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Research paper

Fractional creep and relaxation models of viscoelastic materials via a non-Newtonian time-varying viscosity: physical interpretation

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ABSTRACT

Fractional viscoelastic models have been confirmed to achieve good agreement with experimental data using only a few parameters, in contrast to the classical viscoelastic models in previous studies. With an increasing number of applications, the physical meaning of fractional viscoelastic models has been attracting more attention. This work establishes an equivalent viscoelasticity (including creep and relaxation) between the fractional Maxwell model and the time-varying viscosity Maxwell model to reveal the physical meaning of fractional viscoelastic models. The obtained time-varying viscosity functions are used to interpret the physical meaning of the order of the fractional derivative α from the perspective of rheology. When α changes from 0 to 1, the viscosity functions quantitatively exhibit the transformation of viscoelasticity from elastic solid to Newtonian fluid, which can be considered as an extension of the Deborah number. The infinite viscosity coefficient for $\alpha = 0$ shows the elastic solid property, while the constant viscosity coefficient for $\alpha = 1$ exhibits the Newtonian fluid property. The sharply decreasing viscosity coefficient (versus α) near $\alpha = 0$ indicates that the elastic solid property decays rapidly. In addition, similar viscoelastic responses between the Hausdorff and fractional derivative models are found due to a similar time-varying viscosity.

1. Introduction

Classical viscoelastic rheological models are usually composed of springs and dashpots in series or parallel, such as the Maxwell, Kelvin, Zener, and more complex models (Mainardi, 2010). Because of their simple form, these classical models are easy to use and can help in understanding viscoelastic phenomena. However, the exponential type viscoelastic functions (such as creep and relaxation) deduced from these classical models do not match well with the real mechanical responses for a variety of viscoelastic materials (Di Paola et al., 2011). A number of springs and dashpots need to be added to fit experimental data well. For example, 30 Maxwell models are connected to form the Generalized Maxwell model (see Table 1) for the description of the relaxation behavior of Nafion (Zhang et al., 2017), in which a total of 61 parameters are needed. As claimed by Di Paola et al. (2011), this many parameters cause two main problems: high computational cost and meaningless results (e.g., negative values for the elastic modulus or viscosity coefficient).

Alternatively, a potentially efficient method to remedy these problems is to introduce the fractional derivative, because fractional calculus has been presented that has the ability to capture complex phenomena with very few parameters (Sun et al., 2018). As early as 1936, Gemant (1936) applied a fractional viscoelastic model to study Nutting's relaxation phenomenon. Subsequently, Scott Blair (1944) suggested a fractional constitutive model to describe a viscoelastic material, which is the well-known Scott–Blair (SB) model. Koeller (1984) presented a mathematical expression for the SB model. The fractional viscoelastic model has been broadly applied to cope with a variety of practical viscoelastic materials, as reviewed in a recent paper (Sun et al., 2018). With an increasing number of applications, investigations into the physical meaning of fractional viscoelastic models are significant, and are attracting increasing attention to reveal the hidden physical schemes of the complex phenomena of viscoelastic materials.

From the mathematical perspective, the physical meanings of fractional calculus were generally investigated with relation to non-local and memory effects (Du et al., 2013; Heymans and Podlubny, 2005; Moshrefi–Torbati and Hammond, 1998; Podlubny, 2001; Liang et al., 2019). For example, previous outstanding works from Moshrefi–Torbati and Hammond (1998) and Podlubny (2001) provided a physical interpretation of the Riemann–Liouville fractional derivative as memory. On the other hand, fractional viscoelasticity (see Eq. (1)) was introduced by Scott Blair (1944), who considered viscoelasticity as an intermediate state between the Hooke's elasticity and Newtonian

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Table 1
Schematic diagram for several viscoelastic models and their parameters.

Models	Schematic Diagram	Number of parameters
Fractional Maxwell (FM) model (Jaishankar and Mckinley, 2013)	 fx1.eps	3
Time-varying viscosity Maxwell (TVM) model (Buckingham, 2000)	 fx2.eps	3*
Fractal dashpot model (Cai et al., 2016)	 fx3.eps	2
Generalized Maxwell model (Zhang et al., 2017)	 fx4.eps	$2N+1$

* The number of parameters for the TVM model depends on the form of $\eta(t)$. In this paper, $\eta(t)$ is a function of parameters V and α; therefore, the TVM model has 3 parameters in total.

viscosity, where σ and ε are the stress and strain and α is the order of the fractional derivative. Subsequently, Bagley and Torvik (1983) linked the molecular theory for polymer solids with the fractional viscoelastic model with $\alpha = 0.5$. Wharmby and Bagley (2013) generalized this equivalence for $0 < \alpha < 1$. Metzler et al. (1999) reported that the stress relaxation of fractional viscoelastic models is related to the long-tailed continuous time random walk processes. These works enriched the physical meaning of fractional viscoelasticity.

$$\begin{aligned}&\text{Hooke's elasticity: } \sigma \propto \varepsilon\\ &\text{viscoelasticity: } \sigma \propto \frac{d^\alpha \varepsilon}{dt^\alpha},\ 0 < \alpha < 1\\ &\text{Newtonian viscosity: } \sigma \propto \frac{d\varepsilon}{dt}\end{aligned} \tag{1}$$

In addition, several scholars have bridged the fractional viscoelastic models to the classical viscoelastic models, as the latter have a clear physical meaning. Schiessel and Blumen (1995) and Heymans and Bauwens (1994) demonstrated that the SB model with $\alpha = 0.5$ can be hierarchically represented by springs and dashpots in ladders, trees, or fractal structures. Di Paola and Zingales (2012) established two hereditary models to describe the SB model with $\alpha < 0.5$ and $\alpha > 0.5$. Xiao et al. (2016) numerically validated that the fractional Zener model is equivalent to the generalized Maxwell model by adopting the same relaxation spectra. These prominent lines of research imply that the fractional viscoelastic models are equivalent to the classical models with a large number of viscoelastic elements.

Recently, Pandey and Holm (2016) linked the SB model to the Maxwell model with a linearly time-varying viscosity. Mainardi (2018) noted that the fractional relaxation equation is equivalent to the differential equation with a time-varying coefficient. In addition, viscoelastic models based on the Hausdorff derivative, which is considered as the first-order derivative coupled with a time-varying coefficient, have been reported that exhibit an equivalent viscoelastic response compared with the fractional viscoelastic models (Cai et al., 2016; Su et al., 2017). Therefore, time-varying viscosity may be an inherent property of the fractional viscoelasticity.

This work presents an investigation of the time-varying viscosity property of the fractional Maxwell (FM) model (see Table 1). The FM model is powerful and widely used in the science of viscoelasticity and rheology (Jaishankar and Mckinley, 2013). This is because it can mimic the initial elastic deformation in a creep test and the initial finite stress in a relaxation test, and describe the power-law behavior in a dynamic test for real viscoelastic materials. To reveal the time-varying viscosity property of the FM model, a time-varying viscosity Maxwell (TVM) model (see Table 1) is established. In this work, we obtain the time-varying viscosity functions by establishing equivalent creep and relaxation responses between the FM and the TVM models. The obtained time-varying viscosity functions are used to interpret the physical meaning of fractional viscoelastic models. The main abbreviations in this work are listed in Table 2. The rest of this paper is outlined as follows. Section 2 introduces the FM and the TVM models, and Section 3 establishes the equivalent viscoelasticity including the creep and relaxation between the two models. Section 4 clarifies the physical meaning of the fractional viscoelasticity according to the time-varying viscosity functions. Section 5 discusses the generality of the obtained time-varying viscosity functions and interprets the similar viscoelastic responses between the Hausdorff derivative models and the fractional derivative models. Section 6 presents the conclusions.

Table 2
List of abbreviations.

Abbreviation	Full name
FM	Fractional Maxwell
TVM	Time-varying viscosity Maxwell
SB	Scott–Blair
RL	Riemann–Liouville
ML	Mittag–Leffller

2. Fractional Maxwell (FM) model and time-varying viscosity Maxwell (TVM) model

The FM model (see Table 1) consists of a classical spring and a SB model in series. The constitutive equation of the FM model is

$$\sigma + \frac{V}{E}\frac{d^\alpha \sigma}{dt^\alpha} = V\frac{d^\alpha \varepsilon}{dt^\alpha},\ 0 < \alpha < 1, \tag{2}$$

where V is a material parameter with the SI unit of Pa•s$^\alpha$ (Jaishankar and Mckinley, 2013), E is the elastic modulus of the spring, and α is the order of the fractional derivative. In this paper, the Riemann–Liouville (RL) fractional derivative (Mainardi, 2010) is adopted by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau,\ m-1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t),\ \alpha = m, \end{cases}$$

where d^α/dt^α represents the RL fractional derivative operator, m is an integer not less than 1, and Γ is the Gamma function. The Laplace transform of the RL fractional derivative obeys the following equation if $f(t)$ is integrable near $t = = 0$ (Mainardi, 2010):

$$L\left[\frac{d^\alpha f(t)}{dt^\alpha}\right] = s^\alpha \tilde{f}(s), \tag{3}$$

where $\tilde{f}(s)$ is the transform function of $f(t)$. Taking the Laplace transform for Eq. (2), yields

$$\tilde{\sigma} = \frac{Vs^\alpha}{1 + \frac{V}{E}s^\alpha}\tilde{\varepsilon}. \tag{4}$$

Setting $\varepsilon = \varepsilon_0$, i.e., $\tilde{\varepsilon} = \varepsilon_0/s$ in Eq. (4) and taking the inverse Laplace transform, the relaxation modulus $G(t)$ of the FM model is obtained by

$$G(t) = EM_\alpha\left(-\frac{E}{V}t^\alpha\right), \tag{5}$$

where $M_\alpha(\bullet)$ is the Mittag–Leffller (ML) function with a single parameter (Mainardi, 2010). The creep compliance $J(t)$ of the FM model can also be derived by setting $\sigma = \sigma_0$:

$$J(t) = \frac{1}{E} + \frac{t^{\alpha}}{V\Gamma(1+\alpha)}. \tag{6}$$

The TVM model (see Table 1) is actually the classical Maxwell model with a time-varying viscosity. Time-varying viscosity fluid is a very common non-Newtonian fluid, such as printer ink, yogurt, peanut butter, xanthan gum solutions, pectin gels, and hydrogenated castor oil. To the best of our knowledge, the TVM model was reported for the first time by Buckingham (2000) to investigate the strain-hardening process. Recently, Pandey and Holm (2016) deduced that the TVM model is the bridge between the power-law relaxation and the Lomnitz creep law. The constitutive relation of the TVM model is given by

$$\sigma + \frac{\eta(t)}{E}\dot{\sigma} = \eta(t)\dot{\varepsilon}, \tag{7}$$

where $\eta(t)$ is the time-varying viscosity, and the dots over the variables represent the time difference d/dt. Setting $\sigma = \sigma_0$ in Eq. (7), the creep compliance of the TVM model can be derived by

$$J(t) = \frac{1}{E} + R(t), \tag{8}$$

where

$$R(t) = \int_{\tau=0}^{t} \frac{d\tau}{\eta(\tau)}. \tag{9}$$

On the other hand, substituting a constant strain ε_0 for ε in Eq. (7), the relaxation modulus of the TVM model can be deduced by

$$G(t) = Ee^{-ER(t)}. \tag{10}$$

3. Equivalent viscoelasticity between the FM model and the TVM model

3.1. Equivalent creep and relaxation

First, we investigate the equivalent creep between the FM model and the TVM model and compare the creep compliance of the FM model (see Eq. (6)) with that of the TVM model (see Eq. (8)). If the viscosity function $\eta_c(t)$ is chosen as

$$\eta_c(t) = V\Gamma(\alpha)t^{1-\alpha}, \tag{11}$$

the same creep responses between the FM model and the TVM model are achieved, where the subscript c stands for the creep. The creep viscosity function $\eta_c(t)$ is plotted in Fig. 1.

It can be seen from Fig. 1 that the viscosity coefficient η_c increases with time t in a power-law manner. The small order of fractional derivative α results in a quick increase in the viscosity coefficient η_c versus t. When α is close to zero, the viscosity coefficient η_c is large and tends to infinity. When α equals 1, η_c degenerates into a constant function over time, which corresponds to the Newtonian fluid.

On the other hand, we compare the relaxation modulus of the FM model (see Eq. (5)) with that of the TVM model (see Eq. (10)). If the viscosity function $\eta_r(t)$ is chosen as

$$\eta_r(t) = Vt^{1-\alpha}\frac{M_{\alpha}\left(-\frac{E}{V}t^{\alpha}\right)}{M_{\alpha,\alpha}\left(-\frac{E}{V}t^{\alpha}\right)}, \tag{12}$$

the same relaxation moduli for the two models are obtained. In the derivation process, a useful equation for the relaxation viscosity function η_r is also presented:

$$\exp(-ER_r(t)) = M_{\alpha}\left(-\frac{E}{V}t^{\alpha}\right), \tag{13}$$

where $R_r(t) = \int_{\tau=0}^{t} \frac{d\tau}{\eta_r(\tau)}$. The subscript r represents the relaxation, and $M_{\alpha,\beta}(\bullet)$ is the ML function with two parameters α and β (e.g., β may be equal to α, as in the function $M_{\alpha,\alpha}(\bullet)$ depicted in Eq. (12)) (Mainardi, 2010). It is noted that the following property was used in the above derivation process (Mainardi, 2010):

$$\frac{d}{dz}M_{\alpha}(z^{\alpha}) = z^{\alpha-1}M_{\alpha,\alpha}(z^{\alpha}). \tag{14}$$

The obtained relaxation viscosity function η_r is plotted in Fig. 2. From Fig. 2, it can be seen that the variation of η_r is similar to that of η_c.

3.2. Approximate relaxation and creep

The viscosity functions obtained by the above equivalent viscoelasticity lead to different but approximate relaxation and creep responses. Inserting the creep viscosity function η_c into Eq. (10), the relaxation modulus of the TVM model is obtained by

$$G(t) = E\exp\left(-\frac{Et^{\alpha}}{V\Gamma(1+\alpha)}\right). \tag{15}$$

According to the study from Metzler and Nonnenmacher (2003), the ML function has an asymptotic form similar to the stretched exponential function when t is close to zero:

$$M_{\alpha}(-\lambda t^{\alpha}) \sim \exp\left(-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}\right), \tag{16}$$

where λ is a constant, and $0 < \alpha < 1$. Considering the above property,

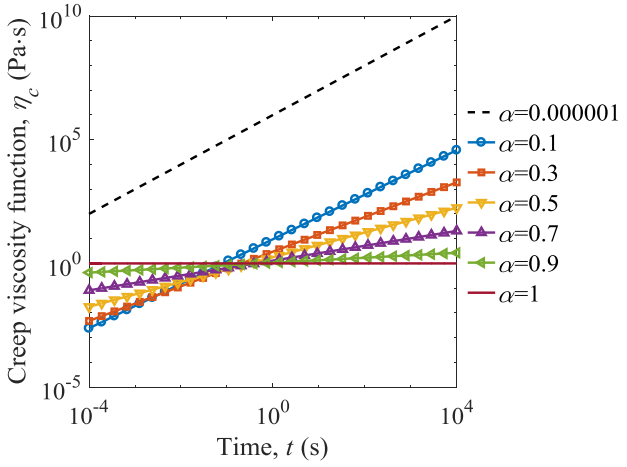

Fig. 1. Creep viscosity function η_c versus time t with different orders of fractional derivative α, where V is set to be 1 Pa•s$^{\alpha}$.

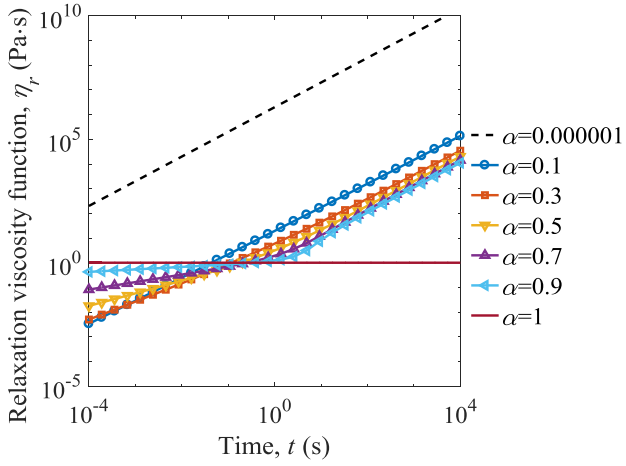

Fig. 2. Relaxation viscosity function η_r versus time t with different orders of fractional derivative α, where $E = = 1$ Pa, and $V = = 1$ Pa•s$^{\alpha}$.

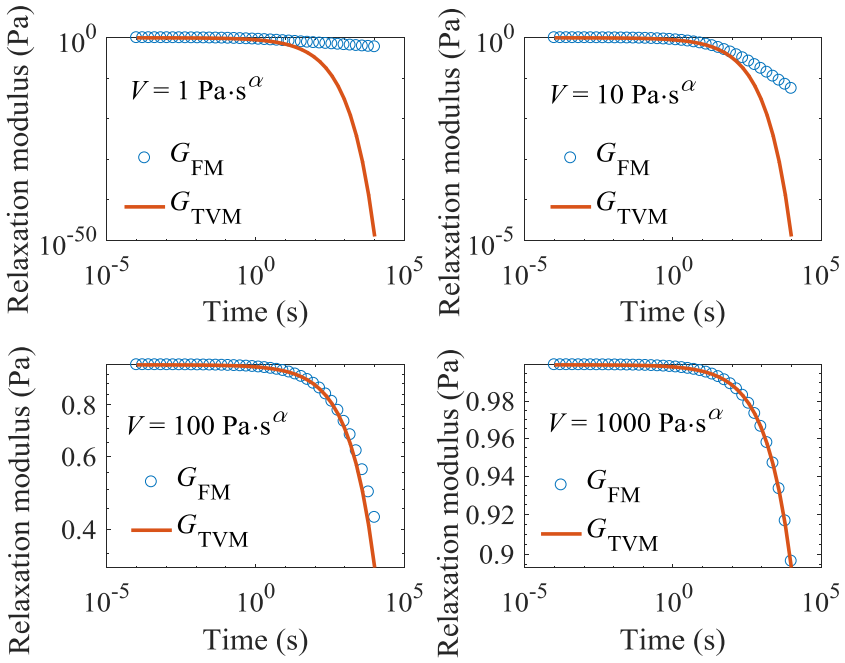

Fig. 3. Comparison of the relaxation modulus of the TVM model (G_{TVM}) and that of the FM model (G_{FM}) with four values of V, where $E = = 1$ Pa, and $\alpha = 0.5$. It is noted that the relaxation modulus of the TVM model is derived by adopting the creep viscosity function η_c. Note the largely distinct ranges of the moduli.

it can be concluded that the relaxation modulus of the TVM model (Eq. (15)) approximates that of the FM model (Eq. (5)) when the value of V is much larger than the value of E or t is close to zero, as shown in Fig. 3.

In addition, setting $\eta(t) = \eta_r$ in Eqs. (8) and (9), the creep compliance of the TVM model is derived as

$$J(t) = \frac{1}{E}\left\{1 - \ln\left[M_{\alpha}\left(-\frac{E}{V}t^{\alpha}\right)\right]\right\}. \tag{17}$$

According to Eq. (16), the creep compliance of the TVM model (Eq. (17)) approximates that of the FM model (Eq. (6)) when the value of V is much larger than the value of E or t is close to zero, as shown in Fig. 4.

3.3. Experimental validation

To validate the equivalent viscoelasticity between the FM model and the TVM model, we choose the rheological data of 'highly anomalous' butyl rubber reported by Scott Blair et al. (1947). The relaxation data is fitted by both the FM and the TVM models, and the obtained parameters are used to predict the corresponding creep data. The results are shown in Fig. 5. From Fig. 5, it can be seen that both the FM

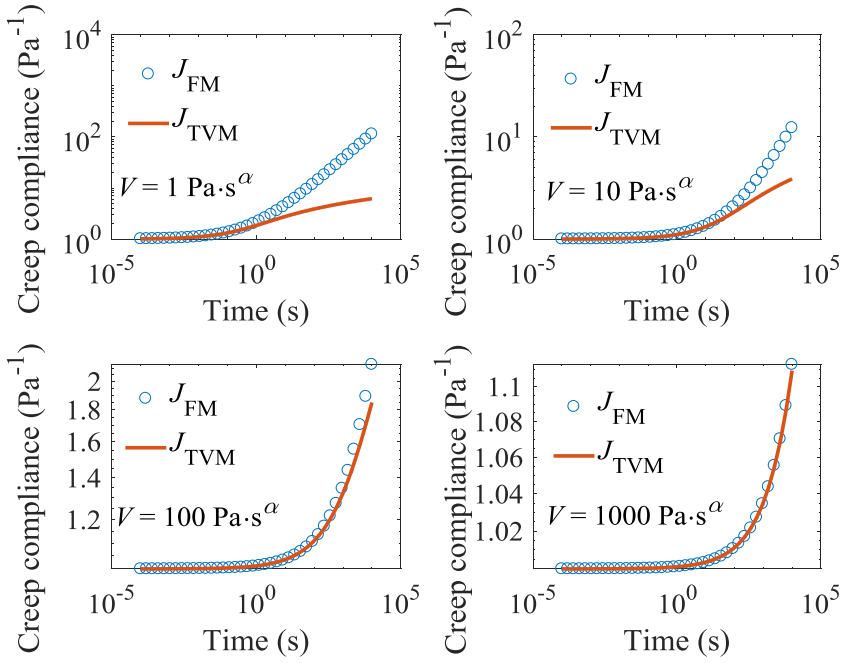

Fig. 4. Comparison of the creep compliance of the TVM model (J_{TVM}) and that of the FM model (J_{FM}) with four values of V, where $E = = 1$ Pa, and $\alpha = 0.5$. It is noted that the creep compliance of the TVM model is derived by adopting the relaxation viscosity function η_r.

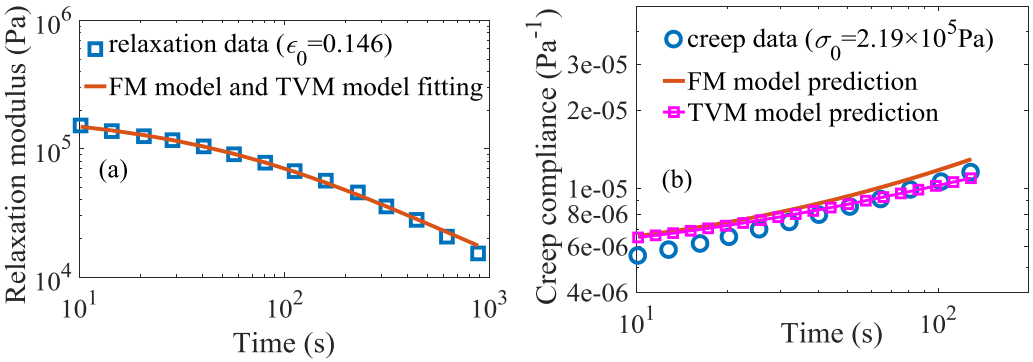

Fig. 5. (a) Relaxation data (Scott Blair et al., 1947) is fitted by the FM model and the TVM model; (b) creep data (Scott Blair et al., 1947) and the predicted values by the two models.

and the TVM models can fit the relaxation data well and capture the tendency of the creep for this viscoelastic material.

4. Time-varying viscosity— the bridge between the FM model and the TVM model

The above section explicitly illustrates that two specific time-varying viscosity functions (η_c and η_r) connect the equivalent viscoelasticity between the FM and the TVM models. As mentioned recently by Mainardi (2018), η_r approximates η_c when the value of V is much larger than the value of E or t is close to zero. Indeed, Fig. 6 shows that the two time-varying viscosity functions are closer to each other with the larger value of V (E is set to be 1 Pa).

4.1. Linking the viscoelasticity with the time-varying viscosity

The increased time-varying viscosity function not only relates to the rheopexy of non-Newtonian fluid but also exhibits a viscoelastic property. For example, creep is a very common viscoelastic property, and its typical curve is plotted in Fig. 7. The strain response under a constant stress σ_0 is usually divided into two parts as follows:

$$\varepsilon = \varepsilon_{\text{elastic}} + \varepsilon_{\text{creep}}, \tag{18}$$

where $\varepsilon_{\text{elastic}}$ and $\varepsilon_{\text{creep}}$ represent the transient elastic strain and the following time-dependent creep strain, respectively. In the field of rheology, the viscosity coefficient reflects the flow resistance (Barnes et al., 1989), which can be obtained by

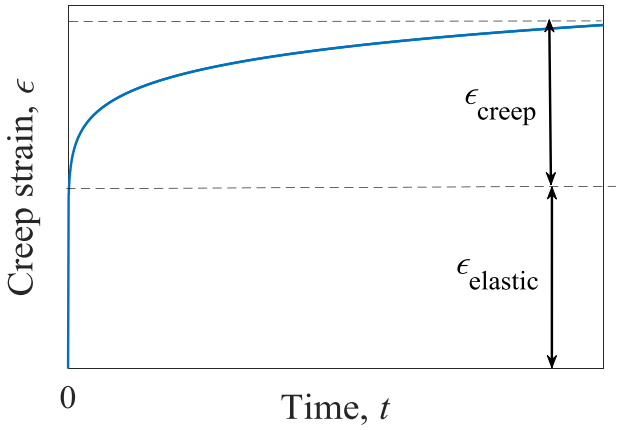

Fig. 7. Typical stable creep curve for a viscoelastic material.

$$\eta = \frac{\sigma}{\dot{\varepsilon}}, \tag{19}$$

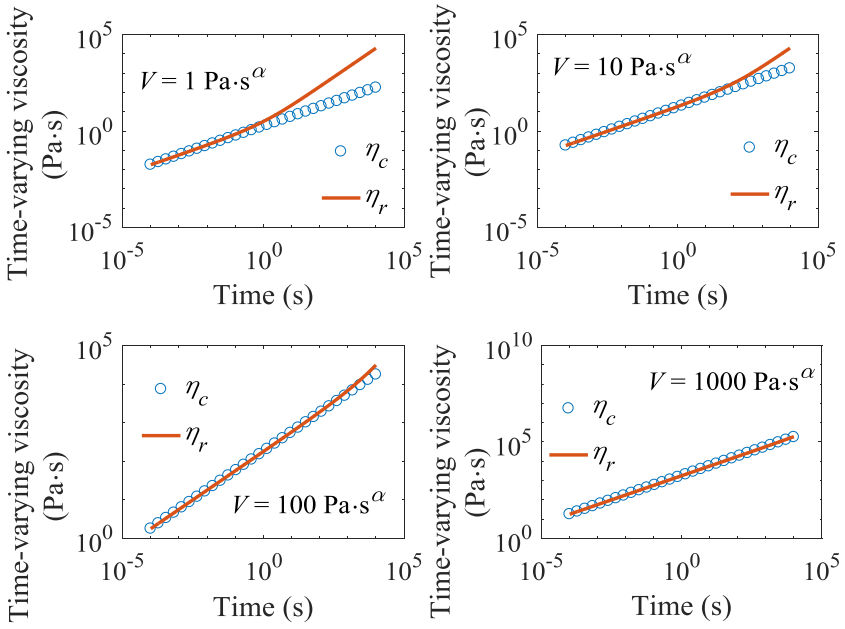

Fig. 6. Comparison of the two time-varying viscosity functions η_c and η_r with four values of V, where E = =1 Pa, and α = 0.5.

where $\dot{\varepsilon}$ is the strain rate. For the typical creep process (see Fig. 7), the initial elastic strain develops in a very short time, and the corresponding strain rate approximates infinity. According to Eq. (19), the initial viscosity coefficient tends to zero. As for the following creep strain, the strain rate decreases with time, which leads to an increased viscosity coefficient.

4.2. Interpreting the order of fractional derivative α from the time-varying viscosity

The order of fractional derivative α is usually regarded as an index of the viscoelastic property (Scott Blair, 1944). Hooke's elasticity and Newtonian viscosity are two limiting cases of the fractional viscoelasticity with $\alpha = 0$ and $\alpha = 1$, respectively. With α varying from 0 to 1, the fractional viscoelasticity is considered to continuously change from Hooke's elasticity to Newtonian viscosity. Di Paola and Zingales (2012) pointed out that elasticity plays the leading role with $\alpha < 0.5$, while viscosity has the major role with $\alpha > 0.5$. Metzler and Nonnenmacher (2003) related the order α to the rheological property with a modified Deborah number for fractional relaxation processes. The Deborah number De is defined as

$$\mathrm{De}=\frac{1}{\alpha}, \tag{20}$$

where the large order α relates to the fluid property, and the small α relates to solid-like materials from the physical meaning of the Deborah number.

Figs. 1 and 2 both show that a small α results in a large value and quick increase of the viscosity coefficient, which corresponds to a large deformation resistance. To illustrate this more clearly, Figs. 8 and 9 exhibit η_c and η_r as the functions of the order of fractional derivative α at $t = = 1$, 10, 100 and 1000s. In addition, Fig. 8 compares η_c at $t = = 1$ s with the Deborah number. From Figs. 8 and 9, it can be seen that the viscosity coefficients η_c and η_r change over α in a similar manner. When α is close to zero, the viscosity coefficient is infinite, i.e., the elastic solid property is indicated. Indeed, real solid materials exhibit very high viscosity coefficients. For example, the viscosity coefficient of the upper mantle is approximately 10^{21} Pa•s (Karato and Wu, 1993), which is 10^{24} times greater than the viscosity coefficient of water (Collier and White, 1990). In addition, the infinite viscosity coefficient corresponding to an elastic solid property can also be deduced from Eq. (19). For an elastic solid, the strain rate is zero when the solid is under a constant stress, which leads to the infinite viscosity coefficient according to Eq. (19). The viscosity coefficient decreases quickly with α when α approaches 0, which means the elastic solid property decays quickly near $\alpha = 0$. With further increases in α, the viscosity coefficient continues to decrease, and its change rate is reduced. This shows that the resistance of flow diminishes, which is a fluid-like property. When α approaches 1, the viscosity coefficient changes little with α and time t, at which point a Newtonian fluid is obtained.

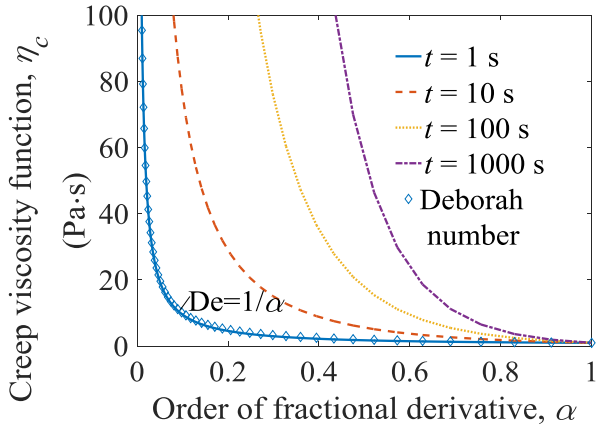

Fig. 8. Creep viscosity function η_c as a function of the order of fractional derivative α at $t = = 1$, 10, 100, and 1000s, and a comparison of η_c at $t = = 1$ s and the Deborah number, where $V = = 1$ Pa•s$^{\alpha}$.

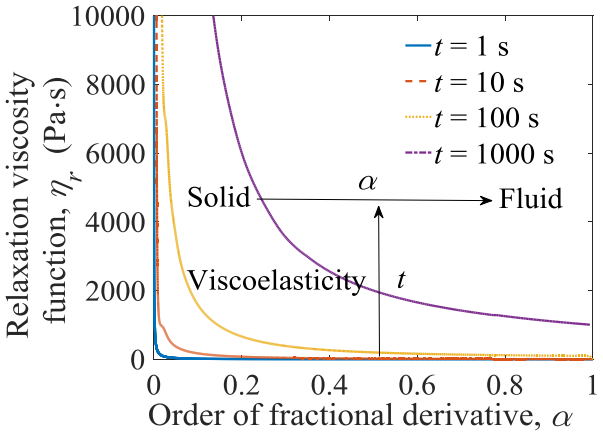

Fig. 9. Relaxation viscosity function η_r as a function of the order of fractional derivative α at $t = = 1$, 10, 100, and 1000s, where $V = = 1$ Pa•s$^{\alpha}$ and $E = = 1$ Pa. It is noted that the viscosity functions exhibit viscoelasticity over time, and the viscoelasticity transforms from an elastic solid to a Newtonian fluid as α changes from 0 to 1.

From Fig. 8, it can be seen that the generalized Deborah number is very close to η_c at $t = = 1$ s, which can be considered as a special case of the viscosity functions. In addition, the increased viscosity coefficient over time exhibits viscoelasticity according to Section 4.1, and when α changes from 0 to 1, the viscosity coefficient reveals the transformation of the viscoelasticity from an elastic solid to a Newtonian fluid, as illustrated in Fig. 9.

5. Discussion

This work aims to investigate the physical meaning of fractional viscoelastic models by establishing the equivalent viscoelastic creep and relaxation responses between the FM model and the TVM model. The obtained time-varying viscosity functions are used to interpret the physical meaning. Although the time-varying viscosity functions are obtained by comparing the FM model with the TVM model, the similar time-varying viscosity functions between other fractional viscoelastic models and classical time-varying viscosity models can also be derived. For example, the equivalent creep response between the fractional Kelvin–Voigt model and the time-varying viscosity Kelvin–Voigt model can be linked by the time-varying viscosity function:

$$\eta_{\mathrm{c\_Kelvin}}(t) = Vt^{1-\alpha}\frac{M_{\alpha}\left(-\frac{E}{V}t^{\alpha}\right)}{M_{\alpha,\alpha}\left(-\frac{E}{V}t^{\alpha}\right)}, \tag{21}$$

which is the same as the relaxation viscosity function η_r in Eq. (12). In addition, $\eta_{\mathrm{c\_Zener}}$ and $\eta_{\mathrm{r\_Zener}}$ can be obtained by comparing the creep and relaxation of the fractional Zener model with that of the time-varying viscosity Zener model, as expressed below:

$$\begin{aligned}\eta_{\mathrm{c\_Zener}} &= Vt^{1-\alpha}\frac{M_{\alpha}\left(-\frac{E_2}{V}t^{\alpha}\right)}{M_{\alpha,\alpha}\left(-\frac{E_2}{V}t^{\alpha}\right)}\\ \eta_{\mathrm{r\_Zener}} &= Vt^{1-\alpha}\frac{M_{\alpha}\left(-\frac{E_1+E_2}{V}t^{\alpha}\right)}{M_{\alpha,\alpha}\left(-\frac{E_1+E_2}{V}t^{\alpha}\right)},\end{aligned} \tag{22}$$

where E_1 and E_2 are the elastic moduli of the two springs. It can be seen that all these time-varying viscosity functions are in similar forms (see

Eqs. (12), (21) and (22) and Fig. 6). Therefore, it is feasible to determine the time-varying viscosity property of fractional viscoelastic models by merely comparing the FM model with the TVM model.

In addition, similar viscoelastic responses between Hausdorff derivative models and fractional derivative models have been reported (Cai et al., 2016; Su et al., 2017). Actually, the basic viscoelastic element fractal dashpot model (see Table 1) exhibits a power-law equivalent viscosity according to the property of the Hausdorff derivative (Su et al., 2017)

$$\eta(t) = \frac{\xi_{\mathrm{H}}}{\beta} t^{1-\beta}, \tag{23}$$

where ξ_{H} and β are two parameters of the fractal dashpot model. This expression is almost the same as the creep viscosity function η_c in Eq. (11), which confirms the similar viscoelastic creep responses between the fractal dashpot model and the SB model. Similar viscoelastic responses of other complex models based on the Hausdorff derivative and the fractional derivative were also discussed by Cai et al. (2016).

This work obtains the time-varying viscosity functions by establishing the static equivalent viscoelastic responses between the FM and the TVM models. By analyzing the obtained time-varying viscosity functions, we manifest the transformation of fractional viscoelasticity from elastic solid to Newtonian fluid when the order of fractional derivative α changes from 0 to 1.

6. Conclusion

This paper established the equivalent creep and relaxation responses between the FM and the TVM models. Two different time-varying viscosity functions were derived from the above equivalences, and they led to approximate relaxation (creep) responses between the FM and the TVM models. On the one hand, the results revealed that the obtained viscosity coefficients increase with time, which exhibits viscoelasticity. On the other hand, the physical meaning of the order of fractional derivative α was given from the viewpoint of rheology. These time-varying viscosity functions quantitatively show the transformation of viscoelasticity from the elastic solid to the Newtonian fluid as α changes from 0 to 1. The infinite viscosity coefficient with $\alpha = 0$ indicates the elastic solid property, and the constant viscosity coefficient with $\alpha = 1$ indicates the Newtonian fluid property. In addition, the similar viscoelastic properties of the Hausdorff derivative models and the fractional derivative models were also presented from the similar equivalent time-varying viscosities. Consequently, the time-varying viscosity enriched the physical meaning of the fractional viscoelasticity, which has enhanced our confidence to use fractional viscoelastic models.

Declaration of Competing interest

There are no conflicts of interest.

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