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Convex Trace Functions and the Wigner–Yanase–Dyson Conjecture

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Several convex mappings of linear operators on a Hilbert space into the real numbers are derived, an example being $A \mapsto -\operatorname{Tr}\exp(L + \ln A)$. Some of these have applications to physics, specifically to the Wigner–Yanase–Dyson conjecture which is proved here and to the strong subadditivity of quantum mechanical entropy which will be proved elsewhere.

1. INTRODUCTION

This paper is concerned with certain convex or concave mappings of linear operators on a Hilbert space into the reals. [$f(A)$ is convex if $f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda)f(B)$ for $0 < \lambda < 1$ and $f(A)$ is concave if $-f(A)$ is convex.] These mappings involve the trace operation which plays a central role in quantum statistical mechanics, and it is not surprising, therefore, that the mappings discussed here were motivated by considerations of physics. In particular, Theorem 1 solves affirmatively a conjecture due to Wigner, Yanase, and Dyson [1] about a certain definition of information. In Section 3 we use Theorem 1 to prove other convexity theorems when the Hilbert space is finite dimensional. One of those, Theorem 6, we extend to infinite dimensional spaces in Section 4. Theorem 6 has a physical application; it is the basis for proving that quantum mechanical entropy is strongly subadditive (cf. Refs. [2, 3, 4]). The proof of that fact will be given in a subsequent paper [5].

From the work of Krauss and Bendat and Sherman ([6] and the references quoted therein) it is known that certain convex functions from

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$\mathbb{R}$ to $\mathbb{R}$ extend to operator-valued convex functions. If $f(x)$ is such a function then $A \mapsto \operatorname{Tr} Kf(A)$ (where Tr means trace) is certainly convex when $K > 0$ and fixed. Simple examples are $f(A) = A^{-p}$ and $f(A) = -A^p$ for $A > 0$ and $0 < p \leqslant 1$. However, $A \mapsto \operatorname{Tr} f(A)$ may be convex even when $f(A)$ is not convex as an operator-valued function. Examples of this are $f(A) = e^A$ for A self-adjoint and $f(A) = A^{-p}$ for $p > 1$ and for $A > 0$ (cf. Theorems 8 and 9).

In this paper we shall be concerned with mappings more complicated than those just mentioned. One example, Theorem 6, is

$$A \mapsto -\operatorname{Tr} \exp[L + \ln A]$$

for $A > 0$ and L self-adjoint.

Theorem 1 is our main theorem and Theorems 2, 3, 6, and 7 are derived from it. Theorems 8 and 9 are a side issue and are independent of and simpler than Theorem 1. In Section 5 we remark briefly on the logical connection of Theorems 1, 2, 3, 6, and 7, namely that they can all be derived simply from each other (at least for finite dimensional Hilbert spaces).

2. The Main Theorem and the Wigner–Yanase–Dyson Problem

We begin by proving our main Theorem 1 which constitutes the basis for Theorems 2, 3, 6, and 7 of the next section. Theorem 1 is also the Wigner–Yanase–Dyson (WYD) conjecture [1] (actually, it is a bit stronger) and at the end of this section we shall explain the WYD problem. We also discuss another problem concerning the WYD definition of information [1] and give a partial solution of it.

Theorem 1 will be proved directly for infinite dimensional Hilbert spaces and our notation is the following:

(1) H is a separable Hilbert space with inner product (x, y) which is linear in y and conjugate linear in x.

(2) $\mathscr{B}(H)$ is the set of bounded linear operators from H to H; $\mathscr{B}^s(H) \subset \mathscr{B}(H)$ are the bounded self-adjoint operators; $\mathscr{B}^+(H) \subset \mathscr{B}^s(H)$ are the positive operators ($A \in \mathscr{B}^+(H) \Rightarrow (x, Ax) \geqslant 0, \forall x$); $\mathscr{B}^{++}(H) \subset \mathscr{B}^+(H)$ are the strictly positive operators ($A \in \mathscr{B}^{++}(H) \Rightarrow (x, Ax) > 0, \forall x \neq 0$).

(3) If $A \in \mathscr{B}^+(H)$ and $z \in \mathbb{C}$, we can use the spectral representation of A to define $A^z \in \mathscr{B}(H)$ for $\operatorname{Re}(z) \geqslant 0$. $A^z \in \mathscr{B}^+(H)$ for $z \geqslant 0$.

(4) The $\mathscr{I}_q$ classes: If $A \in \mathscr{B}(H)$ we form $|A| = (A^\dagger A)^{1/2} \in \mathscr{B}^+(H)$. $A \in \mathscr{I}_q(H) \subset \mathscr{B}(H)$ $(q \geqslant 1)$ if $\| A \|_q \equiv (\operatorname{Tr} |A|^q)^{1/q} < \infty$, where Tr means trace. $\mathscr{I}_1(H)$ is the trace class and $\mathscr{I}_2(H)$ is the Hilbert–Schmidt class. $A \in \mathscr{I}_q(H)$ implies that A is compact and that $A \in \mathscr{I}_1(H)$. $\| A \|_q = (\sum_{j=1}^{\infty} \lambda_j{}^q)^{1/q}$, where the λ_j are the eigenvalues of $|A|$ in decreasing order, including multiplicity. If $A \in \mathscr{B}^+(H)$ but $A \notin \mathscr{I}_1(H)$, it is convenient to define $\operatorname{Tr} A = \infty$.

(5) We recall that if $A \in \mathscr{B}(H)$ and if K is a linear operator (not necessarily bounded) on a dense domain, $D(K)$, in H then AK may have a bounded extension to all of H. If so, it is unique and its adjoint is a bounded extension of $K^\dagger A^\dagger$.

THEOREM 1. *Let K be a linear operator (not necessarily bounded) on H, let $A, B \in \mathscr{B}^+(H)$, and let λ, $0 < \lambda < 1$, be given. Form the convex combination $C = \lambda A + (1 - \lambda)B$. Let p and r be given positive real numbers with $p + r \equiv s \leqslant 1$. If $M \equiv C^{p/2}KC^{r/2}$ has an extension to $\mathscr{I}_2(H)$ then*

(1) *$A^{p/2}KA^{r/2}$ and $B^{p/2}KB^{r/2}$ have extensions to $\mathscr{I}_2(H)$ and*

(2) $$\lambda \operatorname{Tr} A^{r/2}K^\dagger A^p K A^{r/2} + (1 - \lambda) \operatorname{Tr} B^{r/2}K^\dagger B^p K B^{r/2} \leqslant \operatorname{Tr} C^{r/2}K^\dagger C^p K C^{r/2}, \text{ i.e.,}$$

$$A \in \mathscr{B}^+(H) \mapsto \operatorname{Tr} A^{r/2}K^\dagger A^p K A^{r/2} \text{ is concave.}$$

Proof. (a) We recall the theorem [6] that the map $A \in \mathscr{B}^+(H) \mapsto A^q$ is concave on $\mathscr{B}^+(H)$ when $0 < q \leqslant 1$. Thus, $\lambda A^q \leqslant C^q$ and $\operatorname{Ker}(A^q) = \operatorname{Ker}(A) \supset \operatorname{Ker}(C) = \operatorname{Ker}(C^q)$, and similarly for B. As A, B, and C are bounded, their kernels are closed subspaces and $H = \operatorname{Ker}(C) \oplus \operatorname{Ker}(C)^\perp$. The foregoing inequalities show that for $0 < q \leqslant 1$, $\alpha(q) \equiv A^{q/2}C^{-q/2}$ and $\alpha(q)^\dagger = C^{-q/2}A^{q/2}$ can be extended to bounded operators on $\operatorname{Ker}(C)^\perp$ because $\| A^{q/2}C^{-q/2}\Psi \| \leqslant \lambda^{-1/2} \| \Psi \|$, in the dense set $D_C = \{$vectors with support away from zero in the spectral representation of $C\}$. Similarly, we define $\beta(q) \equiv B^{q/2}C^{-q/2}$. Also, $\alpha(q)$ and $\alpha(q)^\dagger$ can be defined to be zero on $\operatorname{Ker}(C)$ and, thus, are defined on all of H. Clearly, $C^{q/2}\alpha(q)^\dagger = A^{q/2} = \alpha(q)C^{q/2}$. Consequently,

$$A^{p/2}KA^{r/2} = \alpha(p)[C^{p/2}KC^{r/2}]\, \alpha(r)^\dagger = \alpha(p)\, M\alpha(r)^\dagger \in \mathscr{I}_2(H),$$

since $M \in \mathscr{I}_2(H)$. Not only is the first part of the theorem thus proved, but we also see that if $\{\Psi_i\}$ and $\{\varphi_i\}$ are orthonormal bases for $\operatorname{Ker}(C)$ and $\operatorname{Ker}(C)^\perp$, respectively, we can compute traces in the basis $\{\Psi_i\} + \{\varphi_i\}$

and all terms involving $\{\Psi_i\}$ will vanish. Thus, Ker(C) is an irrelevant subspace, and we shall, henceforth, assume that $H = \text{Ker}(C)^{\perp}$, i.e., $C > 0$.

(b) With the foregoing definitions, part (2) is equivalent to the following: $\lambda T^A(p, r) + (1 - \lambda) T^B(p, r) \leqslant \text{Tr } M^\dagger M$ for every $M \in \mathscr{I}_2(H)$, where $T^A(p, r) = \text{Tr } \alpha(r) M^\dagger \alpha(p)^\dagger \alpha(p) M \alpha(r)^\dagger$ and similarly for $T^B(p, r)$.

(c) Let $z = x + iy \in \mathbb{C}$ and consider the operator valued function $\alpha(z) \equiv A^{z/2}C^{-z/2} = A^{iy/2}A^{x/2}C^{-x/2}C^{-iy/2} = A^{iy/2}\alpha(x) C^{-iy/2}$. Since $C^{iy/2}$ is unitary and $\| A^{iy/2} \| \leqslant 1$, $\alpha(z)$ is uniformly bounded in $S = \{z \mid 0 \leqslant \text{Re}(z) \leqslant 1\}$. If $\bar{z} = x - iy$, $\alpha(\bar{z})^\dagger = C^{-z/2}A^{z/2}$. For $\Psi \in D_C$, $\alpha(z)\Psi$ is an entire analytic function of z because $C^{-z/2}\Psi$ is entire and $A^{z/2}$ is entire. Hence, by the boundedness of $\alpha(z)$ and a standard density argument, $\alpha(z)\Psi$ is regular on S (continuous on S and analytic in the interior of S) for all $\Psi \in H$. Since weak analyticity implies strong analyticity, we also have that $\alpha(z)$ is strongly continuous on S and is norm analytic in the interior of S. Furthermore, if $A_n \to A$ strongly and if $B \in \mathscr{I}_2(H)$, then $A_n B \to AB$ in the $\mathscr{I}_2(H)$ norm. (This is trivial if B is finite rank, but the finite rank operators are dense in the $\mathscr{I}_2(H)$ norm). Hence, $\alpha(z_1) M \alpha(\bar{z}_2)^\dagger$ is $\mathscr{I}_2(H)$ regular on $S \times S$, which means that

$$T^A(z_1, z_2) \equiv \text{Tr } \alpha(z_2) M^\dagger \alpha(\bar{z}_1)^\dagger \alpha(z_1) M \alpha(\bar{z}_2)^\dagger$$

is bounded and regular on $S \times S$.

(d) We now set $z_1 = z$, $z_2 = s - z$ and consider $T^A(z) \equiv T^A(z, s - z)$ as a regular function on $\{z \mid 0 \leqslant \text{Re}(z) \leqslant s\}$. By (b) we need to show that $f(p) = \lambda T^A(p) + (1 - \lambda) T^B(p) \leqslant \text{Tr } M^\dagger M$. By the maximum modulus principle for bounded regular functions on a strip, $|f(p)| \leqslant \max\{\sup_\theta |f(i\theta)|, \sup_\theta |f(s + i\theta)|\}$. We shall consider only the first case, $p = i\theta$, in detail because the second case, $p = s + i\theta$, is parallel. $|f(i\theta)| \leqslant \lambda |T^A(i\theta)| + (1 - \lambda) | T^B(i\theta)|$. Using the facts that for $A \in \mathscr{B}(H)$ and $B \in \mathscr{I}_2(H)$, AB and $BA \in \mathscr{I}_2(H)$, and for $B, C \in \mathscr{I}_2(H)$, $\text{Tr } BC = \text{Tr } CB$ and $| \text{Tr } BC | \leqslant \frac{1}{2}\text{Tr } B^\dagger B + \frac{1}{2}\text{Tr } C^\dagger C$, we have that

$$\begin{aligned} 2 \mid T^A(i\theta)| \leqslant\ & \text{Tr } \alpha(s - i\theta) M^\dagger \alpha(-i\theta)^\dagger \alpha(-i\theta) M \alpha(s - i\theta)^\dagger \\ & + \text{Tr } \alpha(s + i\theta) M^\dagger \alpha(i\theta)^\dagger \alpha(i\theta) M \alpha(s + i\theta)^\dagger. \end{aligned}$$

However, $\| \alpha(-i\theta)^\dagger \alpha(-i\theta)\| \leqslant 1$, so the first term is at most

$$\begin{aligned} \text{Tr } \alpha(s - i\theta) M^\dagger M \alpha(s - i\theta)^\dagger &= \text{Tr } M \alpha(s - i\theta)^\dagger \alpha(s - i\theta) M^\dagger \\ &= \text{Tr } M C^{-i\theta/2} \alpha(s)^\dagger \alpha(s) C^{i\theta/2} M^\dagger \\ &= \text{Tr } \alpha(s)^\dagger \alpha(s) C^{i\theta/2} M^\dagger M C^{-i\theta/2}. \end{aligned}$$

Likewise, the second term is at most $\operatorname{Tr} \alpha(s)^{\dagger}\alpha(s)\, C^{-i\theta/2}M^{\dagger}MC^{i\theta/2}$. If we add to these the corresponding two terms for $|\, T^{B}(i\theta)|$ we obtain

$$\lambda \,|\, T^{A}(i\theta)| + (1-\lambda)|\, T^{B}(i\theta)| \leqslant \tfrac{1}{2} \operatorname{Tr}[\lambda\alpha^{\dagger}(s)\,\alpha(s) + (1-\lambda)\,\beta^{\dagger}(s)\,\beta(s)]P, \tag{2.1}$$

where $P = C^{i\theta/2}M^{\dagger}MC^{-i\theta/2} + C^{-i\theta/2}M^{\dagger}MC^{i\theta/2} \in \mathscr{B}^{+}(H)$. As we remarked before, $\lambda A^{s} + (1-\lambda)\, B^{s} \leqslant C^{s}$, whence

$$\lambda\alpha(s)^{\dagger}\,\alpha(s) + (1-\lambda)\,\beta(s)^{\dagger}\,\beta(s) = C^{-s/2}[\lambda A^{s} + (1-\lambda)\, B^{s}]\, C^{-s/2} \leqslant \mathbb{1}.$$

Substituting this in (2.1) proves the theorem. Q.E.D.

Remark. If $C^{p/2}K$ has an extension to $\mathscr{I}_{1}(H)$ then so does $A^{p/2}K$ and $B^{p/2}K$ since $\|\, C^{p/2}K\,\| \geqslant \lambda^{1/2}\|\, A^{p/2}K\,\|$. In this case

$$\operatorname{Tr} C^{r/2}K^{\dagger}C^{p}KC^{r/2} = \operatorname{Tr} C^{r}K^{\dagger}C^{p}K = \operatorname{Tr} K^{\dagger}C^{p}KC^{r}$$

and similarly for A and B.

COROLLARY 1.1. *With p and r as in Theorem* 1, *the function from $\mathscr{B}^{+}(H) \times \mathscr{B}^{+}(H) \times \mathscr{B}(H)$ to the nonnegative reals defined by* $(A, B, K) \mapsto F(A, B, K) = \operatorname{Tr} A^{r/2}K^{\dagger}B^{p}KA^{r/2}$

(1) *is jointly concave in* (A, B) *and*

(2) *is convex in* K.

Proof. Consider the Hilbert space $H' \equiv H \oplus H$ and define the following operators in $\mathscr{B}(H')$:

$$k\colon (x, y) \mapsto (0, Kx),$$
$$k^{\dagger}\colon (x, y) \mapsto (K^{\dagger}y, 0),$$
$$a\colon (x, y) \mapsto (Ax, By), \qquad a \in \mathscr{B}^{+}(H').$$

Applying Theorem 1 to $\operatorname{Tr} a^{r/2}k^{\dagger}a^{p}ka^{r/2}$ proves the first part. The second part follows from a Schwartz inequality type of argument since $F(A, B, K)$ is nonnegative and quadratic in K. Q.E.D.

COROLLARY 1.2. *With p and r as in Theorem* 1, $p + r \equiv s \leqslant 1$, *the functions from $\mathscr{B}^{+}(H) \times \mathscr{B}^{+}(H) \times \mathscr{B}(H)$ to the nonnegative reals defined by*

$$(A, B, K) \mapsto F_{q}(A, B, K) = \{\operatorname{Tr} A^{r/2}K^{\dagger}B^{p}KA^{r/2}\}^{q}$$

(1) *are jointly concave in* (A, B) *when* $0 < q \leqslant 1/s$,

(2) *are jointly convex in* (A, B) *when* $q < 0$, *and*

(3) *are convex in* K *when* $q \geqslant \frac{1}{2}$.

Proof. The proof is a standard one for homogeneous concave (or convex) functions [7]. Let $x = (x_1, x_2) \in \mathbb{R}_+^2 = \{(x_1, x_2) \mid x_1 \geqslant 0, x_2 \geqslant 0\}$ and define $f(x) = F_1(x_1 A + x_2 A', x_1 B + x_2 B', K)$ for an arbitrary but, henceforth, fixed choice of $A, A', B, B' \in \mathscr{B}^+(H)$. Parts (1) and (2) are equivalent to showing that for all such choices, $f(x)^q$ is concave (or convex). By Corollary 1.1 $f(x)$ is nonnegative, concave and homogeneous of order s, i.e., $f(\lambda x) = \lambda^s f(x)$ for $\lambda \geqslant 0$. For each $\alpha \geqslant 0$, define $G_\alpha = \{x \mid f(x) \geqslant \alpha, x \in \mathbb{R}_+^2\}$. It is easily seen from the properties of $f(x)$ that G_α is a convex subset of $\mathbb{R}_+^2$ and $G_\alpha = \alpha^{1/s} G_1$ for $\alpha > 0$. Define $k(x) = \sup\{\mu \geqslant 0 \mid x \in G_{\mu^s}\}$ for $x \in \mathbb{R}_+^2$. As $x \in G_{f(x)}$, $k(x)$ is verywhere defined. In fact, since $f(x) = \sup\{\alpha \geqslant 0 \mid x \in G_\alpha\}, f(x) = k(x)^s$. Obviously, $k(x)$ is nonnegative and homogeneous of order 1, and, since $k(x) = \sup\{\mu > 0 \mid x \in \mu G_1\}$ when $f(x) \neq 0$, it is easy to check that $k(x)$ is a concave function. For a nonnegative concave function, $k(x)$, $k(x)^p$ is concave when $0 < p \leqslant 1$ and $k(x)^p$ is convex when $p < 0$. This proves parts (1) and (2). For part (3) we define $f(x) = F_1(A, B, x_1 K + x_2 K')$, with $K, K' \in \mathscr{B}(H)$. $f(x)$ is nonnegative, convex, and homogeneous of order 2. We define: $G_\alpha = \{x \mid f(x) \leqslant \alpha, x \in \mathbb{R}_+^2\}$ which is convex; $k(x) = \inf\{\mu \geqslant 0 \mid x \in G_{\mu^2}\}$. Then $k(x)$ is nonnegative, convex, and homogeneous of order 1 and $f(x) = k(x)^2$. For any nonnegative convex function, $k(x)$, $k(x)^p$ is convex when $p \geqslant 1$. Q.E.D.

The setting for the next corollary is the following: Let H^1 and H^2 be two separable Hilbert spaces and $H^{12} \equiv H^1 \otimes H^2$ their tensor product. If $A_{12} \in \mathscr{B}^+(H^{12})$ and $A_{12} \in \mathscr{I}_1(H^{12})$ we can define $A_1 \in \mathscr{B}^+(H^1)$ by means of the partial trace, i.e.,

$$A_1 = \operatorname{Tr}^1 A_{12},$$

which means that for $x, y \in H^1$

$$(x, A_1 y) = \sum_i (x \otimes e_i, A_{12}[y \otimes e_i])$$

where $\{e_i\}$ is any orthonormal basis in H^2.

COROLLARY 1.3. *Let* H^2 *be of dimension* $d_2 < \infty$, *let* K *be a* *linear operator (not necessarily bounded) on* H^1, *and let* $A_{12} \in \mathscr{B}^+(H^{12})$, $A_{12} \in \mathscr{I}_1(H^{12})$ *with* $A_1 = \operatorname{Tr}^1 A_{12}$. *Let* $p > 0$, $r > 0$, $p + r = s \leqslant 1$. *Define* $L = K \otimes \mathbb{1}_2$ *on* H^{12}. *If* $A_1^{p/2} K A_1^{r/2}$ *has an extension to* $\mathscr{I}_2(H^1)$ *then* $A_{12}^{p/2} L A_{12}^{r/2}$ *has an extension to* $\mathscr{I}_2(H^{12})$ *and*

$$d_2^{1-s} \operatorname{Tr}^1 A_1^{r/2} K^\dagger A_1^{\,p} K A_1^{r/2} \geqslant \operatorname{Tr}^{12} A_{12}^{r/2} L^\dagger A_{12}^{p} L A_{12}^{r/2}. \tag{2.2}$$

Proof. If G is the group of unitary transformations on H^2 and if dU is the normalized Haar measure on G then it is easy to see [8] that $B_{12} \equiv \int_G U^\dagger A_{12} U \, dU = d_2^{-1} A_1 \otimes \mathbb{1}_2$. Let $F(A_{12})$ be the right side of (2.2). By Theorem 1, $A_{12} \mapsto F(A_{12})$ is concave so

$$F(B_{12}) \geqslant \int_G F(U^\dagger A_{12} U) \, dU.$$

But $F(U^\dagger A_{12} U)$ is independent of U since $(U^\dagger A_{12} U)^p = U^\dagger A_{12}^p U$, $U^\dagger L U = L$, and $\operatorname{Tr}^{12} U^\dagger X U = \operatorname{Tr}^{12} X$. However, $F(B_{12})$ is the left side of (2.2). Q.E.D.

Remarks. (i) Theorem 1 can be regarded as a special case of Corollary 1.3 as may be seen by taking $H^2 = \mathbb{C}^2$ and

$$A_{12} = \tfrac{1}{2} A \otimes P^a + \tfrac{1}{2} B \otimes P^b,$$

where P^a and P^b are two orthogonal projections on H^2 and $A, B \in \mathscr{B}^+(H^1)$. Then $A_1 = \tfrac{1}{2}A + \tfrac{1}{2}B = C$.

(ii) Similar to Corollary 1.1, we can extend Corollary 1.3 to the following: Let A_{12}, $B_{12} \in \mathscr{B}^+(H^{12})$, A_{12}, $B_{12} \in \mathscr{I}_1(H^{12})$, and $A_1 = \operatorname{Tr}^1 A_{12}$, $B_1 = \operatorname{Tr}^1 B_{12}$, then

$$d_2^{1-s} \operatorname{Tr}^1 A_1^{r/2} K^\dagger B_1^{\,p} K A_1^{r/2} \geqslant \operatorname{Tr}^{12} A_{12}^{r/2} K^\dagger B_{12}^{p} K A_{12}^{r/2}. \tag{2.3}$$

(iii) When $d_2 = \infty$, Corollary 1.3 makes no sense except when $s = 1$. In that special case, the corollary is true when $d_2 = \infty$. The proof, which we shall not give here, can be constructed in imitation of the proof of Theorem 1 itself. The principal idea is to define $\alpha(q) = A_{12}^{q/2}[A_1^{-q/2} \otimes \mathbb{1}_2] \in \mathscr{B}(H^{12})$ and $M = A_1^{p/2} K A_1^{r/2} \in \mathscr{I}_2(H^1)$.

(iv) If we let $A_{12} = \mathbb{1}_{12}$ in (2.3) and let K be a projection onto an arbitrary one-dimensional subspace of H^1, we obtain the *operator* inequality (since $A_1 = d_2 \mathbb{1}_1$)

$$d_2^{1-p} B_1^{\,p} \geqslant \operatorname{Tr}^2 B_{12}^p$$

for all $B_{12} \in \mathscr{B}^+(H^{12})$ and $0 < p \leqslant 1$.

The Wigner–Yanase–Dyson Conjecture

In quantum mechanics, a density matrix, ρ, on H satisfies $\rho \in \mathscr{B}^+(H)$, $\rho \in \mathscr{I}_1(H)$ and $\mathrm{Tr}\,\rho = 1$. The entropy of ρ, as usually defined, is

$$S(\rho) = -\mathrm{Tr}\,\rho \ln \rho.$$

Wigner and Yanase [1] extended this to the concept of the entropy of ρ relative to a self-adjoint "observable," K, and defined it as $S(\rho, K) \equiv \frac{1}{2}\mathrm{Tr}[\rho^{1/2}, K]^2$, where $[A, B] = AB - BA$. Dyson (cf. Ref. [1]) proposed a generalization of this to

$$S_p(\rho, K) \equiv \tfrac{1}{2}\,\mathrm{Tr}[\rho^p, K][\rho^{1-p}, K]$$

for $0 < p < 1$. [Actually, Wigner and Yanase defined $I_p(\rho, K) \equiv -S_p(\rho, K)$ which they termed skew information.]

It is well known and easy to prove that $S(\rho)$ is concave in ρ, and the WYD conjecture is that $S_p(\rho, K)$ is concave in ρ for each fixed K. They were able to prove this only when $p = \frac{1}{2}$. In physical applications K may be unbounded, but it is always correct to assume that $\rho^p K$ and $K\rho^p$ have unique extensions to $\mathscr{I}_1(H)$ for all $p > 0$. Thus, (cf. the remark after Theorem 1)

$$S_p(\rho, K) = -\mathrm{Tr}\,\rho K^2 + \mathrm{Tr}\,\rho^{1-p} K \rho^p K.$$

The first term is linear and, hence, concave, and the second term is concave by Theorem 1.

Remark. Theorem 1 is stronger than necessary because it allows K to be non-self-adjoint, i.e., $\mathrm{Tr}[\rho^p, K^\dagger][\rho^{1-p}, K]$ is concave in ρ when K is non-self-adjoint. This generalization can be derived from the self-adjoint case when $p = \frac{1}{2}$ by a simple polarization argument, but not when $p \neq \frac{1}{2}$. Baumann and Jost [9, 10] proved the concavity for general p, but for a special class of ρ and H.

Wigner and Yanase properly regarded the concavity of $\rho \mapsto S_p(\rho, K)$ as a necessary requirement in order that $I_p(\rho, K)$ be a sensible definition of information. Another absolute requirement is the subadditivity of $S_p(\rho, K)$. Subadditivity of the ordinary entropy, $S(\rho)$, means (in the terminology preceding Corollary 1.3 and with $\rho_2 \equiv \mathrm{Tr}^2\,\rho_{12}$) that

$$S(\rho_{12}) \leqslant S(\rho_1) + S(\rho_2).$$

This inequality is well known.

For $S_p(\rho, K)$, Wigner and Yanase [1] take the following definition: Let K_1 (resp. K_2) be a self-adjoint operator on H^1 (resp. H^2) and define $L = K_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes K_2$ on H^{12}. The subadditivity condition is that

$$S_p(\rho_{12}, L) \leqslant S_p(\rho_1, K_1) + S_p(\rho_2, K_2). \tag{2.4}$$

It is easy to see that (2.4) is true when $\rho_{12} = \rho_1 \otimes \rho_2$. Wigner and Yanase proved (2.4) when ρ_{12} is a projection onto a one-dimensional subspace of H^{12} and $p = \frac{1}{2}$. In the general case, (2.4) becomes (with $r = 1 - p$)

$$\begin{aligned} &\mathrm{Tr}^1\, \rho_1{}^p K_1 \rho_1{}^r K_1 + \mathrm{Tr}^2\, \rho_2{}^p K_2 \rho_2{}^r K_2 \\ &\quad \geqslant -2\,\mathrm{Tr}^{12} \rho_{12}[K_1 \otimes K_2] + \mathrm{Tr}^{12} \rho_{12}^p L \rho_{12}^r L. \end{aligned} \tag{2.5}$$

We do not have a proof of this, but when K_1 or K_2 is zero, (2.5) is simply Corollary 1.3. Because (2.5) is true in these three special cases, there is reason to believe it is true generally.

3. Some Finite Dimensional Theorems

In this section we confine our attention to finite dimensional Hilbert spaces over the complex numbers, i.e., $H = \mathbb{C}^n$. Some of the results of this section will be generalized to the infinite dimensional case in the next section by approximation arguments. If $C \in \mathscr{B}^{++}(H)$ then $C > \epsilon \mathbb{1}$ for some $\epsilon > 0$. We remark that for $A, B \in \mathscr{B}^{++}(H)$ and $K \in \mathscr{B}(H)$, $\mathrm{Tr}\, AK^\dagger BK = \mathrm{Tr}(B^{1/2}KA^{1/2})^\dagger(B^{1/2}KA^{1/2}) > 0$ for $K \neq 0$.

Theorem 2. *The function from $\mathscr{B}^{++}(H) \times \mathscr{B}(H)$ to the nonnegative reals defined by*

$$(A, K) \mapsto F(A, K) = \mathrm{Tr}\, A^{-r} K^\dagger A^{-p} K \tag{3.1}$$

is jointly convex in (A, K) whenever $p \geqslant 0$, $r \geqslant 0$ and $p + r \leqslant 1$, i.e., for all λ, $0 < \lambda < 1$, $\lambda F(A, K) + (1 - \lambda) F(B, L) \geqslant F(C, M)$ when $A, B \in \mathscr{B}^{+}(H)$; $K, L \in \mathscr{B}(H)$; $C = \lambda A + (1 - \lambda)B$; $M = \lambda K + (1 - \lambda)L$.

Proof. We can think of $\mathscr{B}(H)$ as a complex Hilbert space, $V(H)$, of dimension n^2 with the inner product $K, K' \in \mathscr{B}(H) \mapsto \langle K, K' \rangle = \mathrm{Tr}\, K^\dagger K'$. The linear transformation of $V(H) \to V(H)$ defined by $K \mapsto \alpha K \beta$, with $\alpha, \beta \in \mathscr{B}^{++}(H)$ is Hermitian and positive definite because

$\langle K, \alpha K\beta\rangle > 0$ when $K \neq 0$. Thus, $K \mapsto F(A, K)$, $K \mapsto F(B, K)$, and $K \mapsto F(C, K)$ are positive definite quadratic forms on $V(H)$. Furthermore, we can define $V_2(H) = V(H) \oplus V(H)$ with inner product $\langle (K, L), (K', L')\rangle = \operatorname{Tr} K^\dagger K' + \operatorname{Tr} L^\dagger L'$. Clearly

$$D \equiv \lambda F(A, K) + (1-\lambda) F(A, L) \qquad \text{and} \qquad N \equiv F(C, \lambda K + (1-\lambda)L)$$

are both positive quadratic forms on $V_2(H)$ and D is definite. With A and B fixed, we form the variational quotient N/D and maximize it with respect to $(K, L) \in V_2(H)$. Using the fact that $\operatorname{Tr} AB = 0$ for all $A \in \mathscr{B}(H)$ implies $B = 0$, we find the eigenvalue equations

$$\begin{aligned} \gamma A^{-p} K A^{-r} &= C^{-p}[\lambda K + (1-\lambda)L]\, C^{-r} \equiv M \in \mathscr{B}(H), \\ \gamma B^{-p} L B^{-r} &= C^{-p}[\lambda K + (1-\lambda)L]\, C^{-r} \equiv M, \end{aligned} \tag{3.2}$$

and the problem is to show that the eigenvalue $\gamma \leqslant 1$. If $\gamma \neq 0$, the equation $\gamma A^{-p} K A^{-r} = M$ has the unique solution $K = \gamma^{-1} A^p M A^r$. Likewise, $L = \gamma^{-1} B^p M B^r$ and $\lambda K + (1-\lambda) L = C^p M C^r$. Thus, finding $\gamma \neq 0$ solutions to (3.2) is equivalent to finding an $M \in \mathscr{B}(H)$ such that $M \neq 0$ and $\lambda A^p M A^r + (1-\lambda)\, B^p M B^r = \gamma C^p M C^r$. However, if we multiply this equation on the left by $M^\dagger$ and take the trace, we see that $\gamma \leqslant 1$ by Theorem 1. Q.E.D.

COROLLARY 2.1. *Let $p \geqslant 0$, $r \geqslant 0$, $p + r \equiv s \leqslant 1$, and $q \in \mathbb{R}$, $q \neq 0$ be fixed. Then the functions from $\mathscr{B}^{++}(H) \times \mathscr{B}^{++}(H) \times \mathscr{B}(H)$ to the nonnegative reals defined by*

$$(A, B, K) \to F_q(A, B, K) = (\operatorname{Tr} A^{-p} K^\dagger B^{-r} K)^q$$

is jointly convex in (A, B, K) when $q \geqslant (2-s)^{-1}$.

Proof. The same as for Corollaries 1.1 and 1.2, since $F_1(A, B, K)$ is homogeneous of order $2 - s > 0$. Q.E.D.

We next turn our attention to a function similar to (3.1) from $\mathscr{B}^{++}(H) \times \mathscr{B}(H)$ to the nonnegative reals defined by

$$(A, K) \mapsto Q(A, K) = \operatorname{Tr} \int_0^\infty dx (A + x\mathbb{1})^{-1} K^\dagger (A + x\mathbb{1})^{-1} K.$$

We note that the linear transformation T_A from $V(H)$ (defined in the proof of Theorem 2) to $V(H)$ given by

$$T_A\colon K \mapsto \int_0^\infty dx(A + x\mathbb{1})^{-1} K(A + x\mathbb{1})^{-1} \tag{3.3}$$

is Hermitian and positive definite. In a basis in which A is diagonal it is easy to compute T_A explicitly:

$$T_A\colon \{K_{ij}\} \mapsto \{K_{ij} f(A_i, A_j)\},$$

where the $\{A_i\}$ are the eigenvalues of A and

$$f(x, y) = (x - y)^{-1} \ln(x/y) \qquad \text{if} \quad x \neq y, \quad f(x, x) = x^{-1}.$$

The inverse transformation is

$$T_A^{-1}\colon \{K_{ij}\} \mapsto \{K_{ij}/f(A_i, A_j\},$$

but this is the same as

$$T_A^{-1}\colon K \mapsto \int_0^1 dx\, A^x K A^{1-x} \tag{3.4}$$

as we see by calculating the integral.

THEOREM 3. *The function $Q(A, K)$ from $\mathscr{B}^{++}(H) \times \mathscr{B}(H)$ to the nonnegative reals is jointly convex in (A, K).*

Proof. The proof is exactly the same as for Theorem 2 up to the eigenvalue equation (3.2) which now reads

$$\begin{aligned} \gamma T_A(K) &= T_C(\lambda K + (1 - \lambda)L) \equiv M \in \mathscr{B}(H), \\ \gamma T_B(K) &= T_C(\lambda K + (1 - \lambda)L) = M. \end{aligned} \tag{3.5}$$

If $\gamma \neq 0$ we apply the inversion formula (3.4), whence

$$\int_0^1 dx\, [\lambda A^x M A^{1-x} + (1 - \lambda)\, B^x M B^{1-x} - \gamma C^x M C^{1-x}] = 0.$$

By multiplying this on the left by $M^\dagger$ and using Theorem 1, we see that $\gamma \leqslant 1$. Q.E.D.

COROLLARY 3.1. *The functions*

$$(A, B, K) \mapsto Q_q(A, B, K) = \left[\mathrm{Tr} \int_0^\infty dx\,(A + x\mathbb{1})^{-1} K^\dagger (B + x\mathbb{1})^{-1} K\right]^q$$

from $\mathscr{B}^{++}(H) \times \mathscr{B}^{++}(H) \times \mathscr{B}(H)$ *to the nonnegative reals are*

(1) *jointly convex in* (A, B, K) *when* $q \geqslant 1$,

(2) *convex in* K *when* $q \geqslant \frac{1}{2}$,

(3) *jointly convex in* (A, B) *when* $q > 0$, *and*

(4) *jointly concave in* (A, B) *when* $-1 \leqslant q < 0$ *and* $K \neq 0$.

Proof. The construction given in Corollary 1.1 allows us to replace A by (A, B). We note that $Q_1(A, B, K)$ is homogeneous of order -1 in (A, B), as may be seen by changing the integration variable to λx, and, hence, $Q_1(A, B, K)$ is homogeneous of order 1 in (A, B, K). The proof of parts (1) and (2) is the same as that given in the second half of Corollary 1.2 for nonnegative, homogeneous, convex functions of positive order. To prove parts (3) and (4) we have to modify the proof given in Corollary 1.2. For $x \in \mathbb{R}_+{}^2\backslash\{(0, 0)\}$, we construct $f(x) \equiv Q_1(x_1 A + x_2 A', x_1 B + x_2 B', K)$ with $A, A', B, B' \in \mathscr{B}^{++}(H)$. Assuming that $K \neq 0$ (otherwise there is nothing to prove) we note that $f(x)$ is strictly positive (as remarked before), convex and homogeneous of order $-t = -1$. We shall here give the construction for general $t > 0$ as we shall need it in Corollary 8.1 and Theorem 9. Define the convex sets $G_\alpha = \{x \mid f(x) \leqslant \alpha, x \in \mathbb{R}_+{}^2\backslash\{(0, 0)\}\}$, whence $f(x) = \inf\{\alpha > 0 \mid x \in G_\alpha\}$. Define $k(x) \equiv \sup\{\mu > 0 \mid x \in G_{\mu^{-t}}\} = \sup\{\mu > 0 \mid x \in \mu G_1\} = f(x)^{-1/t}$. As before, $k(x)$ is positive, concave, and homogeneous of order 1. The rest follows from the remarks in Corollary 1.2. Q.E.D.

The mapping $T_A\colon \mathscr{B}(H) \to \mathscr{B}(H)$ defined in (3.3) has a special significance when restricted to the self-adjoint operators, $\mathscr{B}^s(H)$. Let $A \in \mathscr{B}^{++}(H)$ and $K \in \mathscr{B}^s(H)$. Then

$$(d/dx) \ln(A + xK)\,|_{x=0} = T_A(K). \tag{3.6}$$

[To derive this, use the representation

$$\ln A = \int_0^\infty dx\,(1 + x)^{-1}\,(A - \mathbb{1})(A + x\mathbb{1})^{-1} \tag{3.7}$$

which can be verified in a basis in which A is diagonal; then use

$$-\ (d/dx)(A + xK)^{-1}\,|_{x=0} = A^{-1} K A^{-1} \tag{3.8}$$

for any $K \in \mathscr{B}(H)$, $A \in \mathscr{B}^{++}(H)$.] Using (3.6)–(3.8) we have

$$\begin{aligned}\frac{d^2}{dx^2} \ln(A + xK)\,|_{x=0} &= -2 \int_0^\infty dx\, (A + x\mathbb{1})^{-1} K(A + x\mathbb{1})^{-1} K(A + x\mathbb{1})^{-1} \\ &\equiv -R_A(K).\end{aligned} \tag{3.9}$$

We note that $R_A(K) \in \mathscr{B}^{++}(H)$ when $K \neq 0$.

Proposition 4. *For any real number, γ, and $A \in \mathscr{B}^{++}(H)$, $K \in \mathscr{B}^s(H)$, $\gamma^2 R_A(K) + 2\gamma T_A(K) + \mathbb{1} \geqslant 0$.*

Proof.

$$\begin{aligned}0 \leqslant R_A(K + A) &= -(d^2/dx^2) \ln(A + xK + xA)\,|_{x=0} \\ &= -(d^2/dx^2)\{\ln(1 + x)\mathbb{1} + \ln(A + x(1 + x)^{-1} K)\}\,|_{x=0} \\ &= \mathbb{1} + 2T_A(K) + R_A(K).\end{aligned}$$

Now replace K by γK. Q.E.D.

Proposition 4 is not necessary for what follows, but we mention it because it almost, but not quite, implies the following proposition, P: $R_A(K) \geqslant T_A(K)^2$. It does imply that $(\Psi, [R_A(K) - T_A(K)^2]\Psi) \geqslant 0$ when Ψ is any eigenvector of $T_A(K)$. P is false, however, as may be seen by the two-dimensional example: $A = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a > b > 0$ and $K = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Theorem 6 would be trivial to prove if P were true, and, in some sense, it may be taken as a substitute for P.

In order to prove Theorem 6, we need some preliminary remarks and Lemma 5. The remarks are that

$$Q(A, K) = \operatorname{Tr} K^\dagger T_A(K). \tag{3.10}$$

Also, $Q(A + xB, K)$ is differentiable near $x = 0$ for all $B \in \mathscr{B}^s(H)$ and

$$(d/dx)\, Q(A + xB, K)\,|_{x=0} = -\operatorname{Tr} BR_A(K) \tag{3.11}$$

for $K \in \mathscr{B}^s(H)$.

Lemma 5. *Let $\mathscr{C}$ be a convex cone in a vector space and let $F\colon \mathscr{C} \to \mathbb{R}$ be a convex function which is also right differentiable in the sense that*

$$\lim_{x \downarrow 0} x^{-1}\{F(A + xB) - F(A)\} \equiv G(A, B)$$

exists for all $A, B \in \mathscr{C}$. *Assume, also, that* F *is homogeneous of order* 1, *i.e.,* $F(\lambda A) = \lambda F(A)$ *for* $\lambda > 0$. *Then* $G(A, B) \leqslant F(B)$. *Conversely, if* F *is two-sided differentiable in the foregoing sense (with equal left and right derivatives), if* $G(A + xB, B)$ *is measurable on* $\{x \mid x \geqslant 0\}$, *if* $G(A, B) \leqslant F(B)$ *and if* F *is homogeneous of order* 1, *then* F *is convex.*

Proof. For all $x > 0$,

$$\begin{aligned} F(A + xB) &= F((1 + x)[(1 + x)^{-1} A + x(1 + x)^{-1} B]) \\ &= (1 + x) F((1 + x)^{-1} A + x(1 + x)^{-1} B) \\ &\leqslant (1 + x)\{(1 + x)^{-1} F(A) + x(1 + x)^{-1} F(B)\} \\ &= F(A) + xF(B). \end{aligned}$$

Subtract $F(A)$ from both sides, divide by x, and take the limit $x \downarrow 0$. For the second part, let $C = \lambda A + (1 - \lambda)B$, and note that

$$\begin{aligned} F(C) - \lambda F(A) &= F(C) - F(\lambda A) \\ &= \int_0^{1-\lambda} dx\, dF(\lambda A + xB)/dx = \int_0^{1-\lambda} dx\, G(A + xB, B) \\ &\leqslant \int_0^{1-\lambda} dx\, F(B) = (1 - \lambda) F(B). \end{aligned}$$

Q.E.D.

We shall return again to this lemma, but for now we note that since $Q(A, K)$ is homogeneous of order 1 on the convex cone $\mathscr{B}^{++}(H) \times \mathscr{B}^{s}(H)$ then, using (3.10) and (3.11),

$$-\operatorname{Tr} BR_A(K) + 2 \operatorname{Tr} MT_A(K) \leqslant \operatorname{Tr} MT_B(M) \tag{3.12}$$

for $A, B \in \mathscr{B}^{++}(H)$ and $K, M \in \mathscr{B}^{s}(H)$.

THEOREM 6. *Let* $L \in \mathscr{B}^{s}(H)$ *be fixed. Then the function from* $\mathscr{B}^{++}(H)$ *to the nonnegative reals, defined by*

$$A \mapsto F_L(A) = \operatorname{Tr} \exp(L + \ln A),$$

is concave on $\mathscr{B}^{++}(H)$ *for all* L.

Proof. Choose $A \in \mathscr{B}^{++}(H)$ and $K \in \mathscr{B}^{s}(H)$ and consider the function $f(x) = \operatorname{Tr} \exp(L + \ln(A + xK))$ which is defined and differentiable for the real variable x in some neighborhood of $\{0\}$. The theorem is equivalent to the statement that $d^2f/dx^2 \leqslant 0$ when $x = 0$ for all choices of A, L, and K. Using the fact (which can be proved by a power series expansion, for example) that

$$\frac{d}{dx} e^{F+xG} |_{x=0} = \int_0^1 dy \, e^{yF} G e^{(1-y)F} = T^{-1}_{\exp(F)}(G)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} |_{x=0} &= -\operatorname{Tr} BR_A(K) + \operatorname{Tr} \int_0^1 dy \, T_A(K) B^y T_A(K) B^{(1-y)} \\ &= -\operatorname{Tr} BR_A(K) + \operatorname{Tr} T_A(K) T_B^{-1}[T_A(K)], \end{aligned} \tag{3.13}$$

where $B = \exp(L + \ln A)$. Now use inequality (3.12) with $M = T_B^{-1}[T_A(K)]$. Q.E.D.

COROLLARY 6.1. *Let k be a positive integer and $p_1, ..., p_k$ positive real numbers with $p_1 + \cdots + p_k = s \leqslant 1$. Let $L \in \mathscr{B}^s(H)$ and $q \in \mathbb{R}$ be fixed. Then the functions from $\mathscr{B}^{++}(H)^k$ to the positive reals defined by*

$$F_q(A_1, ..., A_k) = \left\{ \operatorname{Tr} \exp \left[L + \sum_{j=1}^{k} p_j \ln(A_j) \right] \right\}^q$$

(1) *are jointly concave in $(A_1, ..., A_k)$ when $0 < q \leqslant s^{-1}$ and*

(2) *are jointly convex in $(A_1, ..., A_k)$ when $q < 0$.*

Proof. F_1 is homogeneous of order s and we have already explained in Corollaries 1.2 and 2.1 how to treat the cases $q \neq 1$. That F_1 is jointly concave seems like a stronger result than Theorem 6, but, surprisingly, it is not. We have to show that for every choice of $A_1, ..., A_k \in \mathscr{B}^{++}(H)$ and $K_1, ..., K_k \in \mathscr{B}^s(H)$, the k-square second-derivative matrix of $f(x_1, ..., x_k) \equiv F(A_1 + x_1K_1, ..., A_k + x_kK_k)$ is positive semidefinite when $x_1 = \cdots = x_k = 0$. If $\partial^2/(\partial x_i \, \partial x_j) f(x_1, ..., x_k)|_{x_1 = \cdots = x_k = 0} \equiv m_{ij} \equiv g_{ij} + \delta_{ij} h_i$, we compute

$$g_{ij} = p_i p_j \operatorname{Tr} T_B^{-1}[T_{A_i}(K_i)] \, T_{A_j}(K_j),$$

$$h_i = -p_i \operatorname{Tr} BR_{A_i}(K_i) \leqslant 0,$$

where $B = \exp[L + \sum_{j=1}^k p_j \ln(A_j)]$. Clearly, for any F, $G \in \mathscr{B}(H)$, $\operatorname{Tr} F T_B^{-1}(G) = \operatorname{Tr} G T_B^{-1}(F)$, and $\operatorname{Tr} F^\dagger T_B^{-1}(F) \geqslant 0$ so, by a Schwarz inequality argument, $|\operatorname{Tr} F^\dagger T_B^{-1}(G)|^2 \leqslant [\operatorname{Tr} F^\dagger T_B^{-1}(F)][\operatorname{Tr} G^\dagger T_B^{-1}(G)]$.

Hence, $|g_{ij}|^2 \leqslant g_{ii}g_{jj}$ and $g_{ii} \leqslant -p_i h_i$ by Theorem 6. Thus, for any $\Psi \in \mathbb{C}^k$,

$$\sum_{i=1}^{k} \sum_{j=1}^{k} \bar{\Psi}_i m_{ij} \Psi_j \leqslant \left\{\sum_{i=1}^{k} (-p_i h_i)^{1/2} \,|\, \Psi_i \,|\right\}^2 + \sum_{i=1}^{k} h_i \,|\, \Psi_i \,|^2$$

$$\leqslant (1-s) \sum_{i=1}^{k} h_i \,|\, \Psi_i \,|^2 \leqslant 0$$

since $h_i \leqslant 0$, all i. Q.E.D.

COROLLARY 6.2. *Let $p_1, \ldots, p_k$ be as in Corollary 6.1, and let $\Psi \in H$ with $\|\Psi\| = 1$. Then*

$$(A_1, \ldots, A_k) \mapsto \exp\left[\sum_{j=1}^{k} p_j(\Psi, \ln A_j \Psi)\right]$$

is a concave function on $\mathscr{B}^{++}(H)^k$.

Proof. Let P be the projection onto the one-dimensional subspace of H spanned by Ψ. Then take the limit $e^L \to P$ in Corollary 6.1. Q.E.D.

Remark. If we write $B = e^L$ in Theorem 6, then, by the Trotter product formula, $F_L(A) = \lim_{n\to\infty} F_L^{(n)}(A)$ with $F_L^{(n)}(A) = \mathrm{Tr}(B^{1/n}A^{1/n})^n$. Now, $F_L^{(1)}(A)$ is concave in A since it is linear, $F_L^{(2)}(A)$ is concave in A by Theorem 1 and $F_L(A)$ is concave in A by Theorem 6. Hence, we are led to the following conjecture.

Conjecture. For each fixed $B \in \mathscr{B}^+(H)$ and $n \in \mathbb{Z}^+$, the positive function $\mathrm{Tr}(B^{1/n}A^{1/n})^n$ is concave in $A \in \mathscr{B}^+(H)$.

We remind the reader of the Golden–Thompson inequality (GT) [11, 12] which is that

$$\mathrm{Tr}\, e^{A+B} \leqslant \mathrm{Tr}\, e^A e^B \tag{3.14}$$

for $A, B \in \mathscr{B}^s(H)$. This theorem can be extended to the infinite dimensional case [13]. The obvious generalization of GT to three operators in the form $\mathrm{Tr}\, e^{A+B+C} \leqslant \mathrm{Tr}\, e^A e^B e^C$ is false. We shall now show that Theorem 6 does provide a correct generalization of GT to three operators (and, hence, provides an alternative proof of GT itself).

THEOREM 7. *Let $A, B, C \in \mathscr{B}^s(H)$.*

$$\mathrm{Tr}\, e^C T_{\exp(-A)}(e^B) \geqslant \mathrm{Tr}\, e^{A+B+C}.$$

If A commutes with B then

$$\operatorname{Tr} e^C e^A e^B \geqslant \operatorname{Tr} e^{A+B+C}.$$

Proof. Define $\alpha, \beta \in \mathscr{B}^{++}(H)$ by $\alpha = e^{-A}$ and $\beta = e^B$ and define $L \in \mathscr{B}^s(H)$ by $L = A + C$. Since $\alpha \mapsto -\operatorname{Tr} e^{L+\ln\alpha}$ is homogeneous of order 1 and convex on the cone $\mathscr{B}^{++}(H)$, we can use Lemma 5 to deduce that for $\beta \in \mathscr{B}^{++}(H)$

$$\operatorname{Tr} e^{A+B+C} = \operatorname{Tr}^{\,L+\ln\beta} \leqslant (d/dx) \operatorname{Tr} e^{L+\ln(\alpha+x\beta)} \big|_{x=0} = \operatorname{Tr} e^C T_\alpha(\beta).$$

For the last part, we note that

$$T_\alpha(\beta) = \int_0^\infty dx\, (e^{-A} + x\mathbb{1})\, e^B (e^{-A} + x\mathbb{1})^{-1} = e^A e^B \qquad \text{if} \quad [A, B] = 0.$$

Q.E.D.

Remark. An alternative formulation of Theorem 7 is this: Let $A, C, D \in \mathscr{B}^s(H)$. Then

$$\operatorname{Tr} e^C e^D \geqslant \operatorname{Tr} \exp \left[C + A + \ln \int_0^1 ds\, e^{-As} e^D e^{-A(1-s)} \right]$$

which follows from the definition $T_{\exp(-A)}(e^B) \equiv e^D$ and the inversion formula (3.4).

Additional Theorems

The Theorems proved thus far all rely on Theorem 1 in an essential way. We shall now prove some theorems which appear to be similar to Theorems 2, 3, 6, and 7, but which in reality are less complicated because they can be proved directly by elementary methods.

THEOREM 8. *Let $K \in \mathscr{B}(H)$ and $1 \geqslant p > 0$, $1 \geqslant r > 0$ be fixed. Then the function from $\mathscr{B}^{++}(H)$ to the nonnegative reals defined by*

$$A \mapsto F(A) = \operatorname{Tr} A^{-p} K^\dagger A^{-r} K$$

is convex in A.

Proof. It is sufficient to show that for all $L \in \mathscr{B}^s(H)$, $g(x) \equiv F(A + xL)$ has a positive second derivative at $x = 0$. Let

$$A(x) = A + xL, \qquad B_p = (d/dx)\, A(x)^{-p} \big|_{x=0}, \qquad C_p = (d/dx)\, A(x)^p \big|_{x=0},$$

$$D_p = (d^2/dx^2)\, A(x)^{-p} \big|_{x=0}$$

and $E_p = (d^2/dx^2)\, A(x)^p \,|_{x=0}$. Using the fact that $A(x)^p A(x)^{-p} = \mathbb{1}$, we compute that $B_p = -A^{-p} C_p A^{-p}$ and $D_p = 2B_p A^p B_p - A^{-p} E_p A^{-p}$. Hence,

$$\begin{aligned}(d^2/dx^2)\, g(x)\,|_{x=0} = {} & -\operatorname{Tr} E_p A^{-p} K^\dagger A^{-r} K A^{-p} - \operatorname{Tr} E_r A^{-r} K A^{-p} K^\dagger A^{-r} \\ & + \operatorname{Tr}(\gamma\delta_p + \delta_r\gamma)(\gamma\delta_p + \delta_r\gamma)^\dagger + \operatorname{Tr}(\gamma\delta_p)(\gamma\delta_p)^\dagger \\ & + \operatorname{Tr}(\delta_r\gamma)(\delta_r\gamma)^\dagger,\end{aligned}$$

where $\gamma = A^{-r/2} K A^{-p/2}$ and $\delta_p = A^{-p/2} C_p A^{-p/2}$. The theorem will be proved if we show that $E_p \leqslant 0$, but this fact follows from the integral representation (valid for $1 > p > 0$)

$$A^p = \pi^{-1} \sin \pi p \int_0^\infty x^{p-1}\, dx\, A(A + x\mathbb{1})^{-1}$$

and (3.8). Q.E.D.

COROLLARY 8.1. *Let $1 \geqslant p > 0$ and $1 \geqslant r > 0$, $p + r \equiv s$ and $q \in \mathbb{R}$, $q \neq 0$ be fixed. Then the functions from $\mathscr{B}^{++}(H) \times \mathscr{B}^{++}(H) \times \mathscr{B}(H)$ to the nonnegative reals, defined by*

$$(A, B, K) \mapsto F_q(A, B, K) = (\operatorname{Tr} A^{-p} K^\dagger B^{-r} K)^q$$

(1) *are convex in K when $q \geqslant \frac{1}{2}$,*

(2) *are jointly convex in (A, B) when $q > 0$, and*

(3) *are jointly concave in (A, B) when $-1/s \leqslant q < 0$.*

Proof. The same as for Corollary 3.1. We note that the degree of homogeneity in K is 2 while in (A, B) it is $-s$. Q.E.D.

Remark. The map $A \mapsto A^{-p}$ is not convex for $p > 1$, but $A \mapsto \operatorname{Tr} A^{-p}$ is convex for $p > 0$. See Theorem 9.

THEOREM 9. *Let k be a positive integer and $p_1, \ldots, p_k$ positive real numbers with $p_1 + \cdots + p_k \equiv s$. Let $L \in \mathscr{B}^s(H)$ and $q \in \mathbb{R}$ be fixed. Then the functions from $\mathscr{B}^{++}(H)^k$ to the positive reals defined by $F_q(A_1, \ldots, A_k) = \{\operatorname{Tr} \exp[L - \sum_{j=1}^k p_j \ln(A_j)]\}^q$*

(1) *are jointly convex in $(A_1, \ldots, A_k)$ when $q \geqslant 0$ and*

(2) *are jointly concave in $(A_1, \ldots, A_k)$ when $-1/s \leqslant q \leqslant 0$.*

Proof. We need only consider the case $q = 1$. The extension to the general case is the same as in Corollary 3.1. When $k = 1$ we define $g(x) = F(A + xK)$, $K \in \mathscr{B}^s(H)$, and compute its second derivative at $x = 0$ to be $p \operatorname{Tr} BR_A(K) + p^2 \operatorname{Tr} T_A(K)\, T_B^{-1}[T_A(K)]$ where $B = \exp(L - p \ln A) > 0$. $R_A(K) \geqslant 0$ and, as remarked in the proof of Corollary 6.1, $\operatorname{Tr} F^\dagger T_B^{-1}(F) \geqslant 0$ for all $F \in \mathscr{B}(H)$. This last fact is the essence of the proof when $k > 1$. The second derivative matrix is positive definite. Q.E.D.

4. Extension of Theorem 6 to Infinite Dimensions

We fix L, which is assumed to be self-adjoint, and $e^L \in \mathscr{I}_1(H)$, which implies that L has purely discrete spectrum. For $A, B \in \mathscr{B}^+(H)$ and $C \equiv \lambda A + (1 - \lambda)B$, $0 < \lambda < 1$, we want to show that

$$\lambda \operatorname{Tr} e^{L+\ln A} + (1 - \lambda) \operatorname{Tr} e^{L+\ln B} \leqslant \operatorname{Tr} e^{L+\ln C}, \tag{4.1}$$

which requires, among other things, giving meaning to these quantities.

Case 1. We first assume that there exist positive numbers ϵ and ω such that $\epsilon\mathbb{1} < A < \omega\mathbb{1}$, $\epsilon\mathbb{1} < B < \omega\mathbb{1}$ so that $\ln A$, etc. can be defined as bounded, self-adjoint operators by means of the spectral representation of A, etc. We define $\alpha = L + \ln A$. Since $\ln A$ is bounded, α is a self-adjoint operator on the domain of L. If we label the eigenvalues of L by $\mu_1(L) \geqslant \mu_2(L) \geqslant \cdots$ we have, by the mini–max principle, that $\mu_k(\alpha) \leqslant \mu_k(L) + \ln \| A \|$ since $\ln A \leqslant \ln \| A \| \mathbb{1}$. The convergence of $\sum_1^\infty \exp(\mu_k(L))$ implies that $\mu_k(L) \to -\infty$, which implies that $\mu_k(\alpha) \to -\infty$, which implies that $\exp(\alpha)$ is compact, and since $t(A) \equiv \operatorname{Tr} e^\alpha \leqslant \| A \| \operatorname{Tr} e^L < \infty$, the trace is finite. Now let P_n be the projection onto R_n, the subspace spanned by the first n eigenvectors of L, and define

$$A_n \equiv P_n A P_n + \epsilon(\mathbb{1} - P_n),$$

$$\alpha_n \equiv L + \ln A_n ,$$

and likewise for B_n and C_n. Clearly,

$$t(A_n) = \operatorname{Tr} \exp(\alpha_n) = \operatorname{Tr} P_n \exp(L + \ln(A \restriction R_n)) + r_n ,$$

where $r_n = (\ln \epsilon) \operatorname{Tr}(\mathbb{1} - P_n)\, e^L$. From this we see that not only is $t(A_n)$ finite, but that the terms involving r_n cancel from both sides of (4.1) leaving an inequality about traces on the finite dimensional space R_n. Thus, by Theorem 6, $\lambda t(A_n) + (1-\lambda)\, t(B_n) \leqslant t(C_n)$. All we have to do is prove that $t(A_n) \to t(A)$, etc. Now $A_n < \omega\mathbb{1}$ and $\alpha_n < L + \mathbb{1} \ln \omega$. Since $\mu_k(\alpha_n) \leqslant \mu_k(L) + \ln \omega$, $t(A_n) \to t(A)$ by the dominated convergence theorem if we can show that $\mu_k(\alpha_n) \to \mu_k(\alpha)$ for each k. As L is bounded above, we can find a constant, d, such that $\alpha_n < (d-1)\mathbb{1}$ and $\alpha < (d-1)\mathbb{1}$. Define

$$G_n = (\alpha_n - d\mathbb{1})^{-1} - (\alpha - d\mathbb{1})^{-1} = (\alpha_n - d\mathbb{1})^{-1}(\alpha - \alpha_n)(\alpha - d\mathbb{1})^{-1}$$

and note that $(\alpha_n - d\mathbb{1})^{-1}$ is uniformly bounded, $\alpha_n - \alpha \to 0$ strongly, and $(\alpha - d\mathbb{1})^{-1}$ is compact since e^L is compact. Hence, $\| G_n \| \to 0$. In general, $| \mu_k(A) - \mu_k(B)| \leqslant \| A - B \|$ by the mini–max principle. Thus, $\mu_k(\alpha_n) \to \mu_k(\alpha)$ since $\mu_k((\alpha - d\mathbb{1})^{-1}) = [\mu_k(\alpha) - d]^{-1}$, and the theorem is proved for Case 1.

Case 2. $0 < A < \omega\mathbb{1}$, $0 < B < \omega\mathbb{1}$. If we replace A by $A_\epsilon = A + \epsilon\mathbb{1}$ and B by $B_\epsilon = B + \epsilon\mathbb{1}$, then C is replaced by

$$\lambda(A + \epsilon\mathbb{1}) + (1-\lambda)(B + \epsilon\mathbb{1}) = C + \epsilon\mathbb{1}.$$

Also, A_ϵ, etc. are decreasing in ϵ. Thus, we can *define*

$$\operatorname{Tr} \exp(L + \ln A) \equiv \lim_{\epsilon \downarrow 0} \operatorname{Tr} \exp(L + \ln A_\epsilon) \tag{4.2}$$

because the right trace decreases as ϵ decreases. Theorem 6 is true with this definition because it is true for every $\epsilon > 0$. The usefulness of this definition stems from the following fact (B. Simon, private communication): $\ln A$ is defined as an unbounded self-adjoint operator in a natural way by its spectral decomposition. If $D(\ln A) \cap D(L)$ is dense in H (where $D(\cdot)$ means domain), and if we define $-\alpha$ to be the Friedrichs extension of $-L - \ln A$, then $\operatorname{Tr} e^\alpha = \lim_{\epsilon\downarrow 0} \exp(L + \ln A_\epsilon)$.

Finally, another case to which Theorem 6 can be extended (B. Simon, private communication) is where A and B are positive, not necessarily bounded, self-adjoint operators and $\ln A$ and $\ln B$ are form bounded perturbations of $-L$.

5. On the "Equivalence" of Theorems 1, 2, 3, 6, and 7

Although Theorem 1 is our main theorem and Theorems 2, 3, 6, and 7 appear to be corollaries of it, the fact is that if we could find an independent proof of any of Theorems 2, 3, 6, or 7 the others could be derived from it in a simple way when the Hilbert space is finite dimensional. Thus, in some sense, all five theorems have equal content. We shall indicate very briefly some of the links among these theorems and the reader can easily supply the missing details as well as establish additional connections.

We have previously established the implications $1 \Rightarrow 2$ and $1 \Rightarrow 3 \Rightarrow 6 \Rightarrow 7$.

(a) $2 \Rightarrow 1$. This is easily seen simply by reading the proof of Theorem 2 backwards, i.e., one maximizes the variational quotient N/D with respect to K, where $N = \lambda \operatorname{Tr} A^r K^\dagger A^p K + (1 - \lambda) \operatorname{Tr} B^r K^\dagger B^p K$ and $D = \operatorname{Tr} C^r K^\dagger C^p K$.

(b) $7 \Rightarrow 6$. Theorem 7 was derived from Theorem 6 by using the derivative inequality, Lemma 5. However, we can retrace our steps by using the second part of Lemma 5.

(c) $6 \Rightarrow (3.12)$. If, in (3.12), we fix A, B, K and minimize $\operatorname{Tr} M[T_B(M) - 2T_A(K)]$ with respect to M, we find $M = T_B^{-1}(T_A(K))$. Hence, (3.12) is equivalent to the fact that the right side of (3.13) is negative and this, in term, is equivalent to Theorem 6.

(d) $(3.12) \Rightarrow 3$. The same remark as in (b) suffices.

(e) $3 \Rightarrow 2$. Take 3 in the form of Corollary 3.1 with $q = 1$ and assume that $p + r \equiv s < 1$. For A (resp. B) substitute $\lambda A + \mathbb{1}$ (resp. $\gamma A + \mathbb{1}$), where $\lambda, \gamma \geqslant 0$. Then integrate

$$\int_0^\infty d\lambda\, \lambda^{p-1} \int_0^\infty d\gamma\, \gamma^{r-1} Q_1(\lambda A + \mathbb{1}, \gamma A + \mathbb{1}, K)$$

to obtain a positive constant times $F(A, K)$ of (3.1). When $p + r = 1$ we can appeal to continuity.

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Note added in proof. After this paper was submitted for publication, H. Epstein [14] found alternative proofs of several of these theorems and a proof of the conjecture given after Corollary 6.2.

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