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Tiling problems, automata, and tiling graphs

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ABSTRACT

This paper continues the investigation of tiling problems via formal languages, which was begun in papers by Merlini, Sprugnoli, and Verri. Those authors showed that certain tiling problems could be encoded by regular languages, which lead automatically to generating functions and other combinatorial information on tilings. We introduce a method of simplifying the DFA's recognizing these language, which leads to bijective proofs of certain tiling identities. We apply these ideas to some other tiling problems, including three-dimensional tilings and tilings with triangles and rhombi. We also study graph-theoretic variations of these tiling problems.

1. Introduction

Tiling problems investigate the ways in which given geometric regions may be covered using a specified collection of *tiling pieces*. Some commonly used tiling pieces are shown in Fig. 1. One famous example of a tiling problem is to find the number of tilings of an $2m \times 2n$ rectangle using horizontal and vertical dominos. The answer, found by Kasteleyn [6] and Fisher and Temperley [4], is given by the following remarkable formula:

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left[\cos^2\left(\frac{j\pi}{2m+1}\right) + \cos^2\left(\frac{k\pi}{2n+1}\right) \right].$$

In two recent papers [7,8], Merlini, Sprugnoli, and Verri introduced a general method for solving certain classes of tiling problems. The first step is to introduce a letter for each tiling piece and to encode a tiling as a string of these letters according to certain rules (described later). One then shows that the set of all words associated to the tilings of interest is a regular language that can be recognized by a deterministic finite automaton (DFA). Given the DFA, it is easy to produce an unambiguous grammar for this language. Using techniques first popularized by Schützenberger [9], this grammar leads automatically to generating functions for the number of tilings of various sizes.

This paper aims to extend the techniques of [7,8] in several directions. First, we introduce a method for simplifying the DFA associated to a given tiling problem. After simplification, we obtain a digraph whose edges are labeled by strings of letters, such that valid tilings correspond naturally to closed walks in the digraph based at a designated starting vertex. It turns out that different tiling problems can produce the same simplified digraphs (disregarding edge labels). This observation leads to *bijective proofs* that two different tiling problems have the same number of solutions. For example, we will use our method to prove the following result.

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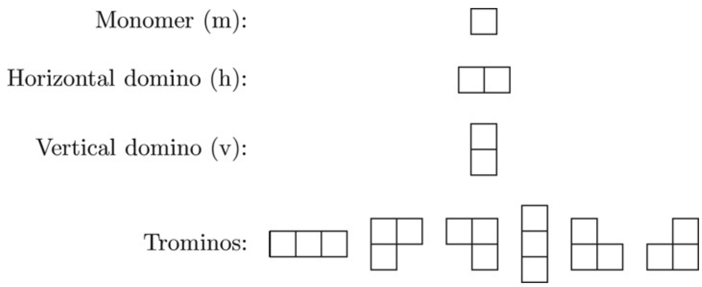


Fig. 1. Some common tiling pieces.

Fig. 2. More tiling pieces.

Theorem 1. *For each $n \geq 1$, the following three tiling problems have the same number of solutions.*

- *the number of tilings of a $3 \times 2n$ space using horizontal and vertical dominos;*
- *the number of tilings of a $2 \times 3n$ space using trominos;*
- *the number of tilings of a $1 \times n$ space using the pieces displayed in Fig. 2.*

While this theorem can certainly be proved by comparing generating functions, our approach has the added benefit of automatically supplying bijections between the different kinds of tilings.

To handle the third tiling problem in Theorem 1, we will need to extend the pivot rules in [7] to more general geometric configurations. As another illustration of such an extension, we consider tilings of a $2 \times 2 \times n$ space using three-dimensional dominos. We apply generating function methods to these situations to obtain information about the number of tilings of various sizes, the number of times a given piece is used in a tiling, and so on. For example, letting a_n be the number of 3-D domino tilings of a $2 \times 2 \times n$ box, we will derive the following generating function in Section 3.2:

$$\sum_{n\geq 0} a_n t^n = \frac{1-t}{1-3t-3t^2+t^3} = 1 + 2t + 9t^2 + 32t^3 + 121t^4 + 450t^5 + 1681t^6 + 6272t^7 + 23\,409t^8 + \cdots .$$

(See [3] for an alternate derivation via perfect matchings on graphs.)

Finally, we investigate a graph-theoretic variation of some of these tiling problems, following a proposal of Anant Godbole [5]. The idea is to convert a tiling into a graph by putting vertices at the corners of each tiling piece, and joining these vertices with edges that follow the boundaries of the pieces in the tiling. One can then seek enumerative and asymptotic information regarding graph invariants such as the number of vertices, diameter, etc.

2. Tilings, automata, and digraphs

This section reviews the method of Merlini et al. [2,6,7] for modeling tiling problems by regular languages. We use this method to associate a deterministic finite automaton (DFA) to a tiling problem. Next, we show how to simplify this DFA by applying certain rules to eliminate or collapse unnecessary states. Comparing the resulting simplified digraphs will lead to bijective proofs of certain tiling results. We also consider extensions of the basic model to other tiling problems.

2.1. The word of a tiling

We first describe a way to encode tilings of strip-like shapes using words. Begin by assigning a distinct letter to each available tiling piece. We will consider a running example in which we tile $2 \times n$ rectangles using monomers and dominos. Let us use the letter v for a vertical domino, h for a horizontal domino, and m for a monomer (see Fig. 1).

In our general discussion, we assume initially that all tiling pieces are polyominos (connected unions of unit squares of fixed orientation), and the containing shapes are also connected unions of unit squares. To convert a given tiling to a word, we visit the pieces in the tiling in a particular order and write down the corresponding letters to form the word. The visitation order is defined recursively using the following *pivot rule*. Suppose $i > 0$ and the first $i - 1$ tiles have already been visited. Consider the unit squares of the containing shape that have not been covered by the first $i - 1$ tiles. Find the leftmost and then topmost square in this set, and call it the *pivot cell*. By definition, the next tile to be visited is the unique tile covering this pivot cell. For example, the tiling shown in Fig. 3 is encoded by the word vmmmhhhhhmhmhhhhm. Clearly, the passage from tilings to words is reversible, assuming the containing shape is fixed and known.

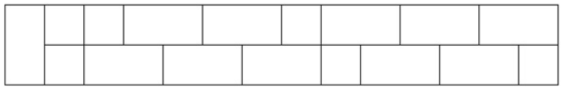


Fig. 3. A tiling with monomers and dominos.

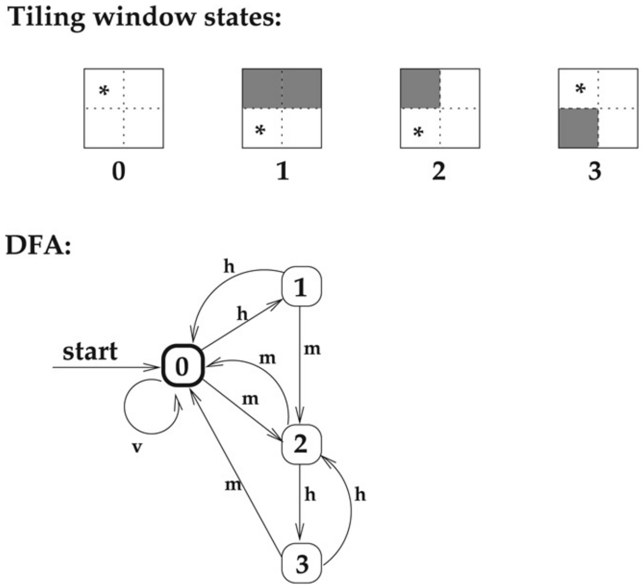


Fig. 4. DFA for a tiling problem.

2.2. From tilings to DFA's

Using the correspondence between tilings and words, we can associate a *language* (set of words) to a set of tilings. In many tiling problems, the set of valid tilings corresponds to a *regular* language, which can be recognized by a DFA or (equivalently) by a regular grammar. We refer the reader to standard references such as [10] for more background on formal languages.

To construct a DFA for a given tiling problem, we need to study *partial tilings* obtained by placing tiles one at a time in the order dictated by the pivot rule. The pivot rule (for rectangular strips of fixed height and variable width) is designed to build a tiling "from left to right". To determine the possible continuations of a partial tiling, one only needs to know the configuration of occupied and unoccupied squares at the right edge of the tiling. These configurations will be the *states* of the DFA.

More precisely, suppose we are tiling $m \times n$ rectangles, where m is fixed and n varies. Let k be the maximum width of all the tiling pieces. We can describe the state of a partial tiling by considering an $m \times k$ "tiling window" whose leftmost column contains the current pivot cell. All cells to the left of the window have already been tiled, and all cells to the right of the window have not yet been tiled. The state tells us which cells in the tiling window have already been tiled; this uniquely determines the next pivot cell.

The *start state* of the DFA corresponds to an empty tiling window with the pivot cell in the upper-left corner. This state is also the unique accepting state of the DFA (for tilings of rectangles). We can generate the remaining DFA states recursively by seeing what new states are reachable from existing states. Given an existing state S and a tile piece labeled x, there is a transition out of S labeled x iff it is possible to place the tile piece onto unfilled cells in the tiling window, so that the pivot cell is covered. If such a placement exists, it is unique, because of the way the pivot cell is defined. By placing the piece and shifting the tiling window so that the new pivot cell is in the leftmost column, we obtain the destination state for this transition. Clearly the number of states is bounded by 2^{mk}, hence is finite.

Fig. 4 shows the DFA we obtain for our example problem of tiling $2 \times n$ rectangles with monomers and dominos. There are four states reachable from the start state, with transitions as indicated. Tilings correspond uniquely to strings accepted by this DFA, which correspond in turn to closed walks through the digraph for the DFA based at the start state.

2.3. Simplifying the DFA

We will now modify the digraph for the DFA to reduce the number of states (vertices in the digraph). After our modifications, there will still exist a correspondence between valid tilings and closed walks through the new digraph based at the start state. However, the labels on the edges of these walks are now allowed to consist of multi-letter strings.

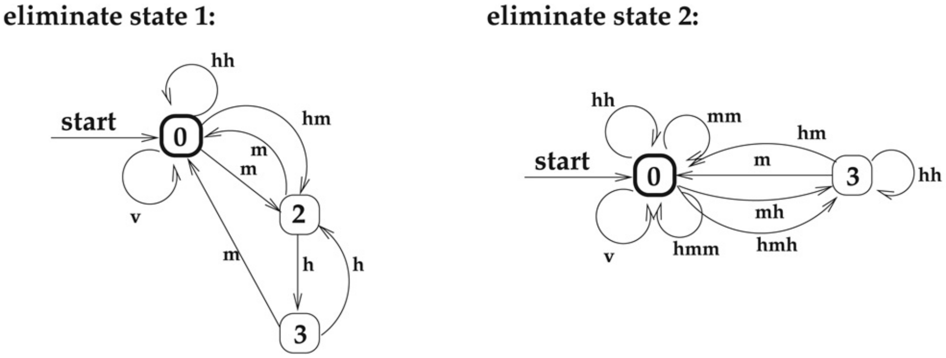


Fig. 5. Eliminating DFA states.

Let s_0 denote the start state. The first simplification is to delete all states t such that there is no directed path from t to s_0. These states are clearly unnecessary, since only closed walks ending at s_0 (which is also the accepting state) will lead to valid tilings.

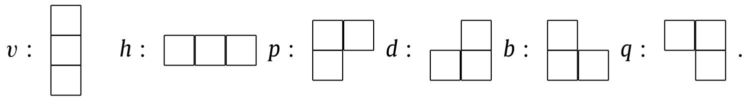
The main simplification step is to eliminate certain other states $u \neq s_0$, one by one, as follows. If there is a "loop transition" leading from state u to itself, then state u cannot be eliminated. Otherwise, we can eliminate u as follows. For each pair of directed edges $y \to u, u \to z$, replace these edges by a single directed edge $y \to z$. If the two original edges were labeled by strings p and q (respectively), label the new edge with the string pq. After making all these replacements, delete the isolated vertex u. Continue to eliminate vertices other than the start state, one at a time, until all remaining vertices have self-loops.

For example, Fig. 5 shows the digraphs obtained from the DFA in Fig. 4 by eliminating state 1 and then state 2. The simplified digraph has two states and nine directed edges.

2.4. Proof of Theorem 1

To prove Theorem 1, we will apply the construction in the preceding subsection to each of the three tiling problems mentioned in the theorem. Fig. 6 shows the tiling window states, DFA, and simplified digraph for tilings of a $3 \times 2n$ rectangle using dominos. We use the letter v for a vertical domino and h for a horizontal domino, as above. After constructing the DFA, we eliminate states 3, 6, 7, 8, 1, and 2 in this order. We obtain a simplified digraph with three states and seven transitions.

Fig. 7 shows the states, DFA, and simplified digraph for tilings of a $2 \times 3n$ rectangle using trominos. We use the following letters to label the trominos:



Of course, tromino v is never usable in this situation. When simplifying the digraph, we first delete state 4 since the accepting state 0 is not reachable from that state. Then we eliminate states 1, 5, and 6, leaving a simplified digraph with three states.

Although the initial DFA's for these two problems are different, the final simplified digraphs are obviously isomorphic (disregarding edge labels), for instance via the map sending 0 to 0, 4 to 2, and 5 to 3. Recall that tilings correspond to closed walks in the simplified digraph based at state 0. We can therefore obtain a bijection between domino tilings and tromino tilings by converting a domino tiling to a closed walk in the first digraph, then traversing the same closed walk in the second digraph, concatenating the edge labels encountered to form the word of a tromino tiling. Fig. 8 illustrates a matched pair of tilings obtained via this bijection. The word of the domino tiling is hhh-hv-v-vh-hhh-v-hv-hhh-v, whereas the word of the tromino tiling is hh-b-q-p-hh-d-b-hh-q.

This bijection maps domino tilings of a $3 \times 2n$ rectangle to tromino tilings of a $2 \times 3n$ rectangle. The reason is that each closed walk labeled by a $3n$-letter string in the first digraph corresponds to a closed walk labeled by a $2n$-letter string in the second digraph. This claim is easily verified by induction on the number r of times the walk returns to vertex zero. The key point is to verify the claim for $r = 1$. Here, the possible walks in the first digraph are $0, 0$ or $0, 4, 4, \ldots, 4, 0$ or $0, 5, 5, \ldots, 5, 0$. The corresponding label strings are hhh or hv-(hhh)^k-v (for some $k \geq 0$) or vh-(hhh)^k-v (for some $k \geq 0$). The lengths of these strings are 3, $3(k+1)$, and $3(k+1)$, respectively. The associated walks in the second digraph have label strings hh or b-(hh)^k-q or p-(hh)^k-d, which have lengths 2, $2(k+1)$, and $2(k+1)$, respectively.

To tackle the third tiling problem in Theorem 1, we first need a slight extension of the pivot rule to handle the tiling pieces in Fig. 2. Divide each unit square into 4 small triangles, as shown at the top of Fig. 9. Given a partial tiling, find the leftmost unit square that has not yet been fully tiled. Then declare the pivot location to be the lowest-numbered small triangle within this square that has not yet been filled, using the numbering in Fig. 9. By definition of the tiling process, the next tiling piece in the tiling order must cover the triangle designated as the pivot location.

We can now generate the tiling windows, DFA, and simplified digraph as usual. We use the lettering of the tiling pieces indicated in Fig. 2. In this particular problem, the initial DFA cannot be further simplified. However, we recognize the same

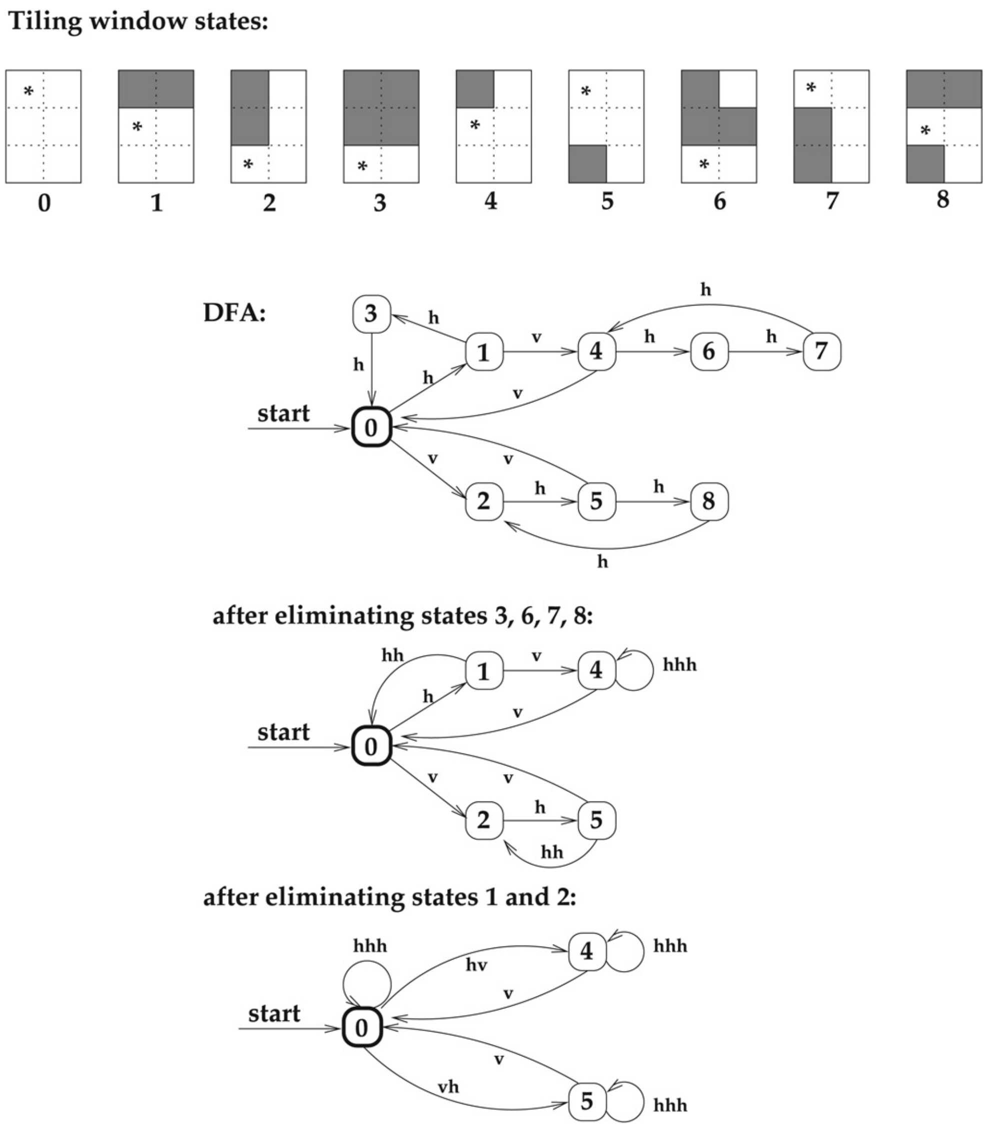


Fig. 6. Tiling $3 \times 2n$ rectangles with dominos.

digraph that appeared in the previous two tiling problems. Hence, we get another tiling bijection by matching closed walks in the isomorphic digraphs. Suppose we use the isomorphism between the domino digraph and the present digraph that maps 0 to 0, 4 to 1, and 5 to 2. Then the domino tiling at the top of Fig. 8 is matched to the tiling shown at the bottom of that figure, whose word is eadcfbagd. As before, one proves that this bijection preserves the parameter n by induction on the number r of returns to vertex 0. For $r = 1$, the walk labeled hhh in the first digraph maps to the walk labeled e in the new digraph (so $n = 1$ for both tilings); the walk labeled hv-(hhh)^k-v maps to the walk a-g^k-d (so $n = k + 1$ for both tilings); and the walk labeled vh-(hhh)^k-v maps to the walk c-f^k-b (so $n = k + 1$ for both tilings). This completes the bijective proof of Theorem 1.

2.5. *Three-dimensional tilings*

The ideas in the preceding section can be extended to more general tiling problems. As an example, we consider the problem of tiling $2 \times 2 \times n$ parallelepipeds in $\mathbb{R}^3$ with three-dimensional dominos. We consider x-dominos ($2 \times 1 \times 1$ blocks), y-dominos ($1 \times 2 \times 1$ blocks), and z-dominos ($1 \times 1 \times 2$ blocks). We label each unit cube in $[0, 2] \times [0, 2] \times [0, \infty)$ with the coordinates (x, y, z) of the corner of the cube closest to the origin. Given a partial tiling, define the *pivot cube* to be the untiled unit cube (x, y, z) in which we first minimize z, then x, then y. The tiling order is determined, as usual, by requiring that the next three-dimensional domino be placed so that the pivot cube is covered.

In this case, the tiling windows are $2 \times 2 \times 2$ cubes, which are displayed schematically in Fig. 10. In each state, the bottom 2×2 square represents the lower half of the tiling window, and the upper square represents the upper half. The DFA for

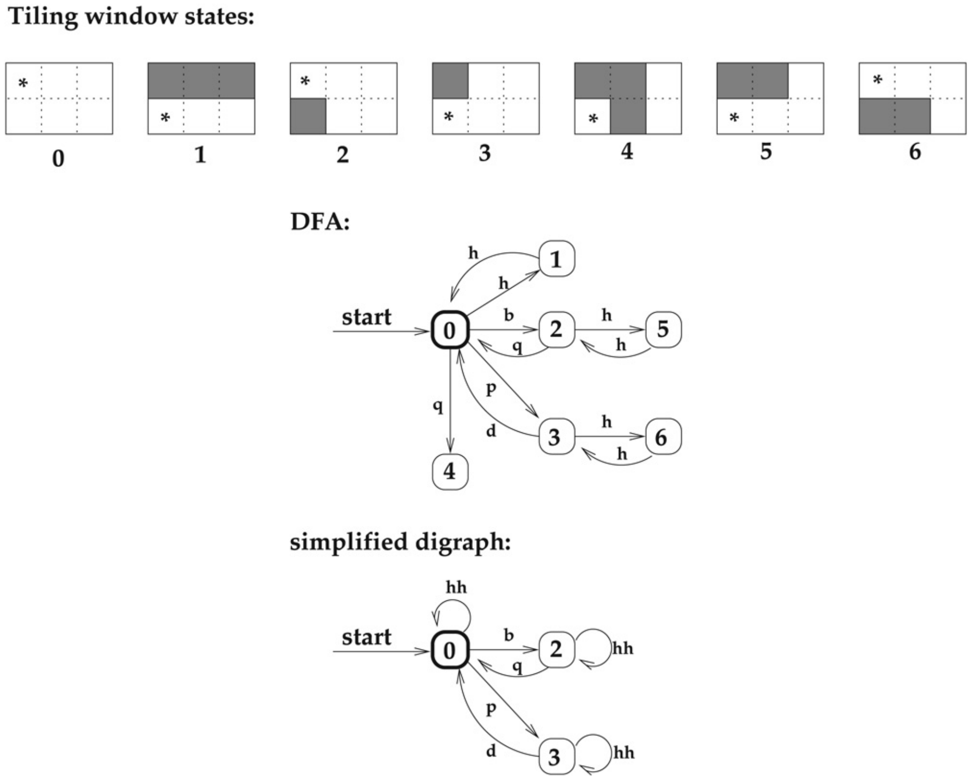


Fig. 7. Tiling $2 \times 3n$ rectangles with trominos.

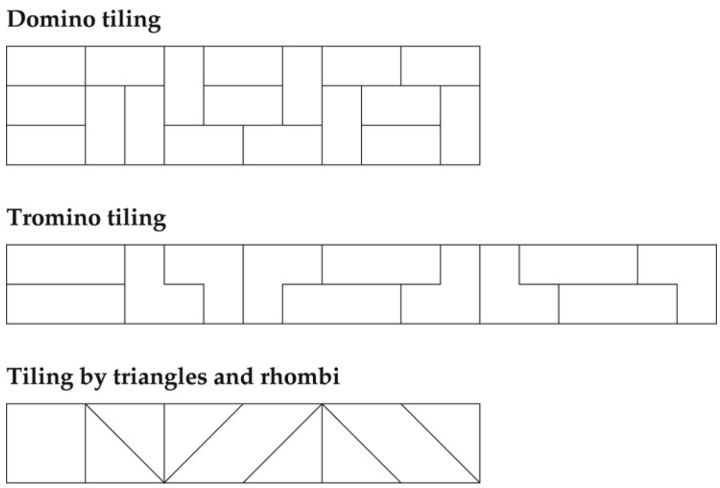


Fig. 8. Example of the tiling bijection.

this tiling problem, which has 12 states, is shown in Fig. 11. We simplify this DFA by eliminating states 4, 5, 10, 11, 6, 7, 9, and 3 (in this order). The resulting digraph is shown in Fig. 12.

3. Generating functions

To obtain generating functions for tiling enumeration problems, Merlini et al. converted DFA's to regular grammars and then applied Schützenberger's technique [9] to derive systems of equations satisfied by the generating functions. We can apply the same methods to our simplified digraphs. As an illustration, we derive generating functions for some of the tiling problems considered in the previous section.

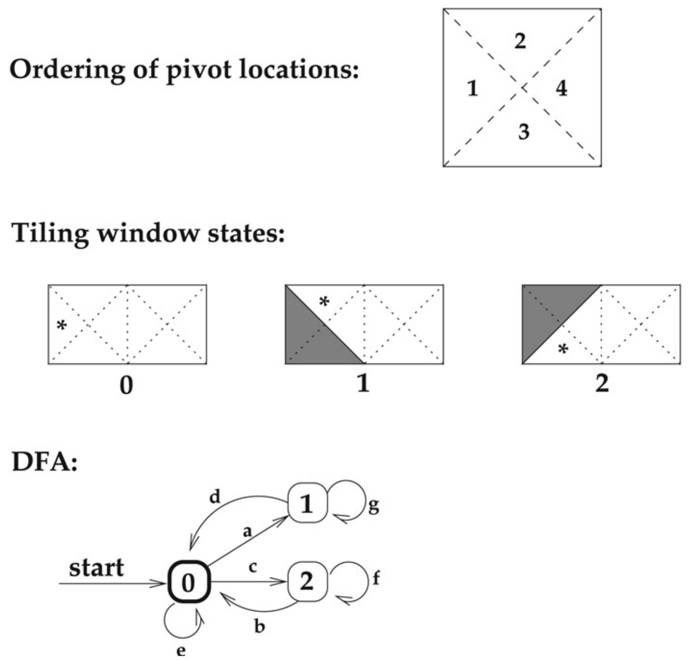


Fig. 9. Analysis of the third tiling problem.

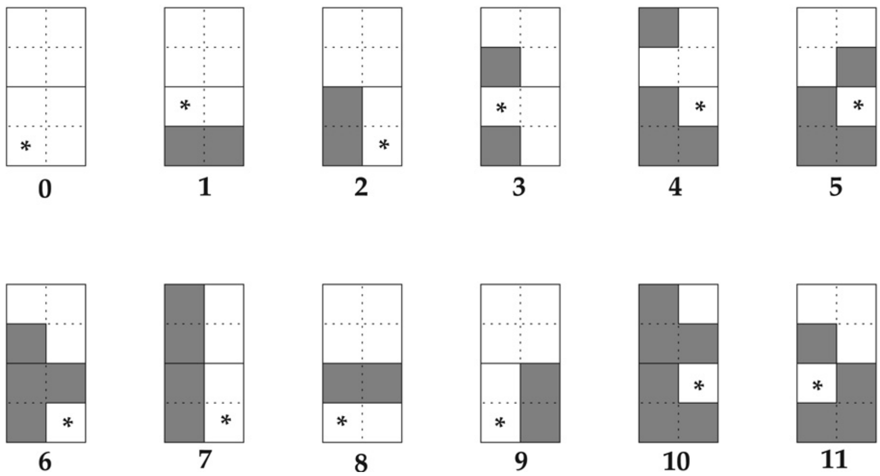


Fig. 10. Tiling windows for a three-dimensional tiling problem.

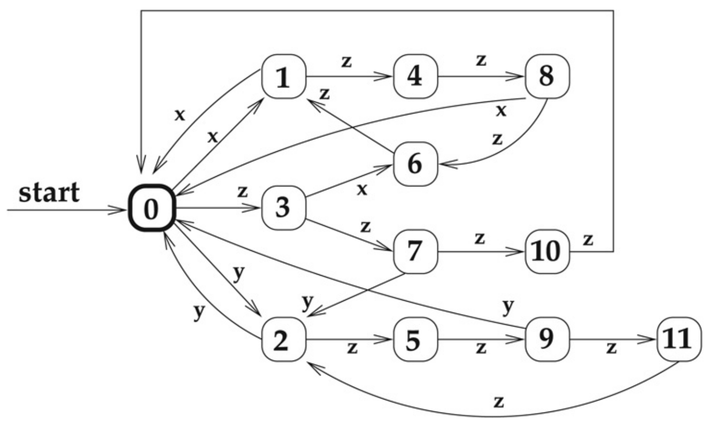


Fig. 11. DFA for tiling $2 \times 2 \times n$ rectangles with dominos.

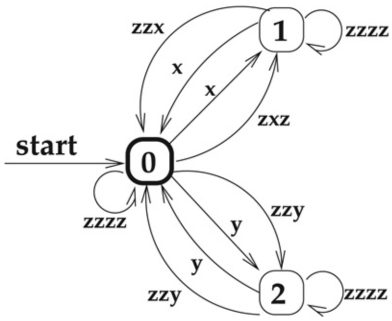


Fig. 12. Simplified digraph for the three-dimensional tiling problem.

3.1. Tilings by triangles and rhombi

We start by reviewing the construction for ordinary (unsimplified) DFA's. Consider the DFA shown in Fig. 9, which models tilings of a $1 \times n$ strip using the pieces shown in Fig. 2. Introduce a grammar with terminal symbols a, b, c, d, e, f, g; nonterminal symbols T_0, T_1, T_2; start symbol T_0; and productions:

$$\begin{aligned} T_0 &\to \mathtt{e}T_0 \mid \mathtt{a}T_1 \mid \mathtt{c}T_2 \mid \epsilon \\ T_1 &\to \mathtt{g}T_1 \mid \mathtt{d}T_0 \\ T_2 &\to \mathtt{f}T_2 \mid \mathtt{b}T_0. \end{aligned}$$

(Here ϵ denotes the empty string.) For every state T_i, let G_i be the formal sum of all strings of terminals that can be produced from the nonterminal T_i. Now we are viewing the letters a through g as noncommuting indeterminates; so each G_i lies in the ring of noncommuting formal power series $\mathbb{R}\langle\langle a, b, c, d, e, f, g\rangle\rangle$. The unambiguous grammar above leads to the following equations for the G_i:

$$\begin{aligned} G_0 &= eG_0 + aG_1 + cG_2 + 1 \\ G_1 &= gG_1 + dG_0 \\ G_2 &= fG_2 + bG_0. \end{aligned}$$

Solving, we find that $G_1 = (1-g)^{-1}dG_0$, $G_2 = (1-f)^{-1}bG_0$, and hence

$$G_0 = \left[1 - e - a(1-g)^{-1}d - c(1-f)^{-1}b\right]^{-1}.$$

We can specialize this formula to obtain generating functions keeping track of whatever information we are interested in. For example, suppose we want a five-variable generating function where the variables record how many a's, b's, e's, f's, and g's are used. Apply an evaluation homomorphism to G_0 such that $a \mapsto v$, $b \mapsto w$, $c \mapsto 1$, $d \mapsto 1$, $e \mapsto x$, $f \mapsto y$, and $g \mapsto z$, where v, w, x, y, z are *commuting* indeterminates. Then the desired generating function is

$$\frac{(y-1)(z-1)}{v(y-1) + (w + x + y - xy - 1)(z-1)}.$$

Now suppose we want the one-variable generating function for these tilings, where the variable keeps track of the width of the tiling (measured along the bottom of the strip). Since each of the five pieces in the previous generating function adds one to the length, whereas c and d add zero to the length, we need only substitute $v = w = x = y = z = t$ to obtain this generating function. Letting a_n be the number of tilings of the $1 \times n$ strip, we therefore have

$$\sum_{n \geq 0} a_n t^n = \frac{1-t}{1 - 4t + t^2}.$$

By Theorem 1, this is also the generating function for the number of domino tilings of $3 \times 2n$ strips, or the number of tromino tilings of $2 \times 3n$ strips.

3.2. Three-dimensional domino tilings

Let us find generating functions for the three-dimensional domino tilings of $2 \times 2 \times n$ boxes considered in Section 2.5. We convert the simplified digraph in Fig. 12 to a grammar in the obvious way. The terminal symbols are x, y, z; the nonterminal symbols are T_0, T_1, T_2; the start symbol is T_0; and the productions are:

$$\begin{aligned} T_0 &\to \mathtt{zzzz}T_0 \mid \mathtt{x}T_1 \mid \mathtt{zxz}T_1 \mid \mathtt{y}T_2 \mid \mathtt{zzy}T_2 \mid \epsilon \\ T_1 &\to \mathtt{x}T_0 \mid \mathtt{zzx}T_0 \mid \mathtt{zzzz}T_1 \\ T_2 &\to \mathtt{y}T_0 \mid \mathtt{zzy}T_0 \mid \mathtt{zzzz}T_2. \end{aligned}$$

The corresponding generating functions satisfy the following equations in the ring $\mathbb{R}\langle\langle x, y, z\rangle\rangle$:

$$\begin{aligned}
G_0 &= z^4G_0 + (x + zxz)G_1 + (y + z^2y)G_2 + 1\\
G_1 &= (1 + z^2)xG_0 + z^4G_1\\
G_2 &= (1 + z^2)yG_0 + z^4G_2.
\end{aligned}$$

Solving for G_0 gives

$$G_0(x, y, z) = \left[1 - z^4 - (x + zxz)(1 - z^2)^{-1}x - (1 + z^2)y(1 - z^2)^{-1}y\right]^{-1}.$$

To find the generating function $\sum_{n\geq 0} a_nt^n$, where a_n is the number of tilings of a $2 \times 2 \times n$ box, first let $x = y = z = t^2$ to find the volume generating function:

$$G_0(t^2, t^2, t^2) = \frac{1 - t^4}{1 - 3t^4 - 3t^8 + t^{12}}.$$

Then replace t^4 by t to get the desired generating function:

$$\sum_{n=0}^{\infty} a_nt^n = \frac{1 - t}{1 - 3t - 3t^2 + t^3}.$$

If we want to keep track of, say, the volume and number of z-dominos used, just compute

$$G_0(t^2, t^2, zt^2) = \frac{1 - t^4z^2}{1 + t^{12}z^6 - t^4(2 + z^2) - t^8z^2(2 + z^2)}.$$

Other variations on these generating functions can be derived similarly.

3.3. Asymptotics

For a fixed n, suppose we pick a random domino tiling of a $2 \times 2 \times n$ box. What is the probability distribution of the number Z_n of z-dominos used in the tiling? We can use the following central limit theorem of Bender [1] to answer this question and others like it.

Theorem 2. *Suppose*

$$f(w, z) = \frac{g(w, z)}{h(w, z)} = \sum_{n,k\geq 0} c_n(k)w^nz^k,$$

where: (a) *$h(w, z)$ is a polynomial in w whose coefficients are continuous functions of z;* (b) *for some r, $h(r, 1) = 0$ and all other roots of $h(w, 1)$ have larger absolute value;* (c) *$g(w, z)$ is analytic for z close to 1 and $w < |r| + \epsilon$; and* (d) *$g(r, 1) \neq 0$. Define random variables X_n by setting $P(X_n = k_0) = c_n(k_0)/\sum_k c_n(k) = g(w, z)|_{w^nz^{k_0}}/g(w, 1)|_{w^n}$. Then the X_n's are asymptotically normal with mean $n\mu$ and variance $n\sigma^2$, where*

$$\mu = \frac{h_z}{rh_w}, \qquad \sigma^2 = \mu^2 + \frac{(h_z/h_w)^2h_{ww} - 2(h_z/h_w)h_{wz} + h_z + h_{zz}}{rh_w},$$

with all partial derivatives being evaluated at $(w, z) = (r, 1)$. We abbreviate the conclusion by writing $X_n \sim N(n\mu, n\sigma^2)$.

To illustrate the use of this theorem, we answer the question posed at the beginning of the last paragraph. Using the results of Section 3.2, the relevant generating function is

$$f(w, z) = \frac{1 - wz^2}{1 + w^3z^6 - w(2 + z^2) - w^2z^2(2 + z^2)}.$$

Here $h(w, 1) = 1 - 3w - 3w^2 + w^3$ has smallest root $r = 2 - \sqrt{3} \approx 0.26795$. Using the preceding formulas, we find $\mu = 0.8453$ and $\sigma^2 = 0.7698n$, so that $Z_n \sim N(0.8453n, 0.7698n)$.

The same technique leads to the following asymptotic normality results for the tiling problems considered so far.

- Let V_n and H_n be the number of vertical and horizontal dominos in a random tiling of a $3 \times 2n$ rectangle. Then

$$V_n \sim N(1.1547n, 0.7698n), \qquad H_n \sim N(1.8453n, 0.7698n).$$

- Let V_n, H_n, M_n respectively denote the number of vertical dominos, horizontal dominos, and monomers in a random tiling of a $2 \times n$ rectangle. Then

$$V_n \sim N(0.2068n, 0.2032n), \quad H_n \sim N(0.3997n, 0.2444n), \quad M_n \sim N(0.7870n, 0.6684n).$$

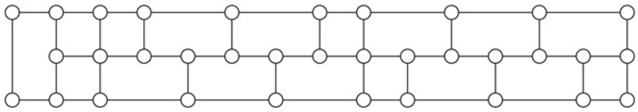


Fig. 13. Example of a tiling graph.

Table 1
Effect of each transition on graph statistics

Transition	Δw	Δx	Δe
start$\to$ 0	0	2	1
$0 \xrightarrow{v} 0$	1	2	3
$0 \xrightarrow{h} 1$	0	3	3
$0 \xrightarrow{m} 2$	0	3	3
$1 \xrightarrow{h} 4$	2	1	2
$1 \xrightarrow{m} 2$	1	2	2
$2 \xrightarrow{h} 3$	1	2	3
$2 \xrightarrow{m} 4$	1	1	2
$3 \xrightarrow{h} 2$	1	2	3
$3 \xrightarrow{m} 4$	1	1	2
$4 \xrightarrow{v} 0$	1	2	3
$4 \xrightarrow{h} 1$	0	2	3
$4 \xrightarrow{m} 2$	0	2	3

- Let H_n, P_n, D_n, B_n, Q_n be the number of trominos of type h, p, d, b, q (respectively) in a random tiling of a $2 \times 3n$ rectangle. Then

$$H_n \sim N(0.4853n, 0.7698n), \qquad P_n, D_n, B_n, Q_n \sim N(0.2887n, 0.19245n).$$

- Let $A_n, \ldots, G_n$ be the number of tiling pieces of type $a, \ldots, g$ when tiling a $1 \times n$ rectangle with the pieces shown in Fig. 2. Then

$$A_n, B_n, C_n, D_n \sim N(0.2887n, 0.19245n), \qquad E_n \sim N(0.2113n, 0.19245n),$$
$$F_n, G_n \sim N(0.10566n, 0.13962n).$$

- Let X_n, Y_n, Z_n be the number of x-dominos, y-dominos, and z-dominos in a random tiling of a $2 \times 2 \times n$ box. Then

$$X_n, Y_n \sim N(0.57735n, 0.7698n), \qquad Z_n \sim N(0.8453n, 0.7698n).$$

4. Tiling graphs

Anant Godbole [5] proposed the following graph-theoretic variation on tiling problems. Suppose we select a random tiling of a given region using a given collection of tiling pieces. Turn the tiling into a graph by placing a vertex at every corner of every tiling piece, and converting the boundaries of the tiling pieces to edges in the obvious way. For example, Fig. 13 shows the tiling graph obtained from the tiling in Fig. 3. We can now seek information on the distribution of various graph-theoretic statistics on these tiling graphs. For example, how many vertices and edges does a random tiling graph have? We show how to extend the techniques of the preceding sections to answer questions like this one.

Continuing our example from Section 2, we consider tiling graphs associated to tilings of $2 \times n$ rectangles using monomers and dominos. Suppose we want to find a generating function that keeps track of the width (w), number of vertices (x), and number of edges (e) in these tiling graphs. As before, we build a tiling one step at a time by placing tiles according to the pivot rule. The placement of each new tile will increase the width, number of vertices, and number of edges by a certain amount. (Here, the *width* of a partial tiling is the number of fully tiled columns.) To predict the increase in x and e caused by the placement of a given tile, we need to know which vertices and edges already exist on the boundary between the tiled region and the untiled region. So, we must modify our "tiling window states" to incorporate this extra information.

In the current example, we obtain the set of states shown in Fig. 14. The states are similar to those in Fig. 4, but now there are two states (0 and 4) corresponding to the old state 0; both of these states are accepting states. We build a DFA just as before, as shown in Fig. 14. We now make a table showing, for each transition in the DFA, how the statistics w, x, and e change when we follow that transition by adding one new tile. See Table 1. The first row of the table specifies an "initial condition" in which an empty tiling is represented by a graph with two vertices and one edge.

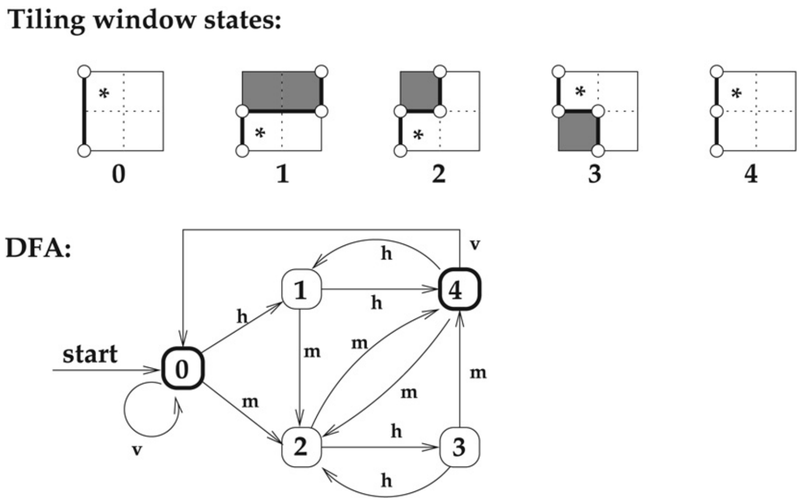


Fig. 14. States and DFA for a tiling graph problem.

For $0 \leq i \leq 4$, let $G_i(w, x, e)$ be the generating function for the partial tilings that correspond to state i of the DFA. Using the information in Table 1, we can immediately write down a system of equations satisfied by the G_i's:

$$\begin{aligned}
G_0 &= x^2e + wx^2e^3G_0 + x^3e^3G_1 + x^3e^3G_2 \\
G_1 &= w^2xe^2G_4 + wx^2e^2G_2 \\
G_2 &= wx^2e^3G_3 + wxe^2G_4 \\
G_3 &= wx^2e^3G_2 + wxe^2G_4 \\
G_4 &= wx^2e^3G_0 + x^2e^3G_1 + x^2e^3G_2.
\end{aligned}$$

The generating function for "complete" tiling graphs on $2 \times n$ rectangles is $G_0 + G_4$, since these are the two accepting states in the DFA. Solving the system, we obtain $G_0 + G_4 = p/q$ where

$$\begin{aligned}
p &= ex^2 + e^7x^6w^2 + e^8x^7w^2 + e^9x^7w^3 + e^6x^5w(1+w), \\
q &= -1 + 2e^3x^2w - e^6x^4w^2 + e^7x^5w^2 + e^{10}(-1+x)x^7w^3 \\
&\quad -e^{11}(-1+x)x^7w^4 + e^5x^3w(1+w) + e^8x^5w^2(-1+x-2w+xw).
\end{aligned}$$

Now we can use Bender's theorem to determine the asymptotic behavior of the number of vertices X_n and the number of edges E_n of a random tiling graph of width n. We simply set $e = 1$ (resp. $x = 1$) in the preceding generating function and proceed as in Section 3.3. For X_n, the smallest root in Bender's theorem is $r = 0.311108$, and we calculate $X_n \sim N(2.47162n, 0.358533n)$. Analogous calculations show that $E_n \sim N(3.8651n, 0.91355n)$.

We can similarly analyze the tiling graphs associated to the other tiling problems considered earlier. The details of these calculations will be omitted, but we record the final results here:

- For tiling graphs based on domino tilings of $3 \times 2n$ rectangles, we have
  $$X_n \sim N(5.3094n, 0.4715n), \qquad E_n \sim N(18.375n, 0.4715n).$$
- For tiling graphs based on tromino tilings of $2 \times 3n$ rectangles, we have
  $$X_n \sim N(5.0981n, 1.0829n), \qquad E_n \sim N(7.0981n, 1.0829n).$$
- For tiling graphs based on tilings of $1 \times n$ rectangles using the pieces in Fig. 2, we have
  $$X_n \sim N(2.5773n, 0.10245n), \qquad E_n \sim N(3.5773n, 0.19245n).$$
- For tiling graphs based on 3-D domino tilings of $2 \times 2 \times n$ boxes, we have
  $$X_n \sim N(6.9074n, 1.1324n), \qquad E_n \sim N(12.7339n, 2.3879n).$$

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