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A non-classical solution to a Hessian equation from Cartan isoparametric cubic

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Abstract

We show how to construct a non-smooth solution to a Hessian fully nonlinear second-order uniformly elliptic equation using the Cartan isoparametric cubic in 5 dimensions.

Keywords: Non-linear elliptic equation; Weak solutions; Isoparametric surfaces

1. Introduction

This paper shows that one can use certain specific minimal cubic cone, namely, the Cartan isoparametric eigencubic [2,5] to construct a non-smooth solution to a Hessian fully nonlinear second-order elliptic equation.

More precisely, we study a class of fully nonlinear second-order elliptic equations of the form

$$F(D^2u) = 0 \tag{1.1}$$

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defined in a domain of $\mathbf{R}^n$. Here D^2u denotes the Hessian of the function u. We assume that F is a Lipschitz function defined on the space $S^2(\mathbf{R}^n)$ of $n \times n$ symmetric matrices satisfying the uniform ellipticity condition, i.e. there exists a constant $C = C(F) \geq 1$ (called an *ellipticity constant*) such that

$$C^{-1}\|N\| \leq F(M+N) - F(M) \leq C\|N\| \tag{1.2}$$

for any non-negative definite symmetric matrix N; if $F \in C^1(S^2(\mathbf{R}^n))$ then this condition is equivalent to

$$\frac{1}{C'}|\xi|^2 \leq F_{u_{ij}}\xi_i\xi_j \leq C'|\xi|^2, \quad \forall \xi \in \mathbf{R}^n. \tag{1.2$'$}$$

Here, u_{ij} denotes the partial derivative $\partial^2 u/\partial x_i \partial x_j$. A function u is called a *classical* solution of (1) if $u \in C^2(\Omega)$ and u satisfies (1.1). Actually, any classical solution of (1.1) is a smooth ($C^{\alpha+3}$) solution, provided that F is a smooth (C^α) function of its arguments.

For a matrix $S \in S^2(\mathbf{R}^n)$ we denote by $\lambda(S) = \{\lambda_i : \lambda_1 \geq \cdots \geq \lambda_n\} \in \mathbf{R}^n$ the (ordered) set of eigenvalues of the matrix S. Eq. (1.1) is called a *Hessian equation* ([20,19] cf. [4]) if the function $F(S)$ depends only on the eigenvalues $\lambda(S)$ of the matrix S, i.e., if

$$F(S) = f(\lambda(S)),$$

for some function f on $\mathbf{R}^n$ invariant under permutations of the coordinates.

In other words Eq. (1.1) is called Hessian if it is invariant under the action of the group $O(n)$ on $S^2(\mathbf{R}^n)$:

$$\forall O \in O(n), \quad F({}^tO \cdot S \cdot O) = F(S). \tag{1.3}$$

The Hessian invariance relation (1.3) implies the following:

(a) F is a smooth (real-analytic) function of its arguments if and only if f is a smooth (real-analytic) function.
(b) Inequalities (1.2) are equivalent to the inequalities
$$\frac{\mu}{C_0} \leq f(\lambda_i + \mu) - f(\lambda_i) \leq C_0\mu, \quad \forall \mu \geq 0,$$
$\forall i = 1, \ldots, n$, for some positive constant C_0.
(c) F is a concave function if and only if f is concave.

Well known examples of the Hessian equations are Laplace, Monge–Ampère, Bellman, Isaacs and Special Lagrangian equations.

Bellman and Isaacs equations appear in the theory of controlled diffusion processes; see [7]. Both are fully nonlinear uniformly elliptic equations of the form (1.1). The Bellman equation is concave in $D^2u \in S^2(\mathbf{R}^n)$ variables. However, Isaacs operators are, in general, neither concave nor convex. In a simple homogeneous form the Isaacs equation can be written as follows:

$$F(D^2u) = \sup_b \inf_a L_{ab}u = 0, \tag{1.4}$$

where L_{ab} is a family of linear uniformly elliptic operators of type

$$L_{ab} = \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{1.5}$$

with an ellipticity C independent on the parameters a, b and a_{ij} being constant coefficients.

Consider the Dirichlet problem

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

We are interested in the problem of existence and regularity of solutions to the Dirichlet problem (1.6) for Hessian equations and the Isaacs equation. Problem (1.6) has always a unique viscosity (weak) solution for fully nonlinear elliptic equations (not necessarily Hessian equations). The viscosity solutions satisfy Eq. (1.1) in a weak sense, and the best known interior regularity [2,5,3,18] for them is $C^{1,\epsilon}$ for some $\epsilon > 0$. For more details see [3,6]. Note, however, that viscosity solutions are $C^{2,\epsilon}$-regular almost everywhere; in fact, it is true on the complement of a close set of Hausdorff dimension strictly less than n [1]. Until recently it remained unclear whether non-smooth viscosity solutions exist. In the recent papers [11–14] two of the present authors first proved the existence of non-classical viscosity solutions to a fully nonlinear elliptic equation, and then of singular solutions to the Hessian uniformly elliptic equation in all dimensions beginning from 12. Those papers use the functions

$$w_{12,\delta}(x) = \frac{P_{12}(x)}{|x|^\delta}, \qquad w_{24,\delta}(x) = \frac{P_{24}(x)}{|x|^\delta}, \quad \delta \in [1, 2[,$$

with $P_{12}(x)$, $P_{24}(x)$ being cubic forms as follows:

$$P_{12}(x) = Re(q_1q_2q_3), \quad x = (q_1, q_2, q_3) \in \mathbf{H}^3 = \mathbf{R}^{12},$$

H being Hamiltonian quaternions,

$$P_{24}(x) = Re((o_1 \cdot o_2) \cdot o_3) = Re(o_1 \cdot (o_2 \cdot o_3)), \quad x = (o_1, o_2, o_3) \in \mathcal{O}^3 = \mathbf{R}^{24}$$

$\mathcal{O}$ being the algebra of Cayley octonions.

As was noted by the second author (V.T.), these are (most symmetric) examples of so-called *radial eigencubics*, which define minimal cubic cones. Since the family of such cubics contains especially interesting isoparametric Cartan cubics [2,5] in dimensions 5, 8, 14 and 26 admitting large automorphism groups, the last two being intimately connected with $P_{12}(x)$ and $P_{24}(x)$, it is but natural to try these Cartan cubics as numerators of tentative non-classical solutions to a Hessian uniformly elliptic equation in their respective dimensions. Our main goal in this paper is to show that in 5 dimensions it really works at least for $C^{1,1}$-solutions, and to prove the following theorem.

Theorem 1.1. *The function*

$$w_5(x) = \frac{P_5(x)}{|x|}$$

is a viscosity solution to a uniformly elliptic Hessian equation (1.1) *with Lipschitz F in a unit ball $B \subset \mathbf{R}^5$ for the isoparametric Cartan cubic form in $x = (x_1, x_2, z_1, z_2, z_3) \in \mathbf{R}^5$*

$$P_5(x) = x_1^3 + \frac{3x_1}{2}\left(z_1^2 + z_2^2 - 2z_3^2 - 2x_2^2\right) + \frac{3\sqrt{3}}{2}\left(x_2z_1^2 - x_2z_2^2 + 2z_1z_2z_3\right).$$

At the time of writing it is not clear that the same is true for the function $w_{5,\delta}(x) = P_5(x)/|x|^\delta$ for $\delta > 1$ (see Remark 4.1 below) and thus the question on the optimality of the interior $C^{1,\alpha}$-

regularity (i.e. of possibility for α to be arbitrary small) for viscosity solutions to fully nonlinear equations is open in dimensions up to 12.

However, the method of [12] permits to construct singular solutions in ten dimensions:

Corollary 1.1. *There exist $\varepsilon > 0$, $M > 0$ such that the homogeneous order $2 - 2\varepsilon$ function*

$$u_{10,\varepsilon,M}(x, y) = \frac{w_5(x) + w_5(y) + M(|x|^2 - |y|^2)}{(|x|^2 + |y|^2)^\varepsilon}$$

in the unit ball $B \subset \mathbf{R}^{10}$ is a viscosity solution to a uniformly elliptic equation (1.1) with smooth F.

For a proof it is sufficient just to repeat the argument of [12] which gives the result for $\varepsilon = 10^{-6}$, $M = 100$.

As in [13] we get also that w_5 is a viscosity solution to a uniformly elliptic Isaacs equation:

Corollary 1.2. *The function*

$$w_5(x) = P_5(x)/|x|$$

is a viscosity solution to a uniformly elliptic Isaacs equation (1.4) in a unit ball $B \subset \mathbf{R}^5$.

Remark 1.1. One could hope that using a minimal cubic cone in 4 dimensions, namely, the Lawson cubic [9] (essentially unique by Perdomo [15])

$$P_4(x) = x_3(x_1^2 - x_2^2) + 2x_1x_2x_4,$$

it is possible to construct a non-smooth solution to a Hessian uniformly elliptic equation in four dimensions, but it does not work. Namely, the function

$$w_4(x) = P_4(x)/|x|$$

is not a solution to such an equation, since it does not verify the conditions of Lemma 2.1 (the corresponding matrix family is not hyperbolic).

The rest of the paper is organized as follows: in Section 2 we recall some necessary preliminary results from [13], then we recall some facts about radial eigencubics and especially the Cartan cubic in Section 3 and we prove our main results in Section 4.

One should note that similar results are valid for three other Cartan cubics (and other isoparametric homogeneous polynomials) and that all of them can be used in some other similar applications. These results will be exposed elsewhere.

2. Preliminary results

Let $w = w_n$ be an odd homogeneous function of order 2, defined on a unit ball $B = B_1 \subset \mathbf{R}^n$ and smooth in $B \setminus \{0\}$. Then the Hessian of w is homogeneous of order 0.

We want to give a criterion for w to be a solution of a uniformly elliptic Hessian equation or a uniformly elliptic Isaacs equation. To do this, recall that a family $\mathcal{A} \subset S^2(\mathbf{R}^n)$ of symmetric matrices A is called *uniformly hyperbolic* if there exists a constant $M > 1$ such that

$$\frac{1}{M} < -\frac{\lambda_1(A)}{\lambda_n(A)} < M$$

for any $A \in \mathcal{A}$, $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$ being the eigenvalues of A.

One can reformulate some results from [13] (namely, Lemma 2.1, six final lines of Section 4, Lemmas 5.1 and 5.2) as follows, in our special case $\delta = 1$:

Lemma 2.1. *Set for $x, y \in S^{n-1}$ and for an orthogonal matrix $O \in O(n)$,*

$$M(x, y, O) := D^2 w(x) - {}^t O D^2 w(y) O.$$

Suppose that the family

$$\mathcal{M} := \{M(x, y, O) : M(x, y, O) \neq 0, x \neq y, x \neq 0, y \neq 0, O \in O(n)\} \subset S^2(\mathbf{R}^n)$$

is uniformly hyperbolic. Then w is a solution to a uniformly elliptic Hessian equation as well as to a uniformly elliptic Isaacs equation.

We need also the following property of the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of real symmetric matrices of order n which is a classical result by Weyl [21]:

Lemma 2.2. *Let $A \neq B$ be two real symmetric matrices with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n$ respectively. Then for the eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_n$ of the matrix $A - B$ we have*

$$\Lambda_1 \geq \max_{i=1,\dots,n} (\lambda_i - \lambda'_i), \qquad \Lambda_n \leq \min_{i=1,\dots,n} (\lambda_i - \lambda'_i).$$

3. Radial eigencubics

Let us recall the Cartan cubic form $P_5(x)$ which is closely related with real algebraic minimal cones, that is, homogeneous polynomial solutions u to the minimal surface equation

$$(1 + |\nabla u|^2)\Delta u - \sum u_{ij} u_i u_j = 0.$$

According to Hsiang [8], the study of those is equivalent to classifying polynomial solutions $f = f(x_1, \dots, x_n) \in \mathbf{R}[x_1, \dots, x_n]$ of the following congruence:

$$L(f) = 0(\bmod f), \qquad (3.1)$$
$$L(f) := |\nabla f|^2 \Delta f - \sum f_{ij} f_i f_j$$

being the normalized mean curvature operator. This condition means that the zero-locus $f^{-1}(0)$ has zero mean curvature everywhere where the gradient $\nabla f \neq 0$. A non-zero polynomial solution of this congruence is called an *eigenfunction* of L. The ratio $L(f)/f$ (a polynomial in x) is called the weight of an eigenfunction f. An eigenfunction f which is a cubic homogeneous is called an *eigencubic*.

Among them the most interesting are solutions of the following non-linear equation:

$$L(f) = \lambda |x|^2 f, \quad \lambda \in \mathbf{R}, \qquad (3.2)$$

which are called *radial eigencubics*. In [8], Hsiang posed the problem to determine all solutions of (3.2) up to a congruence in $\mathbf{R}^n$ (for any degree).

This classification for radial eigencubics is almost completed in [17,16], namely, any radial eigencubic is either a member of the infinite family of eigencubics of Clifford type completely classified in [17], or belongs to one of the exceptional families, the number of these lying between 12 and 20.

The cubic forms $P_{12}(x)$ and $P_{24}(x)$ belong to the Clifford family; the Cartan polynomial $P_5(x)$ is the first, that is of least dimension, in the list of exceptional radial eigencubics. Moreover, it is an isoparametric polynomial, that satisfies the Münzner system [10]:

$$|\nabla f|^2 = 9|x|^4, \quad \Delta f = 0,$$

expressing the fact that all principal curvatures of $f^{-1}(0) \bigcap S^4$ are constant (and different). Since $L(f) = -54|x|^2 f$, $P_5(x)$ is a radial eigencubic as well.

The form $P_5(x)$ admits a three-dimensional automorphism group. Indeed, one easily verifies that the orthogonal transformations

$$A_1(\phi) := \frac{1}{2}\begin{pmatrix} 3\cos(\phi)^2 - 1 & \sqrt{3}\sin(\phi)^2 & \sqrt{3}\sin(2\phi) & 0 & 0 \\ \sqrt{3}\sin(\phi)^2 & 1+\cos(\phi)^2 & -\sin(2\phi) & 0 & 0 \\ -\sqrt{3}\sin(2\phi) & \sin(2\phi) & 2\cos(2\phi) & 0 & 0 \\ 0 & 0 & 0 & 2\cos(\phi) & 2\sin(\phi) \\ 0 & 0 & 0 & -2\sin(\phi) & 2\cos(\phi) \end{pmatrix}$$

$$A_2(\psi) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(2\psi) & 0 & -\sin(2\psi) & 0 \\ 0 & 0 & \cos(\psi) & 0 & -\sin(\psi) \\ 0 & \sin(2\psi) & 0 & \cos(2\psi) & 0 \\ 0 & 0 & \sin(\psi) & 0 & \cos(\psi) \end{pmatrix}$$

$$A_3(\theta) := \frac{1}{2}\begin{pmatrix} 3\cos(\theta)^2 - 1 & -\sqrt{3}\sin(\theta)^2 & 0 & 0 & -\sqrt{3}\sin(2\theta) \\ -\sqrt{3}\sin(\theta)^2 & 1+\cos(\theta)^2 & 0 & 0 & -\sin(2\theta) \\ 0 & 0 & 2\cos(\theta) & -2\sin(\theta) & 0 \\ 0 & 0 & 2\sin(\theta) & 2\cos(\theta) & 0 \\ \sqrt{3}\sin(2\theta) & \sin(2\theta) & 0 & 0 & 2\cos(2\theta) \end{pmatrix}$$

do not change the value of $P_5(x)$.

Moreover, one easily gets the following lemma.

Lemma 3.1. *Let G_P be a subgroup of $SO(5)$ generated by*

$$\{A_1(\phi), A_2(\psi), A_3(\theta) : (\phi, \psi, \theta) \in \mathbf{R}^3\}.$$

Then the orbit $G_P S^1$ of the circle

$$S^1 = \{(\cos(\chi), 0, \sin(\chi), 0, 0) : \chi \in \mathbf{R}\} \subset S^4$$

under the natural action of G_P is the whole S^4.

Proof. Indeed, calculating the differential of the action

$$(S^1)^4 \longrightarrow S^4, \qquad (\phi, \psi, \theta, \chi) \mapsto (\cos(\chi), 0, \sin(\chi), 0, 0)A_1(-\phi)A_2(-\psi)A_3(-\theta)$$

at $(\phi, \psi, \theta, \chi) = (0, 0, 0, 0)$ one sees that its rank is 4 which implies the surjectivity. □

4. Proofs

Let $w_5 = P_5(x)/|x|$. By Lemma 2.1 it is sufficient to prove the uniform hyperbolicity of the family

$$M_5(x, y, O) := D^2 w_5(x) - {}^t O D^2 w_5(y) O.$$

Proposition 4.1. *Let $O \in O(5), x, y \in S^4$, $M_5(x, y, O) \neq 0$ and let $\Lambda_1 \geq \cdots \geq \Lambda_5$ be its eigenvalues. Then*

$$\frac{1}{20} \leq -\frac{\Lambda_1}{\Lambda_5} \leq 20.$$

Proof. We begin with calculating the eigenvalues of $D^2 w_5(x)$.

More precisely, we need the following lemma.

Lemma 4.1. *Let $x \in S^4$, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_5$ be the eigenvalues of $D^2 w_5(x)$, and let $x \in G_P(p, 0, q, 0, 0)$ with $p^2 + q^2 = 1$. Then*

$$\lambda_1 = \frac{p^3 - 6p + 3\sqrt{3(4-p^2)}}{2}, \qquad \lambda_3 = \frac{p^3 + 3p}{2},$$
$$\lambda_5 = \frac{p^3 - 6p - 3\sqrt{3(4-p^2)}}{2}.$$

Proof of Lemma 4.1. Since w_5 is invariant under G_P, we can suppose that $x = (p, 0, q, 0, 0)$. Then $w_5(x) = \frac{p(3-p^2)}{2}$ and we get by a brute force calculation:

$$D^2 w_5(x) := \frac{1}{2} \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

being a block matrix with

$$M_1 := \begin{pmatrix} p(1+2p^2-3p^4) & 3\sqrt{3}p(p^2-1) & 3q(1-p^4) \\ 3\sqrt{3}p(p^2-1) & p^3 - 15p & 3\sqrt{3}q(p^2+1) \\ 3q(1-p^4) & 3\sqrt{3}q(p^2+1) & p^3 + 3p^5 \end{pmatrix},$$
$$M_2 := \begin{pmatrix} p^3 + 3p & 6\sqrt{3}q \\ 6\sqrt{3}q & p^3 - 15p \end{pmatrix}$$

which gives for the characteristic polynomial $F(S) = F_1(S) \cdot F_2(S)$ where

$$F_1(S) = \left(S - \frac{3p}{2} - \frac{p^3}{2}\right)\left(S^2 + 6pS - p^3 S + \frac{63p^2}{4} - 3p^4 + \frac{p^6}{4} - 27\right),$$
$$F_2(S) = \left(S^2 + \frac{15pS}{2} - \frac{5p^3 S}{2} - \frac{45p^2}{4} + \frac{15p^4}{2} - \frac{5p^6}{4} - 9\right)$$

have the roots

$$\lambda_1 = \frac{p^3 - 6p + 3\sqrt{3(4-p^2)}}{2}, \qquad \lambda_3 = \frac{p^3 + 3p}{2},$$
$$\lambda_5 = \frac{p^3 - 6p - 3\sqrt{3(4-p^2)}}{2}$$
$$\lambda_2 = \frac{5p^3 - 15p + 3r}{4}, \qquad \lambda_4 = \frac{5p^3 - 15p - 3r}{4}$$

with $r := \sqrt{5p^6 - 30p^4 + 45p^2 + 16}$.

One needs only to verify that indeed

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$$

which is elementary. For example, let us verify for $p \in [-1, 1]$ the inequality

$$\lambda_1 = \frac{p^3 - 6p + 3\sqrt{3(4 - p^2)}}{2} \geq \lambda_2 = \frac{5p^3 - 15p + 3r}{4},$$

which by symmetry gives $\lambda_4 \geq \lambda_5$ (the two remaining inequalities being simpler). □

Indeed,

$$\lambda_1 - \lambda_2 = 3(r_1 - r)/4 \quad \text{with } r_1 := p - p^3 + 2\sqrt{3(4 - p^2)} > 0,$$
$$r_1^2 - r^2 = 4(1 - p^2)(s_1 + s_2) \geq 0$$

for $s_1 := p^4 - 6p^2 + 8 = (4 - p^2)(2 - p^2) > 0,\ s_2 := p\sqrt{3(4 - p^2)}$ since

$$s_1^2 - s_2^2 = (1 - p)(4 - p^2)(p^5 + p^4 - 7p^3 - 7p^2 + 13p + 16) \geq 0$$

because

$$p^5 + p^4 - 7p^3 - 7p^2 + 13p + 16 = (p + 1)(p^4 - 7p^2 + 13) + 3 \geq 3.$$

End of the proof. Let now $y \in G_P(\bar{p}, 0, \bar{q}, 0, 0)$. If $p = \bar{p}$ but $M_5(x, y, O) \neq 0$, the trace $Tr(M_5(x, y, O)) = 8w_5(y) - 8w_5(x) = 0$ and the conclusion follows as for any traceless matrix in dimension 5. Let then $p > \bar{p}$; by Lemma 2.2, one gets

$$\Lambda_1 \geq \lambda_2(p) - \lambda_2(\bar{p}) = \frac{(p - \bar{p})(p^2 + p\bar{p} + \bar{p}^2 + 3)}{2} \geq \frac{3(p - \bar{p})}{2},$$
$$\begin{aligned} -\Lambda_5 &\geq \max\{\lambda_3(\bar{p}) - \lambda_3(p), \lambda_1(\bar{p}) - \lambda_1(p)\} \\ &\geq (p - \bar{p}) \inf_{p \in [-1,1]} \max\{|\lambda_1'(p)|, |\lambda_3'(p)|\} = 3(p - \bar{p}), \end{aligned}$$
$$\begin{aligned} 0 &\leq -Tr(M_5(x, y, O)) = 8w_5(x) - 8w_5(y) \\ &= 8(p - \bar{p})(3 - p^2 - p\bar{p} - \bar{p}^2) \leq 24(p - \bar{p}). \end{aligned}$$

Therefore,

$$-\Lambda_5 \geq 4\Lambda_1$$

since $Tr(M_5(x, y, O)) \leq 0$ and

$$4\Lambda_1 + \Lambda_5 \geq Tr(M_5(x, y, O)) \geq 24(\bar{p} - p),$$
$$-\Lambda_5 \leq 4\Lambda_1 + 24(p - \bar{p}) \leq 4\Lambda_1 + 16\Lambda_1 = 20\Lambda_1,$$

the case $p < \bar{p}$ being completely parallel which finishes the proof of our results. □

Remark 4.1. One can calculate the eigenvalues of $D^2(P_5(x)/|x|^\delta)$ for $\delta > 1$ as well but then the analogue of Lemma 4.1 does not hold, which makes impossible applying the technique of [13, Section 4] to prove the hyperbolicity of the corresponding matrix family. Thus the question whether $P_5(x)/|x|^\delta$ is for $\delta > 1$ a solution of a uniformly elliptic (Hessian or not) equation remains open.

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