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Equitable list colorings of planar graphs without short cycles[☆]

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ABSTRACT

A graph G is equitably k-choosable if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. A graph G is equitably k-colorable if G has a proper k-vertex coloring such that the sizes of any two color classes differ by at most 1. In this paper, we prove that every planar graph G is equitably k-choosable and equitably k-colorable if one of the following conditions holds: (1) G is triangle-free and $k \geq \max\{\Delta(G), 8\}$; (2) G has no 4- and 5-cycles and $k \geq \max\{\Delta(G), 7\}$.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. A plane graph is a particular drawing of a planar graph in the Euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set, order, maximum degree and minimum degree by $V(G)$, $E(G)$, $F(G)$, $|V(G)|$, $\Delta(G)$ and $\delta(G)$ respectively (V, E, F, $|V|$, Δ and δ for short). For $v \in V(G)$, let $d_G(v)$ ($d(v)$ for short) denote the degree of v in G. For $f \in F(G)$, let $d_G(f)$ ($d(f)$ for short) denote the number of edges on the boundary of f, where each cut edge is counted twice. A vertex v (face f) is called a k-vertex (k-face) if $d(v) = k$ ($d(f) = k$). A vertex v (face f) is called a k^+-vertex (k^+-face) if $d(v) \geq k$ ($d(f) \geq k$). For $f \in F(G)$, we use $b(f)$ and $V(f)$ to denote the boundary walk of f and the vertices on the boundary walk respectively. A face f of G is called a simple face if $b(f)$ forms a cycle. Obviously, each k-face ($k \leq 5$) is a simple face when $\delta \geq 2$. A simple k-face f of G is called a $(d_1, d_2, \ldots, d_k)$-face if the vertices of f are, respectively, of degree $d_1, d_2, \ldots, d_k$. Let $P(v)$ and $Q(v)$ denote the set of 4-faces and 5-faces incident to the vertex v, respectively. Let $n_k(f)$ denote the number of k-vertices incident to the face f. Let $m_3(v)$ denote the number of 3-faces incident to the vertex v. Let $n_2(v)$ denote the number of 2-vertices adjacent to the vertex v. A graph G is called d-degenerate if every induced subgraph H of G has a vertex of degree at most d. A graph G is equitably k-choosable if, for any k-uniform list assignment L, G is L-colorable and each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. A graph G is equitably k-colorable if G has a proper k-vertex coloring such that the sizes of any two color classes differ by at most 1. The smallest integer k for which G is equitably k-colorable is called the equitable chromatic number of G, denoted by $\chi_e(G)$.

Equitable colorings naturally arise in some scheduling, partitioning and load balancing problems. In contrast with ordinary coloring, a graph may have an equitable k-coloring but have no equitable $(k+1)$-coloring. For example, the complete bipartite graph $K_{2n+1,2n+1}$ for $n \geq 1$ has an equitable 2-coloring but has no equitable $(2n+1)$-coloring.

In 1970, Hajnál and Szemerédi [1] proved that every graph has an equitable k-coloring whenever $k \geq \Delta+1$. This bound is sharp for some special graph classes. In 1973, Meyer [2] introduced the notion of equitable coloring and made the following conjecture:

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Conjecture 1. *The equitable chromatic number of a connected graph, which is neither a complete graph nor odd cycle, is at most Δ.*

In 1994, Chen, Lih and Wu [3] put forth the following conjecture:

Conjecture 2. *A connected graph is equitably Δ-colorable if it is different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$ for $m \geq 1$.*

This conjecture has been confirmed for graphs with $\Delta \leq 3$ or $\Delta \geq \frac{|V|}{2}$ [3], trees [4], bipartite graphs [5], outerplanar graphs [6], planar graphs with $\Delta \geq 13$ [7], line graphs [8] and d-degenerate graphs with $\Delta \geq 14d + 1$ [9].

In 2003, Kostochka, Pelsmajer and West [10] introduced the list analogue of equitable coloring. A list assignment L for a graph G assigns to each vertex $v \in V(G)$ a set $L(v)$ of acceptable colors. An L-coloring of G is a proper vertex coloring such that for every $v \in V(G)$ the color on v belongs to $L(v)$. A list assignment L for G is k-uniform if $|L(v)| = k$ for all $v \in V(G)$.

Given a k-uniform list assignment L for a graph G, we say that G is equitably L-colorable if G has an L-coloring such that each color appears on at most $\lceil \frac{|V(G)|}{k} \rceil$ vertices. A graph G is equitably list k-colorable or equitably k-choosable if G is equitably L-colorable whenever L is a k-uniform list assignment for G. In [10], Kostochka, Pelsmajer and West also conjectured the analogue of the Hajnál and Szemerédi Theorem [1]:

Conjecture 3. *Every graph is equitably k-choosable whenever $k \geq \Delta + 1$.*

It has been proved that Conjecture 3 holds for graphs with $\Delta \leq 3$ independently in [11,12].

Conjecture 4. *If G is a connected graph with $\Delta \geq 3$, then G is equitably Δ-choosable unless G is a complete graph or is $K_{2m+1,2m+1}$.*

Kostochka, Pelsmajer and West [10] proved that a graph G is equitably k-choosable if either $G \neq K_{k+1}, K_{k,k}$ (with k odd in the later case) and $k \geq \max\{\Delta, \frac{|V|}{2}\}$, or G is a forest and $k \geq 1 + \frac{\Delta}{2}$, or G is a connected interval graph and $k \geq \Delta$, or G is a 2-degenerate graph and $k \geq \max\{\Delta, 5\}$. Pelsmajer [11] proved that every graph is equitably k-chooable for any $k \geq \frac{\Delta(\Delta-1)}{2} + 2$.

In this paper we prove that every triangle-free plane graph is equitably k-choosable and equitably k-colorable whenever $k \geq \max\{\Delta, 8\}$, and every plane graph without 4- and 5-cycles is equitably k-choosable and equitably k-colorable whenever $k \geq \max\{\Delta, 7\}$.

2. Triangle-free planar graphs

Lemma 1 (*[10]*). *Let G be a graph with a k-uniform list assignment L. Let $S = \{v_1, v_2, \ldots, v_k\}$, where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in G. If $G - S$ has an equitable L-coloring and $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable L-coloring.*

Lemma 2. *Let $S = \{v_1, v_2, \ldots, v_k\}$, where $\{v_1, v_2, \ldots, v_k\}$ are distinct vertices in graph G. If $G - S$ has an equitable k-coloring and $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$, then G has an equitable k-coloring.*

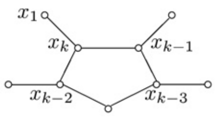
Proof. Let $G_i = G - \{v_{i+1}; v_{i+2}, \ldots, v_k\}$, so that $G - S = G_0$ and $G = G_k$. Let f_0 be an equitable k-coloring of G_0. For $1 \leq i \leq k$, extend f_{i-1} to a k-coloring f_i of G_i by giving v_i a color different from the colors that f_i has used on neighbors of v_i and on the vertices $v_1, v_2, \ldots, v_i$. Condition $|N_G(v_i) - S| \leq k - i$ for $1 \leq i \leq k$ guarantees that this is possible. By construction, the colors used on S are distinct, and hence f_k is an equitable k-coloring of G. $\square$

Lemma 3. *Every triangle-free plane graph is 3-degenerate.*

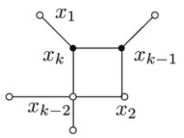
Lemma 4 (*[12]*). *Every graph with $\Delta \leq 3$ is equitably k-choosable whenever $k \geq \Delta + 1$.*

Lemma 5 (*[1]*). *Every graph has an equitable k-coloring whenever $k \geq \Delta + 1$.*

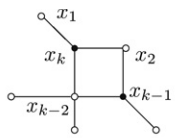
Lemma 6. *Every connected triangle-free plane graph G with order at least 5 has one of the following configurations*



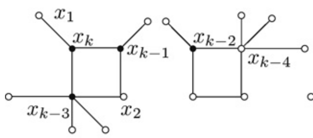
$H_1 2 \leq d(x_k), d(x_{k-1}), d(x_{k-2}), d(x_{k-3}) \leq 3$



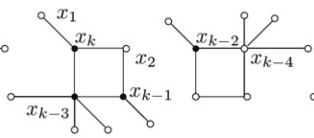
$H_2 3 \leq d(x_{k-2}) \leq 4$



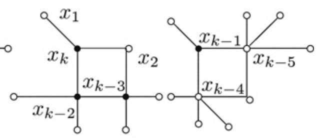
$H_3 3 \leq d(x_{k-2}) \leq 4$



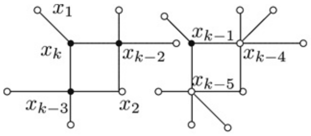
H_4: $3 \leq d(x_{k-4}) \leq 5$



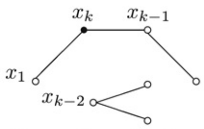
H_5: $3 \leq d(x_{k-4}) \leq 5$



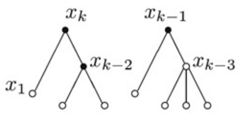
H_6: $4 \leq d(x_{k-4}), d(x_{k-5}) \leq 5$



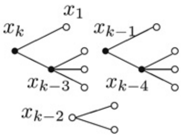
H_7: $4 \le d(x_{k-4}), d(x_{k-5}) \le 5$



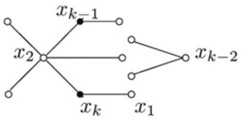
$H_8 1 \le d(x_{k-1}), d(x_{k-2}) \le 2$



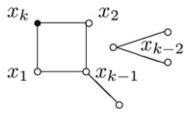
$H_9 3 \le d(x_{k-3}) \le 4$



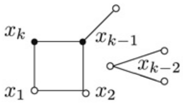
$H_{10} 1 \le d(x_{k-1}) \le 2$



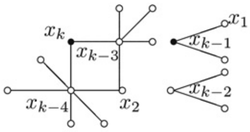
H_{11}: $1 \le d(x_{k-2}) \le 2$



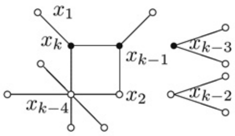
$H_{12} 2 \le d(x_{k-1}) \le 3$
$1 \le d(x_{k-2}) \le 2$



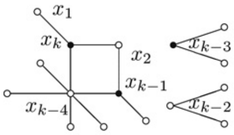
$H_{13} 1 \le d(x_{k-2}) \le 2$



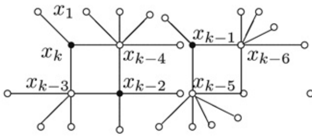
$H_{14} 4 \le d(x_{k-3}) \le 5$
$4 \le d(x_{k-4}) \le 6$
$1 \le d(x_{k-2}) \le 2$



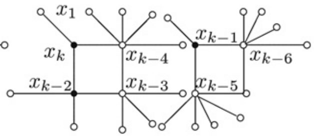
$H_{15} 4 \le d(x_{k-4}) \le 6$
$1 \le d(x_{k-2}) \le 2$



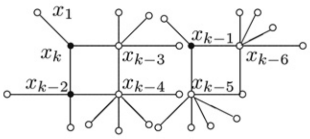
$H_{16} 4 \le d(x_{k-4}) \le 6$
$1 \le d(x_{k-2}) \le 2$



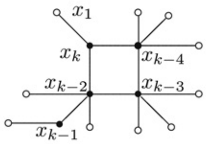
$H_{17} 4 \le d(x_{k-3}) \le 5$
$4 \le d(x_{k-4}), d(x_{k-5}), d(x_{k-6}) \le 6$



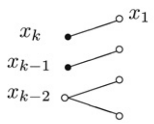
$H_{18} 4 \le d(x_{k-3}) \le 5$
$4 \le d(x_{k-4}), d(x_{k-5}), d(x_{k-6}) \le 6$



$H_{19} 4 \le d(x_{k-3}) \le 5$
$4 \le d(x_{k-4}), d(x_{k-5}), d(x_{k-6}) \le 6$



H_{20}



$H_{21} 1 \le d(x_{k-2}) \le 2$

Remark. In the above, each configuration represents subgraphs for which: (1) the degree of a solid vertex is exactly shown, (2) except for special pointed, the degree of a hollow vertex may be any integer from $[d, \Delta]$, where d is the number of edges incident to the hollow vertex, (3) hollow vertices may be not distinct while solid vertices are distinct.

Proof. Suppose G is a counterexample, then G is a connected triangle-free plane graph with order at least 5 and without configurations $H_1 \sim H_{21}$. We rewrite the Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ into the following equivalent form:

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (d(f) - 6) = -12.$$

We define a weight function w by $w(v) = 2d(v) - 6$ for $v \in V(G)$ and $w(f) = d(f) - 6$ for $f \in F(G)$. Thus $\sum_{x \in V \cup F} w(x) = -12$. We will design appropriate discharging rules and redistribute weights accordingly. Once discharging is finished, a new weight function w' is produced while the total sum of weights is kept fixed. For $x, y \in V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the sum of weights discharged from x to y according to our rules.

By Lemma 3, we have $\delta(G) \le 3$. We consider the following three cases:

Case 1: $\delta(G) = 3$

Our discharging rule is defined as follows:

(R) If $d(v) \ge 4$, then $\tau(v \to f) = \frac{w(v)}{d(v)}$ for each $f \in Q(v) \cup P(v)$.

We give the following obvious properties:

(P1) If $d(v) = 4$, then $\tau(v \to f) = \frac{w(v)}{d(v)} = \frac{2}{4} = \frac{1}{2}$ for each $f \in Q(v) \cup P(v)$.

(P2) If $d(v) = 5$, then $\tau(v \to f) = \frac{w(v)}{d(v)} = \frac{4}{5}$ for each $f \in Q(v) \cup P(v)$.

(P3) If $d(v) \geq 6$, then $\tau(v \to f) = \frac{w(v)}{d(v)} = \frac{2d(v)-6}{d(v)} \geq 1$ for each $f \in Q(v) \cup P(v)$.

Let $v \in V$. If $d(v) = 3$, then $w'(v) = w(v) = 0$. If $d(v) \geq 4$, then $w'(v) \geq 0$ by (R).

Let $f \in F$. If $d(f) \geq 6$, then $w'(f) = w(f) = d(f) - 6 \geq 0$.

If $d(f) = 5$, then $n_3(f) \leq 3$ since G has no H_1. Hence, f is a $(3^+, 3^+, 3^+, 4^+, 4^+)$-face. Thus, $w'(f) \geq 5 - 6 + \frac{1}{2} \times 2 = 0$ by (P1) $\sim$ (P3).

If $d(f) = 4$, then $n_3(f) \leq 2$ since G has no H_2 and H_3. If, furthermore, $n_3(f) = 2$, then $n_4(f) = 0$. Therefore, f is a $(3, 3, 5^+, 5^+)$-face if $n_3(f) = 2$.

Let $n_3(f) = 2$, then f is a $(3, 3, 5^+, 5^+)$-face. Since G has no H_4 and H_5, there is at most one $(3, 3, 5, 5^+)$-face f_1. By (P2) and (P3), $w'(f_1) \geq 4 - 6 + \frac{4}{5} \times 2 = -\frac{2}{5}$. If f is a $(3, 3, 6^+, 6^+)$-face, then $w'(f) \geq 4 - 6 + 1 \times 2 = 0$ by (P3).

Let $n_3(f) = 1$ and $n_4(f) \geq 2$, then f is a $(3, 4, 4, 4^+)$-face. Since G has no H_6 and H_7, there is at most one $(3, 4, 4, 4)$-face f_2 or at most one $(3, 4, 4, 5)$-face f_3 (f_2, f_3 do not exist at the same time). By (P1) and (P2), $w'(f_2) \geq 4 - 6 + \frac{1}{2} \times 3 = -\frac{1}{2}$ and $w'(f_3) \geq 4 - 6 + \frac{1}{2} \times 2 + \frac{4}{5} = -\frac{1}{5}$. If f is a $(3, 4, 4, 6^+)$-face, then $w'(f) \geq 4 - 6 + \frac{1}{2} \times 2 + 1 = 0$ by (P1) and (P3). Let $n_3(f) = 1$ and $n_4(f) \leq 1$, then f is a $(3, 4^+, 5^+, 5^+)$-face. Thus, $w'(f) \geq 4 - 6 + \frac{1}{2} + \frac{4}{5} \times 2 > 0$ by (P1) $\sim$ (P3).

Let $n_3(f) = 0$, then f is a $(4^+, 4^+, 4^+, 4^+)$-face. Thus, $w'(f) \geq 4 - 6 + \frac{1}{2} \times 4 = 0$ by (P1) $\sim$ (P3).

Thus, it follows from the above argument that $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -\frac{2}{5} - \frac{1}{2} = -\frac{9}{10}$, which is a contradiction.

Case 2: $\delta(G) = 2$

Subcase 2.1: There is one 2-vertex in G

The total weights of 2-vertex, 4-faces incident to a 2-vertex and 5-faces incident to a 2-vertex are not less than $(-2) + (-2) \times 2 = -6$. The discharging rule is the same as in Case 1 (2-vertex, 4-faces incident to a 2-vertex and 5-faces incident to a 2-vertex are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -\frac{69}{10}$, which is a contradiction.

Subcase 2.2: There are two 2-vertices in G

If the two 2-vertices are incident to one common face, then the total weights of 2-vertices, 4-faces incident to a 2-vertex and 5-faces incident to a 2-vertex are not less than $(-2) \times 2 + (-2) \times 3 = -10$. The discharging rule is the same as in Case 1 (2-vertices, 4-faces incident to a 2-vertex and 5-faces incident to a 2-vertex are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -\frac{109}{10}$, which is a contradiction.

If two 2-vertices are not incident to one common face, then the discharging rule is the same as in Case 1 (2-vertices and 5-faces incident to a 2-vertex are not considered). If $d(f) = 4$ and $n_2(f) = 1$, then f is a $(2, 4^+, 4^+, 3^+)$-faces since G has no H_{13}. Thus, $w'(f) \geq -2 + \frac{1}{2} \times 2 = -1$ by (P1) $\sim$ (P3). Therefore, the new total weights of 2-vertices, 4-faces incident to a 2-vertex and 5-faces incident to a 2-vertex are not less than $(-2) \times 2 + (-1) \times 4 = -8$. Hence, $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -\frac{89}{10}$, which is a contradiction.

Subcase 2.3: There are at least three 2-vertices in G

Since G has no H_8, there are no two adjacent 2-vertices. Since G has no H_9, there is at most one 2-vertex which is adjacent to a 3-vertex.

If there is one 2-vertex v_1 which is adjacent to a 3-vertex, since G has no H_9, there is no 2-vertex which is adjacent to a 4-vertex other than v_1. Thus, $w(v_1) = -2$.

If there is one 2-vertex which is adjacent to a 4-vertex, since G has no H_{10}, there is at most one 2-vertex v_2 which is adjacent to 4-vertices. Thus, $w(v_2) = -2$.

We will consider 2-vertices which are adjacent to two 5^+-vertices only while the weight of 2-vertex which is adjacent to a 3-vertex or 4-vertex kept fixed in the following.

Our discharging rules are as follows:

(R′1) Every 5^+-vertex sends 1 to each adjacent 2-vertex.

(R′2) If $d(v) = 4$, then $\tau(v \to f) = \frac{1}{2}$ for each $f \in Q(v) \cup P(v)$.

(R′3) If $d(v) \geq 5$, then $\tau(v \to f) = \frac{w(v) - n_2(v)}{d(v)}$ for each $f \in Q(v) \cup P(v)$.

Let v be a 5^+-vertex, since G has no H_{11}, we have $n_2(v) \leq 1$. We give the following obvious properties:

(P′1) If $d(v) = 5$, then $\tau(v \to f) \geq \frac{4-1}{5} = \frac{3}{5}$ for each $f \in Q(v) \cup P(v)$ by (R′3) when $n_2(v) = 1$, otherwise, $\tau(v \to f) \geq \frac{4}{5}$ by (R′3).

(P′2) If $d(v) = 6$, then $\tau(v \to f) \geq \frac{6-1}{6} = \frac{5}{6}$ for each $f \in Q(v) \cup P(v)$ by (R′3).

(P′3) If $d(v) \geq 7$, then $\tau(v \to f) \geq \frac{2d(v)-6-1}{d(v)} \geq 1$ for each $f \in Q(v) \cup P(v)$ by (R′3).

Let $v \in V$. If $d(v) = 2$, then $w'(v) = -2 + 1 \times 2 = 0$ for each 2-vertex which is adjacent to two 5^+-vertices by (R′1).

If $d(v) = 3$, then $w'(v) = w(v) = 0$.

If $d(v) = 4$, then $w'(v) \geq 2 - \frac{1}{2} \times 4 = 0$ by (R′2).

If $d(v) \geq 5$, then $w'(v) \geq 0$ by (P′1)$\sim$ (P′3) and (R′3).

Let $f \in F$. If $d(f) \geq 6$, then $w'(f) = w(f) = d(f) - 6 \geq 0$.

If $d(f) = 5$, then $n_2(f) + n_3(f) \leq 3$ since G has no H_1. Hence, f is a $(2^+, 2^+, 2^+, 4^+, 4^+)$-face. Thus, $w'(f) \geq 5 - 6 + \frac{1}{2} \times 2 = 0$ by (R′2) and (P′1) $\sim$ (P′3).

If $d(f) = 4$, then f is a $(2^+, 3^+, 2^+, 3^+)$-face since there are no two adjacent 2-vertices.

Let $n_2(f) \geq 1$, then f is a $(2, 3^+, 2^+, 3^+)$-face. Since G has no H_{12}, f is $(2, 3^+, 4^+, 3^+)$-face. Furthermore, since G has no H_{13}, f is a $(2, 4^+, 4^+, 4^+)$-face. Since G has no H_{14}, $(2, 4^+, 4^+, 4^+)$-faces are none but $(2, 4, 4^+, 7^+)$-faces f_4, $(2, 5, 4^+, 7^+)$-faces f_5 and $(2, 6^+, 4^+, 6^+)$-faces f_6. By (R′2) and (P′1) ∼ (P′3), $w'(f_4) \geq 4 - 6 + \frac{1}{2} \times 2 + 1 = 0$, $w'(f_5) \geq 4 - 6 + \frac{1}{2} + \frac{3}{5} + 1 > 0$ and $w'(f_6) \geq 4 - 6 + \frac{1}{2} + \frac{5}{6} \times 2 > 0$.

Let $n_2(f) = 0$. Since G has no H_2 and H_3, $n_3(f) \leq 2$.

Let $n_2(f) = 0$ and $n_3(f) = 2$, then f is a $(3, 3, 4^+, 4^+)$-face. Since G has no H_{15} and H_{16}, f is a $(3, 3, 7^+, 7^+)$-face. Thus, $w'(f) \geq 4 - 6 + 1 \times 2 = 0$ by (P′3).

Let $n_2(f) = 0$ and $n_3(f) = 1$, then f is a $(3, 4^+, 4^+, 4^+)$-face. Since G has no $H_{17} \sim H_{19}$, there is at most one 4-face, denoted by f_7, which is incident to one 3-vertex, one 4-vertex, one vertex of degree at most 5 and one vertex of degree at most 6. By (R′2) and (P′1) ∼ (P′2), $w'(f_7) \geq 4 - 6 + \frac{1}{2} \times 3 = -\frac{1}{2}$.

If G has f_7, then the other $(3, 4^+, 4^+, 4^+)$-faces are $(3, 4^+, 4^+, 7^+)$-faces f_8. By (R′2) and (P′1) ∼ (P′3), $w'(f_8) \geq 4 - 6 + \frac{1}{2} \times 2 + 1 = 0$.

If G has no f_7, then $(3, 4^+, 4^+, 4^+)$-faces are none but $(3, 4, 6^+, 6^+)$-faces, $(3, 4, 4^+, 7^+)$-faces and $(3, 5^+, 5^+, 5^+)$-faces.

If f is a $(3, 4, 6^+, 6^+)$-face, then $w'(f) \geq 4 - 6 + \frac{1}{2} + \frac{5}{6} \times 2 > 0$ by (R′2) and (P′2) ∼ (P′3).

If f is a $(3, 4, 4^+, 7^+)$-face, then $w'(f) \geq 4 - 6 + \frac{1}{2} \times 2 + 1 = 0$ by (R′2) and (P′1) ∼ (P′3).

Let f be a $(3, 5, 5, 5)$-face, since G has no H_{20}, there is at least one 5-vertex v on $V(f)$ such that $n_2(v) = 0$. Hence, $w'(f) \geq 4 - 6 + \frac{4}{5} + \frac{3}{5} \times 2 = 0$ by (P′1).

Let f be a $(3, 5^+, 5^+, 6^+)$-face, then $w'(f) \geq 4 - 6 + \frac{3}{5} \times 2 + \frac{5}{6} > 0$ by (P′1) ∼ (P′3).

Let $n_2(f) = 0$ and $n_3(f) = 0$, then f is a $(4^+, 4^+, 4^+, 4^+)$-face. Thus, $w'(f) \geq 4 - 6 + \frac{1}{2} \times 4 = 0$ by (R′2) and (P′1) ∼ (P′3).

Thus, it follows from the above argument that $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -2 - \frac{1}{2} = -\frac{5}{2}$, which is a contradiction.

Case 3: $\delta(G) = 1$

Since G has no H_{21}, there are at most two 1-vertices. Furthermore, there is no 2-vertex while there are two 1-vertices. Since G has no C_3, every k-face ($k \leq 5$) is a simple face.

Subcase 3.1: There are two 1-vertices in G

The total weights of 1-vertices is $(-4) \times 2 = -8$. The discharging rule is the same as in Case 1 (1-vertices are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -\frac{89}{10}$, which is a contradiction.

Subcase 3.2: There is one 1-vertex and at most one 2-vertex in G

The total weights of 1-vertex, 2-vertex and 4-face incident to a 2-vertex or 5-face incident to a 2-vertex is not less than $-4 + (-2) + (-2) \times 2 = -10$. The discharging rule is the same as in Case 1 (1-vertex , 2-vertex and 4-face incident to a 2-vertex or 5-face incident to a 2-vertex are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -\frac{109}{10}$, which is a contradiction.

Subcase 3.3: There is one 1-vertex and at least two 2-vertices in G

Since G has no H_8, there is no 2-vertex which is adjacent a 1-vertex. The total weight of 1-vertex is -4. The discharging rules are the same as in Subcase 2.3 (1-vertex is not considered). If there are exactly two 2-vertices, then the 1-vertex v_0 can be considered as a 2-vertex while the weight of v_0 will kept fixed. Therefore, this case can be also considered as Subcase 2.3. Hence, we have $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -4 - \frac{5}{2} = -\frac{13}{2}$, which is a contradiction. □

Theorem 7. *If G is a triangle-free plane graph and $k \geq \max\{\Delta(G), 8\}$, then G is equitably k-choosable.*

Proof. We use induction on $|V(G)|$. If $|V(G)| \leq k$, then we color all vertices using different colors from their lists. Suppose now that $|V(G)| > k \geq 8$. If every component of G has at most 4 vertices, then $\Delta(G) \leq 3$. By Lemma 4, G is equitably k-choosable. Otherwise, by Lemma 6, G has one of the structures $H_1 \sim H_{21}$. The vertices are labeled as they are in Lemma 6. If there are vertices labeled repeatedly, then we take the larger. (x_i is larger than x_{i-1}). We will find S in Lemma 1.

If G has H_8 or H_{21}, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_1\}$. If G has H_1 or H_9, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_{k-3}, x_1\}$. If G has H_{10} or H_{20}, then let $S' = \{x_k, x_{k-1}, , \ldots, x_{k-4}, x_1\}$. If G has one of $H_{17} \sim H_{19}$, then let $S' = \{x_k, x_{k-1}, , \ldots, x_{k-6}, x_1\}$. If G has one of H_2, H_3 and $H_{11} \sim H_{13}$, then let $S' = \{x_k, x_{k-1}, x_{k-2}, x_2, x_1\}$. If G has one of H_4, H_5 and $H_{14} \sim H_{16}$, then let $S' = \{x_k, x_{k-1}, , \ldots, x_{k-4}, x_2, x_1\}$. If G has H_6 or H_7, then let $S' = \{x_k, x_{k-1}, \ldots, x_{k-5}, x_2, x_1\}$ and $i = 5$. We fill the remaining unspecified positions in S from highest to lowest indices by choosing at each step a vertex with minimum degree in the graph obtained from G by delating the vertices thus far chosen for S. Such a vertex always exists because G is 3-degenerate by Lemma 3. Since $G - S$ is also a triangle-free plane graph and $k \geq \Delta(G) \geq \Delta(G - S)$, by the induction hypothesis, $G - S$ is equitably k-choosable. Hence, by Lemma 1, G is equitably k-choosable. The proof is complete. □

Theorem 8. *If G is a triangle-free plane graph and $k \geq \max\{\Delta(G), 8\}$, then G is equitably k-colorable.*

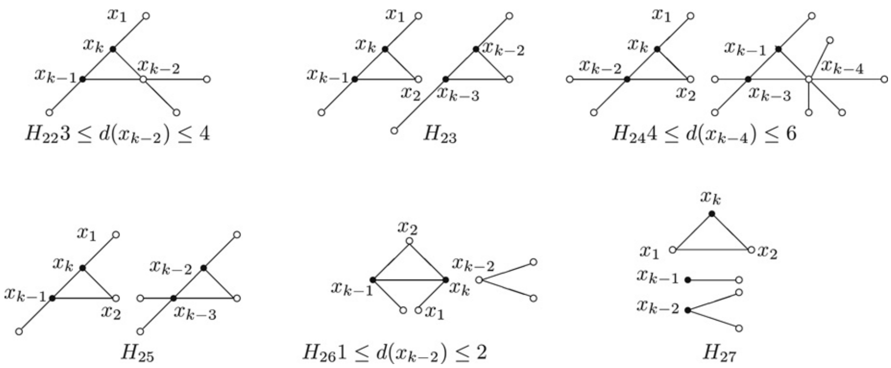
Proof. If every component of G has at most 4 vertices, then $\Delta(G) \leq 3$. By Lemma 5, G is equitably k-colorable. In other cases, we can obtain the desired results applying Lemma 2.

Conjectures 1–4 hold for every triangle-free planar graph G with $\Delta(G) \geq 8$. □

3. Planar graphs without 4-cycles and 5-cycles

Lemma 9 (*[13]*). *Every plane graph without* 5*-cycles is* 3*-degenerate.*

Lemma 10. *Let G be a connected plane graph with order at least* 5*. If G has neither* 4*-cycles nor* 5*-cycles, then G has one of the following configurations*



Remark. In the above, each configuration represents subgraphs for which: (1) the degree of a solid vertex is exactly shown, (2) except for special pointed, the degree of a hollow vertex may be any integer from $[d, \Delta]$, where d is the number of edges incident to the hollow vertex, (3) hollow vertices may be not distinct while solid vertices are distinct.

Proof. Suppose G is a counterexample, then G is a connected plane graph with order at least 5 and without configurations $H_8 \sim H_{11}$, $H_{21} \sim H_{27}$, 4-cycles and 5-cycles. We use the same Euler's formula and define the same weight function as in the proof of Lemma 6. Similarly, we shall derive a contradiction. Since G has no C_4, we have $m_3(v) \le [\frac{d(v)}{2}]$. By Lemma 10, we have $\delta(G) \le 3$. We consider the following three cases:

Case 1: $\delta(G) = 3$ Our discharging rules are as follows:

(R1) Every 4-vertex sends 1 to each of its incident 3-faces.

(R2) Every 5^+-vertex sends 2 to each of its incident 3-faces.

Let $v \in V$. If $d(v) = 3$, then $w'(v) = w(v) = 0$.

If $d(v) = 4$, then $m_3(v) \le 2$. Thus, $w'(v) \ge 2 \times 4 - 6 - 1 \times 2 = 0$ by (R1).

If $d(v) \ge 5$, then $m_3(v) \le [\frac{d(v)}{2}]$. Thus, $w'(v) \ge 2d(v) - 6 - 2 \times [\frac{d(v)}{2}] \ge 0$ by (R2).

Let $f \in F$. If $d(f) \ge 6$, then $w'(f) = w(f) = d(f) - 6 \ge 0$.

If $d(f) = 3$, then $n_3(f) \le 2$ since G has no H_{22}. If, furthermore, $n_3(f) = 2$, then $n_4(f) = 0$.

Let $n_3(f) = 2$, then f is a $(3, 3, 5^+)$- face. Since G contain no H_{23}, there is at most one $(3, 3, 5^+)$- face f_1. By (R2), $w'(f_1) = 3 - 6 + 2 = -1$.

Let $n_3(f) = 1$ and $n_4(f) = 2$, then f is a $(3, 4, 4)$-face. Since G has no H_{24}, there is at most one $(3, 4, 4)$-face f_2. By (R1), $w'(f_2) = 3 - 6 + 1 \times 2 = -1$.

Since G has no H_{25}, f_1, f_2 do not exist at the same time.

Let $n_3(f) = 1$ and $n_4(f) \le 1$, then f is a $(3, 4^+, 5^+)$- face. By (R1) and (R2), $w'(f) \ge 3 - 6 + 1 + 2 = 0$.

Let $n_3(f) = 0$, then f is a $(4^+, 4^+, 4^+)$-face. By (R1) and (R2), $w'(f) \ge 3 - 6 + 1 \times 3 = 0$.

Thus, it follows from the above argument that $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \ge -1$, which is a contradiction.

Case 2: $\delta(G) = 2$

Subcase 2.1: There are at most two 2-vertices in G The total weights of 2-vertices and 3-faces incident to 2-vertices is not less than $(-2) \times 2 + (-3) \times 2 = -10$. The discharging rules are the same as in Case 1 (2-vertices and 3-faces incident to 2-vertices are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \ge -11$, which is a contradiction.

Subcase 2.2: There are at least three 2-vertices in G Since G has no H_{26}, there is no $(3, 3, 2^+)$-face. Since G has no H_8, there are no two adjacent 2-vertices. Since G has no H_9, there is at most one 2-vertex which is adjacent to a 3-vertex.

If there is one 2-vertex v_1 which is adjacent to a 3-vertex, since G has no H_9, there is no 2-vertex which is adjacent to a 4-vertex other than v_1. Thus, $w(v_1) = -2$.

If there is one 2-vertex v_2 which is adjacent to 4-vertices, since G has no H_{10}, there is at most one 2-vertex adjacent to 4-vertices. Thus, $w(v_2) = -2$.

We will consider 2- vertices which are adjacent to two 5^+-vertices while the weight of 2-vertex which is adjacent to a 3-vertex or 4-vertex kept fixed in the following.

Our discharging rules are as follows:

(R′1) Every 5^+-vertex sends 1 to each of its adjacent 2-vertices.

(R′2) Every 4-vertex transfers 1 to each of its incident 3-faces.

(R′3) Every 5^+-vertex transfers $\frac{w(v) - n_2(v)}{m_3(v)}$ to each of its incident 3-faces ($m_3(v) \ne 0$).

Let v be a 5^+-vertex, since G has no H_{11}, we have $n_2(v) \le 1$. We give the following obvious properties:

(P1) Let v be a 5-vertex and f be a 3-face incident to v, then $\tau(v \to f) \ge \frac{4-1}{2} = \frac{3}{2}$ by (R′3).

(P2) Let v be a 6-vertex and f be a 3-face incident to v, then $\tau(v \to f) \geq \frac{6-1}{3} = \frac{5}{3}$ by (R′3).

(P3) Let v be a 7^+-vertex and f be a 3-face incident to v, then $\tau(v \to f) \geq \frac{2d(v)-6-1}{[\frac{d(v)}{2}]} \geq \frac{9}{4}$ by (R′3).

Let $v \in V$. If $d(v) = 2$, then $w'(v) = -2 + 1 \times 2 = 0$ for each 2-vertex which is adjacent to two 5^+-vertices by (R′1).

If $d(v) = 3$, then $w'(v) = w(v) = 0$.

If $d(v) = 4$, then $w'(v) \geq 2 \times 4 - 6 - 1 \times 2 = 0$ by (R′2).

If $d(v) \geq 5$. If v is not incident to 3-faces, then $w'(v) \geq 2d(v) - 6 - 1 > 0$ by (R′1). Otherwise, $w'(v) = 0$ by (P1) $\sim$ (P3) and (R′3).

Let $f \in F$. If $d(f) \geq 6$, then $w'(f) = w(f) = d(f) - 6 \geq 0$.

If $d(f) = 3$, then f is a $(2^+, 3^+, 3^+)$-face since there are no two adjacent 2-vertices.

Let $n_2(f) \geq 1$. Since there is at most one 2-vertex which is adjacent to a 3-vertex or 4-vertex, there is at most one $(2, 3, 3^+)$-face f_3 or $(2, 4, 4^+)$-face f_4 (f_3, f_4 do not exist at the same time). By (R′2) and (P1) $\sim$ (P3), $w'(f_3) \geq 3 - 6 = -3$, $w'(f_4) \geq 3 - 6 + 1 \times 2 = -1$. If f is a $(2, 5^+, 5^+)$-face, then $w'(f) \geq 3 - 6 + \frac{3}{2} \times 2 = 0$ by (P1) $\sim$ (P3).

Let $n_2(f) = 0$ and $n_3(f) \geq 1$. Since G has no $(3, 3, 2^+)$-face, f is a $(3, 4^+, 4^+)$-face.

Since G has no H_{24}, there is at most one $(3, 4, 4)$-face f_2 or $(3, 4, 5)$-face f_5 or $(3, 4, 6)$-face f_6 (at most one of f_2, f_5 and f_6 exists). By (R′2) and (P1) and (P3), $w'(f_2) = 3 - 6 + 1 \times 2 = -1$, $w'(f_5) \geq 3 - 6 + 1 + \frac{3}{2} = -\frac{1}{2}$, $w'(f_6) \geq 3 - 6 + 1 + \frac{5}{3} = -\frac{1}{3}$.

Let f be a $(3, 4, 7^+)$-face, then $w'(f) \geq 3 - 6 + 1 + \frac{9}{4} > 0$ by (R′2) and (P1) $\sim$ (P3).

Let f be a $(3, 5^+, 5^+)$-face, then $w'(f) \geq 3 - 6 + \frac{3}{2} \times 2 = 0$ by (P1) $\sim$ (P3).

Let $n_2(f) = n_3(f) = 0$, then f is a $(4^+, 4^+, 4^+)$-face. Thus $w'(f) \geq 3 - 6 + 1 \times 3 = 0$ by (R′2) and (P1) $\sim$ (P3).

Then, it follows from the above argument that $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -2 - 3 - 1 = -6$, which is a contradiction.

Case 3: $\delta(G) = 1$

Since G has no H_{26}, there is no $(3, 3, 2^+)$-face. Since G has no H_{21}, there are at most two 1-vertices.

Subcase 3.1: There are two 1-vertices in G

Since G has no H_{21}, there is no 2-vertex.

Since G has neither 4-cycles nor 5-cycles, there is no 4-face and at most two 5-faces. The total weights of 1-vertices and 5-faces are $(-4) \times 2 + (-1) \times 2 = -10$. The discharging rules are the same as in Case 1 (1-vertices and 5-faces are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -11$, which is a contradiction.

Subcase 3.2: There is one 1-vertex and at most one 2-vertex in G

Since G has neither 4-cycles nor 5-cycles, there is no 4-face and at most one 5-face. The total weights of 1-vertex, 2-vertex, 5-face and 3-face which is incident to a 2-vertex are not less than $-4 + (-2) + (-1) + (-3) = -10$. The discharging rules are the same as in Case 1 (1-vertices , 2-vertices, 5-face and 3-face which is incident to a 2-vertex are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -11$, which is a contradiction.

Subcase 3.3: There is one 1-vertex and two 2-vertices in G

Since G has neither 4-cycles nor 5-cycles, there is no 4- face and at most one 5-face. Since G has no H_{27}, there is no 3-face which is incident to 2-vertices. The total weights of 1-vertex, 2-vertices and 5-face are $-4 + (-2) \times 2 + (-1) = -9$. The discharging rules are the same as in Case 1(1-vertex, 2-vertices and 5-face are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -10$, which is a contradiction.

Subcase 3.4: There is one 1-vertex and at least three 2-vertices in G

Since G has neither 4-cycles nor 5-cycles, there is no 4-face and at most one 5-face. Since G has no H_8, then there is no 2-vertex which is adjacent to a 1-vertex. The total weights of 1-vertex and 5-face are not less than $-4 + (-1) = -5$. The discharging rules are the same as in Subcase 2.2 (1-vertex and 5-face are not considered), then $-12 = \sum_{x \in V \cup F} w(x) = \sum_{x \in V \cup F} w'(x) \geq -11$, which is a contradiction. □

Theorem 11. *Every plane graph G without 4-cycles and 5-cycles is equitably k-choosable whenever $k \geq \max\{\Delta(G), 7\}$.*

Proof. The proof is similar to the proof of Theorem 7. □

Theorem 12. *Every plane graph G without 4-cycles and 5-cycles is equitably k-colorable whenever $k \geq \max\{\Delta(G), 7\}$.*

Proof. The proof is similar to the proof of Theorem 8. □

Conjectures 1–4 hold for every planar graph G with $\Delta(G) \geq 7$ and without 4-cycles and 5-cycles.

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