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A sharp growth condition for a fast escaping spider's web

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Abstract

We show that the fast escaping set $A(f)$ of a transcendental entire function f has a structure known as a spider's web whenever the maximum modulus of f grows below a certain rate. The proof uses a new local version of the $\cos \pi\rho$ theorem, based on a comparatively unknown result of Beurling. We also give examples of entire functions for which the fast escaping set is not a spider's web which show that this growth rate is sharp. These are the first examples for which the escaping set has a spider's web structure but the fast escaping set does not. Our results give new insight into possible approaches to proving a conjecture of Baker, and also a conjecture of Eremenko.

MSC: primary 37F10; secondary 30D05

Keywords: Fast escaping set; Spiders' webs; $\cos \pi\rho$ theorem

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and denote by f^n, $n = 0, 1, 2, \ldots,$ the nth iterate of f. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$ of f. An introduction to the properties of these sets can be found in [3].

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In recent years, the escaping set defined by

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

has come to play an increasingly significant role in the study of the iteration of transcendental entire functions with much of the research being motivated by a conjecture of Eremenko [5] that all the components of the escaping set are unbounded. For partial results on this conjecture see, for example, [9,15].

The most general result on Eremenko's conjecture was obtained in [10] where it was proved that the escaping set always has at least one unbounded component. This result was proved by considering the fast escaping set $A(f) = \bigcup_{n\in\mathbb{N}} f^{-n}(A_R(f))$, where

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

Here

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|,$$

$M^n(r, f)$ denotes the nth iterate of M with respect to r, and $R > 0$ is chosen so that $M(r, f) > r$ for $r \geq R$. The set $A(f)$ has many nice properties including the fact that all its components are unbounded—these properties are described in detail in [12].

There are many classes of transcendental entire functions for which the fast escaping set has the structure of a spider's web—see [12,8,16]. We say that a set E has this structure if E is connected and there exists a sequence of bounded simply connected domains G_n such that

$$\partial G_n \subset E, \qquad G_n \subset G_{n+1}, \quad \text{for } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}.$$

As shown in [12], if $A_R(f)$ has this structure then so do both $A(f)$ and $I(f)$, and hence Eremenko's conjecture is satisfied. Also, the domains G_n can be chosen so that $\partial G_n \subset A_R(f) \cap J(f)$ and so f has no unbounded Fatou components. This gives a surprising link between Eremenko's conjecture and a conjecture of Baker [1] that all the components of the Fatou set are bounded if f is a transcendental entire function of order less than $1/2$. Recall that the *order* of a transcendental entire function f is defined to be

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

For background and recent results on Baker's conjecture, see [6,7,11,13]. It was shown in [11] (see also [12]) that the techniques used to prove all earlier partial results on Baker's conjecture can in fact be used to prove the stronger result that $A_R(f)$ is a spider's web.

In this paper we show the limitation of these techniques, and demonstrate that they cannot even be used to prove Baker's conjecture for all functions of order zero. To do this we give a *sharp* condition on the growth of the maximum modulus that is sufficient to imply that $A_R(f)$ is a spider's web and hence that Baker's conjecture and Eremenko's conjecture are both satisfied. More precisely, we prove the following sufficient condition.

Theorem 1.1. *Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$. Let*

$$R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log\log M(r)}{\log r}.$$

If

$$\sum_{n\in\mathbb{N}} \varepsilon_n < \infty, \tag{1.1}$$

then $A_R(f)$ *is a spider's web.*

Remark. Theorem 1.1 follows surprisingly easily from a new local version of the classical $\cos \pi\rho$ theorem; see Theorem 2.1. We obtained a closely related result in [11, Theorem 3] with the stronger hypothesis that $\sum_{n\in\mathbb{N}} \sqrt{\varepsilon_n} < \infty$ and remarked there that the square root could be removed by introducing a more sophisticated argument. The method of proof given here is quite different, and more enlightening.

As mentioned earlier, the condition (1.1) in Theorem 1.1 is, in a strong sense, best possible. In particular, the following result shows that (1.1) cannot be replaced by the weaker condition that $\sum_{n\in\mathbb{N}}(\varepsilon_n)^c < \infty$, for some $c > 1$.

Theorem 1.2. *There exist transcendental entire functions of the form*

$$f(z) = z^3 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{2p_n}, \tag{1.2}$$

where $p_n \in \mathbb{N}$, *for* $n \in \mathbb{N}$, *and the sequence* (a_n) *is positive and strictly increasing such that* $A(f) \cap (-\infty, 0] = \emptyset$; *in particular,* $A(f)$ *is not a spider's web.*

Moreover, if (δ_n) *is a positive sequence such that*

$$\sum_{n\in\mathbb{N}} \delta_n = \infty,$$

then we can choose the sequence $(a_n)_{n\in\mathbb{N}}$ *and a value* $R > 0$ *in such a way that, with*

$$p_n = [a_n^{\delta_n/4}/4], \qquad R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \le r \le R_{n+1}} \frac{\log\log M(r)}{\log r},$$

there exists a subsequence (n_k) *such that*

$$\varepsilon_{n_k} \le \delta_k + \frac{1}{2^{n_k}}, \quad \text{for } k \in \mathbb{N}, \tag{1.3}$$

and

$$\varepsilon_{n_k+m} \le \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_k+m}}, \quad \text{for } k \in \mathbb{N},\ 1 \le m < n_{k+1} - n_k. \tag{1.4}$$

To deduce from Theorem 1.2 that the condition (1.1) in Theorem 1.1 cannot be relaxed to $\sum_{n\in\mathbb{N}}(\varepsilon_n)^c < \infty$, for some $c > 1$, it suffices to take a positive sequence (δ_n) such that $\sum_{n\in\mathbb{N}} \delta_n = \infty$ but $\sum_{n\in\mathbb{N}} \delta_n^c < \infty$, and so obtain an entire function f for which $A(f)$ is not a spider's web but $\sum_{n\in\mathbb{N}} \varepsilon_n^c < \infty$, by (1.3) and (1.4) and the estimate $(a+b)^c \le 2^c \max\{a^c, b^c\}$.

Although the proof of Theorem 1.2 is rather complicated, this result has several significant consequences in addition to showing that the condition in Theorem 1.1 is best possible. First, Theorem 1.2 implies that there are functions of order zero for which $A_R(f)$ fails to be a spider's web and so new techniques are needed in order to solve Baker's conjecture. One such technique is introduced in [13] where we show that all real functions of order less than 1/2 with their zeros on the negative real axis satisfy Baker's conjecture.

Second, in [13] we show that all real functions of order less than $1/2$ with their zeros on the negative real axis also satisfy Eremenko's conjecture and, moreover, $I(f)$ is a spider's web. Since functions of the form (1.2) with $\limsup_{n\to\infty} \varepsilon_n < 1/2$ are of this type, this gives the following corollary to Theorem 1.2, which answers a question in [12].

Corollary 1.3. *There exist transcendental entire functions for which $I(f)$ is a spider's web but $A(f)$ is not a spider's web.*

In fact we show in [13] that real functions of order less than $1/2$ with their zeros on the negative real axis have the stronger property that $Q(f)$ contains a spider's web, where $Q(f)$ is the *quite fast escaping set* which is described in more detail in [14] and satisfies $A(f) \subset Q(f) \subset I(f)$. Thus Theorem 1.2 provides examples of functions for which $Q(f) \neq A(f)$; these two sets are equal for many functions, including all functions in the Eremenko–Lyubich class $\mathcal{B}$ as we show in [14].

The paper is arranged as follows. In Section 2 we prove Theorem 1.1 and then, in Section 3, we prove Theorem 1.2.

2. Proof of Theorem 1.1

Let f be a transcendental entire function and $R > 0$ be such that $M(r) > r$ for $r \geq R$. Recall that

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\}$$

and that $A_R(f)$ is a spider's web if $A_R(f)$ is connected and there exists a sequence of bounded simply connected domains G_n such that

$$\partial G_n \subset A_R(f), \qquad G_n \subset G_{n+1}, \quad \text{for } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}.$$

In this section we prove Theorem 1.1 which gives a condition that is sufficient to ensure that $A_R(f)$ is a spider's web. Many earlier results on Baker's conjecture as well as sufficient conditions to ensure that $A_R(f)$ is a spider's web were obtained by using a version of the classical $\cos \pi\rho$ theorem proved by Barry [2, p. 294]. The key ingredient in our proof is the following result which can be viewed as a local version of Barry's theorem. The estimate (2.2) gives more precise information than the corresponding estimate that can be obtained from Barry's theorem and the proof follows easily from a comparatively unknown result of Beurling given in his thesis [4]. This latter result turns out to have other applications; see [14], for example.

Theorem 2.1. *Let f be a transcendental entire function. There exists $r(f) > 0$ such that, if*

$$\log M(r) \leq r^{\alpha} \quad \text{and} \quad r^{1-2\alpha} \geq r(f), \tag{2.1}$$

for some $\alpha \in (0, 1/2)$, then there exists $t \in (r^{1-2\alpha}, r)$ such that

$$\log m(t) > \log M(r^{1-2\alpha}) - 2. \tag{2.2}$$

Proof. We apply the following result of Beurling [4, p. 96].

Let f be analytic in $\{z : |z| < r_0\}$, let $0 \leq r_1 < r_2 < r_0$, and put

$$E = \{t \in (r_1, r_2) : m(t) \leq \mu\}, \quad \text{where } 0 < \mu < M(r_1).$$

Then

$$\log \frac{M(r_2)}{\mu} > \frac{1}{2} \exp\left(\frac{1}{2}\int_E \frac{dt}{t}\right) \log \frac{M(r_1)}{\mu}. \tag{2.3}$$

Taking $r_2 = r, r_1 = r^{1-2\alpha}, \mu = M(r^{1-2\alpha})/e^2$, and $r(f) > 0$ such that $M(r(f)) \geq e^2$, we deduce from (2.1) and (2.3) that, if $m(t) \leq \mu$ for $t \in (r^{1-2\alpha}, r)$, then

$$r^\alpha \geq \log M(r) \geq \log \frac{M(r)}{\mu} > \frac{1}{2} \exp\left(\frac{1}{2}\int_{r^{1-2\alpha}}^{r} \frac{dt}{t}\right) \log \frac{M(r^{1-2\alpha})}{\mu} = r^\alpha.$$

This is a contradiction and so there must exist $t \in (r^{1-2\alpha}, r)$ such that $m(t) > \mu$; that is,

$$\log m(t) > \log \mu = \log M(r^{1-2\alpha}) - 2,$$

as required. □

We also use the following results about spiders' webs proved in [12].

Lemma 2.2 (*[12, Corollary 8.2]*)**.** *Let f be a transcendental entire function and let $R > 0$ be such that $M(r) > r$ for $r \geq R$. Then $A_R(f)$ is a spider's web if there exists a sequence (ρ_n) such that, for $n \geq 0$,*

$$\rho_n > M^n(R) \tag{2.4}$$

and

$$m(\rho_n) \geq \rho_{n+1}. \tag{2.5}$$

Lemma 2.3 (*[12, Lemma 7.1(d)]*)**.** *Let f be a transcendental entire function, let $R > 0$ be such that $M(r) > r$ for $r \geq R$, and let $R' > R$. Then $A_R(f)$ is a spider's web if and only if $A_{R'}(f)$ is a spider's web.*

In addition, we need the following property of the maximum modulus function, which was proved in this form in [11].

Lemma 2.4 (*[11, Lemma 2.2]*)**.** *Let f be a transcendental entire function. Then there exists $R > 0$ such that, for all $r \geq R$ and all $c > 1$,*

$$M(r^c) \geq M(r)^c.$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $R > 0$ be such that, for $r \geq R$, Lemma 2.4 holds and $M(r) > r$. For $n \in \mathbb{N}$, let

$$R_n = M^n(R) \quad \text{and} \quad \varepsilon_n = \max_{R_n \leq r \leq R_{n+1}} \frac{\log\log M(r)}{\log r}.$$

Suppose that $\sum_{n\in\mathbb{N}} \varepsilon_n < \infty$. Then we can take N sufficiently large to ensure that

$$\sum_{n \geq N} \varepsilon_n < \frac{1}{8}, \tag{2.6}$$

and

$$M(R_n)^{1/(8n^2)} = R_{n+1}^{1/(8n^2)} \geq e^2, \quad \text{for } n \geq N, \quad \text{and} \quad R_{N+1}^{1/4} \geq R_N \geq r(f), \tag{2.7}$$

where $r(f)$ is as defined in Theorem 2.1. Note that (2.7) is possible since $\log M(r)/\log r \to \infty$ and so, for large n, we have $\log R_{n+1} > 4 \log R_n$.

Now let

$$r_n = M^{n+1}\left(R_{N+1}^{\prod_{m=N}^{N+n}(1-2\varepsilon_m-1/(8m^2))}\right), \quad \text{for } n \geq 0.$$

We note that, for $n \geq 0$, it follows from (2.6) that

$$\prod_{m=N}^{N+n}\left(1 - 2\varepsilon_m - \frac{1}{8m^2}\right) > 1 - \sum_{m=N}^{N+n} 2\varepsilon_m - \sum_{m=N}^{N+n} \frac{1}{8m^2} \geq \frac{1}{2}$$

and so, by (2.7) and Lemma 2.4,

$$R_{N+n+2} > r_n > M^{n+1}(R_{N+1}^{1/2}) \geq M^{n+1}(R_N^2) \geq R_{N+n+1}^2.$$

We claim that, for $n \geq 0$, there exists $\rho_n \in (R_{N+n+1}, r_n)$ with $m(\rho_n) > r_{n+1}$. Indeed, it follows from Theorem 2.1, (2.6), (2.7) and Lemma 2.4 that, for $n \geq 0$, there exists $\rho_n \in (r_n^{1-2\varepsilon_{n+N+1}}, r_n) \subset (R_{N+n+1}, r_n)$ such that

$$\begin{aligned}
m(\rho_n) &\geq \frac{1}{e^2} M(r_n^{1-2\varepsilon_{n+N+1}}) \\
&\geq M(r_n^{1-2\varepsilon_{n+N+1}})^{1-1/(8(n+N+1)^2)} \\
&\geq M(r_n^{(1-2\varepsilon_{n+N+1})(1-1/(8(n+N+1)^2))}) \\
&\geq M(r_n^{(1-2\varepsilon_{n+N+1}-1/(8(n+N+1)^2))}) \\
&= M\left(\left(M^{n+1}\left(R_{N+1}^{\prod_{m=N}^{N+n}(1-2\varepsilon_m-1/(8m^2))}\right)\right)^{(1-2\varepsilon_{n+N+1}-1/(8(n+N+1)^2))}\right) \\
&\geq M^{n+2}\left(R_{N+1}^{\prod_{m=N}^{N+n+1}(1-2\varepsilon_m-1/(8m^2))}\right) \\
&= r_{n+1}.
\end{aligned}$$

Thus, for $n \geq 0$, there exists $\rho_n > R_{N+n}$ with $m(\rho_n) \geq \rho_{n+1}$ and so, by Lemma 2.2, $A_{R_{N+1}}(f)$ is a spider's web. It now follows from Lemma 2.3 that $A_R(f)$ is a spider's web as claimed. □

3. Proof of Theorem 1.2

Let

$$f(z) = z^3 \prod_{n=1}^{\infty}\left(1 + \frac{z}{a_n}\right)^{2p_n},$$

where the sequence (a_n) is positive and strictly increasing. In addition, let (δ_n) be a positive sequence such that

$$\sum_{n\in\mathbb{N}} \delta_n = \infty,$$

and let

$$p_n = [a_n^{\delta_n/4}/4]. \tag{3.1}$$

Without loss of generality, we assume that

$$\delta_n < 1/2, \quad \text{for } n \in \mathbb{N}. \tag{3.2}$$

Note that $f((-\infty, 0]) \subset (-\infty, 0]$ and that $m(r) = -f(-r)$ and $M(r) = f(r) > r^3$, for $r > 0$. Further, $M(r) > r$ for $r \geq 1$.

We first show that the sequence (a_n) can be chosen so that $A(f) \cap (-\infty, 0] = \emptyset$.

We choose the values of a_n carefully, beginning with a_1, then a_2 and so on. Because of the way in which we choose the values of a_n, it is helpful to introduce the function g defined by

$$g(r) = \begin{cases} r^3, & 0 \leq r < a_1, \\ r^3 \displaystyle\prod_{a_n \leq r} \left(1 + \frac{r}{a_n}\right)^{2p_n}, & r \geq a_1. \end{cases} \tag{3.3}$$

Note that g is a strictly increasing function and that it is discontinuous at a_n, for $n \in \mathbb{N}$. A key property of g which we use repeatedly is that

$$m(r) = -f(-r) < g(r) < M(r), \quad \text{for } r \geq 0. \tag{3.4}$$

Since g is increasing, (3.4) implies that

$$f([-r, 0]) \subset [-g(r), 0], \quad \text{for } r \geq 0. \tag{3.5}$$

We now set $r_0 = 10$ and $r_{n+1} = g(r_n) = g^{n+1}(10)$, for $n \in \mathbb{N}$, and note that

$$r_{n+1} \geq r_n^3, \quad \text{for } n \geq 0. \tag{3.6}$$

Also, it follows from (3.5) that

$$f^n((-r_m, 0]) \subset (-r_{m+n}, 0], \quad \text{for } n, m \in \mathbb{N}. \tag{3.7}$$

We begin by proving the following result.

Lemma 3.1. *If there exists a sequence (N_k) such that,*

$$f^{N_1}((-r_2, 0]) \subset (-r_{N_1}, 0] \tag{3.8}$$

and, for $k \geq 2$,

$$f^{N_k}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\cdots+N_k}, 0], \tag{3.9}$$

then $A(f) \cap (-\infty, 0] = \emptyset$.

Proof. We first note that, if the hypotheses of Lemma 3.1 hold, then it follows from (3.7) and (3.9) that, for $k \in \mathbb{N}$,

$$\begin{aligned} f^{N_1+\cdots+N_k}((-r_{2k}, 0]) &= f^{N_k}(f^{N_1+\cdots+N_{k-1}})((-r_{2k}, 0]) \\ &\subset f^{N_k}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \\ &\subset (-r_{N_1+\cdots+N_k}, 0]. \end{aligned}$$

Thus

$$f^{N_1+\cdots+N_k}((-r_{2k},0]) \subset (-r_{N_1+\cdots+N_k},0]. \tag{3.10}$$

Now let $z \in (-\infty,0]$. There exists $K \in \mathbb{N}$ such that, for $k \geq K$, we have $z \in (-r_k,0]$ and hence, by (3.7), we have $f^k(z) \in (-r_{2k},0]$. Thus, by (3.10) and (3.4), for $k \geq K$,

$$|f^{N_1+\cdots+N_k+k}(z)| < r_{N_1+\cdots+N_k} < M^{N_1+\cdots+N_k}(10)$$

and hence

$$z \notin \{z : |f^{n+k}(z)| \geq M^n(10) \text{ for } n \in \mathbb{N}\}.$$

Thus $A(f) \cap (-\infty,0] = \emptyset$ as required. □

We will show that we can choose the values of a_n in such a way that the hypotheses of Lemma 3.1 hold. In order to do this, it is helpful to set certain restrictions on our choice of values. First, we choose a_1 and a_{n+1}/a_n, $n \in \mathbb{N}$, sufficiently large to ensure that

$$a_1^{\delta_1/4} \geq 4, \qquad a_{n+1} > a_n^2, \qquad a_{n+1}^{\delta_{n+1}/2} > 16 a_n^{\delta_n} \tag{3.11}$$

and

$$a_{n+1}^{\delta_{n+1}/16} > a_n^{\delta_n} \log a_{n+1}. \tag{3.12}$$

We note that (3.11) implies that

$$p_1 \geq 1 \quad \text{and} \quad p_{n+1} \geq 2p_n^2, \quad \text{for } n \in \mathbb{N}. \tag{3.13}$$

We also place certain restrictions on our choice of the values of a_n in relation to the values of r_n:

$$\text{if } a_k \in [r_n, r_{n+1}), \text{ then } a_m \notin [r_n, r_{n+4}) \quad \text{for } k, m \in \mathbb{N},\ m \neq k. \tag{3.14}$$

We now show that, in order to prove that the hypotheses of Lemma 3.1 hold, it is sufficient to prove the following result.

Lemma 3.2. *Suppose that, for some $m \in \mathbb{N}$, we have defined the values of a_n for which $a_n \leq r_m$ in such a way that they satisfy (3.11), (3.12) and (3.14). Then we can choose $N \in \mathbb{N}$ and the values of a_n for which $r_m < a_n \leq r_{m+N-1}$ in such a way that they satisfy (3.11), (3.12) and (3.14) and, no matter how the later values of a_n are chosen,*

$$f^N((-r_{m+1},0]) \subset (-r_{m+N},0].$$

Proving Lemma 3.2 is the key part of the proof that we can choose the sequence (a_n) so as to ensure that $A(f) \cap (-\infty,0] = \emptyset$. Before proving Lemma 3.2, we show that, if this result holds, then the hypotheses of Lemma 3.1 also hold. First, by applying Lemma 3.2 when $m = 1$ we see that there exists $N_{1,1} \in \mathbb{N}$ and a choice of a_n for $r_1 < a_n \leq r_{N_{1,1}}$ such that

$$f^{N_{1,1}}((-r_2,0]) \subset (-r_{N_{1,1}+1},0]. \tag{3.15}$$

We then apply Lemma 3.2 with $m = N_{1,1}$ and deduce that there exists $N_{1,2} \in \mathbb{N}$ and a choice of a_n for $r_{N_{1,1}} < a_n \leq r_{N_{1,1}+N_{1,2}-1}$ such that

$$f^{N_{1,2}}((-r_{N_{1,1}+1},0]) \subset (-r_{N_{1,1}+N_{1,2}},0]. \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$f^{N_{1,1}+N_{1,2}}(-r_2, 0] \subset f^{N_{1,2}}(-r_{N_{1,1}+1}, 0] \subset (-r_{N_{1,1}+N_{1,2}}, 0].$$

Putting $N_1 = N_{1,1} + N_{1,2}$, we deduce that we can choose the values of a_n for which $r_1 < a_n \le r_{N_1-1}$ in such a way that

$$f^{N_1}((-r_2, 0]) \subset (-r_{N_1}, 0].$$

Thus (3.8) holds.

Now suppose that, for some $k \ge 2$, we have defined N_j, for $1 \le j \le k-1$, and defined a_n, for $r_1 < a_n \le r_{N_1+\cdots+N_{k-1}-1}$. We claim that we can use Lemma 3.2 to define $N_k \in \mathbb{N}$ and a_n with $r_{N_1+\cdots+N_{k-1}-1} < a_n \le r_{N_1+\cdots+N_k-1}$ such that (3.9) holds for k. The argument is similar to that given above. First, we apply Lemma 3.2 with $m = N_1 + \cdots + N_{k-1} + 2k - 1$ to construct $N_{k,1}$ and a_n with

$$r_{N_1+\cdots+N_{k-1}+2k-1} < a_n \le r_{N_1+\cdots+N_{k-1}+N_{k,1}+2k-2}$$

such that

$$f^{N_{k,1}}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\cdots+N_{k-1}+N_{k,1}+2k-1}, 0].$$

Then, for $2 \le j \le 2k$, we apply Lemma 3.2 repeatedly with

$$m = N_1 + \cdots + N_{k-1} + N_{k,1} + \cdots + N_{k,j-1} + 2k - j$$

to construct $N_{k,j}$ and a_n with

$$r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j-1}+2k-j} < a_n \le r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j}+2k-j-1}$$

such that

$$\begin{aligned} &f^{N_{k,j}}((-r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j-1}+2k-j+1}, 0]) \\ &\quad \subset (-r_{N_1+\cdots+N_{k-1}+N_{k,1}+\cdots+N_{k,j}+2k-j}, 0]. \end{aligned}$$

Putting $N_k = N_{k,1} + \cdots + N_{k,2k}$, we deduce that a_n can be chosen with

$$r_{N_1+\cdots+N_{k-1}-1} < a_n \le r_{N_1+\cdots+N_k-1}$$

such that

$$f^{N_k}((-r_{N_1+\cdots+N_{k-1}+2k}, 0]) \subset (-r_{N_1+\cdots+N_k}, 0]$$

and hence (3.9) holds for k.

So, it remains to prove Lemma 3.2.

We begin by proving four lemmas. The first describes the extent to which f is small close to a zero at $-a_k$, where $k \in \mathbb{N}$.

Lemma 3.3. *For each $k \in \mathbb{N}$,*

$$|f(z)| < 1, \quad \text{for } z \in (-a_k, -a_k^{1-\delta_k/16}). \tag{3.17}$$

Proof. This holds since, for such a z, it follows from (3.1), (3.11), (3.12) and (3.13) that

$$\begin{aligned}|f(z)| &\le a_k^3\left(1-\frac{a_k^{1-\delta_k/16}}{a_k}\right)^{2p_k}\prod_{m=1}^{k-1}\left(1+\frac{a_k}{a_m}\right)^{2p_m}\prod_{m\ge k+1}\left(1+\frac{a_k}{a_m}\right)^{2p_m}\\ &\le \left(1-\frac{1}{a_k^{\delta_k/16}}\right)^{a_k^{\delta_k/4}} a_k^{3+2p_1+\cdots+2p_{k-1}}\prod_{m\ge k+1}\left(1+\frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}}\\ &\le \exp(-a_k^{\delta_k/16})a_k^{a_{k-1}^{\delta_{k-1}/2}}e^{1+1/2+1/4+\cdots}\\ &\le a_k^{a_{k-1}^{\delta_{k-1}}}\exp(-a_k^{\delta_k/16}) < 1. \quad \square\end{aligned}$$

The second lemma shows that there is a large increase in the size of $g(r)$ at $r = a_k$, where $k \in \mathbb{N}$.

Lemma 3.4. *For each* $k \in \mathbb{N}$,

$$\log g(a_k) \ge p_k^{1/2}\log g(a_k^{1-\delta_k/16}).$$

Proof. For $k \in \mathbb{N}$, it follows from (3.11) that

$$\begin{aligned}g(a_k^{1-\delta_k/16}) &< a_k^3\prod_{m=1}^{k-1}\left(1+\frac{a_k}{a_m}\right)^{2p_m}\\ &< a_k^{3+2\sum_{m=1}^{k-1}p_m} \le a_k^{4p_{k-1}}\end{aligned}$$

and

$$g(a_k) \ge 2^{2p_k}.$$

Thus, by (3.11)–(3.13),

$$\frac{\log g(a_k)}{\log g(a_k^{1-\delta_k/16})} \ge \frac{2p_k\log 2}{4p_{k-1}\log a_k} > \frac{p_k}{3p_{k-1}\log a_k} > p_k^{1/2}. \quad \square$$

The third lemma shows that $\log g$ has a certain convexity property.

Lemma 3.5. *Let* $r > 0$ *and* $t \ge 2$. *Then*

$$\log g(r^t) \ge t\log g(r).$$

Proof. Let $r > 0$ and $t \ge 2$. We have

$$g(r^t) \ge r^{3t}\prod_{a_m\le r}\left(1+\frac{r^t}{a_m}\right)^{2p_m}$$

and

$$g(r)^t = r^{3t} \prod_{a_m \le r} \left(1 + \frac{r}{a_m}\right)^{2p_m t}.$$

Thus it is sufficient to show that

$$\left(1 + \frac{r}{a_m}\right)^t \le \left(1 + \frac{r^t}{a_m}\right),$$

when $a_m \le r$. This is true since it follows from (3.11) that, for $a_m \le r$ and $t \ge 2$,

$$\left(1 + \frac{r}{a_m}\right)^t \le \left(\frac{r}{a_m^{1/2}}\right)^t = \frac{r^t}{a_m^{t/2}} < 1 + \frac{r^t}{a_m}. \quad \square$$

The fourth lemma gives an upper bound on the growth of g on intervals where no point is the modulus of a zero of f.

Lemma 3.6. *Let $r > 0$, $0 < s < 1/2$ and $t > 1$ and suppose that there are no values of $n \in \mathbb{N}$ for which $a_n \in (r^s, r^t]$. Then*

$$\log g(r^t) \le t(1 + 2s) \log g(r).$$

Proof. It follows from (3.11) that

$$g(r^t) = r^{3t} \prod_{a_m \le r^s} \left(1 + \frac{r^t}{a_m}\right)^{2p_m} < r^{3t} \prod_{a_m \le r^s} r^{2p_m t} = r^{t(3+\sum_{a_m \le r^s} 2p_m)}$$

and

$$g(r) > r^3 \prod_{a_m \le r^s} \left(\frac{r}{a_m}\right)^{2p_m} > r^{3+\sum_{a_m \le r^s} 2p_m(1-s)}.$$

Thus

$$\log g(r^t)/\log g(r) < t/(1-s) \le t(1+2s),$$

since $s < 1/2$. $\square$

We are now in a position to prove Lemma 3.2.

Proof of Lemma 3.2. Suppose that $m \in \mathbb{N}$ and that we have defined the values of a_n for which $a_n \le r_m$. We now define a sequence (s_k), $0 \le k \le N$, inductively according to certain rules that we give below. Each time we define a value s_k, we also add a zero of f at $-s_k$ provided this is allowed by (3.11), (3.12) and (3.14); no other zeros of f are added. We choose our values s_k in such a way that

$$r_{m+k} \le s_k \le r_{m+k+1}, \quad \text{for } 0 \le k < N, \tag{3.18}$$

$$s_N \le r_{m+N} \tag{3.19}$$

and

$$f^k((-r_{m+1}, 0]) \subset (-s_k, 0], \quad \text{for } 0 \le k \le N. \tag{3.20}$$

The result of Lemma 3.2 follows directly from (3.19) and (3.20). The difficult part of the proof is to show that there exists an $N \in \mathbb{N}$ for which (3.19) is satisfied.

We define our sequence (s_k) as follows:

- set $s_0 = r_{m+1}$;
- if $s_k > r_{m+k}$ and there is a zero of f at $-s_k$, then we set
$$s_{k+1} = g(s_k^{1-\delta_{n_k}/16}); \tag{3.21}$$
- if $s_k > r_{m+k}$ and there is no zero of f at s_k, then we set
$$s_{k+1} = g(s_k); \tag{3.22}$$
- if $s_k \le r_{m+k}$, then we terminate the sequence (s_k).

It follows from Lemma 3.3 and (3.5) that, with this construction, (3.18)–(3.20) are indeed satisfied.

It remains to prove that there exists $K \in \mathbb{N}$ such that the sequence terminates at s_K; that is, if

$$T_k = \frac{\log s_k}{\log r_{m+k}},$$

then there exists $K \in \mathbb{N}$ such that $T_K \le 1$.

We introduce the following terminology. We let L denote the largest integer for which $a_L \le r_m$ and define a (finite) subsequence (k_n) such that

$$a_{L+n} = s_{k_n}, \quad \text{for } n = 1, 2, \ldots. \tag{3.23}$$

The main idea is to show that, for each $n \ge 2$ we have that T_{k_n+1} is less than T_{k_n}, with k_n defined as above. These decreases counteract the small increases that may occur from T_k to T_{k+1} for other values of k and, for n large enough, they will combine together to cause T_{k_n+1} to drop below 1.

We first estimate some quantities that will be useful in our calculations. We begin by noting that it follows from (3.23), (3.18) and Lemma 3.4 that, for $n \ge 1$,

$$\begin{aligned}\log r_{m+k_n+2} &= \log g(r_{m+k_n+1}) \\ &\ge \log g(s_{k_n}) \ge p_{L+n}^{1/2} \log g(s_{k_n}^{1-\delta_{L+n}/16}).\end{aligned}$$

Thus, by (3.21)

$$\log r_{m+k_n+2} \ge p_{L+n}^{1/2} \log s_{k_n+1}, \quad \text{for } n \ge 1. \tag{3.24}$$

Together with (3.6), (3.24) implies that

$$\log r_{m+k_n+q} \ge 3^{q-2} p_{L+n}^{1/2} \log s_{k_n+1}, \quad \text{for } q \ge 2,\ n \ge 1. \tag{3.25}$$

Together with Lemma 3.5, (3.24) implies that

$$\frac{\log s_{k_n+q}}{\log r_{m+k_n+q+1}} \le \frac{\log g^{q-1}(s_{k_n+1})}{\log g^{q-1}(r_{m+k_n+2})} \le \frac{\log s_{k_n+1}}{\log r_{m+k_n+2}} \le \frac{1}{p_{L+n}^{1/2}},$$
$$\text{for } q \ge 2,\ n \ge 1. \tag{3.26}$$

Now fix $n \ge 2$ and write

$$t_{n,q} = T_{k_n+q} = \frac{\log s_{k_n+q}}{\log r_{m+k_n+q}}, \quad \text{for } q \ge 2.$$

For $2 \le q < k_{n+1} - k_n$, there are no zeros of f with modulus in the interval $(s_{k_n}, s_{k_{n+q}})$ and so it follows from (3.22), Lemma 3.6 and (3.25) that, for such q,

$$\begin{aligned}\log s_{k_n+q+1} &= \log g(s_{k_n+q})\\ &\le t_{n,q}\left(1+2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right)\log g(r_{m+k_n+q})\\ &= t_{n,q}\left(1+2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right)\log r_{m+k_n+q+1}\\ &\le t_{n,q}\left(1+\frac{2}{3^{q-2}p_{L+n}^{1/2}}\right)\log r_{m+k_n+q+1}.\end{aligned}$$

Thus, for $2 \le q < k_{n+1} - k_n$, we have

$$t_{n,q+1} \le t_{n,q}\left(1+\frac{2}{3^{q-2}p_{L+n}^{1/2}}\right). \tag{3.27}$$

For $q = k_{n+1} - k_n$, there are no zeros of f with modulus in the interval $(s_{k_n}, s_{k_{n+q}})$ and so it follows from (3.21), Lemma 3.6 and (3.25) that

$$\begin{aligned}\log s_{k_n+q+1} &= \log g(s_{k_n+q}^{1-\delta_{L+n+1}/16})\\ &\le t_{n,q}\left(1-\frac{\delta_{L+n+1}}{16}\right)\left(1+2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right)\log g(r_{m+k_n+q})\\ &= t_{n,q}\left(1-\frac{\delta_{L+n+1}}{16}\right)\left(1+2\frac{\log s_{k_n}}{\log r_{m+k_n+q}}\right)\log r_{m+k_n+q+1}\\ &\le t_{n,q}\left(1-\frac{\delta_{L+n+1}}{16}\right)\left(1+\frac{2}{3^{q-2}p_{L+n}^{1/2}}\right)\log r_{m+k_n+q+1}.\end{aligned}$$

Thus, for $q = k_{n+1} - k_n$, we have

$$t_{n,q+1} \le t_{n,q}\left(1-\frac{\delta_{L+n+1}}{16}\right)\left(1+\frac{2}{3^{q-2}p_{L+n}^{1/2}}\right). \tag{3.28}$$

Finally, it follows from (3.14) that, if $q = k_{n+1} - k_n + 1$, then $q - 1 \ge 2$. Also, there are no zeros of f with modulus in the interval $(s_{k_{n+1}}, s_{k_{n+1}+1}) = (s_{k_{n+1}}, s_{k_n+q})$ and so it follows from Lemma 3.6 and (3.26) that

$$\begin{aligned}\log s_{k_n+q+1} &= \log g(s_{k_n+q})\\ &\le t_{n,q}\left(1+2\frac{\log s_{k_{n+1}}}{\log r_{m+k_n+q}}\right)\log g(r_{m+k_n+q})\\ &= t_{n,q}\left(1+2\frac{\log s_{k_n+q-1}}{\log r_{m+k_n+q}}\right)\log r_{m+k_n+q+1}\\ &\le t_{n,q}\left(1+\frac{2}{p_{L+n}^{1/2}}\right)\log r_{m+k_n+q+1}.\end{aligned}$$

Thus, for $q = k_{n+1} - k_n + 1$, we have

$$t_{n,q+1} \le t_{n,q}\left(1 + \frac{2}{p_{L+n}^{1/2}}\right). \tag{3.29}$$

It follows from (3.27)–(3.29) and (3.13) that, for $M \ge 2$, we have

$$\begin{aligned}
T_{k_{M+1}+2} &= t_{M,k_{M+1}-k_M+2} \\
&= t_{2,2}\prod_{n=2}^{M}\prod_{q=2}^{k_{n+1}-k_n+1}\frac{t_{n,q+1}}{t_{n,q}} \\
&\le t_{2,2}\prod_{n=2}^{M}\left(1+\frac{2}{p_{L+n}^{1/2}}\right)\left(1-\frac{\delta_{L+n+1}}{16}\right)\prod_{q=2}^{k_{n+1}-k_n}\left(1+\frac{2}{3^{q-2}p_{L+n}^{1/2}}\right) \\
&\le t_{2,2}\prod_{n=2}^{M}\left(\left(1+\frac{2}{p_{L+n}^{1/2}}\right)^3\left(1-\frac{\delta_{L+n+1}}{16}\right)\right).
\end{aligned}$$

It follows from (3.13) that $\sum_{n\in\mathbb{N}}\frac{1}{p_{L+n}^{1/2}} < \infty$ and so, since $\sum_{n\in\mathbb{N}}\delta_{L+n+1} = \infty$, we deduce that, for M sufficiently large, $T_{k_{M+1}+2} \le 1$, as required. □

We have now proved Lemma 3.2. As noted earlier, this is sufficient to imply that the hypotheses of Lemma 3.1 hold and hence that $A(f) \cap (-\infty, 0] = \emptyset$ as required.

We complete the proof of Theorem 1.2 by showing that, in addition, conditions (1.3) and (1.4) are satisfied. That is, we prove the following.

Lemma 3.7. *Let*

$$\varepsilon_n = \max_{R_n \le r \le R_{n+1}} \frac{\log\log M(r)}{\log r}. \tag{3.30}$$

There exists a subsequence (n_k) *such that*

$$\varepsilon_{n_k} \le \delta_k + \frac{1}{2^{n_k}}, \quad \text{for } k \in \mathbb{N}, \tag{3.31}$$

and

$$\varepsilon_{n_k+m} \le \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_k+m}}, \quad \text{for } k \in \mathbb{N},\ 1 \le m < n_{k+1} - n_k. \tag{3.32}$$

Proof. We begin by setting $R_0 = r_0 = 10$ and defining $R_{n+1} = M(R_n)$, for $n \in \mathbb{N}$. Clearly $R_n \ge r_n$ by (3.4) and

$$R_{n+1} \ge R_n^3, \quad \text{for } n \in \mathbb{N}. \tag{3.33}$$

We claim that

$$\text{if } a_k \in [R_n, R_{n+1}), \text{ then } a_m \notin [R_n, R_{n+2}) \quad \text{for } k, m \in \mathbb{N},\ m \ne k. \tag{3.34}$$

In order to deduce this from (3.14), it is sufficient to show that, if $r_p \in [R_n, R_{n+1})$, for some $p, n \in \mathbb{N}$, then $r_{p+2} > R_{n+1}$. We prove this in two steps. First, we note that if $r_p \in [R_n, R_n^3)$, for some $p, n \in \mathbb{N}$, then it follows from (3.6) that $r_{p+1} \geq r_p^3 \geq R_n^3$. Second, if $r_p \in [R_n^3, R_{n+1})$, for some $p, n \in \mathbb{N}$, then we claim that

$$r_{p+1} = g(r_p) \geq g(R_n^3) > M(R_n) = R_{n+1}. \tag{3.35}$$

This is true since, if k is the smallest integer such that $a_k > R_n^3$, then

$$g(R_n^3) = R_n^9 \prod_{m=1}^{k-1} \left(1 + \frac{R_n^3}{a_m}\right)^{2p_m}$$

and so, by (3.1) and (3.11),

$$\begin{aligned} M(R_n) &= f(R_n) = R_n^3 \prod_{m=1}^{\infty} \left(1 + \frac{R_n}{a_m}\right)^{2p_m} \\ &< \frac{g(R_n^3)}{R_n^6} \left(1 + \frac{R_n}{a_k}\right)^{2p_k} \prod_{m \geq k+1} \left(1 + \frac{a_k}{a_m}\right)^{2p_m} \\ &< \frac{g(R_n^3)}{R_n^6} \left(1 + \frac{1}{a_k^{1/2}}\right)^{a_k^{1/2}} \prod_{m \geq k+1} \left(1 + \frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\ &\leq \frac{g(R_n^3)}{R_n^6} e^{1+1+1/2+1/4+\cdots} < g(R_n^3). \end{aligned}$$

Thus (3.35) does indeed hold and, by the reasoning above, this is sufficient to show that (3.34) holds.

Now, for $k \in \mathbb{N}$, we choose $n_k \in \mathbb{N}$ such that $a_k \in [R_{n_k}, R_{n_k+1})$. Then, by (3.34), this defines a sequence (n_k) with $n_j \neq n_k$ for $j \neq k$. Now suppose that $r \in [R_{n_k}, R_{n_k+1}]$, for some $k \in \mathbb{N}$. It follows from (3.11) and (3.34) that

$$\begin{aligned} M(r) &= f(r) \leq r^3 \left(1 + \frac{r}{a_k}\right)^{2p_k} \prod_{m=1}^{k-1} \left(1 + \frac{r}{a_m}\right)^{2p_m} \prod_{m \geq k+1} \left(1 + \frac{r}{a_m}\right)^{2p_m} \\ &\leq \left(1 + \frac{r}{a_k}\right)^{2p_k} r^{3+2p_1+\cdots+2p_{k-1}} \prod_{m \geq k+1} \left(1 + \frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\ &< \left(1 + \frac{r}{a_k}\right)^{a_k^{\delta_k}} r^{a_{k-1}^{\delta_{k-1}}} e^{1+1/2+1/4+\cdots} \end{aligned}$$

and so

$$M(r) < e^2 r^{a_{k-1}^{\delta_{k-1}}} \left(1 + \frac{r}{a_k}\right)^{a_k^{\delta_k}}. \tag{3.36}$$

If $r < a_k^{1/2}$, then it follows from (3.2) and (3.36) that

$$M(r) < e^3 r^{a_{k-1}^{\delta_{k-1}}} < e^3 r^{r^{\delta_k}}$$

and hence, since $r \geq R_1 \geq 1000$,

$$\frac{\log\log M(r)}{\log r} < \frac{\delta_k \log r + 2\log\log r}{\log r} = \delta_k + 2\frac{\log\log r}{\log r} \leq \delta_k + 2\frac{\log\log R_{n_k}}{\log R_{n_k}}.$$

It follows from (3.33) that, in this case,

$$\frac{\log\log M(r)}{\log r} \le \delta_k + 2\frac{\log(3^{n_k}\log 10)}{3^{n_k}\log 10} < \delta_k + \frac{1}{2^{n_k}}. \tag{3.37}$$

If $a_k^{1/2} \le r \le a_k$, then

$$\left(1+\frac{r}{a_k}\right)^{a_k^{\delta_k}} = \left(1+\frac{r}{a_k}\right)^{(a_k/r)^{\delta_k} r^{\delta_k}} < \left(1+\frac{r}{a_k}\right)^{(a_k/r) r^{\delta_k}} \le e^{r^{\delta_k}}$$

and, if $r > a_k$, then

$$\left(1+\frac{r}{a_k}\right)^{a_k^{\delta_k}} < r^{a_k^{\delta_k}} < r^{r^{\delta_k}}.$$

So, if $r \ge a_k^{1/2}$, it follows from (3.36) and (3.11) that

$$M(r) < e^2 r^{a_{k-1}^{\delta_{k-1}}} r^{r^{\delta_k}} < e^2 r^{a_k^{\delta_k/2}} r^{r^{\delta_k}} < e^2 r^{2r^{\delta_k}}$$

and hence

$$\frac{\log\log M(r)}{\log r} < \frac{\delta_k \log r + 2\log\log r}{\log r} = \delta_k + 2\frac{\log\log r}{\log r} \le \delta_k + 2\frac{\log\log R_{n_k}}{\log R_{n_k}}.$$

As before, it follows from (3.33) that

$$\frac{\log\log M(r)}{\log r} \le \delta_k + \frac{1}{2^{n_k}}. \tag{3.38}$$

Together with (3.37), this implies that (3.31) holds.

Now suppose that $r \in [R_{n_k+m}, R_{n_k+m+1})$, for some $k \in \mathbb{N}$, $1 \le m < n_{k+1} - n_k$. It follows from (3.11) and (3.33) that

$$\begin{aligned} M(r) = f(r) &\le r^3 \prod_{m=1}^{k} \left(1+\frac{r}{a_m}\right)^{2p_m} \prod_{m\ge k+1} \left(1+\frac{r}{a_m}\right)^{2p_m} \\ &\le r^{3+2p_1+\cdots+2p_k} \prod_{m\ge k+1} \left(1+\frac{1}{a_m^{1-1/2^{m-k}}}\right)^{a_m^{1/2}} \\ &\le r^{a_k^{\delta_k}} e^{1+1/2+1/4+\cdots} \\ &\le e^2 r^{a_k^{\delta_k}} \le e^2 r^{R_{n_k+1}^{\delta_k}} \\ &< e^2 r^{r^{\delta_k/3^{m-1}}}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\log\log M(r)}{\log r} &< \frac{\delta_k \log r/3^{m-1} + 2\log\log r}{\log r} < \frac{\delta_k}{3^{m-1}} + 2\frac{\log\log r}{\log r} \\ &\le \frac{\delta_k}{3^{m-1}} + 2\frac{\log\log R_{n_k+m}}{\log R_{n_k+m}}. \end{aligned}$$

As before, it follows from (3.33) that

$$\frac{\log\log M(r)}{\log r} \leq \frac{\delta_k}{3^{m-1}} + \frac{1}{2^{n_m+m}} \tag{3.39}$$

and so (3.32) holds. □

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