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Champagne subdomains with unavoidable bubbles

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Abstract

A champagne subdomain of a connected open set $U \neq \emptyset$ in $\mathbb{R}^d$, $d \geq 2$, is obtained by omitting pairwise disjoint closed balls $\overline{B}(x, r_x)$, $x \in X$, the bubbles, where X is an infinite, locally finite set in U. The union A of these balls may be unavoidable, that is, Brownian motion, starting in $U \setminus A$ and killed when leaving U, may hit A almost surely or, equivalently, A may have harmonic measure 1 for $U \setminus A$.

Recent publications by Gardiner and Ghergu ($d \geq 3$) and by Pres ($d = 2$) give rather sharp answers to the question of how small such a set A may be, when U is the unit ball.

In this paper, using a totally different approach, optimal results are obtained, which hold also for arbitrary connected open sets U.

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1. Introduction and the main results

Throughout this paper let U denote a non-empty connected open set in $\mathbb{R}^d$, $d \geq 2$. Let us say that a relatively closed subset A of U is *unavoidable* if Brownian motion, starting in $U \setminus A$

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and killed when leaving U, hits A almost surely or, equivalently, if $\mu_y^{U\setminus A}(A) = 1$, for every $y \in U \setminus A$, where $\mu_y^{U\setminus A}$ denotes the harmonic measure at y with respect to $U \setminus A$ (we note that $\mu_y^{U\setminus A}$ may fail to be a probability measure, if $U \setminus A$ is not bounded).

For $x \in \mathbb{R}^d$ and $r > 0$, let $B(x, r)$ denote the open ball of center x and radius r. Suppose that X is a countable set in U having no accumulation point in U, and let $r_x > 0$, $x \in X$, such that the closed balls $\overline{B}(x, r_x)$, the *bubbles*, are pairwise disjoint, $\sup_{x\in X} r_x/\mathrm{dist}(x, \partial U) < 1$ and, if U is unbounded, $r_x \to 0$ as $x \to \infty$. Then the union A of all $\overline{B}(x, r_x)$ is relatively closed in U, and the connected open set $U \setminus A$ (which is non-empty!) is called a *champagne subdomain of* U.

This generalizes the notions used in [3,8,12–14] for $U = B(0, 1)$; see also [6] for the case where U is $\mathbb{R}^d$, $d \geq 3$. Avoidable unions of randomly distributed balls have been discussed in [11] and, recently, in [5].

It will be convenient to introduce the set X_A for a champagne subdomain $U \setminus A$: X_A is the set of centers of all the bubbles forming A (and r_x, $x \in X_A$, is the radius of the bubble centered at x). It is fairly easy to see that, given a champagne subdomain $U \setminus A$ and a finite subset X' of X_A, the set A is unavoidable if and only if the union of all bubbles $\overline{B}(x, r_x)$, $x \in X_A \setminus X'$, is unavoidable.

The main result of Akeroyd [3] is, for a given $\delta > 0$, the existence of a champagne subdomain of the unit disc such that

$$\sum_{x\in X_A} r_x < \delta \quad \text{and yet } A \text{ is unavoidable.} \tag{1.1}$$

Ortega-Cerdà and Seip [13] improved the result of Akeroyd in characterizing a certain class of champagne subdomains $B(0, 1) \setminus A$, where A is unavoidable and $\sum_{x\in X_A} r_x < \infty$, and hence the statement of (1.1) can be obtained omitting finitely many of the discs $\overline{B}(x, r_x)$, $x \in X_A$.

Let us note that already in [10] the existence of a champagne subdomain of an arbitrary bounded connected open set U in $\mathbb{R}^2$ having property (1.1) was crucial for the construction of an example answering Littlewood's one-circle problem in the negative. In fact, Proposition 3 in [10] is a bit stronger: Even a Markov chain formed by jumps on annuli hits A before it goes to ∂U. The statement about harmonic measure (hitting by Brownian motion) is obtained by the first part of the proof of Proposition 3 in [10] (cf. also [9], where this is explicitly stated at the top of p. 72). This part uses only "one-bubble estimates" for the global Green function and the minimum principle.

Recently, Gardiner and Ghergu [8, Corollary 3] proved the following.

Theorem A. *If $d \geq 3$, then, for all $\alpha > d - 2$ and $\delta > 0$, there is a champagne subdomain $B(0, 1) \setminus A$ such that A is unavoidable and*

$$\sum_{x\in X_A} r_x^\alpha < \delta.$$

Moreover, Pres [14, Corollary 1.3] showed the following for the plane.

Theorem B. *If $d = 2$, then, for all $\alpha > 1$ and $\delta > 0$, there is a champagne subdomain $B(0, 1)\setminus A$ such that A is unavoidable and*

$$\sum_{x\in X_A} \left(\log \frac{1}{r_x}\right)^{-\alpha} < \delta.$$

For capacity reasons both results are sharp in the sense that α cannot be replaced by $d-2$ in Theorem A and α cannot be replaced by 1 in Theorem B. In fact, taking $\alpha = d-2$, $\alpha = 1$, respectively, the corresponding series diverge if A is an unavoidable set of bubbles (see [8, p. 323] and [14, Remark 1.4]). The proofs of Theorems A and B are quite involved and, in addition, use the delicate results [7, Theorem 1] (cf. [2, Corollary 7.4.4]) on minimal thinness of subsets A of $B(0,1)$ at points $z \in \partial B(0,1)$ and [1, Proposition 4.1.1] on quasi-additivity of capacity.

Carefully choosing bubbles centered at concentric spheres, estimating related potentials, and using the minimum principle, we obtain the following optimal result, not only for the unit ball, but even for arbitrary connected open sets.

Theorem 1.1. *Let $U \neq \emptyset$ be a connected open set in $\mathbb{R}^d$, $d \geq 2$, and let $h : (0,1) \to \mathbb{R}^+$ be such that $\liminf_{t\to 0} h(t) = 0$. Then, for every $\delta > 0$, there is a champagne subdomain $U \setminus A$ such that A is unavoidable and*

$$\sum_{x\in X_A} \left(\log \frac{1}{r_x}\right)^{-1} h(r_x) < \delta, \quad \text{if } d = 2,$$
$$\sum_{x\in X_A} r_x^{d-2} h(r_x) < \delta, \quad \text{if } d \geq 3.$$

Moreover, we may treat the cases $d = 2$ and $d \geq 3$ simultaneously. To that end we define functions

$$N(t) := \begin{cases} \log \dfrac{1}{t}, & \text{if } d = 2, \\ t^{2-d}, & \text{if } d \geq 3, \end{cases} \quad \text{and} \quad \varphi(t) := 1/N(t)$$

so that $(x,y) \mapsto N(|x-y|)$ is the global Green function and, for $d \geq 3$, $\varphi(t) = t^{d-2}$ is the capacity of balls with radius t (for $d = 2$, $\varphi(t)$ should be considered for $t \in (0,1)$ only). Using the (capacity) function φ, the two displayed formulas can be simultaneously expressed as

$$\sum_{x\in X_A} \varphi(r_x) h(r_x) < \delta. \tag{1.2}$$

Accordingly, the results of Gardiner and Ghergu and of Pres (Theorems A and B) can be unified as follows.

Theorem C. *If $d \geq 2$, then, for all $\varepsilon > 0$ and $\delta > 0$, there is a champagne subdomain $B(0,1)\setminus A$ such that A is unavoidable and*

$$\sum_{x\in X_A} \varphi(r_x)^{1+\varepsilon} < \delta.$$

Clearly, Theorem C follows from Theorem 1.1 taking $h = \varphi^\varepsilon$. Of course, we may get much stronger statements taking, for example,

$$h(t) = (\log\log\cdots\log(1/\varphi(t)))^{-1}, \quad t > 0 \text{ sufficiently small.}$$

In fact, we shall obtain the following.

Theorem 1.2. *Let $d \geq 2$ and $U := \mathbb{R}^d$ or $U := B(0,L)$, $L > 0$. Further, let $0 < R_1 < R_2 < \cdots$ with $B(0,R_k) \uparrow U$ and $h : (0,1) \to \mathbb{R}^+$ with $\liminf_{t\to 0} h(t) = 0$. Then, for every $\delta > 0$, there exist finite sets X_k in $\partial B(0,R_k)$ and $r_k > 0$ such that, taking*

$$A := \bigcup_{x \in X_k, k \in \mathbb{N}} \overline{B}(x, r_k),$$

the set $B(0,1) \setminus A$ is a champagne subdomain, A is unavoidable and (1.2) holds.

Let us finish this section by explaining in some detail how these results are obtained. Given an exhaustion of an arbitrary domain U by a sequence (V_n) of bounded open subsets, we first present a criterion for unavoidable sets A in U in terms of probabilities for Brownian motion, starting in $\overline{V}_n$, to hit A before leaving V_{n+1} (Section 2).

To apply this criterion we prove the existence of $c > 0$ and $\kappa > 0$ such that the following holds (Sections 3 and 5): Given $R > 0$ and $0 < \rho < (1/3)\min\{R, 1\}$, there exists $0 < \rho_0 \le \rho/3$ such that, for *every* $0 < r < \rho_0$, we may choose a finite subset X_r of $\partial B(0, R)$ satisfying

(i) the product $\#X_r \cdot \varphi(r)$ is bounded by $c\rho^{-1}R^{d-1}$,
(ii) the balls $\overline{B}(x, r)$, $x \in X_r$, are pairwise disjoint,
(iii) starting in $\overline{B}(0, R+\rho)$, Brownian motion hits the union of the balls $\overline{B}(x, r)$, $x \in X_r$, before leaving $B(0, R+2\rho)$ with a probability which is at least κ.

In Section 4 we give a straightforward application of our construction X_r to the unit ball considering an exhaustion $(B(0, R_k))_{k \ge k_0}$ given by $R_{k+1} - R_k = (k \log^2 k)^{-1}$ and a "one-bubble estimate" for the global Green function. The resulting Proposition 4.1 is already fairly close to Theorem C.

The proof of (iii) in Section 5 will be based on a comparison of the sum of the potentials for the points $x \in X_r$ with the equilibrium potential for $\overline{B}(0, R)$ (both with respect to $B(0, R+2\rho)$). The proof of Theorem 1.2 is now easily accomplished by taking $\rho_k := (1/3)\min\{R_{k+1} - R_k, R_k - R_{k-1}, 1/k\}$ and choosing $0 < r_k < \rho_{0,k} \le \rho_k$ with $c{\rho_k}^{-1}R_k^{d-1}h(r_k) < 2^{-k}\delta$ (Section 6). Finally, in Section 7, we prove Theorem 1.1 by covering the boundaries ∂V_n of an arbitrary exhaustion (V_n) with small balls to which we apply the results of Sections 3 and 5.

2. A general criterion for unavoidable sets

Given an open set W in $\mathbb{R}^d$ and a bounded Borel measurable function f on $\mathbb{R}^d$, let $H_W f$ denote the function which extends the (generalized) Dirichlet solution $x \mapsto \int f \, d\mu_x^W$, $x \in W$, to a function on $\mathbb{R}^d$ taking the values $f(x)$ for $x \in \mathbb{R}^d \setminus W$. We shall use the fact that the harmonic kernel H_W has the following property: If W' is an open set in W, then $H_{W'}H_W = H_W$.

Let $U \ne \emptyset$ be a connected open set in $\mathbb{R}^d$, $d \ge 2$, and let $A \subset U$ be relatively closed. Then A is unavoidable if and only if

$$H_{U \setminus A} 1_A = 1 \quad \text{on } U.$$

Proposition 2.1. *Let $0 \le \kappa_j \le 1$ and V_j be bounded open sets in U, $j \ge j_0$, such that $\overline{V}_j \subset V_{j+1}$, $V_j \uparrow U$, and the following holds: For every $j \ge j_0$ and every $z \in \partial V_j \setminus A$, there exists a closed set E in $A \cap V_{j+1}$ such that*

$$H_{V_{j+1} \setminus E} 1_E(z) \ge \kappa_j. \tag{2.1}$$

Then, for all $n, m \in \mathbb{N}$, $j_0 \le n < m$,

$$H_{U \setminus A} 1_A \ge 1 - \prod_{n \le j < m} (1 - \kappa_j) \quad \text{on } \overline{V}_n. \tag{2.2}$$

In particular, A is unavoidable if the series $\sum_{j \ge j_0} \kappa_j$ is divergent.

As we noticed later on, the probabilistic aspect of such a result has already been used in [13] and subsequently in [6,12]: of course, Brownian motion starting in V_n hits ∂V_n before reaching ∂V_{n+1}. Inequality (2.1) implies that a Brownian particle starting at some $z \in \partial V_j \setminus A$, $n \le j < m$, does not hit A before reaching ∂V_{j+1} with probability at most $1-\kappa_j$. By induction and by the strong Markov property, it does not hit A with probability at most $\prod_{n\le j<m}(1-\kappa_j)$ before reaching ∂V_m, and therefore it hits A with probability at least $1-\prod_{n\le j<m}(1-\kappa_j)$ before leaving U.

Proof of Proposition 2.1. For $j \ge j_0$, let $W_{j+1} := V_{j+1} \setminus A$. If E is a closed set in $A \cap V_{j+1}$, then $H_{W_{j+1}} 1_{\partial V_{j+1}} \le 1 - H_{V_{j+1}\setminus E} 1_E$, by the minimum principle. Hence, by (2.1),

$$H_{W_{j+1}} 1_{\partial V_{j+1}} \le 1-\kappa_j \quad \text{on } \partial V_j.$$

Now let $n, m \in \mathbb{N}$, $j_0 \le n < m$. By induction,

$$H_{W_m} 1_{\partial V_m} = H_{W_{n+1}} H_{W_{n+2}} \cdots H_{W_m} 1_{\partial V_m} \le \prod_{n\le j<m} (1-\kappa_j) \quad \text{on } \partial V_n.$$

By the minimum principle, we conclude that

$$H_{U\setminus A} 1_A \ge H_{W_m} 1_A \ge 1 - H_{W_m} 1_{\partial V_m} \ge 1 - \prod_{n\le j<m} (1-\kappa_j) \quad \text{on } \overline{V}_n. \quad \square$$

3. Choice of bubbles

Let $R > 0$, $V := B(0,R)$, and $0 < \rho < (1/3)\min\{R,1\}$. For every $r > 0$ which is sufficiently small, we shall choose an associated finite subset X_r of ∂V and consider the union E_r of all bubbles $\overline{B}(x,r)$, $x \in X_r$. For $r > 0$, we first define

$$\beta := (\varphi(r)\rho)^{1/(d-1)}. \tag{3.1}$$

In other words, we take β satisfying

$$\varphi(r) = \beta^{d-1}\rho^{-1}, \quad \text{that is, } r = \begin{cases} \exp(-\rho/\beta), & \text{if } d=2, \\ \beta^{(d-1)/(d-2)}\rho^{-1/(d-2)}, & \text{if } d \ge 3. \end{cases} \tag{3.2}$$

It is easily seen that $\beta < \rho$ if $r < \rho$. Further, there exists $0 < \rho_0 \le \rho/3$ such that

$$r < \beta/3, \quad \text{whenever } r \in (0,\rho_0). \tag{3.3}$$

Indeed, if $d \ge 3$ and $r < 3^{1-d}\rho$, then $r/\beta = \left(r^{d-1}/(r^{d-2}\rho)\right)^{1/(d-1)} < 1/3$. Assume now that $d=2$ and $r < (1/18)\rho^2$. Then $\rho/\beta = \log(1/r) < \log\{[\rho/(3r)]^2/2\} < \rho/3r$.

Given $0 < r < \rho_0$, we choose a finite subset X_r of ∂V such that the balls $B(x,\beta)$, $x \in X_r$, cover ∂V and the balls $B(x,\beta/3)$, $x \in X_r$, are pairwise disjoint (such a set X_r exists; see [15, Lemma 7.3]). By (3.3), the balls $\overline{B}(x,r)$, $x \in X_r$, forming E_r are pairwise disjoint. A consideration of the areas involved, when intersecting the balls with ∂V, shows that there exists a constant $c = c(d) > 0$ such that

$$c^{-1}(R/\beta)^{d-1} \le \#X_r \le c(R/\beta)^{d-1}, \tag{3.4}$$

and hence, by (3.2),

$$\#X_r \cdot \varphi(r) \le c\rho^{-1}R^{d-1}. \tag{3.5}$$

Thus, given R and ρ, our assumption on h implies that $\liminf_{r\to 0} \#X_r \cdot \varphi(r)h(r) = 0$.

4. A result based on a "one-bubble approach"

It may be surprising that, having Proposition 2.1 and our construction of unions E_r of bubbles centered at spheres $\partial B(0, R)$, a "one-bubble approach", which only uses the global Green function with one pole, may already yield a result which is almost as strong as Theorem C.

For Proposition 4.1, a sequence $(R_k)_{k\ge k_0}$ will be chosen in the following way. We fix $k_0 \ge 3^d$ such that $\sum_{j\ge k_0}(j\log^2 j)^{-1} < 1/2$ and $ke^{-k} < (9\log^2 k)^{-1}$, for $k \ge k_0$. For every $k \ge k_0$, let

$$R_k := 1 - \sum_{j\ge k}(j\log^2 j)^{-1} \quad \text{and} \quad V_k := B(0, R_k).$$

To apply our construction in Section 3 let us, for the moment, fix $k \ge k_0$ and let $R := R_k$, $\rho := (3\log^2 k)^{-1}$ so that $V = V_k$ and $\rho < 1/6 < R/3$. Further, let

$$r := \begin{cases} e^{-k}, & \text{if } d = 2, \\ k^{-(d-1)/(d-2)}\rho, & \text{if } d \ge 3. \end{cases} \tag{4.1}$$

According to (3.1) we have $\beta = \rho/k$, and hence $3\beta = (k\log^2 k)^{-1} = R_{k+1} - R_k$ and $r/\beta = kr/\rho < 1/3$. So we may choose a corresponding finite set X_r and take $X_k := X_r$, $r_k := r$. Let us already notice that $r_k/(1 - R_k) < \rho/(3k) \cdot k\log^2 k = 1/9$.

Proposition 4.1. *Let $\varepsilon > 1/(d-1)$ and $\delta > 0$. Then there exists $K \ge k_0$ such that, taking*

$$A := \bigcup_{x\in X_k, k\ge K} \overline{B}(x, r_k),$$

the set $B(0, 1) \setminus A$ is a champagne subdomain, A is unavoidable, and

$$\sum_{x\in X_A} \varphi(r_x)^{1+\varepsilon} < \delta. \tag{4.2}$$

Proof. Let $k \ge k_0$. By (4.1), $\varphi(r_k) \le k^{1-d}$. Hence, by (3.5),

$$\#X_k\, \varphi(r_k)^{1+\varepsilon} \le c(3\log^2 k)\varphi(r_k)^{\varepsilon} \le c(3\log^2 k)k^{\varepsilon(1-d)}.$$

So (4.2) holds if K is sufficiently large.

We next claim that the union A of all $\overline{B}(x, r_k)$, $x \in X_k$, $k \ge K$, is unavoidable. Indeed, let us fix $k \ge K$ and let β, r be as above. Let $z \in \partial V_k \setminus A$. There exists $x \in X_k$ such that $|z - x| < \beta$. We define $E := \overline{B}(x, r)$ and

$$g(y) := \varphi(r)\,(N(|y - x|) - N(3\beta)), \quad y \in \mathbb{R}^d.$$

Since $3\beta = R_{k+1} - R_k$, we know that $B(x, 3\beta) \subset V_{k+1}$, and hence $g \le 0$ on ∂V_{k+1}. Further, $g \le \varphi(r)N(r) = 1$ on the boundary of E. By the minimum principle,

$$H_{V_{k+1}\setminus E}1_E \ge g \quad \text{on } V_{k+1} \setminus E.$$

Clearly, $N(|z - x|) - N(3\beta) \ge (2/3)\beta^{2-d}$, since $\log 3 \ge 1$ and $1 - 3^{2-d} \ge 2/3$ for $d \ge 3$. Therefore, by (3.2),

$$H_{V_{k+1}\setminus E}1_E(z) \ge g(z) \ge (2/3)\varphi(r)\beta^{2-d} = (2/3)\beta/\rho = (2/3)k^{-1}.$$

By Proposition 2.1, A is unavoidable. Clearly, $B(0, 1) \setminus A$ is a champagne subdomain. □

Remark 4.2. If $d \geq 3$, then $\varphi(r_x)^{1+\varepsilon} = r_x^{(d-2)(1+\varepsilon)}$, where the critical exponent $(d-2)(1+1/(d-1)) = d-1-1/(d-1)$ is strictly smaller than $d-1$.

5. A crucial estimate

Let us now return to the general situation introduced in Section 3. In addition, let

$$V' := B(0, R+\rho), \qquad W := B(0, R+2\rho),$$

and let G be the Green function for W.

Proposition 5.1. *There exists a constant $\kappa = \kappa(d) > 0$ such that*

$$H_{W\setminus E_r} 1_{E_r} \geq \kappa \quad \text{on } \overline{V'}, \quad \text{for every } r \in (0, \rho_0), \tag{5.1}$$

that is, Brownian motion starting in $\overline{V'}$ hits E_r with probability at least κ before leaving W, whatever $0 < r < \rho_0$ is.

Before proving Proposition 5.1 we establish two lemmas.

Lemma 5.2. *There exists a constant $c_1 := c_1(d) > 0$ such that*

$$G(y,z) \leq c_1 G(y,z'), \quad \text{if } y \in W \quad \text{and} \quad z, z' \in \partial V \text{ with } |y-z'| \leq 4|y-z|.$$

Proof. For $y, z \in W$, let $\Psi(y,z) := (R+2\rho-|y|)(R+2\rho-|z|)/|y-z|^2$ and

$$F(y,z) := \begin{cases} \log(1+\Psi(y,z)), & d=2, \\ \min\{1, \Psi(y,z)\}|y-z|^{2-d}, & d \geq 3. \end{cases}$$

If $y \in W$ and $z, z' \in \partial V$ with $|y-z'| \leq 4|y-z|$, then $\Psi(y,z) \leq 4^2 \Psi(y,z')$, and hence $F(y,z) \leq 4^d F(y,z')$. It follows immediately from [4, Theorem 4.1.5] that there exists a constant $c_0 = c_0(d)$ such that ${c_0}^{-1} F \leq G \leq c_0 F$. So it suffices to take $c_1 := 4^d c_0^2$. $\square$

For every measure χ on W, let $G\chi(y) := \int G(y,z)\, d\chi(z)$, $y \in W$. Let σ be the surface measure on ∂V. We note that

$$G\sigma = \|\sigma\| \cdot \min\{N(|\cdot|) - N(R+2\rho), N(R) - N(R+2\rho)\}. \tag{5.2}$$

Now we fix $r \in (0, \rho_0)$ and define

$$\mu := \beta^{d-1} \sum_{x \in X_r} \varepsilon_x.$$

Since $c^{-1} R^{d-1} \leq \|\mu\| \leq c R^{d-1}$, by (3.4), and X_r is distributed on ∂V in a fairly regular way, there is a close relation between $G\mu$ and $G\sigma$. We shall use the following.

Lemma 5.3. *There exists a constant $C = C(d) > 0$ such that $G\sigma \leq CG\mu$ on $\partial V'$ and, for every $x \in X_r$,*

$$G\mu \leq \beta^{d-1} G(\cdot, x) + CG\sigma \quad \text{on } \overline{B}(x,r).$$

Proof. Let us introduce a partition of ∂V corresponding to $X_r = \{x_1, \ldots, x_M\}$. For $1 \leq j \leq M$, let $S'_j := \partial V \cap B(x_j, \beta/3)$, $S''_j := \partial V \cap B(x_j, \beta)$, and let S' be the union of the pairwise disjoint

sets $S'_1, \dots, S'_M$. We recursively define $S_1, S_2, \dots, S_M$ by $S_1 := S'_1 \cup (S''_1 \setminus S')$ and

$$S_j := \Big(S'_j \cup (S''_j \setminus S')\Big) \setminus (S_1 \cup \dots \cup S_{j-1}).$$

Since $S''_1, \dots, S''_M$ cover ∂V, the sets $S_1, \dots, S_M$ form a partition of ∂V such that

$$S'_j \subset S_j \subset S''_j \quad \text{for every } 1 \le j \le M.$$

So there exists a constant $c_2 = c_2(d) > 0$ such that

$$c_2{}^{-1}\beta^{d-1} \le \sigma(S_j) \le c_2\beta^{d-1}, \quad 1 \le j \le M. \tag{5.3}$$

To prove the first inequality of the lemma, we fix $y \in \partial V'$. Let $1 \le j \le M$. For every $z \in S_j$, $|y - z| \ge \rho > \beta > |z - x_j|$, and hence $|y - x_j| \le |y - z| + |z - x_j| < 2|y - z|$. So, by Lemma 5.2, $G(y, \cdot) \le c_1 G(y, x_j)$ on S_j, and hence

$$G(1_{S_j}\sigma)(y) = \int_{S_j} G(y, z)\, d\sigma(z) \le c_1\sigma(S_j)G(y, x_j) \le c_1c_2\beta^{d-1}G(y, x_j).$$

Taking the sum we see that $G\sigma(y) \le c_1c_2G\mu(y)$.

To prove the second inequality let $x := x_{j_0}$, $1 \le j_0 \le M$, and assume that $1 \le j \le M$, $j \ne j_0$. Moreover, let $y \in \overline{B}(x, r)$ and $z' \in S_j$. Clearly, $y \in B(x, \beta/3)$, by (3.3). Since $B(x, \beta/3) \cap B(x_j, \beta/3) = \emptyset$, we see that $|y - x_j| > \beta/3$, whereas $|x_j - z'| < \beta$. So $|y - z'| \le |y - x_j| + |x_j - z'| < 4|y - x_j|$, and therefore $G(y, x_j) \le c_1G(y, \cdot)$ on S_j, by Lemma 5.2. By integration, $\sigma(S_j)G(y, x_j) \le c_1G(1_{S_j}\sigma)(y)$. Thus, using (5.3),

$$\begin{aligned} G\mu &\le \beta^{d-1}G(\cdot, x) + c_2 \sum_{j \ne j_0} \sigma(S_j)G(\cdot, x_j) \\ &\le \beta^{d-1}G(\cdot, x) + c_1c_2 \sum_{j \ne j_0} G(1_{S_j}\sigma) \\ &\le \beta^{d-1}G(\cdot, x) + c_1c_2G\sigma \quad \text{on } \overline{B}(x, r). \end{aligned}$$

Taking $C := c_1c_2$ the proof is finished. □

By (5.2), there exists a constant $c_3 = c_3(d) > 0$ such that

$$c_3{}^{-1}\rho \le G\sigma(y) \le c_3\rho, \quad \text{whenever } y \in W \text{ such that } R - \rho \le |y| \le R + \rho. \tag{5.4}$$

After these preparations we are ready to prove the crucial estimate in Proposition 5.1. We first claim that

$$G\mu \le (2 + c_3C)\rho \quad \text{on } \partial E_r. \tag{5.5}$$

Indeed, let $x \in X_r$ and $y \in \partial B(x, r)$. Since $W \subset B(x, 3)$ and $|y - x| = r < 1/3$, we obtain that $G(y, x) \le N(|y - x|) - N(3) = N(r) - N(3) \le 2N(r)$ (if $d = 2$, then $N(r) - N(3) = \log(1/r) + \log 3 \le 2\log(1/r)$). So, by (3.2),

$$\beta^{d-1}G(x, y) \le 2\beta^{d-1}N(r) = 2\beta^{d-1}\varphi(r)^{-1} = 2\rho.$$

Further, by (5.4), $G\sigma(y) \le c_3\rho$. Therefore (5.5) holds, by Lemma 5.3.

Since $G\mu$ is harmonic on $W \setminus E_r$ and $G\mu$ vanishes at ∂W, we conclude that

$$H_{W \setminus E_r} 1_{E_r} \ge (2 + c_3C)^{-1}\rho^{-1}G\mu \quad \text{on } W \setminus E_r.$$

On the other hand, by Lemma 5.3 and (5.4),

$$G\mu \geq C^{-1}G\sigma \geq (c_3C)^{-1}\rho \quad \text{on } \partial V',$$

whence also on $\overline{V'}$, by the minimum principle. Taking $\kappa := (c_3C(2+c_3C))^{-1}$ we thus obtain that $H_{W\setminus E_r}1_{E_r} \geq \kappa$ on $\overline{V'}$.

6. The main result for $\mathbb{R}^d$ and for open balls

To prove Theorem 1.2, let $0 < R_1 < R_2 < \cdots$ such that $V_k := B(0, R_k) \uparrow U$. Further, let $\delta > 0$ and $h\colon (0,1) \to \mathbb{R}^+$ with $\liminf_{t\to 0} h(t) = 0$.

To apply our construction in Section 3 let us, for the moment, fix $k \in \mathbb{N}$ and take $R := R_k$ (so that $V = V_k$) and $\rho := (1/3)\min\{R_{k+1}-R_k, R_k-R_{k-1}, 1/k\}$ (with $R_0 := 0$). Let $0 < \rho_0 \leq \rho/3$ such that (3.3) holds. There exists $0 < r < \rho_0$ such that $c\rho^{-1}R_k^{d-1}h(r) < 2^{-k}\delta$. We choose X_r as in Section 3 and define $r_k := r$, $X_k := X_{r_k}$. Then, by (3.5),

$$\#X_k \cdot \varphi(r_k)h(r_k) \leq c\rho^{-1}R_k^{d-1}h(r_k) < 2^{-k}\delta. \tag{6.1}$$

By the minimum principle and Proposition 5.1, the union E_k of all $\overline{B}(x, r_k)$, $x \in X_k$, satisfies

$$H_{V_{k+1}\setminus E_k}1_{E_k} \geq H_{B(0,R_k+2\rho)\setminus E_k}1_{E_k} \geq \kappa \quad \text{on } \partial V_k. \tag{6.2}$$

Clearly, $r_k < \min\{1/k, \operatorname{dist}(x, U^c)/9\}$, for every $x \in X_k$, and the balls $\overline{B}(x, r_k)$, $x \in X_k$, $k \in \mathbb{N}$, are pairwise disjoint. Let A be the union of all E_k, $k \in \mathbb{N}$. Then $B(0,1)\setminus A$ is a champagne subdomain. By (6.2) and Proposition 2.1, A is unavoidable. Finally, $\sum_{x\in X_A}\varphi(r_x)h(r_x) < \delta$, by (6.1).

7. The proof for arbitrary connected open sets

Let U be an arbitrary non-empty connected open set in $\mathbb{R}^d$, $d \geq 2$. Let us fix bounded open sets $V_n \neq \emptyset$, $n \in \mathbb{N}$, such that $\overline{V}_n \subset V_{n+1}$ and $V_n \uparrow U$. For every $n \in \mathbb{N}$, we define

$$d_n := \min\{\operatorname{dist}(\partial V_n, \partial V_{n-1} \cup \partial V_{n+1}), 1/n\}$$

(take $V_0 := \emptyset$) and choose a finite subset Y_n of ∂V_n such that the balls $B(y, d_n/2)$, $y \in Y_n$, cover ∂V_n and the balls $B(y, d_n/6)$, $y \in Y_n$, are pairwise disjoint.

For the moment, let us fix $y \in Y_n$. By Section 3, Proposition 5.1, and translation invariance, there exist a finite set X_y in $\partial B(y, d_n/7)$ and $0 < s_y < d_n/42$ such that

$$\#X_y \cdot \varphi(s_y)h(s_y) < (\#Y_n \cdot 2^n)^{-1}\delta \tag{7.1}$$

and the union E_y of all $\overline{B}(x, s_y)$, $x \in X_y$, satisfies

$$H_{B(y,d_n/6)\setminus E_y}1_{E_y} \geq \kappa \quad \text{on } \overline{B}(y, d_n/7). \tag{7.2}$$

For $x \in X_y$, let $r_x := s_y$. Then, for every $x \in X_y$, $\operatorname{dist}(x, U^c) \geq d_n/2$ and hence $r_x < \operatorname{dist}(x, U^c)/21$.

Let X be the union of all X_y, $y \in Y_n$, $n \in \mathbb{N}$, and let A be the union of all $\overline{B}(x, r_x)$, $x \in X$. Of course, X is locally finite in U and, if U is unbounded, $r_x \to 0$ if $x \to \infty$. Hence, $U \setminus A$ is a champagne subdomain. Moreover, by (7.1),

$$\sum_{x\in X}\varphi(s_x)h(s_x) < \sum_{n\in\mathbb{N}}\sum_{y\in Y_n}(\#Y_n \cdot 2^n)^{-1}\delta = \delta.$$

So it remains to prove that A is unavoidable. To that end we define

$$\eta := \inf\bigl\{H_{B(0,1)\setminus\overline{B}(0,1/7)}1_{\overline{B}(0,1/7)}(z)\colon |z| < 1/2\bigr\},$$

so Brownian motion starting in $B(0, 1/2)$ hits $\overline{B}(0, 1/7)$ with probability at least η before leaving $B(0, 1)$. (Of course η is easily determined: it is $\log 2/\log 7$ if $d = 2$, and $(2^{d-2} - 1)/(7^{d-2} - 1)$ if $d \geq 3$.) Let us fix $n \in \mathbb{N}$, $y \in Y_n$, and let $E := E_y$. We claim that

$$H_{V_{n+1}\setminus E}1_E \geq \kappa\eta \quad \text{on } B(y, d_n/2), \tag{7.3}$$

that is, Brownian motion starting in $B(y, d_n/2)$ hits E with probability at least $\kappa\eta$ before leaving V_{n+1}. Since the balls $B(y, d_n/2)$, $y \in Y_n$, cover ∂V_n, then Proposition 2.1 (this time with $\kappa_n := \kappa\eta$) will show that A is unavoidable.

To prove the claim let

$$B := B(y, d_n), \qquad D := B(y, d_n/6), \qquad F := \overline{B}(y, d_n/7).$$

In probabilistic terms we may argue as follows. Starting in $B(y, d_n/2)$, Brownian motion hits F with probability at least η before leaving $B \subset V_{n+1}$. And, continuing from a point in F, it hits E with probability at least κ before leaving D, by (7.2). So Brownian motion starting in $B(y, d_n/2)$ hits E with probability at least $\kappa\eta$ before leaving V_{n+1}.

For an analytic proof, we first observe that, by translation and scaling invariance of harmonic measures, $H_{B\setminus F}1_F \geq \eta$ on $B(y, d_n/2)$. By the minimum principle,

$$H_{V_{n+1}\setminus E}1_E \geq H_{B\setminus E}1_E \geq H_{D\setminus E}1_E,$$

where $H_{D\setminus E}1_E \geq \kappa$ on F, by (7.2), and hence

$$H_{B\setminus E}1_E = H_{B\setminus(E\cup F)}H_{B\setminus E}1_E \geq \kappa H_{B\setminus(E\cup F)}1_{E\cup F} \geq \kappa H_{B\setminus F}1_F.$$

Thus (7.3) holds and our proof is finished.

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