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Fault-free Hamiltonian cycles in twisted cubes with conditional link faults

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ABSTRACT

The n-dimensional twisted cube, denoted by TQ_n, a variation of the hypercube, possesses some properties superior to the hypercube. In this paper, assuming that each vertex is incident with at least two fault-free links, we show that TQ_n can tolerate up to $2n-5$ edge faults, while retaining a fault-free Hamiltonian cycle. The result is optimal with respect to the number of edge faults tolerated.

1. Introduction

The hypercube is a popular interconnection network with many attractive properties such as regularity, symmetry, small diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [17]. The twisted cube [12], as one of the important variations of the hypercube, and derived by changing some connections of the hypercube according to specific rules, possesses some desirable features: its diameter, wide diameter, and fault diameter are about half of those of the comparable hypercube [6]. An n-dimensional twisted cube is $(n-3)$-Hamiltonian connected [13] and $(n-2)$-pancyclic [20], whereas the hypercube is not. Moreover, its performance is superior to that of the hypercube [1].

An *embedding* of one *guest graph* G into another *host graph* H is a one-to-one mapping f from the vertex set of G to the vertex set of H [15]. An edge of G corresponds to a path of H under f. Linear arrays and rings, two of the most fundamental networks for parallel and distributed computation, are suitable for designing simple algorithms with low communication costs. Numerous efficient algorithms designed on linear arrays and rings for solving various algebraic problems and graph problems can be found in [2,14]. Linear arrays and rings can also be used as control/data flow structures for distributed computation in arbitrary networks. There is an application of longest paths to a practical problem that was encountered in the on-line optimization of a complex Flexible Manufacturing System (see [3]). These applications motivate the embedding of paths and cycles in networks.

Since processor or link faults may develop in real world networks, it is important to consider faulty networks. The problems of finding the diameter [7], routing [9], multicasting [16], broadcasting [18], gossiping [8], and embedding [19] have been solved on various faulty networks. The fault-tolerant Hamiltonicity [10] measures the performance of the Hamiltonian property in the faulty networks.

It was shown in [5] (respectively, [4,14]) that if each vertex of an n-dimensional hypercube (respectively, k-ary n-cube and n-dimensional crossed cube) is incident with at least two fault-free edges, then it contains a fault-free Hamiltonian

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cycle, even if there are $2n-5$ (respectively, $4n-5$ and $2n-5$) edge faults. In this paper, we show that if each vertex of an n-dimensional twisted cube is incident with at least two fault-free edges, then it contains a fault-free Hamiltonian cycle, even if there are $2n-5$ edge faults. The rest of this paper is organized as follows. In Section 2, the structure of the twisted cube is elaborated, and some definitions and notations used throughout this paper are introduced. In Section 3, a basic idea is given and some properties of the twisted cube are derived. In Section 4, under the assumption that each vertex is incident with at least two fault-free edges, we show that an n-dimensional twisted cube contains a fault-free Hamiltonian cycle, even if there are up to $2n-5$ edge faults. Moreover, we also show that this result is optimal. In Section 5, this paper concludes with some remarks.

2. Preliminaries

We usually use a graph to represent the topology of an interconnection network. A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its endpoints [21]. We use (u, v) to denote an edge whose endpoints are u and v. The degree of vertex v in G, written as $d_G(v)$, is the number of edges incident to v. In addition, $\delta(G) = \min\{d_G(v) | v \in V(G)\}$.

A path $P_{x_0x_t} = \langle x_0, x_1, \ldots, x_t\rangle$, is a sequence of nodes such that two consecutive nodes are adjacent. In addition, $P_{x_0x_t}$ is a cycle if $x_0 = x_t$. A path $\langle x_0, x_1, \ldots, x_t\rangle$ may contain other subpath, denoted as $\langle x_0, x_1, \ldots, x_i, P_{x_ix_j}, x_j, \ldots x_t\rangle$, where $P_{x_ix_j} = \langle x_i, x_{i+1}, \ldots x_{j-1}, x_j\rangle$. A path (or cycle) in G is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every vertex of G exactly once. G is called *Hamiltonian* if there is a Hamiltonian cycle in G, and *Hamiltonian connected* if there is a Hamiltonian path between every two distinct vertices of G. A Hamiltonian network can embed a longest ring with dilation 1, congestion 1, load 1, and expansion 1.

Consider $F \subset G = V(G) \cup E(G)$, i.e., F is the set of edge faults and vertex faults. We say G is k-Hamiltonian (respectively, k-Hamiltonian connected) if $G - F$ is Hamiltonian (respectively, Hamiltonian connected) for arbitrary F with $|F| \le k$. Note that G cannot be $(\delta(G) - 1)$-Hamiltonian when $\delta(G - F) = 1$ and $|F| = \delta(G) - 1$. However, if $\delta(G - F) > 1$ can be assured, then $G - F$ could be Hamiltonian even when $|F| > \delta(G) - 1$. Actually, regarding F as the set of edge faults, $\delta(G - F) > 1$ means each vertex in $G - F$ is incident with at least two fault-free edges.

The vertex set of the twisted n-cube TQ_n is the set of all binary strings of length n, where n is odd. Let $b = b_{n-1}b_{n-2}\ldots b_0$ denote one vertex in TQ_n. For $i \in \{0, 1, \ldots, n-1\}$, let the i-th *parity function* $P_i(b) = b_i \oplus b_{i-1} \oplus \cdots \oplus b_0$, where $\oplus$ denotes the exclusive-or operation. The TQ_n can be defined recursively as follows: TQ_1 is a complete graph with two vertices 0 and 1. Suppose that $n \ge 3$. We can decompose the vertices of TQ_n into four sets, $TQ^{0,0}_{n-2}$, $TQ^{0,1}_{n-2}$, $TQ^{1,0}_{n-2}$, and $TQ^{1,1}_{n-2}$, where $TQ^{i,j}_{n-2}$ consists of those vertices b with $b_{n-1} = i$ and $b_{n-2} = j$. For each $ij \in \{00, 01, 10, 11\}$, the induced subgraph of $TQ^{i,j}_{n-2}$ in TQ_n is isomorphic to TQ_{n-2}. Edges that connect these four subtwisted cubes can be described as follows: an $(n-1)$-edge joins vertices $b = b_{n-1}b_{n-2}\ldots b_0$ and $b^{(n-1)} = \overline{b_{n-1}}b_{n-2}\ldots b_0$. An $(n-2)$-edge joins vertices b and $b^{(n-2)}$, where $b^{(n-2)} = \overline{b_{n-1}}\,\overline{b_{n-2}}\ldots b_0$ when $P_{n-3}(b) = 0$, and $b^{(n-2)} = b_{n-1}\overline{b_{n-2}}\ldots b_0$ when $P_{n-3}(b) = 1$. Note that $(n-1)$-edges connect $TQ^{0,i}_{n-2}$ and $TQ^{1,i}_{n-2}$ and $(n-2)$-edges connect $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$, where $i = 0$ or 1. Fig. 1 depicts TQ_5, containing four sets, $TQ^{0,0}_3$, $TQ^{0,1}_3$, $TQ^{1,0}_3$, and $TQ^{1,1}_3$. Formally, TQ_n can be defined as follows.

Definition 1. The vertex set of TQ_n is $\{b_{n-1}b_{n-2}\ldots b_0 | b_i \in \{0, 1\}$ for all $0 \le i \le n-1\}$, where n is odd. Vertex $b = b_{n-1}b_{n-2}\ldots b_0$ is adjacent to vertex b^d, for all $0 \le d \le n-1$, where $b^d = b_{n-1}b_{n-2}\ldots \overline{b_d}\ldots b_0$ if (1) d is even or (2) d is odd and $P_{d-1}(b) = 1$, and $b^d = b_{n-1}b_{n-2}\ldots \overline{b_{d+1}}\,\overline{b_d}\ldots b_0$ if d is odd and $P_{d-1}(b) = 0$. The edge joining b and b^d is referred to as a d-edge.

Furthermore, we use b^{ij} to denote $(b^i)^j$. Note it is possible that $b^{ij} \ne b^{ji}$. The following lemma shown in [13] will be used often.

Lemma 1 (*[13]*). *For $n \ge 3$, TQ_n (respectively, $TQ^{0,i}_n \cup TQ^{1,i}_n$ for $i \in \{0, 1\}$) is $(n-2)$-Hamiltonian (respectively, $(n-1)$-Hamiltonian) and $(n-3)$-Hamiltonian connected (respectively, $(n-2)$-Hamiltonian connected).*

3. Basic idea and some properties

Our method is based on a recursive construction. We will partition TQ_n into $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ and $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ (Then $TQ^{0,i}_{n-2} \cup TQ^{1,i}_{n-2}$ can be partitioned into $TQ^{0,i}_{n-2}$ and $TQ^{1,i}_{n-2}$, for $i \in \{0, 1\}$). Let $F \subset E(TQ_n)$, $F_0 = F \cap E(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$, and $F_1 = F \cap E(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$, where $|F| = 2n-5$. A simple idea is to construct a Hamiltonian path for $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2} - F_0$ and a Hamiltonian path for $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2} - F_1$, and then to combine these two paths into a Hamiltonian cycle in $TQ_n - F$. In addition, without loss of generality, we can assume $|F_0| \ge |F_1|$. In general, we will first construct a Hamiltonian cycle C in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2} - F_0$, and select an edge, says (x, y) in C. Then, we construct a Hamiltonian path $P_{y^{(n-2)}x^{(n-2)}}$ in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2} - F_1$. As a result, we can construct a Hamiltonian cycle in $TQ_n - F$ by combining $C - \{(x, y)\}$, $P_{y^{(n-2)}x^{(n-2)}}$, $(x, x^{(n-2)})$, and $(y, y^{(n-2)})$ (see Fig. 2(a)). When $F = F_0$, it is possible that the Hamiltonian cycle in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2} - F_0$ cannot be constructed. However, we can select two edges in F_0, says (u, v) and (x, y), and construct a Hamiltonian

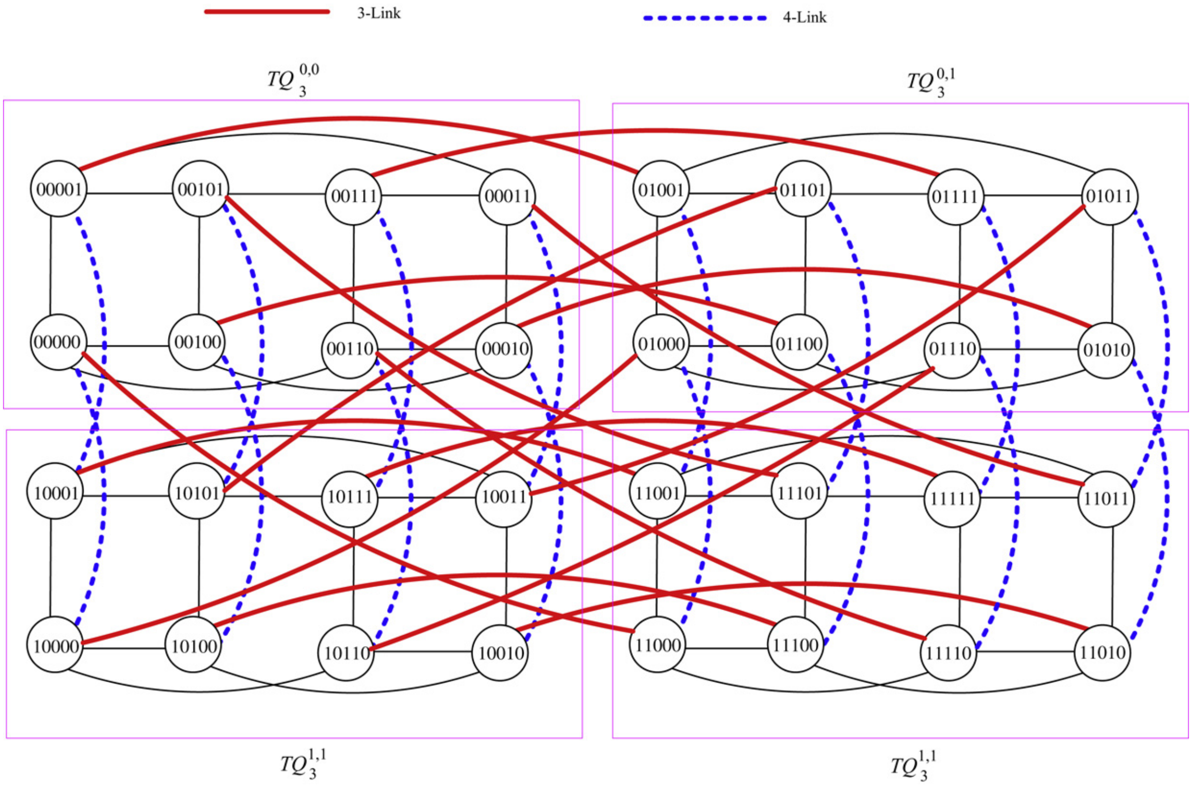

Fig. 1. TQ_5 (contains $TQ_3^{0,0}$, $TQ_3^{0,1}$, $TQ_3^{1,0}$, and $TQ_3^{1,1}$).

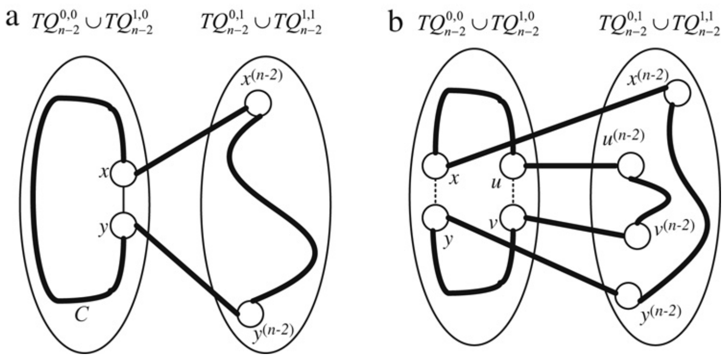

Fig. 2. Construction of a Hamiltonian cycle in $TQ_n - F$.

cycle C in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(u, v), (x, y)\})$. Moreover, we can use following lemma to construct two disjoint paths $P_{u^{(n-2)},v^{(n-2)}}$ and $P_{x^{(n-2)},y^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. Then we can construct a Hamiltonian cycle in $TQ_n - F$ by combining $C - \{(u, v), (x, y)\}$, $P_{u^{(n-2)},v^{(n-2)}}$, $P_{x^{(n-2)},y^{(n-2)}}$, $(x, x^{(n-2)})$, $(y, y^{(n-2)})$, $(u, u^{(n-2)})$, and $(v, v^{(n-2)})$ into a Hamiltonian cycle in TQ_n (see Fig. 2(b)).

Lemma 2. *Let x, y, u, and v be four distinct vertices in TQ_n (respectively, $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ for $i \in \{0, 1\}$), where $n \geq 5$ is an odd integer. There exist P_{uv} and P_{xy} such that $V(P_{uv}) \cap V(P_{xy}) = \emptyset$ and $V(P_{uv}) \cup V(P_{xy}) = V(TQ_n)$ (respectively, $= V(TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i})$).*

Proof. We proceed by induction on n. The lemma holds for $TQ_3^{0,i} \cup TQ_3^{1,i}$ with $i \in \{0, 1\}$ (i.e., $n = 5$), which can be verified by a computer exhausted search program [11]. Two steps can complete the proof. First, for all odd integers $n \geq 5$, we show that if the lemma holds for $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ then the lemma holds for TQ_n, , where $i \in \{0, 1\}$. Secondly, for all odd integers $n \geq 5$, we show that if the lemma holds for TQ_n then the lemma holds for $TQ_n^{0,i} \cup TQ_n^{1,i}$, where $i \in \{0, 1\}$. However, since the proof of the second step is easier than and similar to that of first step, we only show the first step. That is, for all odd integers $n \geq 5$, we assume that there exist $P_{u'v'}$ and $P_{x'y'}$ such that $V(P_{u'v'}) \cap V(P_{x'y'}) = \emptyset$ and $V(P_{u'v'}) \cup V(P_{x'y'}) = V(TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i})$, where x', y', u', and v' are four arbitrary distinct vertices in $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ and $i \in \{0, 1\}$. We want to show that there exist

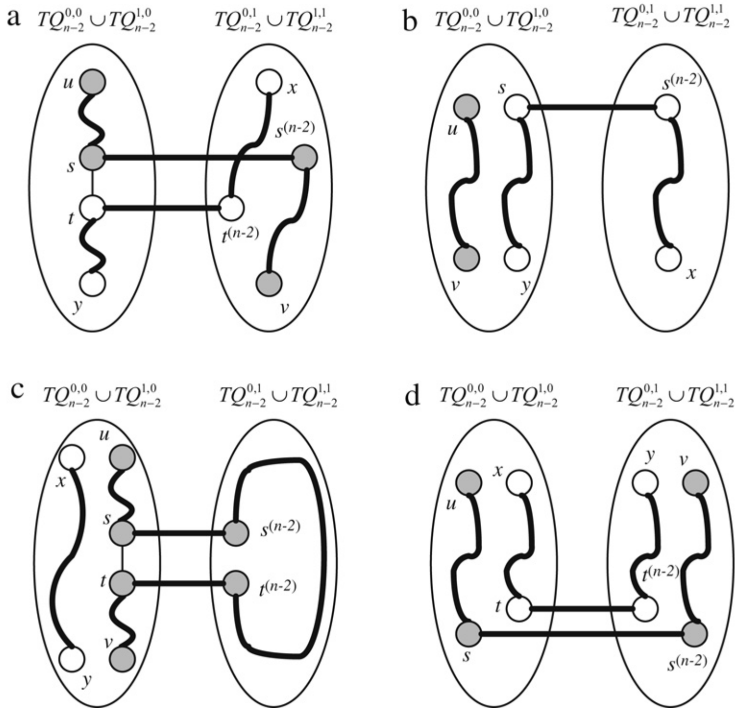

Fig. 3. Construction of P_{uv} and P_{xy} (Lemma 2).

P_{uv} and P_{xy} such that $V(P_{uv}) \cap V(P_{xy}) = \emptyset$ and $V(P_{uv}) \cup V(P_{xy}) = V(TQ_n)$, where x, y, u, and v are four arbitrary distinct vertices in TQ_n. Four cases are considered:

Case 1. $u \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$ and $x \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$. Four cases are further considered:

Case 1.1. $v \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$ and $y \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$. By Lemma 1, there is a Hamiltonian path P_{uv} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ and another Hamiltonian path P_{xy} in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$, which are the desired P_{uv} and P_{xy}, respectively.

Case 1.2. $v \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$ and $y \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$. By Lemma 1, there is a Hamiltonian pathP_{uy} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$. An edge (s, t) in P_{uy} with $s^{(n-2)}, t^{(n-2)} \notin \{x, v\}$ can be found. Let P_{us} and P_{ty} denote two subpaths of P_{uy}. In addition, by the induction hypothesis; there exist $P_{s^{(n-2)}v}$ and $P_{xt^{(n-2)}}$ in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ such that $V(P_{s^{(n-2)}v}) \cap V(P_{xt^{(n-2)}}) = \emptyset$ and $V(P_{s^{(n-2)}v}) \cup V(P_{xt^{(n-2)}}) = V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$. The desired P_{uv} and the desired P_{xy} can be constructed as $\langle u, P_{us}, s, s^{(n-2)}, P_{s^{(n-2)}v}, v\rangle$ and $\langle x, P_{xt^{(n-2)}}, t^{(n-2)}, t, P_{ty}, y\rangle$, respectively (see Fig. 3(a)).

Case 1.3. $v, y \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$. We select a vertex s in $V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$ - $\{u, v, y\}$ such that $s^{(n-2)} \neq x$. By the induction hypothesis, there are P_{uv} and P_{sy} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ such that $V(P_{uv}) \cap V(P_{sy}) = \emptyset$, $V(P_{uv}) \cup V(P_{ys}) = V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$. In addition, by Lemma 1, there exists a Hamiltonian path $P_{xs^{(n-2)}}$ in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$. The P_{uv} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ is the desired P_{uv}. The desired P_{xy} can be constructed as $\langle x, P_{xs^{(n-2)}}, s^{(n-2)}, s, P_{sy}, y\rangle$ (see Fig. 3(b)).

Case 1.4. $v, y \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$. The construction of the desired P_{uv} and P_{xy} is similar to that of Case 1.3.

Case 2. $x \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$ and $u \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$. The construction is similar to that of Case 1.

Case 3. $u, x \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$. If $v \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$ and $y \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$ or $v \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$ and $y \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$, the construction is similar to that of Case 1.3.

If $v, y \in V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$, then by the induction hypothesis, there exist P'_{uv} and P_{xy} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ such that $V(P'_{uv}) \cap V(P_{xy}) = \emptyset$ and $V(P'_{uv}) \cup V(P_{xy}) = V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$. The P_{xy} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ is the desired P_{xy}. We arbitrarily select an edge (s, t) in P'_{uv} and let P_{us} and P_{tv} be two subpaths of P'_{uv}. By Lemma 1, there exist a Hamiltonian path $P_{s^{(n-2)}t^{(n-2)}}$ in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$. The desired P_{uv} can be constructed as $\langle u, P_{us}, s, s^{(n-2)}, P_{s^{(n-2)}t^{(n-2)}}, t^{(n-2)}, t, P_{tv}, v\rangle$ (see Fig. 3(c)).

If $v, y \in V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$, then select two vertices s, t from $V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}) - \{x, u\}$ such that $s^{(n-2)}, t^{(n-2)} \notin \{v, y\}$. By the induction hypothesis, there exist P_{us} and P_{xt} in $TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2}$ such that $V(P_{us}) \cap V(P_{xt}) = \emptyset$ and $V(P_{us}) \cup V(P_{xt}) = V(TQ^{0,0}_{n-2} \cup TQ^{1,0}_{n-2})$. Also, by the induction hypothesis, there exist $P_{t^{(n-2)}y}$ and $P_{s^{(n-2)}v}$ in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2}$ such that $V(P_{t^{(n-2)}y}) \cap V(P_{s^{(n-2)}v}) = \emptyset$ and $V(P_{t^{(n-2)}y}) \cup V(P_{s^{(n-2)}v}) = V(TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2})$. The desired P_{uv} and the desired P_{xy} can be constructed as $\langle u, P_{us}, s, s^{(n-2)}, P_{s^{(n-2)}v}, v\rangle$ and $\langle x, P_{xt}, t, t^{(n-2)}, P_{t^{(n-2)}y}, y\rangle$, respectively (see Fig. 3(d)).

Case 4. $u, x \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. The construction of the desired P_{uv} and P_{xy} is similar to that of Case 3. □

Remember that when we use the method of Fig. 2(a), we need to construct a Hamiltonian path $P_{y^{(n-2)}x^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. We can certainly construct $P_{y^{(n-2)}x^{(n-2)}}$ when $|F_1| \le n-4$ because $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ is $(n-4)$-Hamiltonian connected (by Lemma 1). However it is possible that $|F_1| = n-3$ and Lemma 1 can not be used. It is impossible that $|F_1| \ge n-2$ since $2n+5 \ge |F| \ge |F_0| + |F_1|$ and $|F_0| \ge |F_1|$. Lemmas 3–5 are needed to construct $P_{y^{(n-2)}x^{(n-2)}}$ when $|F_1| = n-3$.

Lemma 3. *Let C be a Hamiltonian cycle in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ (respectively, $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$). There exist two edges (x, y) and (u, v) in C such that $(x^{(n-2)}, y^{(n-2)})$ and $(u^{(n-2)}, v^{(n-2)})$ are also two edges in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ (respectively, $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$), where vertices x, y, u, and v are distinct.*

Proof. Suppose that C is a Hamiltonian cycle in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$, there are at least two edges (x, y) and (u, v) in C such that $x, u \in TQ_{n-2}^{0,0}$ and $y, v \in TQ_{n-2}^{1,0}$. Let $x = 00a_{n-3}a_{n-4}\ldots a_0$, $y = 10a_{n-3}a_{n-4}\ldots a_0$, $u = 00b_{n-3}b_{n-4}\ldots b_0$ and $v = 10b_{n-3}b_{n-4}\ldots b_0$. Clearly, we have $P_{n-3}(x) = P_{n-3}(y)$ and $P_{n-3}(u) = P_{n-3}(v)$. If $P_{n-3}(x) = 0$, then according to the Definition 1, we have $x^{(n-2)} = 11a_{n-3}a_{n-4}\ldots a_0$ and $y^{(n-2)} = 01a_{n-3}a_{n-4}\ldots a_0$. Additionally, if $P_{n-3}(x) = 1$, then $x^{(n-2)} = 01a_{n-3}a_{n-4}\ldots a_0$ and $y^{(n-2)} = 11a_{n-3}a_{n-4}\ldots a_0$. Therefore, $(x^{(n-2)}, y^{(n-2)})$ is an edge in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. We can show that $(u^{(n-2)}, v^{(n-2)})$ is an edge in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ by similar discussion. In addition, the discussion is similar when C is in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. □

Lemma 4. *Let (x, y) be an arbitrary edge in TQ_n (respectively, $TQ_n^{0,i} \cup TQ_n^{1,i}$ for $i \in \{0, 1\}$) and let $F \subset E(TQ_n)$ with $|F| \le n-2$ (respectively, $F' \subset E(TQ_n^{0,i} \cup TQ_n^{1,i})$ with $|F'| \le n-1$), where $n \ge 3$ is odd. Then, there exists a Hamiltonian path P_{xy} in $TQ_n - F$ (respectively, $TQ_n^{0,i} \cup TQ_n^{1,i} - F'$).*

Proof. We proceed by induction on n. It is not difficult to check that the lemma holds for TQ_3. Two steps can complete the proof. First, for all odd integers $n \ge 3$, we show that if the lemma holds for TQ_n, then the lemma holds for $TQ_n^{0,i} \cup TQ_n^{1,i}$, where $i \in \{0, 1\}$. Secondly, for all odd integers $n \ge 5$, we show that if the lemma holds for $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$, then the lemma holds for TQ_n, where $i \in \{0, 1\}$. In addition, since the proof of the first step is easier than and similar to that of the second step, we only show the second step. That is, for all odd integers $n \ge 5$, we assume that there is a Hamiltonian path $P_{x'y'}$ in $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i} - F'$ if $F' \subset E(TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i})$ and $|F'| \le n-3$ where $(x', y') \in E(TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i})$ and $i \in \{0, 1\}$. We will show that there is a Hamiltonian path P_{xy} in $TQ_n - F$ if $F \subset E(TQ_n)$ and $|F| \le n-2$, where $(x, y) \in E(TQ_n)$.

Let $F_0 = F \cap E(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, $F_1 = F \cap E(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$, and $F_c = F \cap \{(u, u^{(n-2)}) | u \in TQ_n\}$. Additionally, let $f_0 = |F_0|, f_1 = |F_1|, f_c = |F_c|$. Without loss of generality, we assume $f_0 \ge f_1$. Two cases are considered:

Case 1. $f_0 \le n-3$. Note that $f_1 \le n-4$ since $f_0 \ge f_1$, $f_0 + f_1 \le n-2$ and $n-4 \ge \lfloor (n-2)/2 \rfloor$ (remember that $n \ge 5$). Three cases are further considered:

Case 1.1. $x \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$ and $y \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$; that is, $y = x^{(n-2)}$. Since $f_c \le n-2$ and there are $n-1$ vertices in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ adjacent to x, we can find a vertex $s \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$ such that s is a neighbor of x and $(s, s^{(n-2)}) \notin F_c$. By the induction hypothesis (since $f_0 \le n-3$), there is a Hamiltonian path P_{xs} in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Additionally, by Lemma 1 (since $f_1 \le n-4$), there exists a Hamiltonian path $P_{s^{(n-2)}y}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired P_{xy} can be constructed as $\langle x, P_{xs}, s, s^{(n-2)}, P_{s^{(n-2)}y}, y\rangle$ (see Fig. 4(a)).

Case 1.2. $x, y \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$. By the induction hypothesis (since $f_0 \le n-3$), there is a Hamiltonian path P'_{xy} in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Moreover, there exists an edge (u, v) in P'_{xy} such that $(u, u^{(n-2)}), (v, v^{(n-2)}) \notin F_c$ since there are $2^{n-1} - 1$ edges in P'_{xy} and $2^{n-1} - 1 > 2(n-2) \ge 2f_c$. (This is because an edge in F_c eliminates two choices in P'_{xy}.) Additionally, by Lemma 1 (since $f_1 \le n-4$), there is a Hamiltonian path $P_{u^{(n-2)}v^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. Let P_{xu} and P_{vy} be two subpaths of P'_{xy}. The desired P_{xy} can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vy}, y\rangle$ (see Fig. 4(b)).

Case 1.3. $x, y \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. When $f_0 \le n-4$, the construction is similar to that of Case 1.2. When $f_0 = n-3$, then $f_1 + f_c = 1$. If $f_c = 0$, then by the induction hypothesis (since $f_1 = 1$), there is a Hamiltonian path P'_{xy} in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. Clearly, $P'_{xy} \cup \{(x, y)\}$ is a Hamiltonian cycle in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. By Lemma 3, there exists an edge (u, v) in P'_{xy} such that $(u^{(n-2)}, v^{(n-2)})$ is also an edge in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$, where vertices x, y, u, and v are distinct. By the induction hypothesis (since $f_0 = n-3$), there is a Hamiltonian path $P_{u^{(n-2)}v^{(n-2)}}$ in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Let P_{xu} and P_{vy} be two subpaths of P'_{xy}. The desired P_{xy} can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vy}, y\rangle$.

If $f_1 = 0$, then we have $f_c = 1$. Let (u, v) be an edge in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$ such that $u^{(n-2)}, v^{(n-2)} \notin \{x, y\}$ and $(x, x^{(n-2)}), (x, x^{(n-2)}) \notin F_c$. By Lemma 2, there exist two paths $P_{xu^{(n-2)}}$ and $P_{v^{(n-2)}y}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ such that $V(P_{xu^{(n-2)}}) \cap V(P_{v^{(n-2)}y}) = \emptyset$ and $V(P_{xu^{(n-2)}}) \cup V(P_{v^{(n-2)}y}) = V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. In addition, by the induction hypothesis (since $f_0 = n-3$), there is a Hamiltonian path P_{uv} in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. The desired P_{xy} can be constructed as $\langle x, P_{xu^{(n-2)}}, u^{(n-2)}, u, P_{uv}, v, v^{(n-2)}, P_{v^{(n-2)}y}, y\rangle$ (see Fig. 4(c)).

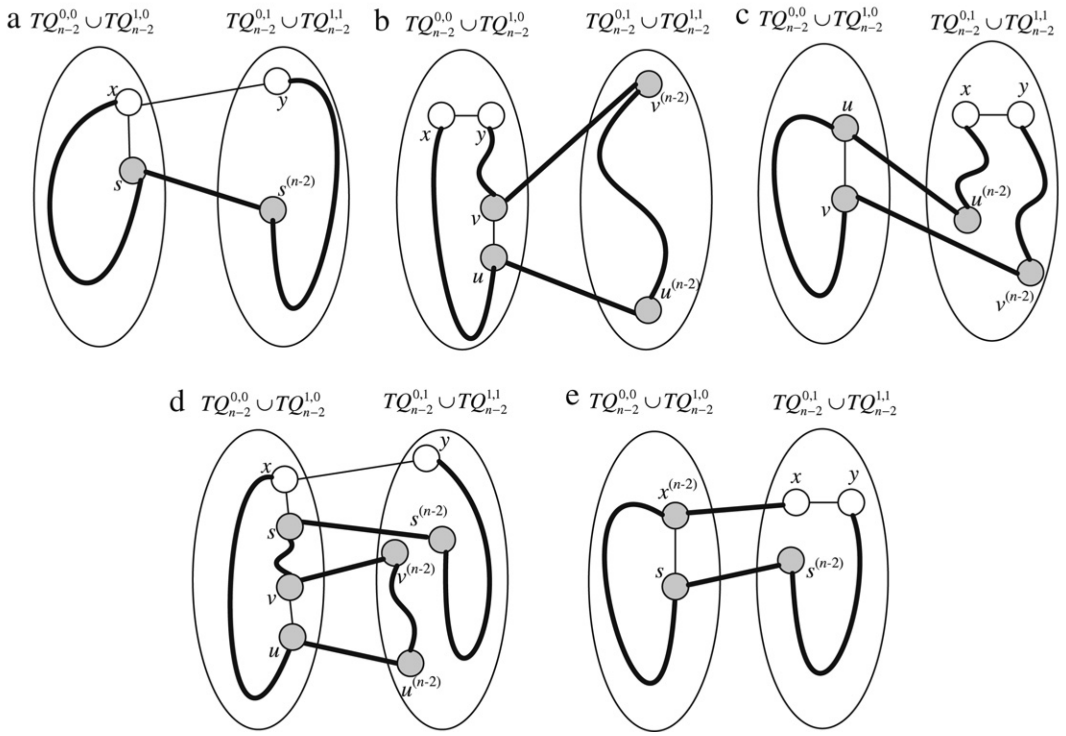

Fig. 4. Construction of P_{xy} in $TQ_n - F$ (Lemma 4).

Case 2. $f_0 = n-2$. We have $f_1 = f_c = 0$. Three cases are further considered:
Case 2.1. $x \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$ and $y \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. Suppose that $(u, v) \in F_0$. If $x \in \{u, v\}$ (suppose that $(x, s) = (u, v)$), then $|F_0 - \{(x, s)\}| = n-3$ and the construction is similar to that of Case 1.1. In the rest of the proof, we assume $x \notin \{u, v\}$. Let $s \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}) - \{u, v\}$ and s is a neighbor of x. Since $|F_0 - \{(u, v)\}| = n-3$, by the induction hypothesis, there exists a Hamiltonian path P_{xs} in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(u, v)\})$. If P_{xs} contains (u, v), then by Lemma 2, there exist two paths $P_{s^{(n-2)}y}$ and $P_{u^{(n-2)}v^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ such that $V(P_{s^{(n-2)}y}) \cap V(P_{u^{(n-2)}v^{(n-2)}}) = \emptyset$ and $V(P_{s^{(n-2)}y}) \cup V(P_{u^{(n-2)}v^{(n-2)}}) = V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. The desired P_{xy} can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vs}, s^{(n-2)}, P_{s^{(n-2)}y}, y\rangle$ (see Fig. 4(d)). If P_{xs} does not contain (u, v), the construction is similar to that of Case 1.1.
Case 2.2. $x, y \in V(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$. Let $(u, v) \in F_0$. Since $|F_0 - \{(u, v)\}| = n-3$, by the induction hypothesis, there exists a Hamiltonian path P'_{xy} of $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(u, v)\})$. If P'_{xy} contains (u, v), then let P_{xu} and P_{vv} be two subpaths of P'_{xy}. By Lemma 1, there is a Hamiltonian path $P_{u^{(n-2)}v^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. The desired P_{xy} can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vy}, y\rangle$ (see Fig. 4(b)). If P'_{xy} does not contain (u, v), then select arbitrary edge (r, z) in P'_{xy}. Let P_{xr} and P_{zy} be two subpaths of P'_{xy}. By Lemma 1, there is a Hamiltonian path $P_{r^{(n-2)}z^{(n-2)}}$ in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$. The desired P_{xy} can be constructed as $\langle x, P_{xr}, r, r^{(n-2)}, P_{r^{(n-2)}z^{(n-2)}}, z^{(n-2)}, z, P_{zy}, y\rangle$.
Case 2.3. $x, y \in V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. Let $(u, v) \in F_0$ and $\{u, v\} \neq \{x^{(n-2)I}, y^{(n-2)}\}$. If $x^{(n-2)} \in \{u, v\}$, then $y^{(n-2)} \notin \{u, v\}$. Let $(x^{(n-2)}, s) = (u, v)$. Then $|F_0 - \{(x^{(n-2)}, s)\}| = n-3$ is obtained. By the induction hypothesis, there is a Hamiltonian path $P_{x^{(n-2)}s}$ in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x^{(n-2)}, s)\})$. In addition, by Lemma 1, there is a Hamiltonian path $P_{s^{(n-2)}y}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - \{x\}$ (since $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ is $(n-4)$-Hamiltonian connected and $n-4 \geq 1$). The desired P_{xy} can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}s}, s, s^{(n-2)}, P_{s^{(n-2)}y}, y\rangle$ (see Fig. 4(e)). If $y^{(n-2)} \in \{u, v\}$, then $x^{(n-2)} \notin \{u, v\}$ and the construction is similar.

If $x^{(n-2)}, y^{(n-2)} \notin \{u, v\}$, then by Lemma 2, there exist two paths $P_{xu^{(n-2)}}$ and $P_{v^{(n-2)}y}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$ such that $V(P_{xu^{(n-2)}}) \cap V(P_{v^{(n-2)}y}) = \emptyset$ and $V(P_{xu^{(n-2)}}) \cup V(P_{v^{(n-2)}y}) = V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. In addition, since $|F_0 - \{(u, v)\}| = n-3$, by the induction hypothesis, there is a Hamiltonian path P_{uv} in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(u, v)\})$. The desired P_{xy} can be constructed as $\langle x, P_{xu^{(n-2)}}, u^{(n-2)}, u, P_{uv}, v, v^{(n-2)}, P_{v^{(n-2)}y}, y\rangle$ (see Fig. 4(c)). $\square$

Lemma 5. *Let $F \subset E(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$ with $|F| \leq n-3$ and let u be an arbitrary vertex in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0}$, where $n \geq 7$ is odd. There exists an integer $d \in \{0, 1, 2, \ldots, n-3\}$ such that there is a Hamiltonian path $P_{u^{(n-2)}u^{d(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F$.*

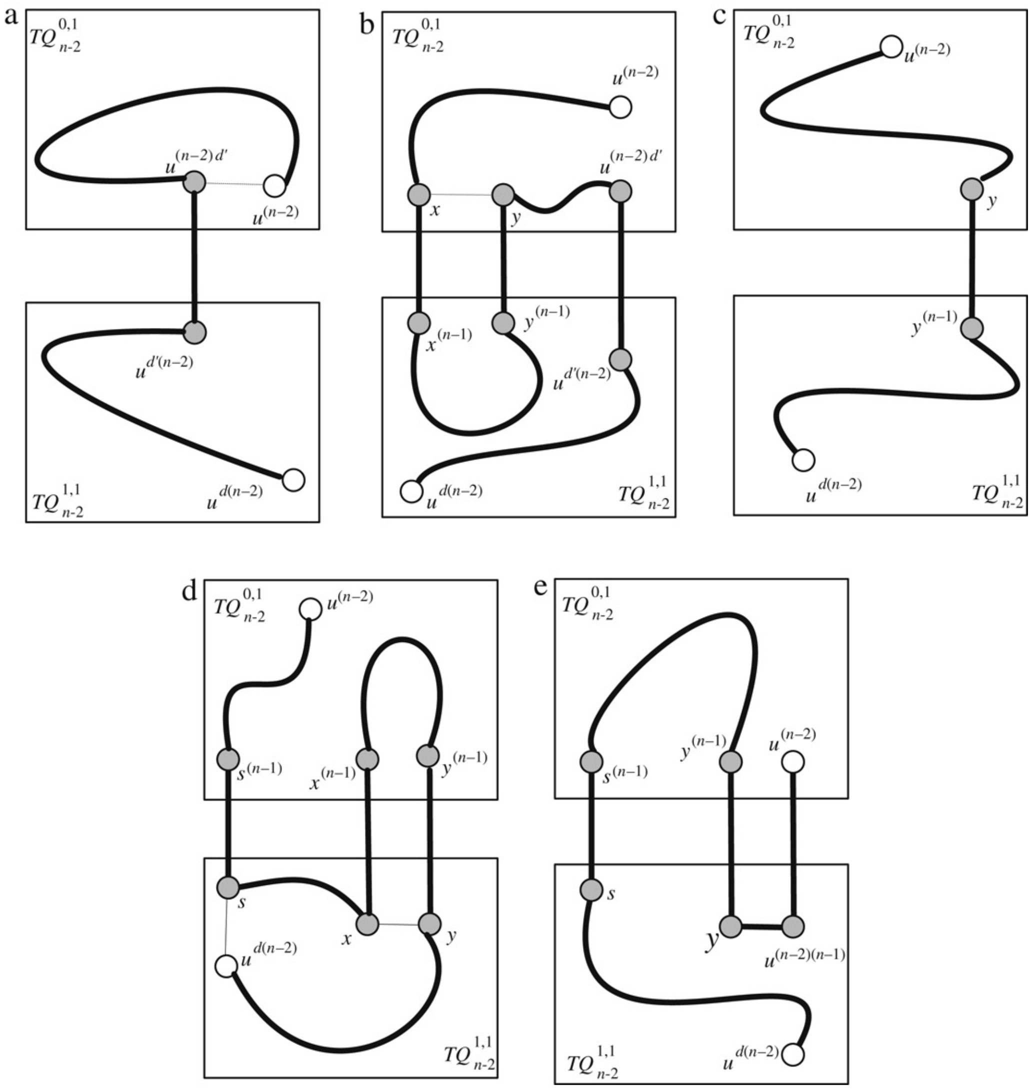

Fig. 5. Construction of $P_{u^{(n-2)}u^{d(n-2)}}$ in $TQ^{0,1}_{n-2} \cup TQ^{1,1}_{n-2} - F$ (Lemma 5).

Proof. Clearly, a d-edge joins $u^{(n-2)}$ and $u^{(n-2)d}$, where $d \in \{0, 1, 2, \ldots, n-3\}$. We claim that $u^{(n-2)d} = u^{d(n-2)}$ or an $(n-1)$-edge joins $u^{(n-2)d}$ and $u^{d(n-2)}$. (The claim proof is placed after the main proof). If there exists an integer $d \in \{0, 1, 2, \ldots, n-3\}$ such that $u^{(n-2)d} = u^{d(n-2)}$, then $u^{(n-2)}$ and $u^{d(n-2)}$ are adjacent. By Lemma 4, the lemma holds. In the rest of the proof, we assume that an $(n-1)$-edge joins $u^{(n-2)i}$ and $u^{i(n-2)}$ for all $i \in \{0, 1, 2, \ldots, n-3\}$. Let $F_0 = F \cap E(TQ^{0,1}_{n-2})$, $F_1 = F \cap E(TQ^{1,1}_{n-2})$, and $F_c = F \cap \{(v, v^{(n-1)}) | v \in V(TQ^{0,1}_{n-2})\}$. Additionally, let $f_0 = |F_0|, f_1 = |F_1|, f_c = |F_c|$. Without loss of generality, assume that $u^{(n-2)} \in TQ^{0,1}_{n-2}$. Therefore, we have $u^{(n-2)i} \in TQ^{0,1}_{n-2}$ and $u^{i(n-2)} \in TQ^{1,1}_{n-2}$, for all $i \in \{0, 1, 2, \ldots, n-3\}$ (because an $(n-1)$-edge joins $u^{(n-2)i}$ and $u^{i(n-2)}$). Five cases are considered:

Case 1. $f_0 = n-3$. Thus, $f_1 + f_c = 0$. First, suppose that $(u^{(n-2)}, u^{(n-2)d'}) \in F_0$ for some $d' \in \{0, 1, 2, \ldots, n-3\}$. Since $|F_0 - \{(u^{(n-2)}, u^{(n-2)d'})\}| = n-4$, by Lemma 4, there exists a Hamiltonian path $P_{u^{(n-2)}u^{(n-2)d'}}$ in $TQ^{0,1}_{n-2} - (F_0 - \{(u^{(n-2)}, u^{(n-2)d'})\})$. Select an integer d from $\{0, 1, 2, \ldots, n-3\} - \{d'\}$. By Lemma 1, there is a Hamiltonian path $P_{u^{d'(n-2)}u^{d(n-2)}}$ in $TQ^{1,1}_{n-2}$. Remember that an $(n-1)$-edge joins $u^{(n-2)i}$ and $u^{i(n-2)}$ for all $i \in \{0, 1, 2, \ldots, n-3\}$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}u^{(n-2)d'}}, u^{(n-2)d'}, u^{d'(n-2)}, P_{u^{d'(n-2)}u^{d(n-2)}}, u^{d(n-2)} \rangle$ (see Fig. 5(a)).

Then, suppose that $(u^{(n-2)}, u^{(n-2)i}) \notin F_0$, for all $i \in \{0, 1, 2, \ldots, n-3\}$. Let $(x, y) \in F_0$. Since $n \geq 7$, we can select an integer d' from $\{0, 1, 2, \ldots, n-3\}$ such that $u^{(n-2)d'} \notin \{x, y\}$. Since $|F_0 - \{(x, y)\}| = n-4$, by Lemma 4, there exists a Hamiltonian path $P_{u^{(n-2)}u^{(n-2)d'}}$ in $TQ^{0,1}_{n-2} - (F_0 - \{(x, y)\})$. Also, since $n \geq 7$, we can select an integer d from $\{0, 1, 2, \ldots, n-3\} - \{d'\}$ such that $u^{d(n-2)} \notin \{x^{(n-1)}, y^{(n-1)}\}$. If $P_{u^{(n-2)}u^{(n-2)d'}}$ does not contain (x, y), then by Lemma 1, there is a Hamiltonian path $P_{u^{d'(n-2)}u^{d(n-2)}}$ in $TQ^{1,1}_{n-2}$. The desired Hamiltonian path can be constructed as

$\langle u^{(n-2)}, P_{u^{(n-2)}u^{(n-2)d'}}, u^{(n-2)d'}, u^{d'(n-2)}, P_{u^{d'(n-2)}u^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(a)). If $P_{u^{(n-2)}u^{(n-2)d'}}$ contains (x, y), then let $P_{u^{(n-2)}x}$ and $P_{yu^{(n-2)d'}}$ be two subpaths of $P_{u^{(n-2)}u^{(n-2)d'}}$. By Lemma 2, there exist $P_{u^{d'(n-2)}u^{d(n-2)}}$ and $P_{x^{(n-1)}y^{(n-1)}}$ in $TQ^{1,1}_{n-2}$ such that $V(P_{u^{d'(n-2)}u^{d(n-2)}}) \cap V(P_{x^{(n-1)}y^{(n-1)}}) = \emptyset$ and $V(P_{u^{d'(n-2)}u^{d(n-2)}}) \cup V(P_{x^{(n-1)}y^{(n-1)}}) = V(TQ^{1,1}_{n-2})$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}x}, x, x^{(n-1)}, P_{x^{(n-1)}y^{(n-1)}}, y^{(n-1)}, y, P_{yu^{(n-2)d'}}, u^{(n-2)d'}, u^{d'(n-2)}, P_{u^{d'(n-2)}u^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(b)).

Case 2. $f_0 = n-4$. Thus, $f_1 + f_c = 1$. Since $f_c \le 1$, we can select an integer d' from $\{0, 1, 2, \ldots, n-3\}$ such that $(u^{d'(n-2)}, u^{(n-2)d'}) \notin F_c$. Since $f_0 = n-4$, by Lemma 4, there exists a Hamiltonian path $P_{u^{(n-2)}u^{(n-2)d'}}$ in $TQ^{0,1}_{n-2} - F_0$. Then, we select an integer d from $\{0, 1, 2, \ldots, n-3\} - \{d'\}$. Since $f_1 \le 1$, by Lemma 1, there is a Hamiltonian path $P_{u^{d'(n-2)}u^{d(n-2)}}$ for $TQ^{1,1}_{n-2}$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}u^{(n-2)d'}}, u^{(n-2)d'}, u^{d'(n-2)}, P_{u^{d'(n-2)}u^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(a)).

Case 3. $f_0 \le n-5$ and $f_1 \le n-5$. Select a vertex $y \in TQ^{0,1}_{n-2} - \{u^{(n-2)}\}$ such that $(y, y^{(n-1)}) \notin F_c$. Then, select an integer d from $\{0, 1, 2, \ldots, n-3\}$ such that $u^{d(n-2)} \neq y^{(n-1)}$. By Lemma 1, there is a Hamiltonian path $P_{u^{(n-2)}y}$ (respectively, $P_{y^{(n-1)}u^{d(n-2)}}$) in $TQ^{0,1}_{n-2} - F_0$ (respectively, $TQ^{1,1}_{n-2} - F_1$). The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}y}, y, y^{(n-1)}, P_{y^{(n-1)}u^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(c)).

Case 4. $f_1 = n-4$. Thus, $f_0 + f_c = 1$. Since $f_c \le 1$, we can select an integer d' from $\{0, 1, 2, \ldots, n-3\}$ such that $(u^{d'(n-2)}, u^{(n-2)d'}) \notin F_c$. Since $f_0 \le 1$, by Lemma 4, there exists a Hamiltonian path $P_{u^{(n-2)}u^{(n-2)d'}}$ in $TQ^{0,1}_{n-2} - F_0$. Then we select an integer d from $\{0, 1, 2, \ldots, n-3\} - \{d'\}$. Since $f_1 = n-4$, by Lemma 1, there is a Hamiltonian path $P_{u^{d'(n-2)}u^{d(n-2)}}$ for $TQ^{1,1}_{n-2}$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}u^{(n-2)d'}}, u^{(n-2)d'}, u^{d'(n-2)}, P_{u^{d'(n-2)}u^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(a)).

Case 5. $f_1 = n-3$. Thus, $f_0 + f_c = 0$. First, suppose that there exists an edge $(x, y) \in F_1$ such that $u^{(n-2)(n-1)} \notin \{x, y\}$. We can choose an edge $(u^{d(n-2)}, s)$ in $TQ^{1,1}_{n-2}$ such that $u^{d(n-2)}, s \notin \{x, y, u^{(n-2)(n-1)}\}$, where $d \in \{0, 1, 2, \ldots, n-3\}$. (We can first select $u^{d(n-2)}$, which has at least $(n-2) - 3 \ge 2$ choices, and then, we have $(n-2) - 3 \ge 2$ ways to select s, which is the neighbor of $u^{d(n-2)}$.) Since $|F_1 - \{(x, y)\}| = n-4$, by Lemma 4, there exists a Hamiltonian path $P_{su^{d(n-2)}}$ in $TQ^{1,1}_{n-2} - (F_1 - \{(x, y)\})$. If $P_{su^{d(n-2)}}$ does not contain (x, y), then by Lemma 1, there is a Hamiltonian path $P_{u^{(n-2)}s^{(n-1)}}$ in $TQ^{0,1}_{n-2}$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}s^{(n-1)}}, s^{(n-1)}, s, P_{su^{d(n-2)}}, u^{d(n-2)}\rangle$. If $P_{su^{d(n-2)}}$ contains (x, y), then let P_{sx} and $P_{yu^{d(n-2)}}$ be two subpaths of $P_{su^{d(n-2)}}$. In addition, by Lemma 2, there exist $P_{u^{(n-2)}s^{(n-1)}}$ and $P_{x^{(n-1)}y^{(n-1)}}$ in $TQ^{0,1}_{n-2}$ such that $V(P_{u^{(n-2)}s^{(n-1)}}) \cap V(P_{x^{(n-1)}y^{(n-1)}}) = \emptyset$ and $V(P_{u^{(n-2)}s^{(n-1)}}) \cup V(P_{x^{(n-1)}y^{(n-1)}}) = V(TQ^{0,1}_{n-2})$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, P_{u^{(n-2)}s^{(n-1)}}, s^{(n-1)}, s, P_{sx}, x, x^{(n-1)}, P_{x^{(n-1)}y^{(n-1)}}, y^{(n-1)}, y, P_{yu^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(d)).

Then, suppose that there is no edge $(x', y') \in F_1$ such that $u^{(n-2)(n-1)} \notin \{x', y'\}$. That is, $F_1 \subseteq \{(u^{(n-2)(n-1)}, u^{(n-2)(n-1)i}) | i \in \{0, 1, 2, \ldots, n-3\}\}$. Since $f_1 = n-3$, and $|\{(u^{(n-2)(n-1)}, u^{(n-2)(n-1)i}) | i \in \{0, 1, 2, \ldots, n-3\}\}| = n-2$, we have $(u^{(n-2)(n-1)}, u^{(n-2)(n-1)d'}) \notin F_1$ for some $d' \in \{0, 1, 2, \ldots, n-3\}$. Let $y = u^{(n-2)(n-1)d'}$. In addition, let $u^{d(n-2)} \in V(TQ^{1,1}_{n-2}) - \{y, u^{(n-2)(n-1)}\}$ for some $d \in \{0, 1, 2, \ldots, n-3\}$ and let $s \in V(TQ^{1,1}_{n-2}) - \{y, u^{(n-2)(n-1)}, u^{d(n-2)}\}$. By Lemma 1, there are a Hamiltonian path $P_{y^{(n-1)}s^{(n-1)}}$ in $TQ^{0,1}_{n-2} - \{u^{(n-2)}\}$ and a Hamiltonian path $P_{su^{d(n-2)}}$ in $TQ^{1,1}_{n-2} - \{y, u^{(n-2)(n-1)}\}$ (since TQ_{n-2} is $(n-5)$-Hamiltonian connected and $n-5 \ge 2$). Moreover, since $F_1 \subseteq \{(u^{(n-2)(n-1)}, u^{(n-2)(n-1)i}) | i \in \{0, 1, 2, \ldots, n-3\}\}$, we have $P_{su^{d(n-2)}} \cap F_1 = \emptyset$. The desired Hamiltonian path can be constructed as $\langle u^{(n-2)}, u^{(n-2)(n-1)}, y, y^{(n-1)}, P_{y^{(n-1)}s^{(n-1)}}, s^{(n-1)}, s, P_{su^{d(n-2)}}, u^{d(n-2)}\rangle$ (see Fig. 5(e)).

Claim Proof. Suppose that $u = u_{n-1}u_{n-2}\ldots u_0$. When $P_{n-3}(u) = 0$, according to Definition 1, we have $u^{(n-2)} = \overline{u_{n-1}}\,\overline{u_{n-2}}u_{n-3}\ldots u_0$. If d is odd and $P_{d-1}(u) = 0$, then $u^d = u_{n-1}u_{n-2}\ldots\overline{u_{d+1}}\,\overline{u_d}\ldots u_0$. Hence, $P_{n-3}(u^d) = P_{n-3}(u) = 0$. As a result, $u^{d(n-2)} = \overline{u_{n-1}}\,\overline{u_{n-2}}u_{n-3}\ldots\overline{u_{d+1}}\,\overline{u_d}\ldots u_0$. Additionally, since $P_{d-1}(u^{(n-2)}) = P_{d-1}(u) = 0$, we have $u^{(n-2)d} = \overline{u_{n-1}}\,\overline{u_{n-2}}u_{n-3}\ldots\overline{u_{d+1}}\,\overline{u_d}\ldots u_0 = u^{d(n-2)}$. If (1) d is even or (2) d is odd and $P_{d-1}(u) = 1$, then $u^d = u_{n-1}u_{n-2}\ldots\overline{u_d}\ldots u_0$. Hence, $P_{n-3}(u^d) = 1$. As a result, $u^{d(n-2)} = u_{n-1}\overline{u_{n-2}}u_{n-3}\ldots u_{d+1}\overline{u_d}\ldots u_0$. Additionally, we have $u^{(n-2)d} = \overline{u_{n-1}}\,\overline{u_{n-2}}u_{n-3}\ldots u_{d+1}\overline{u_d}\ldots u_0$ since (1) d is even or (2) d is odd and $P_{d-1}(u^{(n-2)}) = P_{d-1}(u) = 1$. Clearly, $u^{(n-2)d}$ is connected to $u^{d(n-2)}$ by an $(n-1)$-edge.

When $P_{n-3}(u) = 1$, according to Definition 1, we have $u^{(n-2)} = u_{n-1}\overline{u_{n-2}}u_{n-3}\ldots u_0$. If d is odd and $P_{d-1}(u) = 0$, then $u^d = u_{n-1}u_{n-2}\ldots\overline{u_{d+1}}\,\overline{u_d}\ldots u_0$. Hence, $P_{n-3}(u^d) = P_{n-3}(u) = 1$. As a result, $u^{d(n-2)} = u_{n-1}\overline{u_{n-2}}u_{n-3}\ldots\overline{u_{d+1}}\,\overline{u_d}\ldots u_0$. Additionally, since $P_{d-1}(u^{(n-2)}) = P_{d-1}(u) = 0$, we have $u^{(n-2)d} = u_{n-1}\overline{u_{n-2}}u_{n-3}\ldots\overline{u_{d+1}}\,\overline{u_d}\ldots u_0 = u^{d(n-2)}$. If (1) d is even or (2) d is odd and $P_{d-1}(u) = 1$, then $u^d = u_{n-1}u_{n-2}\ldots\overline{u_d}\ldots u_0$. Hence, $P_{n-3}(u^d) = 0$. As a result, $u^{d(n-2)} = \overline{u_{n-1}}\,\overline{u_{n-2}}u_{n-3}\ldots u_{d+1}\overline{u_d}\ldots u_0$. Additionally, we have $u^{(n-2)d} = u_{n-1}\overline{u_{n-2}}u_{n-3}\ldots u_{d+1}\overline{u_d}\ldots u_0$ since (1) d is even or (2) d is odd and $P_{d-1}(u^{(n-2)}) = P_{d-1}(u) = 1$. Clearly, $u^{(n-2)d}$ is connected to $u^{d(n-2)}$ by an $(n-1)$-edge. $\square$

4. Longest fault-free cycles with edge faults

In this section, we will show that if each vertex of TQ_n is incident with at least two fault-free edges, then it contains a fault-free Hamiltonian cycle, even if there are $2n-5$ edge faults. Regard F as the set of edge faults; $TQ_n - F$ represents a faulty TQ_n, and $\delta(TQ_n - F) \geq 2$ means that each vertex in $TQ_n - F$ is incident with at least two fault-free edges. Hence, we format the main theorem as follows.

Theorem 1. *If $F \subset E(TQ_n)$ (respectively, $F' \subset E(TQ_n^{0,i} \cup TQ_n^{1,i})$ for $i \in \{0, 1\}$) with $|F| \leq 2n - 5$ (respectively, $|F'| \leq 2n - 3$) and $\delta(TQ_n - F) \geq 2$ (respectively, $\delta(TQ_n^{0,i} \cup TQ_n^{1,i} - F') \geq 2$), then $TQ_n - F$ (respectively, $TQ_n^{0,i} \cup TQ_n^{1,i} - F'$) is Hamiltonian, where $n \geq 3$ is an odd integer.*

Proof. We proceed by induction on n. By Lemma 1, the theorem holds for TQ_3 since $2n-5 = n-2$ when $n = 3$. In addition, the theorem holds for $TQ_3^{0,i} \cup TQ_3^{1,i}$, which can be verified by a computer exhausted search program [11]. Two steps can complete the proof. First, for all odd integers $n \geq 3$, we show that if the theorem holds for TQ_n, then the theorem holds for $TQ_n^{0,i} \cup TQ_n^{1,i}$. Secondly, for all odd integers $n \geq 5$, we show that if the theorem holds for $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i}$ then the theorem holds for TQ_n. However, since the proof of the first step is easier than and similar to that of second step, we only show the second step. That is, we assume that $TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i} - F'$ is Hamiltonian if $F' \subset E(TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i})$, $\delta(TQ_{n-2}^{0,i} \cup TQ_{n-2}^{1,i} - F') \geq 2$, and $|F'| \leq 2n-7$, where $i \in \{0, 1\}$ and $n \geq 5$. We will show that $TQ_n - F$ is Hamiltonian if $F \subset E(TQ_n)$, $\delta(TQ_n - F) \geq 2$, and $|F| \leq 2n-5$. Let $F_0 = F \cap E(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0})$, $F_1 = F \cap E(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$, and $F_c = F \cap \{(u, u^{(n-2)}) | u \in V(TQ_n)\}$. Additionally, let $f_0 = |F_0|, f_1 = |F_1|, f_c = |F_c|$. Without loss of generality, we assume $f_0 \geq f_1$. Therefore, we have $f_1 \leq \lfloor (2n-5)/2 \rfloor = n-3$. Three cases are considered:

Case 1. $f_0 \leq 2n - 7$. There is at most one vertex with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$, for otherwise $f_0 \geq 2n - 5$, which is a contradiction. Two cases are further considered:

Case 1.1. $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0) \geq 2$. By the induction hypothesis, there exists a Hamiltonian cycle C in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$.

If $f_1 = n - 3$, then $f_c \leq 2n - 5 - 2(n - 3) = 1$. By Lemma 3, there exist two edges (x, y) and (u, v) in C such that $(x^{(n-2)}, y^{(n-2)})$ and $(u^{(n-2)}, v^{(n-2)})$ are also two edges in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$, where x, y, u, v are distinct. Clearly, at most one of $(x, x^{(n-2)})$, $(y, y^{(n-2)})$, $(u, u^{(n-2)})$, and $(v, v^{(n-2)})$ is in F_c. Without loss of generality, we assume $(x, x^{(n-2)})$, $(y, y^{(n-2)}) \notin F_c$. Let $P_{yx} = C - \{(x, y)\}$. In addition, by Lemma 4 (since $f_1 \leq n - 3$), there exists a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired fault-free Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

If $f_1 \leq n - 4$, then there exists an edge (x, y) in C such that $(x, x^{(n-2)})$ and $(y, y^{(n-2)}) \notin F_c$ since there are 2^{n-1} edges in C and $2^{n-1} > 2(2n - 5) \geq 2|F| \geq 2f_c$. (This is because an edge in F_c eliminates two choices in C.) Let $P_{yx} = C - \{(x, y)\}$. In addition, by Lemma 1, there exists a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired fault-free Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

Case 1.2. $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0) = 1$. Recall that there is at most one vertex with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$, let x be such a vertex. Hence, $f_0 \geq n - 2$ (thus $f_1 + f_c \leq n - 3$) and $(x, x^{(n-2)}) \notin F_c$.

First, suppose that $f_1 = n - 3$ (then $f_c = 0$ and $f_0 = n - 2$). When $(x, x^{(n-1)}) \in F_0$, let $y = x^{(n-1)}$. We claim that $(x^{(n-2)}, y^{(n-2)})$ is an edge in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. (The claim proof is placed after the main proof). Since $f_1 = n - 3$, by Lemma 4, there exists a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. Moreover, since $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})) = 2$, by the induction hypothesis, there is a Hamiltonian cycle C in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$. Additionally, since (x, y) is one of the two edges incident with x in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$, it is not difficult to see that C contains (x, y). Let $P_{yx} = C - \{(x, y)\}$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

When $(x, x^{(n-1)}) \notin F_0$, we have $(x, x^i) \in F_0$ for all $i \in \{0, 1, 2, \ldots, n-3\}$. If $n = 5$, the desired Hamiltonian cycles are constructed by using a computer program [11]. If $n \geq 7$, then by Lemma 5, there exists an integer $d \in \{0, 1, 2, \ldots, n-3\}$ such that there is a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$, where $y = x^d$. Moreover, since $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})) = 2$, by the induction hypothesis, there is a Hamiltonian cycle C in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$. Additionally, since (x, y) is one of the two edges incident with x in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$, it is not difficult to see that C contains (x, y). Let $P_{yx} = C - \{(x, y)\}$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

Now, suppose that $f_1 \leq n - 4$. Since $f_c \leq n - 3$ and $|\{(x, x^i) | i \in \{0, 1, 2, \ldots, n-3, n-1\}\} \cap F_0| = n - 2$, we have $(x, x^d) \in F_0$ with $(x^d, x^{d(n-2)}) \notin F_c$, for some $d \in \{0, 1, 2, \ldots, n-3, n-1\}$. Let $y = x^d$. By Lemma 1, there exists a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. In addition, by the induction hypothesis, there is a Hamiltonian cycle C in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$. Additionally, since (x, y) is one of the two edges incident with x in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$, it is not difficult to see that C contains (x, y). Let $P_{yx} = C - \{(x, y)\}$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

Case 2. $f_0 = 2n - 6$. We have $f_1 + f_c \leq 1$. Similarly, there is at most one vertex with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$.

First, suppose that $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0) \geq 2$. Select an edge $(x, y) \in F_0$ such that $(x, x^{(n-2)})$, $(y, y^{(n-2)}) \notin F_c$. Since $|F_0 - \{(x, y)\}| = 2n-7$, by the induction hypothesis, there exists a Hamiltonian cycle C for $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$. If C contains (x, y), then let $P_{yx} = C - \{(x, y)\}$. In addition, since $f_1 \leq 1$, by Lemma 1, there is a Hamiltonian path

$P_{x^{(n-2)}y^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)). If C does not contain (x, y), then select an edge (x', y') in C such that $(x', x'^{(n-2)}), (y', y'^{(n-2)}) \notin F_c$. Let $P_{y'x'} = C - \{(x', y')\}$. By Lemma 1, there is a Hamiltonian path $P_{x'^{(n-2)}y'^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired Hamiltonian cycle can be constructed as $\langle x', x'^{(n-2)}, P_{x'^{(n-2)}y'^{(n-2)}}, y'^{(n-2)}, y', P_{y'x'}, x'\rangle$.

Then, suppose that $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0) = 1$. Let x be the vertex with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Select an edge $(x, y) \in F_0$ such that $(y, y^{(n-2)}) \notin F_c$. Since $|F_0 - \{(x, y)\}| = 2n - 7$ and $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})) = 2$, by the induction hypothesis, there exists a Hamiltonian cycle C for $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\}$. Since (x, y) is one of the two edges incident with x in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\})$, C contains (x, y). Let $P_{yx} = C - \{(x, y)\}$. In addition, by Lemma 1, there is a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

Case 3. $f_0 = 2n - 5$. We have $f_1 = f_c = 0$. Clearly, there are at most two vertices with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Three cases are further considered:

Case 3.1. There are no vertices with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Clearly, there exist two edges (x, y) and (u, v) in F_0 such that $\{x, y\} \cap \{u, v\} = \emptyset$. Apparently, $|F_0 - \{(x, y), (u, v)\}| = 2n - 7$. By the induction hypothesis, there exists a Hamiltonian cycle C for $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y), (u, v)\})$. If C contains both (x, y) and (u, v), then let P_{xu} and P_{vy} be two subpaths of C. In addition, by Lemma 2, there exist two paths $P_{y^{(n-2)}x^{(n-2)}}$ and $P_{u^{(n-2)}v^{(n-2)}}$ with $V(P_{y^{(n-2)}x^{(n-2)}}) \cap V(P_{u^{(n-2)}v^{(n-2)}}) = \emptyset$ and $V(P_{y^{(n-2)}x^{(n-2)}}) \cup V(P_{u^{(n-2)}v^{(n-2)}}) = V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. The desired Hamiltonian cycle can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vy}, y, y^{(n-2)}, P_{y^{(n-2)}x^{(n-2)}}, x^{(n-2)}, x\rangle$ (refer to Fig. 2(b)). If C contains only one of them, say (x, y), then let $P_{yx} = C - \{(x, y)\}$. In addition, by Lemma 1, there is a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)). If C contains none of them, then select an edge (x', y') in C. Let $P_{y'x'} = C - \{(x', y')\}$. By Lemma 1, there is a Hamiltonian path $P_{x'^{(n-2)}y'^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired Hamiltonian cycle can be constructed as $\langle x', x'^{(n-2)}, P_{x'^{(n-2)}y'^{(n-2)}}, y'^{(n-2)}, y', P_{y'x'}, x'\rangle$.

Case 3.2. There is only one vertex x with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Let $(u, v), (x, y) \in F_0$ such that $\{u, v\} \cap \{x, y\} = \emptyset$. In addition, we have $|F_0 - \{(x, y), (u, v)\}| = 2n - 7$ and $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y), (u, v)\}) = 2$. By the induction hypothesis, there exists a Hamiltonian cycle C for $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y), (u, v)\})$. Since (x, y) is one of the two edges incident with x in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y), (u, v)\})$, it is not difficult to see that C contains (x, y). If C contains (u, v), then let P_{xu} and P_{vy} be two subpaths of C. In addition, by Lemma 2, there exist two paths $P_{y^{(n-2)}x^{(n-2)}}$ and $P_{u^{(n-2)}v^{(n-2)}}$ with $V(P_{y^{(n-2)}x^{(n-2)}}) \cap V(P_{u^{(n-2)}v^{(n-2)}}) = \emptyset$ and $V(P_{y^{(n-2)}x^{(n-2)}}) \cup V(P_{u^{(n-2)}v^{(n-2)}}) = V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. The desired Hamiltonian cycle can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vy}, y, y^{(n-2)}, P_{y^{(n-2)}x^{(n-2)}}, x^{(n-2)}, x\rangle$ (refer to Fig. 2(b)). If C does not contain (u, v), then let $P_{yx} = C - \{(x, y)\}$. In addition, by Lemma 1, there is a Hamiltonian path $P_{x^{(n-2)}y^{(n-2)}}$ for $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1} - F_1$. The desired Hamiltonian cycle can be constructed as $\langle x, x^{(n-2)}, P_{x^{(n-2)}y^{(n-2)}}, y^{(n-2)}, y, P_{yx}, x\rangle$ (refer to Fig. 2(a)).

Case 3.3. There are two vertices x and u with degree one in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - F_0$. Notice that $(x, u) \in F_0$, for otherwise it will have $f_0 \geq 2n - 4$, which is a contradiction. Let $(u, v), (x, y) \in F_0$ such that $\{u, v\} \cap \{x, y\} = \emptyset$. In addition, we have $|F_0 - \{(x, y), (u, v)\}| = 2n - 7$ and $\delta(TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y)\}, \{u, v\})) = 2$. By the induction hypothesis, there exists a Hamiltonian cycle C for $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y), (u, v)\})$. Since (x, y) is one of the two edges incident with x and (u, v) is one of the two edges incident with u in $TQ_{n-2}^{0,0} \cup TQ_{n-2}^{1,0} - (F_0 - \{(x, y), (u, v)\})$, it is not difficult to see that C contains both (x, y) and (u, v). Let P_{xu} and P_{vy} be two subpaths of C. In addition, by Lemma 2, there exist two paths $P_{y^{(n-2)}x^{(n-2)}}$ and $P_{u^{(n-2)}v^{(n-2)}}$ with $V(P_{y^{(n-2)}x^{(n-2)}}) \cap V(P_{u^{(n-2)}v^{(n-2)}}) = \emptyset$ and $V(P_{y^{(n-2)}x^{(n-2)}}) \cup V(P_{u^{(n-2)}v^{(n-2)}}) = V(TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1})$. The desired Hamiltonian cycle can be constructed as $\langle x, P_{xu}, u, u^{(n-2)}, P_{u^{(n-2)}v^{(n-2)}}, v^{(n-2)}, v, P_{vy}, y, y^{(n-2)}, P_{y^{(n-2)}x^{(n-2)}}, x^{(n-2)}, x\rangle$ (refer to Fig. 2(b)).

Claim Proof. Suppose that $x = x_{n-1}x_{n-2}\dots x_0$ (therefore $y = \overline{x_{n-1}}x_{n-2}\dots x_0$). If $P_{n-3}(x) = 0$ (therefore $P_{n-3}(y) = 0$), then $x^{(n-2)} = \overline{x_{n-1}}\,\overline{x_{n-2}}\dots x_0$ and $y^{(n-2)} = x_{n-1}\overline{x_{n-2}}\dots x_0$. If $P_{n-3}(x) = 1$ (therefore $P_{n-3}(y) = 1$), then $x^{(n-2)} = x_{n-1}\overline{x_{n-2}}\dots x_0$ and $y^{(n-2)} = \overline{x_{n-1}}\,\overline{x_{n-2}}\dots x_0$. As a result, $(x^{(n-2)}, y^{(n-2)})$ is an edge in $TQ_{n-2}^{0,1} \cup TQ_{n-2}^{1,1}$. □

Our result is optimal with respect to the number of edge faults tolerated since there are distributions of $2n - 4$ edge faults over a TQ_n such that no fault-free Hamiltonian cycle can be found in the faulty TQ_n. Consider that two vertices $u = 0^n$ (n consecutive 0's) and $v = 0^{n-3}101$ of TQ_n. Suppose that (u, u^d) and (v, v^d) are fault-free if $d \in \{0, 2\}$, and are faulty if $d \in \{1, 3, 4, \dots, n-1\}$. Refer to Fig. 6, any fault-free cycle containing nodes 0^n and $0^{n-3}101$ must contain edges $(0^n, 0^{n-1}1)$, $(0^n, 0^{n-3}10^2)$, $(0^{n-3}101, 0^{n-1}1)$, and $(0^{n-3}101, 0^{n-3}10^2)$. This is because edges $(0^n, 0^{n-1}1)$ and $(0^n, 0^{n-3}10^2)$ (respectively, $(0^{n-3}101, 0^{n-1}1)$ and $(0^{n-3}101, 0^{n-3}10^2)$) are the only two fault-free edges incident with 0^n (respectively, $0^{n-3}101$). Since $\langle 0^n, 0^{n-1}1, 0^{n-3}101, 0^{n-3}10^2, 0^n\rangle$ is a cycle, it is easy to see that no fault-free Hamiltonian cycle exists in faulty TQ_n.

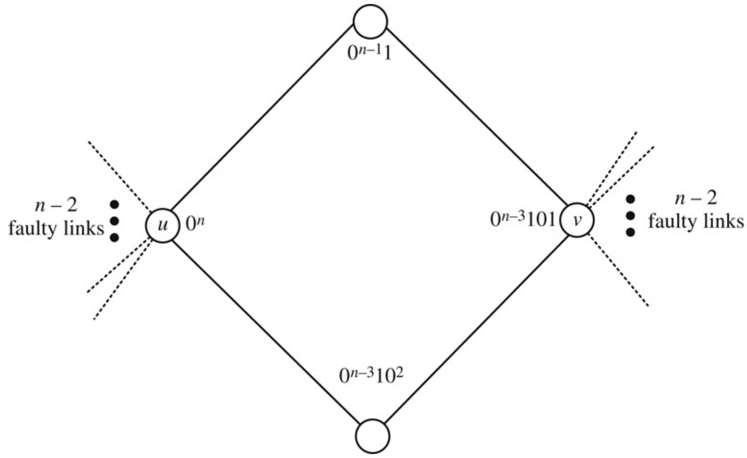

Fig. 6. A distribution of $2n-4$ edge faults in TQ_n.

5. Discussion and conclusion

In this paper, with the assumption of at least two fault-free edges incident with each vertex, we have shown that there exists a fault-free Hamiltonian cycle in an n-dimensional twisted cube (TQ_n) with up to $2n-5$ edge faults. A recursive algorithm for constructing the fault-free Hamiltonian cycle can easily result from the proof of Theorem 1.

With the same discussion in [14], we can verify that the assumption is practically meaningful by evaluating its probability of occurrence, which is very close to one, even if n is small.

Many properties of $TQ_3^{0,i} \cup TQ_3^{1,i}$ are hard to derive. Therefore, it seems that using an exhaustive algorithm to justify these properties is necessary. It just took several seconds to run the program by a personal computer with a 2.7 GHz CPU. Additionally, for the same reason, we still need to use a program to construct fault-free Hamiltonian cycles in a faulty TQ_5 when one specific situation is confronted (refer to Case 1.2 in the proof of Theorem 1). There are 32 nodes in TQ_5 and the degree of each node is five, thus, the size of the search tree is about $4^{32} = 1.84467441 \times 10^{19}$. In this situation, it needs to traverse $16 \times \binom{32}{2} = 7936$ search trees. It took about six hours to run the program. If we construct fault-free Hamiltonian cycles in a faulty TQ_5 in all cases, we need to traverse $\binom{80}{5} = 24\,040\,016$ search trees. As a result, it will need about $(24\,040\,016/7936) \times 6 = 18175$ h to run this program.

In addition, exploration of the conditional fault-tolerant Hamiltonicity of other networks such as arrangement graphs, star graphs, and pancake graphs are our topics for further research.

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