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# A proof of equivalence of two force methods for active structural control

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ABSTRACT

This paper gives a proof of equivalence between two existing force methods (FM) for structural analysis: The Integrated Force Method (IFM) and a force method based on singular value decomposition (SVD) of the equilibrium conditions here named as SVD-FM. Recently, these methods have been employed to design and control active structures. Actuation is employed to counteract the effect of external loading by modifying internal forces and the external geometry in order to meet strength and serviceability requirements. Both IFM and SVD-FM offer an effective way to estimate the combined effect of external loading and that of actuation. Generally, the SVD-FM has a lower degree of computational complexity with respect to the IFM, the more so as the structure static indeterminacy increases. However, the IFM has a more intuitive formulation that is preferable pedagogically and it is of value for future extensions to kinematically indeterminate configurations and to geometric non-linear cases.



## 1. Introduction

Active structural control through integrated sensing and actuation has been investigated and experimentally tested to reduce the structure response under extreme loading events such as strong winds, earthquakes and unusual crowds [1–4]. The ability to control internal forces and the external geometry allows a structure to operate closer to design limits. For example, a change of geometry can result in a significant shift of the structure natural frequencies. This strategy has been investigated to reduce the dynamic response of tensegrity structures [5,6], of an underslung cable-stayed beam bridge under pedestrian loading [7] and of frames equipped with variable stiffness joints made of shape memory polymer composites [8].

It has been shown that through integrated structure-control optimization [9–12] it is possible to design adaptive structures with a significantly better material utilization compared to weight-optimized passive structures. Senatore et al. [13] formulated a new integrated structure-control optimization method that produces structures with minimum 'whole-life' energy. The whole-life energy includes the energy embodied in the material and the operational energy for actuation. Numerical and experimental studies have shown that well-designed adaptive structures save up to 70% of the energy with respect to weight-optimized passive structures [14,15]. Recent studies [16,17] have investigated structural adaptation through large shape changes. In this case the structure is designed to be controlled into a shape that is optimal to counteract the effect of the external load. The optimal shape changes as the load changes. This way a structure can be designed with a lower embodied energy (i.e. better material utilization) compared to an adaptive structure limited to small shape changes.

When designing adaptive structures, the actuation system must be strategically placed. The action of the actuators is to modify internal forces and displacements actively in order to meet strength and/or serviceability requirements. If the location of the actuators is not well chosen, it might be impossible to achieve the required force and geometry correction or it might require a much higher energy than needed [13]. Many methods exist to obtain an optimal actuator layout some of which based on efficient enumerations, ranking and continuous relaxation [13,18–20]. However, actuator placement remains a challenging task due to the combinatorial nature of the problem.

Actuation has been modeled as an imposed strain distribution caused by the active elements [21–23]. For example, in reticular structures, the structure internal forces and shape are controlled through length changes of linear actuators which are fitted within some of the structural elements. A convenient method to compute the effect of an imposed strain is the Integrated Force Method

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**Nomenclature**

| | |
|---|---|
| $\mathbf{A}$ | equilibrium matrix |
| $E$ | Young's modulus |
| $\mathbf{G}$ | flexibility matrix |
| $\mathbf{J}$ | IFM deformation coefficient matrix |
| $\mathbf{S}$ | IFM governing matrix |
| $\mathbf{S}_d$ | shape influence matrix |
| $\mathbf{S}_f$ | force influence matrix |
| $\mathbf{S}_d^*$ | reduced shape influence matrix |
| $\mathbf{S}_f^*$ | reduced force influence matrix |
| $[\mathbf{U}_r \quad \mathbf{U}_q]$ | left singular vectors |
| $\mathbf{U}_q$ | States of infinitesimal mechanisms |
| $\mathbf{V}_r$ | diagonal matrix of singular values |
| $[\mathbf{W}_r \quad \mathbf{W}_s]$ | right singular vectors |
| $\mathbf{W}_s$ | states of self-stress |
| $\boldsymbol{\alpha}$ | element cross section areas |
| $\Delta\mathbf{d}$ | change of node positions (i.e. change of shape) |
| $\Delta\mathbf{f}$ | change of internal forces |
| $m$ | degree of kinematic indeterminacy |
| $\mathbf{l}$ | element lengths |
| $\Delta\mathbf{l}$ | actuator length changes |
| $\Delta\mathbf{l}^e$ | element length changes (all elements are active) |
| $n^{act}$ | number of actuators |
| $n^{cd}$ | number of controlled degrees of freedom |
| $n^d$ | number of degrees of freedom |
| $n^e$ | number of elements |
| $n^n$ | number of nodes |
| $\Delta\mathbf{p}$ | change of external load |
| $s$ | degree of static indeterminacy |
| $\boldsymbol{\eta}$ | coefficient vector of force solution |
| $\boldsymbol{\xi}$ | coefficient matrix of decomposed generalized inverse |

(IFM) [24]. The IFM formulation is a generalization of the standard force method (SFM) because it does not require the choice of any redundant and basis determinate structures [25]. The IFM takes a familiar form of matrix-vector product similar to the classical equilibrium formulation. The IFM allows an initial deformation to be assigned directly as part of the external load in order to model a non-elastic strain which is usually produced by a lack of fit or by thermal strain. On the other hand, a non-elastic strain can be thought of as caused by the action of actuators and it has been referred as eigenstrain [13,26].

Another force method that has been used for control of adaptive structures was first presented in the work of Pellegrino and Calladine [27,28]. In common with the IFM, this method also allows to assign a non-elastic strain as part of the external load and it requires the determination of the self-stress basis to compute the internal forces in statically indeterminate structures. The self-stress basis can be obtained through singular value decomposition (SVD) of the equilibrium conditions. This method, which is here referred as SVD-FM, has been generalized to structural systems with static as well as kinematic indeterminacies. The SVD-FM formulation has also been extended to control structures with geometric non-linear behavior through an iterative scheme [29,30].

The work presented in this paper shows that the IFM and SVD-FM are equivalent. For simplicity the proof of equivalence between these two methods will be given referring to reticular structures with no kinematic indeterminacy under quasi-static loading. However, the IFM can be generalized to continuous structures [31] and to compute the dynamic response [32]. Similar considerations apply to the SVD-FM.

## 2. Proof of equivalence

Consider a reticular structure with $s$-degree of static indeterminacy. The structure is a set of $n^n$ nodes connected by $n^e$ elements. There are $n^d = n^n \cdot dim$ degrees of freedom; where $dim$ is either 2 or 3 for a planar and a spatial structure, respectively. Force-equilibrium conditions are:

$$\mathbf{A}\Delta\mathbf{f} = \Delta\mathbf{p}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n^d \times n^e}$, $\Delta\mathbf{f}$ and $\Delta\mathbf{p}$ are the equilibrium matrix, internal forces and external load. The symbol $\Delta$ denotes a change or a difference. The IFM and SVD-FM formulations make use of some terms resulting from the singular value decomposition of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix}\mathbf{U}_r & \mathbf{U}_q\end{bmatrix}\begin{bmatrix}\mathbf{V}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{W}_r & \mathbf{W}_s\end{bmatrix}^T. \tag{2}$$

$[\mathbf{U}_r \quad \mathbf{U}_q] \in \mathbb{R}^{n^d \times n^d}$, $[\mathbf{W}_r \quad \mathbf{W}_s] \in \mathbb{R}^{n^e \times n^e}$ and $\mathbf{V}_r \in \mathbb{R}^{n^e \times n^e}$ are the left singular vectors, right singular vectors and singular values for matrix $\mathbf{A}$, respectively. $\mathbf{U}_r \in \mathbb{R}^{n^d \times (n^d - m)}$ is the basis of the column space of $\mathbf{A}$ which can be interpreted as the range of all loads that can be taken by the structure. These loads are in equilibrium with the forces that lie in the space spanned by $\mathbf{W}_r \in \mathbb{R}^{n^e \times (n^e - s)}$ (the row space of $\mathbf{A}$). The term $\mathbf{W}_s \in \mathbb{R}^{n^e \times s}$ is the null space of $\mathbf{A}$ ($\mathbf{W}_s = \ker(\mathbf{A})$), $s$ is the degree of static indeterminacy. The columns of $\mathbf{W}_s$ are the $s$ states of self-stress i.e. all possible combination of internal forces which are in equilibrium with zero load. In the absence of external load (i.e. $\Delta\mathbf{p} = \mathbf{0}$), there exist an infinite number of non-trivial solutions for the homogeneous equation $\mathbf{A}\Delta\mathbf{f} = \mathbf{0}$ which are linear combination of the self-stress basis:

$$\Delta\mathbf{f} = \mathbf{W}_s\boldsymbol{\eta}. \tag{3}$$

The term $\mathbf{U}_q \in \mathbb{R}^{n^d \times m}$ is the left null space of $\mathbf{A}$ ($\mathbf{U}_q = \mathrm{coker}(\mathbf{A})$), $m$ is the degree of kinematic indeterminacy. The columns of $\mathbf{U}_q$ are $m$ independent nodal displacements which do not cause any strain i.e. the inextensional mechanism basis. For kinematically determinate structures, $\mathbf{U}_q \in \mathbb{R}^{n^d \times m}$ does not exist. This work only considers kinematically determinate structures. For further details regarding the static and kinematic interpretation of the terms in Eq. (2) the reader is referred to [28].

### 2.1. Solution for forces

The IFM formulation can be written as follows:

$$\begin{bmatrix}\mathbf{A}\\ \mathbf{W}_s^T\mathbf{G}\end{bmatrix}\Delta\mathbf{f} = \begin{Bmatrix}\Delta\mathbf{p}\\ -\mathbf{W}_s^T\Delta\mathbf{l}\end{Bmatrix}, \tag{4}$$

or in compact form as:

$$\mathbf{S}\Delta\mathbf{f} = \Delta\mathbf{p}^*, \tag{5}$$

where:

$$\mathbf{S} = \begin{bmatrix}\mathbf{A}\\ \mathbf{W}_s^T\mathbf{G}\end{bmatrix};\ \Delta\mathbf{p}^* = \begin{Bmatrix}\Delta\mathbf{p}\\ -\mathbf{W}_s^T\Delta\mathbf{l}\end{Bmatrix}. \tag{6}$$

The concatenation of $r$ equations of compatibility:

$$\mathbf{W}_s^T\mathbf{G}\Delta\mathbf{f} = -\mathbf{W}_s^T\Delta\mathbf{l}, \tag{7}$$

makes the IFM governing matrix $\mathbf{S} \in \mathbb{R}^{n^e \times n^e}$ a square matrix for kinematically determinate structures. $\mathbf{G} \in \mathbb{R}^{n^e \times n^e}$ is the member flexibility matrix, for example for a reticular structures $\mathbf{G}$ is a diagonal matrix with components $l_i/(E_i\alpha_i)$ where $E_i$ and $\alpha_i$ are the Young's modulus and the cross section area of the $i$th member of the structure. A simple derivation of the compatibility conditions in Eq. (7) is given in [13]. $\Delta\mathbf{p}^* \in \mathbb{R}^{n^e}$ is the concatenation of the external load vector $\Delta\mathbf{p} \in \mathbb{R}^{n^d}$ and the eigenstrain load vector $-\mathbf{W}_s^T\Delta\mathbf{l}$ (as defined in [13]) where $\Delta\mathbf{l} \in \mathbb{R}^{n^e}$ is the prescribed strain which is thought of as caused by the active element length changes. For a statically determinate system $\mathbf{W}_s$ does not exist, therefore in this case the actuator length changes can only be used to control the shape of the structure (i.e. $\Delta\mathbf{l}$ does not cause any change of the internal forces). The assignment of $\Delta\mathbf{l}$ implies that the actuator locations are known. In Senatore et al. [13] optimization of the actuator layout is based on a control efficacy index computed through the IFM. This index measures the contribution of each candidate actuator to attain required force and geometry changes.

Solving Eq. (4) gives the change of internal forces caused by the external load in combination with the effect of the actuator length changes $\Delta\mathbf{l}$:

$$\Delta\mathbf{f} = \begin{bmatrix} \mathbf{A} \\ \mathbf{W}_s^T\mathbf{G} \end{bmatrix}^{-1} \begin{Bmatrix} \Delta\mathbf{p} \\ -\mathbf{W}_s^T\Delta\mathbf{l} \end{Bmatrix}. \tag{8}$$

Following the SVD-FM formulation given in [28], the internal forces caused by the external load and actuator length changes is:

$$\Delta\mathbf{f} = \left(\mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T\right)\Delta\mathbf{p} + \mathbf{W}_s\boldsymbol{\eta}, \tag{9}$$

where:

$$\boldsymbol{\eta} = -\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\mathbf{W}_s^T\left[\Delta\mathbf{l} + \mathbf{G}\left(\mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T\right)\Delta\mathbf{p}\right]. \tag{10}$$

Different to the IFM, SVD-FM does not have a simple form and it is more difficult to give it a physical interpretation. The reader is referred to [28] for the derivation of Eqs. (9) and (10). However, Eq. (8) is equivalent to Eq. (9). Knowing that $\mathbf{A}^+ = \mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T$ [33] where $(\cdot)^+$ denotes the Moore-Penrose pseudoinverse, Eq. (9) appears as the expanded expression of Eq. (8). Lemma 1, which gives the decomposition of a generalized inverse of a vertically partitioned rectangular matrix [34], is employed to prove the equivalence of Eq. (8) with Eq. (9).

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**Lemma 1.** *If* $\mathbf{C} \in \mathbb{R}^{m\times p}$ *and* $\mathbf{D} \in \mathbb{R}^{n\times p}$ *then:*

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}^{-} = \left[\, \mathbf{C}^- - \xi\mathbf{D}\mathbf{C}^- \mid \xi \,\right], \tag{11}$$

*where:*

$$\boldsymbol{\xi} = \left(\mathbf{I} - \mathbf{C}^-\mathbf{C}\right)\left[\mathbf{D}\left(\mathbf{I} - \mathbf{C}^-\mathbf{C}\right)\right]^-. \tag{12}$$

The notation $(\cdot)^-$ denotes the generalized inverse and $[\cdot|\cdot]$ column concatenation. If $\mathbf{C}$ and $\mathbf{D}$ are such that $m + n = p$, the inverse which is denoted as $(\cdot)^{-1}$ is identical to the generalized inverse. A decomposition of the inverse of a vertically partitioned square matrix is thus given as:

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}^{-1} = \left[\, \mathbf{C}^+ - \xi\mathbf{D}\mathbf{C}^+ \mid \xi \,\right], \tag{13}$$

where:

$$\boldsymbol{\xi} = \left(\mathbf{I} - \mathbf{C}^+\mathbf{C}\right)\left[\mathbf{D}\left(\mathbf{I} - \mathbf{C}^+\mathbf{C}\right)\right]^+. \tag{14}$$

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Through Lemma 1, Eq. (8) can be decomposed into:

$$\Delta\mathbf{f} = \left[\, \mathbf{A}^+ - \xi\left(\mathbf{W}_s^T\mathbf{G}\right)\mathbf{A}^+ \mid \xi \,\right] \begin{Bmatrix} \Delta\mathbf{p} \\ -\mathbf{W}_s^T\Delta\mathbf{l} \end{Bmatrix}, \tag{15}$$

where:

$$\boldsymbol{\xi} = \left(\mathbf{I} - \mathbf{A}^+\mathbf{A}\right)\left[\mathbf{W}_s^T\mathbf{G}\left(\mathbf{I} - \mathbf{A}^+\mathbf{A}\right)\right]^+. \tag{16}$$

Through the identity $\mathbf{I} - \mathbf{A}^+\mathbf{A} = \mathbf{W}_s\mathbf{W}_s^T$ [33], $\boldsymbol{\xi}$ can be rewritten as:

$$\boldsymbol{\xi} = \mathbf{W}_s\mathbf{W}_s^T\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\mathbf{W}_s^T\right)^+. \tag{17}$$

Since $\mathbf{W}_s\mathbf{W}_s{}^T(\mathbf{W}_s{}^T)^+ = \mathbf{W}_s$ [33], and owing to the fact that $\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s$ is an invertible square matrix, $\boldsymbol{\xi}$ can be rearranged into:

$$\boldsymbol{\xi} = \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}. \tag{18}$$

Expanding Eq. (15) and substituting $\boldsymbol{\xi}$:

$$\Delta\mathbf{f} = \left[\mathbf{A}^+ - \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\mathbf{W}_s^T\mathbf{G}\mathbf{A}^+\right]\Delta\mathbf{p} - \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\mathbf{W}_s^T\Delta\mathbf{l}. \tag{19}$$

Using the identity $\mathbf{A}^+ = \mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T$ [33] and then tiding up:

$$\Delta\mathbf{f} = \left(\mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T\right)\Delta\mathbf{p} - \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\mathbf{W}_s^T\left[\Delta\mathbf{l} + \mathbf{G}\left(\mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T\right)\Delta\mathbf{p}\right]. \tag{20}$$

Finally, $\Delta\mathbf{f}$ can be expressed as:

$$\Delta\mathbf{f} = \left(\mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T\right)\Delta\mathbf{p} + \mathbf{W}_s\boldsymbol{\eta}, \tag{21}$$

where:

$$\boldsymbol{\eta} = -\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\mathbf{W}_s^T\left[\Delta\mathbf{l} + \mathbf{G}\left(\mathbf{W}_r\mathbf{V}_r^{-1}\mathbf{U}_r^T\right)\Delta\mathbf{p}\right]. \tag{22}$$

Eq. (8) is equivalent to Eq. (9). ■

### 2.2. Solution for displacement

Consider the IFM formulation to compute the displacements due to the external load $\Delta\mathbf{p}$ and the actuator length changes $\Delta\mathbf{l}$:

$$\Delta\mathbf{d} = \left\{\begin{bmatrix} \mathbf{A} \\ \mathbf{W}_s^T\mathbf{G} \end{bmatrix}^{-1}\right\}^T (\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}), \tag{23}$$

or in compact form:

$$\Delta\mathbf{d} = \mathbf{J}(\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}), \tag{24}$$

where $\mathbf{J}$, the deformation coefficient matrix is:

$$\mathbf{J} = n^d \text{ rows of } \left[\mathbf{S}^{-1}\right]^T. \tag{25}$$

The same task can be carried out via the SVD-FM as follows:

$$\Delta\mathbf{d} = \mathbf{U}_r\mathbf{V}_r^{-1}\mathbf{W}_r^T(\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}). \tag{26}$$

Recalling Lemma 1 and Eq. (15), Eq. (23) can be rewritten as:

$$\Delta\mathbf{d} = \left[\, \mathbf{A}^+ - \xi\left(\mathbf{W}_s^T\mathbf{G}\right)\mathbf{A}^+ \mid \xi \,\right]^T (\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}), \tag{27}$$

where:

$$\boldsymbol{\xi} = \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}. \tag{28}$$

Therefore by substitution of $\boldsymbol{\xi}$ in Eq. (27):

$$\Delta\mathbf{d} = \left[\, \mathbf{A}^+ - \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\left(\mathbf{W}_s^T\mathbf{G}\right)\mathbf{A}^+ \mid \mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1} \,\right]^T (\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}). \tag{29}$$

By taking only the first $n^d$ rows of the matrix term in Eq. (29):

$$\Delta\mathbf{d} = \left[\mathbf{A}^+ \boxed{-\mathbf{W}_s\left(\mathbf{W}_s^T\mathbf{G}\mathbf{W}_s\right)^{-1}\left(\mathbf{W}_s^T\mathbf{G}\right)\mathbf{A}^+}\right]^T (\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}). \tag{30}$$

The terms containing $\mathbf{W}_s$ vanish because the product:

$$\mathbf{W}_s^T(\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}) = \mathbf{0}, \tag{31}$$

which expresses compatibility conditions in the absence of external load. For this reason, the only non-zero term in Eq. (30) is:

$$\Delta\mathbf{d} = \left(\mathbf{A}^+\right)^T(\mathbf{G}\Delta\mathbf{f} + \Delta\mathbf{l}). \tag{32}$$

Since $(\mathbf{A}^+)^T = \mathbf{U}_r\mathbf{V}_r^{-1}\mathbf{W}_r^T$ [33], Eq. (23) is equivalent to Eq. (26). ■

### 2.3. Shape and force control

Both force methods discussed in this paper offers an effective formulation for shape and force control. Within the assumption of small deformations, the effect of a non-elastic element length change $\Delta\mathbf{l}^e$ is to cause a change of the internal forces $\Delta\mathbf{f}$ and shape $\Delta\mathbf{d}$ which can be expressed as a matrix-vector product:

$$\mathbf{S}_f \Delta\mathbf{l}^e = \Delta\mathbf{f}, \tag{33}$$

$$\mathbf{S}_d \Delta\mathbf{l}^e = \Delta\mathbf{d}, \tag{34}$$

where $\mathbf{S}_f \in \mathbb{R}^{n^e \times n^e}$ and $\mathbf{S}_d \in \mathbb{R}^{n^d \times n^e}$ are the force influence matrix and the shape influence matrix, respectively. Note that in Eq. (33) and (34), $\Delta\mathbf{l}^e \in \mathbb{R}^{n^e}$ is length change of all the elements if they were all active.

From the IFM formulation, $\mathbf{S}_f$ and $\mathbf{S}_d$ are:

$$\mathbf{S}_f = \begin{bmatrix} \mathbf{A} \\ \mathbf{W}_s^T \mathbf{G} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{0} \\ -\mathbf{W}_s^T \mathbf{I} \end{Bmatrix}, \tag{35}$$

$$\mathbf{S}_d = \mathbf{J}\left(\mathbf{G}\mathbf{S}_f + \mathbf{I}\right), \tag{36}$$

where $\mathbf{I}$ is the identity matrix of dimensions of $n^e \times n^e$. Eq. (35) and (36) are equivalent to iterating Eq. (8) and (24) for a unitary length change of each active element and no external load i.e. $\Delta\mathbf{p}$ is a zero vector.

Using the SVD-FM formulation, the force and shape influence matrices are:

$$\mathbf{S}_f = -\mathbf{W}_s\left(\mathbf{W}_s{}^T \mathbf{G}\mathbf{W}_s\right)^{-1} \mathbf{W}_s{}^T, \tag{37}$$

$$\mathbf{S}_d = \mathbf{U}_r \mathbf{V}_r^{-1} \mathbf{W}_r^T \left(\mathbf{G}\mathbf{S}_f + \mathbf{I}\right), \tag{38}$$

which are Eq. (9) and (26) when the external load $\Delta\mathbf{p}$ is a zero vector. From the proof of equivalence of Eq. (8) with (9) and Eq. (23) with (26) it follows immediately the equivalence of Eqs. (35) with (37) and Eq. (36) with (38).

In practice, only selected elements are replaced by actuators, and only selected degrees of freedoms are controlled. Without loss of generality, there are $n^{act} \leq n^e$ number of actuators and $n^{cd} \leq n^d$ number of controlled degrees of freedom. The force influence matrix is reduced to $\mathbf{S}_f^* \in \mathbb{R}^{n^e \times n^{act}}$, which contains only the columns corresponding to the active elements. Similarly, the shape influence matrix is reduced to $\mathbf{S}_d^*$, which contains only the rows and columns corresponding to the controlled degrees of freedom and active elements, respectively. A target (required) shape $\Delta\mathbf{d}^{t*} \in \mathbb{R}^{n^{cd}}$, only contains the values corresponding to the controlled degrees of freedom. It is usually desirable to control large structures with a simple actuation system, and thus $\mathbf{S}_f$ and $\mathbf{S}_d$ are generally rectangular with significantly more rows than columns (i.e. an overdetermined linear system). The inverse problem to compute the actuator length changes $\Delta\mathbf{l}$ in order to obtain a required change of internal forces $\Delta\mathbf{f}^t$ and shape $\Delta\mathbf{d}^t$ can be formulated as a least square optimization:

$$\min_{\Delta\mathbf{l}} \left\| \mathbf{S} \cdot \Delta\mathbf{l} - \begin{Bmatrix} \Delta\mathbf{d}^t \\ \Delta\mathbf{f}^t \end{Bmatrix} \right\|_2, \tag{39}$$

where $\mathbf{S}$ is:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_d^* & \mathbf{S}_f^* \end{bmatrix}^T. \tag{40}$$

For small deformations, the minimum number of actuators $n^{act}$ to control the required displacements exactly is equal to the sum of $n^{cd}$ and the static indeterminacy $s$ [13]. This way, Eqs. (39) can be solved exactly. As the number of actuators decreases, control accuracy also decreases. The formulation stated in Eqs. (39) and (40) has been extended to geometrically non-linear problems in [29]. Other formulations exist that include buckling constraints [17].

The computational cost of the IFM and SVD-FM is dominated by the SVD of the equilibrium matrix $\mathbf{A}$. In the IFM the terms $\mathbf{U}_r$, $\mathbf{V}_r$ and $\mathbf{W}_r$ are discarded. Hence, it may appear desirable to replace the SVD with a more efficient process to determine the null space of matrix $\mathbf{A}$. However, SVD offers the best precision and stability for this task [35]. Without considering the SVD routine, the computational complexity of IFM and SVD-FM are $O((n^e)^3)$ and $O(n^e(n^d)^2)$, respectively. For kinematically determinate structures, $n^d = n^e - s$ holds, and thus the SVD-FM has a lower degree of computational complexity as the degree of static indeterminacy increases.

## 3. Conclusions

This work has presented a proof of equivalence of two force method formulations, namely the IFM and SVD-FM. These formulations have been employed as part of design and control methods for active structures because they offer an effective way to control internal forces and the external geometry in order to counteract the effect of loading. The SVD-FM has a lower degree of computational complexity as the degree of static indeterminacy increases. However, the IFM offers a more intuitive formulation that is preferable pedagogically and is of value for future extensions to kinematically indeterminate configurations and to geometric non-linear cases.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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