \cdot u'$, because both $p \cdot u$ and $p \cdot u'$ are left derivatives of L. Hence $(p \cdot u') \cdot v = (p \cdot u) \cdot v$ and we can conclude that $w \in p \cdot u'v = p \cdot (uv)$.

Now, assume that $L \notin \mathscr{N}(A)$. This means that there exist two words $u, \bar{u}$ such that $u^{-1}L$ and $\bar{u}^{-1}L$ are different derivatives which are not disjoint. I.e. without loss of generality there are words $v, \bar{v}$ such that $v \notin u^{-1}L$, $v \in \bar{u}^{-1}L$ and $\bar{v} \in u^{-1}L \cap \bar{u}^{-1}L$. If we denote $p = L$ in $\mathsf{U}(L)$, then $\lambda \in p \cdot u\bar{v}$, $\lambda \in p \cdot \bar{u}\bar{v}$, $\lambda \in p \cdot \bar{u}v$ but $\lambda \notin p \cdot uv$. This implies $\lambda \in p \cdot (u\bar{v} \vee \bar{u}\bar{v} \vee \bar{u}v)$ and $\lambda \notin p \cdot uv$, which means that the pseudoidentity $xy \le xt \vee zt \vee zy$ is not D_{mi}-satisfied in the syntactic semiring homomorphism $\varphi(L)$. □

Remark. It is easy to see that if the pseudoidentity $xy \le xt \vee zt \vee zy$ is D_{mi}-satisfied in a semiring homomorphism then it is D_{nk}-satisfied in the same semiring homomorphism.

7.3. *The conjunctive* $\mathsf{D}_{\mathsf{mlm}}$*-variety* $\mathscr{C}$

Proposition 23. *Let L be a language over an alphabet A. The language L belongs to $\mathscr{C}(A)$ if and only if all pseudoidentities from the set*

$$\Pi = \{z_1 \dots z_m \le x_1 \dots x_k \vee y_1 \dots y_l \mid k, l, m \le 0, k \ne l\}$$

are $\mathsf{D}_{\mathsf{mlm}}$-satisfied in the syntactic semiring homomorphism $\varphi(L)$.

Proof. If a language L belongs to $\mathscr{C}(A)$ and $p \in \mathsf{U}(L) \setminus \{\top\}$, $k \neq l$, $a_1, \ldots, a_k, b_1, \ldots b_l \in A$ then $p \cdot (a_1 \ldots a_k \vee b_1 \ldots b_l)$ is the hell $\emptyset$ by Proposition 16.3. Hence the identities from Π are $\mathsf{D}_{\mathsf{mlm}}$-satisfied.

Let all pseudoidentities from the set Π be $\mathsf{D}_{\mathsf{mlm}}$-satisfied in the syntactic semiring homomorphism $\varphi(L)$. We want to show that $q = p \cdot (a_1 \ldots a_k \vee b_1 \ldots b_l)$ is the hell for each $p \in \mathsf{U}(L) \setminus \{\top\}$ and $a_1, \ldots, b_l \in A$, $k, l \geq 0$, $k \neq l$. Because $p \in \mathsf{U}(L) \setminus \{\top\}$ we have $p = L \cdot \{v_1, \ldots, v_n\}$ for some $\emptyset \neq \{v_1, \ldots, v_n\} \in A^{\square}$. If $v_1 = c_1 \ldots c_r$, where $c_1, \ldots, c_r \in A$, then we have $q = L \cdot \{v_1, \ldots v_n\}(a_1 \ldots a_k \vee b_1 \ldots b_l) \leq L \cdot (c_1 \ldots c_r a_1 \ldots a_k \vee c_1 \ldots c_r b_1 \ldots b_l) \leq L \cdot d_1 \ldots d_m$ for each $d_1, \ldots, d_m \in A$ by the pseudoidentity $x_1 \ldots x_{r+k} \vee y_1 \ldots y_{r+l} \geq z_1 \ldots z_m$. Hence $q = \perp$. Moreover, for each $a \in A$ we have $q \cdot a \leq L \cdot (c_1 \ldots c_r a_1 \ldots a_k a \vee c_1 \ldots c_r b_1 \ldots b_l a) \leq L \cdot d_1 \ldots d_m$ for each $d_1, \ldots, d_m \in A$ by the pseudoidentity $x_1 \ldots x_{r+k+1} \vee y_1 \ldots y_{r+l+1} \geq z_1 \ldots z_m$. Hence $q \cdot a = \perp$ too. This implies $q \cdot a = q$ for each $a \in A$, i.e. q is the hell. $\square$

Note that we can put into the set Π only pseudoidentities for which $k < l$ are relatively prime. One can also show that each finite subset of pseudoidentities of Π is not equivalent to Π, i.e. does not characterize the variety $\mathscr{C}$.

8. Final remarks

1. In the forthcoming paper [9] the authors study classes of meet automata defined by splitting pairs (defined there). It is a far reaching generalization of the condition (†). We mention here only the simplest case :

Let $n \geq 0$, $d \geq 1$. The class $\mathsf{Split}(x^{n+d}, x^n)$ consists of all meet automata whose minimalizations satisfy

$$(\forall\, q \in Q,\ u \in A^*) \quad (q \cdot u^{n+d} \neq q \cdot u^n \implies q \cdot u^{n+d} \wedge q \cdot u^n \text{ is the hell }).$$

It is shown that $\mathsf{Split}(x^{n+d}, x^n)$ is a D_{mi}-variety of meet automata and that different pairs (n, d) lead to different varieties.

2. Our techniques allow us to consider further classes of languages. There are many natural examples of disjunctive D_{mi}-varieties of languages (i.e. the classes consisting of complements of members of a conjunctive variety), for instance:

(a) finite sums of $A^* a_1 A^* \ldots a_l A^*$, $l \leq k$, k fixed,

(b) finite sums of $u(v_1 + \cdots + v_k)^* w$, $u, v_1, \ldots, v_k, w \in A^*$, $k \geq 1$.

Here join automata (= the dualizations of meet automata) are useful.

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