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Parking spaces ☆

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ABSTRACT

Let W be a Weyl group with root lattice Q and Coxeter number h. The elements of the finite torus $Q/(h+1)Q$ are called the W-**parking functions**, and we call the permutation representation of W on the set of W-parking functions the (standard) W-**parking space**. Parking spaces have interesting connections to enumerative combinatorics, diagonal harmonics, and rational Cherednik algebras. In this paper we define two new W-parking spaces, called the **noncrossing parking space** and the **algebraic parking space**, with the following features:

- They are defined more generally for real reflection groups.
- They carry not just W-actions, but $W \times C$-actions, where C is the cyclic subgroup of W generated by a Coxeter element.
- In the crystallographic case, both are isomorphic to the standard W-parking space.

Our Main Conjecture is that the two new parking spaces are isomorphic to each other as permutation representations of $W \times C$. This conjecture ties together several threads in the Catalan combinatorics of finite reflection groups. Even the

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weakest form of the Main Conjecture has interesting combinatorial consequences, and this weak form is proven in all types except E_7 and E_8. We provide evidence for the stronger forms of the conjecture, including proofs in some cases, and suggest further directions for the theory.

Contents

1. Introduction

Let W be a finite Coxeter group (finite real reflection group). (We refer to the standard references [10,30].) The main goal of this paper is to define two new objects that deserve to be called "parking spaces" for W. First we will describe the origin of the term "parking space".

1.1. *Classical parking functions and spaces*

A classical parking function is a map $f : [n] := \{1, 2, \dots, n\} \to \mathbb{N}$ for which the increasing rearrangement $(b_1 \leq b_2 \leq \cdots \leq b_n)$ of the sequence $(f(1), f(2), \dots, f(n))$ satisfies $b_i \leq i$.[1] Let Park_n denote the set of parking functions, which carries a permutation representation of the symmetric group $W = \mathfrak{S}_n$ via $(w.f)(i) := f(w^{-1}(i))$. We will call this the classical parking space. We note that Park_n has size $(n+1)^{n-1}$ and its W-orbits are counted by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Each orbit is represented by an increasing parking function, but one could also parametrize the orbits by other Catalan objects, for example the *nonnesting partitions* defined in Definition 2.4 below.

Example 1.1. Below we exhibit the $(3+1)^{3-1} = 4^2 = 16$ elements of Park_3, grouped into $5 = \frac{1}{4}\binom{2\cdot 3}{3}$ $\mathfrak{S}_3$-orbits by rows. The increasing orbit representatives are on the left.

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

The permutation action of $\mathfrak{S}_3$ on Park_3 decomposes over the orbits. Furthermore, note that the action of $\mathfrak{S}_3$ on the orbit $\{122, 212, 221\}$ is isomorphic to the action on cosets of a subgroup of type $\mathfrak{S}_2 \times \mathfrak{S}_1$, which has Frobenius characteristic given by the complete homogeneous symmetric function $h_{2,1} = h_2 h_1 = (\sum_{1 \leq i_1 \leq i_2} x_{i_1} x_{i_2})(\sum_{1 \leq i} x_i)$. Hence the action of $\mathfrak{S}_3$ on Park_3 has Frobenius characteristic

$$h_3 + 3h_{2,1} + h_{1,1,1}.$$

1.2. *Weyl group and real reflection group parking spaces*

Now we review how the action of $\mathfrak{S}_n$ on Park_n can be generalized to other finite Weyl groups W (crystallographic real reflection groups). Let W act irreducibly on $V \cong \mathbb{R}^n$. Since W is crystallographic one can choose

$$\Delta \subseteq \Phi^+ \subseteq \Phi \subseteq Q \subseteq V, \tag{1.1}$$

[1] The name "parking function" comes from a combinatorial interpretation due to Konheim and Weiss [31]. Suppose n cars want to parking in n linearly-ordered parking spaces and that car i wants to park in spot $f(i)$. At step i, car i tries to park in space $f(i)$. If the spot is full then i parks in the first available spot $\geq f(i)$. If no such spot exists then i leaves the parking lot. The function f is called a parking function if every car is able to park.

where Δ, Φ^+, Φ, Q are the simple roots, positive roots, root system, and root lattice, respectively. Given $\Delta = \{\alpha_1, \dots, \alpha_n\}$, let s_i denote the reflection through the hyperplane $H_i = \alpha_i^\perp$. This endows W with set of Coxeter generators $S = \{s_1, \dots, s_n\}$. The product $s_1 s_2 \cdots s_n$ (taken in any order) is called a standard Coxeter element of W. A Coxeter element of W is any W-conjugate of a standard Coxeter element. The Coxeter elements form a single conjugacy class [30, §3.16] and the multiplicative order h of any Coxeter element is called the Coxeter number of W.

The role of W-parking functions is played by the quotient $Q/(h+1)Q$, having cardinality $(h+1)^n$, of the two nested rank n lattices $(h+1)Q \subset Q$. The W-action on V induces a permutation W-action on $Q/(h+1)Q$ which we call the standard W-parking space; see Haiman [29, §2.4 and §7.3]. Recall that there is a standard partial order on positive roots Φ^+ defined by setting $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{N}\Phi^+$. Then the W-orbits on $Q/(h+1)Q$ can be parametrized by antichains in the root poset $(\Phi^+, \leq)$ (also called W-nonnesting partitions); see Cellini and Papi [15, §4], Sommers [46, §5], and Shi [44]. Finally, we note that the number of W-nonnesting partitions equals the W-Catalan number,

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{h + d_i}{d_i},$$

where $\{d_1, d_2, \dots, d_n\}$ is the multiset of degrees for W. By definition, these are the degrees of any set of homogeneous algebraically-independent generators $f_1, \dots, f_n$ for the invariant subalgebra $\mathbb{R}[V]^W = \mathbb{R}[f_1, \dots, f_n]$ when W acts on the algebra of polynomial functions $\mathbb{R}[V] = \mathrm{Sym}(V^*) \cong \mathbb{R}[x_1, \dots, x_n]$ (see [41]).

Example 1.2. In type A_{n-1} one has $W = \mathfrak{S}_n$ generated by $S = \{s_1, \dots, s_{n-1}\}$, where the adjacent transposition $s_i = (i, i+1)$ corresponds to reflection through the hyperplane $H_{i,i+1} = \{x \in \mathbb{R}^n : \langle x, e_i - e_{i+1} \rangle = 0\} \subseteq \mathbb{R}^n$. Thus $\mathfrak{S}_n$ acts irreducibly on the subspace $V = \mathbf{1}^\perp \subseteq \mathbb{R}^n$ perpendicular to $\mathbf{1} = (1, 1, \dots, 1)$. A Coxeter element $c = s_1 s_2 \cdots s_{n-1}$ is an n-cycle, with order $h = n$.

Here the root lattice Q is the $\mathbb{Z}$-sublattice $\{x \in \mathbb{Z}^n : \sum_i x_i = 0\}$ inside $\mathbb{Z}^n$. Hence $Q/(n+1)Q \cong \{x \in \mathbb{Z}_{n+1}^n : \sum_{i=1}^n x_i \equiv 0 \bmod n+1\}$, using the abbreviation $\mathbb{Z}_{n+1} := \mathbb{Z}/(n+1)\mathbb{Z}$. Haiman [29, Proposition 2.6.1] noted that the parking functions $\mathsf{Park}_n \subseteq \mathbb{Z}^n$ descend to give coset representatives for $\mathbb{Z}_{n+1}^n / \mathbb{Z}_{n+1}\mathbf{1}$. We claim that this implies the $\mathfrak{S}_n$-actions on the sets Park_n and on $Q/(n+1)Q$ are isomorphic: the finite abelian groups $\mathbb{Z}_{n+1}^n / \mathbb{Z}_{n+1}\mathbf{1}$ and $\{x \in \mathbb{Z}_{n+1}^n : \sum_{i=1}^n x_i \equiv 0 \bmod n+1\}$ are naturally *Pontrjagin dual*, so that they carry *contragredient* $\mathfrak{S}_n$-representations, and permutation representations are always self-contragredient.

In this paper we will propose two new W-parking spaces. The first space is defined combinatorially, in terms of the W-noncrossing partitions, and the second space is defined algebraically, as a quotient of the polynomial ring. These new spaces have the following features:

- For crystallographic W, both are isomorphic as W-representations to the standard W-parking space $Q/(h+1)Q$. However,
- both are defined more generally for *non-crystallographic* finite reflection groups, and
- both carry an additional $W \times C$-action, where $C = \langle c \rangle$ is the cyclic subgroup of W generated by a Coxeter element c.

Our Main Conjecture states that the two new parking spaces are *isomorphic to each other* as $W \times C$-permutation representations, which has several consequences for the Catalan combinatorics of finite reflection groups. In the next section we will define the new parking spaces, state the Main Conjecture, and explore some of its consequences. After that we will give evidence for the conjecture, prove some special cases, and suggest problems for the future.

2. Definitions and Main Conjecture

First we will review the concepts of W-noncrossing and W-nonnesting partitions. For a full treatment, see [1].

Let W be a finite Coxeter group (finite real reflection group) acting irreducibly on $V \cong \mathbb{R}^n$ and orthogonally with respect to the inner product $\langle \cdot,\cdot \rangle$. Since W may not be crystallographic there is no root lattice Q. However, one can still discuss the (noncrystallographic) root system $\Phi = \{\pm\alpha\}$ of unit normals to the reflecting hyperplanes H_α for the set of all reflections $T = \{t_\alpha\} \subseteq W$. Let $\mathsf{Cox}(\Phi) = \{H_\alpha\}_{\alpha\in\Phi}$ denote the arrangement of reflecting hyperplanes and consider its lattice $\mathcal{L}$ of intersection subspaces (called *flats* X), ordered by reverse-inclusion.

Example 2.1. Again, consider the irreducible action of $W = \mathfrak{S}_n$ on the codimension-one subspace $V = \mathbf{1}^\perp \subseteq \mathbb{R}^n$. The reflections of $\mathfrak{S}_n$ are precisely the transpositions $T = \{t_{i,j} = (i,j)\}$, where $t_{i,j}$ switches the i and j coordinates in $\mathbb{R}^n$ and hence reflects in the hyperplane $H_{i,j} = \{x \in V : \langle x, e_i - e_j \rangle = 0\}$. A typical intersection flat X is defined by setting several blocks of coordinates equal, and corresponds to a set partition $\pi = \{B_1, \ldots, B_\ell\}$ of $[n]$ for which one has $H_{i,j} \subseteq X$ if and only if i and j occur in the same block of π. For example, in the case $W = \mathfrak{S}_9$, the flat X defined by equations

$$\{x_1 = x_3 = x_6 = x_7,\ x_4 = x_5,\ x_8 = x_9\}$$

corresponds to the set partition

$$\pi = \{B_1, B_2, B_3, B_4\} = \bigl\{\{1,3,6,7\},\{2\},\{4,5\},\{8,9\}\bigr\}.$$

2.1. Noncrossing partitions for W

The reflecting hyperplanes $\mathsf{Cox}(\Phi)$ decompose V into connected components called chambers, on which W acts simply-transitively. Choose a fundamental chamber c_0 for this action and for each reflecting hyperplane say that c_0 lies in the "positive" half-space. This induces a choice of positive roots Φ^+ – the "positive" normal vectors – and simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\} \subseteq \Phi^+$ – corresponding to the hyperplanes that bound c_0. The reflections $S = \{s_1, \dots, s_n\}$ defined by the simple roots Δ endow W with a Coxeter system (W, S). Fix once and for all a Coxeter element $c = s_1 s_2 \cdots s_n$ by ordering S. The Coxeter number h is the multiplicative order of c, and it turns out to equal the maximum of the fundamental degrees $d_1, \dots, d_n$ mentioned earlier; [30, §3.17].

For all $w \in W$, let $\ell_T(w)$ denote the minimal value of k such that $w = t_1 \cdots t_k$ with $t_i \in T$ for all i. One calls this the reflection length on W. It turns out (see Carter [14]) that $\ell_T(w)$ is equal to the codimension of the W-fixed space $V^w \subseteq V$, that is, $\ell_T(w) = \operatorname{codim} V^w = \dim V - \dim V^w$. Then one defines the absolute order $\leq_T$ on W by setting

$$u \leq_T v \quad \Longleftrightarrow \quad \ell_T(v) = \ell_T(u) + \ell_T\big(u^{-1}v\big) \tag{2.1}$$

for all $u, v \in W$. This poset is graded with rank function ℓ_T. It has a unique minimal element $1 \in W$ but in general many maximal elements (the elements with trivial fixed space), among which is the conjugacy class of Coxeter elements. Following Brady and Watt [12, §4] and Bessis [8, Definition 2.1.3], make the following definition.

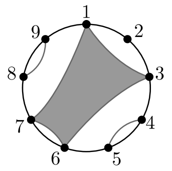
Definition 2.2. Define the poset $NC(W)$ of W-noncrossing partitions as the interval $[1, c]_T$ in absolute order. Brady and Watt [13, Theorem 2] showed that this poset embeds into the partition lattice

$$\begin{aligned} NC(W) &\hookrightarrow \mathcal{L} \\ w &\mapsto V^w \end{aligned}$$

We will sometimes identify $NC(W)$ with its image under this embedding, and refer to the elements of $NC(W)$ as the noncrossing flats $X \in \mathcal{L}$.

Because Coxeter elements form a conjugacy class, and because conjugation by $w \in W$ is a poset isomorphism $[1, c]_T \cong [1, wcw^{-1}]_T$, we do not include c in the notation for $NC(W)$. Furthermore, the map $w \mapsto c^d w c^{-d}$ is a poset automorphism, which corresponds to the action $X \mapsto c^d X$ on noncrossing flats.

Example 2.3. For $W = \mathfrak{S}_n$ one can choose as Coxeter generators the set of adjacent transpositions $S = \{s_1, \dots, s_{n-1}\}$, where $s_i = (i, i+1)$. A natural choice of Coxeter element is the n-cycle $c = s_1 s_2 \cdots s_{n-1} = (1, 2, \dots, n)$. In this case the noncrossing flats $X \in \mathcal{L}$ correspond to partitions $\pi = \{B_1, \dots, B_\ell\}$ that are *noncrossing* in the sense of Kreweras [32] and Poupard [36]. That is, if one draws the numbers $[n] = \{1, 2, \dots, n\}$ clockwise around a circle then the convex hulls of its blocks B_i are pairwise-disjoint (i.e. "noncrossing"). For example the set partition $\pi = \{\{1, 3, 6, 7\}, \{2\}, \{4, 5\}, \{8, 9\}\}$ is noncrossing, as shown here:



Given such a noncrossing partition/flat $\pi = \{B_1, \ldots, B_\ell\}$, one recovers a permutation $w \in NC(W)$ by converting each block B_i into a cycle, oriented clockwise, and then multiplying these. For example the noncrossing partition π shown above corresponds to $w = (1, 3, 6, 7)(2)(4, 5)(8, 9)$.

2.2. Nonnesting partitions for W

The concept of nonnesting partitions is more recent than that of noncrossing partitions, and their original definition was completely general. The following definition is due to Postnikov [38, Remark 2].

Definition 2.4. Let W be a Weyl group,[2] so that there exists a root poset $(\Phi^+, \leq)$, defined by setting $\alpha \leq \beta$ if and only if $\beta - \alpha \in \mathbb{N}\Phi^+$. The set $NN(W)$ of W-nonnesting partitions is the collection of antichains (sets of pairwise-incomparable elements) in $(\Phi^+, \leq)$. Athanasiadis and Reiner [6, Corollary 6.2] showed that there is an embedding

$$\begin{aligned} NN(W) &\hookrightarrow \mathcal{L} \\ A &\mapsto \bigcap_{\alpha \in A} H_\alpha, \end{aligned}$$

which defines a partial order on $NN(W)$. We will sometimes identify $NN(W)$ with its image under this embedding, and speak about nonnesting flats $X \in \mathcal{L}$.

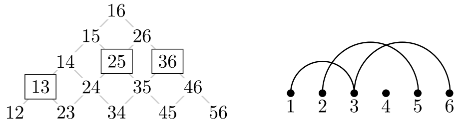
Example 2.5. For $W = \mathfrak{S}_n$, one can choose the root system and positive roots as follows:

$$\begin{aligned} \Phi &:= \{e_i - e_j : 1 \leq i \neq j \leq n\}, \\ \Phi^+ &:= \{e_i - e_j : 1 \leq i < j \leq n\}. \end{aligned}$$

Draw the numbers $[n] = \{1, \ldots, n\}$ in a line. Then one has $\alpha = e_i - e_j \leq \beta = e_r - e_s$ in the root poset if and only if the semicircular arc connecting i and j is enclosed (i.e. "nested") inside the arc connecting r and s. Thus an antichain A in Φ^+ gives rise to a set of arcs whose transitive closure is a nonnesting set partition of $[n]$. For example, let $W = \mathfrak{S}_6$. The antichain $A = \{e_1 - e_3, e_2 - e_5, e_3 - e_6\}$ corresponds to the nonnesting

[2] Note that the same definition can be made for noncrystallographic types, but that the resulting structures do not have nice properties. It is an open problem to generalize "nonnesting partitions" to this case.

flat $X = \bigcap_{\alpha \in A} H_\alpha = \{x_1 = x_3 = x_6,\ x_2 = x_5\}$ and the nonnesting set partition $\pi = \{\{1,3,6\},\{2,5\},\{4\}\}$. Below we show the antichain A (where ij stands for $e_i - e_j$). The corresponding nonnesting partition is on the right.



2.3. Noncrossing and nonnesting parking functions

Now we define a family of W-modules, two of which are the standard W-parking space, and the new W-noncrossing parking space.

Define an equivalence relation on the set of ordered pairs

$$W \times \mathcal{L} = \{(w, X) : w \in W,\ X \in \mathcal{L}\}$$

by setting $(w, X) \sim (w', X')$ when one has both

- $X = X'$, that is, the flats are equal, and
- $wW_X = w'W_X$ where W_X is the pointwise W-stabilizer of the flat X.

Let $[w, X]$ denote the equivalence class of (w, X), and note that the left-regular action of W on itself in the first coordinate descends to a W-action on equivalence classes: $v.[w, X] := [vw, X]$.

Definition 2.6. Define the W-nonnesting and W-noncrossing parking functions as the following W-stable subsets of $(W \times \mathcal{L})/\sim$:

$$\mathsf{Park}_W^{NN} := \{[w, X] : w \in W \text{ and } X \in NN(W)\}$$
$$\mathsf{Park}_W^{NC} := \{[w, X] : w \in W \text{ and } X \in NC(W)\}.$$

One could alternately phrase[3] Park_W^{NN} (resp. Park_W^{NC}) as the set of pairs (wW_X, X) with X in $NN(W)$ (resp. in $NC(W)$) and wW_X in W/W_X a coset modulo W_X.

Both of these subsets inherit the W-action $v.[w, X] = [vw, X]$, but the second set Park_W^{NC} also has a $W \times C$-action, defined by letting C act on the right:

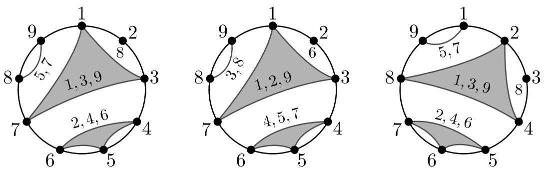
$$(v, c^d).[w, X] := [vwc^{-d}, c^d(X)]$$

Example 2.7. One can think of type A_{n-1} noncrossing parking functions pictorially. Consider $[w, X] \in \mathsf{Park}_{\mathfrak{S}_n}^{NC}$. Then the noncrossing flat X corresponds to a noncrossing

[3] Thanks to an anonymous referee for suggesting this phrasing.

partition $\pi = \{B_1, \dots, B_\ell\}$ of $[n]$ and the permutation $w \in \mathfrak{S}_n$ is considered only up to its coset wW_X for the Young subgroup $W_X = \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_\ell}$. Thus one can think of w as a function that assigns to each block a set of labels $w(B_i) = \{w(j)\}_{j \in B_i}$. Visualize this by labeling each block B_i in the noncrossing partition diagram by the set $w(B_i)$. For example, the leftmost picture in the figure below shows the noncrossing parking function $[w, X] \in \mathsf{Park}^{NC}_{\mathfrak{S}_9}$, where X corresponds to $\pi = \{\{1,3,7\},\{2\}\{4,5,6\},\{8,9\}\}$, and

$$wW_X = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 8 & 3 & 2 & 4 & 6 & 9 & 5 & 7 \end{pmatrix} W_X$$
$$= \left(\begin{array}{ccc|c|ccc|cc} 1 & 3 & 7 & 2 & 4 & 5 & 6 & 8 & 9 \\ 1 & 3 & 9 & 8 & 2 & 4 & 6 & 5 & 7 \end{array}\right) W_X.$$



The $W \times C$-action is also easy to visualize. If one chooses the standard n-cycle for our Coxeter element, $c = s_1 s_2 \cdots s_{n-1} = (1, 2, \dots, n)$, then the action $[w, X] \mapsto [vwc^{-d}, c^d(X)]$ of the element $(v, c^d) \in \mathfrak{S}_n \times C$ just permutes the labels by $v \in \mathfrak{S}_n$ and rotates the picture by $2\pi d/n$ clockwise. For example, let $v = (1)(2,7,3)(4)(5,8,6)(9) \in \mathfrak{S}_9$. The middle and right pictures in the figure show $[vx, X]$ and $[wc^{-1}, c(X)]$, respectively.

2.4. The coincidence of W-representations

We consider several *W-sets* (and $W \times C$-sets), that is, sets with a W-action, such as Park^{NN}_W, Park^{NC}_W, or the set of cosets $W/W_X = \{wW_X : w \in W\}$ under left-translation.

Definition 2.8. Given a finite group G and two G-sets A_1, A_2, write $A_1 \cong_G A_2$ to mean there is a G-equivariant bijection $A_1 \to A_2$. Let $\mathbb{C}[A]$ denote the G-permutation representation associated to the G-set A. Given two G-representations V_1, V_2, write $V_1 \cong_{\mathbb{C}[G]} V_2$ if they are isomorphic as $\mathbb{C}[G]$-modules, or equivalently if they have the same character.

Note that $A_1 \cong_G A_2$ implies $\mathbb{C}[A_1] \cong_{\mathbb{C}[G]} \mathbb{C}[A_2]$, but the converse need not hold.[4] As an example of the notations, $\mathbb{C}[W/W_X] \cong_{\mathbb{C}[G]} \mathrm{Ind}_{W_X}^W \mathbf{1}_{W_X}$, and by definition

[4] Although see Proposition 3.1 for a situation where the converse does hold.

$$\mathsf{Park}_W^{NN} \cong_W \coprod_{X \in NN(W)} W/W_X,$$
$$\mathsf{Park}_W^{NC} \cong_W \coprod_{X \in NC(W)} W/W_X. \tag{2.2}$$

Recall that Shi [44] and Cellini and Papi [15, §4] established a bijection between antichains $NN(W)$ in the root poset and W-orbits on the finite torus $Q/(h+1)Q$. It turns out that this bijection sends a nonnesting flat $X \in NN(W)$ to a W-orbit whose stabilizer is conjugate to W_X; see Athanasiadis [4, Lemma 4.1, Theorem 4.2]. Hence one has an isomorphism of W-sets:

$$Q/(h+1)Q \cong_W \mathsf{Park}_W^{NN}.$$

Furthermore, it was checked **case-by-case** in [6, Theorem 6.3] that $NN(W)$ and $NC(W)$ contain the same number of flats in each W-orbit on the partition lattice $\mathcal{L}$. (This generalizes an observation of Stanley in type A; see [49, Proposition 2.4].) Hence one also has this isomorphism of W-sets:

$$\mathsf{Park}_W^{NN} \cong_W \mathsf{Park}_W^{NC}. \tag{2.3}$$

Example 2.9. In the case of the symmetric group $W = \mathfrak{S}_n$ with $n \leq 3$, **every** partition of $[n]$ is both noncrossing and nonnesting, hence in these cases one has an equality $\mathsf{Park}_{\mathfrak{S}_n}^{NN} = \mathsf{Park}_{\mathfrak{S}_n}^{NC}$. For $n = 4$ there is exactly one crossing set partition, namely $\pi_1 = \{\{1,3\},\{2,4\}\}$, and exactly one nesting set partition, namely $\pi_2 = \{\{1,4\},\{2,3\}\}$. However, note that π_1 and π_2 correspond to flats X_1, X_2 in the same W-orbit on $\mathcal{L}$, hence one still has an isomorphism because

$$\mathsf{Park}_{\mathfrak{S}_4}^{NN} \cong_W \coprod_{X \in \mathcal{L} - \{X_2\}} W/W_X \cong_W \coprod_{X \in \mathcal{L} - \{X_1\}} W/W_X \cong_W \mathsf{Park}_{\mathfrak{S}_4}^{NC}.$$

2.5. The algebraic W-parking space

Next we will define a new W-parking space *algebraically*. Extend scalars from $V \cong \mathbb{R}^n$ to $\mathbb{C}^n$ and let W act on the polynomial algebra isomorphic to the symmetric algebra of the dual space V^*

$$\mathbb{C}[V] = \mathrm{Sym}\big(V^*\big) = \mathbb{C}[x_1, \ldots, x_n],$$

where $x_1, \ldots, x_n$ is a $\mathbb{C}$-basis for V^*. It is a subtle consequence of the representation theory of rational Cherednik algebras (see Berest, Etingof and Ginzburg [7], Gordon [26], and Etingof [19], [20, §4]), that there exist homogeneous systems of parameters $\Theta = (\theta_1, \ldots, \theta_n)$ of degree $h+1$ inside $\mathbb{C}[V]$, with the following property:

The $\mathbb{C}$-linear isomorphism defined by

$$\begin{aligned} V^* &\to \mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n \\ x_i &\mapsto \theta_i \end{aligned} \tag{2.4}$$

is W-equivariant. In particular, the linear span $\mathbb{C}\theta_1 + \cdots + \mathbb{C}\theta_n$ carries a copy of the dual[5] reflection representation V^*.

In some cases (for example in types B/C and D, and in cases of rank ≤ 2) one can choose the coordinate functionals $x_1, \ldots, x_n$ in V^* such that $(\theta_1, \ldots, \theta_n) = (x_1^{h+1}, \ldots, x_n^{h+1})$; see Sections 6 and 7 below. However, already in type A the construction of such an hsop is somewhat tricky – see Haiman [29, Proposition 2.5.4] or Chmutova and Etingof [16, §3] – and for the exceptional real reflection groups we know of no simple construction.

When one has such a Θ it is natural to consider the quotient ring $\mathbb{C}[V]/(\Theta)$. For example, this quotient occurs in the rational Cherednik theory [26,7,19,20] as the finite-dimensional irreducible module $L_{(h+1)/h}(\mathbf{1}_W)$ corresponding to the trivial representation $\mathbf{1}_W$ of W, and taken at the rational parameter $\frac{h+1}{h}$. In the crystallographic case it is known (see [26, §5], [9, Eq. (5.5)]) that $\mathbb{C}[V]/(\Theta)$ is isomorphic to the permutation representation $\mathbb{C}[Q/(h+1)Q] \cong_{\mathbb{C}[W]} \mathbb{C}[\mathsf{Park}_W^{NN}]$. We wish to **deform** the quotient $\mathbb{C}[V]/(\Theta)$ somewhat, considering instead the following.

Definition 2.10. Let W be a real irreducible reflection group W, with $\Theta = (\theta_1, \ldots, \theta_n)$ and $(x_1, \ldots, x_n)$ chosen as in (2.4). Consider the ideal

$$(\Theta - \mathbf{x}) := (\theta_1 - x_1, \ldots, \theta_n - x_n) \tag{2.5}$$

and define the algebraic W-parking space as the quotient ring

$$\mathsf{Park}_W^{alg} := \mathbb{C}[V]/(\Theta - \mathbf{x}).$$

This quotient has the structure of a $W \times C$-representation since the ideal $(\Theta - \mathbf{x})$ is stable under the following two commuting actions on $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]$:

- the action of W by linear substitutions, and
- the action of $C = \langle c \rangle$ by scalar substitutions $c^d(x_i) = \omega^{-d} x_i$, with $\omega := e^{\frac{2\pi i}{h}}$.

Note that we have not included the choice of Θ in the notation Park_W^{alg}. This is justified by the following proposition, which we will prove in Section 4 below.

[5] Since the reflection groups considered in this paper are real, one always has a $\mathbb{C}[W]$-module isomorphism $V \cong_{\mathbb{C}[W]} V^*$. However, we use V^* to be consistent with the case of complex reflection groups. We hope that some of our conjectures can be extended to complex reflection groups, at least in the well-generated case considered by Bessis and Reiner [9], and perhaps even in the arbitrary case considered by Gordon and Griffeth [27].

Proposition 2.11. *For every irreducible real reflection group W, and for* **any** *choice of Θ satisfying (2.4), one has an isomorphism of $W \times C$-representations*

$$\mathsf{Park}_W^{alg} := \mathbb{C}[V]/(\Theta - \mathbf{x}) \cong_{\mathbb{C}[W\times C]} \mathbb{C}[V]/(\Theta).$$

While it is conceivable that the ring structures of $\mathbb{C}[V]/(\Theta)$ and $\mathbb{C}[V]/(\Theta - \mathbf{x})$ may depend on Θ, we will only be interested in the $\mathbb{C}[W \times C]$-module structures of these objects, and hence denote them both by Park_W^{alg}, suppressing reference to Θ. This proposition has several important consequences. For example, forgetting the C-action, one obtains the following string of $\mathbb{C}[W]$-module isomorphisms:

$$\begin{aligned} \mathsf{Park}_W^{alg} &:= \mathbb{C}[V]/(\Theta - \mathbf{x}) \\ &\cong_{\mathbb{C}[W]} \mathbb{C}[V]/(\Theta) \\ &\cong_{\mathbb{C}[W]} \mathbb{C}\big[Q/(h+1)Q\big] \\ &\cong_{\mathbb{C}[W]} \mathbb{C}\big[\mathsf{Park}_W^{NN}\big] \\ &\cong_{\mathbb{C}[W]} \mathbb{C}\big[\mathsf{Park}_W^{NC}\big]. \end{aligned} \tag{2.6}$$

2.6. The Main Conjecture

Our Main Conjecture is a stronger and more direct geometric connection between the first and last terms in (2.6), eliminating the crystallographic hypothesis. Here we consider the subvariety/zero locus V^Θ cut out in V by the ideal $(\Theta - \mathbf{x})$. The notation V^Θ is meant to be suggestive of the fact that this subvariety is the subset of V fixed pointwise under the map $\Theta : V \to V$ that sends the element of V having coordinates $(x_1, \dots, x_n)$ to the element of V having coordinates $(\theta_1, \dots, \theta_n)$. Note that Proposition 2.11 implies $\mathbb{C}[V]/(\Theta - \mathbf{x})$ is a $\mathbb{C}$-vector space of dimension $(h+1)^n$, since $\mathbb{C}[V]/(\Theta)$ is. In particular, this subvariety $V^\Theta \subseteq V$ can contain at most $(h+1)^n$ points, and contains exactly this many points when counted with multiplicity. Note that the $W \times C$-action on V, in which an element c^d in C scales V by the root-of-unity ω^d, restricts to one on V^Θ.

Main Conjecture. *Let W be an irreducible real reflection group.*

(strong version)	**For all choices** *of Θ as in (2.4), one has that ...*
(intermediate version)	**There exists a** *choice of Θ as in (2.4) such that ...*

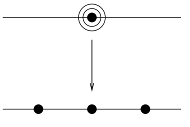
... the subvariety V^Θ inside V consists of $(h+1)^n$ distinct points, that have a $W \times C$-equivariant bijection to the set Park_W^{NC}, that is, $V^\Theta \cong_{W\times C} \mathsf{Park}_W^{NC}$.

In other words, the ideal (Θ) has its zero locus supported at the origin in V, with scheme structure as a fat point of multiplicity $(h+1)^n$, while $(\Theta - \mathbf{x})$ cuts out a subvariety V^Θ that deforms this fat point at the origin, blowing it apart into $(h+1)^n$ reduced points, carrying a $W \times C$-representation identifiable as[6] that of the noncrossing parking functions Park_W^{NC}.

Appendix A to this paper contains a uniform argument communicated to the authors by Etingof [18] which implies that for any irreducible real reflection group W, there exists an hsop Θ as in (2.4) such that V^Θ has $(h+1)^n$ distinct points. Unfortunately, the hsop Θ involved arises from deep results in the theory of rational Cherednik algebras and determining the $W \times C$-set structure of V^Θ has been so far intractable.

Example 2.12. To gain intuition, we check the strong version of the Main Conjecture in type A_1, that is, when W has rank 1. Here $W = \{1, s\}$ acts on $V = \mathbb{C}^1$ by $s.v = -v$. Hence W acts on the basis element x for V^*, as well as on $\mathbb{C}[V] = \mathrm{Sym}(V^*) = \mathbb{C}[x]$, via $s(x) = -x$. The only choice of a Coxeter element is $c = s$, with Coxeter number $h = 2$. An hsop θ of degree $h+1 = 3$ must take the form $\theta = \alpha x^3$ for some α in $\mathbb{C}^\times$. Any such θ has the map $x \mapsto \alpha x^3$ being W-equivariant as required by (2.4).

As depicted schematically here



the subvariety of $V = \mathbb{C}^1$ cut out by $\theta = \alpha x^3$ is the origin $\{0\}$, cut out with multiplicity $3(= h+1)$. Meanwhile, the subvariety V^θ cut out by

$$\theta - x = \alpha x^3 - x = \alpha x\left(x - \alpha^{-\frac{1}{2}}\right)\left(x + \alpha^{-\frac{1}{2}}\right)$$

consists of 3 $\mathsf{distinct}$ points $\{0, \pm\alpha^{-\frac{1}{2}}\}$. Furthermore, one can check that the $W \times C$-permutation module structure on these 3 points of $\mathsf{Park}_W^{alg} = V^\theta$ matches that of Park_W^{NC}. In this case, there are only two flats X in the intersection lattice $\mathcal{L}$, namely the origin $\{0\}$ and the whole space V, both of which are noncrossing flats, since $c = s$ has $V^s = \{0\}$ and 1 has $V^1 = V$. Thus $NC(W) = [1, s] \cong \{\{0\}, V\}$.

- The singleton $W \times C$-orbit $\{0\}$ within V^θ matches the singleton $W \times C$-orbit $\{[1, \{0\}]\}$ in Park_W^{NC}, both carrying trivial $W \times C$-action.
- The two-element $W \times C$-orbit $\{\pm\alpha^{-\frac{1}{2}}\}$ within V^θ matches the two-element $W \times C$-orbit $\{[1, V], [s, V]\}$ in Park_W^{NC}: in both cases either element in the orbit has $W \times C$-stabilizer subgroup equal to $\{(1,1), (s,s)\}$.

It can be shown that certain $W \times C$-subsets of V^Θ carry actions which provide evidence for the Main Conjecture. Define the $\mathsf{dimension}$ of a point $p \in V^\Theta$ to be the

[6] Actually, this would show that the two $W \times C$-representations are contragredient, since $\mathbb{C}[V]/(\Theta - \mathbf{x})$ is the space of functions on V^Θ. But permutation representations are self-contragredient, so this does not affect the $W \times C$-isomorphism statement.

minimal dimension of a flat $X \in \mathcal{L}$ such that $p \in X$. Then one has a disjoint union decomposition

$$V^\Theta = V^\Theta(0) \uplus V^\Theta(1) \uplus \cdots \uplus V^\Theta(n),$$

where $n = \dim(V)$ is the rank of W and $V^\Theta(d)$ denotes the set of d-dimensional points in V^Θ. Moreover, each of the sets $V^\Theta(d)$ is $W \times C$-stable.

Note that since $NC(W) = [1, c]_T$ has unique bottom and top elements 1, c, inside Park_W^{NC} there are two corresponding $W \times C$-orbits: the W-*trivial orbit* $\{[1, \{0\}]\}$, carrying the trivial $W \times C$-representation, and the W-*regular orbit* $\{[w, V]\}_{w \in W}$ carrying the coset representation $(W \times C)/\langle(c, c)\rangle$. Section 5 proves the following result on the counterparts[7] in V^Θ, showing that $V^\Theta(d)$ for $d \in \{0, 1, n\}$ are all as described in the Main Conjecture.

Proposition 2.13. *For W an irreducible real reflection group of rank n, and for all choices of Θ as in (2.4), one has the following.*

- (i) *The set $V^\Theta(0) = \{0\}$ of 0-dimensional points in V^Θ is the unique $W \times C$-orbit in V^Θ carrying the trivial $W \times C$-representation.*
- (ii) *There exists a $W \times C$-equivariant injection $V^\Theta(1) \hookrightarrow \mathsf{Park}_W^{NC}$ whose image is precisely the set of noncrossing parking functions of the form $[w, X]$ with $\dim(X) = 1$.*
- (iii) *The set $V^\Theta(n)$ of n-dimensional points is the unique W-regular orbit of points in V^Θ.*

Furthermore, every point in the subsets $V^\Theta(0)$, $V^\Theta(1)$, and $V^\Theta(n)$ of V^Θ is cut out by the ideal $(\Theta - \mathbf{x})$ in a reduced fashion, that is, with multiplicity one.

When $n \leq 2$, the sets $V^\Theta(0)$, $V^\Theta(1)$, and $V^\Theta(n)$ exhaust the variety V^Θ, and Proposition 2.13 implies that V^Θ consists of $(h+1)^n$ distinct points with the same $W \times C$-action as Park_W^{NC}. This completes the proof of the strong version of the Main Conjecture in rank ≤ 2. We remark that it is possible to prove the strong version of the Main Conjecture in rank 2 directly by explicitly computing V^Θ for all relevant hsops Θ.

Corollary 2.14. *The strong version of the Main Conjecture holds in rank ≤ 2.*

The intermediate version is verified for the Weyl groups of type B/C in Section 6 and type D in Section 7, using representations of W via signed permutation matrices, and picking the simple hsop $\Theta = (x_1^{h+1}, \ldots, x_n^{h+1})$.

In fact, all the important consequences of the Main Conjecture are even implied by the following weakest version, on the level of $W \times C$-characters, generalizing the W-isomorphism (2.3). This weak form is just shy of being a theorem; it has been verified

[7] See Haiman [29, Corollary 7.4.1] for the counterpart to (iii) for Park_W^{NN}, in the guise of $Q/(h+1)Q$.

for all irreducible real reflection groups except E_7, E_8 (where the computations became too big for our computing power).

Main Conjecture. *(Weak version.) Let W be an irreducible real reflection group and consider the cyclic subgroup $C := \langle c \rangle \leq W$ generated by a Coxeter element $c \in W$. Then one has an isomorphism of $W \times C$-representations*

$$\mathbb{C}\big[\mathsf{Park}_W^{alg}\big] \cong_{\mathbb{C}[W \times C]} \mathbb{C}\big[\mathsf{Park}_W^{NC}\big]. \tag{2.7}$$

This weakest version of the conjecture is approachable via a simple explicit formula for the $W \times C$-character for Park_W^{alg}, explained next. Proposition 2.11 tells us that Park_W^{alg} has the same $W \times C$-character as that of the graded vector space $\mathbb{C}[V]/(\Theta)$. The latter space has c acting by the scalar c^d on the dth homogeneous component $(\mathbb{C}[V]/(\Theta))_d$. Thus taking advantage of the known[8] *graded W-character* on $\mathbb{C}[V]/(\Theta)$

$$\sum_d \chi_{(\mathbb{C}[V]/(\Theta)_d)}(w) q^d = \frac{\det(1 - q^{h+1} w)}{\det(1 - qw)} \tag{2.8}$$

one immediately deduces the following.

Proposition 2.15. *For any element w in an irreducible real reflection group W, and any d in $\mathbb{Z}$, one can evaluate the $W \times C$-character of Park_W^{alg} explicitly as*

$$\chi_{\mathsf{Park}_W^{alg}}\big(w, c^d\big) = \lim_{q \to \omega^d} \frac{\det(1 - q^{h+1} w)}{\det(1 - qw)} = (h+1)^{\mathrm{mult}_w(\omega^d)}$$

where $\mathrm{mult}_w(\omega^d)$ denotes the multiplicity of the eigenvalue ω^d when w acts on the (complexified) reflection representation V.

One can then approach the weak version of the Main Conjecture by comparing this formula to an explicit computation of the $W \times C$-character $\chi_{\mathsf{Park}_W^{NC}}$. Section 8 verifies the weak version of the Main Conjecture in type A by such a combinatorial argument. The authors also thank Christian Stump for writing software in SAGE that verified this character equality for the irreducible exceptional real reflection groups H_3, H_4, F_4, E_6. Types E_7 and E_8 are still unverified.

The status of the various versions of the Main Conjecture is summarized in Table 1, together with the locations of the corresponding proofs.[9] The reader may wonder why we were able to prove the intermediate form of the Main Conjecture in types B, C, D but only the weak form in type A. This is because in types B, C, D there exist simple

[8] Calculated already in [7, Theorem 1.11], [9, §4], [26, §5] using a Koszul resolution for (Θ) and a famous result of Solomon [45] on W-invariant differential forms with polynomial coefficients.

[9] While this paper was under review, the third author was able to prove the intermediate form in type A_{n-1}, as well as the strong form in type A_3. The writeup is in preparation.

Table 1
Status of the Main Conjecture.

Reflection group W	Strongest version of the Main Conjecture proven for W
rank ≤ 2	Strong; Corollary 2.14
type A_{n-1}	Weak; Section 8
type B_n/C_n	Intermediate; Section 6
type D_n	Intermediate; Section 7
type H_3, H_4, F_4, E_6	Weak; computer verification
type E_7, E_8	Open

hsops satisfying the conditions of the intermediate form which make the locus V^Θ easy to work with; indeed, when one lets these groups act on $V = \mathbb{C}^n$ by their standard matrix representations, one can let $(\theta_1, \ldots, \theta_n)$ be given by $\theta_i = x_i^{h+1}$, where x_i is the standard coordinate function. With this choice of hsop, we have that

$$V^\Theta = \{(v_1, \ldots, v_n) \in \mathbb{C}^n : v_i = 0 \text{ or } v_i^h = 1 \text{ for } 1 \leq i \leq n\}.$$

Analyzing the action of W on this set is straightforward. Unfortunately, constructing hsops in type A is substantially more difficult. While explicit hsops have been written down in type A, their complicated forms make it difficult to get a handle on the loci V^Θ. This is a somewhat anomalous case where the theory in type A is more difficult than in the other classical types.

3. Consequences of the Main Conjecture

3.1. *First consequence: the W-action gives $\mathsf{Park}_W^{NN} \cong_W \mathsf{Park}_W^{NC}$*

As mentioned earlier, the W-set isomorphism $\mathsf{Park}_W^{NN} \cong_W \mathsf{Park}_W^{NC}$ in (2.3) was checked case-by-case. However, this would follow immediately from even the weak form of the Main Conjecture by forgetting the C-action, and using only the $\mathbb{C}[W]$-module isomorphism $\mathbb{C}[\mathsf{Park}_W^{NN}] \cong_{\mathbb{C}[W]} \mathbb{C}[\mathsf{Park}_W^{NC}]$, via the following proposition.

Proposition 3.1. *For real reflection groups W and finite W-sets A_1, A_2 whose W-orbits are all of the form W/W_X for varying reflection subgroups W_X, one has $\mathbb{C}[A_1] \cong_{\mathbb{C}[W]} \mathbb{C}[A_2]$ if and only if $A_1 \cong_W A_2$.*

Proof. Only the forward direction is nontrivial, so assume $\mathbb{C}[A_1] \cong_{\mathbb{C}[W]} \mathbb{C}[A_2]$. One must show for each W-conjugacy class of reflection subgroups W_X, or equivalently, for each W-orbit $W.X$ of intersection subspaces X, that the number of W-orbits in A_i having stabilizers conjugate to W_X is the same for $i = 1, 2$. This follows after showing that the characters $\mathrm{Ind}_{W_X}^W \mathbf{1}_{W_X}$ corresponding to different W-orbits $W.X$ are linearly independent, via a triangularity[10] argument: picking for each W-orbit $W.X$ an element w_X

[10] The authors thank A.R. Miller for suggesting this argument, similar to [33, Proof of Theorem 1].

in W whose fixed space is X, the character value $\chi_{\mathrm{Ind}_{W_X}^W \mathbf{1}_{W_X}}(w_Y) \neq 0$ exactly when orbits $W.X$, $W.Y$ have a choice of nested representatives $X \subseteq Y$. $\square$

3.2. Second consequence: the C-action is a cyclic sieving phenomenon

Another consequence of even the weak form of the Main Conjecture comes from comparing the residual C-representations carried by the W-invariant subspaces in the two sides of the $W \times C$-isomorphism (2.7).

The definition of $\mathbb{C}[\mathsf{Park}_W^{NC}]$ shows that its W-invariant subspace carries the usual permutation representation of $C = \langle c \rangle$ acting on the noncrossing flats $NC(W)$, that is, for any noncrossing flat $X = V^w$ with w in $[1, c]_T$, it has image $c^d(X) = V^{c^d w c^{-d}}$, another noncrossing flat. In particular, the trace of c^d in this representation is the number of noncrossing flats X in $NC(W)$ with $c^d(X) = X$.

On the other hand, the C-representation structure on the W-invariant subspace of Park_W^{alg} can be deduced from the fact that c acts via the scalar ω^m on the mth graded component of the graded vector space $(\mathbb{C}[V]/(\Theta))^W$. This means that c^d will act with trace on Park_W^{alg} given by substituting $q = \omega^d$ into the Hilbert series in q for $(\mathbb{C}[V]/(\Theta))^W$, which is known to be the *q-Catalan number for W* (see [7,9,26]):

$$\mathrm{Cat}(W, q) = \prod_{i=1}^{n} \frac{1 - q^{h+d_i}}{1 - q^{d_i}}. \tag{3.1}$$

Thus whenever the weak form the Main Conjecture is true, it re-proves (the *real* reflection group case of) this main result from [9]:

Theorem 3.2. *(See [9, Theorem 1.1].) For irreducible real reflection groups W,*

$$\big(NC(W), \mathrm{Cat}(W, q), C\big)$$

exhibits the ***cyclic sieving phenomenon*** *defined in [39]: the number of elements X in $NC(W)$ with $c^d(X) = X$ is the evaluation of* $\mathrm{Cat}(W, q)$ *at $q = \omega^d$.*

We remark that, modulo the weak form of the Main Conjecture, our proof of Theorem 3.2 is both uniform and follows from a character computation rather than direct enumeration. The proof of [9, Theorem 1.1] had to rely on certain facts that had been checked case-by-case, and used a counting argument.

3.3. Third consequence: Kirkman and Narayana numbers for W

Recall that ignoring the C-action in the weak form of the Main Conjecture gives the $\mathbb{C}[W]$-module isomorphism that follows from (2.3), which has previously only been checked case-by-case in [6, Theorem 6.3]. This $\mathbb{C}[W]$-module isomorphism already has an interesting corollary for the *Narayana and Kirkman polynomials for W*. These can be defined by

$$\begin{aligned}
\mathrm{Nar}_W(t) &:= \sum_{X \in NC(W)} t^{\dim_{\mathbb{C}}(X)} \\
&= \sum_{w \in [1,c]_T} t^{\dim V^w} \\
\mathrm{Kirk}_W(t) &:= \mathrm{Nar}_W(t+1) \\
&:= \sum_A t^{n-|A|} \qquad (3.2)
\end{aligned}$$

where in the last sum, A ranges over all *clusters* in the *cluster complex of finite type* associated to W by Fomin and Zelevinsky [24] in the crystallographic case, or over all subsets A of rays that form the cones of the *Cambrian fan* associated to W by Reading [37] in the real reflection group case. When W is crystallographic, it is also known (see [23, Theorem 5.9]) that $\mathrm{Kirk}_W(t) = \sum_F t^{n-\dim(F)}$ where F ranges over faces in the interior of the dominant region of the Shi arrangement for W, discussed in Section 10 below.

Let V be the geometric representation of W with dimension n. Recall that the exterior powers $\bigwedge^k V$ for $k \in \{0,\dots,n\}$ are irreducible and pairwise inequivalent, with $\bigwedge^0 V$ equal to the trivial representation and $\bigwedge^n V$ equal to the determinant representation [25, Theorem 5.1.4]. In the case of the symmetric group $\mathfrak{S}_m$ the irreducible representation $\bigwedge^k V$ corresponds to the hook-shaped partition $(m-k, 1^k) \vdash m$. Denote by $\mathsf{Park}(W)$ either of the equivalent $\mathbb{C}[W]$-representations $\mathbb{C}[\mathsf{Park}_W^{NC}] \cong_{\mathbb{C}[W]} \mathbb{C}[\mathsf{Park}_W^{alg}]$.

Corollary 3.3. *The Kirkman numbers are the multiplicities of the exterior powers $\bigwedge^k V$ in the irreducible decomposition of the parking space $\mathsf{Park}(W)$. That is, for W irreducible one has*

$$\mathrm{Kirk}_W(t) = \sum_{k=0}^{n} \langle \chi_{\bigwedge^k V}, \chi_{\mathsf{Park}(W)} \rangle_W \cdot t^k.$$

The type A_{n-1} special case of this corollary is essentially an observation of Pak and Postnikov [35, Eq. (1-6)]. Section 9 deduces Corollary 3.3 from the isomorphism (2.3). It also explains how calculations of Gyoja, Nishiyama, and Shimura[11] [28] can be used to give explicit formulas for natural q-analogues of the coefficients of $\mathrm{Kirk}_W(t)$, that we call *q-Kirkman numbers*, in types A, B/C, D.

Section 10 explains how, for crystallographic W, the set of nonnesting parking functions Park_W^{NN} labels in a natural way the regions of the *Shi arrangement* of affine

[11] The authors thank Eric Sommers for pointing them to this reference.

hyperplanes, giving a labeling closely related to one in the type A case defined by Athanasiadis and Linusson [5].

Section 11 closes with some open problems.

4. Proof of Proposition 2.11

Recall the statement here.

Proposition 2.11. *For every irreducible real reflection group W, and for* **any** *choice of Θ satisfying (2.4), one has an isomorphism of $W \times C$-representations*

$$\mathsf{Park}_W^{alg} := \mathbb{C}[V]/(\Theta - \mathbf{x}) \cong_{\mathbb{C}[W\times C]} \mathbb{C}[V]/(\Theta).$$

In fact, this is really a commutative algebra statement having little to do with the $W \times C$-action, as we now explain.

Consider the polynomial ring $S := k[x_1, \ldots, x_n]$ with its standard grading in which each x_i has degree 1, so a monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ has degree $|\alpha| = \sum_i \alpha_i$. For any not necessarily homogeneous polynomial $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ of top degree d, let $\mathrm{in}(f) := \sum_{\alpha:|\alpha|=d} c_\alpha \mathbf{x}^\alpha$ be its initial form with respect to the degree. Given a finite set of polynomials $\{f_1, \ldots, f_\ell\}$ in S, denote by

$$\begin{aligned} J &= (f_1, \ldots, f_\ell) \\ I &= \big(\mathrm{in}(f_1), \ldots, \mathrm{in}(f_\ell)\big) \end{aligned}$$

the ideals J, I in S generated by $\{f_i\}_{i=1}^\ell$, and by their initial forms $\{\mathrm{in}(f_i)\}_{i=1}^\ell$. Thus the quotient S/I inherits the standard grading, but $R = S/J$ generally does not. We wish to compare them via the increasing filtration of k-subspaces

$$\{0\} \subseteq R_{(0)} \subseteq R_{(1)} \subseteq R_{(2)} \subseteq \cdots \subseteq R \tag{4.1}$$

where $R_{(d)}$ is the image under the composite map $S_{\leq d} \hookrightarrow S \twoheadrightarrow S/J$ of the subspace $S_{\leq d} := \bigoplus_{i \leq d} S_i$; here S_d is the dth graded component of S. As $\bigcup_{d=0}^\infty R_{(d)} = R$, one has a k-vector space isomorphism

$$R \cong \mathfrak{gr}(R) := \bigoplus_{d \geq 0} R_{(d)}/R_{(d+1)}. \tag{4.2}$$

Lemma 4.1. *With the above notations, assume that* $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_\ell)$ *form an S-regular sequence. Then the following composite map φ_d is surjective,*

$$S_d \hookrightarrow S_{\leq d} \to R^{(d)} \to R^{(d)}/R^{(d-1)}$$

with $\ker(\varphi_d) = I_d := I \cap S_d$*, and hence induces* k*-vector space isomorphisms*

$$(S/I)_d \cong R^{(d)}/R^{(d-1)},$$
$$S/I \cong \mathfrak{gr}(R).$$

Assuming Lemma 4.1 for the moment, we explain how it implies Proposition 2.11. Combining the last assertion of the lemma with (4.2) gives a k-vector space isomorphism $S/I \to R$. Furthermore, note that if the ideals I, J happen to both be G-stable for a finite group G of grade-preserving automorphisms of S, then all of the maps involved will be G-equivariant. This holds in the set-up of Proposition 2.11, where $S = \mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$, with $G = W \times C$, acting as before, taking

$$f_i = \theta_i - x_i, \quad \text{so that} \quad R = \mathbb{C}[V]/(\theta - \mathbf{x}),$$
$$\operatorname{in}(f_i) = \theta_i, \quad \text{so that} \quad S/I = \mathbb{C}[V]/(\theta).$$

Lastly, note that $\theta_1, \dots, \theta_\ell$ form an S-regular sequence because they are a system of parameters and S is a Cohen–Macaulay ring. Hence the hypotheses of Lemma 4.1 are satisfied, and this would complete the proof of Proposition 2.11.

Proof of Lemma 4.1. The map $S_d \xrightarrow{\varphi_d} R_{(d)}/R_{(d-1)}$ surjects since $S_{\le d} = S_d \oplus S_{\le d-1}$ and $S_{\le d-1}$ is annihilated by the composite surjection $S_{\le d} \to R_{(d)} \to R_{(d)}/R_{(d-1)}$.

Any f in I_d will lie in the kernel of φ_d, as there exist homogeneous h_i expressing

$$f = \sum_{i=1}^{\ell} h_i \operatorname{in}(f_i) = \sum_{i=1}^{\ell} h_i f_i - \sum_{i=1}^{\ell} h_i \big(f_i - \operatorname{in}(f_i)\big).$$

The first sum is in $\ker(\varphi_d)$ as it is in J, so it maps to zero in $R = S/J$ and hence maps to zero in $R_{(d)}$. The second sum lies in $S_{\le d-1} \subset \ker(\varphi_d)$. Hence $I_d \subset \ker(\varphi_d)$.

To show $\ker(\varphi_d) = I_d$, it remains to show $S_d \cap (S_{\le d-1} + J) \subseteq I_d$. Assume f lies in $S_d \cap (S_{\le d-1} + J)$, so $\deg(f) = d$, and one can find an expression

$$f = g + \sum_{i=1}^{\ell} g_i f_i \tag{4.3}$$

with g in $S_{\le d-1}$, and g_i in S. Choose this expression in such a way that the quantity $d_0 := \max\{\deg(g_i f_i)\}_{i=1}^{\ell}$ is *minimized*.

Note that $\deg(f) = d$ forces $d_0 \ge d$. If $d_0 = d$, then taking the degree d_0 component in (4.3) gives the expression $f = \sum_{i=1}^{\ell} \operatorname{in}(g_i) \operatorname{in}(f_i)$, showing that f lies in I_d, and we are done.

On the other hand, if $d_0 > d$, then taking the degree d_0 component in (4.3) gives $0 = \sum_{i=1}^{\ell} \operatorname{in}(g_i) \operatorname{in}(f_i)$. In other words, when one creates the free graded S-module S^ℓ having ordered S-basis elements $(e_1, \ldots, e_\ell)$ with $\deg(e_i) := \deg(f_i)$, the vector $\sum_{i=1}^{\ell} \mathrm{in}(g_i)e_i$ is in the kernel of the S-module map defined S-linearly by

$$\begin{aligned} S^\ell &\to S \\ e_i &\mapsto \mathrm{in}(f_i) \end{aligned}$$

Because the $\{\mathrm{in}(f_i)\}_{i=1}^{\ell}$ are a regular sequence, this vector must be an S-linear combination of *Koszul syzygies*,[12] that is, one can write

$$\sum_{i=1}^{\ell} \mathrm{in}(g_i)e_i = \sum_{1\le j\le k\le \ell} \gamma_{jk}\big(\mathrm{in}(f_k)e_j - \mathrm{in}(f_j)e_k\big) \tag{4.4}$$

for some homogeneous γ_{jk} in S that satisfy

$$\deg(g_j) + \deg(f_j) = \deg(\gamma_{jk}) + \deg(f_j) + \deg(f_k) \tag{4.5}$$

Now apply to (4.4) the different S-module map $S^\ell \to S$ that sends $e_i \mapsto f_i$, giving

$$\sum_{i=1}^{\ell} \mathrm{in}(g_i)f_i = \sum_{1\le j<k\le \ell} \gamma_{jk}\big(\mathrm{in}(f_k)f_j - \mathrm{in}(f_j)f_k\big).$$

This allows one to rewrite (4.3) as follows:

$$\begin{aligned} f &= g + \sum_{i=1}^{\ell} g_i f_i \\ &= g + \sum_{i=1}^{\ell} \mathrm{in}(g_i)f_i && + \sum_{i=1}^{\ell}\big(g_i - \mathrm{in}(g_i)\big)f_i \\ &= g + \sum_{1\le j<k\le \ell} \gamma_{jk}\big(\mathrm{in}(f_k)f_j - \mathrm{in}(f_j)f_k\big) && + \sum_{i=1}^{\ell}\big(g_i - \mathrm{in}(g_i)\big)f_i \\ &= g + \sum_{1\le j<k\le \ell} \gamma_{jk}\big(\big(\mathrm{in}(f_k) - f_k\big)f_j - \big(\mathrm{in}(f_j) - f_j\big)f_k\big) && + \sum_{i=1}^{\ell}\big(g_i - \mathrm{in}(g_i)\big)f_i \end{aligned}$$

in which the last line added $0 = (-f_k)f_j - (-f_j)f_k$ for $1 \le j < k \le \ell$ to the previous line. Using (4.5) and the fact that $\deg(\mathrm{in}(f_i) - f_i) < \deg(f_i)$, one finds that this last expression for f contradicts the minimality of d_0 in (4.3). $\square$

[12] We only use this small part of the regular sequence hypothesis, that first syzygies are Koszul.

5. Proof of Proposition 2.13

Recall the statement here.

Proposition 2.13. *For W an irreducible real reflection group of rank n, and for all choices of Θ as in (2.4), one has that*

(i) *the set $V^\Theta(0) = \{0\}$ of 0-dimensional points in V^Θ is the unique $W \times C$-orbit in V^Θ carrying the trivial $W \times C$-representation,*
(ii) *there exists a $W \times C$-equivariant injection $V^\Theta(1) \hookrightarrow \mathsf{Park}_W^{NC}$ whose image is precisely the set of noncrossing parking functions of the form $[w, X]$ with $\dim(X) = 1$, and*
(iii) *the set $V^\Theta(n)$ of n-dimensional points is the unique W-regular orbit of points in V^Θ.*

Furthermore, every point in the subsets $V^\Theta(0)$, $V^\Theta(1)$, and $V^\Theta(n)$ of V^Θ is cut out by the ideal $(\Theta - \mathbf{x})$ in a reduced fashion, that is, with multiplicity one.

The proof of the proposition will show that the injection $V^\Theta(1) \hookrightarrow \mathsf{Park}_W^{NC}$ in part (ii) can be made more explicit as follows. Lemma 5.2 below will show that one can choose $W \times C$ orbit representatives $\{\beta_i\}_{i=1}^r$ in $V^\Theta(1)$ in such a way that for each $i = 1, 2, \ldots, r$, the unique smallest subspace X_i in $\mathcal{L}$ containing β_i is a noncrossing line. The injection $V^\Theta(1) \hookrightarrow \mathsf{Park}_W^{NC}$ is then determined by sending β_i to $[1, X_i]$.

The proofs of the three parts (i), (ii), (iii) have differing levels of difficulty and different flavors.

5.1. Proof of Proposition 2.13(i)

Note that the origin 0 is contained in V^Θ, so that $V^\Theta(0) = \{0\}$. Also, since W acts irreducibly on V, the origin is the *only* W-fixed point of V, or of V^Θ. For the multiplicity one assertion, note that the Jacobian matrix J for $(\Theta - \mathbf{x}) = (\theta_1 - x_1, \ldots, \theta_n - x_n)$ looks like

$$J = \left[\frac{\partial(\theta_i - x_i)}{\partial x_j}\right]_{\substack{i=1,2,\ldots,n\\ j=1,2,\ldots,n}} = \left[\frac{\partial \theta_i}{\partial x_j}\right] - I_{n\times n}$$

and will be nonsingular when evaluated $\mathbf{x} = 0$: each θ_i has degree $h+1 \geq 2$, so the above expression shows $J|_{\mathbf{x}=0} = -I_{n\times n}$.

This completes the proof of Proposition 2.13(i).

5.2. Proof of Proposition 2.13(iii)

Observe that a point in V^Θ is in $V^\Theta(n)$ if and only if it is W-regular. Therefore, it suffices to show that there exists a unique W-regular orbit of points in V^Θ and that the points in this orbit have multiplicity one.

We first exhibit a W-regular orbit of points inside V^Θ. It is known that any Coxeter element c in W has a W-regular eigenvector v in V, spanning a simple ω-eigenspace for c, where $\omega = e^{\frac{2\pi i}{h}}$, as usual; see, e.g., Humphreys [30, §3.19]. Starting with the eigenvector equation for v, and applying the W-equivariant map $\Theta : V \mapsto V$ discussed in Section 2.6, one obtains the following:

$$\begin{aligned} c(v) &= \omega v \\ \Theta\big(c(v)\big) &= \Theta(\omega v) \\ c\Theta(v) &= \omega^{h+1}\Theta(v) = \omega\Theta(v) \end{aligned}$$

where the third equality used the fact that Θ is homogeneous of degree $h+1$. Due to the simplicity of the ω-eigenspace for c, one concludes that $\Theta(v) = \lambda v$ for some λ in $\mathbb{C}$. Furthermore, $\lambda \neq 0$, else the hsop (Θ) would vanish on the line spanned by the nonzero vector v. This allows one to rescale v to $v_0 := \lambda^{-\frac{1}{h}} v$, giving another W-regular ω-eigenvector for c, but which now lies in V^Θ:

$$\begin{aligned} \Theta(v_0) &= \Theta\big(\lambda^{-\frac{1}{h}} v\big) \\ &= \lambda^{-\frac{h+1}{h}}\Theta(v) \\ &= \lambda^{-\frac{h+1}{h}}\lambda v = \lambda^{-\frac{1}{h}} v = v_0. \end{aligned}$$

Thus the entire regular W-orbit of v_0 lies inside the W-stable subvariety V^Θ, as desired. As $c(v_0) = \omega v_0$, one also knows that v_0 has $W \times C$-stabilizer $\langle (c,c)\rangle$.

To see v_0 and its W-images are the *only* W-regular points in V^Θ, and cut out by the ideal $(\Theta - \mathbf{x})$ with multiplicity one, we rely on the following, proven below.

Lemma 5.1. *For an irreducible real reflection group W, there is exactly one copy of the W-alternating character* det *in the W-representation $\mathbb{C}[V]/(\Theta)$.*

By Proposition 2.11, this implies det occurs only once in $\mathbb{C}[V]/(\Theta - \mathbf{x})$. The idea is to use the copy of det inside the regular representation of W to exhibit multiple copies of det in the $\mathbb{C}[V]/(\Theta - \mathbf{x})$ if there were more than one W-regular orbit in V^Θ or if there were a W-regular orbit with points of multiplicity at least two.

To make this rigorous, consider the *primary decomposition*

$$(\Theta - \mathbf{x}) = \bigcap_{v \in V^\Theta} I_v$$

where I_v is the $\mathfrak{m}_v$-primary component of $(\Theta - \mathbf{x})$ for the maximal ideal

$$\mathfrak{m}_v = \big(x_1 - x_1(v), \ldots, x_n - x_n(v)\big)$$

of $\mathbb{C}[V]$ corresponding to v. Because $\mathbb{C}[V]/(\Theta - \mathbf{x})$ is Artinian and Noetherian, the Chinese Remainder Theorem [17, §2.4] gives a ring isomorphism

$$\mathbb{C}[V]/(\Theta - \mathbf{x}) \xrightarrow{\varphi} \bigoplus_{v \in V^\Theta} \mathbb{C}[V]/I_v := A. \tag{5.1}$$

By definition, $(\Theta - \mathbf{x})$ cuts out the point v in V^Θ with multiplicity $\dim_{\mathbb{C}} \mathbb{C}[V]/I_v$.

Note that the isomorphism φ also respects W-actions, if one lets $w \in W$ map the summands of A via isomorphisms $w : \mathbb{C}[V]/I_v \to \mathbb{C}[V]/I_{w(v)}$, induced from the action of w on $\mathbb{C}[V]$ (sending I_v isomorphically to $I_{w(v)}$). As φ is a W-isomorphism, the W-alternating (det-isotypic) component $A^{W,\det}$ is 1-dimensional.

Now given any W-regular vector v_0 in V^Θ, any nonzero element f_{v_0} in $\mathbb{C}[V]/I_{v_0}$ can be W-antisymmetrized to give a (nonzero) element

$$\delta_{v_0} := \sum_{w \in W} \det(w) w(f_{v_0})$$

lying in the 1-dimensional space $A^{W,\det}$.

If v_0, v_0' are W-regular vectors lying in *different* W-orbits, then these two elements δ_{v_0}, $\delta_{v_0'}$ would be $\mathbb{C}$-linearly independent in $A^{W,\det}$, because they are supported in different summands of A; a contradiction.

Similarly if the multiplicity $\dim_{\mathbb{C}} \mathbb{C}[V]/I_{v_0} \geq 2$, then one could pick f_{v_0}, f_{v_0}' which are $\mathbb{C}$-linearly independent in $\mathbb{C}[V]/I_{v_0}$, and obtain δ_{v_0}, δ_{v_0}' that are $\mathbb{C}$-linearly independent in $A^{W,\det}$; again, a contradiction.

This completes the proof of Proposition 2.13(iii), except for the proof of Lemma 5.1.

Proof of Lemma 5.1. Note that (2.8) or Proposition 2.15 implies that $\mathbb{C}[V]/(\Theta)$ has W-character $\chi(w) = (h+1)^{\dim V^w}$. Thus the multiplicity of det in $\mathbb{C}[V]/(\Theta)$ is

$$\langle \det, \chi \rangle_W = \frac{1}{|W|} \sum_{w \in W} \det(w)(h+1)^{\dim V^w}. \tag{5.2}$$

On the other hand, well-known results of Orlik and Solomon [34, §1, §4] show that for complex reflection groups,

$$\sum_{w \in W} \det(w) t^{\dim V^w} = \prod_{i=1}^{n} (t - e_i) \tag{5.3}$$

where $e_1, \ldots, e_n$ are the *exponents* of W. Combining (5.2) and (5.3) gives

$$\langle \det, \chi \rangle_W = \frac{1}{|W|} \prod_{i=1}^{n} (h + 1 - e_i) = \frac{1}{|W|} \prod_{i=1}^{n} (e_i + 1) = 1$$

where the second equality uses the *exponent duality* $e_i + e_{n-1-i} = h$ known for real reflection groups [34, §5], and the last equality uses the Shephard and Todd formula $|W| = \prod_{i=1}^{n} (e_i + 1)$; cf. the proof of [29, Corollary 7.4.1]. $\square$

5.3. Some noncrossing geometry

Before delving into the proof of Proposition 2.13(ii) in the next subsection, we collect here three results (Lemmas 5.2, 5.3, 5.4 below) on noncrossing subspaces, particularly noncrossing lines, needed in the proof, and of possible interest in their own right.

Lemma 5.2. *Every flat in $\mathcal{L}$ is a W-translate of a noncrossing flat, so the map*

$$\begin{aligned} C\backslash NC(W) &\xrightarrow{\phi} W\backslash\mathcal{L} \\ C.X &\mapsto W.X \end{aligned}$$

is a well-defined surjection.

Proof. Let X in $\mathcal{L}$ be a flat and let $W_X = \{w \in W : wx = x \text{ for all } x \in X\}$ be the isotropy subgroup of W corresponding to X. Then W_X is a parabolic subgroup of W; i.e. there exists $w \in W$ and $I \subseteq S$ such that $W_X = wW_Iw^{-1}$. It follows that $X = V^{W_I}$. Then by Carter [14, Lemma 2] one has that $V^{W_I} = V^x$, where x is the product of the generators I taken in any order. Note that this $x \in W$ is noncrossing with respect to the Coxeter element $c = xy$, where y is the product of the generators $S - I$ in any order. Since all Coxeter elements are conjugate in W, we are done. □

One can be much more precise about the map ϕ in Lemma 5.2 when one restricts to 1-dimensional flats X, that is, when X is a line. For the remainder of this section, a *line* will always mean a 1-dimensional flat in the intersection lattice $\mathcal{L}$ for W.

Lemma 5.3. *Let X in $\mathcal{L}$ be a noncrossing line. Then the set of noncrossing lines in the W-orbit of X is precisely the C-orbit of X, or in other words, the surjection ϕ from Lemma 5.2 restricts to a bijection*

$$C\backslash\{\textit{noncrossing lines}\} \xrightarrow{\phi} W\backslash\{\textit{lines}\}.$$

Proof. The map ϕ occurs as the top horizontal map in the following larger diagram of maps, explained below, involving the orbits of various groups acting on the sets of lines, noncrossing lines, half-lines, reflections T, roots Φ, and simple roots Π:

$$\begin{tikzcd} C\backslash\{\text{noncrossing lines}\} \arrow[r, "\phi"] & W\backslash\{\text{lines}\} \\ C\backslash T \arrow[u, "f_1"] & \langle W, -1\rangle\backslash\{\text{half-lines}\} \arrow[u, "g_1"'] \\ \langle C, -1\rangle\backslash\Phi \arrow[u, "f_2"] & \langle -w_0\rangle\backslash\Pi \arrow[u, "g_2"'] \arrow[l, "\psi"'] \end{tikzcd} \tag{5.4}$$

We have seen in Lemma 5.2 that ϕ is well-defined and surjective. Our strategy will be to show that

- each of the vertical maps f_1, f_2, g_1, g_2 is bijective, and
- the other horizontal map ψ is surjective.

Cardinality considerations will then imply that both ϕ, ψ are also bijective.

The bijection f_1. The map f_1 is induced by the map from the set of *reflections* T in W to the set of noncrossing lines

$$\begin{aligned} T &\to \{\text{noncrossing lines}\} \\ t &\mapsto V^{ct} \end{aligned}$$

which is bijective by work of Bessis [8, §2.4], and equivariant for the action of C via conjugation on T and via translation on noncrossing lines:

$$c^d tc^{-d} \mapsto V^{c\cdot c^d tc^{-d}} = V^{c^d \cdot ct\cdot c^{-d}} = c^d\bigl(V^{ct}\bigr).$$

The bijection f_2. The map f_2 is induced from map $\Phi \to T$ that sends a root α to the reflection t_α through the hyperplane perpendicular to α. The latter map is two-to-one, with fibers $\{\pm\alpha\}$, inducing a bijection $\langle -1\rangle\backslash\Phi \to T$. It is also equivariant for the W-action as well as the C-action by restriction:

$$c(\pm\alpha) \mapsto t_{\pm c(\alpha)} = ct_\alpha c^{-1}.$$

Hence f_2 is a bijection.

The bijection g_1. A *half-line* is either of the two rays emanating from the origin inside any of the lines for W. There is a two-to-one map $\{\text{half-lines}\} \to \{\text{lines}\}$ that sends a half-line to the line that it spans, inducing the bijection g_1.

The bijection g_2. Normalize the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$, to have unit length, and denote by $\{\delta_1, \dots, \delta_n\}$ their dual basis, defined by

$$(\alpha_i, \delta_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.5}$$

Then the map $\Pi \to \{\text{half-lines}\}$ sending $\alpha_i \mapsto \mathbb{R}_+\delta_i$ gives a well-defined map

$$\langle -w_0\rangle\backslash\Pi \xrightarrow{g_2} \langle W, -1\rangle\backslash\{\text{half-lines}\}$$

where w_0 is the unique *longest element* of W, because $-w_0$ lies in $\langle W, -1\rangle$. Note that $-w_0$ always takes a simple root α_i to another simple root α_j, corresponding to the fact that conjugation by w_0 sends the simple reflection s_i to another simple reflection $w_0 s_i w_0 = s_j$, via a *diagram automorphism*[13]; see [10, §2.3].

The usual geometry of *Tits cones* and *Coxeter complexes* [30, §1.15, 5.13] says

- each half-line is in the W-orbit of *exactly one* half-line $\mathbb{R}_+\delta_1, \dots, \mathbb{R}_+\delta_n$, and
- these are the only half-lines lying within the *dominant cone*

$$F := \{x \in V : (\alpha_i, x) \geq 0 \text{ for } i = 1, 2, \dots, n\} = \mathbb{R}_+\delta_1 + \cdots + \mathbb{R}_+\delta_n$$

In particular, this shows that g_2 is surjective. It remains to show that g_2 is also injective, that is, assuming that $\mathbb{R}_+\delta_j$ lies in $\langle W, -1\rangle.\mathbb{R}_+\delta_i$, we must show that this forces $\alpha_j \in \langle -w_0\rangle.\alpha_i$. If $\mathbb{R}_+\delta_j = w(\mathbb{R}_+\delta_i)$ for some w in W, then the above-mentioned Tits cone geometry implies $i = j$ and hence $\alpha_j = \alpha_i$. So assume without loss of generality that $\mathbb{R}_+\delta_j = -w(\mathbb{R}_+\delta_i)$ for some $w \in W$. Note that both -1 and w_0 send the dominant cone F to its negative $-F$, and therefore $-w_0(F) = F$. Hence $-w_0(\mathbb{R}_+\delta_j) = w_0 w(\mathbb{R}_+\delta_i)$ lies in F. The above-mentioned Tits cone geometry forces $-w_0(\mathbb{R}_+\delta_j) = \mathbb{R}_+\delta_i$. Since $-w_0$ is an isometry, this forces $-w_0\delta_j = \delta_i$, and also $-w_0\alpha_j = \alpha_i$, as desired.

The surjection ψ. We will show that the inclusion $\Pi \hookrightarrow \Phi$ induces a well-defined surjection $\langle -w_0\rangle \backslash \Pi \xrightarrow{\psi} \langle C, -1\rangle \backslash \Phi$, whenever the Coxeter element c is chosen to be *bipartite*, that is, one has properly 2-colored the Coxeter diagram, giving a decomposition of the simple reflections into two mutually commuting sets

$$S = \{s_1, s_2, \dots, s_m\} \sqcup \{s_{m+1}, s_{m+2}, \dots, s_n\}$$

so that

$$c = c_+ c_- = \underbrace{s_1 s_2 \dots s_m}_{c_+} \cdot \underbrace{s_{m+1} s_{m+2} \dots s_n}_{c_-}. \tag{5.6}$$

It is well-known (see [11, Chapter V, §6, Exercise 2], [30, §3.19, Exercise 2]) that in this setting, the longest element w_0 can be expressed as

$$w_0 = \begin{cases} c^{\frac{h}{2}} & \text{if } h \text{ is even,} \\ c^{\frac{h-1}{2}} c_+ & \text{if } h \text{ is odd.} \end{cases}$$

We first show that ψ is well defined, that is, for any α_i in Π, the root $-w_0(\alpha_i)$ lies in $\langle C, -1\rangle.\alpha_i$. If h is even, then $w_0 = c^{\frac{h}{2}}$ and this is obvious, so assume without loss of generality that h is odd. Then $-w_0(\alpha_i) = -c^{\frac{h-1}{2}} c_+(\alpha_i)$. If the subscript i in α_i lies

[13] Although not needed in the sequel, we remark that the diagram (5.4) will also show the following: both sets of orbits $C\backslash\{\text{noncrossing lines}\}$ and $W\backslash\{\text{lines}\}$ biject with the orbits of the simple reflections S under the Coxeter diagram automorphism $s \mapsto w_0 s w_0$. Thus their cardinality is at most $n = |S|$, with equality if and only if -1 lies in W.

in $\{1, 2, \ldots, m\}$, so that s_i appears in c_+, then $c_+(\alpha_i) = -\alpha_i$, and hence $-w_0(\alpha_i) = c^{\frac{h-1}{2}}(\alpha_i)$ lies in $\langle C, -1\rangle.\alpha_i$. On the other hand, if the subscript i lies in $\{m+1, m+2, \ldots, n\}$ so that s_i appears in c_-, then $c_-(\alpha_i) = -\alpha_i$ and

$$-w_0(\alpha_i) = -c^{\frac{h-1}{2}}c_+(\alpha_i) = c^{\frac{h-1}{2}}c_+c_-(\alpha_i) = c^{\frac{h+1}{2}}(\alpha_i)$$

which lies in $\langle C, -1\rangle.\alpha_i$. This proves that ψ is well defined.

To show that ψ is surjective, it suffices to show that every root in Φ is C-conjugate to plus or minus a simple root. Starting with the factorization $c = c_1c_2 \ldots c_n$ it is known that one can define a sequence of roots

$$\theta_i := s_n s_{n-1} \cdots s_{i+1}(\alpha_i)$$

for $i = 1, 2, \ldots, n$, giving a complete set of representatives for the C-orbits on Φ; see [11, Chapter VI, §1, No. 11, Proposition 33]. Thus it suffices to check that for each $i = 1, 2, \ldots, n$, one has θ_i lying in $\pm C.\alpha_i$.

If i lies in $\{m+1, m+2, \ldots, n\}$, so s_i appears in c_-, then $s_n, s_{n-1}, \ldots, s_{i+1}$ all fix α_i. Therefore $\theta_i = s_n s_{n-1} \cdots s_{i+1}\alpha_i = \alpha_i$ and we are done. If i lies in $\{1, 2, \ldots, m\}$, so s_i appears in c_+, then $s_1, s_2, \ldots, s_{i-1}$ all fix α_i. Therefore

$$c(\theta_i) = s_1s_2 \cdots s_n \cdot s_n s_{n-1} \cdots s_{i+1}(\alpha_i) = s_1s_2 \cdots s_i(\alpha_i) = -s_1s_2 \cdots s_{i-1}(\alpha_i) = -\alpha_i.$$

Hence $\theta_i = -c^{-1}(\alpha)$ lies in $-C.\alpha_i$, as desired. $\square$

We remark that Lemma 5.3 was originally verified case-by-case along these lines:

- it is trivial to check it in the dihedral types $I_2(m)$,
- it can be checked by computer in the exceptional types, and
- it is easy to argue directly in the classical types A, B/C, D.

For example, in type A_{n-1}, the W-orbits of noncrossing lines correspond to set partitions with precisely two blocks of fixed sizes k and $n-k$. The set of noncrossing partitions with block sizes k and $n-k$ forms a single C-orbit. The analysis in the other classical types B/C and D is similar.

The last lemma needed for the proof of Proposition 2.13(ii) pertains to the *Coxeter plane* $P \subset V$ corresponding to a bipartite Coxeter element $c = c_+c_-$. This P is a 2-dimensional subspace of V, acted on by the dihedral group $\langle c_+, c_-\rangle$. The subgroup of this dihedral group generated by c acts on P by h-fold rotation; see [30, Chapter 3].

Lemma 5.4. *A noncrossing line cannot be orthogonal to the Coxeter plane P.*

Proof. Suppose that X in $\mathcal{L}$ is a noncrossing line, spanned over $\mathbb{R}$ by a vector v, and that v is perpendicular to P. Since X is noncrossing, there exists a reflection t in T such that $X = V^{tc}$. Let H be the reflecting hyperplane for t.

One has that $t(v) = t(tc(v)) = c(v)$, and therefore $v \neq t(v)$ (else $V^c \neq \{0\}$). Moreover, as $c(P) = P$ and c is an orthogonal transformation, the vector $t(v) = c(v)$ is also perpendicular to P. It follows that the difference $v - t(v)$ is perpendicular to P and nonzero. The vector $v - t(v)$ is perpendicular to H, which shows that $P \subseteq H$. But by the discussion in [30, Section 3.18], no reflecting hyperplane contains the Coxeter plane P, so this is a contradiction. $\square$

5.4. Proof of Proposition 2.13(ii)

We will assume, without loss of generality, that our Coxeter element c is bipartite. To prove (ii), we begin by analyzing the set $V^{\Theta}(1)$. Consider the map $\Theta : V \to V$ discussed in Section 2.6. Because any flat $X \in \mathcal{L}$ has the form $X = V^w$ for some $w \in W$ and Θ is W-equivariant, one has that $\Theta(X) \subseteq X$ for all $X \in \mathcal{L}$ and the restriction $\Theta|_X$ is a well-defined map $X \to X$ given by homogeneous polynomials of degree $h + 1$.

If the flat X is 1-dimensional, choosing a coordinate x for X, so that its coordinate ring is $\mathbb{C}[x]$, one can express $(\Theta - \mathbf{x})|_X = \alpha x^{h+1} - x$ for some $\alpha \in \mathbb{C}^{\times}$. Its zero set X^{Θ} consists of the $h + 1$ points

$$\{0\} \sqcup \underbrace{\{\alpha^{-\frac{1}{h}}, \omega\alpha^{-\frac{1}{h}}, \ldots, \omega^{h-1}\alpha^{-\frac{1}{h}}\}}_{V^{\Theta}(1)\cap X},$$

where ω is a primitive hth root of unity, and each point occurs with multiplicity one. Recall that the cyclic group C acts via scaling by ω, and so its action on these $h+1$ points decomposes into the two orbits $\{0\}$, $V^{\Theta}(1) \cap X$ shown above. Since X is 1-dimensional and W is a real reflection group, it follows that any element w in W which stabilizes X must act as 1 or -1 on X. Since C acts by scaling by ω, the $W \times C$-stabilizer of the typical point $\alpha^{-\frac{1}{h}}$ in the orbit $V^{\Theta}(1) \cap X$ equals

$$\left\{ (w, c^d) : \begin{array}{c} w \text{ acts as } 1 \text{ on } X \text{ and } c^d = 1, \text{ or} \\ w \text{ acts as } -1 \text{ on } X \text{ and } d \equiv \frac{h}{2} \bmod h \end{array} \right\}. \tag{5.7}$$

Suppose in addition that the flat X is noncrossing. It remains to show that the $W \times C$-stabilizer of the noncrossing parking function $[1, X] \in \mathsf{Park}_W^{NC}$ equals the same subgroup as in (5.7). It follows directly from the definition of Park_W^{NC} that this stabilizer equals

$$\{(w, c^d) : c^d \text{ stabilizes } X \text{ and } c^d w^{-1} \text{ acts as } 1 \text{ on } X\}.$$

Let $\pi : V \twoheadrightarrow P$ be orthogonal projection onto the Coxeter plane. If c^d stabilizes X for some $d \geq 0$, since c^d acts orthogonally, one has that c^d stabilizes the subspace $\pi(X)$ of P, as well, and acts by the same scalar on both X and $\pi(X)$. By Lemma 5.4, this subspace $\pi(X)$ is a line inside P, that is, $\pi(X) \neq 0$. Since c acts on P by h-fold rotation, this forces either

- $d \equiv 0 \bmod h$, so that c^d fixes the line $\pi(X)$ (and X) pointwise, or
- $d \equiv \frac{h}{2} \bmod h$, so that c^d acts as -1 on the line $\pi(X)$ (and X).

It follows that the $W \times C$-stabilizer of $[1, X] \in \mathsf{Park}_W^{NC}$ is as described in (5.7).

Let $\beta_1, \ldots, \beta_r$ be a complete set of representatives for the $W \times C$-orbits in $V^\Theta(1)$. By Lemma 5.2, without loss of generality we can assume that $\beta_i \in X_i$ for all i, where $X_1, \ldots, X_r \in \mathcal{L}$ are noncrossing and are a complete set of representatives for the W-orbits of lines in $\mathcal{L}$. Since the $W \times C$-stabilizers of β_i and $[1, X_i]$ are equal for all i, we get a well defined $W \times C$-equivariant map $V^\Theta(1) \to \mathsf{Park}_W^{NC}$ induced by $\beta_i \mapsto [1, X_i]$.

We claim that the map $V^\Theta(1) \to \mathsf{Park}_W^{NC}$ described above is a bijection onto $\mathsf{Park}_W^{NC}(1) := \{[w, X] \in \mathsf{Park}_W^{NC} : \dim(X) = 1\}$. By stabilizer equality, this map restricts to a bijection from the $W \times C$-orbit of β_i to the $W \times C$-orbit of $[1, X_i]$ for all i. Moreover, the image of the $W \times C$-orbit of β_i is $\{[w, c^d X_i] : w \in W,\ c^d \in C\}$. Since the $W \times C$-orbit of β_i inside $V^\Theta(1)$ is $V^\Theta(1) \cap \{wX_i : w \in W\}$ and the W-orbit of X_i contains the C-orbit of X_i, the sets $\{[w, c^d X_i] : w \in W,\ c^d \in C\}$ are disjoint for distinct flats X_i and the map $V^\Theta(1) \to \mathsf{Park}_W^{NC}(1)$ is injective. Lemma 5.3 shows that this map is also surjective.

This completes the proof of Proposition 2.13(ii).

6. Type B/C

Before proceeding with the proof, we review from [38] how to visualize the elements of $NC(W)$ for W of type B/C and extend this to similarly visualize the elements of Park_W^{NC}, analogous to the description for type A given in Example 2.7.

6.1. Visualizing type B/C

The Weyl group W of type B_n or C_n is the *hyperoctahedral group* $\mathfrak{S}_n^{\pm}$ of $n \times n$ signed permutation matrices, that is, matrices having one nonzero entry equal to ± 1 in each row and column. These signed permutation matrices act on $V = \mathbb{C}^n$, but we will also often think of w in W as a permutation of the set

$$\pm[n] := \{1, 2, \ldots, n, -1, -2, \ldots, -n\}$$

satisfying $w(-i) = -w(i)$. The reflections $T \subseteq W = \mathfrak{S}_n^{\pm}$ are tabulated here:

Reflection	Reflecting hyperplane
$t_{ij}^{+} = (i, j)(-i, -j)$	$x_i = +x_j$
$t_{ij}^{-} = (i, -j)(-i, j)$	$x_i = -x_j$
$t_{ii}^{-} = (+i, -i)$	$x_i = 0 (= -x_i)$

A typical intersection flat X has several blocks of coordinates set equal to each other or their negatives, and possibly one block (called the *zero block*, if present) where the coordinates are all zero. This corresponds to a set partition $\pi = \{B_1, \ldots, B_\ell\}$ of the set $\pm[n]$ with $\pi = -\pi$, in the sense that for each block B_i there exists some block $B_j = -B_i$, and at most one zero block $B_k = -B_k$. For example, when $n = 9$, the flat X defined by the equations

$$x_1 = -x_6 = -x_9$$
$$x_2 = x_5 = 0$$
$$x_3 = x_4$$
$$x_7 = x_8$$

has zero coordinates x_2, x_5, corresponding to the zero block $\{\pm 2, \pm 5\}$ in the associated set partition:

$$\pi = \big\{\{+1, -6, -9\}, \{-1, +6, +9\}, \{\pm 2, \pm 5\}, \\ \{+3, +4\}, \{-3, -4\}, \{+7, +8\}, \{-7, -8\}\big\}. \tag{6.1}$$

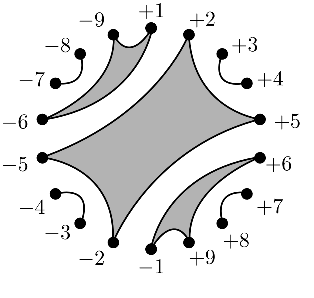
One can take as Coxeter generators for W the set $S = \{s_0, s_1, ..., s_{n-1}\}$, where

$$s_i = t^+_{i,i+1} = (i, i+1) \quad \text{for } i = 1, 2, \ldots, n-1, \quad \text{and}$$
$$s_0 = t^-_{ii} = (-1, +1).$$

The Coxeter element $c = s_0 s_1 \cdots s_{n-1}$ acts as the $2n$-cycle

$$c = (+1, +2, \ldots, +n, -1, -2, \ldots, -n)$$

on $\pm[n]$, and hence has order $h = 2n$. Drawing these values in the $2n$-cycle clockwise around a circle, one can depict set partitions of $\pm[n]$ as before, by drawing the polygonal convex hulls of their blocks. The flats X in $\mathcal{L}$ have the extra property that they are *centrally symmetric*, and at most one block is sent to itself by the central symmetry. Furthermore a flat X in $\mathcal{L}$ lies in $NC(W)$ if and only if its blocks are noncrossing. For example, the element X or partition π of $\pm[9]$ from (6.1) is noncrossing, and is depicted here:



As in type A, such a noncrossing partition π or flat X bijects with the signed permutation w whose associated permutation of $\pm[n]$ is obtained by orienting the edges of each block B_i counterclockwise. One can check that the conjugation action of $C = \langle c \rangle$ on $NC(W) = [1, c]_T$ again corresponds to clockwise rotation through $\frac{2\pi}{h} = \frac{2\pi}{2n}$ in the picture.

Also as in type A (see Example 2.7), when $W = \mathfrak{S}_n^{\pm}$ is of type B_n/C_n, an element $[w, X]$ of Park_W^{NC} is an equivalence class where X is a noncrossing flat, corresponding to some noncrossing partition π of $\pm[n]$, say with nonzero blocks $\{B_1, -B_1, \ldots, B_\ell, -B_\ell\}$, and w is a signed permutation in $\mathfrak{S}_n^{\pm}$ considered only up to its coset wW_X for the parabolic subgroup

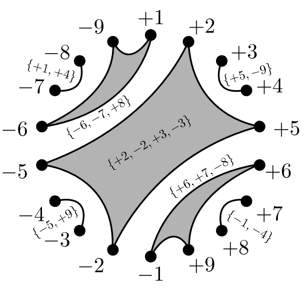
$$W_X = \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_\ell} \times \mathfrak{S}_{B_0}^{\pm}.$$

Here $\mathfrak{S}_{B_i}$ is isomorphic to a symmetric group on $|B_i|$ letters that permutes the elements of B_i, and $\mathfrak{S}_{B_0}^{\pm}$ is a hyperoctahedral group acting on the coordinates in the zero block B_0, if present. One can think of the coset wW_X as representing a function w assigning to each block B_i the (unordered) subset $w(B_i) := \{w(j)\}_{j \in B_i} \subseteq \pm[n]$. It is convenient to visualize $[w, X]$ as labeling the blocks B_i by the sets $w(B_i)$ in the noncrossing partition diagram for π.

For example, when $n = 9$, consider the element $[w, X]$ of Park_W^{NC} in which X is represented by the partition π in (6.1), with

$$\begin{aligned} W_X &= \mathfrak{S}_{\{+1,-6,-9\}} \times \mathfrak{S}_{\{+3,+4\}} \times \mathfrak{S}_{\{+7,+8\}} \times \mathfrak{S}_{\{2,5\}}^{\pm} \\ wW_X &= \begin{pmatrix} +1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 \\ -6 & -3 & +5 & -9 & +2 & -8 & -1 & -4 & +7 \end{pmatrix} W_X \\ &= \left(\begin{array}{ccc|cc|cc|cc} +1 & -6 & -9 & +7 & +8 & +3 & +4 & +2 & +5 \\ -6 & +8 & -7 & -1 & -4 & +5 & -9 & -3 & +2 \end{array}\right) W_X. \end{aligned} \tag{6.2}$$

This $[w, X]$ in Park_W^{NC} is depicted here:



The $W \times C$-action on Park_W^{NC} has $W = \mathfrak{S}_n^{\pm}$ permuting the values within the sets that label the blocks of the partition π, and has the generator c for C rotating the picture $\frac{2\pi}{2n}$ clockwise.

6.2. Proof of Main Conjecture (intermediate version) in type B/C

When the hyperoctahedral group $W = \mathfrak{S}_n^{\pm}$ acts on $V = \mathbb{C}^n$, with $x_1, \ldots, x_n$ the standard coordinate functionals in V^*, we choose

$$(\Theta) = (\theta_1, \ldots, \theta_n) := \left(x_1^{2n+1}, \ldots, x_n^{2n+1}\right).$$

It is easily seen that this is an hsop of degree $h+1$ in $\mathbb{C}[V]$, and that the map Θ sending x_i to θ_i is W-equivariant, so that (2.4) is satisfied. In this setting, the subvariety V^{Θ} cut out by the ideal

$$\begin{aligned}(\Theta - \mathbf{x}) &= \left(x_1^{2n+1} - x_1, \ldots, x_n^{2n+1} - x_n\right) \\ &= \left(x_1\left(x_1^{2n} - 1\right), \ldots, x_n\left(x_n^{2n} - 1\right)\right)\end{aligned}$$

consists exactly of the points $(v_1, \ldots, v_n) \in \mathbb{C}^n$ such that each v_i is either 0 or a $2n$th root of unity, that is, a power of $\omega := e^{\frac{2\pi i}{2n}}$. We now describe a $W \times C$-equivariant bijection

$$\begin{aligned}\mathsf{Park}_W^{NC} &\xrightarrow{f} V^{\Theta} \\ [w, X] &\mapsto v\end{aligned}$$

followed by a description of its inverse bijection.

The forward bijection f. Given $[w, X]$, one starts with the noncrossing partition π of $\pm[n]$ corresponding to X, and re-encodes it via a bijection from [38, Proposition 6] as a periodic parenthesization of the doubly infinite string

$$\cdots - (n-1), -n, +1, +2, \ldots, +(n-1), +n, -1, -2, \ldots, -n, +1, +2, \ldots \tag{6.3}$$

For example, when $[w, X]$ and π are as in (6.2), this parenthesization is

$$\begin{aligned}\cdots - 8) - 9 + 1\big) + 2(+3 + 4) + 5\big(+6(+7 + 8) + 9 - 1\big) \\ - 2(-3 - 4) - 5\big(-6(-7 - 8) - 9 + 1\big) \cdots\end{aligned} \tag{6.4}$$

The idea of the encoding from [38] is that each nonzero block B of π corresponds to a pair of parentheses, having its left parenthesis appearing just to the left of some value j in $\pm[n]$ that we will call the *opener* of block B. The innermost parentheses pairs enclose nonzero blocks that nest no other blocks in π, whose elements can then be removed, and then one continues to pair off the nonzero blocks enclosed with parentheses, inductively.

The map f then sends $[w, X]$ to the vector v whose kth coordinate v_k depends upon the unique block $B_{i(k)}$ of π for which $+k$ lies in $w(B_{i(k)})$:

- if $B_{i(k)}$ is the zero block of π, then $v_k = 0$,
- if $B_{i(k)}$ is a nonzero block, with positive opener $+j$, then $v_k = +\omega^j$,
- if $B_{i(k)}$ is a nonzero block, with negative opener $-j$, then $v_k = -\omega^j$.

Continuing the example of $[w, X]$ and π from (6.2), one determines the coordinates of $v = f([w, X])$ as follows. The zero block $B_0 = \{\pm 2, \pm 5\}$ is labeled $w(B_0) = \{\pm 2, \pm 3\}$, so $v_2 = v_3 = 0$. The nonzero blocks B_i have openers and images $w(B_i)$ tabulated here

Nonzero block B_i	Opener	Image $w(B_i)$
$\{+1, -6, -9\}$	-6	$\{-6, +8, -7\}$
$\{-1, +6, +9\}$	$+6$	$\{+6, -8, +7\}$
$\{+3, +4\}$	$+3$	$\{+5, -9\}$
$\{-3, -4\}$	-3	$\{-5, +9\}$
$\{+7, +8\}$	$+7$	$\{-1, -4\}$
$\{-7, -8\}$	-7	$\{+1, +4\}$

and from this one concludes that

$$\begin{aligned} v = \big(\ & v_1, & v_2, && v_3, & v_4, & v_5, & v_6, & v_7, & v_8, & v_9\ \big) \\ \big(& -\omega^7, & 0, && 0 & -\omega^7, & +\omega^3, & +\omega^6, & +\omega^6, & -\omega^6, & -\omega^3\big). \end{aligned} \tag{6.5}$$

We leave it to the reader to check that f is $W \times C$-equivariant.

The inverse bijection f^{-1}. Given v in V^Θ, one can first determine the noncrossing partition π of $\pm[n]$, via its infinite parenthesization, as follows. There will be left parentheses located just to the left of both $+j$, $-j$ if and only if either of $\pm\omega^j$ occurs at least once among the coordinates v_k. Furthermore, the sizes of the blocks having openers $+j$ and $-j$ are both equal to the multiplicity m_j with which the two values $\pm\omega^j$ occur among the coordinates v_k. Given any such sequence of multiplicities $(m_1, \dots, m_n)$ having $\sum_{j=1}^n m_j \le n$, proceeding from the smallest m_j to the largest, one can recover the location of the right parentheses closing the block opened by $+j$, for each j. The zero block of π then contains the unparenthesized letters left in the infinite string.

For example, given $n = 9$ and the vector v in V^Θ from (6.5), v has multiplicities

$$\begin{aligned} & m_3 = 2 \\ & m_6 = 3 \\ & m_7 = 2 \\ & m_1 = m_2 = m_4 = m_5 = m_8 = m_9 = 0. \end{aligned}$$

One concludes that the infinite parenthesization has openers at ± 3, ± 6, ± 7, enclosing blocks of sizes 2, 3, 2, respectively. From this one can recover the parenthesization in (6.4) by first closing off the blocks of size 2 around $(+3, +4)$, $(-3, -4)$, and the blocks of size 2 around $(+7, +8)$, $(-7, -8)$. Then one removes the values ± 3, ± 4, ± 7, ± 8 from consideration, and next closes off the blocks of size 3 opened by ± 6, namely $(+6+9-1)$, $(-6-9+1)$.

Once one has recovered the parenthesization, and hence π, one determines the set $w(B_i)$ labeling the block B_i opened by $+j$ as follows: $\pm k$ lies in the set $w(B_i)$ if and only if $v_k = \pm\omega^j$. This gives w up to its coset wW_X, determining $[w, X] = f^{-1}(v)$.

7. Type *D*

We begin with a review from [6] on visualizing the elements of $NC(W)$ for W of type D, and extend this to similarly visualize the elements of Park_W^{NC}, analogous to the description of types A and B/C.

7.1. Visualizing type D

The Weyl group W of type D_n is the subgroup of the hyperoctahedral group $\mathfrak{S}_n^{\pm}$ consisting of $n \times n$ signed permutation matrices having an even number of -1 entries, acting on $V = \mathbb{C}^n$. Its reflections T are these:

$$t_{ij}^{+} = (i, j)(-i, -j),$$
$$t_{ij}^{-} = (i, -j)(-i, j).$$

The intersection subspaces X in $\mathcal{L}$ of type D_n correspond, as in type B_n, to set partitions $\pi = \{B_1, \ldots, B_\ell\}$ of the set $\pm[n]$ having $\pi = -\pi$, except for an additional restriction: if the zero block is present, not only is it unique, but it must involve at least two different coordinates $x_i = x_j = 0$. In other words, a zero block $B_0 = -B_0$, if present in π, must contain at least four elements $\{+i, +j, -i, -j\}$ of $\pm[n]$.

One can take as Coxeter generators for W the set $S = \{s_1, ..., s_{n-1}, s_n\}$, where

$$s_i = t_{i,i+1}^{+} = (i, i+1) \quad \text{for } i = 1, 2, \ldots, n-1, \quad \text{and}$$
$$s_n = t_{n-1,n}^{-} = \big(+(n-1), -n, -(n-1), +n\big).$$

The Coxeter element $c = s_1 \cdots s_{n-1} s_n$ permutes the set $\pm[n]$ as the following simultaneous $2(n-1)$-cycle and transposition

$$c = \big(+1, +2, \ldots, +(n-1), -1, -2, \ldots, -(n-1)\big)(+n, -n)$$

having order $h = 2(n-1)$. Drawing the labels

$$\big(+1, +2, \ldots, +(n-1), -1, -2, \ldots, -(n-1)\big)$$

clockwise around a circle, and drawing two vertices labeled $+n$, $-n$ at the center of this circle, one can depict set partitions of $\pm[n]$ as before by drawing the polygonal convex hulls of their blocks. These pictures will always be centrally symmetric. Furthermore, it turns out (see [6, §3]) that the noncrossing flats X in $NC(W)$ again biject with set partitions π for which the polygonal convex hulls of two different blocks have no intersection of their relative interiors. In particular, this implies that if π has a zero block $B_0 = -B_0$, then $\{+n, -n\} \subseteq B_0$. One bijects such a type D_n noncrossing partition π or flat X with the element w in $NC(W) = [1, c]_T$, whose associated permutation of $\pm[n]$ has cycles

- coming from each *nonzero* block B by orienting the edges of the polygonal convex hull of B in the cyclic order (6.3), and
- if there is a zero block B_0 present, then including the 2-cycle $(+n, -n)$, as well as the $2(|B_0| - 1)$-cycle containing the elements of $B_0 \backslash \{\pm n\}$ oriented in the cyclic order (6.3).

Such an element X has pointwise stabilizer

$$W_X = \mathfrak{S}_{B_1} \times \cdots \times \mathfrak{S}_{B_\ell} \times D_{B_0}$$

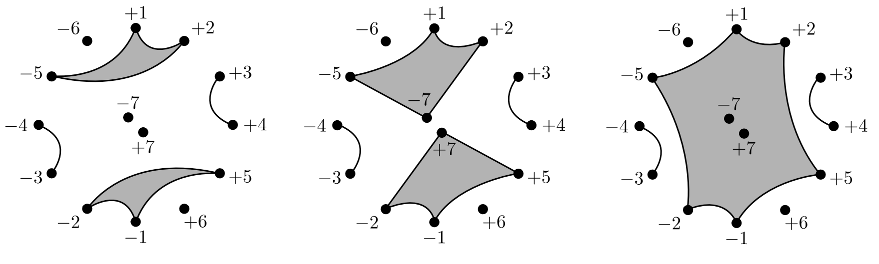
where D_{B_0} is present if and only if the partition π of $\pm[n]$ corresponding to X has a zero block. For example, these three elements X_1, X_2, X_3 of $NC(W)$ for $W = D_7$,

$$X_1 = \{x_1 = x_2 = -x_5,\ x_3 = x_4\}$$
$$X_2 = \{x_1 = x_2 = -x_5 = -x_7,\ x_3 = x_4\}$$
$$X_3 = \{x_1 = x_2 = -x_5 = x_7 = 0,\ x_3 = x_4\}$$

are equal to V^{w_1}, V^{w_2}, V^{w_3} for these elements w_1, w_2, w_3 in $[1, c]_T$:

$$w_1 = (+1, +2, -5)(-1, -2, +5)(+3, +4)(-3, -4),$$
$$w_2 = (+1, +2, -7, -5)(-1, -2, +7, +5)(+3, +4)(-3, -4),$$
$$w_3 = (+1, +2, +5, -1, -2, -5)(+7, -7)(+3, +4)(-3, -4).$$

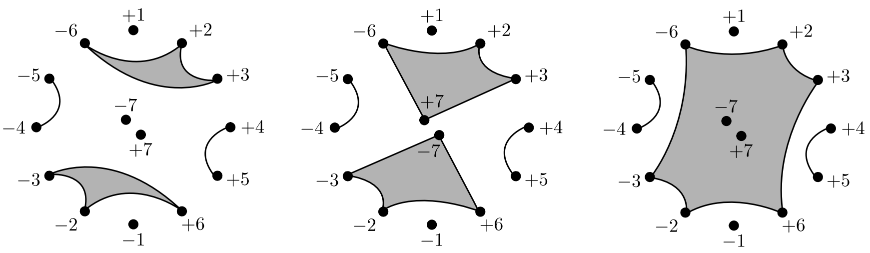
Their partitions π_1, π_2, π_3 of $\pm[n]$ are depicted here



and have pointwise stabilizers

$$
\begin{aligned}
W_{X_1} &= \mathfrak{S}_{\{+1,+2,-5\}} \times \mathfrak{S}_{\{+3,+4\}} \\
W_{X_2} &= \mathfrak{S}_{\{+1,+2,-5,-7\}} \times \mathfrak{S}_{\{+3,+4\}} \\
W_{X_3} &= \mathfrak{S}_{\{+3,+4\}} \times D_{\{+1,+2,-5,-7\}}.
\end{aligned}
$$

The action of $C = \langle c \rangle$ on $NC(W) = [1, c]_T$ corresponds to a clockwise rotation through $\frac{2\pi}{2(n-1)}$ along with a simultaneous swap of the labels $+n$, $-n$. For example, here are the pictures for $c(X_i)$ for X_i with $i = 1, 2, 3$ shown above:

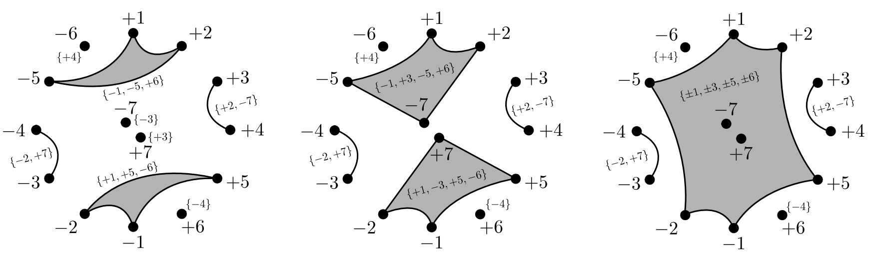


As in type B/C, an element $[w, X]$ of Park_W^{NC} is an equivalence class where X is a noncrossing flat, and w represents a coset wW_X. One can again visualize this by labeling the blocks in the drawing of X by their images under w.

For example, the element w in D_7 given by

$$
w = \begin{pmatrix} +1 & +2 & +3 & +4 & +5 & +6 & +7 \\ -1 & -5 & +2 & -7 & -6 & -4 & -3 \end{pmatrix}
$$

gives rise to three elements $[w, X_1]$, $[w, X_2]$, $[x, X_3]$ of Park_W^{NC} shown here:



7.2. Proof of Main Conjecture (intermediate version) in type D

When $W = D_n$ acts on $V = \mathbb{C}^n$, with $x_1, \ldots, x_n$ the standard coordinate functions in V^*, we will choose

$$
(\Theta) = (\theta_1, \ldots, \theta_n) := \left(x_1^{2n-1}, \ldots, x_n^{2n-1}\right).
$$

It is easily seen that this is an hsop of degree $h+1$ in $\mathbb{C}[V]$, and that the map Θ sending x_i to θ_i is W-equivariant, so that (2.4) is satisfied. In this setting, the subvariety V^Θ cut out by the ideal

$$\begin{aligned}(\Theta - \mathbf{x}) &= \left(x_1^{2n-1} - x_1, \ldots, x_n^{2n-1} - x_n\right) \\ &= \left(x_1\left(x_1^{2(n-1)} - 1\right), \ldots, x_n\left(x_n^{2(n-1)} - 1\right)\right)\end{aligned}$$

consists of the points $(v_1, \ldots, v_n) \in \mathbb{C}^n$ such that each v_i is either 0 or a $2(n-1)$st root of unity, that is, a power of $\omega := e^{\frac{2\pi i}{2(n-1)}}$. We now describe a $W \times C$-equivariant bijection

$$\begin{aligned}\mathsf{Park}_W^{NC} &\xrightarrow{f} V^\Theta \\ [w, X] &\mapsto v\end{aligned}$$

followed by a description of its inverse bijection.

The forward bijection f. Given $[w, X]$, start with the noncrossing partition π of $\pm[n]$ corresponding to X, and ignore the elements $+n$, $-n$ to obtain a type B_{n-1} noncrossing partition of $\pm[n-1]$. Encode X with $\pm n$ removed via its parenthesization of the string

$$\cdots, -(n-1), +1, +2, \ldots, +(n-2), +(n-1), -1, -2, \ldots -(n-1), +1, +2, \ldots$$

As before, every nonzero block will have an opener, $+j$ or $-j$, and note that this opener will lie in the set $\pm[n-1]$.

The map f then sends $[w, X]$ to the vector v whose kth coordinate v_k depends upon the unique block $B_{i(k)}$ of π for which $+k$ lies in $w(B_{i(k)})$:

- if $B_{i(k)}$ is the zero block of π, then $v_k = 0$,
- if $B_{i(k)}$ is a singleton block of either form $\{+n\}$ or $\{-n\}$, then $v_k = 0$,
- if $B_{i(k)}$ is a nonzero block, with positive opener $+j$, then $v_k = +\omega^j$,
- if $B_{i(k)}$ is a nonzero block, with negative opener $-j$, then $v_k = -\omega^j$.

Continuing the example with $n = 7$ and $[w, X_1]$, $[w, X_2]$, $[w, X_3]$ and π from before, one has parenthesizations corresponding to X_1, X_2, X_3 with ± 7 removed

$$\begin{aligned}X_1 &\leftrightarrow \cdots \big(-5(-6) + 1 + 2\big)(+3 + 4)\big(+5(+6) - 1 - 2\big)(-3 - 4)\big(-5(-6) + 1 + 2\big) \cdots \\ X_2 &\leftrightarrow \cdots \big(-5(-6) + 1 + 2\big)(+3 + 4)\big(+5(+6) - 1 - 2\big)(-3 - 4)\big(-5(-6) + 1 + 2\big) \cdots \\ X_3 &\leftrightarrow \cdots - 5(-6) + 1 + 2(+3 + 4) + 5(+6) - 1 - 2(-3 - 4) - 5(-6) + 1 + 2 \cdots\end{aligned}$$

from which one can compute

$$
\begin{array}{rccccccc}
 & (\; v_1, & v_2, & v_3, & v_4, & v_5, & v_6, & v_7 \;) \\
f\big([w,X_1]\big) = & \big(+\omega^5, & +\omega^3, & 0, & -\omega^6, & +\omega^5, & -\omega^5, & -\omega^3\big) \\
f\big([w,X_2]\big) = & \big(+\omega^5, & +\omega^3, & -\omega^5, & -\omega^6, & +\omega^5, & -\omega^5, & -\omega^3\big) \\
f\big([w,X_3]\big) = & \big(\; 0, & +\omega^3, & 0, & -\omega^6, & 0, & 0, & -\omega^3\big).
\end{array}
$$

We leave it to the reader to check that f is $W \times C$-equivariant.

The inverse bijection f^{-1}. Given v in V^{Θ}, we determine $f^{-1}(v) = [w, X]$ based on three cases for the number of coordinates j with $v_j = 0$.

Case 1. There is a unique coordinate $v_{j_0} = 0$.

In this case, by the definition of f, the partition π of $\pm[n]$ corresponding to X has no zero block, and has $\{+n\}$, $\{-n\}$ as singleton blocks, with either $w(+n) = +j_0$ or $w(+n) = -j_0$. Treat the remaining $n-1$ coordinates $\hat{v}$ of v, which are all nonzero, as a type B_{n-1} element of V^{Θ}, and use this to recover the other blocks $\hat{\pi}$ beside $\{+n\}$, $\{-n\}$ of π, along with their labeling by $\hat{w}$, taking values in $[n]\backslash\{j_0\}$. Finally, one has $w(+n) = +j_0$ or $w(+n) = -j_0$ depending upon whether the number of negative values taken by $\hat{w}$ is even or odd.

Case 2. The number of coordinates j with $v_j = 0$ is $z \geq 2$.

In this case, by the definition of f, the element X in $NC(W)$ must have a zero block containing z coordinates, and one of them must be the coordinate x_n. Note that v has length n, and its multiplicities m_j of $\pm\omega^j$ give a composition $(m_1, m_2, \ldots, m_{n-1})$, whose sum is $n - z \leq n - 2$. Thus these multiplicities can be used as in the type B inverse bijection to recover the blocks of a type B_{n-1}/C_{n-1}-noncrossing parking function and labeling of its blocks. Each of its nonzero blocks B will be labeled by a set of the same cardinality as B, but its zero block $B_0 \subseteq \pm[n]$ will include only $z-1$ coordinates, and be labeled by a set of coordinates of size z (namely those j for which $v_j = 0$). Augmenting this zero block B_0 to include the extra coordinate x_n recovers the correct $[w, X] \in \mathsf{Park}_W^{NC}$.

Case 3. Every coordinate of v is nonzero.

In this case, by the definition of f, the element X in $NC(W)$ must have only nonzero blocks, one of which will be of size at least two and contain $+n$; call it B, so that $-B$ will contain $-n$. Because the multiplicities $(m_1, \ldots, m_{n-1})$ satisfy $\sum_{j=1}^{n} m_j = n$, one knows that, for each j having $m_j \geq 2$, the adjusted multiplicities $(m_1, \ldots, m_{j-1}, m_j - 1, m_{j+1}, \ldots, m_{n-1})$ will sum to $n-1$, and therefore recover a valid type B_{n-1}/C_{n-1} noncrossing partition $\hat{X}$. One can check that *exactly one* choice of j (call it j_0) has the property that if one adds in one of the central vertices $\pm n$ to the block opened by $+j_0$, the picture remains noncrossing for type D_n. Once we know whether to add $+n$ or $-n$ to this block opened by $+j_0$, we will uniquely define an element X in $NC(W)$, and the actual coordinates of v let one recover the labeling of the blocks of X to get $[w, X]$. But now this determines whether $+n$ or $-n$ should go in the block opened by j_0, since only one of the two choices will make w have an even number of negative signs.

8. Proof of Main Conjecture (weak version) in type A

Recall that the weak version of the Main Conjecture asserts the isomorphism $\mathsf{Park}_W^{alg} \cong \mathsf{Park}_W^{NC}$ as $W \times C$-representations. As mentioned in the Introduction, Proposition 2.15 lets one rephrase this as an equality of $W \times C$-character values: for every u in W and every integer ℓ,

$$\begin{array}{ccc} \chi_{\mathsf{Park}_W^{alg}}(u, c^\ell) & = & \chi_{\mathsf{Park}_W^{NC}}(u, c^\ell) \\ \| & & \| \\ (h+1)^{\mathrm{mult}_u(\omega^\ell)} & & \left|\left(\mathsf{Park}_W^{NC}\right)^{(u,c^\ell)}\right| \end{array} \tag{8.1}$$

where $\omega := e^{2\pi i h}$ as usual, and $\mathrm{mult}_u(\omega^\ell)$ is the multiplicity of ω^ℓ as an eigenvalue of u acting on V.

Note that when $c^\ell = 1$, but u varies over all of W, equality in (8.1) is equivalent to the isomorphism $\mathsf{Park}_W^{alg} \cong \mathsf{Park}_W^{NC}$ of W-representations, already known case-by-case, and in particular, known for type A; see Sections 2.4, 2.5. Consequently, to prove (8.1), it only remains to consider the case where $c^\ell \neq 1$. In this case, c^ℓ has multiplicative order $d := \frac{n}{\gcd(\ell,n)} \geq 2$, and we will denote $\hat{n} := \frac{n}{d}$, so that $\langle c^\ell \rangle = \langle c^{\hat{n}} \rangle$.

Recall that in type A_{n-1}, one has $W = \mathfrak{S}_n$, and we have chosen as Coxeter element the n-cycle $c = (1, 2, \ldots, n)$. For this choice, the combinatorial model for the parking functions $[w, X]$ in Park_W^{NC} was discussed in Example 2.7: one has X corresponding to a noncrossing set partition $\pi = \{B_1, B_2, \ldots\}$ of $[n]$, and w gives a labeling of each of its blocks B_i by a set $w(B_i)$ of the same cardinality $|w(B_i)| = |B_i|$, giving another set partition $\pi = \{w(B_1), w(B_2), \ldots\}$ of $[n]$.

It is easily checked that a permutation u acting on V has $\omega^\ell \neq 1$ as an eigenvalue with multiplicity $\mathrm{mult}_u(\omega^\ell)$ equal to

$$r_d(u) := \left|\{\text{cycles of } u \text{ whose size is divisible by } d\}\right|.$$

Thus the equality (8.1) to be proven can be rephrased as follows: one must show that, for every permutation u in $\mathfrak{S}_n$ and integer ℓ with $c^\ell \neq 1$, one has

$$(n+1)^{r_d(u)} = \left|\left(\mathsf{Park}_W^{NC}\right)^{(u,c^\ell)}\right|. \tag{8.2}$$

Our approach in proving (8.2) will be to show both sides count the following objects.

Definition 8.1. Let u be in $\mathfrak{S}_n$ and $c^\ell \neq 1$, as above. Extend the action of c^ℓ permuting $[n]$ to an action on $[n] \cup \{0\}$ by letting $c^\ell(0) = 0$, and say that a function $f : [n] \to [n] \cup \{0\}$ is (u, c^ℓ)-equivariant if

$$f\big(u(j)\big) = c^\ell f(j)$$

for every j in $[n]$. Equivalently, whenever $f(j) \neq 0$, then $f(u(j)) \equiv f(j) \bmod n$.

Seeing that the (u, c^ℓ)-equivariant f are counted by the left side of (8.2) is easy.

Proposition 8.2. *For any u in $\mathfrak{S}_n$ and $c^\ell \neq 1$, with notation as above, $(n+1)^{r_d(u)}$ counts the number of (u, c^ℓ)-equivariant functions $f : [n] \to [n] \cup \{0\}$.*

Proof. The (u, c^ℓ)-equivariance implies that such a function f is completely determined by its values on one representative j from each cycle of u. If the cycle has size divisible by d, this value $f(j)$ can be an arbitrary element of $[n] \cup \{0\}$, giving $n+1$ choices. If the cycle has size not divisible by d then this value $f(j)$ must be a fixed point of c^ℓ, that is, $f(j) = 0$. □

The first step in showing that (u, c^ℓ)-equivariant functions f are also counted by the right side of (8.2) is similar in spirit to the proof of a Stirling number identity presented in the Twelvefold Way [50, Chapter 1, Eq. (24d), pp. 34–35]: one classifies functions f according to how their fibers $f^{-1}(j)$ partition the domain $[n]$. To this end, make the following definition.

Definition 8.3. Given u in $\mathfrak{S}_n$ and $d \geq 2$, say that a set partition $\pi = \{A_1, A_2, \ldots\}$ of $[n]$ is (u, d)-admissible if

- π is u-stable in the sense that $u(\pi) = \{u(A_1), u(A_2), \ldots\} = \pi$, so for each A_i there exists some j with $u(A_i) = A_j$, and
- at most one block A_{i_0} is itself u-stable in the sense that $u(A_{i_0}) = A_{i_0}$, and
- all other blocks beside the u-stable block A_{i_0} (if present) are permuted by u in orbits of length d, that is, $\{A_i, u(A_i), u^2(A_i), \ldots, u^{d-1}(A_i)\}$ are distinct, but $u^d(A_i) = A_i$.

Proposition 8.4. *With notation as above, the number of (u, c^ℓ)-equivariant functions $f : [n] \to [n] \cup \{0\}$ is*

$$\sum_\pi n(n-d)(n-2d)\cdots\big(n-(k_\pi-1)d\big)$$

where k_π is the number of u-orbits of blocks of π having length d.

Proof. Associate to each (u, c^ℓ)-equivariant function f a partition $\pi = \{A_1, A_2, \ldots\}$ of the domain $[n]$, by letting the blocks of π be the nonempty fibers $f^{-1}(j)$. Note that equivariance forces π to be (u, d)-admissible. Note also that π contains a u-stable block $A_{i_0} = f^{-1}(0)$ if and only if f takes on the value 0.

On the other hand, if one fixes a (u, d)-admissible partition π of $[n]$, one can count how many (u, c^ℓ)-equivariant f are associated to it as follows. If π has a u-stable block A_{i_0}

then set $f(A_{i_0}) = 0$; otherwise f does not take on the value 0. To determine the rest of f, choose representative blocks $A_1, A_2, \dots, A_{k_\pi}$ from each of the u-orbits of blocks of π having length d. After choosing the value $f(A_1) = j_1$ from the n choices in $[n]$, this forces (working modulo n) that

$$\begin{aligned} f\big(u(A_1)\big) &= j_1 + \ell, \\ f\big(u^2(A_1)\big) &= j_1 + 2\ell, \\ &\vdots \\ f\big(u^{d-1}(A_1)\big) &= j_1 + (d-1)\ell. \end{aligned}$$

This then leaves $n - d$ choices for $f(A_2) = j_2$, forcing

$$\begin{aligned} f\big(u(A_2)\big) &= j_2 + \ell, \\ f\big(u^2(A_2)\big) &= j_2 + 2\ell, \\ &\vdots \\ f\big(u^{d-1}(A_2)\big) &= j_2 + (d-1)\ell. \end{aligned}$$

There are then $n - 2d$ choices for $f(A_3) = j_3$, etc. $\square$

Equality (8.2) now follows from Propositions 8.2, 8.4 and the next proposition.

Proposition 8.5. *Fix u in $\mathfrak{S}_n$ and $c^\ell \neq 1$, with notations as above.*

If $[w, X]$ in Park_W^{NC} is fixed by (u, c^ℓ), and has $X = \{B_1, B_2, \dots\}$, then the associated set partition $\pi = \{A_1, A_2, \dots\}$ of $[n]$ having blocks $A_i := w(B_i)$ will be (u, d)-admissible.

Conversely, fixing a (u, d)-admissible partition π, there will be exactly

$$n(n-d)(n-2d)\cdots\big(n-(k_\pi-1)d\big)$$

$[w, X]$ in Park_W^{NC} fixed by (u, c^ℓ) that are associated to π in this way.

Proof. For the first assertion, note that

$$[w, X] = \big(u, c^\ell\big)[w, X] = \big[uwc^{-\ell}, c^\ell(X)\big]$$

implies, in particular, that $c^\ell(X) = X$, so X is a noncrossing partition of $[n]$ that has d-fold rotational symmetry. This means that X can have at most one d-fold symmetric block B_{i_0} (if any at all), and all of its other blocks come in d-fold rotation orbits of length d. The fact that $uwc^{-\ell}W_X = wW_X$ then means that the partition $\pi = \{A_1, A_2, \dots\}$ defined by $A_i := w(B_i)$ is (u, d)-admissible. In fact, one has that the

d-fold symmetric block of X (if present) gives the unique u-stable block $A_{i_0} = w(B_{i_0})$ of π (if present). Also, if one chooses representatives $B_1, B_2, \ldots, B_k$ for the d-fold rotation orbits of length d among the blocks of X, then their corresponding w-images $A_1, A_2, \ldots, A_k$ give representatives for the u-orbits of π of length d. In particular, one must have $k = k_\pi$.

For the second assertion, define the type of a (u, d)-admissible partition π of $[n]$ to be the sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_{\hat{n}})$ where π has μ_j different u-orbits of blocks of length d in which the blocks have size j. Similarly define the type μ of a d-fold symmetric noncrossing partition X of $[n]$ to mean that X has μ_j different d-fold rotation orbits of length d consisting of blocks having size j. When π is associated to X as above, note that they have the same type μ.

Having fixed a (u, d)-admissible partition π, say of type μ, one can count how many $[w, X]$ in Park_W^{NC} fixed by (u, c^ℓ) will be associated to it in the following way. A result of Athanasiadis [3, Theorem 2.3] counts the number of d-fold symmetric[14] noncrossing partitions X of $[n]$ having type μ as

$$\frac{\hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-(k-1))}{\mu_1!\mu_2!\cdots\mu_{\hat{n}}!}. \tag{8.3}$$

For each such X, and for each fixed block size j, the permutation w must match the μ_j different d-fold rotation orbits having blocks of size j

$$\big(B, c^\ell(B), c^{2\ell}(B), \ldots, c^{(d-1)\ell}(B)\big) \tag{8.4}$$

with the μ_j different u-orbits of π having blocks of size j

$$\big(A, u(A), u^2(A), \ldots, u^{d-1}(A)\big). \tag{8.5}$$

There are $\mu_j!$ ways to do this matching. Having picked such a matching, for each of the $k = \mu_1 + \mu_2 + \cdots + \mu_{\hat{n}}$ different matched orbit pairs as in (8.4), (8.5), w has d choices for how to align them cyclically: w specifies which of the d possible sets $\{A, u(A), u^2(A), \ldots, u^{d-1}(A)\}$ will be the image $w(B)$.

Hence there are $d^k\mu_1!\mu_2!\cdots\mu_{\hat{n}}!$ ways to choose the image sets $w(B_i)$ after making the choice of $X = \{B_1, B_2, \ldots\}$. Together with (8.3), this gives the desired count:

$$\begin{aligned} &d^k\mu_1!\mu_2!\cdots\mu_{\hat{n}}! \cdot \frac{\hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-(k-1))}{\mu_1!\mu_2!\cdots\mu_{\hat{n}}!} \\ &\quad = n(n-d)(n-2d)\cdots\big(n-(k-1)d\big). \qquad \square \end{aligned}$$

[14] Actually, he counts the **centrally**-symmetric noncrossing partitions X of type μ, but these have an easy bijection to those which are d-fold symmetric, for any $d \geq 2$.

9. Narayana and Kirkman polynomials

After proving a statement equivalent to Corollary 3.3 on the Kirkman numbers for W, we explain how calculations of Gyoja, Nishiyama, and Shimura [28] give product formulas for the Kirkman numbers in classical types, and even their graded q-analogues.

9.1. Proof of Corollary 3.3

The following result is an equivalent version of Corollary 3.3; see the discussion in Section 3.3.

Proposition 9.1. *For W an irreducible real reflection group, letting $\mathsf{Park}(W)$ denote either of the equivalent W-representations $\mathsf{Park}_W^{NC} \cong \mathsf{Park}_W^{alg}$, one has*

$$\sum_{k=0}^{n} \langle \chi_{\bigwedge^k V}, \chi_{\mathsf{Park}(W)} \rangle_W \cdot (t-1)^k = \sum_{X \in NC(W)} t^{\dim_{\mathbb{C}} X}.$$

Proof. Note that

$$\begin{aligned} \langle \chi_{\bigwedge^k V}, \chi_{\mathsf{Park}(W)} \rangle_W &= \sum_{X \in NC(W)} \big\langle \chi_{\bigwedge^k V}, \mathrm{Ind}_{W_X}^{W} \mathbf{1}_{W_X} \big\rangle_W \\ &= \sum_{X \in NC(W)} \big\langle \mathrm{Res}_{W_X}^{W} \chi_{\bigwedge^k V}, \mathbf{1}_{W_X} \big\rangle_{W_X} \end{aligned}$$

using (2.2) and Frobenius Reciprocity. For purposes of later multiplying this by $(t-1)^k$ and summing on k, note that when w acts on V and on $\bigwedge^k V$, one has

$$\sum_{k=0}^{n} t^k \chi_{\bigwedge^k V}(w) = \det(1+tw). \tag{9.1}$$

However, note that for w that happen to lie in the pointwise stabilizer W_X of X inside W, there will be $\dim_{\mathbb{C}} X$ extra $+1$-eigenvalues for w when it is considered as an element acting on V, rather than as an element w/X of W_X acting on V/X. Hence

$$\begin{aligned} \sum_{k=0}^{n} (t-1)^k \chi_{\bigwedge^k V}(w) &= \det\big(1+(t-1)w\big) \\ &= \big(1+(t-1)\big)^{\dim_{\mathbb{C}} X} \det\big(1+(t-1)(w/X)\big) \\ &= t^{\dim_{\mathbb{C}} X} \sum_{k=0}^{n} (t-1)^k \chi_{\bigwedge^k (V/X)}(w/X) \end{aligned}$$

and therefore

$$
\begin{aligned}
\sum_{k=0}^{n}(t-1)^k\langle\chi_{\bigwedge^k V},\chi_{\mathsf{Park}(W)}\rangle_W &= \sum_{X\in NC(W)}\sum_{k=0}^{n}(t-1)^k\left\langle \mathrm{Res}^W_{W_X}\chi_{\bigwedge^k V},\mathbf{1}_{W_X}\right\rangle_{W_X}\\
&= \sum_{X\in NC(W)} t^{\dim_{\mathbb{C}} X}\sum_{k=0}^{n}(t-1)^k\left\langle \chi_{\bigwedge^k (V/X)}(w/X),\mathbf{1}_{W_X}\right\rangle_{W_X}\\
&= \sum_{X\in NC(W)} t^{\dim_{\mathbb{C}} X}
\end{aligned}
$$

where the very last equality uses the fact that the W_X-representations $\{\bigwedge^k(W/X)\}$ for $k=0,1,2,\dots\dim W/X$ are inequivalent W_X-irreducibles [11, Chapter 5, §2, Exercise 3], with $\bigwedge^0(W/X)=\mathbf{1}_{W_X}$. $\square$

9.2. Formulas for Kirkman and q-Kirkman numbers

Define the *Kirkman* and *q-Kirkman numbers* for a real irreducible reflection W acting on $V=\mathbb{C}^n$ and $0\le k\le n$ by

$$
\begin{aligned}
\mathrm{Kirk}(W,k) &:= \langle\chi_{\bigwedge^k V},\chi_{\mathsf{Park}(W)}\rangle_W\\
\mathrm{Kirk}(W,k;q) &:= \sum_{d\ge 0}\langle\chi_{\bigwedge^k V},\chi_{(\mathbb{C}[V]/(\Theta))_d}\rangle_W\cdot q^d,
\end{aligned}
\tag{9.2}
$$

so that $[\mathrm{Kirk}(W,k;q)]_{q=1}=\mathrm{Kirk}(W,k)$ has the combinatorial interpretation from cluster theory given by Corollary 3.3. One calculates them via

$$
\begin{aligned}
\sum_{k=0}^{n}\mathrm{Kirk}(W,k;q)t^k &= \left\langle\sum_{k=0}^{n}\chi_{\bigwedge^k V}t^k,\sum_{d\ge 0}\chi_{(\mathbb{C}[V]/(\Theta))_d}q^d\right\rangle_W\\
&= \frac{1}{|W|}\sum_{w\in W}\frac{\det(1+tw)\det(1-q^{h+1}w)}{\det(1-qw)}
\end{aligned}
\tag{9.3}
$$

using (9.1) and Proposition 2.15. We wish to record here some more explicit product formulas for these numbers using character calculations due to Gyoja, Nishiyama, and Shimura [28]. To state these, first recall the q-number $[n]_q:=1+q+\cdots+q^{n-1}$, the q-factorial $[n]!_q:=[1]_q[2]_q\cdots[n]_q$, and the q-binomial $\left[{n \atop k}\right]_q:=\frac{[n]!_q}{[k]!_q[n-k]!_q}$.

Proposition 9.2. *For W an irreducible real reflection group acting on $V=\mathbb{C}^n$, with root system Φ, positive roots Φ^+, fundamental degrees $d_1\le\cdots\le d_n$, and Coxeter number $h:=d_n$, one has*

$$
\mathrm{Kirk}(W,n;q)=q^{|\Phi^+|}\quad\textit{and}\quad \mathrm{Kirk}(W,0;q)=\prod_{i=1}^{n}\frac{[h+d_i]_q}{[d_i]_q}=:\mathrm{Cat}(W,q)
\tag{9.4}
$$

as well as these explicit formulas for $\mathrm{Kirk}(W,k;q)$ *in the classical infinite families*

W	$\mathrm{Kirk}(W,k;q)$
$A_{n-1}(=\mathfrak{S}_n)$	$\frac{q^{\binom{k+1}{2}}}{[n]_q}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}2n-k\\n-k-1\end{bmatrix}_q$
$B_n=C_n$	$q^{k^2}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-k\\n-k\end{bmatrix}_{q^2}$
D_n	$q^{k^2}\begin{bmatrix}n-1\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-k-1\\n-k\end{bmatrix}_{q^2}+q^{k^2-2k+n}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\begin{bmatrix}2n-k-2\\n-k\end{bmatrix}_{q^2}$

and this formula for rank 2 *dihedral groups* $W = I_2(m)$*:*

$$\mathrm{Kirk}\big(I_2(m),1;q\big) = q^1[m]_{q^2} + q^{m-1}[2]_{q^2} = q^1[m+2]_q\frac{[2]_{q^{m-2}}}{[2]_q}. \tag{9.5}$$

Proof. The first formula in (9.4) can be proven via a character computation, but one can also argue it as follows. Since $\bigwedge^n V$ carries the alternating W-character det, the formula asserts that the unique copy of this character det occurring in $\mathbb{C}[V]/(\Theta)$, whose existence is proven in Lemma 5.1, occurs in degree $|\Phi^+|$. In other words, it is carried by the smallest nonzero W-alternating polynomial $\prod_{H\in\mathsf{Cox}(\Phi)}\ell_H$, where ℓ_H is a linear form in V^* that vanishes on the reflecting hyperplane H. This follows because the same is well-known to be true for the *coinvariant algebra* [30, Chapter 3], and the coinvariant algebra is known [9, Proposition 5.2] to be a quotient of $\mathbb{C}[V]/(\Theta)$.

The second formula in (9.4) is simply re-asserting the discussion surrounding (3.1), since $\bigwedge^0 V$ carries the trivial W-character.

For the classical Weyl groups W and for any irreducible W-character χ, Gyoja, Nishiyama, and Shimura [28] compute explicit formulas for a generating function

$$\tilde{\tau}_W(\chi;q,u) := \frac{1}{|W|}\sum_{w\in W}\chi(w)\frac{\det(1+uw)}{\det(1-qw)}. \tag{9.6}$$

Comparing with (9.3) one sees that

$$\mathrm{Kirk}(W,k;q) = \big[\tilde{\tau}_W(\chi_{\bigwedge^k};q,u)\big]_{u=-q^{h+1}}.$$

For type A_{n-1}, one must divide the result by $[n+1]_q$, due to the fact that they are not using the irreducible action of $W=\mathfrak{S}_n$ on $V\cong\mathbb{C}^{n-1}$, but rather the action on $\mathbb{C}^n$ by permuting coordinates.[15] One needs to know that the exterior power $\bigwedge^k V$ corresponds in their indexing of irreducible characters, to

- χ^α with $\alpha=(n-k,1^k)$ in type A_{n-1} [28, formula (3.2)],
- $\chi^{(\alpha,\beta)}$ with $(\alpha,\beta)=((n-k),(1^k))$ in type B_n/C_n [28, formula (3.9)],
- $\chi^{\{\alpha,\beta\}}$ with[16] $\{\alpha,\beta\}=\{(n-k),(1^k)\}$ in type D_n [28, formula (3.15)].

[15] Alternatively, in type A_{n-1}, this same calculation can be reproduced by combining the *q-hook-content formula* for the principal specialization of a Schur function [51, Theorem 7.21.2] with a calculation from Haiman [29, Proposition 2.5.2].

[16] One must also remove a stray factor of $\frac{1}{2}$ appearing on the right in [28, formula (3.15)].

Formula (9.5) can be verified as follows. The reflection representation V for $W = I_2(m)$ turns out to be induced from the character χ_ω of its rotation subgroup $C = \langle c\rangle$ that sends the generator c to $\omega = e^{\frac{2\pi i}{m}}$. Thus Frobenius Reciprocity gives

$$\begin{aligned}\mathrm{Kirk}(W, 1; q) &= \sum_{d\ge 0}\langle \chi_V, \chi_{(\mathbb{C}[V]/(\Theta))_d}\rangle_W \cdot q^d \\ &= \sum_{d\ge 0}\big\langle \chi_\omega, \mathrm{Res}_C^W\, \chi_{(\mathbb{C}[V]/(\Theta))_d}\big\rangle_W \cdot q^d.\end{aligned}$$

In other words, we wish to compute the graded Hilbert series in the variable q for the χ_ω-isotypic component of $\mathbb{C}[V]/(\Theta)$, considered as a C-representation by restriction from W. If one chooses the convenient hsop $(\Theta) = (x^{m+1}, y^{m+1})$ inside $\mathbb{C}[V] = \mathbb{C}[x, y]$ then $\mathbb{C}[V]/(\Theta)$ has $\mathbb{C}$-basis $\{x^i y^j\}_{i,j=0,1,2,\dots,m}$. Hence its χ_ω isotypic component has $\mathbb{C}$-basis

$$\begin{aligned}&\big\{x^i y^j : 0 \le i, j \le m,\ \text{and } i - j \equiv 1 \bmod m\big\} \\ &\quad = \big\{x^1y^0, x^2y^1, x^3y^2, \dots, x^m y^{m-1}\big\} \cup \big\{x^0y^{m-1}, x^1y^m\big\}\end{aligned}$$

and its Hilbert series is

$$\big(q^1 + q^3 + q^5 + \cdots + q^{2m-1}\big) + \big(q^{m-1} + q^{m+1}\big) = q^1[m]_{q^2} + q^{m-1}[2]_{q^2}. \qquad \square$$

We close this section with two remarks on the formulas in Proposition 9.2.

Remark 9.3. One can readily check that the formula for $\mathrm{Kirk}(W, k; q)$ when W is of type A_{n-1} given above is, up to a power of q, the same as the *q-hook-formula* $f^\lambda(q)$ or *fake degree polynomial* for $\lambda = (n-k, n-k, 1^k)$ which q-counts standard Young tableaux of shape λ by their major index; see [51, Corollary 7.21.5].

At $q = 1$, this coincidence between type A Kirkman numbers and tableaux numbers is an observation of K. O'Hara and A.V. Zelevinsky; see Stanley [48] for a bijective proof. We do not have a good algebraic explanation for the coincidence when the variable q is unspecialized, even though both polynomials in q have Hilbert series interpretations.

Remark 9.4. Other versions of q-Kirkman numbers $\mathrm{Kirk}(W, k; q)$ have been considered recently in the literature in types A, B/C, D and $I_2(m)$, but defined in ad hoc ways. They have appeared in conjunction with *cyclic sieving phenomena* (*CSP's*) involving the set X of clusters for W of a fixed cardinality – recall from Corollary 3.3 that $\mathrm{Kirk}(W, k)$ counts the clusters of cardinality $n - k$. This set X carries a natural action of a cyclic group $C = \langle\tau\rangle$ of order $h + 2$ introduced by Fomin and Zelevinsky [24], generated by their *deformed Coxeter element* τ.

We compare here this type-by-type with the other version of q-Kirkman numbers with a renormalized version of our $\mathrm{Kirk}(W, k; q)$ defined in (9.2), where one divides by the smallest power of q to make the constant term 1; call this renormalized polynomial $\mathrm{Kirk}_0(W, k; q)$.

Type A. In type A, it was shown in [39, Theorem 7.1], by brute force evaluation and enumeration that one has a CSP for this triple $(X, X(q), C)$, with X and C as described above, and $X(q) = \mathrm{Kirk}_0(W, k; q)$.

Type B/C. In type B/C, the analogous assertion was proven using similar methods, in work of Eu and Fu [21, Theorem 4.1 at $s = 1$], again using $X(q) = \mathrm{Kirk}_0(W, k; q)$.

Type I (dihedral). For type $I_2(m)$, Eu and Fu prove the analogous assertion in [21, §6, $s = 1$], except that their choice of polynomials $X(q)$, while agreeing with $\mathrm{Kirk}_0(W, k; q)$ for $k = 0, 2$, will disagree at $k = 1$: they use instead

$$X(q) = \begin{cases} [a+2]_q & \text{if } a \text{ is odd,} \\ [a+2]_{q^2} & \text{if } a \text{ is even.} \end{cases}$$

However, one readily checks that their choice of $X(q)$ is congruent to $\mathrm{Kirk}_0(W, 1; q)$ modulo $q^{h+2} - 1$. Thus either polynomial has the same evaluation when q is any $(h+2)$th root-of-unity, and hence the CSP holds for either choice.

Type F. Eu and Fu do not suggest $X(q)$ in general for type F_4, but they do give data in [21, Fig. 15] on the orbit sizes for the action of the deformed Coxeter element τ on the sets of clusters of a fixed cardinality $n - k$. For each $k = 0, 1, 2, 3, 4$, this data shows all orbits of size $7 = \frac{h+2}{2}$, and we checked that $\mathrm{Kirk}_0(W, k; q)$ would predict this correctly for $k = 0, 1, 2, 3, 4$.

Type D. In type D, Eu and Fu prove in [21, Theorem 5.1 at $s = 1$] an analogous CSP, but using polynomials $X(q)$ that disagree more fundamentally with $\mathrm{Kirk}_0(W, k; q)$. Their formula (16) at $s = 1$ (and replacing k by $n - k$) defines a polynomial

$$X(q) = \begin{bmatrix} 2n-k-1 \\ n-k \end{bmatrix}_{q^2} \left(\begin{bmatrix} n-1 \\ n-k \end{bmatrix}_{q^2} + q^n \begin{bmatrix} n-2 \\ n-k-1 \end{bmatrix}_{q^2} \right) + \left(\begin{bmatrix} 2n-k-1 \\ n-k \end{bmatrix}_{q^2} + q^n \begin{bmatrix} 2n-k-2 \\ n-k \end{bmatrix}_{q^2} \right) \begin{bmatrix} n-2 \\ k \end{bmatrix}_{q^2}$$

which they prove gives a triple $(X, X(q), C)$ as above exhibiting the CSP.

Although this $X(q)$ equals $\mathrm{Kirk}_0(W, k; q)$ whenever $k = 0$ or $k = n$, one can check that, starting with $n = 5$ and $k = 2, 3$, they are $X(q) \not\equiv \mathrm{Kirk}_0(W, k; q) \bmod q^{h+2} - 1$. In fact, $X(q)$ and $\mathrm{Kirk}_0(W, k; q)$ disagree at $q = -1$. Thus if one assumes the result [21, Theorem 5.1 at $s = 1$] is correct, which we have not checked, then one reaches the somewhat surprising conclusion that $\mathrm{Kirk}_0(W, k; q)$ fails to give a CSP.

Type E. We did not check E_7, E_8. However, in type E_6, the data given by Eu and Fu in [21, Fig. 15] on the orbit sizes for the action of τ agrees with the CSP prediction of $\mathrm{Kirk}_0(W, k; q)$ for $k = 0, 1, 3, 5, 6$, but fails to give a CSP for $k = 2, 4$.

10. Inspiration: nonnesting parking functions label Shi regions

The original motivation for this paper came from the work of the first and third author on the Shi arrangement and the Ish arrangement of hyperplanes [2]. In type A there is a natural labeling of the regions of the Shi arrangement due to Athanasiadis and Linusson [5] (which is a variant of a description due to Shi [43]) and these labels are essentially the nonnesting parking functions. Upon generalizing this labeling to other Weyl groups, the authors realized that the same process could be used to define **noncrossing parking functions** and that this notion extends beyond crystallographic types.

As motivation, we explain here how (for crystallographic W) the set of nonnesting parking functions Park_W^{NN} naturally labels the regions of the Shi arrangement.

Let W be a (by definition crystallographic) Weyl group with positive roots Φ^+ inside the root system $\Phi \subseteq V$, and for each $\alpha \in \Phi$ and $k \in \mathbb{R}$ define an affine hyperplane

$$H_{\alpha,k} := \{v \in V : \langle v, \alpha\rangle = k\}. \tag{10.1}$$

As a special case one has the linear hyperplanes $H_{\alpha,0} = H_\alpha$ from the Coxeter arrangement $\mathsf{Cox}(\Phi)$.

Definition 10.1. The Shi arrangement $\mathsf{Shi}(\Phi)$ is the arrangement of affine hyperplanes

$$\mathsf{Shi}(\Phi) := \{H_{\alpha,k} : \alpha \in \Phi^+,\ k = 0, 1\}. \tag{10.2}$$

This arrangement arose originally in work of Shi [42] on Kazhdan–Lusztig cells for affine Weyl groups. He later showed in [43] that $\mathsf{Shi}(\Phi)$ dissects V into $(h+1)^n$ connected components R, which we will call *regions*; these correspond to Shi's *admissible sign types*. Given a region R of $\mathsf{Shi}(\Phi)$, and $\alpha \in \Phi^+$, say that the hyperplane $H_{\alpha,1}$ is a ceiling of R if the inequality $\langle v, \alpha\rangle \le 1$ is one of the irredundant facet inequalities defining its closure $\overline{R}$ as a (sometimes unbounded) polyhedron.

Note that since $\mathsf{Cox}(\Phi) \subseteq \mathsf{Shi}(\Phi)$, each region R lies in a unique chamber $\mathsf{Cox}(\Phi)$, of the form wC for a uniquely defined w in W, where C is the (open) dominant region of $\mathsf{Cox}(\Phi)$ defined by $\langle v, \alpha\rangle > 0$ for all $\alpha \in \Phi^+$. The Shi regions R lying within C correspond to the $\oplus$-*sign types* from [44].

The following lemma about the interaction of $\mathsf{Cox}(\Phi)$ and $\mathsf{Shi}(\Phi)$ is surely well-known, but we include the easy proof for the sake of completeness.

Lemma 10.2. *For $\beta \in \Phi^+$ a positive root, its associated affine Shi hyperplane H_{β}, 1 intersects the (open) chamber wC if and only if $\beta = w(\alpha)$ for some $\alpha \in \Phi^+$.*

Proof. Fix β in Φ^+. If $\beta = w(\alpha)$ for α in Φ^+, then choosing v in C with $\langle v, \alpha\rangle = 1$ gives $v' := w(v)$ in wC with

$$\langle v', \beta\rangle = \langle w(v), w(\alpha)\rangle = \langle v, \alpha\rangle = 1. \tag{10.3}$$

So $H_{\beta,1}$ intersects wC.

Conversely, if $H_{\beta,1}$ intersects wC, then there exists some v' in wC with $\langle v', \beta\rangle = 1$. The same calculation (10.3) shows that the point $v := w^{-1}(v')$ lying in C, and the root $\alpha \in \Phi$ for which $\beta = w(\alpha)$, have pairing $\langle v, \alpha\rangle = 1 > 0$. On the other hand, since v lies in C, it pairs positively with positive roots and negatively with negative roots. Thus α must be a positive root, i.e. $\beta = w(\alpha)$ for some $\alpha \in \Phi^+$. $\square$

Proposition 10.3. *The labeling map $\lambda : R \mapsto [w, X]$, where $R \subseteq wC$ and*

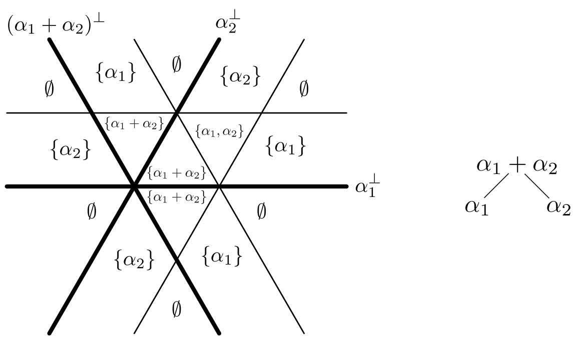
$$X := \bigcap_{\substack{\text{ceilings } H_{\alpha,1} \\ \text{of } R}} H_{w^{-1}(\alpha),0},$$

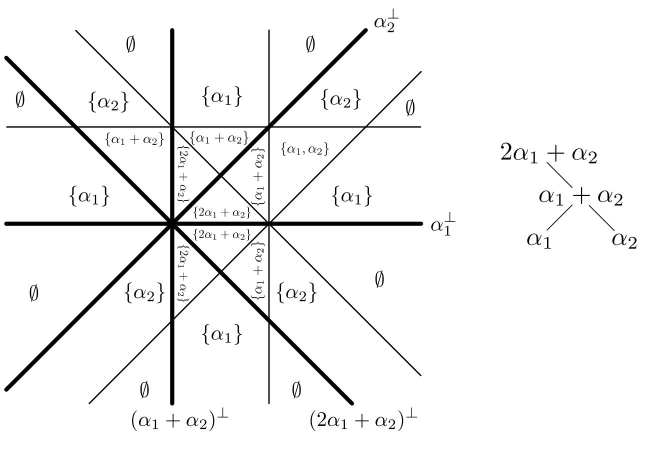
gives a well-defined bijection from the regions of $\mathsf{Shi}(\Phi)$ to Park_Φ^{NN}.

The poset of positive roots Φ^+, along with the labeling of regions R by $[w, X]$ for $\mathsf{Shi}(A_2)$, $\mathsf{Shi}(B_2)$, are illustrated below. Here the $\mathsf{Cox}(\Phi)$ chamber wC containing R is labeled toward the periphery, with the subset of roots

$$A := \left\{w^{-1}(\alpha) : \text{ceilings } H_{\alpha,1} \text{ of } R\right\} \tag{10.4}$$

whose normals intersect in X labeled in the interior of R. This labeling should be compared with Shi [43, Figs. 1 and 2].





Proof. We first check that λ is well-defined. Note that when $R \subseteq wC$, the subset $A \subseteq \Phi$ defined in (10.4) actually lies in Φ^+, due to Lemma 10.2: a ceiling hyperplane $H_{\alpha,1}$ for $R \subseteq wC$ has $H_{\alpha,1}$ intersecting wC, so $w^{-1}(\alpha)$ lies in Φ^+.

Well-definition also requires checking that A forms an antichain in the root ordering on Φ^+. Suppose, for the sake of contradiction, that A contains two positive roots $w^{-1}(\alpha) < w^{-1}(\beta)$ in Φ^+ such that both $H_{\alpha,1}$, $H_{\beta,1}$ are ceilings for R. Note that $\beta - \alpha$ has $w^{-1}(\beta - \alpha) = w^{-1}(\beta) - w^{-1}(\alpha)$, a nonnegative sum of positive roots. Therefore $\langle v, w^{-1}(\beta - \alpha)\rangle \geq 0$ is a valid inequality for v in the closure $\overline{C}$ of the fundamental chamber. Then

$$\langle v', \beta - \alpha\rangle \geq 0 \tag{10.5}$$

is valid for v' in the closed chamber $\overline{wC}$, and hence also on the closed region $\overline{R}$. But then the valid inequality $\langle v', \alpha\rangle \leq 1$ on $\overline{R}$ is already implied by (10.5) together with the valid inequality $\langle v', \beta\rangle \leq 0$ on $\overline{R}$. Thus one cannot have both $H_{\alpha,1}$ and $H_{\beta,1}$ as ceilings for R.

Since one knows that the number of regions R in $\mathsf{Shi}(\Phi)$ is the same as the cardinality of Park_W^{NN}, namely $(h+1)^n$, it only remains to show that the map $\lambda : R \mapsto [w, X]$ defined above is injective. To this end, define a map backward $\mu : [w, X] \mapsto R$ as follows. Given an equivalence class $[w, X]$ in Park_W^{NN}, pick the unique coset representative w for wW_X having $w(\alpha) \in \Phi^+$ for all α in Φ^+ with $H_{\alpha,0} \supset X$. Then μ maps $[w, X]$ to the intersection R of the chamber wC with the open half-spaces defined by $\langle v', w(\alpha)\rangle < 1$ as α runs through the (unique) antichain A of positive roots having $\bigcap_{\alpha\in A} H_{\alpha,0} = X$.

To see that $\mu \circ \lambda$ is the identity map on Shi regions, we first claim that $\mu(\lambda(R))$ and R lie in the same chamber wC. This requires a fact due to Sommers [46, §2]: an antichain A in the positive root poset Φ^+ is always the W-image of a subset of some choice of *simple roots* for W. This implies that the antichain A of positive roots defined by R as in (10.4) above will always form a system of simple roots for the subgroup W_X. Since Lemma 10.2 implies $w(\alpha) \in \Phi^+$ for all α in A, this then implies that $w(\alpha) \in \Phi^+$ for all α in Φ^+ with $H_\alpha \supset X$. Thus if $R \subseteq wC$ and $\lambda(R) = [w, X]$, then w *is* always the unique coset representative for wW_X chosen by the map μ when applied to $\lambda(R)$.

Finally, note that a Shi region R is completely determined by knowing which chamber wC contains it along with its set of ceilings $H_{\alpha,1}$, so $\mu(\lambda(R)) = R$. □

11. Open problems

11.1. Two basic problems

Problem 11.1. Prove the strong or intermediate version of the Main Conjecture in a case-free fashion.

Short of this, check that the weakest version of the Main Conjecture holds via computer calculation in the remaining types E_7, E_8.

Problem 11.2. Extend the Main Conjecture to well-generated complex reflection groups, or even all complex reflection groups.

In this regard the authors do not know, for example, how far the statement and proof of Etingof's Theorem A.1 below generalizes.

11.2. Nilpotent orbits, q-Kreweras and q-Narayana numbers

Recall from the Introduction that the Kirkman numbers and Narayana numbers for W give the f-vector and h-vector of the Fomin–Zelevinsky cluster complex. In Section 9 we related them to the multiplicities of the W-irreducible exterior powers $\bigwedge^k V$ in the W-parking space, and the graded multiplicities allowed one to define q-Kirkman numbers.

In recent work on geometry of nilpotent orbits, E. Sommers [47] has suggested how to define q-analogues for Weyl groups W of the *Kreweras numbers*, which in type A count elements of $NC(W)$ according to their parabolic type; they are the polynomials $[f_{e,\phi}(q,t)]_{t=h+1}$, in the notation of [47, §5.3]. Sommers and the second author have observed that these can be grouped into q-analogues of *Narayana numbers*.

Problem 11.3. What is the relation, if any, between these q-Narayana numbers suggested by Sommers's work, and our q-Kirkman numbers $\mathrm{Kirk}(W,k;q)$? Is there a well-behaved q-analogue of the f-vector to h-vector transformation[17]?

11.3. The Fuss parameter

We discuss here some problems in the known direction of generalization for Catalan objects that replaces the parameter $p = h+1$ with its *Fuss* analogue $p = mh+1$, or more generally, with a parameter p assumed to satisfy $\gcd(p,h) = 1$ (or weakenings of this assumption). Under these hypotheses, the ungraded and graded virtual W-characters

$$\chi(w) = p^{\dim V^w} \quad \text{and} \quad \chi(w;q) = \frac{\det(1-q^p w)}{\det(1-qw)}$$

actually come from genuine W-representations that one might call *p-parking spaces*. The ungraded character corresponds to the W-action permuting Q/pQ, as observed by Haiman [29, Proposition 7.4.1]. The graded character corresponds (via Koszul complex calculations as in [7, Theorem 1.11], [9, §4], [26, §5]) to the quotient $\mathbb{C}[V]/(\Theta)$ where Θ is a homogeneous system of parameters of degree p carrying a certain Galois twist of the reflection representation V, and whose existence is provided by the rational Cherednik theory (see, e.g., Etingof [19]).

[17] While this article was under review, an affirmative answer to this question was found by Sommers and the second author (in preparation).

One can start with product formulas for the cases of (9.6) corresponding to the trivial and determinant characters (calculated as usual via Solomon's theorem [45])

$$\tilde{\tau}_W(\mathbf{1}; q, u) = \prod_{i=1}^{n} \frac{1 + q^{e_i} u}{1 - q^{e_i+1}} \quad \text{and} \quad \tilde{\tau}_W(\det; q, u) = \prod_{i=1}^{n} \frac{u + q^{e_i}}{1 - q^{e_i+1}}. \tag{11.1}$$

From these one deduces, by setting $u = -q^p$, that the Hilbert series of the W-fixed spaces and the W-det-isotypic spaces within $\mathbb{C}[V]/(\Theta)$ have Hilbert series

$$\mathrm{Cat}^{(p)}(W; q) := \prod_{i=1}^{n} \frac{[e_i + p]_q}{[e_i + 1]_q} \quad \text{and} \quad q^{|\Phi^+|} \prod_{i=1}^{n} \frac{[p - e_i]_q}{[e_i + 1]_q},$$

generalizing the extreme cases (9.4) of the q-Kirkman number formulas. The $q = 1$ specializations then give [29, Theorem 7.4.2] the total number of W-orbits and the number of W-regular orbits on Q/pQ. For $p = mh + 1$, one has that $\mathrm{Cat}^{(mh+1)}(W; q)$ is the *q-Fuss–Catalan number*, whose $q = 1$ specialization counts m-element multichains in the noncrossing partitions $NC(W)$, dominant regions in the m-extended Shi arrangement, and facets in the m-generalized cluster complex; see [1, Chapter 5].

One can again define Narayana and Kirkman polynomials with this parameter p, which have interpretations when $p = mh + 1$. For example, one can define Narayana numbers at parameter p that count the W-orbits x in Q/pQ according to the parabolic corank[18] of their W-stabilizer W_x:

$$\sum_{x \in W \backslash Q/pQ} t^{n - \mathrm{rank}(W_x)} = \frac{1}{|W|} \sum_{w \in W} \det\bigl(t + (1 - t)w\bigr) \cdot p^{\dim V^w} \tag{11.2}$$

One finds in the crystallographic case that when $p = mh + 1$, this is the *h-polynomial* of the m-generalized cluster complex of Fomin and Reading [22], that is, replacing t by $t + 1$ in (11.2) gives the generating function for the face numbers of this simplicial complex.

Problem 11.4. Given an irreducible real reflection group W and positive integer p with $\gcd(p, h) = 1$, find definitions of noncrossing and nonnesting partitions and noncrossing parking functions, and generalize the Main Conjecture to this context.

In the 'Fuss case' $p = mh + 1$, the third author defined Fuss analogs $\mathsf{Park}_W^{NC}(m)$ and $\mathsf{Park}_W^{alg}(m)$ of Park_W^{NC} and Park_W^{alg}, as well as a Fuss analog of the nonnesting parking space when W is crystallographic [40]. Fuss analogs of the strong, intermediate, and weak versions of the Main Conjecture are presented and are proven in nearly the generality of the $m = 1$ case. On the other hand, when $\gcd(p, h) = 1$ but $p \not\equiv 1 \bmod h$, one encounters the complication that the hsop guaranteed by the rational Cherednik algebra theory need not carry the representation V^*, but rather some Galois conjugate of this representation.

[18] In the usual case $p = h + 1$, it would be immaterial whether one uses parabolic rank or corank, as the Narayana polynomial will have symmetric coefficient sequence in t.

11.4. The near boundary cases of Kirkman numbers

As mentioned above, the product formulas (11.1) for (9.6) when χ is either the character of the trivial and determinant representations $\bigwedge^0 V$, $\bigwedge^n V$ of W yield the product formulas (9.4) for the boundary cases $k = 0, n$ of the q-Kirkman numbers $\mathrm{Kirk}(W, k; q)$, and even allowing the general parameter p, not just $p = h + 1$.

We discuss here a conjectural[19] formula for (9.6) in the *near-boundary* cases where χ comes either from $\bigwedge^1 V = V$ or $\bigwedge^{n-1} V \cong \det \otimes V$, along with its consequences for the (q-)Kirkman numbers $\mathrm{Kirk}(W, k; q)$, even with the general parameter p. For the sake of stating the conjecture, define a different q-analogue of the rank n by

$$n_q := \sum_{i=1}^{n} q^{e_i - 1}.$$

In other words, n_q is the generating function for the *codegrees* of W.

Conjecture 11.5. *For a real reflection group W acting irreducibly on $V = \mathbb{C}^n$, with exponents $(e_1, \ldots, e_n)$ indexed so that the Coxeter number $h = e_n + 1$, one has the following two equivalent*[20] *formulas:*

$$\tilde{\tau}(\chi_V; q, u) = n_q \cdot \frac{q + u}{1 + q^{h-1}u} \cdot \prod_{i=1}^{n} \frac{1 + q^{e_i}u}{1 - q^{e_i+1}}$$

$$\tilde{\tau}(\chi_{\bigwedge^{n-1} V}; q, u) = n_q \cdot \frac{1 + qu}{u + q^{h-1}} \cdot \prod_{i=1}^{n} \frac{u + q^{e_i}}{1 - q^{e_i+1}}.$$

Setting $u = -q^p$ for a positive integer p would imply that the graded character $\chi(w; q) = \frac{\det(1-q^p w)}{\det(1-qw)}$ has its graded multiplicities of the irreducibles V and $\bigwedge^{n-1} V$ given by, respectively,

$$q^1 \cdot n_q \cdot \frac{[p-1]_q}{[p+h-1]_q} \cdot \mathrm{Cat}^{(p)}(W; q)$$

$$q^{|\Phi^+|-h+1} \cdot n_q \cdot \frac{[p+1]_q}{[p-h+1]_q} \cdot \prod_{i=1}^{n} \frac{[p-e_i]_q}{[e_i+1]_q}. \qquad (11.3)$$

This would have two interesting consequences.

Firstly, setting $p = h + 1$ in (11.3) would give

$$\mathrm{Kirk}(W, 1, q) = q^1 \cdot n_q \cdot \frac{[h]_q}{[2h]_q} \cdot \mathrm{Cat}(W; q)$$

[19] While this article was under review, this conjecture (Conjecture 11.5 or 11.5′ below) was resolved affirmatively in joint work of A. Shepler and the second author (in preparation).

[20] The equivalence of the two formulas comes from the observation (see [28, Eq. (1.24)]) that $\tilde{\tau}(\det \cdot \chi; q, u) = u^n \tilde{\tau}(\chi; q, u^{-1})$, combined with $V^* \cong V$ and $\bigwedge^{n-1} V \cong V^* \otimes \bigwedge^n V \cong \det \otimes V$.

$$\mathrm{Kirk}(W, n-1, q) = q^{|\Phi^+|-h+1} \cdot n_q \cdot \frac{[h+2]_q}{[2]_q} \tag{11.4}$$

where the second equation used the fact that $h - e_i = e_{n-i}$.

Secondly, Eq. (11.3) at $q = 1$ says that the ratio of the multiplicities of the reflection and trivial characters in the character $\chi(w) = p^{\dim V^w}$ is $\frac{n(p-1)}{p+h-1}$. This turns out to be equivalent, by taking $\frac{d}{dt}$ in (11.2), to the following curious statement: a W-orbit x in $W \backslash Q/pQ$ chosen uniformly at random has the expected value for the parabolic corank of the W-stabilizer subgroup W_x equal to $\frac{n(p-1)}{p+h-1}$, or equivalently, its expected rank is $\frac{nh}{p+h-1}$.

Lastly, we mention the algebraic interpretation of Conjecture 11.5. Denote the symmetric and exterior algebras of V^* by $S = \mathbb{C}[V] = \mathrm{Sym}(V^*)$ and $\wedge = \wedge V^*$. Their tensor product $S \otimes V$ becomes a W-representation, bigraded by polynomial and exterior degree. Its W-intertwiner space with V^*

$$M := \mathrm{Hom}_W\big(V^*, S \otimes \wedge\big) \cong (S \otimes \wedge \otimes V)^W$$

will have bigraded Hilbert series given by the left side of the conjecture, using the variables, q, u for the polynomial, exterior gradings, respectively.

Furthermore, $S \otimes \wedge$ becomes a bigraded module over the invariant subalgebra S^W, via multiplication in the left tensor factor S. Since $S^W = \mathbb{C}[f_1, \dots, f_n]$ is a polynomial algebra, S will be a free S^W-module. Hence $S \otimes \wedge$ is also S^W-free, and the same holds for M. Conjecture 11.5 then predicts the bidegrees for any choice of bihomogeneous S^W-basis elements of M, or equivalently, the Hilbert series of the quotient $M/S_+^W M$. Using the fact that $1/\mathrm{Hilb}(S^W, q) = \prod_{i=1}^{\ell}(1-q^{e_i+1})$, Conjecture 11.5 becomes equivalent to the following.

Conjecture 11.5′.

$$\mathrm{Hilb}\big(M/S_+^W M; q, u\big) = n_q \cdot (q+u) \cdot \prod_{i=1}^{\ell-1}\big(1 + q^{e_i} u\big).$$

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Appendix A. Etingof's proof of reducedness for a Certain hsop

Let W be an irreducible real reflection group of rank n, with (complexified) reflection representation V, Coxeter number h, and let p be any positive integer coprime to h.

We present here a uniform argument due to P. Etingof [18] showing the existence of an hsop of degree p carrying V^* which satisfies the reducedness condition of the strong or intermediate versions of the Main Conjecture from Section 2.6.

Theorem A.1. *(Etingof) There exists an hsop $\Theta = (\theta_1, \dots, \theta_n)$ of degree p carrying V^* such that the subvariety $V^\Theta \subseteq V$ consists of p^n distinct points.*

The choice of hsop in Theorem A.1 arises from the theory of rational Cherednik algebras, and to explain it requires some notation. Let $\langle -,- \rangle$ be a W-invariant positive definite form on V, unique up to scaling. For any $v \in V$, let $v^\vee \in V^*$ be the linear functional given by $\langle -, v\rangle : V \to \mathbb{C}$. Also let $\partial_v : \mathbb{C}[V] \to \mathbb{C}[V]$ be the partial derivative operator in the direction of v. The Dunkl operator $D_v : \mathbb{C}[V] \to \mathbb{C}[V]$ with parameter $\frac{p}{h}$ is given by

$$D_v := \partial_v - \frac{p}{h} \sum_{t \in T} \frac{\langle \alpha_t, v\rangle}{\alpha_t^\vee}(1-t), \tag{A.1}$$

where $\{\alpha_t : t \in T\}$ is the root system associated to W, normalized to satisfy $\langle \alpha_t, \alpha_t\rangle = 2$ for all t in T. For any t in T and $f \in \mathbb{C}[V]$, one has $(1-t).f$ divisible by $\alpha_t^\vee$ in $\mathbb{C}[V]$, so the operator D_v maps polynomials to polynomials.

It follows from Gordon's work[21] on the rational Cherednik algebras [26] that there exists a W-equivariant injective linear map $\widehat{\Theta} : V^* \hookrightarrow \mathbb{C}[V]_p$ whose image $\widehat{\Theta}(V^*)$

(i) is annihilated by all Dunkl operators $\{D_b : b \in V\}$, and
(ii) generates an ideal I in $\mathbb{C}[V]$ whose quotient $\mathbb{C}[V]/I$ carries a p^n-dimensional simple module for the rational Cherednik algebra at parameter $\frac{p}{h}$.

In particular, since the quotient $\mathbb{C}[V]/I$ is finite-dimensional, for any basis $x_1, \dots, x_n$ of V^*, the elements $\theta_i = \widehat{\Theta}(x_i)$ for $i = 1, 2, \dots, n$ form an hsop $\Theta = (\theta_1, \dots, \theta_n)$ as in the setup (2.4) for the Main Conjecture. One then has an associated degree p polynomial map $\Theta : V \to V$ characterized by $a(\Theta(v)) = (\widehat{\Theta}(a))(v)$ for $v \in V$, $a \in V^*$, with fixed point locus

$$V^\Theta = \{x \in V : \Theta(x) = x\} = \{x \in V : a(x) = (\widehat{\Theta}(a))(x) \text{ for all } a \in V^*\}.$$

Proof of Theorem A.1. To avoid trivialities, assume $p \neq 1$ without loss of generality. Let $x \in V^\Theta$ and let $d\Theta_x : V \to V$ be the linear map given by the differential of the polynomial map Θ at x. Since the generators $\theta_i - x_i$ for the ideal $(\Theta - \mathbf{x})$ that cuts out V^Θ have Jacobian $d\Theta_x - 1_V$, it is enough to show that $d\Theta_x$ does not have the eigenvalue 1.

[21] Although there are subtleties to this story realized only later than [26] – see Gordon and Griffeth [27, §1.5, 2.11, 2.12] for how this is circumvented via results of Malle and of Rouquier.

To do this, we compute the map $d\Theta_x$ explicitly. We will make frequent use of the fact that for $a, b, x \in V$ one has

$$\begin{aligned}\langle a, d\Theta_x(b)\rangle = \big(\partial_b \widehat{\Theta}(a^\vee)\big)(x) &= \frac{p}{h}\sum_{t\in T}\langle \alpha_t, b\rangle \left(\frac{(1-t)\widehat{\Theta}(a^\vee)}{\alpha_t^\vee}\right)(x) \\ &= \frac{p}{h}\sum_{t\in T}\langle \alpha_t, b\rangle\langle \alpha_t, a\rangle \left(\frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}\right)(x) \qquad \text{(A.2)}\end{aligned}$$

where the second-to-last equality used the fact that $\widehat{\Theta}$ is annihilated by the Dunkl operator D_b, and the last equality derives from $\widehat{\Theta}$ being linear and W-equivariant.

Given x in V^Θ, let $W_x \subseteq W$ be its isotropy subgroup, which is itself a reflection group having reflection representation V/V^{W_x}. Decompose

$$W_x = W_1 \times \cdots \times W_m$$

as a product of irreducible reflection groups. For $1 \le i \le m$, let $T_i \subseteq T$ be the reflections in W_i and let $T_0 = T - (T_1 \cup \cdots \cup T_m)$, giving a disjoint union of sets

$$T = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_m.$$

One also has a W_x-stable orthogonal direct sum decomposition

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$$

in which $V_0 = V^{W_x}$, and V_i is the (complexified) irreducible reflection representation of W_i for $1 \le i \le m$. Since $d\Theta_x$ is a W_x-invariant operator, it preserves these direct summands. In fact, we will show that this is an eigenspace decomposition: since W_i acts irreducibly on V_i for $i = 1, 2, \ldots, m$, by Schur's Lemma, the restriction $d\Theta_x|_{V_i}$ acts as a scalar c_i (to be determined in Case 2 below), and we will show in Case 1 below that $d\Theta_x|_{V_0}$ acts as the scalar $p > 1$. The following fact will be useful.

Lemma A.2. *For any x in V^Θ and t in T, one has*

$$\left(\frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}\right)(x) = \begin{cases} 1 & \text{if } t \in T_0, \\ c_i & \text{if } t \in T_i \text{ with } i \in \{1, 2, \ldots, m\}. \end{cases} \qquad \text{(A.3)}$$

Proof. Since x lies in V^Θ, one has $\widehat{\Theta}(\alpha_t^\vee)(x) = \alpha_t^\vee(\Theta(x)) = \alpha_t^\vee(x)$. Thus the first case in the lemma follows, as $\alpha_t^\vee(x) \neq 0$ if t lies in T_0, since such t does not lie in W_x. To derive the second case, write $\widehat{\Theta}(\alpha_t^\vee) = \alpha_t^\vee \cdot f$ with f in $\mathbb{C}[V]$. Then one has

$$\begin{aligned}\partial_{\alpha_t}\widehat{\Theta}(\alpha_t^\vee) &= \partial_{\alpha_t}(\alpha_t^\vee \cdot f) \\ &= \partial_{\alpha_t}(\alpha_t^\vee) \cdot f + \alpha_t^\vee \cdot \partial_{\alpha_t}(f) \\ &= 2f + \alpha_t^\vee \cdot \partial_{\alpha_t}(f).\end{aligned}$$

Hence $\partial_{\alpha_t}\widehat{\Theta}(\alpha_t^\vee(x)) = 2f(x)$, as $\alpha_t^\vee(x) = 0$ when t lies in T_i for $i \geq 1$, and therefore

$$\begin{aligned} c_i &= \frac{1}{2}\langle \alpha_t, c_i\alpha_t\rangle = \frac{1}{2}\big\langle \alpha_t, d\Theta_x(\alpha_t)\big\rangle = \frac{1}{2}\partial_{\alpha_t}\widehat{\Theta}\big(\alpha_t^\vee\big)(x) \\ &= \frac{1}{2}\big(2f(x)\big) = f(x) = \frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}(x). \qquad \square \end{aligned} \tag{A.4}$$

We can now determine the operators $d\Theta_x|_{V_i}$ for $i = 0, 1, 2, \ldots, m$.

Case 1. $i = 0$. We claim that for any $a, b \in V_0$, one has $\langle a, d\Theta_x(b)\rangle = p \cdot \langle a, b\rangle$ and hence that $d\Theta_x|_{V_0}$ acts as the scalar $p > 1$. To see this, start with (A.2):

$$\begin{aligned} \big\langle a, d\Theta_x(b)\big\rangle &= \frac{p}{h}\sum_{t\in T}\langle \alpha_t, b\rangle\langle \alpha_t, a\rangle\left(\frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}\right)(x) \\ &= \frac{p}{h}\sum_{t\in T_0}\langle \alpha_t, b\rangle\langle \alpha_t, a\rangle\left(\frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}\right)(x) \\ &= \frac{p}{h}\sum_{t\in T_0}\langle \alpha_t, b\rangle\langle \alpha_t, a\rangle = \frac{p}{h}\sum_{t\in T}\langle \alpha_t, b\rangle\langle \alpha_t, a\rangle. \end{aligned} \tag{A.5}$$

Here the second and fourth equality used the fact that any t in $T\backslash T_0 = \bigcup_{i=1}^m T_i$ fixes a in V_0, and hence has $\langle \alpha_t, a\rangle = 0$, while the third equality used Lemma A.2. The claim then follows from this lemma.

Lemma A.3. *For W an irreducible real reflection group with Coxeter number h and reflections T acting on V, any $a, b \in V$ will satisfy*

$$\sum_{t\in T}\langle a, \alpha_t\rangle\langle b, \alpha_t\rangle = h\langle a, b\rangle. \tag{A.6}$$

Proof. The left side is a W-invariant bilinear function of a, b, and hence equal to some scalar multiple λ of $\langle a, b\rangle$. Letting $a = b = e_i$ for any orthonormal basis $e_1, \ldots, e_n$ of V and summing over i, one has $\sum_{t\in T}\langle \alpha_t, \alpha_t\rangle = \lambda n$, so that $2|T| = \lambda n$. However, it is well-known [30, §3.18] that $2|T| = hn$, so $\lambda = h$. $\square$

Case 2. i lies in $\{1, 2, \ldots, m\}$. We will compute the scalar c_i by which $d\Theta_x|_{V_i}$ acts. Assume a, b lie in V_i, and start with (A.2):

$$\begin{aligned} \big\langle a, d\Theta_x(b)\big\rangle &= \frac{p}{h}\sum_{t\in T}\langle \alpha_t, a\rangle\langle \alpha_t, b\rangle\frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}(x) \\ &= \frac{p}{h}\left(\sum_{t\in T}\langle \alpha_t, a\rangle\langle \alpha_t, b\rangle + \sum_{t\in T}\langle \alpha_t, a\rangle\langle \alpha_t, b\rangle\left(\frac{\widehat{\Theta}(\alpha_t^\vee)}{\alpha_t^\vee}(x) - 1\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{p}{h}\left(h\langle a,b\rangle + (c_i-1)\sum_{t\in T_i}\langle \alpha_t, a\rangle\langle \alpha_t, b\rangle\right) \\
&= \frac{p}{h}\big(h\langle a,b\rangle + (c_i-1)h_i\langle a,b\rangle\big) \qquad \text{(A.7)}
\end{aligned}
$$

where h_i is the Coxeter number for W_i. Here the third equality comes from applying Lemma A.3 to W in the first sum, and using Lemma A.2 in the second sum, along with the fact that $\langle \alpha_t, a\rangle = 0$ if $t \notin T_0 \cup T_i$. The fourth equality applies Lemma A.3 to W_i in the second sum.

Now given $t \in T_i$, specializing (A.7) at $a = b = \alpha_t$ gives

$$
\begin{aligned}
2c_i &= \langle \alpha_t, c_i\alpha_t\rangle = \big\langle \alpha_t, d\Theta_x(\alpha_t)\big\rangle = p\cdot\langle \alpha_t, \alpha_t\rangle + \frac{p}{h}(c_i-1)h_i\cdot\langle \alpha_t,\alpha_t\rangle \\
&= 2p + 2\frac{p}{h}(c_i-1)h_i
\end{aligned}
$$

yielding $c_i = \frac{ph-ph_i}{h-ph_i}$. Therefore $c_i \neq 1$, since $p, h > 1$. $\square$

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