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Boundary Behavior of Harmonic Functions in Non-tangentially Accessible Domains

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1. INTRODUCTION

The basic aim of this paper is to extend classical results on the boundary behavior of harmonic functions in $\mathbb{R}^{m+1}_+ = \{(x, y): x \in \mathbb{R}^m, y \in \mathbb{R}, y > 0\}$ to domains D of as general a type as possible. Many of our results also hold for solutions of uniformly elliptic equations, in divergence form, with bounded measurable coefficients. We begin by recalling some classical theorems and more recent results that form the background of our paper.

A classical theorem of Fatou says that if $u(z)$ is a bounded harmonic function on the unit disc, $|z| < 1$, then u has non-tangential boundary values almost everywhere on the unit circle, S^1. The same conclusion holds if u is only bounded from below. These results have a local analogue due to Privalov [44]. In fact, if $u(z)$ is harmonic in the unit disc, and at each point $e^{i\theta}$ of a measurable subset E of the unit circle there is an $\alpha > 0$ such that $u(z)$ is bounded from below on the set $\Gamma_\alpha(e^{i\theta}) = \{z: |z| < 1, |z - e^{i\theta}| < (1+\alpha)\operatorname{dist}(z, S^1)\}$, then $u(z)$ has a non-tangential limit at almost every $e^{i\theta}$ in E, i.e., u restricted to $\Gamma_\beta(e^{i\theta})$ has a limit as $z \to e^{i\theta}$ for any β. This local result was first proved using conformal mapping; thus the extension to higher dimensions required new ideas.

In 1950, Calderón showed [7] that if u is a harmonic function on $\mathbb{R}^{m+1}_+$, which is non-tangentially bounded at every point x of a measurable set $E \subset \mathbb{R}^m = \partial\mathbb{R}^{m+1}_+$ (i.e., such that given $x \in E$, there exist α, h, M such that $|u(x)| \leqslant M$ for all $X \in \Gamma^h_\alpha(x) = \{Y \in \mathbb{R}^{n+1}_+: |Y - x| \leqslant (1+\alpha)\operatorname{dist}(Y, \mathbb{R}^n),$

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$|Y-x|\leqslant h\}$), then u has a non-tangential limit at almost every x in E. In 1962, Carleson [10] obtained the same conclusion, but with the hypothesis of non-tangential boundedness replaced by non-tangential boundedness from below. Both proofs used a so-called sawtooth region over E, namely, $R=\bigcup_{x\in E}\Gamma_\alpha^h(x)$. In this region, the harmonic function u is bounded, or bounded from below, thus reducing the local question on a nice domain ($\mathbb{R}_+^{n+1}$) to a global question in the domain R, which is a Lipschitz domain. In Carleson's proof, the harmonic measure (defined below (1.1)) for the domain R appears for the first time. Carleson's technique played a key part in subsequent developments.

Inspired, in part, by the recurring appearance of the regions R mentioned above, Stein [46] posed the question of extending these results (as well as others) to the most general domains D "for which non-tangential behavior is meaningful." In 1968 and 1970, Hunt and Wheeden [25, 26] took up this question and proved that if D is a bounded Lipschitz domain in $\mathbb{R}^n$, and u is a harmonic function on D that is non-tangentially bounded from below at every Q in a subset E of D, then u has non-tangential limits at almost every point of E relative to harmonic measure. This result implies the theorem of Calderón and Carleson, and its proof was based on ideas developed by Carleson in [10].

One of the results of this paper is an extension of the theorem above to a class of domains in $\mathbb{R}^n$ that we call non-tangentially accessible (NTA) domains. Their main property is that every boundary point is accessible from inside and outside the domain by means of non-tangential balls. We call the union of these twisting non-tangential balls corkscrews (see (3.1)). The boundaries of these domains are not necessarily rectifiable and need not have tangent planes at any point. Examples of such domains are Lipschitz domains, Zygmund domains, and quasispheres. (See [25, (2.6); 53] for precise definitions.) Our first result is:

(6.4) THEOREM. *Let D be a bounded, non-tangentially accessible domain, and let u be harmonic in D. The set of points of D where u is non-tangentially bounded from below equals a.e. with respect to harmonic measure the set where u has a non-tangential limit.*

We now turn to a discussion of other ways of describing the boundary behavior of harmonic functions. The property of being non-tangentially bounded, or bounded from below, is of an elusive nature, and difficult to pin down analytically. In an effort to overcome this problem, Marcinkiewicz and Zygmund [40] and Spencer [45] showed that, in the case of the circle, $u(z)$ has a nontangential limit a.e. $d\theta$ in E if and only if the area integral, $A_\alpha(u)(e^{i\theta})^2=\int_{\Gamma_\alpha(e^{i\theta})}|\nabla u(x+iy)|^2\,dx\,dy$, is finite a.e. $d\theta$ in E. Also, in 1950, Calderón [8] showed that if u is non-tangentially bounded on $E\subset\mathbb{R}^n$, then

$A_\alpha(u)(x)^2 = \int_{\Gamma^h_\alpha(x)} \operatorname{dist}(X, \mathbb{R}^m)^{1-m} |\nabla u(x)|^2 \, dX$ is finite a.e. dx in E. In 1961, Stein [46] obtained the converse result, and gave applications to conjugate harmonic functions.

In this paper we extend these results to *NTA* domains. We obtain:

(6.6) THEOREM. *Let D be a bounded, non-tangentially accessible domain, and let u be harmonic in D. The set of points of ∂D where the area integral of u is finite equals a.e. with respect to harmonic measure the set of points where u has a non-tangential limit.*

In 1978, in [16], Dahlberg took up the study of the area integral of a harmonic function, in a Lipschitz domain D contained in $\mathbb{R}^n$. (See also [33] for results when $n = 2$.)

The area integral is defined by $A_\alpha(u)^2\,(Q) = \int_{\Gamma_\alpha(Q)} \operatorname{dist}(X, \partial D)^{2-n} |\nabla u(X)|^2 \, dX$. The non-tangential maximal function is defined by $N_\alpha(u)(Q) = \sup_{X \in \Gamma_\alpha(Q)} |u(X)|$. One of Dahlberg's result is that $N_\alpha(u)$ and $A_\alpha(u)$ have comparable $L^p(\partial D, d\sigma)$ norms, $0 < p < \infty$, where $d\sigma$ denotes surface area of ∂D. He accomplished this by proving so-called "good λ" inequalities between $N_\alpha(u)$ and $A_\alpha(u)$ (see [3] for the corresponding results in $\mathbb{R}^{n+1}_+$). A key tool in Dahlberg's proof, is the fact, established by him in [15], that harmonic measure and surface measure are mutually absolutely continuous on Lipschitz domains; moreover, harmonic measure belongs to the Muckenhoupt class $A_\infty(d\sigma)$ (see [12]). Recall that a measure belongs to $A_\infty(d\nu)$, where $d\nu$ is a measure on ∂D, if there exist $\alpha, \beta \in (0, 1)$, such that for all Borel sets $E \subset \Delta$, Δ a surface ball of ∂D (i.e., Δ is the intersection of a Euclidean ball with center at a boundary point, with ∂D), $\nu(E)/\nu(\Delta) < \alpha \Rightarrow \mu(E)/\mu(\Delta) < \beta$. It was shown in [12] that μ belongs to $A_\infty(d\nu)$ if and only if ν belongs to $A_\infty(d\mu)$.

This brings us to another source of ideas for this paper. When $n = 2$, the property that harmonic measure belongs to $A_\infty(d\sigma)$ holds for an even more general class of domains than Lipschitz, namely, chord-arc domains. A domain is called a chord-arc domain if it is bounded by a rectifiable Jordan curve C in $\mathbb{R}^2$, and there exists a constant M, such that for any z_1, z_2 in C, the length of the arc in C between z_1 and z_2 of smaller diameter is less than $M|z_1 - z_2|$. The A_∞ property of harmonic measure, for chord-arc domains, was shown in 1936 by Lavrentiev [36], many years before the A_∞ condition was systematically studied. The A_∞ property of harmonic measure is a main ingredient in the proof of the theorem of Calderón [9] on the Cauchy integral on Lipschitz curven. A generalization, due to Coifman and Meyer [13], to curves satisfying the chord-arc condition (provided the constant M is sufficiently close to 1), also relies on A_∞. Coifman and Meyer also give a new proof of Lavrentiev's result, in the case when M is sufficiently close to 1. On the other hand, it is shown in [27, 28] that if D is an L^p_1 domain in

$\mathbb{R}^n$, with $p > n-1$, then harmonic measure on ∂D and surface measure on ∂D are mutually absolutely continuous. (An L_1^p domain is a domain whose boundary is given locally in some C^∞ coordinate system as the graph of a function ϕ, with $\nabla\phi$ in L^p.) It is natural then to seek an analogue of the chord-arc domains in higher dimensions. This analogue is provided by the observation of Coifman and Meyer that if $\phi' \in BMO$ then $(x, \phi(x))$, $x \in \mathbb{R}$ satisfies the chord-arc condition. Taking this point of view, we define BMO_1 domains in $\mathbb{R}^n$ as domains whose boundaries are given locally in some C^∞ coordinate system as the graph of a function ϕ with $\nabla\phi \in BMO$. BMO_1 domains turn out to be a subclass of NTA domains. For them, we have:

(10.1) THEOREM. *Let D be a BMO_1 domain. Then the harmonic measure for D belongs to $A_\infty(d\sigma)$.*

This theorem has applications to the Dirichlet problem on BMO_1 domains (see (10.1)) and to area integral estimates generalizing those of Dahlberg.

Before describing further results and background material, we need to give the precise definition of harmonic measure. Let D be any bounded domain in $\mathbb{R}^n$. Let f be defined on ∂D. The upper class of functions $U_f = \{u;\, u$ is either identically $+\infty$ on D, or u is superharmonic in D, u is bounded from below, and $\liminf_{X \in Q} u(X) \geqslant f(Q)$ for all $Q \in \partial D\}$. The lower class $L_f = \{-u;\, u$ is an upper function of $-f\}$. For any such f, define $\overline{H}f(X) = \inf\{u(X), u \in U_f\}$, the upper solution of the generalized Dirichlet problem for f. Also $\underline{H}f(X) = \sup\{u(X), u \in L_f\}$ is the corresponding lower solution. If $\overline{H}f(X) = \underline{H}f(X)$ for every X on D, and $\Delta(\overline{H}f) = 0$ on D, f is called a resolutive boundary function. In that case, we set $Hf(X) = \overline{H}f(X) = \underline{H}f(X)$. Wiener [54] showed that every continuous real valued function on ∂D is resolutive. This fact, and the maximum principle makes it possible to define harmonic measure.

(1.1) DEFINITION. The unique probability Borel measure on ∂D, denoted ω^X, such that for all continuous functions f on ∂D, $Hf(X) = \int_{\partial D} f\, d\omega^X$, is called the *harmonic measure* of D, evaluated at X.

(1.2) DEFINITION. A bounded domain D is called *regular* for the Dirichlet problem if, given any $f \in C(\partial D)$, $Hf(X) \in C(\overline{D})$, and $Hf(Q) = f(Q)$ for every $Q \in \partial D$.

We note that as a consequence of Harnack's inequality, for any X_1, $X_2 \in D$, and any domain D, the measures ω^{X_1} and ω^{X_2} are mutually absolutely continuous.

(1.3) DEFINITION. Fix a point $X_0 \in D$, and denote $\omega = \omega^{X_0}$. Then, the *kernel function* $K(A, Q) = (d\omega^A/d\omega)(Q)$, the Radon–Nikodym derivative, which exists by the previous remark.

A priori, $K(A, Q)$ is only defined for almost every $(\omega)\, Q$. However, for *NTA* domains we have:

(7.1) THEOREM. *Let D be a bounded NTA domain. Then for fixed $A \in D$, $K(A, Q)$ is a Hölder continuous function of Q on ∂D.*

This result has applications to the theory of Hardy spaces H^p on *NTA* domains *D*. It is new even for C^1 and Lipschitz domains. Fabes has also, independently, obtained this result for C^1 domains. His proof, however, does not generalize to Lipschitz or *NTA* domains. We now explain the background for the applications to H^p spaces.

In 1960, in [49], Stein and Weiss generalized, to the context of $\mathbb{R}^{m+1}_+$, the classical notion of Hardy spaces of analytic functions in the unit disc. They defined $H^p(\mathbb{R}^{m+1}_+)$ to be the set of vectors $\mathbf{u} = (u_0, u_1, ..., u_m)$ of harmonic functions, satisfying the generalized Cauchy–Riemann equations (these equations are equivalent to $\mathbf{u} = \nabla U$, U harmonic in $\mathbb{R}^{m+1}_+$), such that the nontangential maximal function of u belongs to $L_p(\mathbb{R}^m, dx)$. To each such vector u, they associated the trace of u_m on $\mathbb{R}^m$, and denoted by $H^p(\mathbb{R}^m)$ the set of traces. It is well known that for $p > 1$, $H^p(\mathbb{R}^m) = L^p(\mathbb{R}^m)$. For in 1970, Burkholder *et al.* [4] showed that if $0 < p < \infty$ and u has non-tangential maximal function in $L^p(\mathbb{R})$, then its harmonic conjugate has the same property. Thus, $f \in H^p(\mathbb{R})$ if and only if its harmonic extension u has a non-tangential maximal function in $L^p(\mathbb{R})$. In 1972, Fefferman and Stein [21, 22] extended this result to $n > 1$, and showed that $H^1(\mathbb{R}^m)^* = BMO$, the space of functions of bounded mean oscillation introduced by John and Nirenberg [30]. This duality result led Fefferman to a further characterization of $H^1(\mathbb{R}^m)$, in terms of atoms. An atom a is a function supported in a ball Δ, with $\int a\, dx = 0$, and $\|a\|_\infty \leqslant 1/|\Delta|$. Fefferman showed that $f \in H^1(\mathbb{R}^m)$ if and only if $f = \sum \lambda_i a_i$, where the a_i's are atoms, and $\sum |\lambda_i| < +\infty$. Coifman [11] for $m = 1$, and Latter [35] for $m > 1$ proved the corresponding result for $H^p(\mathbb{R}^m)$, $p < 1$. Recently, the atomic theory of H^p spaces has been extended to the abstract setting of spaces of homogeneous type. (See [14, 39, 52].)

The theory of H^p spaces on non-smooth domains began in [33, 34], for Lipschitz domains in two dimensions. There, they were considered in the context of analytic functions, and the analogous of the results mentioned above were found. A new feature that arose in this study is that, if one takes the point of view of the Burkholder–Gundy–Silverstein theorem, and defines H^1 in terms of harmonic functions whose non-tangential maximal function is in L^1, there are two natural measures to consider, the arc length measure and the harmonic measure. In the first case, the dual space is not the analogue of *BMO*, but a weighted *BMO* space (see [34, 42]), while the dual of the second one is *BMO*. Thus, Hardy spaces relative to harmonic measure are more natural.

For a higher dimensional version, denote $H^1(D, d\omega) = \{u: N_\alpha(u) \in L^1(d\omega)\}$. ($H^p(D, d\omega)$ is defined analogously.) In [19], an atomic decomposition for the set of $f = u|_{\partial D}$, $u \in H^1(D, d\omega)$ as well as a duality theorem with BMO, for C^1 domains, was proved. The atomic decomposition of $H^1(D, d\omega)$ and the BMO duality, for D merely Lipschitz, and $n > 2$, thus remained open. Other notions of H^1 are studied in [17, 18, 33].

In this paper we study $H^1(D, d\omega)$ for an arbitrary *NTA* domain D. This is the appropriate notion of H^1 in these domains because their boundaries need not have a surface measure. We prove here the atomic decomposition of $H^1(D, d\omega)$ and duality with BMO for any *NTA* domain (and hence in particular for any Lipschitz domain). We have:

(8.13) THEOREM. *Let D be a bounded NTA domain. Then, $u \in H^1(D, d\omega)$ if and only if there exists $f \in L^1(d\omega)$, with $f = \sum \lambda_i a_i$, $\sum |\lambda_i| < +\infty$, $\operatorname{supp} a_i \subset \Delta_i$, $\int a_i\, d\omega = 0$ and $\|a_i\|_\infty \leqslant 1/\omega(\Delta_i)$, such that $u(X) = \int_{\partial D} f\, d\omega^X$. Moreover, $H^1(D, d\omega)^* = BMO$.*

We also prove that if $f \in L^1(d\omega)$, and $u(X) = \int f\, d\omega^X$, then, there exists $p_0 < 1$, depending only on D, such that for $p_0 < p \leqslant 1$, the $H^p(D, d\omega)$ norm of u is comparable to the inf of $(\sum |\lambda_i|^p)^{1/p}$, over all decomposition of f as $\sum \lambda_i a_i$, where the a_i's are p-atoms (see (8.6) for the definition of p-atoms).

NTA domains are closely related to the theory of quasiconformal mappings. In this connection, we refer to the work of Ahlfors [1] for $n = 2$, John [29], Jones [31, 32], and Gehring and Osgood [24] for higher dimensions.

We will now discuss some of the main methods and lemmas used in our paper. A key property of *NTA* domains is their dilation invariance, i.e., if D is an *NTA* domain contained in $\mathbb{R}^n$, and we dilate $\mathbb{R}^n$, the resulting domain will also be an *NTA* domain with the same *NTA* constant as D. This fact is exploited by means of the following geometric localization theorem, due to Jones [32]:

(3.11) THEOREM. *If D is a bounded NTA domain, then there exists an r_0, depending only on D, such that for any $Q \in \partial D$, and $r < r_0$, there exists an NTA domain $\Omega \subset D$, such that $B(Q, M^{-1}, r) \cap D \subset \Omega \subset B(Q, Mr) \cap D$. (Here $B(Q, s)$ is the ball of radius s centered at Q.) Furthermore, the constant M in the NTA definition for Ω (see Section 3) is independent of Q and r.*

This theorem extends previous versions due to the authors in the case of Zygmund domains and in the case of quasispheres (see Appendix). This geometric localization replaces in our work, the local starshapedness of Lipschitz domains, which was a key ingredient in the work of Carleson, Hunt and Wheeden, and Dahlberg. (See [10, 15, 25].)

The main lemmas on harmonic measures that we use are the following:

(4.8) LEMMA. *If* $2r < r_0$ *(here* r_0 *is some fixed constant that depends only on* D*), and* $X \in D \backslash B(Q, 2r)$*, then* $M^{-1} < \omega^X(\Delta(Q, r))/r^{n-2} |G(A_r(Q), X)| < M$*, where* $\Delta(Q, r) = B(Q, r) \cap \partial D$*,* $G(X, Y)$ *is the Green function of* D*, and* $A_r(Q)$ *a point in* D*, with* $M^{-1}r < |A_r(Q) - Q| < Mr$*, and* $\operatorname{dist}(A_r(Q), \partial D) > M^{-1}r$ *(see* (3.1)).

This lemma is the generalization to *NTA* domains of a lemma of Dahlberg for Lipschitz domains [15]. It is used to prove the doubling condition for harmonic measure (see below) and the boundary Harnack principle (see below). It is also very useful in the proof of the area integral theorem (see (6.6) and (9.1)).

(4.9) LEMMA (Doubling condition). $\omega^X(\Delta(Q, 2r)) \leqslant C_X \omega^X(\Delta(Q, r))$.

This lemma is needed (for example) in order to have the usual weak type (1, 1) and L^p estimates for the Hardy–Littlewood maximal function associated to the measure ω. It also implies that ∂D is a space of homogeneous type (see (8.5)).

(4.11) LEMMA (Carleson–Hunt–Wheeden lemma). *Let* $\Delta = \Delta(Q_0, r)$*,* $r < r_0$*. Let* $\Delta' = \Delta(Q, s) \subset \Delta(Q_0, r/2)$*. If* $X \in D \backslash B(Q_0, 2r)$*, then* $\omega^{A_r(Q_0)}(\Delta') \simeq \omega^X(\Delta')/\omega^X(\Delta)$*. (The notations are as in the first lemma stated here.)*

This lemma was proved by Carleson for sawtooth regions (see 3.3 of [10]), and by Hunt and Wheeden [25, 26] for Lipschitz domains. It is the tool that allows us to pass from quantitative global results to quantitative local results, by means of iterations and geometric localization. For example, it allows us to deduce, from the continuity in Q of the kernel function $K(A, Q)$ (5.5), its Hölder continuity in Q (see (7.1)), and from the absolute continuity of harmonic measure with respect to surface measure on a chord-arc or BMO_1 domain, it allows us to deduce that harmonic measure belongs to the class $A_\infty(d\sigma)$ (see (10.1) and (2.1)).

The idea of proving the lemmas above in the stated order comes from Caffarelli *et al.* [6].

Other results which follow from the localization technique are the following:

(5.1) THEOREM (Boundary Harnack principle). *Let* D *be an NTA domain, and let* V *be an open set. For any compact set* $K \subset V$*, there exists a constant* C *such that for all positive harmonic functions* u *and* v *in* D *that vanish continuously on* $\partial D \cap V$*,* $u(X_0) = v(X_0)$ *for some* $X_0 \in D \cap K$ *implies that* $C^{-1}u(X) < v(X) < Cu(X)$ *for all* $X \in K \cap \bar{D}$*.*

This theorem was proved in [15, 55] for Lipschitz domains. We also obtain the following refinement of it, which is new even in the case of C^1 and Lipschitz domains, and which answers a question posed by Wu.

(7.9) THEOREM. *Let D be an NTA domain, and let V be an open set. Let K be a compact subset of V. There exists a number $\alpha > 0$, such that for all positive harmonic functions u and v in D, that vanish continuously on $\partial D \cap V$, the function $u(X)/v(X)$ is Hölder continuous of order α on $K \cap \bar{D}$. In particular, for every $Q \in K \cap \partial D$, $\lim_{X \to Q}(u(X)/v(X))$ exists.*

Another useful result is the following:

(5.9) THEOREM. *If D is an NTA domain, then the Martin boundary of D is the Euclidean boundary of D.*

From the above, and the general representation theorem of Martin [41], we obtain:

(5.10) THEOREM. *Let D be an NTA domain. If u is a positive harmonic function in D, there exists a unique positive Borel measure μ on ∂D such that $u(X) = \int_{\partial D} K(X, Q)\, d\mu(Q)$, for $X \in D$.*

In addition to the lemmas above, and the localization technique mentioned before, we use many of the techniques of [6, 10, 15, 25, 26].

The plan of the paper is as follows: In Section 2 we explain how we were led to the notion of *NTA* domain and illustrate the localization technique. We also give several characterizations of *NTA* domains in two dimensions. Section 3 is devoted to the definition of *NTA* domains, and some geometric consequences of the definition. We also prove in this section that every Zygmund domain is an *NTA* domain. In Section 4 we establish the main lemmas on harmonic measure. Section 5 is devoted to the global boundary behavior of harmonic functions. We prove that if u is positive and harmonic in the *NTA* domain D, then u has non-tangential limits a.e. (ω) on D. We also establish a global theorem (5.14) related to the area integral. In Section 6 we prove the local analogue of Fatou's theorem, as well as the local area integral theorem, for *NTA* domains. Section 7 is devoted to the study of the Hölder continuity properties of ratios of harmonic functions that vanish on a piece of the boundary. In particular, we prove here the Hölder continuity of the kernel function. Section 8 is devoted to the atomic decomposition of $H^1(D, d\omega)$, duality with $BMO(\partial D)$, and results on $H^p(D, d\omega)$, $p_0 < p \leqslant 1$. Section 9 treats $L^p(d\omega)$, $1 < p < \infty$, estimates for the area integral, as well as a Carleson measure characterization of $BMO(\partial D)$. In Section 10 we show that in a BMO_1 domain, harmonic measure is in $A_\infty(d\sigma)$. We also give applications to the Dirichlet problem in BMO_1 domains, with boundary data in $L^p(d\sigma)$, p sufficiently large. Section 11 deals with some further results and open questions. Finally, in the Appendix we give a proof of the geometric localization theorem (3.11) for quasispheres and Zygmund domains. In the case of Zygmund domains, our construction is quite delicate, but the region Ω we obtain has the additional advantage of being homeomorphic to a ball.

2. Motivation

In this section we will describe how we were led to the notion of a nontangentially accessible domain and illustrate in two dimensions the idea of localization. We will also show that in two dimensions, a simply connected domain is non-tangentially accessible if and only if harmonic measure for the domain and its complement satisfies a doubling condition.

We begin with a version of Lavrentiev's theorem (see Introduction).

(2.1) Theorem. *Let D be a simply connected chord-arc domain in the plane. Let ω denote harmonic measure in D for some fixed point $z_0 \in D$ and let σ denote arc-length of ∂D. Then $\omega \in A_\infty(d\sigma)$.*

Proof. The procedure followed is to localize a global estimate of absolute continuity. It is interesting to note that the condition A_∞ is just a uniform version of absolute continuity at every scale.

Let Ω be a simply connected domain in the plane containing the unit disc, $|z| < 1$. Suppose that $\partial\Omega$ is rectifiable and denote arc-length measure of $\partial\Omega$ by σ. Denote harmonic measure at the origin by ω_Ω. The global estimate we need is as follows. Let E denote Borel subset, $E \subset \partial\Omega$.

(2.2) For any $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma(E) < \delta$ implies $\omega_\Omega(E) \leqslant \varepsilon\sigma(\partial\Omega)$. (The point is that δ does not depend on Ω.)

Estimate (2.2) follows from [36, Theorem 6]

$$\omega_\Omega(E) \leqslant C\sigma(\partial\Omega)/|\log \sigma(E)| \tag{2.3}$$

for some absolute constant C. To prove (2.3), consider the conformal mapping ϕ from the unit disc to Ω sending the origin to the origin. Since $\int_0^{2\pi} |\phi'(e^{i\theta})|\, d\theta = \sigma(\partial\Omega)$, a crude estimate yields

$$\int_0^{2\pi} \log^+ |\phi'(e^{i\theta})|\, d\theta \leqslant \sigma(\partial\Omega).$$

Moreover, since Ω contains the unit disc,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\phi'(e^{i\theta})|\, d\theta \geqslant \log|\phi'(0)| \geqslant 0.$$

Thus,

$$\int_{\phi^{-1}(E)} \log\ |\phi'(e^{i\theta})|\,d\theta \geqslant \int_0^{2\pi} \log^-\ |\phi'(e^{i\theta})|\,d\theta \geqslant -\sigma(\partial\Omega).$$

Hence, denoting $\eta = \omega(E) = (1/2\pi)$ arc-length $(\phi^{-1}(E))$,

$$\sigma(E) = \int_{\phi^{-1}(E)} |\phi'(e^{i\theta})|\,d\theta > \eta e^{-\sigma(\partial\Omega)/\eta}.$$

Consequently, $\eta < (\sigma(\partial\Omega) + \eta\log\eta)/|\log\sigma(E)|$, and (2.3) follows if we recall that $\eta \leqslant 1$.

Denote a disc of radius r and center Q by $B(Q, r)$. The version of the geometric localization required is:

(2.4) LEMMA. *Let D be a chord-arc domain. There exists r_0 such that for any $r < r_0$ and any $Q \in \partial D$, there is a simply connected domain Ω such that*

$$B(Q, r) \cap D \subset \Omega \subset B(Q, Mr) \cap D.$$

Furthermore, $\sigma(\partial\Omega) \leqslant Mr$ and there is a point A of Ω whose distance from $\partial\Omega$ exceeds $M^{-1}r$. (M depends only on the chord-arc constant of D.)

A chord-arc domain is a quasicircle (see [1, 37]) and hence an NTA domain. It follows that one can construct Ω with all the properties above except possibly the bound on $\sigma(\partial\Omega)$. (See (3.11), (A.3).) In fact, one can even arrange that the length of $\partial\Omega\backslash\partial D$ is bounded by a multiple of r. This is an easy exercise using the fact that D is an NTA domain. Finally, the chord-arc condition implies that $\sigma(\partial\Omega \cap \partial D)$ is bounded by a multiple of r.

The fact that a chord-arc domain is an NTA domain also implies that the lemma of Carleson, Hunt and Wheeden is valid. We state it in a slightly different form (see (4.18)).

(2.5) LEMMA. *With the notations of* (2.1), (2.4), *there exists $r_0 > 0$ such that for all $r < r_0$ and all Borel sets $E \subset \Delta = B(Q, r) \cap \partial D$,*

$$\omega_\Omega^A(E) \simeq \omega(E)/\omega(\Delta).$$

We shall prove $\omega \in A_\infty(d\sigma)$ by finding $\alpha > 0$, $\beta < 1$ such that

$$\frac{\sigma(E)}{\sigma(\Delta)} < \alpha \qquad \text{implies} \qquad \frac{\omega(E)}{\omega(\Delta)} < \beta.$$

(Recall that $\omega \in A_\infty(d\sigma)$ if and only if $\sigma \in A_\infty(d\omega)$.) Assume $\sigma(E)/\sigma(\Delta) < \alpha$;

then $\sigma(E)/\sigma(\partial\Omega) < C\alpha$. We can apply (2.2) to the region Ω dilated by a factor M/r and translated so that A coincides with the origin. We conclude that if $C\alpha < \delta$, then $\omega_\Omega^A(E) \leqslant \varepsilon(M/r)\,\sigma(\partial\Omega) < C\varepsilon$. By Lemma (2.5), $\omega(E)/\omega(\Delta) \leqslant C\omega_\Omega^A(E)$. Therefore, if α is sufficiently small, $\omega(E)/\omega(\Delta) < C^2\varepsilon = \beta < 1$.

$BMO_1(\mathbb{R}^m)$ $(m = n-1)$ is the class of functions ϕ on $\mathbb{R}^m$ such that $\nabla\phi \in BMO(\mathbb{R}^m)$. The *Zygmund class* $\Lambda_1(\mathbb{R}^m)$ is the class of functions ϕ on $\mathbb{R}^m$ such that

$$\sup_{x,z} \frac{|\phi(x+z) + \phi(x-z) - 2\phi(x)|}{|z|} < \infty.$$

Stein and Zygmund [51] showed that $BMO_1 \subset \Lambda_1$.

(2.6) Definition. We call a domain D a *Zygmund domain* (respectively, *BMO_1 domain*) when for every $Q \in \partial D$ there is a ball B containing Q and a smooth diffeomorphism $\eta\colon B \to \mathbb{R}^{m+1}$ such that

$$\eta(D \cap B) = \{(x, y)\colon y > \phi(x), x \in \mathbb{R}^m\} \cap \eta(B),$$

where ϕ belongs to the Zygmund class, Λ_1 (respectively, BMO_1).

In order to prove the analogue of Lavrentiev's theorem for BMO_1 domains we need analogues of Lemmas (2.2), (2.4), (2.5). The analogue of (2.2) (see (10.2)) follows from the Calderón–Zygmund decomposition and Dahlberg's estimate $\omega \in A_\infty(d\sigma)$ for Lipschitz domains. On the other hand, the analogue of (2.5), the lemma of Carleson, Hunt and Wheeden is not a simple consequence of known results. First of all, the geometry of a BMO_1 domain is just as complicated as that of a Zygmund domain. Indeed, it is easy to show that any function of $\Lambda_1(\mathbb{R}^{m-1})$ can be extended to $BMO_1(\mathbb{R}^m)$. (Conversely, the restriction of $\Lambda_1(\mathbb{R}^m)$ to $\mathbb{R}^{m-1}$ is $\Lambda_1(\mathbb{R}^{m-1})$.) The boundary of a Zygmund domain need not be rectifiable, but fortunately the estimate in (2.5) concerns harmonic measure only. In order to prove (2.5) a variant of the geometric localization (2.4) is needed. In this case Ω, while it need not have rectifiable boundary, must be something like a Zygmund domain with constant comparable to D. It is evidently impossible for Ω to be a Zygmund domain because at the place where $\partial\Omega$ joins ∂D, $\partial\Omega$ may not even be given locally in smooth coordinates as the graph of a function. For this reason, we introduce non-tangentially accessible (*NTA*) domains. These domains are even wilder than Zygmund domains. In fact, the boundaries of *NTA* domains in $\mathbb{R}^n$ can have positive Hausdorff dimension α for every $\alpha < n$.

It turns out that a quasisphere (the image under a global quasiconformal mapping of a ball in $\mathbb{R}^n$) is a non-tangentially accessible domain. In fact, the image of an *NTA* domain under a global quasiconformal mapping is an *NTA* domain (see [24]). In two dimensions there is a close relationship between the various classes introduced. A simple closed curve in the plane is said to satisfy *Ahlfors' three point condition* if for any points z_1, z_2 of the curve and any z_3 on the arc between z_1 and z_2 of smaller diameter the distance between z_1 and z_3 is bounded by a constant times the distance between z_1 and z_2.

(2.7) THEOREM. *Let D be a bounded, simply connected domain in the plane. The following are equivalent:*

(a) *D is a quasicircle (the image under a global quasiconformal mapping of a disc).*

(b) *∂D satisfies Ahlfors' three point condition.*

(c) *D is a non-tangentially accessible domain (see Section 2).*

(d) *The harmonic measure for D and ${}^cD$ satisfy the doubling condition (in the form stated below).*

The only new aspect of this is the equivalence with (d). The equivalence of (a) and (b) is due to Ahlfors [1]. The equivalence of (a) and (c) is due to Jones ([31]; see also [24]). The fact that (c) implies (d) is proved in (4.9). We will now prove that (d) implies (a).

A homeomorphism f from the unit circle S^1 to itself is said to satisfy the *doubling condition* if for any pair of adjacent arcs I_1 and I_2 of S^1, of equal length, we have $|f(I_1 \cup I_2)| \leqslant C\,|f(I_1)|$. ($|\ |$ denotes arc length on S^1.)

(2.8) LEMMA. *Denote by ϕ (respectively, ψ) a conformal mapping from the unit disc to D (respectively, the complement of the unit disc to ${}^cD$). Define $f: S^1 \to S^1$ by $f(e^{i\theta}) = \phi^{-1}\psi(e^{i\theta})$. If f satisfies the doubling condition, then D is a quasicircle.*

This lemma (see [37]) is a key step in one proof of Ahlfors' theorem [1].

Suppose that D satisfies (d). Denote by ω harmonic measure for D with pole at $\phi(0)$. If V is an arc of the unit circle, then $\omega(\phi(V)) = |V|/2\pi$. Thus we can state the doubling condition for interior and exterior harmonic measure of D in the following form. Let V_1 and V_2 be consecutive arcs of S^1.

(2.9) If $\operatorname{diam} \phi(V_1) \leqslant \operatorname{diam} \phi(V_2)$, then $|V_1| \leqslant N\,|V_2|$.

(2.10) If $\operatorname{diam} \psi(V_1) \leqslant \operatorname{diam} \psi(V_2)$, then $|V_1| \leqslant N\,|V_2|$.

Now suppose that I_1 and I_2 are consecutive arcs of S^1 and $|I_1| = |I_2|$. It suffices to show

$$\operatorname{diam} \psi(I_2) \leqslant C \operatorname{diam} \psi(I_1). \tag{2.11}$$

In fact, (2.11) and (2.9) imply the hypothesis on f in Lemma (2.8).

Denote $\psi(I_1)$ by (z_1, z_2), where z_1 and z_2 are the end points of $\psi(I_1)$. Similarly, $\psi(I_2) = (z_2, z_3)$. There is a natural ordering of D inherrited from the circle via the mapping ψ. Let $r = \operatorname{diam} \psi(I_1)$. Choose the first point w_1 on (z_2, z_3) at distance r from z_2. If no such point exists, then $\operatorname{diam} \psi(I_2) \leqslant 2r$. For $j \geqslant 2$, choose the first point w_j on the interval (w_{j-1}, z_3) of ∂D at distance $2^{j-1}r$ from w_{j-1}. If none exists, then stop. This procedure always ends. If it stops at stage k, then $\operatorname{diam} \psi(I_2) \leqslant 2^k r$. Moreover, for each j, $\operatorname{diam}(z_1, w_j) \leqslant 2^j r = \operatorname{diam}(w_j, w_{j+1})$. Now by (2.10), $N^{-1} |I_1| \leqslant N^{-1} |\psi^{-1}(z_1, w_j)| \leqslant |\psi^{-1}(w_j, w_{j+1})|$. Since $|I_1| = |I_2|$, it follows that $k \leqslant N + 1$, and thus (2.11) holds with $C = 2^{N+1}$.

One further remark that is special to two dimensions is in order. A domain satisfying only the interior conditions for a non-tangentially accessible domain is called an (ε, δ) domain or uniform domain [24, 32]. When the domain is simply connected, the interior conditions imply that the exterior is also a uniform domain. (And thus the domain is an *NTA* domain.) However, the doubling condition for harmonic measure in D above does not imply the doubling condition for the complement of D as simple examples show.

In higher dimensions we mention one result that will be stated more precisely and proved in Section 11. Let f be a continuous function on the unit interval $f(0) = 0$, $f(y) > 0$ for $y > 0$. Denote $D = \{(x, y): |x| < f(y), x \in \mathbb{R}^m, 0 < y < 1\}$. If harmonic measure in D satisfies the doubling condition, then D contains a cone at the origin, i.e., if $0 \leqslant y < \frac{1}{2}$, $f(y) > cy$ for some $c > 0$.

For the reader's convenience we will chart the known relations between the various classes of domains that have been introduced.

In the plane, for simply connected domains we have:

(i) $BMO_1 \subsetneq$ chord-arc.

(ii) Chord-arc $\subsetneq$ quasicircle.

(iii) Zygmund class $\subsetneq$ quasicircle.

(iv) Quasicircle = non-tangentially accessible.

In $\mathbb{R}^n$ for $n > 2$, for domains homeomorphic to a ball, we have:

(i) $BMO_1 \subset$ Zygmund class.

(ii) Zygmund class $\subset NTA$.

(iii) Quasisphere $\subset NTA$.

Moreover, any Zygmund domain can be locally obtained as the intersection of a BMO_1 domain with a hyperplane. It is an open question in dimensions greater than two whether a Zygmund domain is a quasisphere.

3. Non-tangentially Accessible Domains

For a point P and subset S of $\mathbb{R}^n$, denote the Euclidean distance between P and S by $d(P, S) = \inf\{|P - Q| : Q \in S\}$. Likewise, let $d(S_1, S_2)$ denote the distance between two subsets of $\mathbb{R}^n$. A ball will be denoted $B(A, r) = \{P : |P - A| < r\}$. An *M-non-tangential ball* in a domain D is a ball in D whose distance from ∂D is comparable to its radius: $Mr > d(B(A, r), \partial D) > M^{-1}r$. In what follows M will be some fixed constant that will depend only on D. For P_1 and P_2 in D, a *Harnack chain* from P_1 to P_2 in D is a sequence of M-non-tangential balls such that the first ball contains P_1, the last contains P_2, and such that consecutive balls intersect. Notice that consecutive balls must have comparable radius. By Harnack's principle, if u is a positive harmonic function in D, then $C^{-1}u(P_2) < u(P_1) < Cu(P_2)$, where C depends only on M and the length of the Harnack chain between P_1 and P_2.

A bounded domain D in $\mathbb{R}^n$ is called *non-tangentially accessible* (abbreviated *NTA*) when there exist constants M and $r_0 > 0$ such that:

(3.1) *Corkscrew condition.* For any $Q \in \partial D$, $r < r_0$, there exists $A = A_r(Q) \in D$ such that $M^{-1}r < |A - Q| < r$ and $d(A, \partial D) > M^{-1}r$. (The ball $B(A, \frac{1}{2}M^{-1}r)$ is $3M$-non-tangential.)

(3.2) ${}^cD$ satisfies the corkscrew condition.

(3.3) *Harnack chain condition.* If $\varepsilon > 0$ and P_1, P_2 belong to D, $d(P_j, \partial D) > \varepsilon$ and $|P_1 - P_2| < C\varepsilon$, then there exists a Harnack chain from P_1 to P_2 whose length depends on C, but not ε.

Remarks. It is easy to see that Lipschitz domains are *NTA* domains. We will prove below that Zygmund domains (see (2.6), (3.6)) are also *NTA* domains. Another example of an *NTA* domain is a quasisphere (see Section 2, [24, 32]). In fact, the class of *NTA* domains is invariant under quasiconformal mappings of $\mathbb{R}^n$.

The corkscrew condition is so named because the union of non-tangential balls of radius $\frac{1}{2}M^{-1}r$ as r tends to zero forms a non-tangential approach region tending toward Q, which is a twisting (possibly disconnected) replacement for the usual conical approach region in Lipschitz domains. The exterior corkscrew condition (3.2) implies that *NTA* domains are regular for the Dirichlet problem (1.2). The main use of the exterior corkscrew condition is to construct uniform barrier functions (see (4.1)).

Condition (3.3) allows us to connect the interior corkscrews. This can be done in dyadic fashion. The points $A_r(Q)$ and $A_{r/2}(Q)$ from (3.1) are connected by a Harnack chain whose length depends only on M and whose balls all have radius approximately r. A similar argument, which is left to the reader, shows that (3.1) and (3.3) combined are equivalent to a single condition:

(3.4) If $\varepsilon > 0$, P_1, P_2 belong to D, $d(P_j, \partial D) > \varepsilon$, and $|P_1 - P_2| < 2^k\varepsilon$, then there is a Harnack chain from P_1 to P_2 of length Mk. Moreover, for each ball B in the chain, radius $(B) \geqslant M^{-1} \min(d(P_1, B), d(P_2, B))$.

Harnack's principle implies (replacing M by a larger constant that depends only on M and which is denoted again by M):

(3.5) If P_1 and P_2 are as in (3.4), then every positive harmonic function in D satisfies

$$M^{-k}u(P_2) < u(P_1) < M^k u(P_2).$$

Condition (3.4) has also arisen in the work of Jones on extension domains and quasiconformal mapping (see [24, 31, 32]).

(3.6) PROPOSITION. *A Zygmund domain is a non-tangentially accessible domain.*

Proof. The properties of Zygmund domains we shall verify are local properties, invariant under any mapping that preserves distance in $\mathbb{R}^n$ up to a bounded factor. Hence, we need only consider the special domain $D = \{(x, y): y > \phi(x), x \in \mathbb{R}^m\}$ with ϕ in the Zygmund class, Λ_1. Moreover, replacing ϕ by $\varepsilon\phi$, we may assume that $\|\phi\|_{\Lambda_1}$ is small.

Let θ be a smooth, non-negative even function on $\mathbb{R}^m$ with compact support, $\int \theta(x)\,dx = 1$. Denote $\theta_r(x) = r^{-m}\theta(r^{-1}x)$ and $\phi_r(x) = \phi * \theta_r(x)$.

(3.7) LEMMA. *If* $|z - x| < 10r$, *then* $|\phi(z) - \phi(x) - \nabla\phi_r(x)(z - x)| < \alpha r$, *where* α *denotes an absolute constant times* $\|\phi\|_{\Lambda_1}$.

(3.8) *Remark.* This estimate characterizes the Zygmund class. For this fact and a far more general result, see Nagel and Stein [43]. The characterization provides an excellent geometric picture of ϕ: For each x and each $r > 0$, there is an approximate tangent plane to the graph $(z, \phi(z))$ given by the graph of the linear function of z, $\phi(x) + \nabla\phi_r(x) \cdot (z - x)$.

To prove (3.7), we first observe that

$$|\phi_r(x) - \phi(x)| < \alpha r. \tag{3.9}$$

In fact, because θ is even

$$|\phi_r(x)-\phi(x)| = \tfrac{1}{2}\left|\int (\phi(x+z)+\phi(x-z)-2\phi(x))\,\theta_r(z)\,dz\right|$$
$$\leqslant \tfrac{1}{2}\|\phi\|_{\Lambda_1}\int |z|\,\theta_r(z)\,dz$$
$$= \tfrac{1}{2}\|\phi\|_{\Lambda_1}\, r\int |z|\,\theta(z)\,dz,$$

which proves (3.9).

Next,

$$|\phi(z)-\phi(x)-\nabla\phi_r(x)\cdot(z-x)|$$
$$\leqslant |\phi(z)-\phi_r(z)| + |\phi_r(x)-\phi(x)| + |\phi_r(z)-\phi_r(x)-\nabla\phi_r(x)\cdot(z-x)|.$$

The first two terms are controlled by (3.9). The final term equals

$$\left|\frac{1}{2}\sum_{i,j}(z_i-x_i)(z_j-x_j)\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\phi_r(\xi)\right|,$$

where ξ is some point on the segment joining x to z. It is therefore enough to show

$$\left|\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\phi_r(\xi)\right| < \alpha r^{-1}. \tag{3.10}$$

Denote $\psi(x) = (\partial/\partial x_i)(\partial/\partial x_j)\,\theta(x)$. ψ is even and has mean value zero. Hence,

$$\left|\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\phi_r(\xi)\right| = \left|\int \phi(z)\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j}\theta_r(\xi-z)\,dz\right|$$
$$= \frac{1}{2}r^{-2}\left|\int (\phi(\xi+z)+\phi(\xi-z)-2\phi(\xi))\,\psi_r(z)\,dz\right|$$
$$\leqslant \frac{1}{2}r^{-2}\|\phi\|_{\Lambda_1}\int |z|\,|\psi_r(z)|\,dz$$
$$= \left(\frac{1}{2}\|\phi\|_{\Lambda_1}\int |z|\,|\psi(z)|\,dz\right) r^{-1} \leqslant \alpha r^{-1}.$$

We now check the corkscrew condition (3.1) in the domain D. Denote $Q=(x,\phi(x))$ and $A_r(Q)=(x,\phi(x))+rN$, where

$$N = (-\nabla\phi_r(x),1)/|(-\nabla\phi_r(x),1)|.$$

$A_r(Q)$ is the point above the graph of ϕ at distance r from Q along N, the unit normal to the approximate tangent plane $\{(z, \phi(x) + \nabla\phi_r(x)(z-x))$: $z \in \mathbb{R}^m\}$. Thus $|A_r(Q) - Q| = r$. If $|z-x| < 10r$, then

$$\begin{aligned} |A_r(Q) - (z, \phi(z))| &= |(x, \phi(x)) + rN - (z, \phi(z))| \\ &\geqslant |rN - (z-x, \nabla\phi_r(x)\cdot(z-x))| \\ &\quad - |(0, \phi(x) - \phi(z) + \nabla\phi_r(x)\cdot(z-x))| \\ &\geqslant r - ar, \end{aligned}$$

by Lemma (3.7) and the fact that N and $(z-x, \nabla\phi_r(x)\cdot(z-x))$ are perpendicular. If $|z-x| \geqslant 10r$, then $|A_r(Q) - (z, \phi(z))| \geqslant 9r$. In all, $r \geqslant d(A_r(Q), \partial D) \geqslant (1-\alpha) r$. This proves (3.1). The exterior condition (3.2) has a similar proof.

To obtain a Harnack chain between points P_1 and P_2 satisfying the hypothesis of (3.3), let Q_j be points of D closest to P_j. Denote $t_j = |P_j - Q_j|$, $j = 1, 2$.

Case 1. $t_1 > \frac{1}{100} C\varepsilon$, $t_2 > \frac{1}{100} C\varepsilon$.

For sufficiently small α, Lemma (3.7) implies that the segment joining P_1 to P_2 is farther than $10^{-3} C\varepsilon$ from D. Thus, a sequence of 2×10^3 balls of radius $\frac{1}{20}\min(t_1, t_2)$ with centers on that segment form a Harnack chain from P_2 to P_2.

Case 2. $t_1 \leqslant \frac{1}{100} C\varepsilon$.

Then $t_2 \leqslant t_1 + C\varepsilon \leqslant (1 + \frac{1}{100}) C\varepsilon$. For sufficiently small α, the chain of balls $B(A_{2^j t_1}(Q_1), (9/10)\, 2^j t_1)$, $j = 0, 1, \dots, k_1$, where $C\varepsilon \leqslant 2^{k_1} t_1 < 2C\varepsilon$, links P_1 to the point $\tilde{P}_1 = A_{2^{k_1} t_1}(Q)$. Note that $|\tilde{P}_1 - Q_1| \leqslant 2C\varepsilon$, and $d(\tilde{P}_1, \partial D) \geqslant \frac{1}{2} C\varepsilon$. Also, $k_1 \leqslant \log_2 C + 1$, so the length of the chain depends only on C. Similarly, there is a chain from P_2 to $\tilde{P}_2$ such that $|\tilde{P}_2 - Q_2| \leqslant 2C\varepsilon$ and $d(\tilde{P}_2, \partial D) \geqslant \frac{1}{2} C\varepsilon$. Hence, $|\tilde{P}_1 - \tilde{P}_2| \leqslant |\tilde{P}_1 - Q_1| + |Q_1 - Q_2| + |\tilde{P}_2 - Q_2| \leqslant 10 C\varepsilon$. Therefore, $\tilde{P}_1$ and $\tilde{P}_2$ can be linked in the same way as in Case 1.

We will need a local version of non-tangential accessibility.

(3.11) THEOREM. *If D is an NTA domain, then for any $Q \in \partial D$ and $r < r_0$, there exists an NTA domain $\Omega \subset D$ such that*

$$B(Q, M^{-1} r) \cap D \subset \Omega \subset B(Q, Mr) \cap D.$$

Furthermore, the constant M in the NTA definition for Ω is independent of Q and r.

This theorem is due to Jones (see [32]). The region Ω is the analogue of the inverted cone or Carleson box frequently used in Lipschitz domains. A direct construction of Ω in the case of Zygmund domains and quasispheres will be presented in the Appendix. The proof for quasispheres is quite simple, but the one for Zygmund domains is somewhat delicate. The key point of Jones' theorem, which obviates our constructions, is not to demand that Ω be homeomorphic to a ball.

4. Estimates for Harmonic Measure

In this section we will prove several lemmas about harmonic measure. First, we make a comparison of harmonic measure to the Green function (4.8). Next, we deduce the doubling condition for harmonic measure (4.9). Finally, we obtain the Carleson–Hunt–Wheeden lemma (4.11) and the estimate (4.14) on the kernel function $K(P, Q)$ that implies that the non-tangential maximal function is dominated by the Hardy–Littlewood maximal function with respect to harmonic measure (see Theorem (5.8)). The approach here is similar to that used Caffarelli *et al.* [6] to prove estimates for divergence class operators with bounded, measurable coefficients in smooth domains. In particular, the proof of Lemma (4.4) is taken from there. This lemma is the key one in Carleson's paper [10, p. 398].

We will assume throughout this section that D is a bounded non-tangentially accessible domain. The constants involved in the definition ((3,1), (3.2), (3.3)) and its consequences ((3.4), (3.5)) will usually be denoted M. But occasionally M will be replaced by a larger constant that depends only on the previous value of M.

(4.1) Lemma. *There exists $\beta > 0$ such that for all $Q \in \partial D$, $r < r_0$, and every positive harmonic function u in D, if u vanishes continuously on $\partial D \cap B(Q, r)$, then for $X \in D \cap B(Q, r)$,*

$$u(X) \leqslant M(|X-Q|\, r^{-1})^{\beta}\, C(u),$$

where $C(u) = \sup\{u(Y)\colon Y \in \partial B(Q, r) \cap D\}$.

Proof. Define the harmonic function $v(X)$ in $B(Q, r) \cap D$ with boundary values

$$\begin{aligned} v(X) &= 1, \qquad X \in \partial B(Q, r) \cap \bar{D}, \\ &= 0, \qquad X \in B(Q, r) \cap \partial D. \end{aligned}$$

By the maximum principle, $u(X) \leqslant v(X)$, so it suffices to show $v(X) \leqslant M(|X-Q|\, r^{-1})^{\beta}$. The exterior corkscrew condition (3.2) implies that for some r', $M^{-1}r < r' < r$, $\partial B(Q, r') \cap {}^{c}D$ contains at least some fixed fraction of the full surface measure of $\partial B(Q, r')$. By the maximum principle,

v is dominated by the Poisson integral on the ball $B(Q, r')$ of a function that is 1 on $\partial B(Q, r') \cap D$ and 0 on $\partial B(Q, r') \cap {}^cD$. An easy lower bound for the Poisson kernel of the ball yields $v(X) < 1 - \varepsilon$ for $X \in \partial B(Q, \frac{1}{2}r') \cap D$, where ε depends only on M. In particular, since $r' > M^{-1}r$, $v(X) \leqslant 1 - \varepsilon$ for $X \in \partial B(Q, \frac{1}{2}M^{-1}r) \cap D$. Iterating this procedure, we obtain

$$v(X) \leqslant (1-\varepsilon)^k \qquad \text{for } X \in B(Q, (\tfrac{1}{2}M^{-1})^k\, r) \cap D.$$

In other words, for some $\beta > 0$ depending on M, $v(X) \leqslant M(|X - Q|\, r^{-1})^{\beta}$, as desired.

Recall that a surface ball is defined by

$$\Delta(Q, r) = (\partial D) \cap B(Q, r).$$

(4.2) LEMMA. *Let $r < r_0$. If $Q \in \partial D$ and $A_r(Q)$ is given by* (3.1), *then*

$$\omega^X(\Delta(Q, \partial r)) \geqslant M^{-1} \qquad \textit{for all } X \in B(A_r(Q), \tfrac{1}{2}M^{-1}r).$$

Proof. Let v be as in (4.1). By the maximum principle, $\omega^X(\Delta(Q, \partial r)) \geqslant 1 - v(X)$ for $X \in B(Q, r) \cap D$. In particular, let A' be a $3M$-non-tangential point such that $\frac{1}{2}M^{-2}r < |A' - Q| < \frac{1}{2}M^{-1}r$; then $\omega^{A'}(\Delta(Q, 2r)) \geqslant 1 - v(A') \geqslant \varepsilon$. Since A' can be connected to $B(A_r(Q), \frac{1}{2}M^{-1}r)$ by a Harnack chain of length depending only on M, the lemma follows from Harnack's principle. (The value of M appearing in the conclusion of the lemma may be larger than the previous value.)

Let $G(X, Y)$ denote the Green function of D. (We will use the convention that the Green function is subharmonic and negative.) If $r < r_0$, then for $X \in D \backslash B(A_{r/2}(Q), r/4M)$

$$r^{n-2}\, |G(X, X_{r/2}(Q))| \leqslant M\omega^X(\Delta(Q, r)). \tag{4.3}$$

This follows from (4.2), the estimate $|G(X, A_{r/2}(Q))| \leqslant M\,|X - A_{r/2}(Q)|^{2-n}$, and the maximum principle in $D \backslash B(A_{r/2}(Q), r/4M)$. The estimate $|G(X, A_{r/2}(Q))| \leqslant M\,|X - A_{r/2}(Q)|^{2-n}$ is valid for $X \in D \backslash B(A_{r/2}(Q), r/4M)$ even when $n = 2$, by comparison with the Green function of the region exterior to a ball provided by the exterior corkscrew condition.

(4.4) LEMMA ([6; 10, p. 398]). *If u is harmonic in D, $u \geqslant 0$, and u vanishes continuously on $\Delta(Q, 2r)$, then*

$$u(X) \leqslant M' u(A_r(Q))$$

for all $X \in B(Q, r) \cap D$ for some M' depending only on M.

Proof. Let $Q_0 \in \partial D$. If u vanishes on $\Delta(Q_0, s)$ then by Lemma (4.1), there is M_1 depending only on M so that

$$\sup\{u(X): X \in B(Q_0, M_1^{-1}s) \cap D\} \leqslant \tfrac{1}{2} \sup\{u(X): X \in B(Q_0, s) \cap D\}. \tag{4.5}$$

Normalize u so that $u(A_r(Q)) = 1$. By (3.5) there is a constant M_2 depending only on M_1 such that if $u(Y) > M_2^h$ and $Y \in B(Q, r) \cap D$, then $d(Y, \partial D) < M_1^{-h} r$. Choose N so that $2^N > M_2$. Finally, let $M' = M_2^h$, where $h = N + 3$.

Suppose that there exists $Y_0 \in B(Q, r) \cap D$ such that $u(Y_0) > M'u(A_r(Q)) = M_2^h$. Then $d(Y_0, \partial D) < M_1^{-1} r$ and if Q_0 is a point of ∂D nearest to Y_0, $|Q - Q_0| < r + M_1^{-h} r < \frac{3}{2} r$. Applying (4.5):

$$\begin{aligned}\sup\{u(X)&: X \in B(Q_0, M_1^{-h+N} r) \cap D\} \\ &\geqslant 2^N \sup\{u(X): X \in B(Q_0, M_1^{-h} r)\} \geqslant M_2^{h+1}.\end{aligned}$$

Hence, we can choose $Y_1 \in B(Q_0, M_1^{-h+N}) \cap D$ such that $u(Y_1) \geqslant M_2^{h+1}$. As before for Y_0, $d(Y_1, \partial D) < M_1^{-h-1} r$. Let Q_1 be a point of ∂D closest to Y_1. Continuing in this manner we obtain two sequences, $\{Y_k\}$ and $\{Q_k\}$ such that $u(Y_k) \geqslant M_2^{h+k}$, $d(Y_k, \partial D) = |Y_k - Q_k| < M_1^{-h-k} r$, and $Y_k \in B(Q_{k-1}, M_1^{-h-k+N}) \cap D$. This contradicts the continuity of u at $\Delta(Q, 2r)$ provided we can show that the sequence $B(Q_{k-1}, M_1^{-h-k+N} r)$ is contained in $B(Q, 2r)$. In fact,

$$\begin{aligned}|Y_k - Q| &\leqslant |Y_k - Q_k| + |Q_k - Y_{k-1}| + |Y_{k-1} - Q| \\ &\leqslant (M_1^{-h-k} + M_1^{-h-k+N}) r + |Y_{k-1} - Q|.\end{aligned}$$

Since $|Y_0 - Q| < r$, we have

$$|Y_k - Q| \leqslant r + \sum_{j=1}^{k} (M_1^{-h-j} + M_1^{-h-j+N}) r < 2r,$$

because $h = N + 3$.

$$\omega^X(\Delta(Q, r)) \leqslant M' r^{n-2} |G(X, A_r(Q))| \tag{4.6}$$

for $X \in D \backslash B(Q, 2r)$, $2r < r_0$.

To prove (4.6) fix $X \in D \backslash B(Q, 2r)$ and define

$$\begin{aligned} g(P) &= G(X, P), \qquad & P \in D, \\ &= 0, & P \in {}^cD. \end{aligned}$$

g is continuous in $\mathbb{R}^n\backslash\{X\}$ and subharmonic. For all $P \in \mathbb{R}^n\backslash(\partial D \cup \{X\})$,

$$g(P) = -c_n\left(|X-P|^{2-n} - \int_{\partial D} |Q-P|^{2-n}\, d\omega^X(Q)\right). \qquad (4.7)$$

($|X-P|^{2-n}$ is replaced by $\log|X-P|$ when $n=2$.)

By Fatou's lemma, $\int_{\partial D} |Q-P|^{2-n}\, d\omega^X(Q) < \infty$ for all $P \in \partial D$. Moreover, if we choose $P_j \to P$ non-tangentially, that is, $|P_j - P| \leqslant Md(P_j, \partial D)$, then for all $Q \in \partial D$,

$$|Q-P| \leqslant |Q-P_j| + |P_j - P| \leqslant (M+1)\,|Q-P_j|.$$

Therefore, by dominated convergence (4.7) is valid for all $P \in \mathbb{R}^n\backslash\{X\}$.

For any $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\phi(X) = 0$,

$$\begin{aligned}
&\int_D g(P)\, \Delta\phi(P)\, dP \\
&\quad = -c_n \int_{\mathbb{R}^n} \left\{|X-P|^{2-n} - \int_{\partial D} |P-Q|^{2-n}\, d\omega^X(Q)\right\} \Delta\phi(P)\, dP \\
&\quad = c_n \int_{\partial D}\int_{\mathbb{R}^n} |P-Q|^{2-n}\, \Delta\phi(P)\, dP\, d\omega^X(Q) \\
&\quad = -\int_{\partial D} \phi(Q)\, d\omega^X(Q).
\end{aligned}$$

Choose $\phi \geqslant 0$ so that $\phi = 1$ on $\Delta(Q, r)$, ϕ vanishes outside $B(Q, \frac{3}{2}r)$ and $|(\partial/\partial x_i)(\partial/\partial x_j)\,\phi| \leqslant Mr^{-2}$. Then

$$\begin{aligned}
\omega^X(\Delta(Q,r)) &\leqslant \int_{\partial D} \phi(Q)\, d\omega^X(Q) \\
&= -\int_D g(P)\, \Delta\phi(P)\, dP \leqslant \int_{B(Q,2r)} |g(P)|\, |\Delta\phi(P)|\, dP \\
&\leqslant M'\, |G(X, A_r(Q))|\, r^{n-2}, \qquad \text{by Lemma (4.4).}
\end{aligned}$$

Combining (4.6) and (4.3), we obtain the generalization of a lemma of Dahlberg [15].

(4.8) LEMMA. *If* $2r < r_0$ *and* $X \in D\backslash B(Q, 2r)$, *then*

$$M^{-1} < \omega^X(\Delta(Q,r))/r^{n-2}\, |G(X, A_r(Q))| < M.$$

(We have replaced M' by M, since M' is just a constant depending on the previous value of M.)

Observe that in (4.8) the estimate is uniform as X tends to $\partial D \backslash \Delta(Q, 2r)$.

(4.9) LEMMA (Doubling condition). $\omega^X(\Delta(Q, 2r)) \leqslant C_X \omega^X(\Delta(Q, r))$.

For $2r < r_0$, (4.9) follows from (4.8) and Harnack's principle. For large r it follows from (4.2) and Harnack's principle.

(4.10) LEMMA. *Let r be such that $Mr < r_0$. Suppose that u and v are positive harmonic functions in D vanishing continuously on $\Delta(Q, Mr)$ for some $Q \in \partial D$ and that $u(A_r(Q)) = v(A_r(Q))$. Then $M^{-1} < u(X)/v(X) < M$ for all $X \in B(Q, M^{-1}r) \cap D$.*

Proof. Let Ω be the NTA domain of Theorem (3.11) such that $B(Q, 2M^{-1}r) \cap D \subset \Omega \subset B(Q, \frac{1}{2}Mr) \cap D$.

Denote

$$L_1 = \{P \in \partial\Omega \backslash \partial D : d(P, \partial D) < M^{-1}r\},$$
$$L_2 = \{P \in \partial\Omega : d(P, \partial D) \geqslant M^{-1}r\}.$$

L_2 clearly contains a surface ball (of Ω) of radius comparable to r. Covering L_1 with a finite union of balls of size a small constant times r in order to apply (4.8), we find that $\omega_\Omega^X(L_1 \cup L_2) \leqslant M\omega_\Omega^X(L_2)$ for $X \in B(Q, M^{-1}r) \cap D$. Lemma (4.4) implies $u(X) \leqslant Mu(A_r(Q))$ for $X \in \bar{\Omega}$. By the maximum principle, since u vanishes on $\partial\Omega \cap \partial D$, $u(X) \leqslant M\omega_\Omega^X(L_1 \cup L_2)\, u(A_r(Q))$. On the other hand, Harnack's principle and (3.3) imply $v(X) \geqslant Mv(A_r(Q))$ for all $X \in L_2$. Hence, $v(X) \geqslant Mv(A_r(Q))\, \omega_\Omega^X(L_2)$ for all $X \in \Omega$, and Lemma (4.10) follows.

(4.11) LEMMA. *Let $\Delta = \Delta(Q_0, r)$, $r < r_0$. Let $\Delta' = \Delta(Q, s) \subset \Delta(Q_0, r/2)$. ($Q, Q_0 \in \partial D$.) If $X \in D \backslash B(Q_0, 2r)$, then*

$$\omega^{A_r(Q_0)}(\Delta') \simeq \omega^X(\Delta')/\omega^X(\Delta).$$

($C_1 \simeq C_2$ means that the ratio of C_1 and C_2 is bounded above and below by a constant depending only on M.)

Proof. By (4.8),

$$\omega^X(\Delta) \simeq r^{n-2}\, |G(X, A_r(Q_0))|,$$
$$\omega^X(\Delta') \simeq s^{n-2}\, |G(X, A_s(Q))|,$$

and

$$\omega^{A_r(Q_0)}(\Delta') \simeq s^{n-2}\, |G(A_r(Q_0), A_s(Q))|,$$

Thus it suffices to prove

$$|G(A_r(Q_0), A_s(Q_0))| \simeq r^{2-n} |G(X, A_s(Q))| \, |G(X, A_r(Q_0))|. \tag{4.12}$$

Let $u(Y) = G(A_r(Q_0), Y)$; $v(Y) = G(X, Y)$. Choose a point A such that $|A - A_r(Q)| = \frac{1}{2}M_r^{-1}$ and $d(A, \partial D) \geqslant \frac{1}{2}M_r^{-1}$. Then $u(A) \simeq r^{2-n}$ and $v(A) \simeq |G(X, A_r(Q))|$. Now apply (4.10) to appropriate multiples of u and v, and let $Y = A_s(Q)$ to obtain (4.12).

(4.13) *Notations.* Fix a point $X_0 \in D$ and denote $\omega = \omega^{X_0}$. Denote $K(A, Q) = (d\omega^A/d\omega)(Q)$, the Radon–Nikodym derivative, which exists by Harnack's principle.

The doubling condition (4.9) implies $K(A, Q) = \lim \omega^A(\Delta')/\omega(\Delta')$ a.e. (ω), as Δ' shrinks to Q. A priori, $K(A, Q)$ is only defined for almost every (ω) Q. Actually, $K(A, Q)$ is a Hölder continuous function of Q as we will see later ((5.5), (7.1)).

(4.14) Lemma. *Let* $A = A_r(Q_0)$, $Q_0 \in \partial D$, $\Delta_j = \Delta(Q_0, 2^j r)$ *and* $R_j = \Delta_j \backslash \Delta_{j-1}$. *Then* $\sup\{K(A, Q): Q \in R_j\} \leqslant c_j/\omega(\Delta_j)$, *with* $c_j \leqslant CM2^{-\alpha j}$. $\alpha > 0$, *M depend only on D, (C depends only on the choice of X_0.)*

Proof. First consider j such that $2^j r < r_0$. Pick a small surface ball $\Delta' \subset R_j$. Denote $A_j = A_{2^j r}(Q_0)$. By (4.11), $\omega^{A_j}(\Delta') \simeq \omega(\Delta')/\omega(\Delta_j)$. By Lemmas (4.1) and (4.4),

$$\omega^A(\Delta') \leqslant M\omega^{A_j}(\Delta') \left(\frac{|A - Q_0|}{2^j r}\right)^\beta \leqslant M \frac{\omega(\Delta')}{\omega(\Delta_j)} 2^{-j\beta}.$$

Hence $\omega^A(\Delta')/\omega(\Delta') \leqslant M2^{-j\beta}/\omega(\Delta_j)$, and $c_j \leqslant M2^{-j\beta}$.

There is only a finite number of j for which R_j is non-empty and $2^j r > r_0$. Thus it is enough to show that $\sup\{K(A, Q): Q \in \partial D \backslash \Delta(Q_0, r_0)\} \leqslant C$.

Choose $\Delta' \subset \partial D \backslash \Delta(Q_0, r_0)$. By (4.4) and Harnack's inequality $\omega^A(\Delta') \leqslant M\omega^{A_{r_0}(Q_0)}(\Delta') \leqslant M^2\omega(\Delta')$.

(4.15) Lemma. *Let* $\Delta = \Delta(Q_0, r)$, $r < r_0$. *Then* $\sup\{K(X, Q): Q \in \partial D \backslash \Delta\} \to 0$ *as* $X \to Q_0$.

Proof. Let Δ' be a small surface ball about Q. As in the proof of (4.14), $\omega^X(\Delta') \leqslant M^2\omega(\Delta')$ for $X \in B(Q_0, r/2) \cap D$. Since $\omega^X(\Delta')$ vanishes continuously on $B(Q_0, r/2) \cap \partial D$ we deduce from (4.1) the stronger estimate: $\omega^X(\Delta') \leqslant M^3\omega(\Delta')(|X - Q_0| r^{-1})^\beta$. Thus $K(X, Q) \leqslant M^3(|X - Q_0| r^{-1})^\beta$, and the lemma follows.

Let us add a few variants on the preceding lemmas. Let Ω be as in (3.11) for the distance r and boundary point Q. Choose $A \in \Omega$ such that $d(A, \partial\Omega) \geqslant M^{-1}r$. Let $G_D(X)$ denote the Green function for D with pole at

X_0 (see (4.13)) and denote by $G_\Omega(Y, X)$ the Green function for Ω. Let $\Delta = B(Q, M^{-2}r) \cap \partial D$. Choose Y so that $M^{-1}r \geqslant |A - Y| \geqslant M^{-2}r$ and $d(Y, \partial\Omega) \geqslant M^{-2}r$. By comparison with the Green function for the exterior and interior of a ball we find that $|G_\Omega(Y, A)| \simeq r^{2-n}$. By Lemma (4.8) and Harnack's principle, $|G_D(Y)| \simeq r^{2-n}\omega(\Delta)$. Therefore, by Lemma (4.10)

$$G_\Omega(A, X) \simeq G_D(X)/\omega(\Delta) \qquad \text{for } X \in B(Q, M^{-2}r) \cap D. \tag{4.16}$$

Using (4.16) and (4.8) we obtain

$$\omega_\Omega^A(\Delta') \simeq \omega(\Delta')/\omega(\Delta) \qquad \text{for any surface ball } \Delta' \subset \Delta. \tag{4.17}$$

From (4.17) and (4.11), we deduce that

$$\omega_\Omega^A(E) \simeq \omega(E)/\omega(\Delta) \qquad \text{for any Borel set } E \subset \Delta. \tag{4.18}$$

5. Global Boundary Behavior of Harmonic Functions

(5.1) Theorem (Boundary Harnack principle). *Let D be an NTA domain and let V be an open set. For any compact set $K \subset V$, there exists a constant C such that for all positive harmonic functions u and v in D that vanish continuously on $\partial D \cap V$, $u(X_0) = v(X_0)$ for some $X_0 \in D \cap K$ implies*

$$C^{-1}u(X) < v(X) < Cu(X) \qquad \textit{for all } X \in K \cap \bar{D}.$$

See [15, 55] for the case of Lipschitz domains. Theorem (5.1) is an immediate consequence of Lemma (4.10).

For the rest of the theorems of this section we need a better understanding of $K(X, Q)$ defined in (4.13).

(5.2) Lemma. *Let D be an NTA domain. Let u be harmonic and positive in D and continuous in $\bar{D} \backslash \{Q_0\}$, where $Q_0 \in \partial D$. If $u = 0$ on $\partial D \backslash \Delta(Q_0, r)$, then for all $X \in D \backslash B(Q_0, Mr)$,*

$$u(X) \simeq u(A_r(Q_0))\, \omega^X(\Delta(Q_0, r)).$$

Proof. By the Harnack chain condition (3.3), we can replace $A_r(Q_0)$ with a (non-tangential) point A of $\partial B(Q_0, Mr)$. Cover $(\partial B(Q_0, Mr)) \cap \partial D$ with a finite collection of surface balls of ∂D of size roughly r, disjoint from $\Delta(Q_0, r)$. Both $u(X)$ and $u(A)\, \omega^X(\Delta(Q_0, r))$ vanish continuously on these surface balls. Lemma (4.10) and (4.2) and Harnack's principle imply the desired estimate for all $X \in \bar{D} \cap \partial B(Q_0, Mr)$. The full estimate then follows from the maximum principle.

A *kernel function* in D at $Q \in \partial D$ is a positive harmonic function u in D that vanishes continuously on $\partial D \backslash \{Q\}$ and such that $u(X_0) = 1$. (X_0 is fixed; see (4.13).)

(5.3) LEMMA. *Let D be an NTA domain. There exists a kernel function u at every boundary point.*

Proof. Let $Q \in \partial D$, and denote $u_m(X) = \omega^X(\Delta(Q, 2^{-m}))/\omega(\Delta(Q, 2^{-m}))$. Note that by (4.13), $u_m(X_0) = 1$. By Harnack's inequality, there exists a non-zero harmonic function u such that $u_{m_j} \to u$ uniformly on compact subsets of D. As in the proof of (5.2), $u_{m_j}(X) \leqslant M u_{m_j}(A_r(Q))\, \omega^X(\Delta(Q, r))$, for $r > 2^{-m_j}$. Fix r and let $m_j \to \infty$. We conclude that $u(X)$ vanishes on $\partial D \backslash \Delta(Q, 2r)$ for any $r > 0$. Thus u is a kernel function at Q.

(5.4) LEMMA. *Suppose that u_1 and u_2 are two kernel functions for D at Q. Then*

$$M^{-1} \leqslant u_1(X)/u_2(X) \leqslant M \qquad \textit{for all } X \in D.$$

Proof. Since $u_1(X_0) = 1$, (5.2) implies $u_1(A_r(Q))\, \omega(Q, r)) \simeq 1$ for all r. Applying (5.2) again, we have $u_1(X) \simeq \omega^X(\Delta(Q, r))/\omega(\Delta(Q, r))$ for all $X \in D \backslash B(Q, Mr)$. Choose a subsequence r_j as in (5.3) so that

$$\lim_{j \to \infty} \omega^X(\Delta(Q, r_j))/\omega(\Delta(Q, r_j)) = u(X)$$

uniformly on compact subsets of D. Then $u_1(X) \simeq u(X)$. Similarly, $u_2(X) \simeq u(X)$, and (5.4) follows.

Using (5.4) and the general theory of Martin boundaries [41], one can now deduce:

(5.5) THEOREM. *Let D be an NTA domain. There is exactly one kernel function at $Q \in \partial D$. It is given by $K(X, Q) = \lim_{r \to 0} \omega^X(\Delta(Q, r))/\omega(\Delta(Q, r))$. The limit exists for all Q. $K(X, Q)$ is a positive harmonic function of X for $X \in D$ and a continuous function of $Q \in \partial D$.*

Direct proofs of (5.5) using (5.4) can also be found in [6, 26].

(5.6) *Notations.* A *non-tangential region* (or *corkscrew*) at $Q \in \partial D$ is denoted by $\Gamma_\alpha(Q) = \{P \in D : |P - Q| < (1 + \alpha)\, d(P, \partial D)\}$. The *non-tangential maximal function* is denoted $N_\alpha(u)(Q) = \sup\{|u(P)| : P \in \Gamma_\alpha(Q)\}$ for u defined in D. The value of α is often of little importance, in which case we write $N(u)(Q)$ to mean $N_\alpha(u)(Q)$ for some (or any) value of α. The usual

Hardy–Littlewood maximal function with respect to $\omega = \omega^{X_0}$ is for $f \in L^1(d\omega)$,

$$M_\omega(f)(Q) = \sup_r \frac{1}{\omega(\Delta(Q, r))} \int_{\Delta(Q,r)} |f(Q')| \, d\omega(Q').$$

We say that u converges to f *non-tangentially at* Q if for any α, $u(X)$ restricted to $\Gamma_\alpha(Q)$ converges to $f(Q)$ as $X \to Q$.

As a consequence of the doubling condition (4.9), the usual estimates on the maximal function hold:

$$\omega\{Q: M_\omega f(Q) > \lambda\} \leqslant (C/\lambda) \|f\|_{L^1(d\omega)},$$
$$\|M_\omega f\|_{L^p(d\omega)} \leqslant C_p \|f\|_{L^p(d\omega)}, \qquad 1 < p \leqslant \infty. \tag{5.7}$$

(5.8) THEOREM. *Let D be an NTA domain, $f \in L^1(d\omega)$ and define $u(X) = \int_{\partial D} f(Q) \, K(X, Q) \, d\omega(Q)$. Then $N_\alpha(u)(Q) \leqslant C_\alpha M_\omega(f)(Q)$, and u converges to f non-tangentially a.e. ω. Thus u is the harmonic extension of f. Moreover, if μ is a finite Borel measure on ∂D and $d\mu = f \, d\omega + d\nu$, ν is singular w.r.t. ω, and $u(X) = \int_{\partial D} K(X, Q) \, d\mu(Q)$, then u converges non-tangentially to f a.e. ω.*

Proof. The estimate $N(u)(Q) \leqslant C_\omega(f)(Q)$ follows from Lemma (4.14). Nontangential convergence a.e. ω then follows from a well-known argument using Lemmas (4.9) and (4.15). (See for example, Hunt and Wheeden [25].)

The Martin boundary [41] of a domain D is identified with the collection of kernel functions on D. A consequence of Theorem (5.5) is:

(5.9) THEOREM. *If D is an NTA domain, then the Martin boundary of D is the Euclidean boundary of D.*

The representation theorem of Martin [41] can be expressed in our case as:

(5.10) THEOREM. *Let D be an NTA domain. If u is a positive harmonic function in D, then there exists a unique positive Borel measure μ on ∂D such that*

$$u(X) = \int_{\partial D} K(X, Q) \, d\mu(Q).$$

(5.11) COROLLARY. *If u is a positive harmonic function in D, then u has non-tangential limits a.e. ω.*

(5.12) *Remark.* If u is bounded in D, then $u(X) = \int_{\partial D} K(X, Q) \, g(Q) \, d\omega$ with $g \in L^\infty(d\omega)$.

In fact, adding a constant to u we may assume that $0 \leqslant u \leqslant C$. Thus u is represented in (5.10) by a positive measure μ. Let $\Delta = \Delta(Q, r)$ and $A = A_r(Q)$. By (4.11) and (4.2), $\omega(\Delta) K(A, Q') \geqslant M^{-1}$ for $Q' \in \Delta$. Hence

$$\frac{\mu(\Delta)}{\omega(\Delta)} \leqslant M \int_\Delta K(A, Q')\, d\mu(Q') \leqslant Mu(A) \leqslant MC.$$

Therefore, μ is absolutely continuous w.r.t. ω and $g = d\mu/d\omega \leqslant MC$.

To conclude this section, we prove a result related to the area integral (see Theorem (6.6)). We will make use of a theorem of Riesz [2, Chap. IX, paragraph 5]:

(5.13) THEOREM. *Suppose that* $v(X)$ *is subharmonic in* D. *Then* $\int_D \Delta(X)\, |G(X, X_0)|\, dX < \infty$ *for some* $X_0 \in D$ *if and only if* v *has a harmonic majorant. Moreover, if* v^* *denotes the least harmonic majorant of* v, *then for all* $Y \in D$,

$$v(Y) = v^*(Y) + \int_D \Delta v(X)\, G(X, Y)\, dX.$$

From now on we will denote $G(X) = G(X, X_0)$. (Recall that $\omega = \omega^{X_0}$.)

(5.14) THEOREM. *Let* D *be an NTA domain. If* $f \in L^2(d\omega)$, $\int f\, d\omega = 0$ *and* $u(X) = \int_{\partial D} f(Q)\, d\omega^X(Q)$, *then*

$$\int_D |\nabla u|^2\, |G(X)|\, dX = \tfrac{1}{2} \int_{\partial D} f(Q)^2\, d\omega(Q) < \infty.$$

Conversely, if u *is harmonic*, $u(X_0) = 0$, *and* $\int_D |\nabla u|^2\, |G(X)|\, dX < \infty$, *then there exists* $f \in L^2(d\omega)$ *such that* $u(X) = \int_{\partial D} f(Q)\, d\omega^X(Q)$ *and* u *approaches* f *non-tangentially a.e.* ω.

Proof. Suppose that f is continuous on ∂D and $\int f\, d\omega = 0$. Let $u(X) = \int_{\partial D} f(Q)\, d\omega^X(Q)$. Let $v(X) = u(X)^2$. Then $\Delta v = 2\, |\nabla u|^2 \geqslant 0$. v has a harmonic majorant because it is bounded. Therefore, by Theorem (5.13),

$$v(X_0) = v^*(X_0) - 2 \int_D |\nabla u(X)|^2\, |G(X)|\, dX.$$

Note that $v(X_0) = 0$ and $v^*(X_0) = \int_{\partial D} f(Q)^2\, d\omega$. Hence,

$$\tfrac{1}{2} \int_{\partial D} f(Q)^2\, d\omega(Q) = \int_D |\nabla u(X)|^2\, |G(X)|\, dX.$$

Now let $f \in L^2(d\omega)$ with $\int f\, d\omega = 0$. Choose continuous functions f_j

approaching f in L^2 norm. If u and u_j denote the harmonic extensions of f and f_j, respectively, then ∇u_j approaches ∇u uniformly on compact subsets of D. Therefore

$$\int_D |\nabla u|^2 |G(X)| \, dX \leqslant \sup_K \lim_{j\to\infty} \int_K |\nabla u_j(X)|^2 |G(X)| \, dX$$

$$\leqslant \sup_j \tfrac{1}{2} \int_{\partial D} f_j(Q)^2 \, d\omega < \infty.$$

Now a simple limit procedure implies that

$$\int_D |\nabla u(X)|^2 |G(X)| dX = \tfrac{1}{2} \int_{\partial D} f(Q)^2 \, d\omega(Q).$$

For the converse, let $v = u^2$. Since $\int_D |\nabla u|^2 |G(X)| \, dX < \infty$, v has a least harmonic majorant v^*. Both $u + (1 + v^*)$ and $(1 + v^*)$ are positive harmonic functions in D. Therefore, by Theorem (5.10),

$$u(X) = u + (1 + v^*) - (1 + v^*) = \int_{\partial D} K(X, Q) \, d\mu(Q)$$

for some finite Borel measure μ. If all we wanted to conclude was that u has non-tangential limits a.e. ω, we could stop here and apply Theorem (5.8). However, if $d\mu = g \, d\omega + dv$, where v is singular with respect to ω, we wish to show that $v = 0$ and $g \in L^2(d\omega)$.

Step 1. $v = 0$.

(5.15) LEMMA. *If $u(X) = \int_{\partial D} K(X, Q) \, d\mu(Q)$, then for any $\varepsilon > 0$, there exists a closed set $F \subset \partial D$ and a (sawtooth) NTA domain Ω_ε such that $\omega(\partial D \backslash F) < \varepsilon$, $\partial\Omega_\varepsilon \cap \partial D = F$, u is bounded on Ω_ε. If $\omega_\varepsilon = \omega_\varepsilon^{X_0}$ denotes harmonic measure on $\partial\Omega_\varepsilon$ for a point X_0 of Ω_ε, then ω_ε and ω are mutually absolutely continuous on F. Moreover, $\omega_\varepsilon(\partial\Omega_\varepsilon \backslash F) < M\varepsilon$.*

Proof. Let F be a set on which u is non-tangentially uniformly bounded. The sawtooth domain is constructed exactly as in Lemma (6.3). The only additional property we have stated here is that $\omega_\varepsilon(\partial\Omega_\varepsilon \backslash F) < M\varepsilon$. To prove this recall from (6.3) that $\omega^X(\partial D \backslash F) > M^{-1}$ for all $X \in \partial\Omega_\varepsilon \backslash F$. By the maximum principle,

$$\omega_\varepsilon(\partial\Omega_\varepsilon \backslash F) < M\omega(\partial D \backslash F) < M\varepsilon.$$

It is easy to see from the proof that we can arrange for Ω_ε to increase to D as $\varepsilon \to 0$.

(5.16) PROPOSITION [2]. *If Ω_ε increase to D and f is a continuous function on $\bar{D}$, then*

$$\int_{\partial D} f(Q)\, d\omega^X(Q) = \lim_{\varepsilon\to\infty} \int_{\partial\Omega_\varepsilon} f(Q)\, d\omega_\varepsilon^X(Q).$$

We now proceed with Step 1. By (5.12), there exists $f_\varepsilon \in L^\infty(d\omega_\varepsilon)$ such that $u(X) = \int_{\partial\Omega_\varepsilon} f_\varepsilon(Q)\, d\omega_\varepsilon^X(Q)$. Denote the Green function at X_0 of Ω_ε by $G_\varepsilon(X) = G_\varepsilon(X, X_0)$. By the maximum principle, $|G_\varepsilon(X)| \leqslant |G(X)|$. By the first half of Theorem (5.14),

$$\begin{aligned}\int_{\partial\Omega_\varepsilon} f_\varepsilon(Q)^2\, d\omega_\varepsilon(Q) &= 2\int_{\partial\Omega_\varepsilon} |\nabla u(X)|^2\, |G_\varepsilon(X)|\, dX \\ &\leqslant 2\int_D |\nabla u(X)|^2\, |G(X)|\, dX = C < \infty.\end{aligned}$$

Fix $X \in D$; then

$$u(X) = \int_{\partial\Omega_\varepsilon\cap\partial D} f_\varepsilon(Q)\, d\omega_\varepsilon^X(Q) + \int_{\partial\Omega_\varepsilon\setminus\partial D} f_\varepsilon(Q)\, d\omega_\varepsilon^X(Q).$$

$$\begin{aligned}\left|\int_{\partial\Omega_\varepsilon\setminus\partial D} f_\varepsilon(Q)\, d\omega_\varepsilon^X(Q)\right| &\leqslant \left(\int_{\partial\Omega_\varepsilon} f(Q)^2\, d\omega_\varepsilon^X\right)^{1/2} \omega_\varepsilon^X(\partial\Omega_\varepsilon\setminus\partial D)^{1/2} \\ &\leqslant C^{1/2}(M\varepsilon)^{1/2}.\end{aligned}$$

Furthermore, $f_\varepsilon|_{\partial\Omega_\varepsilon\cap\partial D} = g|_{\partial\Omega_\varepsilon\cap\partial D}$ since both are the non-tangential limit a.e. ω of u. Hence, $u(X) = \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon\cap\partial D} g(Q)\, d\omega_\varepsilon^X(Q)$. Let g_δ be continuous on ∂D and $\int_{\partial D} |g - g_\delta|\, d\omega < \delta$. Denote $u_\delta(X) = \int_{\partial D} g_\delta(Q)\, d\omega^X(Q)$. Since u_δ is continuous in $\bar{D}$, (5.16) implies

$$u_\delta(X) = \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} u_\delta(Q)\, d\omega_\varepsilon^X(Q).$$

Hence,

$$\begin{aligned}u_\delta(X) &= \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon\cap\partial D} u_\delta(Q)\, d\omega_\varepsilon^X(Q) \\ &= \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon\cap\partial D} g_\delta(Q)\, d\omega_\varepsilon^X(Q).\end{aligned}$$

Thus,

$$|u(X) - u_\delta(X)| = \left| \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon \cap \partial D} (g(Q) - g_\delta(Q))\, d\omega_\varepsilon^X(Q) \right|$$

$$\leqslant \sup_\varepsilon \int_{\partial\Omega_\varepsilon \cap \partial D} |g(Q) - g_\delta(Q)|\, d\omega_\varepsilon^X(Q)$$

$$\leqslant \int_{\partial D} |g(Q) - g_\delta(Q)|\, d\omega^X(Q) \leqslant C_X \delta.$$

Thus, $u_\delta \to u$ uniformly on compact sets. And therefore, $u(X) = \int_{\partial D} g(Q)\, d\omega^X(Q)$. This concludes Step 1.

Step 2. $g \in L^2(d\omega)$.

(5.17) LEMMA. *Let D be an NTA domain. For any $\varepsilon > 0$ there exists $D_\varepsilon \subset\subset D$ such that $D_\varepsilon \supset \{P: d(P, \partial D) < \varepsilon\}$, and D_ε is an NTA domain with bounds independent of ε.*

This lemma can be proved in the same way as Lemma (6.3) or Theorem (3.11), but is somewhat simpler.

Denote $T = \{X: d(X, \partial D) < \frac{1}{10} r\}$. Choose a family of balls $B_j = B(Q_j, r)$ with $Q_j \in \partial D$ such that $\bigcup B_j \supset T$ and such that at most 10^n balls intersect at a point. Let ϕ_j be a continuous partition of unity of T subordinate to $\{B_j\}$. Denote $A_j = A_r(Q_j)$. Denote $h_r(Q) = \sum_j u(A_j)\, \phi_j(Q)$. For D_ε as in (5.17), let $\omega_\varepsilon = \omega_\varepsilon^{X_0}$ denote harmonic measure at X_0 for D_ε. By (5.16),

$$\int_{\partial D} h_r(Q)^2\, d\omega(Q) = \lim_{\varepsilon\to 0} \int_{\partial D_\varepsilon} h_r(Q)^2 d\omega_\varepsilon(Q).$$

For $\varepsilon < M^{-1} r$,

$$\int_{\partial D_\varepsilon} h_r(Q)\, d\omega_\varepsilon(Q) \leqslant C \sum_j \int_{\partial D_\varepsilon} u(A_j)^2\, \phi_j(Q)^2\, d\omega_\varepsilon(Q)$$

$$\leqslant C \int_{\partial D_\varepsilon} N^{(\varepsilon)}(u)^2\, (Q)\, d\omega_\varepsilon(Q).$$

$N^{(\varepsilon)}$ denotes the non-tangential maximal function for D_ε. The second inequality follows from the observation that for sufficiently large α, if $Q \in \operatorname{supp} \phi_j$, then $A_j \in \Gamma_\alpha(Q)$.

By Theorem (5.8) and (5.7), applied to D_ε,

$$\int_{\partial D_\varepsilon} N^{(\varepsilon)}(u)^2\,(Q)\,d\omega_\varepsilon(Q) \leqslant M \int_{\partial D_\varepsilon} u(Q)^2\,d\omega_\varepsilon(Q)$$

$$= 2M \int_{D_\varepsilon} |\nabla u(X)|^2\,|G_\varepsilon(X)|\,dX$$

$$\leqslant 2M \int_{D} |\nabla u(X)|^2\,|G(X)|\,dX < \infty.$$

(G_ε denotes the Green function of D_ε at X_0.)

The equality above is from the first part of (5.14). It is valid because u is continuous in $\bar{D}_\varepsilon$.

Combining the inequalities above we conclude that $\|h_r(Q)\|_{L^2(d\omega)}$ is uniformly bounded as $r \to 0$. Choose $h \in L^2(d\omega)$ and a subsequence $h_{r_k} \to h$ weakly in $L^2(d\omega)$. Notice that u approaches g non-tangentially uniformly except on a set of arbitrarily small harmonic measure. It is then easy to see that $g = h$ a.e. ω. Thus, $g \in L^2(d\omega)$ as desired.

6. Local Boundary Behavior of Harmonic Functions

Throughout this section D will denote a non-tangentially accessible domain. A truncated non-tangential approach region (or corkscrew) at $Q \in \partial D$ is denoted $\Gamma_\alpha^h(Q) = \Gamma_\alpha(Q) \cap B(Q, h)$. (See (5.6).) If more than one domain is under consideration, we will display the dependence on the domain by $\Gamma_{\alpha,D}^h(Q)$. We say that a function u defined on D is non-tangentially bounded from below at $Q \in \partial D$ if there exist α, h, M such that $u(X) \geqslant -M$ for all $X \in \Gamma_\alpha^h(Q)$. If $F \subset \partial D$, we say that u is non-tangentially bounded from below on F if u is non-tangentially bounded from below at every point of F.

Denote $S_\alpha(F) = \bigcup_{Q \in F} \Gamma_\alpha(Q)$ and $S_\alpha^h(F) = \bigcup_{Q \in F} \Gamma_\alpha^h(Q)$. We will write $S_{\alpha,D}(F)$ if the former notation is ambiguous.

(6.1) Lemma. *Let $F \subset \partial D$. For every h, $\varepsilon > 0$, $\alpha_2 > \alpha_1 > 0$, there exists a closed set $F_0 \subset F$ and a number $k > 0$ such that $\omega(F \backslash F_0) < \varepsilon$ and $S_{\alpha_2}^k(F_0) \subset S_{\alpha_1}^h(F)$.*

Proof. This lemma is proved by a well-known point of density argument [48, p. 202]. The weak (1, 1) estimate for M_ω (5.7) implies that the set of points of density of F

$$\tilde{F} = \left\{ Q \in F\colon \lim_{r \to 0} \frac{\omega(\Delta(Q, r) \cap F)}{\omega(\Delta(Q, r))} = 1 \right\}$$

satisfies $\omega(F \backslash \tilde{F}) = 0$.

Let $s = \alpha_1 + \alpha_2 + 1$. Choose a closed set $F_0 \subset \tilde{F}$ and k such that $0 < k < h/(1 + \alpha_1)$, $\omega(\tilde{F} \backslash F_0) < \varepsilon$, and for all $Q \in F_0$, $r < sk$,

$$\omega(\Delta(Q, r) \cap F) > (1 - \eta)\, \omega(\Delta(Q, r)).$$

(η will be chosen later.)

Let $X \in \Gamma^k_{\alpha_2}(Q)$ for some $Q \in F_0$. Denote by $\tilde{X}$ a point of ∂D closest to X. By the doubling condition (4.9), $\omega(\Delta) < M\omega(\Delta')$, where

$$\Delta = \Delta(Q, s\,|X - \tilde{X}|) \qquad \text{and} \qquad \Delta' = \Delta(\tilde{X}, \alpha_1\,|X - \tilde{X}|).$$

Therefore,

$$\begin{aligned} \omega(\Delta' \cap F) &\geqslant \omega(\Delta \cap F) - \omega(\Delta \backslash \Delta') \\ &> ((1 - \eta) - (1 - M^{-1}))\, \omega(\Delta) > 0, \end{aligned}$$

provided $\eta < M^{-1}$. Thus, $\Delta' \cap F \neq \varnothing$. It is easy to see that for $Q' \in \Delta'$, $X \in \Gamma^h_{\alpha_1}(Q')$. Hence, $X \in S^h_{\alpha_1}(F)$.

(6.2) Lemma. *Let $F \subset \partial D$. Suppose that u is continuous in D and non-tangentially bounded from below on F. For any $\alpha, \varepsilon > 0$, there exists a closed set $F_0 \subset F$ with $\omega(F \backslash F_0) < \varepsilon$, and a constant C such that $u(X) \geqslant -C$ for all $X \in S_\alpha(F_0)$.*

Proof. Since u is bounded from below, for any $\varepsilon > 0$, there exist constants α, h, and C_1 and a set $F_1 \subset F$ with $u(X) \geqslant -C_1$ on $S^h_{\alpha_1}(F_1)$ and $\omega(F \backslash F_1) < \varepsilon/2$. By (6.1), for any $\alpha = \alpha_2$, there exists $F_0 \subset F$, $\omega(F_1 \backslash F_0) < \varepsilon/2$ and $k > 0$ such that $S^k_\alpha(F_0) \subset S^h_{\alpha_1}(F_1)$. Note that $S_\alpha(F_0) \backslash S^k_\alpha(F_0)$ is relatively compact in D, and thus the continuity of u gives the lemma.

(6.3) Lemma. *For any $\alpha > 0$, there exist β, $\gamma > 0$ such that for any closed set $F \subset \partial D$ there exists an NTA domain $\Omega \subset D$ with $\partial D \cap \partial \Omega = F$, $\bar{\Omega} \subset S_{\beta, D}(F)$ and $\Gamma_{\alpha, D}(Q) \subset \Gamma_{\gamma, \Omega}(Q)$ for every $Q \in F$. Moreover, ω_Ω and ω_D are mutually absolutely continuous on F.*

Proof. The construction is along the lines of the one in [32]. Only a few extra remarks are needed. Let $\{I_j\}$ be the dyadic Whitney decomposition of D, by closed cubes I_j. Given a cube I, $(1 + \delta) I$ denotes the cube with the same center as I, expanded $(1 + \delta)$ time. If $\delta = \frac{1}{32}$, and $(1 + \delta) I_j \cap (1 + \delta) I_k \neq \varnothing$, then $I_j \cap I_k \neq \varnothing$. In this case, $(1 + \delta) I_j \cup (1 + \delta) I_k$ is an *NTA* domain. Let $D = \{I_j : I_j \cap S_{2\alpha}(F) \neq \varnothing\}$. Fix a Whitney cube I_0 of D. For every $I_j \in D$, make a pipe P (see [32] for the definition) connecting I_j to I_0. These are called primary pipes. Let $Z = \{z : z$ is the center of a Whitney cube I with $I \cap P \neq \varnothing$ for some primary pipe $P\}$. Let I_j correspond to z_j. If z_j, z_k satisfy $\frac{1}{4} \leqslant \operatorname{diam}(I_j)/\operatorname{diam}(I_k) \leqslant 4$ and $|z_j - z_k| < M \operatorname{diam}(I_j)$, make a pipe connecting z_j to z_k. These are called secondary pipes. Note that if a pipe does not intersect a Whitney cube I, it does not intersect the expanded cube $(1+\delta)I$. Finally, let

$$\Omega = \bigcup \{(1+\delta)I : I \in D\} \cup \{P : P \text{ is a primary or secondary pipe}\}.$$

The verification that Ω satisfies the required geometric properties is now completely analogous to the one in [32].

To check that ω_Ω and ω_D are mutually absolutely continuous on F, first observe that $\omega_\Omega \leqslant \omega_D$ on F by the maximum principle. Next, take $E \subset F$ with $\omega_\Omega(E) = 0$.

Claim. There exists $C_0 > 0$ such that

$$\omega_D^X({}^cF) > C_0 \qquad \text{for all } X \in (\partial\Omega)\backslash F.$$

In fact, because $S_{\alpha,D}(F) \subset \Omega \cup F$, $(\partial\Omega)\backslash F \subset {}^cS_{\alpha,D}(F)$. Denote a point of D nearest to X by $\tilde{X}$. For every $Q \in \Delta(\tilde{X}, \alpha|X-\tilde{X}|)$, $X \in \Gamma_\alpha(Q)$. Hence, $\Delta(\tilde{X}, \alpha|X-\tilde{X}|) \subset {}^cF$. And thus $\omega_D^X({}^cF) \geqslant \omega_D^X(\Delta(\tilde{X}, \alpha|X-\tilde{X}|)) > C_0$, by (4.2).

A lower function for χ_E in D is a function $\Phi(X)$ that is subharmonic in D, such that $\limsup_{X\to Q} \Phi(X) \leqslant \chi_E(Q)$ for all $Q \in \partial D$. Recall that $\omega_D^X(E) = \sup\{\Phi(X): \Phi \text{ is a lower function for } \chi_E\}$. Hence $\Phi(X) \leqslant \omega_D^X(E) \leqslant 1 - C_0$. Therefore, $\Phi(X) - 1 + C_0$ is a lower function for χ_E in Ω. Thus $\Phi(X) - 1 + C_0 \leqslant \omega_\Omega^X(E) = 0$, and so $\Phi(X) \leqslant 1 - C_0$ for all $X \in \Omega$. But then $\omega_D^X(E) \leqslant 1 - C_0$ for all $X \in \Omega$. This shows that $\omega_D(E) = 0$, because Theorem (5.8) says that $\omega_D^X(E)$ converges non-tangentially to 1 a.e. ω on E.

(6.4) THEOREM. *Assume that u is harmonic in D and non-tangentially bounded from below on $F \subset \partial D$. Then u has non-tangential limits a.e. (ω) on F.*

Proof. Given $\alpha > 0$, we will show that u has limits in $\Gamma_\alpha(Q)$ for a.e. Q in F. Let β correspond to α as in (6.3). Given $\varepsilon > 0$, choose, by (6.2), a closed set $F_0 \subset F$ such that $u(X) \geqslant -C$ on $S_\beta(F_0)$ and $\omega(F\backslash F_0) < \varepsilon$. Construct Ω as in (6.3) corresponding to α and F_0. Then $u \geqslant -C$ on Ω. By (5.11), u has non-tangential limits a.e. (ω_Ω) on $\partial\Omega$. The theorem now follows because $\Gamma_{\alpha,D}(Q) \subset \Gamma_{\delta,\Omega}(Q)$ for $Q \in F_0$, and ω_Ω and ω_D are mutually absolutely continuous on F_0.

(6.5) DEFINITION. *Let $u \in C^1(D)$, $\alpha > 0$. The* area integral *of u in D is given by*

$$A_\alpha(u)(Q)^2 = \int_{\Gamma_\alpha(Q)} |\nabla u(X)|^2\, d(X)^{2-n}\, dX \qquad \text{for } Q \in \partial D.$$

Here, $d(X) \equiv d(X, D)$. When there is more than one domain under consideration, we use the notation $A_{\alpha, D}(u)(Q)$ for the area integral of u in D.

We say that the area integral of u is finite on F if for any $Q \in F$, there exists $\alpha > 0$ (depending on Q) such that $A_\alpha(u)(Q) < \infty$.

(6.6) THEOREM. *Let u be harmonic in an NTA domain D. The set of points of D where the area integral is finite equals a.e. (ω) the set of points where u has non-tangential limits.*

(6.7) LEMMA. *Assume that $u \in C^1(D)$. Then*

$$\int_{\partial D} A_\alpha(u)(Q)^2 \, d\omega(Q) \leqslant M \int_D |\nabla u(X)|^2 \, |G(X)| \, dX,$$

where $G(X) = G(X, X_0)$, $\omega = \omega^{X_0}$.

Proof.

$$\int_{\partial D} A_\alpha(u)(Q) \, d\omega(Q) = \int_{\partial D} \left(\int_{\Gamma_\alpha(Q)} |\nabla u(X)|^2 \, d(X)^{2-n} \, dX \right) d\omega(Q)$$

$$= \int_D d(X)^{2-n} \, |\nabla u(X)|^2 \, \omega\{Q \in \partial D: X \in \Gamma_\alpha(Q)\} \, dX.$$

Denote by $\tilde{X}$ a point of ∂D closest to X. Then $\{Q \in \partial D: X \in \Gamma_\alpha(Q)\} \subset \Delta(\tilde{X}, (\alpha+2)|X - \tilde{X}|)$. But (4.8) and (4.9) show that

$$d(X)^{2-n} \, \omega(\Delta(\tilde{X}, (\alpha+2)|\tilde{X} - X|)) \leqslant M |G(X)|,$$

and the lemma is established.

We now turn to the proof of (6.6). Assume first that u is non-tangentially bounded on F. As in the proof of Theorem (6.4), we can insert an NTA domain Ω on which u is bounded and reduce matters to showing that the area integral of u on Ω is finite a.e. (ω_Ω). By (5.12), $u(X) = \int_{\partial\Omega} g(Q) \, d\omega_\Omega^X(Q)$ for some $g \in L^\infty(D\omega)$ for all $X \in \Omega$. Thus by (5.14),

$$\int_\Omega |\nabla u(X)|^2 \, |G_\Omega(X)| \, dX = \int_{\partial\Omega} |g(Q) - u(X_0)|^2 \, d\omega_\Omega < \infty,$$

where $G_\Omega(X) = G_\Omega(X, X_0)$, $\omega_\Omega = \omega_\Omega^{X_0}$. Thus, by (6.7) the area integral of u on Ω is finite.

For the converse, assume that the area integral of u in D is finite on F.

For any $\varepsilon > 0$, we can choose $F_0 \subset F$, $\alpha > 0$, and C such that $\omega(F \backslash F_0) < \varepsilon$ and $A_\alpha(u)(Q) \leqslant C$ for all $Q \in F_0$. It follows that

$$\int_{S_\alpha(F_0)} d(X)^{2-n} |\nabla u(X)|^2 \, \omega\{Q \in F_0 : X \in \Gamma_\alpha(Q)\} \, dX$$

$$\leqslant \int_{F_0} A_\alpha(u)(Q)^2 \, d\omega(Q) \leqslant C^2.$$

There exists a constant N and a set $F_1 \subset F_0$ such that $\omega(F_0 \backslash F_1) < \varepsilon$ and for all $Q \in F_1$, $\omega(F_0 \cap \Delta(Q, r)) \geqslant N^{-1} \omega(\Delta(Q, r))$. Let $X \in S_{\alpha/2}(F_1)$. Choose $Q_1 \in F_1$ such that $X \in \Gamma_{\alpha/2}(Q_1)$.

$$\{Q \in F_0 : X \in F_\alpha(Q)\} \supset F_0 \cap \Delta(Q_1, (\alpha/2) \, d(X)).$$

Hence $\omega\{Q \in F_0 : X \in \Gamma_\alpha(Q)\} \geqslant N^{-1} \omega(\Delta(Q, (\alpha/2) \, d(X)))$. Moreover, by the doubling condition (4.9),

$$\omega(\Delta(Q_1, (\alpha/2) \, d(X))) \geqslant N^{-1} \omega(\Delta(\tilde{X}, d(X))).$$

Finally, (4.8) implies

$$\int_{S_{\alpha/2}(F_1) \backslash B(X_0, (1/2) d(X_0))} |\nabla u(X)|^2 \, |G(X)| \, dX$$

$$\leqslant M \int_{S_{\alpha/2}(F_1)} |\nabla u(X)|^2 \, d(X)^{2-n} \, \omega(\Delta(\tilde{X}, d(X))) \, dX$$

$$\leqslant MN^2 \int_{S_{\alpha/2}(F_1)} |\nabla u(X)|^2 \, d(X)^{2-n} \, \omega\{Q \in F_0 : X \in \Gamma_\alpha(Q)\} \, dX.$$

Furthermore, $\int_{B(X_0, (1/2) d(X_0))} |\nabla u(X)|^2 \, |G(X)| \, dX < \infty$. Using (6.1) and (6.3) insert an *NTA* domain Ω in $S_{\alpha/2}(F_1)$, with $\omega(F_1 \backslash \partial\Omega) < \varepsilon$. We may as well assume that $X_0 \in \Omega$. $|G_\Omega(X)| \equiv |G_\Omega(X, X_0)| \leqslant |G(X, X_0)|$. Hence, $\int_\Omega |G_\Omega(X)| \, |\nabla u(X)|^2 \, dX < \infty$. By Theorem (5.14), u has non-tangential limits in Ω a.e. ω_Ω, and the theorem follows.

The following corollary is a version of a theorem of Stein for *NTA* domains (see [46, 48]).

(6.8) Corollary. *Suppose that $u_1, \ldots, u_n$ are harmonic in D and satisfy the generalized Cauchy–Riemann equations $\sum_{i=1}^n (\partial u_i / \partial x_i) = 0$, $\partial u_i / \partial u_j = \partial u_j / \partial u_i$. If $(n-1)$ of the u_i's are non-tangentially bounded on a set $F \subset \partial D$, then the remaining one has non-tangential limits a.e. (ω) on F.*

Proof. The area integral of each u_i is bounded by the sum of the area integrals of the others, so the result follows from (6.6) and (6.4).

7. Hölder Continuity of the Kernel Function

We shall prove an estimate on the kernel function $K(X, Q)$ that is needed for the theory of H^p, $p \leqslant 1$, on non-tangentially accessible domains (see Section 8).

(7.1) THEOREM. *There is a constant M depending only on the NTA constat of D such that if $Q_0, Q_1 \in \partial D$, $X \in D$ and $|X - Q_0| \geqslant M^j |Q_1 - Q_0|$, then*

$$\left| \frac{K(X, Q_1)}{K(X, Q_0} - 1 \right| \leqslant M(1 - M^{-1})^j.$$

COROLLARY. *$K(X, Q)$ is Hölder continuous as a function of Q: $|K(X, Q) - K(X, Q')| < C_X |Q - Q'|^\alpha$ for some $\alpha > 0$, depending only on the NTA constant of D.*

The corollary is an immediate consequence of (7.1), if we multiply the inequality in (7.1) by $K(X, Q_0)$.

Proof of (7.1).

(7.2) LEMMA. *Let $M > 2$. Let μ be a positive finite measure on a set S, and let θ be a measurable function on S such that $0 < a \leqslant \theta \leqslant A$. Denote*

$$B(\theta) = \sup \left\{ \frac{\int_S \theta(x)\, w(x)\, d\mu(x)}{\int_S w(x)\, d\mu(x)} : M^{-1} < w < M \right\},$$

$$b(\theta) = \inf \left\{ \frac{\int_S \theta(x)\, w(x)\, d\mu(x)}{\int_S w(x)\, d\mu(x)} : M^{-1} < w < M \right\}.$$

Then

$$\frac{B(\theta)}{b(\theta)} - 1 \leqslant \left(1 - \frac{1}{4} M^{-2}\right) \left(\frac{A}{a} - 1\right).$$

Proof. Without loss of generality, we may assume that $a = 1$ and $\mu(S) = 1$. Denote $\gamma = (1 - \frac{1}{4} M^{-2})(A - 1)$.

Case 1. $B(\theta) \leqslant 1 + \gamma$.
Observe that $b(\theta) \geqslant 1 = a$. Then

$$\frac{B(\theta)}{b(\theta)} - 1 \leqslant \frac{1 + \gamma}{b(\theta)} - 1 \leqslant (1 + \gamma) - 1 = \gamma.$$

Case 2. $B(\theta) > 1 + \gamma$.

Denote $S_1 = \{x \in S: \theta(x) > 1 + \frac{1}{2}\gamma\}$. Let $m = \mu(S_1)$. Define

$$\begin{aligned} \theta_1(x) &= A, & x &\in S_1, \\ &= 1 + \tfrac{1}{2}\gamma, & x &\in S \backslash S_1. \end{aligned}$$

Since $\theta \leqslant \theta_1$, $B(\theta) \leqslant B(\theta_1)$. It is easy to calculate that

$$B(\theta_1) = \frac{AMm + (1 + \frac{1}{2}\gamma)(1 - m) M^{-1}}{Mm + (1 - m) M^{-1}}.$$

Thus

$$1 + \gamma < \frac{AMm + (1 + \frac{1}{2}\gamma)(1 - m) M^{-1}}{Mm + (1 - m) M^{-1}}.$$

Therefore, $\frac{1}{2}\gamma < ((A - 1) M^2 + \frac{1}{2}\gamma - \gamma M^2) m$, and hence $m > \frac{1}{2}$. Denote

$$\begin{aligned} \theta_2(x) &= 1 + \tfrac{1}{2}\gamma, & x &\in S_1, \\ &= 1, & x &\in S \backslash S_1. \end{aligned}$$

$\theta_2 \leqslant \theta$ implies $b(\theta_2) \leqslant b(\theta)$. It is easy to calculate

$$b(\theta_2) = \frac{M^{-1}(1 + \frac{1}{2}\gamma) m + M(1 - m)}{M^{-1} m + M(1 - m)} > \frac{(1 + \frac{1}{2}\gamma) M^{-1} + M}{M^{-1} + M}$$

since $m > \frac{1}{2}$. Observe that $B(\theta) \leqslant A$. Therefore,

$$\frac{B(\theta)}{b(\theta)} - 1 \leqslant \frac{A(M^{-1} + M)}{(1 + \frac{1}{2}\gamma) M^{-1} + M} - 1 = (A - 1)\left(1 - \frac{1 - \frac{1}{4}M^{-2}}{2M(M + M^{-1})}\right) < \gamma.$$

(7.3) LEMMA. *For any point $Q \in \partial D$, and distance $r < r_0$, there is $\Omega \subset D$ such that $B(Q, r) \cap \Omega = \varnothing$ and $\Omega \supset D \backslash B(Q, Mr)$ and Ω is an NTA domain with constant depending only on that of D.*

Proof. This is a variant of Jones' theorem. (See (3.11) and [32].) Consider all Whitney cubes of D that do not intersect $B(Q, \frac{1}{10} Mr)$. Enlarge this union of cubes by adding primary and secondary pipes as in [32]. The *NTA* domain we obtain, Ω, has *NTA* constant depending only on that of D. Because the pipes added are non-tangential, $B(Q, r) \cap \Omega = \varnothing$.

Let Q_0, Q_1 belong to ∂D. Let $r = |Q_1 - Q_0|$. Denote by Ω_j the region of Lemma (7.3) corresponding to Q_0 and the distance $M^j r$, $j = 1, 2, \ldots$; $M^j r < r_0$.

$$D \backslash B(Q_0, M^j r) \subset \Omega_j \subset D \backslash B(Q_0, 2M^{j-1} r).$$

(As usual, we replace M with a larger value than in (7.3).)

Denote $S_j = (\partial\Omega_j)\backslash\partial D$. We can assume that $X_0 \in \Omega_j$ and $d(X_0, \partial\Omega_j) \simeq d(X_0, \partial D)$. Denote harmonic measure for Ω_j by ω_j^X and $K_j(X, Q) = (d\omega_j^X/d\omega_j^{X_0})(Q)$.

Lemma (4.11) can be restated as follows. Let $A = A_r(Q_0)$. For all $Q_1 \in \partial D$, such that $|Q_1 - Q_0| \leqslant r$, $M^{-1} < K(A, Q_1)/\omega(\Delta) < M$, where $\Delta = \Delta(Q_0, r)$. Hence, replacing M with a larger value,

$$M^{-1} < K(A, Q_1)/K(A, Q_0) < M.$$

By the boundary Harnack principle (Lemma (4.10)) again replacing M by a larger constant we have

$$M^{-1} < K(X, Q_1)/K(X, Q_0) < M \qquad \text{for all } X \in D\backslash B(Q_0, 2r). \tag{7.4}$$

Similarly, we have

$$\begin{aligned} M^{-1} &< K_j(X, Q')/K_j(X, Q'') \\ &< M \qquad \text{for } X \in D\backslash B(Q_0, 2M^j r) \text{ and } Q', Q'' \in S_j. \end{aligned} \tag{7.5}$$

Denote $u_0(X) = K(X, Q_0)$, $u_1(X) = K(X, Q_1)$. Define

$$b_j = \inf\left\{\frac{u_1(X)}{u_0(X)} : X \in S_j\right\},$$

$$B_j = \sup\left\{\frac{u_1(X)}{u_0(X)} : X \in S_j\right\}.$$

The maximum principle implies

$$b_j u_0(X) \leqslant u_1(X) \leqslant B_j u_0(X) \qquad \text{for all } X \in \Omega_j.$$

Since $u_0(X_0) = 1 = u_1(X_0)$ we see that

$$b_j \leqslant 1 \leqslant B_j. \tag{7.6}$$

Next, (7.4) implies that

$$b_1 > M^{-1} \qquad \text{and} \qquad B_1 < M. \tag{7.7}$$

Define $\theta(Q) = u_1(Q)/u_0(Q)$ for $Q \in S_j$. Define $d\mu(Q) = u_0(Q)\, d\omega_j(Q)$ for $Q \in S_j$. Let $X \in D\backslash B(Q_0, 2M^j r)$ and $Q_2 \in S_j$. Denote $w(Q) = K_j(X, Q)/K_j(X, Q_2)$ for $Q \in S_j$. By (7.5), $M^{-1} < w < M$.

Note that since u_0 and u_1 are harmonic and continuous in $\bar{\Omega}_j$,

$$u_1(X) = K_j(X, Q_2) \int_{S_j} \theta(Q)\, w(Q)\, d\mu(Q),$$

$$u_0(X) = K_j(X, Q_2) \int_{S_j} w(Q)\, d\mu(Q).$$

Therefore,

$$B_{j+1} = \sup \left\{ \frac{u_1(X)}{u_0(X)} : X \in S_{j+1} \right\}$$
$$\leqslant \sup \left\{ \frac{\int_{S_j} \theta(Q)\, w(Q)\, d\mu(Q)}{\int_{S_j} w(Q)\, d\mu(Q)} : M^{-1} < w < M \right\} = B(\theta),$$

in the notation of Lemma (7.2). Similarly, $b_{j+1} \geqslant b(\theta)$. Notice that $b_j \leqslant \theta(Q) \leqslant B_j$. Therefore, by Lemma (7.2),

$$\left(\frac{B_{j+1}}{b_{j+1}} - 1\right) \leqslant \left(1 - \frac{1}{4} M^{-2}\right) \left(\frac{B_j}{b_j} - 1\right). \tag{7.8}$$

Combining (7.7) and (7.8), we have

$$\left| \frac{B_j}{b_j} - 1 \right| \leqslant M^2 (1-\varepsilon)^j \qquad \left(\varepsilon = \frac{1}{4} M^{-2}\right).$$

We deduce from (7.6) that

$$\left| \frac{u_1(X)}{u_0(X)} - 1 \right| \leqslant M^2 (1-\varepsilon)^j \qquad \text{for } X \in S_j,$$

and hence, by the maximum principle, for all $X \in \Omega_j$.

We will now use the same reasoning to prove a refinement of the boundary Harnack principle (5.1).

(7.9) THEOREM. *Let D be an NTA domain, and let V be an open set. Let K be a compact subset of V. There exists a number $\alpha > 0$ such that for all positive harmonic functions u and v in D that vanish continuously on $\partial D \cap V$, the function $u(X)/v(X)$ is Hölder continuous of order α in $K \cap \bar{D}$. In particular $\lim_{X \to Q} u(X)/v(X)$ exists for every $Q \in \partial D \cap K$.*

Proof. Multiplying v by a constant, we may as well assume that $u(X_0) = v(X_0)$. By the boundary Harnack principle (5.1), we conclude that $M^{-1} < u(X)/v(X) < M$. Choose r_0 so that $B(Q, 2r_0) \subset V$ for all $Q \in K$. Let $Q \in K \cap \partial D$. Denote by $\{\Omega_j\}_{j=1}^{\infty}$ a sequence of regions corresponding to Q at distance $M^{-j} r_0$, given by Theorem (3.11). Thus, $B(Q, 2M^{-j} r_0) \cap$

$D \subset \Omega_j \subset B(Q, M^{-j+1}r_0) \cap D$. Denote $B_j = \sup\{(u(X)/v(X)) : X \in \Omega_j\}$, $b_j = \inf\{(u(X)/v(X)) : X \in \Omega_j\}$. The sequence B_j is decreasing, and the sequence b_j is increasing. Moreover, by the same argument as in the proof of Theorem (7.1), $|B_j/b_j - 1| \leqslant M^2(1-\varepsilon)^j$. The Hölder continuity of $u(X)/v(X)$ at Q follows immediately.

For Hölder continuity at interior points, that is, points of $K \cap D$, the extra estimate needed is Hölder continuity on Whitney cubes of D that intersect K. This follows from the estimate $|\nabla(u/v)(X)| \leqslant Cd(X)^{\alpha-1}$, which can be verified on Whitney cubes using Hölder continuity at the boundary and a change of scale. We will give instead a direct proof that is valid for solutions to elliptic divergence class equations with bounded, measurable coefficients as well as the usual Laplace equation.

Let U be a solution to an elliptic divergence class equation with bounded, measurable coefficients. Let I be the unit cube. The de Giorgi–Nash estimate is [6]

$$|U(X) - U(Y)| \leqslant C\,|X-Y|^\alpha \sup_{Z \in I} |U(Z)| \tag{7.10}$$

for all X, $Y \in cI$ for some $c < 1$. (The constant c does not depend on the divergence class operator, but only its ellipticity constant.)

Now consider u and v as above restricted to a Whitney cube W of diameter r. Let $Q \in \partial D$ be a point at distance roughly r from W. Replace u by a constant multiple of u (between M^{-1} and M) so that $\lim_{X \to Q}(u(X)/v(X)) = 1$. Then the Hölder continuity at Q says that $|u(X)/v(X) - 1| \leqslant Cr^\alpha$ for all $X \in W$. Denote $L = \sup_{X \in W} u(X)$. By Harnack's principle, $L \simeq u(X) \simeq v(X)$ for all $X \in W$. Moreover, the estimate on $u(X)/v(X)$ can be rewritten as

$$|u(X) - v(X)| \leqslant Cr^\alpha L \qquad \text{for all } X \in W. \tag{7.11}$$

Observe that $u(X)/v(X) - u(Y)/v(Y) = A_1 + A_2$, where

$$A_1 = (u(X) - u(Y))(v(Y) - u(Y))/v(X)\,v(Y),$$

$$A_2 = u(Y)((u-v)(X) - (u-v)(Y))/v(X)\,v(Y).$$

Applying (7.10) to u on W with a change of scale r^{-1},

$$|u(X) - u(Y)| \leqslant CL(r^{-1}\,|X-Y|)^\alpha, \qquad X, Y \text{ in } W.$$

Therefore by (7.11), for X, Y in W,

$$A_1 \leqslant CL(r^{-1}\,|X-Y|)^\alpha\,(Cr^\alpha L)/L^2 \leqslant C\,|X-Y|^\alpha.$$

Applying (7.10) to $(u-v)$ with a change of scale, and then applying (7.11), we have for X, Y in W

$$|(u-v)(X)-(u-v)(Y)| \leqslant C \sup_{Z \in W} |u(Z)-v(Z)| \, (r^{-1} |X-Y|)^{\alpha}$$
$$\leqslant C(r^{\alpha} L)(r^{-1} |X-Y|)^{\alpha} = CL\, |X-Y|^{\alpha}.$$

Therefore, $A_2 \leqslant C|X-Y|^{\alpha}$. In all $|u(X)/v(X)-u(Y)/v(Y)| \leqslant C|X-Y|^{\alpha}$ for X, Y in W, as desired.

8. Atomic Decomposition of H^p, $p_0 < p \leqslant 1$

In this section we treat the H^p theory of NTA domains.

(8.1) Definition. For $0<p<\infty$, $H^p(D, d\omega) = \{u$ harmonic in D: $N_\alpha(u) \in L^p(d\omega)\}$.

We first show that this definition is independent of α. This follows from the following standard lemma (see, for example, [3]):

(8.2) Lemma. *Assume $0<\alpha<\infty$, $\alpha_0>\alpha$. Then, there exists a constant C_{α,α_0}, such that $\omega\{Q \in \partial D: N_{\alpha_0}(u) > \lambda\} \leqslant C_{\alpha,\alpha_0}\omega\{Q \in \partial D: N_\alpha(u) > \lambda\}$.*

(8.3) Lemma. *Let $u \in H^1(D, d\omega)$. Then, there exists $f \in L^1(d\omega)$ with $u(X) = \int_{\partial D} f\, d\omega^X$ for all $X \in D$. Also, $u \in H^p(D, d\omega)$, $p>1$, if and only if $u(X) = \int_{\partial D} f\, d\omega^X$, $f \in L^p(d\omega)$.*

Proof. The second statement follows from (5.8) and the proof of the first statement. The proof of the first statement follows the same strategy as in (5.14). Choose $\alpha>0$. As $N_\alpha(u)$ is in $L^1(d\omega)$, u has non-tangential limit f a.e. (ω). Obviously, $f \in L^1(d\omega)$. Choose now β associated to α as in the construction of sawtooth domains (6.3). Let $F_\lambda = \{Q \in \partial D: N_\beta(u)(Q) \leqslant \lambda\}$, and construct the sawtooth region Ω_λ, corresponding to α, β and F_λ. In particular $\Omega_\lambda \subset S_\beta(F_\lambda)$, and so $|u| \leqslant \lambda$ on Ω_λ. As in (5.15), $\omega_\lambda(\partial\Omega_\lambda \backslash F_\lambda) \leqslant M\omega(\partial D \backslash F_\lambda)$, and the Ω_λ increase to D. Thus, by (5.12), there exists a function f_λ in $L^\infty(\partial\Omega_\lambda, d\omega_\lambda)$ such that for all $X \in \Omega_\lambda$, $u(X) = \int_{\partial\Omega_\lambda} f_\lambda\, d\omega_\lambda^X$. Since ω_λ and ω are mutually absolutely continuous on F_λ, it follows that $f = f_\lambda$ a.e. (ω) on F_λ, and $u(X) = \int_{F_\lambda} f(Q)\, d\omega_\lambda^X(Q) + \int_{\partial\Omega_\lambda \backslash F_\lambda} u(Q)\, d\omega_\lambda^X(Q)$. But,

$$\int_{\partial\Omega_\lambda\setminus F_\lambda} u(Q)\, d\omega_\lambda^X(Q) \leqslant M_X \lambda\omega(\partial DF_\lambda)$$

$$= M_X \lambda\omega\{Q \in \partial D: N_\beta(u) > \lambda\}$$

$$\leqslant M_X \int_{\{Q\in\partial D: N_\beta(u)>\lambda\}} N_\beta(u)\, d\omega \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

Thus, we see that $u(X) = \lim_{\lambda\to\infty} \int_{F_\lambda} f(Q)\, d\omega_\lambda^X(Q)$. Arguing as in (5.14), we see that $u(X) = \int_{\partial D} f(Q)\, d\omega^X(Q)$, and the lemma is established.

(8.4) Definition. $BMO(\partial D) = \{f \in L^1(d\omega)\colon \sup_\Delta (1/\omega(\Delta)) \int_\Delta |f - m_\Delta f|\, d\omega < +\infty$, where Δ is a surface ball, and $m_\Delta(f) = (1/\omega(\Delta)) \int_\Delta f\, d\omega\}$. As is well known, by the theorem of John–Nirenberg [30] this is the same as the set of $f \in L^2(d\omega)$, with $\sup_\Delta((1/\omega(\Delta)) \int_\Delta |f - m_\Delta(f)|^2\, d\omega)^{1/2} < +\infty$, with comparable norms.

Before turning to the main of this section, we need to recall several definitions and results from the general theory of spaces of homogeneous type [14, 38, 39, 52].

(8.5) Definition. The triple (X, D, μ) is called a *space of homogeneous type* if X is a topological space, whose topology is given by a quasi-distance d (i.e., $d(x, y) \leqslant K(d(x, z) + d(z, y))$; $d(x, y) = d(y, x)$; $d(x, y) = 0$ if and only if $x = y$), with a Borel measure μ such that $\mu(\Delta_{2r}(x)) \leqslant C\mu(\Delta_r(x))$, where $\Delta_r(x) = \{y; d(x, y) < r\}$. On such a space one defines the so-called *measure distance* $m(x, y) = \inf\{\mu(\Delta)\colon \Delta$ is a ball with $x, y \in \Delta\}$, $m(x, y)$ is also a quasi-distance (see [14]), and if $\Delta_r^m(x) = \{y; m(x, y) < r\}$, then $cr \leqslant \mu(\Delta_r^m(x)) \leqslant r$.

Remark. For a bounded *NTA* domain D, the triple $X = \partial D$, $d =$ Euclidean distance, $\mu = \omega$ is a space of homogeneous type. Also, for $Q_1, Q_2 \in \partial D$, $m(Q_1, Q_2) \simeq \omega(\Delta(Q_1, |Q_1 - Q_2|))$.

Let (X, d, μ) be a space of homogeneous type such that X is compact, $\mu(X) = 1$, and such that the balls Δ are open sets.

(8.6) Definition. $a \in L^1(X)$ is a *p-atom* if $\int a\, d\mu = 0$ and there is a ball Δ containing the support of a, with $\|a\|_\infty \leqslant 1/\mu(\Delta)^{1/p}$. The constant function $a \equiv 1$ is also assumed to be a p-atom. It can be verified (see [39]) that p-atoms on (X, d, μ) are the same as p-atoms on (X, m, μ), up to a bounded multiple.

(8.7) Definition. Let $\alpha > 0$ be given. A function $\Phi(X)$ belongs to $\mathrm{Lip}_m(\alpha)$ if $L(\Phi, \alpha, m) = \sup_{x\in X, y\in X, x\neq y}(|\Phi(X) - \Phi(y)|/m(x, y)\, \alpha) < +\infty$. The norm in $\mathrm{Lip}_m(\alpha)$ is $\|\Phi\|_{\alpha, m} = L(\Phi, \alpha, m) + \int_X |\Phi|\, d\mu$. A function Φ in $L^1(X)$ belongs to BMO if there exists a constant C such that

$\mu(\Delta)^{-1} \int_\Delta |\Phi(X) - m_\Delta(\Phi)| \, d\mu \leqslant C$ for all balls Δ in the quasi-distance m (or d), where $m_\Delta(\Phi) = (1/\mu(\Delta)) \int_\Delta \Phi \, d\mu$.

A function $\Phi(X)$ belongs to $\mathrm{Lip}_d(\alpha, q)$ for $1 \leqslant q < +\infty$, if there exists a constant C, such that for all d-balls Δ we have

$$\left(\frac{1}{\mu(\Delta)} \int_\Delta |\Phi(X) - m_\Delta(\Phi)|^q \, d\mu\right)^{1/q} \leqslant c\mu(\Delta)^\alpha.$$

The norm in $\mathrm{Lip}_d(\alpha, q)$ is the least C above plus $\int_X |\Phi| \, d\mu$.

Remark. It is easy to see from the definitions that a p-atom of (X, d, μ) defines a linear functional of norm less or equal to 1 on $\mathrm{Lip}_d(1/p - 1, 1)$ for $p < 1$, and on BMO for $p = 1$. Also, a p-atom of (X, m, μ) defines a linear functional on $\mathrm{Lip}_m(1/p - 1), p < 1$, or BMO for $p = 1$, with norm less than or equal to 1. It was shown in [38] that $\Phi \in \mathrm{Lip}_d(\alpha, q)$ if and only if it coincides almost everywhere with a function on $\mathrm{Lip}_m(\alpha)$. Moreover, the norms are equivalent.

(8.8) DEFINITION. For $0 < p \leqslant 1$ and $f \in \mathrm{Lip}_d(1/p - 1, 1)^*$, or BMO for $p = 1$, let $\|f\|_{H^p_{at}(X)} = \inf\{(\sum_{i=1}^\infty |\lambda_i|^p)^{1/p}$; there exists a sequence of p-atoms $\{a_i(X)\}$, such that $f = \sum \lambda_i a_i$ in $\mathrm{Lip}_d(1/p - 1, 1)^*$, or BMO^* for $p = 1\}$.

If no such sequence exists, we let $\|f\|_{H^p_{at}(X)} = +\infty$. Then,

$$H^p_{at}(X) = \{f : \|f\|_{H^p_{at}(X)} < +\infty\}.$$

Now, for $f \in L^1(X)$, let $f^*(x) = \sup |1/r \int_X f(y) \phi(y) \, d\mu(y)|$, $0 < r \leqslant 1$, and $\operatorname{supp} \phi \subset \Delta_r^m(x)$, $L(\phi, \gamma, m) \leqslant r^{-\gamma}$, $\|\phi\|_\infty \leqslant 1$.

Let $K(r, x, y)$ be a continuous, non-negative function defined on $(0, 1] \times X \times X$, with the following properties:

(8.9) (a) $K(r, x, y) \leqslant (1 + m(x, y)/r)^{-1-\gamma}$,

(b) $K(r, x, x) \geqslant A^{-1}$,

(c) $|K(r, x, y) - K(r, x, z)| \leqslant (m(y, z)/r)^\gamma ((1 + m(x, y))/r)^{-1-2\gamma}$ for $m(y, z) \leqslant (r + m(x, y)/4A)$

for some $\gamma > 0$, $A > 0$.

For $f \in L^1(X)$, define $f^+(X) = \sup_{0 < r \leqslant 1} |(1/r) \int_X K(r, x, y) f(y) \, d\mu(y)|$.

The main results of [39] and [52] combined, give (Corollary 1′ in [52]):

(8.10) THEOREM. *There exists a $p_0 < 1$, depending only on X, such that for any $f \in L^1(X)$, and any p, $p_0 < p \leqslant 1$, we have*

$$\|f^+\|_{L^p(X)} \simeq \|f^*\|_{L^p(X)} \simeq \|f\|_{H^p_{at}(X)}.$$

We will now apply these general results to the case $X = \partial D$, $d =$ Euclidean distance, $\mu = \omega$. In our particular case, a p-atom is an $L^1(d\omega)$ function on ∂D, with $\int_{\partial D} a d\omega = 0$, supp $a\Delta$, $\|a\|_\infty \leqslant 1/\omega(\Delta)$, or the constant function 1. BMO coincides with $BMO(\partial D)$ (see (8.4)). The main result needed to apply the general theory is the following:

(8.11) LEMMA. *There exists a non-negative continuous function $K(r, P, Q)$ on $(0, 1] \times \partial D \times \partial D$, satisfying* (8.9), *and constants α and C, depending only on D such that whenever $f \in L^1(d\omega)$ and $u(X) = \int_{\partial D} f d\omega^X$, then $f^+(P) \leqslant CN_\alpha(u)(P)$, for every $P \in \partial D$.*

Proof. It is easy to see that all we have to produce is a function $K(r, P, Q)$ as in the statement of the lemma, but which satisfies, instead of (8.9),

(8.9)′ (a) $K(r, P, Q) \leqslant a(1 + m(P, Q)/r)^{-1-\gamma_1}$,

(b) $K(r, P, Q) \geqslant b^{-1}$,

(c) $|K(r, P, Q_1) - K(r, P, Q_2)| \leqslant c(m(Q_1, Q_2)/r)^{\gamma_2}(1 + m(P, Q_1)/r)^{-1-\gamma_3}$ for $m(Q_1, Q_2) \leqslant r/4B$,

for some positive constants γ_1, γ_2, γ_3, a, b, c and B.

We will start by establishing all the required properties for a function $H(r, P, Q)$, not necessarily continuous in r and P.

For fixed $P \in \partial D$ and $0 < r \leqslant 1$, we pick s so that $\omega(\Delta(P, s)) = r$. s and r will be fixed until we verify the desired properties of $H(r, P, Q)$. Let A_s be a point such that $|A_s - P| \simeq s$ and $\text{dist}(A_s, \partial D) \geqslant M^{-1}s$. Then, we let $H(r, P, Q) = rK(A_s, Q)$. Notice that for H, we automatically have $f^+(P) \leqslant N_\alpha(u)(P)$ for every $P \in \partial D$, for sufficiently large α. The verification that H satisfies all the required properties is routine, using the results in Sections 4 and 7. We include all the details for completeness. (8.9′b) is immediate since $K(A_s, P) \simeq 1/\omega(\Delta(P, s))$ by (4.11). We now turn to (8.9′a). If C is small and $m(P, Q) \leqslant Cr$, then $\omega(P, |P - Q|) \leqslant r$, and hence $|P - Q| \leqslant s$. But then, $rK(A_s, Q) \simeq 1$, and so (a) holds in this case. Next assume that $\omega(P, |P - Q|) \geqslant r$, and so $|P - Q| \geqslant s$. Let $|P - Q| \simeq 2^j s$, for some $j \geqslant 1$. (4.14) shows that $rK(A_s, Q) \leqslant Mrc_j/\omega(\Delta(P, 2^j s))$, where $c_j = 2^{-\alpha j}$. But, the doubling condition (4.9) shows that $\omega(\Delta(P, 2^j s)) \leqslant 2^{\beta j}\omega(\Delta(P, s)) = 2^{\beta j} r$. Thus, $2^{-\alpha j} = (r/\omega(\Delta(P, 2^j s)))^{\alpha/\beta}$, and so

$$rK(A_s, Q) \leqslant M\left(\frac{r}{\omega(\Delta(P, |P - Q|))}\right)^{1+\alpha/\beta}$$

$$\leqslant M\left(1 + \frac{\omega(\Delta(P, |P - Q|))}{r}\right)^{-1-\alpha/\beta}$$

and so (a) follows.

We now turn to (c). The main estimate needed for establishing (c) is that if $|Q_1 - Q_2| = t$, $|X - Q_1| \geqslant 2^j t$, $j \geqslant 2$, then $|K(X, Q_1) - K(X, Q_2)| \leqslant M2^{-\lambda j} K(X, Q_1)$ (7.1). From now on, let $|Q_1 - Q_2| = t$. Assume first that $m(P, Q_1) \leqslant Cr$, so that $\omega(\Delta(P, |P - Q_1|)) \leqslant r$, and hence $|P - Q_1| \leqslant s$. If $|A_s - Q_1| \leqslant 2t$, as $\operatorname{dist}(A_s, \partial D) \geqslant M^{-1}s$, we see that $|Q_1 - Q_2| \geqslant M^{-1}s$. Thus, $\Delta(Q_1, M|Q_1 - Q_2|) \supset \Delta(P, s)$, and hence, $\omega(\Delta(Q_1, |Q_1 - Q_r|)) \geqslant Cr$. Thus, for (c) to hold in this case, all we have to show is that $rK(A_s, Q_1) \leqslant C$, and $rK(A_s, Q_2) \leqslant C$. Both estimates follow from (a). Assume then that $|A_s - Q_1| \simeq 2^{\alpha} t$, $j > 2$. Then, $|rK(A_s, Q_1) - rK(A_s, Q_2)| \leqslant rM2^{-\lambda j} K(A_s, Q_1) \leqslant CM2^{-\lambda j}$. The desired estimate then reduces to showing that $2^{-\lambda j} \leqslant C(\omega(\Delta(Q_1, |Q_1 - Q_2|))/\omega(\Delta(P, s)))^{\gamma_2}$. But, as $|A - Q_1| \geqslant M^{-1}s$, $|P - Q_1| \leqslant s$, $\Delta(P, s) \subset \Delta(Q_1, C|A_s - Q_1|)$, and so the required estimate follows from the doubling property of ω.

Assume now that $m(P, Q_1) \geqslant Cr$, so that $\omega(\Delta(P, |Q_1 - P|)) \geqslant r$, and thus $|Q_1 - P| \geqslant s$. As before, assume first that $|A_s - Q_1| \leqslant 2t$. Then, $|Q_1 - Q_2| \geqslant cs$. We claim that $t \leqslant 2s$. If not, $\Delta(P, |Q_1 - Q_2|) \subset \Delta(Q_1, 5|Q_1 - Q_2|)$ and thus $\omega(\Delta(P, |Q_1 - Q_2|)) \leqslant C\omega(\Delta(Q_1, |Q_1 - Q_2|)) \leqslant C(r/4B) \leqslant r$ if B is large, and so, $|Q_1 - Q_2| \leqslant s$, a contradiction. Thus, $|P - Q_1| \leqslant s + 2t \leqslant 5s$, and hence $\omega(Q_1, |Q_1 - Q_2|) \simeq r$. Thus, for (c) to hold in this case, we need the estimate in γ_3 only. This will follow from (a), provided $m(P, Q_2) \geqslant Cm(P, Q_1)$. But, $m(P, Q_2) \geqslant Cr$, and $m(P, Q_1) \leqslant Km(P, Q_2) + Km(Q_1, Q_2) \leqslant Km(P, Q_2) + Kr/4B$, and so the estimate holds if B is large enough. The last case is when $|A_s - Q_1| \simeq 2^j t$, $j > 2$. Then $|rK(A_s, Q_1) - rK(A_s, Q_2)| \leqslant M2^{-\lambda j} K(A_s, Q_1) \leqslant M2^{-\lambda j}(1 + m(P, Q_1)/r)^{1-\gamma}$ by (a). All we have to check then is that $2^{-\lambda j} \leqslant C(\omega(\Delta(Q_1, |Q_1 - Q_2|))/\omega(\Delta(P, s)))^{\gamma_2}$. This will follow from the doubling condition for ω, if we can show that $\Delta(P, s) \subset \Delta(Q_1, C|A_s - Q_1|)$. This easily follows from the fact that $|A_s - Q_1| \geqslant M^{-1}s$. Thus, (8.9)$'$ is established for $H(r, P, Q)$.

We will now modify H so as to make it continuous. We will first define $K(2^{-i}, P, Q)$, $i \geqslant 0$. For each $P \in \partial D$, choose $s(P)$ so that $\omega(\Delta(P, s(P))) = 2^{-i}$. Using the compactness of ∂D, and the Besicovitch covering lemma (see, for example, [50, Lemma 3.3, p. 54]), we can select a finite collection of points $\{P_j\}$ in ∂D, so that $\partial D = \bigcup_j \Delta(P_j, s(P_j))$, and every point $P \in \partial D$ belongs to at most N of the $\Delta(P_j, s(P_j))$, where N is a number depending only on the dimension n. We note that the doubling property of ω implies that if $P \in \Delta(P_j, s(P_j))$, then $\omega(\Delta(P, s(P_j))) \simeq 2^{-i}$, and $s(P_j) \simeq s(P)$. Let now $\{\phi_j\}$ be a continuous partition of unity, subordinate to the cover $\{\Delta(P_j, s(P_j))\}$ of ∂D. Let $A_j \in D$ be such that $|A_j - P_j| \simeq s(P_j)$, and $\operatorname{dist}(A_j, \partial D) \geqslant M^{-1}s(P_j)$. Let $K(2^{-i}, P, Q) = 2^{-i} \sum_j \phi_j(P) K(A_j, Q)$. This is obviously continuous in P and Q. Also, for a given P, there are at most N indices j so that $\phi_j(P) \neq 0$. For each such j, $|A_j - P| \simeq s(P)$, $s(P) \simeq s(P_j)$, $\omega(\Delta(P, s(P))) = 2^{-i}$, and thus $K(2^{-i}, P, Q)$ verifies (8.9)$'$ by our previous estimates on H. Also, for α large, $A_j \in \Gamma_\alpha(P)$, and thus $|(1/2^{-i}) \int_{\partial D} K(2^{-i}, P, Q) f(Q)\, d\omega(Q)| \leqslant CN_\alpha(u)(P)$. We now define

$K(r, P, Q)$ for $2^{-i-1} < r \leqslant 2^{-i}$ by linearly interpolating $K(2^{-i-1}, P, Q)$ and $K(2^{-i}, P, Q)$. The resulting $K(r, P, Q)$ satisfies all the required properties.

An immediate consequence of (8.3), (8.10) and (8.11) is:

(8.12) COROLLARY. (i) *Assume* $u \in H^1(D, d\omega)$. *Then* $u(X) = \int_{\partial D} f \, d\omega^X$, $f \in L^1(d\omega)$. *Then*,

$$\|f\|_{H^1_{at}(\partial D)} \simeq \|f^*\|_{L^1(d\omega)} \simeq \|f^+\|_{L^1(d\omega)} \leqslant C \|u\|_{H^1(D, d\omega)}.$$

(ii) *There exists a* $p_0 < 1$, *depending only on* D, *such that, if* $f \in L^1(d\omega)$, *and* $u(X) = \int_{\partial D} f \, d\omega^X$, *then for any* $p_0 < p \leqslant 1$, *we have*

$$\|f\|_{H^p_{at}(\partial D)} \simeq \|f^*\|_{L^p(d\omega)} \simeq \|f^+\|_{L^p(d\omega)} \leqslant C \|u\|_{H^p(D, d\omega)}.$$

The main results of this section are simple consequences of this corollary.

(8.13) THEOREM. *u is in $H^1(D, d\omega)$ if and only if $u(X) = \int_{\partial D} f \, d\omega^X$, with $f \in H^1_{at}(\partial D)$. Moreover,* $\|u\|_{H^1(D, d\omega)} \simeq \|f\|_{H^1_{at}(\partial D)} \simeq \|f^*\|_{L^1(d\omega)} \simeq \|f^+\|_{L^1(d\omega)}$. *Also,* $H^1(D, d\omega)^* = BMO(\partial D)$.

Proof. By (8.12), it remains to show only that if $f \in H^1_{at}(\partial D)$, then $u(X) = \int_{\partial D} f \, d\omega^X$ is in $H^1(D, d\omega)$. Thus, all we have to show is that if a is a 1-atom, and $u_a(X) = \int_{\partial D} a \, d\omega^X$, then $\|N_\alpha(u)\|_{L^1(d\omega)} \leqslant C$.

More generally, we will show that if a is a p-atom, $p_0 < p \leqslant 1$, then $\|N_\alpha(u_a)\|_{L^p(d\omega)} \leqslant C$, if p_0 is sufficiently close to 1. In fact, let a be a p-atom, supported on a surface ball Δ centered at Q_0, of radius r. Let $\Delta_j = \Delta(Q_0, 2^j r)$. We first observe that $N_\alpha(u_a) \leqslant 1/\omega(\Delta)^{1/p}$, and thus, $\int_{\Delta_1} N_\alpha(u)^p \, d\omega \leqslant \omega(\Delta_1)/\omega(\Delta) \leqslant C$. Next, assume $P \in \partial D \backslash \Delta_1$, and $X \in \Gamma_\alpha(P)$. Then, $|X - Q_0| \geqslant C 2^j r$.

$u_a(X) = \int_{\partial D} a(Q)[K(X, Q) - K(X, Q_0)] \, d\omega(Q)$, and so, by (7.1),

$$\begin{aligned} |u_a(X)| &\leqslant \frac{C}{\omega(\Delta)^{1/p}} K(X, Q_0) \, 2^{-\lambda j} \omega(\Delta) \\ &\leqslant C \frac{2^{-\lambda j}}{\omega(\Delta)^{1/p}} \frac{\omega(\Delta)}{\omega(\Delta(Q_0, |P - Q_0|))}. \end{aligned}$$

Thus, $\int_{\partial D \backslash \Delta_1} N_\alpha(u_a)^p \, (P) \, d\omega \leqslant C \sum_{j=1}^{\infty} 2^{-p\lambda j} (\omega(\Delta)/\omega(\Delta_j))^{(p-1)}$. By the doubling condition, $\omega(\Delta_j) \leqslant C 2^{\beta j} \omega(\Delta)$, and so the sum converges to a value independent of Δ as soon as $\lambda/\beta > (1/p - 1)$.

(8.14) THEOREM. *There exists $p_0 < 1$, depending only on D, such that, for every $f \in L^1(d\omega)$, and $u(X) = \int_{\partial D} f \, d\omega^X$, we have, for $p_0 < p \leqslant 1$*

$$\|u\|_{H^p(D, d\omega)} \simeq \|f^*\|_{L^p(d\omega)} \simeq \|f^+\|_{L^p(d\omega)} \simeq \|f\|_{H^p_{at}(\partial D)}.$$

Proof. The theorem follows from (8.12ii), the proof of (8.13), and the fact (7.1) that there exists $\gamma > 0$ such that, for all $x \in D$, $K(X, -)$ is in $\mathrm{Lip}_d(\gamma, 1)$.

Remark. If we knew that the set of functions $u(X) = \int_{\partial D} f\, d\omega^X$, $f \in L^1(d\omega)$, is dense in $H^p(D, d\omega)$, $p_0 < p < 1$, it would follow that $u \in H^p(D, d\omega)$, $p_0 < p \leqslant 1$, if and only if there exists $f \in H^p_{at}(\partial D)$ such that $u(X) = \int_{\partial D} f(Q)\, K(X, Q)\, d\omega(Q)$, where the integral is interpreted as the action of $f \in \mathrm{Lip}_d(1/p - 1, 1)^*$ on $K(X, -)$. This density easily follows on star-shaped Lipschitz domains by considering the functions $u_r(X) = u(rX)$ (we assume that the star-center of the domain is the origin). Thus, we have:

(8.15) THEOREM. *Let D be a star-shaped Lipschitz domain. Then, there exists $p_0 < 1$, depending only on D, such that for $p_0 < p \leqslant 1$, $u \in H^p(D, d\omega)$ if and only if there exists $f \in H^p_{at}(\partial D)$ such that $u(X) = \int_{\partial D} f(Q)\, K(X, Q)\, d\omega(Q)$ (the integral is interpreted as the action of $f \in \mathrm{Lip}_d(1/p - 1, 1)^*$ on $K(X, -)$). Moreover,*

$$\|u\|_{H^p(D, d\omega)} \simeq \|f\|_{H^p_{at}(\partial D)}.$$

9. AREA INTEGRAL ESTIMATES AND *BMO*

In this section we prove $L^p(d\omega)$ estimates $(1 < p < \infty)$ for the area integral. We also establish a Carleson measure characterization of $BMO(\partial D)$.

(9.1) THEOREM. *Let D be a bounded NTA domain. Assume $1 < p < \infty$, and $f \in L^p(d\omega)$. Let $u(X) = \int_{\partial D} f\, d\omega^X$, and $A_\alpha(u)^2(Q) = \int_{\Gamma_\alpha(Q)} d(X)^{2-n} |\nabla u(X)|^2\, dX$. Then,*

$$\|A_\alpha u\|_{L^p(d\omega)} \leqslant C_p \|f\|_{L^p(d\omega)}. \tag{9.2}$$

Moreover, if $u(X_0) = 0$ (where $\omega = \omega^{X_0}$), then

$$C_p^{-1} \|f\|_{L^p(d\omega)} \leqslant \|A_\alpha(u)\|_{L^p(d\omega)}. \tag{9.3}$$

Proof. The method of proof is taken from [47]. The main difference is that we use (4.8). Also, the pole of the Green function causes some minor technical difficulties. To overcome them, we will also consider the truncated version of the area integral. It is enough to prove (9.2) for $f > 0$, $f \in C(\partial D)$. Fix a small h, and write $A^2_\alpha(u)(Q) = \iint_{\Gamma^h_\alpha(Q)} d(X)^{2-n} |\nabla u(X)|^2\, dX + \iint_{\Gamma_\alpha \setminus \Gamma^h_\alpha(Q)} |\nabla u(X)|^2\, dX = A^2_{\alpha,h}(u)(Q) + B^2_{\alpha,h}(u)(Q)$. Harnack's principle implies that $B_{\alpha,h}(u)(Q) \leqslant C u(X_0)$. Thus, $B^p_{\alpha,h}(u)(Q) \leqslant C \int_{\partial D} f^p\, d\omega$, and we only have to deal with $A_{\alpha,h}(u)$. The proof of the case $p=2$ illustrates the role of (4.8) in this theorem:

$$\int_{\partial D} A^2_{\alpha,h}(u)(Q)\,d\omega = \int_{\partial D}\left(\int_{\Gamma^h_\alpha(Q)} d(X)^{2-n}\,|\nabla u(X)|^2\,dX\right)d\omega$$

$$\leqslant C\int_D |G(X)|\,|\nabla u(X)|^2\,dX$$

$$= \int_{\partial D} |f-u(X_0)|^2\,d\omega \leqslant C\int_{\partial D} |f|^2 d\omega \qquad \text{by (4.8) and (5.14).}$$

In the case $1<p<2$, the proof is the same as the one given in [47], using (5.8) and the fact that if $I_{\alpha,h}(Q)=\int_{\Gamma^h_\alpha(Q)} d(X)^{2-n}\,\Delta(u^p)(X)\,dX$, then $\int_{\partial D} I_{\alpha,h}(Q)\,d\omega(Q)\leqslant C\int_{\partial D} f^p\,d\omega$. This last inequality holds because as $u>0$, u^p is subharmonic, and its least harmonic majorant is $\int f^p\,d\omega^X$. Thus, (5.13) shows that $u^p(X_0)=\int_{\partial D} f^p\,d\omega+\int_D G(X)\,\Delta(u^p)\,dX$. Also, (4.8) and an interchange in the order of integration show that

$$\int_{\partial D} I_{\alpha,h}(Q)\,d\omega(Q)\leqslant C\int_D |G(X)|\,\Delta(u^p)(X)\,dX.$$

We now turn to $4\leqslant p\leqslant\infty$. Once (9.2) is established in this range, it will follow in general from the Marcinkiewicz interpolation theorem (see [50]). Let $1/q+2/p=1$, so that $1<q\leqslant 2$. Let $g>0$, $g\in C(\partial D)$, and $v(X)=\int_{\partial D} g\,d\omega^X$. Let $D_h=\bigcup_{Q\in\partial D}\Gamma^h_\alpha(Q)$. Following [47] once more, we first show that

$$\int_{\partial D} A^2_{\alpha,h}(u)(Q)\,g(Q)\,d\omega(Q)\leqslant C\int_{D_h} |G(X)|\,|\nabla u(X)|^2\,v(X)\,dX. \tag{9.4}$$

This follows again, interchanging the order of integration on the left-hand side, from (4.8), the fact that $\{Q\in\partial D: X\in\Gamma^h_\alpha(Q)\}\subset\Delta(X^*,cd(X))$, and that, by (4.11), $1/\omega(\Delta(X^*,cd(X)))\int_{\Delta(X^*,cd(X))} g\,d\omega\leqslant Cv(X)$, for $X\in D_h$. From (9.4), we will deduce, using the identity $\Delta(u^2\cdot v)=\Delta u^2\cdot v+2\nabla u^2\cdot\nabla v$, that

$$\int_{\partial D} A^2_{\alpha,h}(u)(Q)\,g(Q)\,d\omega(Q)\leqslant C\left\{\int_{\partial D} f^2\cdot g\,d\omega+\int_{\partial D} f^2\,d\omega\int_{\partial D} g\,d\omega\right.$$

$$\left.+\int_{\partial D} N_\alpha(u)\cdot A_{\alpha,h}(u)\cdot A_{\alpha,h}(v)\,d\omega\right\}. \tag{9.5}$$

As in [47], the case $1<q<2$, (5.8) and Hölder's inequality, imply (9.2) in general. The main technical difficulty in establishing (9.5) comes from the pole of the Green function. The right-hand side of (9.4) equals

$$C \int_{D_h} |G(X)| \{\Delta(u^2 \cdot v) - 2\nabla u^2 \cdot \nabla v\} \, dX.$$

Notice that the integrand is non-negative. Let D_ε be a family of C^∞ domains $\bar{D}_\varepsilon \subset D$, which increase to D, and G_ε then corresponding Green functions, with pole at X_0. By monotone convergence, the integral above equals $\lim_{\varepsilon \to 0} \int_{D_\varepsilon \cap D/h} \{\Delta(u^2 \cdot v) - 2\nabla u^2 \cdot \nabla v\} |G_\varepsilon(X)| \, dX$. On $D \backslash D_h$, by Harnack's principle $|\nabla u^2| |\nabla v| \leqslant C(\int_{\partial D} f^2 \, d\omega)(\int_{\partial D} g \, d\omega)$. Thus,

$$\int_{\partial D} A^2_{\alpha,h}(u)(Q) \, g(Q) \, d\omega(Q) \leqslant C \left\{ \overline{\lim_{\varepsilon \to 0}} \left| \int_{D_\varepsilon} \Delta(u^2 v) \, |G_\varepsilon(X)| \, dX \right| \right.$$
$$+ \overline{\lim_{\varepsilon \to 0}} \int_{D_\varepsilon \cap D_h} |\nabla u^2| \, |\nabla v| \, |G_\varepsilon(X)| \, dX$$
$$\left. + \left(\int_{\partial D} f^2 \, d\omega \right) \left(\int_{\partial D} g \, d\omega \right) \right\}.$$

Integrating by parts on D_ε, we see that

$$\left| \int_{D_\varepsilon} \Delta(u^2 v) \, |G_\varepsilon(X)| \, dX \right|$$
$$= \left| \int_{\partial D_\varepsilon} u^2 v \, d\omega_\varepsilon - u^2(X_0) \, v(X_0) \right|$$
$$\leqslant \int_{\partial D_\varepsilon} u^2 v \, d\omega_\varepsilon + \left(\int_{\partial D} f^2 \, d\omega \right) \left(\int_{\partial D} g \, d\omega \right) \xrightarrow[\varepsilon \to 0]{} \int_{\partial D} f^2 \cdot g \, d\omega$$
$$+ \left(\int_{\partial D} f^2 \, d\omega \right) \left(\int_{\partial D} g \, d\omega \right).$$

Also,

$$\int_{D_\varepsilon \cap D_h} |\nabla u^2| \, |\nabla v| \, |G_\varepsilon(X)| \, dX$$
$$\leqslant \int_{D_h} |\nabla u^2| \, |\nabla v| \, |G(X)| \, dX$$
$$\leqslant C \int_{\partial D} \left(\int_{\Gamma^h_\alpha(Q)} |u| \, |\nabla u| \, |\nabla v| \, d(X)^{2-n} \, dX \right) d\omega(Q), \qquad \text{by (4.8)},$$

because X is away from the pole X_0. Thus (9.5) follows and (9.2) is established.

We turn to (9.3). We use the identity $\int_{\partial D} f \cdot g \, d\omega = -\int_{\partial D} G(X) \nabla u(X) \cdot \nabla v(X) \, dX$, where $u(X) = \int_{\partial D} f \, d\omega^X$, $v(X) = \int_{\partial D} g \, d\omega^X$, $u(X_0) = 0$ (see (5.14)).

Isolating the pole, we see that

$$\int_{\partial D} f \cdot g \, d\omega \leqslant \int_{D \setminus B_\delta(X_0)} |G(X)| \, |\nabla u| \, |\nabla v| \, dX + \left| \int_{B_\delta(X_0)} G(X) \nabla u \cdot \nabla v \, dX \right|.$$

By (4.8) and an interchange in the order of integration, we see that the first term is majorized by $C \int_{\partial D} A_\alpha(u) A_\alpha(v) \, d\omega$.

For the second term, using the fact that $u(X_0) = v(X_0) = 0$, and expressing u and v as integrals of their gradients (in $B_\delta(X_0)$), an integration by parts shows that

$$\begin{aligned} \left| 2 \int_{B_\delta(X_0)} G(X) \nabla u \cdot \nabla v \, dX \right| &= \left| \int_{B_\delta(X_0)} G(X) \Delta(u \cdot v) \, dX \right| \\ &\leqslant C \int_{B_{2\delta}(X_0)} |\nabla u| \, |\nabla v| \, dX \\ &\leqslant C \int_{\partial D} A_\alpha(u) A_\alpha(v) \, d\omega. \end{aligned}$$

Thus, $\left| \int_{\partial D} f \cdot g \, d\omega \right| \leqslant C \int_{\partial D} A_\alpha(u) A_\alpha(v) \, d\omega$, and so (9.3) follows.

We now turn to our characterization of $BMO(\partial D)$ (see (8.4) for the definition).

(9.6) THEREOM. *Let D be a bounded NTA domain, and u harmonic in D. Then, $u(X) = \int_{\partial D} f \, d\omega^X$, with $f \in BMO(\partial D)$ if and only if there exists a constant C such that, for all balls B centered at points $Q \in \partial D$, $(1/\omega(\Delta)) \int_\Delta |G(X)| \, |\nabla u(X)|^2 \, dX \leqslant C$, where $\Delta = B \cap D$.*

This theorem was established in [19] for Lipschitz domains, our approach here is parallel to [19, 20].

We need three lemmas in order to prove this theorem.

(9.7) LEMMA. *Assume $f \in BMO(\partial D)$, and $r < r_0$ (see (3.1)). Then, there exists a constant C, independent of r and f, such that if $B = B(Q_0, r)$, and $A_r \in D$, $d(A_r, \partial D) \geqslant M^{-1} r$, $M^{-1} r < |A_r - Q_0| < Mr$, and $\Delta = B \cap D$, then*

$$v(A_r) = \int_{\partial D \setminus \Delta_1} |f - m_{\Delta_1}(f)| \, d\omega^{A_r} \leqslant C \|f\|_{BMO(\partial D)},$$

where Δ_1 is the surface ball with same center as Δ, and twice the radius.

Proof. Let Δ_j be the surface ball with same center as Δ, and 2^j times the radius. Let $R_j = \Delta_j \backslash \Delta_{j-1}$. Since

$$|m_\Delta)f) - m_{\Delta_1}(f)| \leqslant C\,\|f\|_{BMO}, \qquad |m_{\Delta_j}(f) - m_{\Delta_1}(f)| \leqslant C \cdot j \cdot \|f\|_{BMO}.$$

$$v(A_r) \leqslant \sum_j \int_{R_j} |f - m_{\Delta_1}(f)|\, K(A_r, Q)\, d\omega$$

$$\leqslant \sum_j M \frac{c_j}{\omega(\Delta_j)} \int_{\Delta_j} |f - m_{\Delta_1}(f)|\, d\omega \qquad \text{by (4.14).}$$

On the other hand,

$$\frac{1}{\omega(\Delta_j)} \int_{\Delta_j} |f - m_{\Delta_1}(f)|\, d\omega$$

$$\leqslant \frac{1}{\omega(\Delta_j)} \int_{\Delta_j} |f - m_{\Delta_j}(f)|\, d\omega + Cj\,\|f\|_{BMO} \leqslant Cj\,\|f\|_{BMO}.$$

Since $c_j \leqslant M 2^{-2j}$, the lemma is established.

(9.8) Lemma. *Let D be a bounded NTA domain, and $r < r_0$. Then, there exists a constant C, independent of r, such that if $B = B(Q_0, r)$, and $\Delta = B \cap \partial D$, then*

$$\frac{r^{2(n-2)}}{\omega(\Delta)^3} \int_{B \cap D} \frac{G^3(X)}{d^2(X)}\, dX \leqslant C.$$

Proof. Let I_j be a dyadic Whitney cube of D, which touches $B \cap D$. Then $\operatorname{diam}(I_j) \leqslant Cr$. Let $\mathscr{F} = \{\text{all such } I_j\}$, and $\mathscr{F}_k = \{I_j \in \mathscr{F} : l(I_j) = 2^{-k}\}$. Then,

$$\int_{B \cap D} \frac{G^3(X)}{d^2(X)}\, dX \leqslant \sum_k \sum_{I_j \in \mathscr{F}_k} \int_{I_j} \frac{G^3(X)}{d^2(X)}\, dX.$$

Now let Z_j = center of I_j. Fix k, and look at $I_j \in \mathscr{F}_k$. Let $Z_j^* \in \partial D$, be such that $|Z_j - Z_j^*| = \operatorname{dist}(Z_j, \partial D)$, and $\Delta_j = \Delta(Z_j^*, 2^{-k})$. By Harnack's principle, for $X \in I_j$, $G^3(X) \simeq G^3(Z_j)$. Thus,

$$\int_{I_j} \frac{G^3(X)}{d^2(X)}\, dX \simeq G^3(Z_j)\, 2^{-k(n-2)} \simeq C^2(Z_j)\, \omega(\Delta_j),$$

by (4.8). Moreover, there exists a constant C such that, for every j, $\Delta_j \subset \Delta(Q_0, Cr)$, and also $\sum_j \chi_{\Delta_j} \leqslant N$, where N depends only on the dimension

n. This last fact holds because if $Q \in \Delta_i \cap \Delta_j$, then $|Z_i - Z_j| \leqslant C2^{-k}$, and the claim follows because I_i and I_j are dyadic cbes of side 2^{-k}. Also, since $\Delta(Z_j^*, Cr) \supset \Delta(Q_0, 2r)$, (4.1) shows that

$$G(Z_j) \leqslant M\left(\frac{2^{-k}}{r}\right)^{\alpha} \cdot r^{2-n} \cdot \omega(\Delta).$$

Hence, $\sum_{I_j \in \mathscr{F}_k} \int_{I_j} (G^3(X)/d^2(X))\, dX \leqslant M^2 \sum_{I_j \in \mathscr{F}_k} (2^{-k}/r)^{2\alpha}\, r^{2(2-n)} \omega^2(\Delta)\omega(\Delta_j)$ $\leqslant CR^{2(2-n)}(2^{-k}/r)^{2\alpha}\, \omega^3(\Delta)$. Adding all over k such that $2^{-k} \leqslant Cr$, we get the desired estimate.

(9.9) LEMMA. *Assume that u is harmonic in the NTA domain D, and there exists a constant C such that, for all balls B centered at points $Q \in \partial D$, $(1/\omega(\Delta)) \int_{B \cap D} |G(X)|\, |\nabla u(X)|^2\, dX \leqslant C$, where $\Delta = B \cap D$. Let $\|u\|^2$ be the least constant in the inequality above. Let $D_\delta = \{X \in D\colon \operatorname{dist}(X, \partial D) \leqslant \delta\}$. Then, there exists $\delta > 0$, and $M > 0$ such that, for all $X \in D_\delta$, $|\nabla u(X)| \leqslant M\, \|u\|/d(X)$.*

Proof. Fix $\eta > 0$ so that $B_{2\eta}(X_0) \subset D$. Pick δ so small that if $X \in D_\delta$, then $B(X) = \{Y\colon |X - Y| \leqslant d(X)/2\} \subset D \backslash B_{2\eta}(X_0)$. For each $X \in D_\delta$, we have $|\nabla u(X)|^2 \leqslant (1/|B(X)|) \int_{B(X)} |\nabla u(Y)|^2\, dX$. Also, for each $Y \in B(X)$, $d(Y) \simeq d(X)$, and by Harnack's principle $G(X) \simeq G(Y)$. Thus, by (4.8), $G(Y) \simeq d(X)^{2-n}\, \omega(\Delta(X^*, d(X)))$ for all $Y \in B(X)$. Letting $\bar{B}(X) = B(X^*, 2d(X))$, we see that

$$|\nabla u(X)|^2 \leqslant c d(X)^{-n} \cdot d(X)^{n-2} \cdot \frac{1}{\omega(\Delta(X^*, d(X)))}$$
$$\times \int_{\bar{B}(X) \cap D} |G(Y)|\, |\nabla u(Y)|^2\, dY,$$

and thus the proof of the lemma follows.

Proof of (9.6). Assume $f \in BMO(\partial D)$. Since $\int_{\partial D} |f - u(X_0)|^2\, d\omega \leqslant \|f\|_{BMO}$, by (5.14) all we have to worry about is balls B with $r < r_0$. Fix one such ball B, and let $f_1 = (f - m_{\Delta_1}(f))\, \chi_{c_{\Delta_1}}$, $f_2 = (f - m_{\Delta_1}(f))\, \chi_{c_{\Delta_1}}$, with Δ_1 as in (9.7). Let u_1 and u_2 be their respective harmonic extensions. Then, $\nabla u = \nabla u_1 + \nabla u_2$, and (5.14) shows that $(1/\omega(\Delta)) \int_{B \cap D} |G(X)|\, |\nabla u_1(X)|^2\, dX \leqslant (1/\omega(\Delta)) \int_D |G(X)|\, |\nabla u_1(X)|^2\, dX = (1/\omega(\Delta)) \int_D |f_1|^2\, d\omega \leqslant \|f\|^2_{BMO}$.

By Harnack's inequality, splitting f_2 into its positive and negative parts, we see that

$$|\nabla u_2(X)| \leqslant C\frac{v(X)}{d(X)}, \qquad \text{where} \qquad v(X) = \int_{c_{\Delta_1}} |f - m_{\Delta_1}(f)|\, d\omega^X.$$

Applying (4.10) in $B \cap D$ to $v(X)$ and $C(X)$, using (9.7) and (4.8), we see that for $X \in B \cap D$,

$$v(X) \leqslant C \|f\|_{BMO} \cdot \frac{r^{(n-2)}}{\omega(\Delta)} G(X).$$

Thus, by (9.8), $(1/\omega(\Delta)) \int_{B \cap D} |G(X)| |\nabla u_2(X)|^2 \, dX \leqslant C \|f\|_{BMO}$.

Conversely, assume with the notation of Lemma (9.9), that $\|u\|^2 < +\infty$ and $u(X_0) = 0$. Evidently, $\int_D |G(X)| |\nabla u(X)|^2 \, dX < +\infty$. Thus, by (5.14), there exists $f \in L^2(d\omega)$ such that $u(X) = \int_{\partial D} f \, d\omega^X$, and $\int_{\partial D} f \, d\omega = 0$. To check that $f \in BMO(\partial D)$ we need only consider surface balls Δ with small radius. Let $\Delta = \Delta(Q_0, r)$, and let Ω be the "cap" of (3.11) such that $B(Q_0, r) \cap D \subset \Omega \subset B(Q_0, M^2 r) \cap D$. Let A be a point in Ω with $B(A, r/2) \subset \Omega$. We first claim that $\int_\Omega |\nabla u(X)|^2 |G_\Omega(A, X)|^2 \, dX \leqslant C \|u\|^2$. In fact, in $B(A, r/2)$, $|G_\Omega(A, X)| \leqslant 1/|A - X|^{n-2}$, while, by (9.9), $|\nabla u(X)|^2 \leqslant C \|u\|^2 r^{-2}$. Thus,

$$\int_{B(A, r/2)} |\nabla u(X)|^2 |G_\Omega(A, X)| \, dX \leqslant C \frac{\|u\|^2}{r^2} \int_{B(A, r/2)} \frac{dX}{|A - X|^{n-2}} \leqslant C \|u\|^2.$$

On the other hand, in $\Omega \backslash B(A, (M/2) r)$, $G_\Omega(A, X) \leqslant C(G(X)/\omega(\Delta))$, by (4.16), and so our claim follows.

Next, (5.14) shows that $\int_\Delta |f - u(A)|^2 \, d\omega_\Omega^A \leqslant C \|u\|^2$. Using (4.18), we see that $(1/\omega(\Delta)) \int_\Delta |f - u(A)|^2 \, d\omega \leqslant C \|u\|^2$, as desired.

10. BMO_1 Domains

Throughout this section D denotes a BMO_1 domain (see (2.6)). Fix $X_0 \in D$. We will denote $\omega = \omega^{X_0}$, harmonic measure for D at X_0. σ denotes surface measure of ∂D. Recall that $\omega \in A_\infty(d\sigma)$ if there exist α, β, $0 < \alpha$, $\beta < 1$ such that for all surface balls Δ and all Borel sets $E \subset \Delta$, $\omega(E)/\omega(\Delta) < \alpha$ implies $\sigma(E)/\delta(\Delta) < \beta$.

(10.1) THEOREM. *Let D be a BMO_1 domain. Then:*

(a) $\omega \in A_\infty(d\sigma)$. *(In particular, ω and σ are mutually absolutely continuous.)*

(b) *There exists $p_0 < \infty$ such that for all $p > p_0$, and all $f \in L^p(d\sigma)$, there is a unique harmonic function u in D such that u approaches f non-tangentially and $\|N(u)\|_{L^p(d\sigma)} \leqslant C \|f\|_{L^p(d\sigma)}$. ($N(u)$ denotes the non-tangential maximal function* (5.6).*)*

Proof. Suppose that we have already proved $\omega \in A_\infty(d\sigma)$. The fact that

ω and σ are mutually absolutely continuous follows from [28]. In fact, $\omega = g\, d\sigma$ for some $g \in L^q(d\sigma)$, $q > 1$. Hence, if $p > q'$ and $f \in L^p(d\sigma)$, then Hölder's inequality implies $f \in L^1(d\omega)$. By Theorem (5.8), f has a harmonic extension u and $N(u)(Q) \leqslant CM_\omega f(Q)$. By a theorem of Muckenhoupt [12], $\|M_\omega f\|_{L^p(d\sigma)} \leqslant C\|f\|_{L^p(d\sigma)}$ for sufficiently large p.

It remains to prove that $\omega \in A_\infty(d\sigma)$. An immediate consequence of Lemma (4.11) is that if $\Delta = \Delta(Q, r)$, $A = A_r(Q)$, $E \subset \Delta$, then $\omega^A(E) \simeq \omega(E)/\omega(\Delta)$. Therefore, it suffices to show that $\omega^A(E) < \alpha$ implies $\sigma(E)/\sigma(\Delta) < \beta$ for some α, β between 0 and 1. The main lemma is:

(10.2) LEMMA. *There exists a Lipschitz domain $\tilde{D} \subset D$ such that if M is a constant depending only on the BMO_1 constant of D,*

(a) $A \in \tilde{D}$, $d(A, \partial\tilde{D}) \simeq r$,

(b) $\sigma(\partial\tilde{D} \cap \Delta) > M^{-1}\sigma(\Delta)$,

(c) *the Lipschitz constant of $\tilde{D}$ is dominatd by M.*

To prove $\omega \in A_\infty(d\sigma)$ using the lemma, let $\tilde{\omega}^A$ denote harmonic measure at A for $\tilde{D}$. The maximum principle implies $\tilde{\omega}^A(E \cap \partial\tilde{D}) \leqslant \omega^A(E) < \alpha$. An estimate of Dahlberg on Lipschitz domains [15, 27] says that there exists $\gamma > 0$ depending only on the Lipschitz constant of $\tilde{D}$ such that $\tilde{\omega}^A(E \cap \partial\tilde{D}) < \alpha$ implies $\sigma(E \cap \partial D)/\sigma(\Delta) < M\alpha^\gamma$. Hence, for sufficiently small α,

$$\frac{\sigma(E)}{\sigma(\Delta)} = \frac{\sigma(E \backslash \partial\tilde{D})}{\sigma(\Delta)} + \frac{\sigma(E \cap \partial\tilde{D})}{\sigma(\Delta)} < 1 - M^{-1} + M\alpha^\gamma < \beta < 1.$$

We need only Lemma (10.2) in a special domain $D = \{(x, y) | y > \phi(x), x \in \mathbb{R}^m, y \in \mathbb{R}\}$ with $\phi \in BMO_1$. Furthermore, after a bounded dilation in y, we may assume that $\|\phi\|_{BMO_1} < \varepsilon_0$, for some small fixed ε_0. Finally, because the BMO_1 norm is invariant under $\phi(x) \to r\phi(r^{-1}x)$, i.e., the dilation $(x, \phi(x)) \mapsto (rx, r\phi(x))$, we may assume that $\sigma(\Delta) = 1$.

Let ψ be a Lipschitz function on $\mathbb{R}^m$. Denote by $\nu(\psi)(x)$ the unit normal to the graph $(x, \psi(x))$: $\nu(\psi)(x) = (-\nabla\psi(x), 1)/|(-\nabla\psi(x), 1)|$ which exists for almost all x. Let I denote the unit cube in $\mathbb{R}^m$ with center 0. Let $\theta \in C_0^\infty(I)$ be even, $\theta \geqslant 0$, $\int \theta(x)\, dx = 1$, and $\theta = 1$ on $\frac{1}{2}I$. An approximate unit normal to the graph over I of a function $\phi \in BMO_1(\mathbb{R}^m)$ is given by $\nu(\phi * \theta)(0)$.

(10.3) LEMMA. *Suppose that $\|\phi\|_{BMO_1} < \varepsilon_0$. Then there exists a Lipschitz function ψ and an open set $\mathcal{O} \subset I$ such that*

(a) *$\psi \geqslant \phi$ on I and $\phi = \psi$ on $I \backslash \mathcal{O}$.*

(b) $\int_{\mathcal{O}} (|\nabla\phi(x)|^2 + 1)^{1/2}\, dx < C\varepsilon_0^{1/2}$.

(c) *$\nu(\phi * \theta)(0) \cdot \nu(\psi)(x) > \frac{1}{2}$ for $x \in I$.*

In order to see that (10.3) implies (10.2), observe that if $\nu_0 = (0,\ldots, 0, 1)$, the unit vector in $\mathbb{R}^{m+1}$, then $\nu_0 \cdot \nu(\psi)(x) = 1/(|\nabla\psi(x)|^2 + 1)^{1/2}$. Thus, in general, the Lipschitz norm of a surface in rectilinear coordinates with ν_1 as vertical direction is dominated by $|\nu_1 \cdot \nu(\psi)(x)|^{-1}$. Let $\nu_1 = \nu(\phi * \theta)(0)$. We conclude from (10.3c) that ψ is the graph in rectilinear coordinates with ν_1 as a vertical direction of a Lipschitz function with Lipschitz norm <2. The Lipschitz domain $\tilde{D}$ with diameter roughly 1 and bounded Lipschitz constant is constructed by dropping a cone from the vertex $(0, \phi(0)) + \nu_1$. Let $\Delta = \Delta((0, \phi(0)), 1)$. Denote $V = \{(x, \phi(x)) \colon x \in \mathcal{O}\}$. Then by (10.3a) and (b), $\sigma(\Delta \cap \partial\tilde{D}) \geqslant \sigma(\Delta) - \sigma(V) > 1 - C\varepsilon_0^{1/2} > \frac{1}{2} = \frac{1}{2}\sigma(\Delta)$. This proves (10.2).

Proof of (10.3). Denote $a = \nabla(\phi * \theta)(0)$. $R = (|a|^2 + 1)^{1/2}$. Recall that $\nu_1 = (-a, 1)/R$. Let $\eta \in C_0^\infty(\mathbb{R}^m)$; $\eta = 1$ on I. Let $h(x) = (\phi(x) - a \cdot x)\,\eta(x)$. Note that since $\|\phi\|_{\Lambda_1} \leqslant \|\phi\|_{BMO_1} < \varepsilon_0$ and $\phi(0) = 0$, $\int_I |\nabla h(x)|\,dx < \varepsilon_0$ (see (3.7)). Denote the ordinary Hardy–Littlewood maximal function on $\mathbb{R}^m$ $Mf(x) = \sup\{(1/|J|) \int_J |f(y)|\,dy;\ J \text{ is a cube containing } x\}$. Define $\mathcal{O} = \{x \in I \colon M(|\nabla h|)(x) > \varepsilon_0^{1/2} R\}$. The weak type estimate for M tells us that

$$|\mathcal{O}| \leqslant C\varepsilon_0^{1/2} R^{-1}. \tag{10.4}$$

Denote $\delta(x) = d(x, {}^c\mathcal{O})$, and $\theta_\varepsilon(x) = \varepsilon^{-m}\theta(\varepsilon^{-1}x)$. Define

$$\begin{aligned} \bar{h}(x) &= h * \theta_{\delta(x)}(x), && x \in \mathcal{O}, \\ &= h(x), && x \notin \mathcal{O}. \end{aligned}$$

Finally, let $\psi(x) = a \cdot x + \bar{h}(x) + C\varepsilon_0^{1/2}\delta(x)$.

The main estimate is that $\nabla\bar{h}(x)$ exists a.e. and

$$|\nabla\bar{h}(x)| \leqslant C\varepsilon_0^{1/2} R. \tag{10.5}$$

To prove this, denote $v(x, \varepsilon) = h * \theta_\varepsilon(x)$. $\mathscr{R} = \{(x, \varepsilon) \colon \varepsilon > \delta(x)\}$. For $(x, \varepsilon) \in \mathscr{R}$,

$$\begin{aligned} |\nabla_{(x,\varepsilon)} v(x, \varepsilon)| &\leqslant \left| \nabla_x \int h(x-y)\,\theta_\varepsilon(y)\,dy \right| + \left| \frac{\partial}{\partial\varepsilon} \int h(x-y)\,\theta_\varepsilon(y)\,dy \right| \\ &\leqslant \left| \int |\nabla h(y)|\,\theta_\varepsilon(x-y)\,dy \right| \\ &\quad + \left| \int \left(\frac{1}{2} h(x-y) + \frac{1}{2} h(x+y) - h(x) \right) \frac{\partial}{\partial\varepsilon} \mathcal{O}_\varepsilon(y)\,dy \right| \\ &\leqslant C\varepsilon_0^{1/2} R + C\varepsilon_0 \int |y| \left| \frac{\partial}{\partial\varepsilon} \theta_\varepsilon(y) \right| dy \\ &\leqslant C\varepsilon_0^{1/2} R + C\varepsilon_0. \end{aligned}$$

The second to last inequality holds because $\varepsilon > \delta(x)$ implies the ball of radius ε about x contains a point of ${}^c\mathcal{O}$. The final inequality holds because $\int |y|\, |(\partial/\partial\varepsilon)\, \theta_\varepsilon(y)|\, dy \leqslant C$.

Notice that $v(x, \varepsilon)$ is continuous in $\bar{\mathscr{R}}$ and the estimate above says $\sup_{\mathscr{R}} |\nabla u(x, \varepsilon)| \leqslant C\varepsilon_0^{1/2} R$. Therefore [48] $v|_{\mathscr{R}}$ can be extended to all of $\mathbb{R}^{m+1}$ as a Lipschitz function with norm $C\varepsilon_0^{1/2}R$. Finally, $\bar{h}(x)$ is the composition $x \to (x, \delta(x)) \to v(x, \delta(x))$. Because $\delta(x)$ is Lipschitz with bound 1, $\bar{h}(x)$ is Lipschitz with bound $C\varepsilon_0^{1/2}R$.

We can now prove (c). By (7.5),

$$\begin{aligned} \nabla\psi(x) &= a + \nabla\bar{h}(x) + \nabla(C\varepsilon_0\delta(x)) \\ &= a + O(\varepsilon_0^{1/2}R). \end{aligned}$$

Part (c) now follows for sufficiently small ε_0.

For part (a),

$$\begin{aligned} |\bar{h}(x) - h(x)| &\leqslant \left| \int \left(\tfrac{1}{2}h(x+y) + \tfrac{1}{2}h(x-y) - h(x)\right) \theta_{\delta(x)}(y)\, dy \right| \\ &\leqslant \varepsilon_0 \int |y|\, \theta_{\delta(x)}(y)\, dy \leqslant C\varepsilon_0\delta(x). \end{aligned}$$

Hence, $\bar{h}(x) + C\varepsilon_0\delta(x) \geqslant h(x)$. Thus $\psi(x) \geqslant \phi(x)$. Also, we clearly have $\psi = \phi$ on $I \backslash \mathcal{O}$, so (a) is proved.

For part (b), $\mathcal{O} = \bigcup I_j$, where I_j is the Whitney decomposition of $\mathcal{O}$.

$$\begin{aligned} \int_{\mathcal{O}} (|\nabla\phi(x)|^2 + 1)^{1/2}\, dx &\leqslant \sum_j \int_{I_j} (1 + |\nabla\phi(x)|^2)^{1/2}\, dx \\ &\leqslant \sum_j |I_j| + \int_{I_j} |\nabla\phi(x)|\, dx \\ &\leqslant \sum_j (|I_j| + C\varepsilon_0^{1/2}R\, |I_j| + CR\, |I_j|). \end{aligned}$$

The final inequality holds because $|\nabla\phi(x)| \leqslant |\nabla h(X)| + |a| = |\nabla h(x)| + R$, and there is a point of ${}^c\mathcal{O}$ in the expanded cube I_j^*. Thus by (10.4),

$$\int_{\mathcal{O}} (1 + |\nabla\phi(x)|^2)^{1/2}\, dx \leqslant CR \sum_j |I_j| = CR\, |\mathcal{O}| \leqslant C\varepsilon_0^{1/2}.$$

This concludes the proof of (10.3) and the theorem.

Let us compare Theorem (10.1) with a result of [28]. We call a domain D an L_1^p domain if the boundary of D is given locally as the graph of continuous functions ϕ with $\nabla\phi \in L^p$.

(10.6) THEOREM [28]. *If $D \subset \mathbb{R}^n$ is an L_1^p domain for some $p > n - 1$, then ω and σ are mutually absolutely continuous. Moreover, if $p \geqslant 2$, and $f \in L^2(d\sigma)$ for some $q \geqslant 2(p-1)/(p-2)$, then $f \in L^1(d\omega)$ and the harmonic extension u of f converges to f non-tangentially a.e. (ω) or (σ).*

The domains treated in (10.6) are far more general than BMO_1 domains: $BMO \subset L_{\text{loc}}^p$ for all $p < \infty$. However, the convergence we obtain is merely pointwise convergence almost everywhere. Theorem (10.1) has the advantage that the non-tangential maximal function $N(u)(Q)$ is controlled. Thus the convergence is dominated.

11. FURTHER RESULTS AND OPEN QUESTIONS

(11.1) THEOREM. *Let D be a quasicircle. Then, u belongs to $H^1(D, d\omega)$ if and only if $u(X) = \int_{\partial D} f\, d\omega^X$ for some $f \in L^1(d\omega)$, and the harmonic conjugate of u, $\tilde{u}(X) = \int_{\partial D} g\, d\omega^X$ for some $g \in L^1(d\omega)$.*

Proof. Suppose that $u(X) = \int_{\partial D} f\, d\omega^X$, with $f \in L^1(d\omega)$, and $\tilde{u}(X) = \int_{\partial D} g\, d\omega^X$, with $g \in L^1(d\omega)$. Denote $F = u + i\tilde{u}$. Then, $|F|^{1/2}$ is subharmonic, and its boundary values belong to $L^2(d\omega)$. By subharmonic majorization, and Theorem (5.8), the non-tangential maximal function of $|F|^{1/2}$ belongs to $L^2(d\omega)$. Therefore, $u \in H^1(D, d\omega)$. Conversely, suppose that $u \in H^1(D, d\omega)$. Then, $u(X) = \int_D f\, d\omega^X$, where $f \in L^1(d\omega)$ and f has an atomic decomposition (Theorem (8.13)). Let ϕ denote a conformal mapping from the unit disc to D. $u \circ \Phi(Y) = \int_{S^1} f \circ \Phi(e^{i\theta})\, d\omega_{S^1}^Y$, and $f \circ \Phi$ has an ordinary atomic decomposition as a function in the unit circle (relative to arc-length ds). Therefore, by the boundedness of the conjugation operator on H^1 of the circle, $(u \circ \Phi)^{\sim}(Y) = \int_{S^1} h(e^{i\theta})\, d\omega_{S^1}^Y$, where $h \in L^1(ds)$. But, $\tilde{u} \circ \Phi = (u \circ \Phi)^{\sim}$; thus,

$$\tilde{u}(X) = \int_{\partial D} g\, d\omega^X, \qquad \text{where} \qquad g = h \circ \Phi^{-1} \in L^1(d\omega).$$

(11.2) THEOREM. *Let D be a chord-arc domain with chord-arc constant sufficiently close to 1. Then, u belongs to $H^1(D, d\omega)$ with $u(X_0) = 0$, if and only if $u(X) = \int_{\partial D} f\, d\omega^X$ for some $f \in L^1(d\omega)$ with $\int f\, d\omega = 0$, and such that the Cauchy integral of $f \cdot (d\omega/d\sigma)$ (i.e., $p \cdot v \cdot \int_{\partial D} f(z) \cdot (d\omega/d\sigma)(z) \cdot (z - \zeta)^{-1}\, d\sigma(z) = \int_{\partial D} (f(z)/(z - \zeta))\, d\omega(z)$) belongs to $L^1(d\sigma)$.*

Proof. We will only give a sketch of the proof. First, choose the chord-arc constant so close to 1 that the Cauchy integral is bounded in every $L^p(d\sigma)$, $1 < p < \infty$. (See [13].) Now, introduce the space $H_{at}^1(\partial D, d\sigma)$ of $f = \sum \lambda_i a_i$, $\sum |\lambda_i| < +\infty$, such that $\operatorname{supp} a_i \subset \Delta_i$, Δ_i a surface ball, $\int_{\partial D} a_i\, d\sigma = 0$, and $\|a_i\|_\infty \leqslant 1/\sigma(\Delta_i)$. It is easy to see that the L^p boundedness of the Cauchy integral (which we are going to denote by C from now on) implies that if $f \in H^1_{at}(\partial D, d\omega)$, then $Cf \in L^1(\partial D, d\sigma)$. The fact that $\omega \in A_\infty(d\sigma)$ implies that $f \in H^1_{at}(\partial D, d\omega)$, $\int f\, d\omega = 0$ if and only if $f \cdot (d\omega/d\sigma) \in H^1_{at}(\partial D, d\sigma)$ (see [14]). It follows that if $u \in H^1(\partial D, d\omega)$, $u(X_0) = 0$, then $C(f \cdot (d\omega/d\sigma)) \in L^1(d\sigma)$, where $u(X) = \int_{\partial D} f\, d\omega^X$ with $f \in H^1_{at}(\partial D, d\omega)$ by (8.13).

For the converse, we follow the lines of Theorem 2.13 of [18]. Let $Kf(z) = p \cdot v \cdot \int_{\partial D} (\langle z - \phi, N_\phi \rangle / |\phi - z|^2) f(\phi)\, d\sigma(\phi)$, where N_ϕ is the inward pointing unit normal to ∂D. It is easy to see that the boundedness of Cf on L^p, $1 < p < \infty$, implies the same result for K. Moreover, arguing as in Theorem 1.1 of [18], we can show that if the chord-arc constant of D is sufficiently close to 1, then K is not only bounded on $BMO(\partial D)$, but, in addition $K = \tilde{K} + E$, where $\tilde{K}$ is compact on $BMO(\partial D)$, and the operator norm of E on $BMO(\partial D)$ is small. Because of this, an argument similar to the one given in Theorem 1.2 of [18] shows that $\frac{1}{2}I + K$ is invertible on $BMO(\partial D)$, and $\frac{1}{2}I - K^*$ is invertible on $H^1_{at}(\partial D, d\sigma)$. From this, the argument of Theorem 2.13 of [18] (with some modifications) shows that if $f \in L^1(\partial D, d\sigma)$, and $Cf \in L^1(d\sigma)$, then $f \in H^1_{at}(\partial D, d\sigma)$.

Thus, if $f \in L^1(d\omega)$, $\int f\, d\omega = 0$, and $C(f \cdot (d\omega/d\sigma)) \in L^1(d\sigma)$, then $f \cdot (d\omega/d\sigma) \in H^1_{at}(\partial D, d\sigma)$, and so $f \in H^1_{at}(\partial D, d\sigma)$. Hence, $u(X) = \int_{\partial D} f\, d\omega^X$ belongs to $H^1(D, d\omega)$.

An examination of the proof of the doubling condition (4.9) shows that it can be stated in the following form. Let $Q \in \partial D$, $\Delta = B(Q, r) \cap \partial D$, $2\Delta = B(Q, 2r) \cap \partial D$. Then there is a constant C independent of X such that if $d(X, 2\Delta) > \frac{1}{4}r$, $\omega^X(2\Delta) \leqslant C\omega^X(\Delta)$.

(11.3) Proposition. *Let f be a continuous function on the unit interval, $f(0) = 0$, $f(y) > 0$ for $y > 0$. Denote $D = \{(x, y): |x| < f(y), x \in \mathbb{R}^m, 0 < y < 1\}$. If harmonic measure in D satisfies the doubling condition (as stated above), then there exists $\varepsilon > 0$ depending only on C such that $f(y) \geqslant \varepsilon y$ for $y < \frac{1}{4}$. In other words, D contains a cone at the origin.*

Proof. Suppose that for some $y < \frac{1}{4}$, $f(y) < \varepsilon y$. We will deduce a contradiction for ε sufficiently small. Let $Q = (0, 0)$, $X_0 = (0, y)$, $X_1 = (0, \frac{3}{2}y)$. Denote $\Delta = B(Q, y) \cap \partial D$, $B = B(X_1, \frac{1}{4}y)$. Choose N sufficiently large that $B(X_0, N) \supset D$. Denote $H = \{(x, t): t > y\}$. Denote by $u_1(X)$ the harmonic function in $H \cap B(X_0, N)$ such that

$$\begin{aligned} u_1(X) &= 1, \qquad X = (x, y),\ |x| \leqslant f(y), \\ &= 0, \qquad \text{elsewhere on the boundary.} \end{aligned}$$

By the maximum principle, $u_1(X) \geqslant \omega^X(\Delta)$ for all X in $H \cap B(X_0, N) \cap D$. In particular, $u_1(X) \geqslant \omega^X(\Delta)$ for all $X \in B \cap D$. There is a constant C_m such that $u(X) \leqslant C_m \varepsilon^m$ for $X \in B$. Thus, $\omega^X(\Delta) \leqslant C_m \varepsilon^m$ for all $X \in B \cap D$.

Denote by $u_2(X)$ the harmonic function in $H \cap B(X_0, y)$ such that

$$\begin{aligned} u_2(X) &= 1, \qquad X = (x, y),\ |x| > f(y), \\ &= 0, \qquad \text{elsewhere on the boundary.} \end{aligned}$$

By the maximum principle, $u_2(X) \leqslant \omega^X(2\Delta)$ for X in $H \cap B(X_0, y) \cap D$. In particular, $u_2(X) \leqslant \omega^X(2\Delta)$ for $X \in B \cap D$. There is a constant $a_m > 0$ such that $u_2(X) > a_m$ for $X \in B$. Thus, $\omega^X(2\Delta) > a_m$ for all $X \in B \cap D$. In all, $\omega^X(2\Delta) \geqslant a_m C_m^{-1} \varepsilon^{-m} \omega^X(\Delta)$ for $X \in B \cap D$.

Case 1. $B \subset B$. In this case, $d(X_1, 2\Delta) > \frac{1}{4} y$, so for sufficiently small ε the estimate above contradicts the doubling condition at X_1.

Case 2. $B \not\subset D$. Then there exists y_1, $(5/4(y < y_1 < (7/4)y$, such that $f(y_1) < (1/4)y$, The estimate above shows that $\omega^X(2\Delta) \geqslant a_m C_m^{-1} \varepsilon^{-m} \omega^X(\Delta)$ for all X on the boundary of $D \cap \{(x, t) \colon t > y_1\}$. By the maximum principle, the same estimate is valid on the interior of $D \cap \{(x, t) \colon t > y_1\}$. For example, take the interior point $X_2 = (0, 3y)$. X_2 satisfies $d(X_2, 2\Delta) > \frac{1}{4} y$, but for small ε, the doubling condition at X_2 is violated.

We would now like to make some remarks about elliptic operators $\mathscr{L}$, in divergence form, with bounded measurable coefficients, i.e., $\mathscr{L} = \partial_i a_{ij} \partial_j$, where $\lambda \sum |\xi_i|^2 \leqslant \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leqslant \Delta \sum_i |\xi_i|^2$ (see [6]). If we replace harmonic functions on *NTA* domains, by solutions to operators $\mathscr{L}$ as above, on *NTA* domains, many of our theorems remain valid.

Specifically, all the lemmas in Section 4 hold, with similar proofs, using in some instances the results and techniques of [6]. All the global results of Section 5 then also remain valid, except for (5.13) and (5.14), which use specific properties of the Laplacian. (However, we believe that some analogue of (5.13) and (5.14) must also hold for this general class of operators.) The theorems in Section 6 which do not deal with the area integral, also remain valid for the general class of operators, as well as Sections 7 and 8 in their entirety. The proofs of the theorems in Section 9 use special properties of the Laplacian, and thus cannot be claimed to hold for the general class of divergence form operators. The results of Section 10, as they involve surface measure, cannot be true for general divergence form operators, because surface measure and elliptic measure need not be mutually absolutely continuous for these operators, even in smooth domains (see [5]).

We would now like to list some open questions.*

(1) Does the analogue of the local area integral theorem (6.6) remain valid for divergence form operators with bounded measurable coefficients? (See also (4) below.)

* Since this paper was written in June 1980, considerable progress has been made on several of these questions. See the note at the end of the paper.

(2) By the methods of [5], it can be shown that every measure on S^1, which satisfies the doubling condition, arises as the elliptic measure of an operator in divergence form. Is this true for S^n, $n > 1$?

(3) The idea behind the proof of [5] is that quasiconformal mappings in two dimensions preserve the class of operators in divergence form, with bounded coefficients. This, however, is not true in higher dimensions. Is there a class of elliptic operators in higher dimensions, which is invariant under quasiconformal mappings of $\mathbb{R}^n$, and such that our results hold for this class of operators on *NTA* domains (or even smooth domains)? A natural class to consider is operators $\mathscr{L} = \partial_i a_{ij} \partial_j$, with

$$\lambda(x) \sum |\xi_i|^2 \leqslant \sum_{i,j} a_{ij}(x)\, \xi_i \xi_j$$

$$\leqslant \Lambda(x) \sum_i |\xi_i|^2, \qquad \text{with} \quad \frac{\Lambda(x)}{\lambda(x)} \leqslant C,$$

and $\lambda(x), \Lambda(x) \in A_\infty(dx)$.

(4) Distribution function inequalities for the area integral in *NTA* domains: The proposed inequalities are of the form

$$\omega\{Q \in \in D: A_\alpha(u) > 2\lambda;\, N_\beta(u) < \gamma\lambda\} \leqslant C(\gamma)\, \omega\{Q \in \partial D: A_\alpha(u) > \lambda\},$$

where $C(\gamma) \to 0$ as $\gamma \to 0$, and $\alpha < \beta$. The same inequality, with the rôles of A and N interchanged, with the assumption that $u(X_0) = 0$ is also desired. As is well known (see [3]), those inequalities imply the equivalence of the $L^p(d\omega)$ norms of A and N, $0 < p < \infty$. As mentioned in the Introduction, Dahlberg [16] proved those inequalities for Lipschitz domains. The key point in his proof is the existence of constants $a, b > 0$ such that if $F \subset \Delta \subset \partial D$ and Δ is a surface ball of radius r, and we construct the sawtooth region Ω over F, of diameter r (see (6.3)), then

$$\frac{\omega(F)}{\omega(\Delta)} \leqslant C[\omega_\Omega(F)]^a, \qquad \text{and} \qquad \left[\frac{\omega(F)}{\omega(\Delta)}\right]^b \geqslant C^{-1} \omega_\Omega(F).$$

We have not been able to establish these inequalities for general *NTA* domains. On Lipschitz domains, they are an easy consequence of the fact that $\omega \in A_\infty(d\sigma)$. Since on BMO_1 domains (see (10.1)) $\omega \in A_\infty(d\sigma)$, the distribution function inequalities hold in this case.

In the case $n = 2$, for any *NTA* domain one can use a subharmonic majorization argument like the one used in Theorem (11.1) to show the equivalence in $L^1(d\omega)$ of A and N (see [48, Chap. 7]).

The existence of distribution function inequalities of the type described above is also an open question (for exactly the same reason cited above) for divergence class operators, even on smooth domains.

(5) In connection with (8.14) we would like to pose the question of whether the set of functions $u(X) = \int_{\partial D} f\,d\omega^X$, $f \in L^1(d\omega)$, is dense in $H^p(D, d\omega)$, for $p_0 < p < 1$, on any *NTA* domain. This fact, and the a priori estimate of (8.14) would complete the proof of the atomic decomposition of $H^p(D, d\omega)$, $p_0 < p < 1$.

(6) The following generalization of chord-arc domains to higher dimensions seems natural. Let D be an *NTA* domain such that $\sigma(\Delta(Q, r)) \leqslant Cr^{n-1}$ for all surface balls $\Delta(Q, r)$ in ∂D. (Of course, BMO_1 domains satisfy this property.) Does harmonic measure belong to $A_\infty(d\sigma)$ in this case? Conversely, if harmonic measure belongs to $A_\infty(d\sigma)$, does σ satisfy $\sigma(\Delta(Q, r)) \leqslant Cr^{n-1}$? This question is open even when $n = 2$.

(7) Jones asked whether the Corona theorem holds on multiply connected *NTA* domains in the plane. This might be a consequence of the duality of H^1 and BMO (8.13).

The last questions are of a geometric nature.

(8) Does the conformal mapping between two simply connected quasicircles preserve corkscrews? This would give a more direct proof of (11.1). Perhaps (2.7) is relevant here.

(9) Suppose D is an *NTA* domain in $\mathbb{R}^n$, $n > 2$, which is homeomorphic to a ball. Can the "caps" of (3.11) be constructed so as to also be homeomorphic to balls? This is of course true for Zygmund domains, and quasispheres (see Appendix).

(10) Is every Zygmund domain D in $\mathbb{R}^n$, $n > 2$ a quasisphere? (The answer for $n = 2$ is yes, and follows from Ahlfors' three point condition, see Section 2).

APPENDIX: A LOCALIZATION

We will now prove Theorem (3.11) in the special cases where D is a quasisphere or a Zygmund domain. Recall that in both cases D itself is an *NTA* domain.

Notations. $\Phi: \mathbb{R}^n \to \mathbb{R}^n$ and Φ^{-1} will denote global quasiconformal mappings. $d(B)$ ($d(E)$) denotes the diameter of the ball B (respectively, a set E) and $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^n$. As usual $d(B_1, B_2)$ denotes the distance between balls.

We need two properties of quasiconformal mappings. Throughout, M will denote a constant that depends only on the dilatation constant of Φ (and Φ^{-1}).

(A.1) Let $0 < C < M$; if $d(B_1) < Cd(B_2)$ and $d(B_1, B_2) < Cd(B_1)$, then $d(\Phi(B_1)) < MC^\alpha d(\Phi(B_2))$.

(A.2) Let B be a ball with center X; then there exist balls B', B'' with center $\Phi(X)$ such that $B'' \subset \Phi(B) \subset B'$ and $d(B') < Md(B'')$.

These properties are easy consequences of theorems of Gehring and Väisälä.

THEOREM 1 [23]. *$|J\Phi|$, the Jacobian of Φ, satisfies A_∞ with respect to Lebesgue measure. In other words, there exists $\alpha > 0$ such that*

$$\frac{|\Phi(E)|}{|\Phi(B)|} < M\left(\frac{|E|}{|B|}\right)^\alpha \qquad \textit{for any } E \subset B.$$

THEOREM 2 [53]. *For any ball $B \subset \mathbb{R}^n$, there exists a ball B' such that $\Phi(B) \subset B'$ and $d(B') < M|\Phi(B)|^{1/n}$.*

(A.3) PROPOSITION. *If $D = \Phi(B_1)$, where B_1 is the unit ball, then for every $r > 0$ and every $Q \in \partial D$, there exists an NTA domain Ω such that $B(Q, M^{-1}r) \cap D \subset \Omega \subset B(Q, Mr) \cap D$. In fact Ω is a quasisphere with dilatation constant comparable to that of D.*

Proof. Let $B = B(Q, M^{-1}r)$. Choose $B'' \subset \Phi^{-1}(B) \subset B'$ according to (A.2) so that $d(B') < Md(B'')$. Let $\Omega = \Phi(B') \cap D = \Phi(B' \cap B_1)$. Obviously Ω is a quasisphere with dilatation comparable to D and $\Omega \supset B \cap D$. Moreover,

$$d(\Omega) \leqslant d(\Phi(B')) < M|\Phi(B')|^{1/n} \leqslant Md(\Phi(B')).$$

Now by (A.1) and the inclusion $\Phi(B'') \subset B$,

$$Md(\Phi(B')) < M^2 d(\Phi(B'')) \leqslant M^2 d(B) = 2Mr.$$

(A.4) PROPOSITION. *Let D be a Zygmund domain. For every $r > 0$ and every $Q \in \partial D$ there exists an NTA domain Ω such that $B(Q, M^{-1}r) \cap D \subset \Omega \subset B(Q, Mr) \cap D$. M and the NTA constant of Ω depend only on the Zygmund class constant of D.*

Proof. Let $\phi \in \Lambda_1$, the Zygmund class. As in (3.6), it will suffice to verify (A.4) in a special domain $D = \{(x, y): y > \phi(x), x \in \mathbb{R}^m\}$ with $\|\phi\|_{\Lambda_1}$ small. We will use slightly different notations from those in Section 3. If $\theta \in C_0^\infty(\mathbb{R}^m)$ is a non-negative even function with $\int \theta(x)\,dx = 1$, we will denote $\theta_k(x) = 2^{-km}\theta(2^{-k}x)$, $\phi_k(x) = \phi * \theta_k(x)$, $a^{(k)}(x) = \nabla\phi_k(x)$. Let α denote an absolute constant times $\|\phi\|_{\Lambda_1}$. Recall that ((3.7), (3.9))

$$|\phi_k(x) - \phi(x)| < \alpha 2^{-k}.$$

$$|\phi(z) - \phi(x) - a^{(k)}(x)\cdot(z - x)| < \alpha 2^{-k} \qquad \text{for } |x - z| < 10\cdot 2^{-k}. \tag{A.5}$$

It is easy to deduce from (A.5) that

$$|a^k(x) - a^{(k-1)}(x)| < \alpha.$$

$$|a^k(z) - a^{(k)}(x)| < \alpha \qquad \text{for } |x - z| < 10 \cdot 2^{-k}. \tag{A.6}$$

We will need a quantitative version of the inverse function theorem, whose proof is left to the reader.

(A.7) LEMMA. *Suppose that F is a Lipschitz mapping $\mathbb{R}^n \to \mathbb{R}^n$; $F(0) = 0$. Let S be a convex set containing 0 and T be an orthogonal matrix. If $\|F'(x) - T\| < \alpha < \frac{1}{2}$, for all $x \in S$, then:*

(a) $(1 - \alpha)|x - z| \leqslant |F(x) - F(z)| \leqslant (1 + \alpha)|x - z|$.

(b) *If $B(0, (1 + \alpha) r) \subset S$, then $F(B(0, r)) \supset B(0, (1 - 2\alpha) r)$.*

($\|\ \|$ denotes the operator norm of a matrix.)

A mapping F satisfying (A.7) will be called a *near isometry*.
Denote

$$\begin{aligned} \eta_k(t) &= 2^{k+1}t - 1, && 2^{-k-1} \leqslant f \leqslant 2^{-k}, \\ &= 2 - 2^k t, && 2^{-k} \leqslant t \leqslant 2^{-k+1}, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Observe that

$$\sum_{k=-\infty}^{\infty} \eta_k(t) = 1 \qquad \text{for } t > 0.$$

$$\sum_{k=-\infty}^{\infty} 2^{-k}\eta_k(t) = t \qquad \text{for } t > 0. \tag{A.8}$$

Let $\{\psi_j^k(x)\}$ be a smooth partition of unity on $\mathbb{R}^m$ subordinate to the doubles of dyadic cubes of side 2^{-k}. Denote the center of each cube by x_j^k; $R^{(k)}(x) = |(-a^{(k)}(x), 1)|$, $A^{(k)}(x) = (-a^{(k)}(x), 1)/R^{(k)}(x)$, $A_j^k = A^{(k)}(x_j^k)$. $v_k(x) = (x, \phi_k(x)) + 2^{-k} \sum_j \psi_j^k(x) A_j^k$.

Finally, $F(x, t) = \sum_{k=-\infty}^{\infty} \eta_k(t)\, v_k(x)$. ($F$: $\mathbb{R}_+^{m+1} \to D$.) F preserves distance to the boundary:

(A.9) LEMMA. *For sufficiently small α*

(a) $|F(x, t) - (x, \phi(x))| = t + O(\alpha t)$,

(b) $d(F(x, t), \partial D) = t + O(\alpha t)$.

Proof. Choose i so that $2^{-i} \leqslant t < 2^{-i+1}$. Denote $J = \{i-1, i\}$. For $|z-x| < 10 \cdot 2^{-i}$, by (A.5), (A.6), (A.8),

$$\begin{aligned} F(x,t) - (z, \phi(z)) &= \sum_{k \in J} \eta_k(t)(v_k(x) - (z, \phi(z))) \\ &= \sum_{k \in J} \eta_k(t)\{(0, \phi(x) - \phi(z) - a^{(i)}(x)(x-z)) \\ &\quad + (0, \phi_k(x) - \phi(x)) + 2^{-k} \sum_j \psi_j^k(x)(A_j^k - A^{(i)}(x)) \\ &\quad + (x - z, a^{(i)}(x) \cdot (x-z)) + 2^{-k} A^{(i)}(x)\} \\ &= t((x-z, a^{(i)}(x) \cdot (x-z)) + A^{(i)}(x)) + O(\alpha t). \end{aligned}$$

In particular, when $z = x$, $F(x,t) - (x, \phi(x)) = tA^{(i)}(x) + O(\alpha t)$ and (a) follows since $A^{(i)}(x)$ has unit length. For part (b), we need only check the lower bound. Denote $(w, y) = F(x,t)$ with $w \in \mathbb{R}^m$, $y \in \mathbb{R}$. If $|z - x| < 10 \cdot 2^{-i}$, then the formula above shows $|F(x,t) - (z, \phi(z))| \geqslant t - O(\alpha t)$ because the vectors $(x - z, a^{(i)}(x) \cdot (x-z))$ and $A^{(i)}(x)$ are perpendicular.

Part (a) for small α implies $|w - x| < 2t$. If $|z - x| \geqslant 10 \cdot 2^{-i}$, then $|F(x,t) - (z, \phi(z))| \geqslant |z - w| \geqslant |z - x| - |x - w| > t$.

Define a *block* $B_i(x) = \{(z,t): |z - x| < 100 \cdot 2^{-i}$, and $\frac{1}{10} 2^{-i} < t < 10 \cdot 2^{-i}\}$. Denote I_m the $m \times m$ identity matrix. $R_i = R^{(i)}(x)$, $a = a^{(i)}(x)$ and ${}^t a$ the transpose (row) vector. It is not hard to calculate, for $(z,t) \in B_i(x)$,

$$F'(z,t) = \underline{J}_i + O(\alpha), \qquad \text{where } \underline{J}_i = \begin{pmatrix} I_m & -a/R_i \\ {}^t a & 1/R_i \end{pmatrix}. \tag{A.10}$$

For any $m \times m$ matrix U, denote $\underline{U} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$, the $(m+1) \times (m+1)$ matrix. Let U_i denote the rotation of $\mathbb{R}^m$ that sends $a/|a|$ to the unit vector $(1, 0, \ldots, 0)$ and fixes the orthogonal complement of these two vectors. (If $a = 0$, let $U_i = I_m$.) An easy calculation yields:

(A.11) $\underline{J}_i = \underline{T}_i \underline{S}_i$, *orthogonal*, $\underline{S}_i$ *symmetric and* $S_i = {}^t U_i D_i U_i$, $D_i = \begin{pmatrix} R_i & 0 \\ 0 & I_{m-1} \end{pmatrix}$.

Remark. The geometric meaning of calculations (A.10) and (A.11) is that if we translate x to the origin $F \circ \underline{S}_i^{-1}$ is a near isometry on $S_i(B_i(x))$, in the sense of (A.7). Furthermore, (A.9) shows that the image under F of disjoint blocks is disjoint. Thus F is a global homeomorphism.

Denote $E_i(x) = \{z: |S_i(z - x)| < C\}$. $E_i(x)$ is an ellipsoid in $\mathbb{R}^m$ containing the rectangle $\{z: |z - x| < 10 \cdot 2^{-i}, |a_{(i)}(x) \cdot (z - x)| < 10 \cdot 2^{-i}\}$ and contained in a multiple of this rectangle. The *NTA* domain of size 2^{-1} at $Q = (x_0, \phi(x_0))$ is $\Omega = F(E_i(x_0) \times (0, C \cdot 2^{-i}))$. The reader may verify, using (A.5) and (A.9), that $B(Q, 2^{-i}) \cap D \subset \Omega \subset B(Q, C \cdot 2^{-i}) \cap D$.

The key point in the verification that Ω is an NTA domain is to find a non-tangential ball at height t and distance t from the sides of Ω. $F(\partial E_i(x_0) \times (0, C \cdot 2^{-i}))$. Without loss of generality, assume $i = 1$.

(A.12) LEMMA. *Let* $Q = F(x, t)$, $0 < t < \frac{1}{2}$ *and* $x \in \partial E$. $(E = E_1(x_0).)$ *Then there exists* $P \in \Omega$ *such that* $|P - Q| = t$ *and* $d(P, \partial\Omega) > (1 - \frac{1}{100})\, t$.

Proof. Choose k so that $2^{-k} \leqslant t < 2^{-k+1}$. We need only be concerned with the block $B_k(x)$. Make a translation so that $x = 0$. Then $0 \in \partial E$. $F \circ \underline{S}_k^{-1}$ is a near isometry on $\underline{S}_k(B_k(0))$. Therefore, the problem reduces to a question in $\mathbb{R}^m$. It suffices to prove that one can inscribe a sphere inside the ellipsoid $S_k(E)$ of radius at least t at $0 \in \partial(S_k(E))$. (Here we are using the notations of (A.10) and (A.11).)

Recall that $E = \{z : |S_1(z - x_0)| < C\}$; thus $S_k(E) = \{z : |S_1 S_k^{-1} z - S_1 x_0| < C\}$. For any ellipsoid $\{z : |Az - z_0| < C\}$, the least radius of an inscribed sphere at any boundary point is $C/\|A\|^2\, \|A^{-1}\|$. Hence we must prove that $\|S_1 S_k^{-1}\|^2\, \|S_k S_1^{-1}\| < Ct^{-1} \simeq C2^k$.

Denote $H_k = U_k - U_1$. (A.6) implies that $\|H_k\| \leqslant ka/R_1$ and $|R_k - R_1| \leqslant ka$. Therefore,

$$\begin{aligned} \|S_1 S_k^{-1}\| &= \|{}^tU_1 D_1 U_1 {}^tU_k D_k^{-1} U_k\| \\ &= \|{}^tU_1 D_1 D_k^{-1} U_k + {}^tU_1 D_1 U_1 {}^tH_k D_k^{-1} U_k\| \\ &\leqslant \|D_1 D_k^{-1}\| + \|D_1\|\, \|{}^tH_k\| \leqslant 2(1 + ka). \end{aligned}$$

Similarly, $\|S_k S_1^{-1}\| \leqslant 2(1 + ka)$. Hence, $\|S_k S_1^{-1}\|\, \|S_1 S_k^{-1}\|^2 \leqslant 8(1 + ka)^3 \leqslant C2^k$, as desired.

The remainder of the proof that Ω is an NTA domain is similar to the proof of (3.6) and will be left to the reader. The only tools needed are Lemma (A.12), the remark following (A.11), and the fact that $E_i(x_0) \times (0, C \cdot 2^{-i})$ is convex.

Note added in proof. Questions 1 and 4 of Section 11 have been resolved in the affirmative by B. Dahlberg, D. Jerison, and C. Kenig in a forthcoming article: "Area integral estimates for elliptic differential operators with non-smooth coefficients." An answer to Question 3 can be found in: "The local regularity of solutions of degenerate elliptic equations," *Comm. in P.D.E.*, 7 (1) (1982), 77–116 by E. Fabes, C. Kenig, and R. Serapioni and in two articles by E. Fabes, D. Jerison, and C. Kenig: "The Wiener test for degenerate elliptic equations," *Ann. Inst. Fourier*, Grenoble and "Boundary behavior of solutions to degenerate elliptic equations," *Proc. Conf. in honor of A. Zygmund* (1981). The two-dimensional part of Question 6 is treated by D. Jerison and C. Kenig in "Hardy spaces, A_∞ and singular integrals on chord-arc domains," *Math. Scand.*, in press. Finally, the positive answer to Question 8 is well known (although not to us at the time). It follows easily from the fact that the conformal mapping between quasicircles has a global quasiconformal extension.

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