



Bigraded equivariant cohomology of real quadrics

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Abstract

We give a complete description of the bigraded Bredon cohomology ring of smooth projective real quadrics, with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. These invariants are closely related to the integral motivic cohomology ring, which is not known for these varieties. Some of the results and techniques introduced can be applied to other geometrically cellular real varieties.

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1. Introduction

This paper provides a complete description the *ordinary equivariant cohomology* ring of smooth projective real quadrics, with coefficients in the constant MacKey functor \mathbb{Z} ; cf. [17]. From the geometric point of view, the main motivation for this work is that these cohomology rings can detect algebraic geometric invariants that are invisible to ordinary cohomology. From the topological point of view, the results and techniques presented here can be extended to a vast class of spaces and constitute a rather non-trivial family of examples in ordinary equivariant cohomology.

Let \mathfrak{S} denote the Galois group $Gal(\mathbb{C}/\mathbb{R})$ and let $RO(\mathfrak{S}) = \mathbb{Z} \cdot \mathbf{1} \oplus \mathbb{Z} \cdot \sigma$ be its real representation ring, generated by the trivial representation $\mathbf{1}$ and the *sign* representation σ . The ordinary equivariant cohomology of a \mathfrak{S} -space Y with coefficients in \mathbb{Z} is an $RO(\mathfrak{S})$ -graded ring, written additively as

$$H_{\mathfrak{S}}^*(Y; \mathbb{Z}) = \bigoplus_{\alpha \in RO(\mathfrak{S})} H_{\mathfrak{S}}^{\alpha}(Y; \mathbb{Z}), \tag{1}$$

whose multiplication sends $H_{\mathfrak{S}}^{\alpha}(Y; \mathbb{Z}) \otimes H_{\mathfrak{S}}^{\beta}(Y; \mathbb{Z})$ into $H_{\mathfrak{S}}^{\alpha+\beta}(Y; \mathbb{Z})$, where $\alpha + \beta$ denotes addition in the representation ring. We adopt the *motivic notation* under which we denote $H_{\mathfrak{S}}^{(r-s)\mathbf{1}+s\sigma}(Y; \mathbb{Z})$ by $H_{Br}^{r,s}(Y; \mathbb{Z})$, for integers r and s .

We must emphasize that ordinary equivariant cohomology is not obtained as the singular cohomology of the Borel construction $E\mathfrak{S} \times_{\mathfrak{S}} Y$. The bigraded version stems from the $RO(G)$ -graded equivariant cohomology theories, for arbitrary compact Lie groups G , developed by J. Peter May et al. in [17–19]. With weight $p = 0$ ordinary equivariant cohomology first appeared [3] and hence, for simplicity, we call it *Bredon cohomology* even with non-zero weights, thus explaining the notation $H_{Br}^{n,p}(X; \mathbb{Z})$.

When X is a real algebraic variety, \mathfrak{S} acts via complex conjugation on the space $X(\mathbb{C})$ of complex points of X endowed with the analytic topology. In this context, an important property of this bigraded Bredon cohomology is the existence of natural *cycle maps*

$$\gamma : H_{\mathcal{M}}^{r,s}(X; \mathbb{Z}) \rightarrow H_{Br}^{r,s}(X(\mathbb{C}); \mathbb{Z}), \tag{2}$$

from the *motivic cohomology* $H_{\mathcal{M}}^{r,s}(X; \mathbb{Z})$ of X to the Bredon cohomology of $X(\mathbb{C})$, cf. [7], which assemble into a bigraded ring homomorphism

$$\gamma : H_{\mathcal{M}}^{*,*}(X; \mathbb{Z}) \rightarrow H_{Br}^{*,*}(X(\mathbb{C}); \mathbb{Z}). \tag{3}$$

In particular, this map can become a useful tool to detect non-trivial motivic cohomology classes of real varieties.

Another important feature of bigraded Bredon cohomology theory is the existence of a natural map

$$H_{\text{Br}}^{r,s}(X(\mathbb{C}); \mathbb{Z}) \rightarrow H^r(X(\mathbb{C}); \mathfrak{S}, \mathbb{Z}(s)), \tag{4}$$

where the latter group denotes equivariant Borel cohomology with coefficients in the constant sheaf \mathbb{Z} with a twisted \mathfrak{S} -action. In fact, one can show $H^r(X(\mathbb{C}); \mathfrak{S}, \mathbb{Z}(s))$ is naturally isomorphic to $H_{\text{Br}}^{r,s}(X(\mathbb{C}) \times E\mathfrak{S}; \mathbb{Z})$ [19] and the map (4) is just the pull-back under the projection $X(\mathbb{C}) \times E\mathfrak{S} \rightarrow X(\mathbb{C})$. This means that the theory $H^*(-; \mathfrak{S}, \mathbb{Z}(*))$ is the Borel version of bigraded Bredon cohomology in the sense of [19].

The cohomology theory $H^*(-; \mathfrak{S}, \mathbb{Z}(*))$ has been widely used in real algebraic geometry (see, for example, [14,15,21,22]) because it enjoys many important properties such as the existence of cycle class maps [22] from Chow groups:

$$CH^p(X) \rightarrow H^{2p}(X(\mathbb{C}); \mathfrak{S}, \mathbb{Z}(p)). \tag{5}$$

It is easy to check that this map coincides with the composition of (2) and (4) in the case where $(r, s) = (2p, p)$, therefore the cycle map to Bredon cohomology generalizes the cycle class map (5).

In the case when X has no real points these equivariant cohomology theories coincide but, in general, the bigraded Bredon cohomology is a finer invariant (see Example 2.6) and a more natural one when studying real algebraic varieties (see Remark 2.8). In Section 2 we explain briefly the relation between the two equivariant cohomology theories.

The primary example in this paper is the n -dimensional real quadric $X_{\mathbf{q}}$ associated to a non-degenerate real quadratic form \mathbf{q} of rank $n + 2$ and Witt index s . Surprisingly, even the classical Chow ring $CH^*(X_{\mathbf{q}}) \cong \bigoplus_{p \geq 0} H_{\mathcal{M}}^{2p,p}(X_{\mathbf{q}}; \mathbb{Z})$ of $X_{\mathbf{q}}$ as a real variety seems to be unknown. In recent work [23], N. Yagita computes the motivic cohomology of certain real quadrics with $\mathbb{Z}/2$ -coefficients. The results in [23, Section 6] and [20] should be contrasted with our Theorems A, B and Corollary C below.

Let \mathcal{B} denote the Bredon cohomology ring $\mathcal{B}^{*,*} := H_{\text{Br}}^{*,*}(pt; \mathbb{Z})$ of a point; see Section 2.1. Its subring $\mathcal{B}_+ = \bigoplus_{r \geq 0, s \geq 0} \mathcal{B}^{r,s}$ generated by the non-negatively graded elements is isomorphic to a polynomial ring $\mathfrak{N}[\tau]$, where $\mathfrak{N} = \mathbb{Z}[\varepsilon]$ with $2\varepsilon = 0$. Under this identification, ε is the generator of $\mathcal{B}^{1,1} \cong \mathbb{Z}/2\mathbb{Z}$ and τ is the generator of $\mathcal{B}^{0,2} \cong \mathbb{Z}$.

Given indeterminates h, y and x , let $\langle p_1, \dots, p_k \rangle$ denote the ideal generated by elements p_1, \dots, p_k in the polynomial ring $\mathcal{B}[h, x, y]$. Now, given $0 \leq s \in \mathbb{Z}$, define a subring $\mathcal{B}_s[h, x, y]$ of $\mathcal{B}[h, x, y]$ by

$$\mathcal{B}_s[h, x, y] := \mathcal{B}[h] + \langle h^s \rangle. \tag{6}$$

If p_1, \dots, p_k are elements in $\mathcal{B}_s[h, x, y]$, define

$$[p_1, \dots, p_k] := \langle p_1, \dots, p_k \rangle \cap \mathcal{B}_s[h, x, y] \tag{7}$$

and denote $\mathcal{B}_s[h, y] := \mathcal{B}_s[h, x, y] \cap \mathcal{B}[h, y]$.

Setting $\deg h = (2, 1)$, define for each nonnegative integer m the following bihomogeneous polynomial of degree $(2m + 1, 2m + 1)$:

$$\mathbf{f}_m := \sum_{a+2b=m} \binom{a+b}{b} \varepsilon^{2a+1} \tau^b h^{2b} \in \mathcal{B}_+[h], \tag{8}$$

whose properties are studied in Section 4. These polynomials are used in the description of the main result below, where $\Lambda(\eta)$ denotes the exterior algebra over \mathbb{Z} on a single element η .

Theorem A. *Let $X_{\mathbf{q}}$ be the n -dimensional real quadric associated to a non-degenerate quadratic form \mathbf{q} of rank $n + 2$ and Witt index s , with $n \geq 2s$. If $n = 2m - \delta$, with $\delta \in \{0, 1\}$ and $\deg \eta = (2(n - s + 1), n - s + 1)$, $\deg h = (2, 1)$, $\deg X = (n, -1)$ and $\deg y = (0, -2)$, one has a ring isomorphism*

$$H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z}) \cong \mathcal{B}_s[h, X, y] \otimes \Lambda(\eta) / \mathcal{J}_{n,s},$$

where the ideal $\mathcal{J}_{n,s}$ can be written as

$$\mathcal{J}_{n,s} = [h^s] \cdot \tilde{\mathcal{J}}_{n-2s} \otimes \Lambda(\eta) + [h^s] \otimes \langle \eta \rangle + \langle h^{n-s+1} \otimes 1 - 2(1 \otimes \eta) \rangle,$$

where $\tilde{\mathcal{J}}_{n-2s} = [g_1, g_2, g_3, g_4, g_5] \subset \mathcal{B}[h, X, y]$ is the ideal generated by the elements

$$\begin{aligned} g_1 &= \mathbf{f}_{m-s}, \\ g_2 &= \varepsilon^{1-\delta} \tau^{m-s} X - h^{1-\delta} \mathbf{f}_{m-s-1}, \\ g_3 &= hX, \\ g_4 &= h^{2(m-s)} - \delta \{ (-1)^{m-s} \tau^{m-s+1} X^2 \}, \quad \text{and} \\ g_5 &= \tau y - 1. \end{aligned}$$

The elements h, X and η have explicit geometric origin, described in detail in Sections 5 and 6.

In Section 6 we discuss this ring structure from another perspective more suitable for explicit calculations. In some particular cases, such as Pfister quadrics or general anisotropic quadrics, this presentation acquires a simpler form. Let us first introduce the associated Borel cohomology ring $\mathcal{A} := H_{\text{Br}}^{*,*}(E\mathfrak{S}; \mathbb{Z})$, where $E\mathfrak{S}$ is the classifying space of \mathfrak{S} . This ring is obtained from \mathcal{B} by inverting the element τ , i.e. $\mathcal{A} \cong \mathcal{B}[y]/\langle y\tau - 1 \rangle$, and is isomorphic to $\aleph[\tau, \tau^{-1}]$; see Remark 2.9(ii).

Theorem B. *Let \mathbf{q} be an anisotropic real quadratic form of rank n . Then:*

(1) *For $n = 2m - 1$ one has an isomorphism of \mathcal{B} -algebras:*

$$H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z}) \cong \mathcal{A}[h] / I_{2m-1},$$

where $I_{2m-1} = \langle \mathbf{f}_m, h\mathbf{f}_{m-1}, h^{2m} \rangle$, with $\deg h = (2, 1)$.

(2) For $n = 2m$ one has a ring isomorphism:

$$H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z}) \cong \mathcal{A}[h, X]/J_{2m},$$

where $J_{2m} = \langle \mathbf{f}_m; \varepsilon \tau^m X - h \mathbf{f}_{m-1}; hX; h^{2m} - (-1)^m \tau^{m+1} X^2 \rangle$, with $\text{deg } h = (2, 1)$ and $\text{deg } X = (2m, -1)$.

Corollary C. If X is a Pfister quadric of dimension $2^{r+1} - 2$, then

$$H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z}) \cong \mathcal{A}[h, X]/J_{2^{r+1}-2},$$

where $J_{2^{r+1}-2}$ is the ideal

$$\left\langle \varepsilon^{2^r-1}; \varepsilon \tau^{2^r-1} X - h \varepsilon \sum_{j=0}^{r-2} (\varepsilon^4)^{2^r-2^j} \tau^{2^j-1} (h^2)^{2^j-1}; hX; h^{2^{r+1}-2} + \tau^{2^r} X^2 \right\rangle.$$

The paper is organized as follows. Section 2 provides the necessary topological background, for the reader less familiar with Bredon cohomology and its associated Borel cohomology. In Section 3 we introduce spectral sequences

$$E_*^{i,j}(p) \Rightarrow H_{\text{bor}}^{i+j,p}(Y, \mathbb{Z})$$

converging to the associated Borel cohomology—see Definition 2.10—of a \mathfrak{S} -space Y . These sequences have a “tri-graded” multiplicative structure and acquire a particularly interesting form when $Y = X(\mathbb{C})$ where X is a *geometrically cellular* real variety; cf. Definition 2.10. Given a real variety X , let $\mathcal{C}\mathcal{H}^k(X_{\mathbb{C}})$ denote the Chow group $CH^k(X_{\mathbb{C}})$ of codimension- k algebraic cycles modulo rational equivalence, seen as a \mathfrak{S} -module under the action of the Galois group on cycles. Also, for $m \in \mathbb{Z}$, let $\mathbb{Z}(m)$ denote the \mathfrak{S} -submodule $(2\pi i)^m \mathbb{Z} \subset \mathbb{C}$, and for a \mathfrak{S} -module \mathcal{N} let $H^*(\mathfrak{S}; \mathcal{N})$ denote group cohomology with coefficients in \mathcal{N} .

Proposition D. Let X be a geometrically cellular real variety and let $X_{\mathbb{C}}$ be the complex variety obtained by base-change. Then there is a family of spectral sequences $\{E_r^{*,*}(p), d_r\}$ converging to $H_{\text{bor}}^{*,p}(X(\mathbb{C}), \mathbb{Z})$, with

$$E_r^{i,j}(p) = \begin{cases} H^i(\mathfrak{S}; \mathcal{C}\mathcal{H}^k(X_{\mathbb{C}}) \otimes \mathbb{Z}(q - k)), & \text{if } j = 2k \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

It must be noted that B. Kahn has constructed in [12] a spectral sequence converging within a range to Lichtenbaum’s étale motivic cohomology of a real variety whose E_2 -term coincides with ours. A comparison between these two spectral sequences is yet to be done.

Our strategy is to start with anisotropic quadrics in Section 5. In this case Bredon cohomology coincides with its associated Borel theory and one can use the spectral sequences above to aid the computation. After the identification of the appropriate generators h and X in Definition 5.1, we use the various algebraic results from Section 4 to completely determine the relations defining the ideal J_n , proving Theorem B above.

In Section 6 we deal with arbitrary isotropic quadrics. In this case, the additive structure of the cohomology has a classical decomposition which, in current terminology, follows directly from

the decomposition of the *motivic* of the quadric. Roughly speaking, if $\mathbf{q} = \mathbf{q}' + \mathbf{h}$ is a quadratic form of rank $n + 2$, where \mathbf{q}' is anisotropic and \mathbf{h} is hyperbolic of rank s , then $H_{\text{Br}}^{*,*}(\mathbf{X}_{\mathbf{q}}(\mathbb{C}); \mathbb{Z})$ is isomorphic to

$$\left(\bigoplus_{j=0}^{s-1} \mathcal{B} \cdot \mathbf{h}^j \right) \oplus H_{\text{Br}}^{*-2s, *-s}(\mathbf{X}_{\mathbf{q}'}; \mathbb{Z}) \oplus \left(\bigoplus_{j=0}^{s-1} \mathcal{B} \cdot \eta \mathbf{h}^j \right),$$

where \mathbf{h} is the first Chern class in Bredon cohomology of the hyperplane bundle and η is the Poincaré dual of a maximal real linear subspace of dimension $s - 1$ contained in $\mathbf{X}_{\mathbf{q}}$. One should contrast this decomposition with Theorem A. The multiplicative structure involves a careful study of the maps involved in this motivic splitting, along with the relationship between the coefficient rings \mathcal{A} and \mathcal{B} , respectively.

2. Background

This section contains a brief summary of the main properties of $RO(\mathfrak{S})$ -graded Bredon cohomology and its associated Borel version.

Definition 2.1. Given $q \in \mathbb{Z}$, define $\mathbb{Z}(q) := (2\pi i)^q \mathbb{Z} \subset \mathbb{C}$ with the \mathfrak{S} -module structure induced by complex conjugation. If M is a \mathfrak{S} -module, denote $M(q) := M \otimes_{\mathbb{Z}} \mathbb{Z}(q)$, and let $\underline{M}(q)$ be the associated Mackey functor. For simplicity, write $\underline{M} := \underline{M}(0)$. We denote by $\mathbb{Z}[\xi, \xi^{-1}]$ the $\mathbb{Z}[\mathfrak{S}]$ -subalgebra of \mathbb{C} generated by $\xi := 2\pi i$. It has a natural graded ring structure defined by setting $\deg \xi = 1$. As a $\mathbb{Z}[\mathfrak{S}]$ -module, we have $\mathbb{Z}[\xi, \xi^{-1}] \cong \bigoplus_{q \in \mathbb{Z}} \mathbb{Z}(q)$. If M is a $\mathbb{Z}[\mathfrak{S}]$ -module, let $M[\xi, \xi^{-1}]$ denote the $\mathbb{Z}[\mathfrak{S}]$ -module $M \otimes_{\mathbb{Z}} \mathbb{Z}[\xi, \xi^{-1}]$.

Given a \mathfrak{S} -module M and a \mathfrak{S} -space X , the $RO(\mathfrak{S})$ -graded Bredon cohomology of X with coefficients in \underline{M} is an $RO(\mathfrak{S})$ -graded abelian group

$$H_{\mathfrak{S}}^*(X; \underline{M}) = \bigoplus_{\alpha \in RO(\mathfrak{S})} H_{\mathfrak{S}}^{\alpha}(X; \underline{M}).$$

We adopt the *motivic notation* under which $H_{\mathfrak{S}}^{(r-s)\mathbf{1}+s\sigma}(X; \mathbb{Z})$ is denoted $H_{\text{Br}}^{r,s}(X; \mathbb{Z})$, where $r, s \in \mathbb{Z}$, $\mathbf{1}$ denotes the trivial representation of dimension 1 and σ denotes the *sign representation*.

Remark 2.2. Throughout the paper, whenever we want to emphasize the action of \mathfrak{S} on some cohomology group, we use calligraphic symbols. For example, if X is a real variety then the Chow groups $CH^*(X_{\mathbb{C}})$ of the complex variety $X_{\mathbb{C}}$ obtained from X by extension of scalars have a natural \mathfrak{S} -action and we denote by $\mathcal{C}\mathcal{H}^*(X_{\mathbb{C}})$ the corresponding $\mathbb{Z}[\mathfrak{S}]$ -modules. Similarly, we denote by $\mathcal{H}_{\text{sing}}^*(X_{\mathbb{C}}; \mathbb{Z}(p))$ the $\mathbb{Z}[\mathfrak{S}]$ -module obtained by the simultaneous action of \mathfrak{S} on the integral singular cohomology of $X_{\mathbb{C}}$ and on $\mathbb{Z}(p)$.

Property 2.3. Fix a \mathfrak{S} -module M .

- (i) There is a *forgetful functor* $\varphi : H_{\text{Br}}^{p,q}(X; \underline{M}) \rightarrow H^p(X; M(q))$ to ordinary singular cohomology. This maps factors as

$$H_{\text{Br}}^{p,q}(X; \underline{M}) \rightarrow \mathcal{H}_{\text{sing}}^p(X; M(q))^{\mathfrak{S}} \hookrightarrow H_{\text{sing}}^p(X; M(q)),$$

where $\mathcal{H}_{\text{sing}}^p(X; M(q))^{\mathfrak{S}}$ denotes the invariants of $\mathcal{H}_{\text{sing}}^p(X; M(q))$.

- (ii) There is a transfer functor $\mathbf{t}: H_{\text{sing}}^p(X; M) \rightarrow H_{\text{Br}}^{p,q}(X; \underline{M})$ such that the composite $\mathbf{t} \circ \varphi: H_{\text{Br}}^{p,q}(X; \underline{M}) \rightarrow H_{\text{Br}}^{p,q}(X; \underline{M})$ is multiplication by 2.
- (iii) If A is a commutative $\mathbb{Z}[\mathfrak{S}]$ -algebra, the multiplication $A \otimes_{\mathbb{Z}[\mathfrak{S}]} A \rightarrow A$ induces a structure of $(\mathbb{Z} \times \mathbb{Z})$ -graded ring on

$$H_{\text{Br}}^{*,*}(X; \underline{A}) := \bigoplus_{(p,q) \in \mathbb{Z} \times \mathbb{Z}} H_{\text{Br}}^{p,q}(X; \underline{A}).$$

In this case, $H^*(X; A[\xi, \xi^{-1}])$ has a natural structure of $\mathbb{Z}[\mathfrak{S}]$ -algebra and the forgetful functor becomes a map of (bigraded) $\mathbb{Z}[\mathfrak{S}]$ -algebras

$$\varphi: H_{\text{Br}}^{*,*}(X; \underline{A}) \rightarrow H_{\text{sing}}^*(X; A[\xi, \xi^{-1}]),$$

whose image lies in the invariant subring $\mathcal{H}_{\text{sing}}^*(X; A[\xi, \xi^{-1}])^{\mathfrak{S}}$.

- (iv) If M has the trivial \mathfrak{S} -action there is a natural isomorphism $H_{\text{Br}}^{p,0}(X; \underline{M}) \cong H_{\text{sing}}^p(X/\mathfrak{S}; M)$.
- (v) If \mathfrak{S} acts freely on X and M is $\mathbb{Z}/2$ -algebra with trivial \mathfrak{S} -action there is a natural isomorphism $H_{\text{Br}}^{p,q}(X; \underline{M}) \cong H_{\text{sing}}^p(X/\mathfrak{S}; M(q))$.
- (vi) There is a natural isomorphism $H_{\text{Br}}^{p,q}(X \times \mathfrak{S}; \underline{M}) \cong H_{\text{sing}}^p(X; M)$. Under this isomorphism the forgetful functor φ is identified with the map $pr_1^*: H_{\text{Br}}^{p,q}(X; \underline{M}) \rightarrow H_{\text{Br}}^{p,q}(X \times \mathfrak{S}; \underline{M})$ induced by the projection onto the first factor $pr_1: X \times \mathfrak{S} \rightarrow X$.
- (vii) (Poincaré duality [6, Section 1.6], [4]). If X is a smooth compact Real manifold (e.g. the complex points of a projective real variety) of dimension n then, for each p, q there is an isomorphism

$$H_{\text{Br}}^{p,q}(X; \underline{\mathbb{Z}}) \rightarrow H_{2n-p, n-q}^{\text{Br}}(X; \underline{\mathbb{Z}}),$$

where the latter denotes bigraded Bredon homology. This isomorphism is induced by the cap product with the fundamental class $[X] \in H_{2n, n}^{\text{Br}}(X; \underline{\mathbb{Z}})$, which is dual to $\mathbf{1} \in H_{\text{Br}}^{0,0}(X; \underline{\mathbb{Z}})$.

Definition 2.4. The associated Borel cohomology to $H_{\text{Br}}^{*,*}(-; \underline{M})$ is defined as

$$H_{\text{bor}}^{p,q}(X, \underline{M}) := H_{\text{Br}}^{p,q}(X \times E\mathfrak{S}; \underline{M});$$

cf. [19, p. 35].

Remark 2.5. In [6] it is shown that $H_{\text{bor}}^{p,q}(X, \underline{\mathbb{Z}})$ is naturally isomorphic to the twisted Borel equivariant cohomology group $H^p(X; \mathfrak{S}, \mathbb{Z}(q))$, which is what real algebraic geometers usually refer to as equivariant cohomology with $\mathbb{Z}(q)$ coefficients [15,21,22].

Example 2.6. If X has no fixed points then the associated Borel cohomology agrees with the bigraded Bredon cohomology because X is \mathfrak{S} -homotopy equivalent to $X \times E\mathfrak{S}$. However, in general, the Borel theory loses information, as can be seen by considering the spaces $S^{2,2}$ and $S^{2,0}$. Indeed, it is easy to check that $H_{\text{bor}}^{*,*}(S^{2,2}, \underline{\mathbb{Z}}) \cong H_{\text{bor}}^{*,*}(S^{2,0}, \underline{\mathbb{Z}})$ but $H_{\text{Br}}^{*,*}(S^{2,2}; \underline{\mathbb{Z}}) \not\cong H_{\text{Br}}^{*,*}(S^{2,0}; \underline{\mathbb{Z}})$.

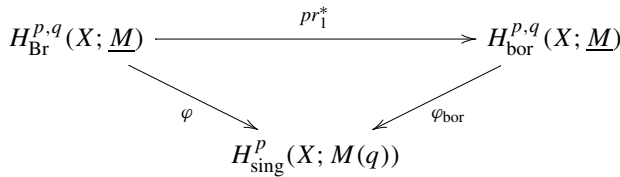
Property 2.7. Fix an abelian group M .

- (i) The projection $pr_1 : X \times E\mathfrak{S} \rightarrow X$ induces a natural transformation

$$pr_1^* : H_{\text{Br}}^{p,q}(X; \underline{M}) \rightarrow H_{\text{bor}}^{p,q}(X, \underline{M}),$$

which is a ring homomorphism whenever $M = A$ is a $\mathbb{Z}[\mathfrak{S}]$ -algebra.

- (ii) When \mathfrak{S} acts freely on X , pr_1^* is an isomorphism. In particular, if X is a finite CW-complex of dimension m then $H_{\text{Br}}^{r,s}(X; \underline{M}) = 0$, for all $r > m$. Furthermore, whenever $M = A$ is a $\mathbb{Z}[\mathfrak{S}]$ -algebra then $H_{\text{bor}}^{*,*}(X; \underline{A})$ becomes an algebra over $H_{\text{Br}}^{*,*}(pr; \underline{A})$.
- (iii) There is a forgetful functor $\varphi_{\text{bor}} : H_{\text{bor}}^{p,q}(X; \underline{M}) \rightarrow H_{\text{sing}}^p(X; M(q))$ making the following diagram commute



- (iv) If A is a $\mathbb{Z}/2$ -algebra with the trivial \mathfrak{S} -action then for every \mathfrak{S} -space X one has a natural isomorphism of bigraded rings

$$H_{\text{bor}}^{*,*}(X; \underline{A}) \cong H_{\text{sing}}^*(E\mathfrak{S} \times_{\mathfrak{S}} X; A) \otimes \mathbb{Z}[\xi, \xi^{-1}],$$

where $\alpha \in H_{\text{sing}}^r(X; A)$ has bidegree $(r, 0)$, $\deg \xi = (0, 1)$ and $\deg \xi^{-1} = (0, -1)$. In particular, if X is a free \mathfrak{S} -space, this induces an isomorphism of bigraded rings

$$H_{\text{Br}}^{*,*}(X; \underline{A}) \cong H_{\text{sing}}^*(X/\mathfrak{S}; A) \otimes \mathbb{Z}[\xi, \xi^{-1}].$$

Remark 2.8. One of the important applications of the Borel version of bigraded Bredon cohomology in real algebraic geometry consists in using the natural map

$$\beta^k : H_{\text{bor}}^{2p,p}(X(\mathbb{C}); \underline{\mathbb{Z}}) \rightarrow H_{\text{sing}}^k(X(\mathbb{R}); \mathbb{Z}/2),$$

(induced by restriction to $X(\mathbb{R})$, reduction of coefficients and the Künneth formula) to define topological restrictions on the group of algebraic cohomology classes

$$H_{\text{alg}}^k(X(\mathbb{R}); \mathbb{Z}/2) \subset H_{\text{sing}}^k(X(\mathbb{R}); \mathbb{Z}/2)$$

(see [15,22]). For example, in [22] the subgroup of potentially algebraic cohomology classes is defined as follows:

$$H_{\geq 1/2}^k(X(\mathbb{R}); \mathbb{Z}/2) = \beta^k(\{\omega \in H_{\text{bor}}^{2k,k}(X(\mathbb{C}), \underline{\mathbb{Z}}) \mid \beta^m(\omega) = 0, \text{ for } m < k\}).$$

Now, the classes in $H_{\text{bor}}^{2k,k}(X(\mathbb{C}), \underline{\mathbb{Z}})$ coming from $H_{\text{Br}}^{2k,k}(X(\mathbb{C}); \underline{\mathbb{Z}})$ are equivariant homotopy classes of maps to an Eilenberg–MacLane space, which we denote by $K(\mathbb{Z}(k), 2k)$. The

computation of the homotopy type of $K(\mathbb{Z}(k), 2k)^{\mathfrak{S}}$ in [16] gives $\pi_m(K(\mathbb{Z}(k), 2k)^{\mathfrak{S}}) = 0$, for $m < k$. This shows that the image of $H_{\text{Br}}^{2k,k}(X(\mathbb{C}); \mathbb{Z})$ in $H_{\text{bor}}^{2k,k}(X(\mathbb{C}), \mathbb{Z})$ lies inside the group $H_{\geq 1/2}^k(X(\mathbb{R}); \mathbb{Z}/2)$. Thus the computation of the bigraded Bredon cohomology ring of $X(\mathbb{C})$ provides more precise information about the algebraic cohomology classes of $X(\mathbb{R})$.

2.1. Cohomology ring of a point

Let $\mathcal{B} := H_{\text{Br}}^{*,*}(pt; \mathbb{Z})$ denote the coefficient ring of the cohomology theory $H_{\text{Br}}^{*,*}(-; \mathbb{Z})$, and let $\mathcal{A} := H_{\text{bor}}^{*,*}(pt; \mathbb{Z})$ denote the coefficient ring of the associated Borel theory $H_{\text{bor}}^{*,*}(-; \mathbb{Z})$. Denote by $\pi : \mathcal{B} \rightarrow \mathcal{A}$ the natural ring homomorphism induced by the projection $\pi : E\mathfrak{S} \rightarrow pt$; cf. Property 2.7(i).

Consider $\mathfrak{S} = \mathbb{Z}[\varepsilon] := \mathbb{Z}[X]/\langle 2X \rangle$, where ε is an indeterminate of bidegree $(1, 1)$, satisfying $2\varepsilon = 0$. It follows from Property 2.7 and basic computations in group cohomology that \mathcal{A} has the following bigraded ring structure.

$$\mathcal{A} \cong \mathbb{Z}[\varepsilon, \tau, \tau^{-1}] = \mathfrak{S}[\tau, \tau^{-1}], \tag{9}$$

where τ has bidegree $(0, 2)$.

In order to describe \mathcal{B} , first consider indeterminates $\varepsilon, \varepsilon^{-1}, \tau, \tau^{-1}$ satisfying $\deg \varepsilon = (1, 1)$, $\deg \varepsilon^{-1} = (-1, -1)$, $\deg \tau = (0, 2)$ and $\deg \tau^{-1} = (0, -2)$. Henceforth, ε and ε^{-1} will always satisfy $2\varepsilon = 0 = 2\varepsilon^{-1}$.

As an abelian group, \mathcal{B} can be written as a direct sum

$$\mathcal{B} := \mathbb{Z}[\varepsilon, \tau] \cdot 1 \oplus \mathbb{Z}[\tau^{-1}] \cdot \alpha \oplus \mathbb{F}_2[\varepsilon^{-1}, \tau^{-1}] \cdot \theta \tag{10}$$

where each summand is a free bigraded module over the indicated ring. The bidegrees of the generators $1, \alpha$ and θ are, respectively, $(0, 0)$, $(0, -2)$ and $(0, -3)$.

The product structure on \mathcal{B} is completely determined by the following relations

$$\alpha \cdot \tau = 2, \quad \alpha \cdot \theta = \alpha \cdot \varepsilon = \theta \cdot \tau = \theta \cdot \varepsilon = 0, \tag{11}$$

and the bigraded ring homomorphism $\pi : \mathcal{B} \rightarrow \mathcal{A}$ is determined by

$$\begin{aligned} \varepsilon &\mapsto \varepsilon, & \tau &\mapsto \tau, \\ \tau^{-j}\alpha &\mapsto 2\tau^{-j-1}, & \varepsilon^{-j-1}\alpha &\mapsto 0, \\ \varepsilon^{-j}\theta &\mapsto 0, & \tau^{-j}\theta &\mapsto 0, \end{aligned}$$

for $j \geq 0$.

Remark 2.9.

- (i) Note that \mathcal{B} is not finitely generated as a ring, and that \mathcal{B} has no homogeneous elements in degrees (p, q) when $p \cdot q < 0$.
- (ii) The ring \mathcal{A} is obtained from \mathcal{B} by inverting the element τ , i.e. $\mathcal{A} \cong \mathcal{B}[y]/\langle y\tau - 1 \rangle$.

2.2. Bredon cohomology of real algebraic varieties

Given a real algebraic variety X , we denote by $X(\mathbb{C})$ its set of *complex points* endowed with the analytic topology. It is a \mathfrak{S} -space under the action of complex conjugation and we can consider the its Bredon cohomology ring $H_{\text{Br}}^{*,*}(X(\mathbb{C}); \underline{\mathbb{Z}})$. It is related to the motivic cohomology ring by a homomorphism called the *cycle map* $\gamma : H_{\mathcal{M}}^{r,s}(X; \mathbb{Z}) \rightarrow H_{\text{Br}}^{r,s}(X(\mathbb{C}); \underline{\mathbb{Z}})$ which generalizes the classical map

$$cl : CH^n(X) = H_{\mathcal{M}}^{2n,n}(X; \mathbb{Z}) \rightarrow H_{\text{sing}}^{2n}(X(\mathbb{C}); \mathbb{Z}(n)).$$

Definition 2.10. An algebraic variety X defined over a field k is *cellular* if there is a filtration $X = X_n \supset X_{n-1} \supset \dots \supset X_0 \supset X_{-1} = \emptyset$ by closed subvarieties such that $X_i - X_{i-1}$ is isomorphic to a disjoint union of affine spaces $\mathbb{A}^{n_{ij}}$. Whenever \bar{k} is an algebraic closure for k and $X_{\bar{k}}$ is cellular we say that X is *geometrically cellular*.

For cellular varieties over \mathbb{R} the Bredon cohomology ring has a simple description relating it to the *Chow ring*.

Proposition 2.11. *Let X be a cellular real variety. Then there is a natural ring isomorphism $H_{\text{Br}}^{*,*}(X(\mathbb{C}); \underline{\mathbb{Z}}) \cong CH^*(X) \otimes_{\mathbb{Z}} \mathcal{B}$, where the elements of $CH^n(X)$ are given degree $(2n, n)$.*

Proof. From Definition 2.10 it is easy to show that $H_{\text{Br}}^{*,*}(X(\mathbb{C}); \underline{\mathbb{Z}})$ is a free \mathcal{B} -module generated by elements with bidegrees of the form $(2r, r)$, with $r \geq 0$. Set $E := \bigoplus_{n \geq 0} H_{\text{Br}}^{2n,n}(X(\mathbb{C}); \underline{\mathbb{Z}})$. Since $\bigoplus_{n \geq 0} \mathcal{B}^{2n,n} = \mathcal{B}^{0,0} \cong \mathbb{Z}$ it follows that the inclusion $E^{*,*} \subset H_{\text{Br}}^{*,*}(X(\mathbb{C}); \underline{\mathbb{Z}})$ induces a graded ring isomorphism $E^{*,*} \otimes_{\mathbb{Z}} \mathcal{B}^{*,*} \cong H_{\text{Br}}^{*,*}(X(\mathbb{C}); \underline{\mathbb{Z}})$.

Using Definition 2.10 and basic properties of motivic and Bredon cohomology, it follows that for a real variety X the cycle map $\gamma : CH^*(X) = \bigoplus_{n \geq 0} H_{\mathcal{M}}^{2n,n}(X; \mathbb{Z}) \rightarrow \bigoplus_{n \geq 0} H_{\text{sing}}^{2n,n}(X(\mathbb{C}); \underline{\mathbb{Z}})$ is a ring isomorphism. Hence $E \cong CH^*(X)$ and the result follows. \square

3. Descent spectral sequence

For each $\mathbb{Z}[\mathfrak{S}]$ -module M and each integer q there is a spectral sequence

$$E_2^{r,s}(q) := H^r(\mathfrak{S}; \mathcal{H}_{\text{sing}}^s(X; M(q))) \Rightarrow H_{\text{bor}}^{r+s,q}(X; \underline{M}),$$

where $H^r(\mathfrak{S}; \mathcal{H}_{\text{sing}}^s(X; M(q)))$ denotes group cohomology of \mathfrak{S} with coefficients in the $\mathbb{Z}[\mathfrak{S}]$ -module $\mathcal{H}_{\text{sing}}^s(X; M(q))$; cf. Remark 2.2. This sequence can be seen as the spectral sequence for the homotopy groups of a homotopy limit, cf. [2]. If $M = A$ is a $\mathbb{Z}[\mathfrak{S}]$ -algebra, its multiplication gives rise to a pairing of spectral sequences

$$E_2^{r,s}(q) \otimes E_2^{r',s'}(q') \rightarrow E_2^{r+r',s+s'}(q+q'), \tag{12}$$

for every $q, q' \in \mathbb{Z}$. This makes $\{E_2^{r,s} := \bigoplus_{q \in \mathbb{Z}} E_2^{r,s}(q)\}$ into a spectral sequence of algebras converging to $H_{\text{bor}}^{*,*}(X; \underline{A})$.

3.1. Spectral sequences for geometrically cellular varieties

Given a smooth projective variety X defined over a field $k \subset \mathbb{R}$, the cycle map

$$cl : \mathcal{CH}^*(X_{\mathbb{C}}) \rightarrow \bigoplus_{j \geq 0} \mathcal{H}_{\text{sing}}^{2j}(X(\mathbb{C}); \mathbb{Z}(j)) \tag{13}$$

is a homomorphism of $\mathbb{Z}[\mathfrak{S}]$ -algebras from the intersection ring $\mathcal{CH}^*(X_{\mathbb{C}})$ to the singular cohomology ring $\mathcal{H}_{\text{sing}}^{2*}(X(\mathbb{C}); \mathbb{Z}(*))$.

When X is a smooth geometrically cellular variety, this map is an isomorphism. Hence, the E_2 -term of the spectral sequence described in Property 2.7(iv) becomes

$$E_2^{r,s}(q) \cong \begin{cases} H^r(\mathfrak{S}; \mathcal{CH}^j(X_{\mathbb{C}}) \otimes \mathbb{Z}(q - j)), & \text{if } s = 2j, 0 \leq j \leq \dim X, \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Lemma 3.1. *Let X be a geometrically cellular real variety. If $j \not\equiv 3 \pmod{4}$ then the differential d_j for the spectral sequence $E_*^{*,*}(q)$ vanishes.*

Proof. It follows from (14), the freeness of $CH^j(X_{\mathbb{C}})$ for geometrically cellular varieties and basic computations in group cohomology (see [8, p. 6]) that if $E_2^{r',s'}(q) \neq 0$, then

$$\begin{cases} s' \equiv 0 \pmod{2}, & \text{and} \\ 2r' + 2q + s' \equiv 0 \pmod{4}. \end{cases} \tag{15}$$

Now, if $d_j \neq 0$ one can find (r, s) such that both $E_j^{r,s}(q)$ and $E_j^{r+j,s-j+1}(q)$ are non-zero, and (15) shows that $2r + 2q + s \equiv 0 \pmod{4}$, and $2(r + j) + 2q + s - j + 1 \equiv 0 \pmod{4}$. Therefore, $j \equiv -1 \pmod{4}$. \square

In order to better understand the ring structure in (12) for a geometrically cellular variety X , denote

$$\mathcal{E} := \bigoplus_{k \in \mathbb{Z}} \mathcal{E}^k, \quad \text{where } \mathcal{E}^k := \bigoplus_{r,j} E_2^{r,2j}(j - k). \tag{16}$$

Then \mathcal{E} is a graded ring under the pairing described in Property 2.7(iii) and, under this structure, \mathcal{E}^0 is a subring which inherits its bigrading from the group cohomology ring $H^*(\mathfrak{S}; \mathcal{CH}^*(X_{\mathbb{C}}))$ of \mathfrak{S} with coefficients in $\mathcal{CH}^*(X_{\mathbb{C}})$. The elements $\tau \in E^{0,0}(2) \subset \mathcal{E}^{-2}$ and $\tau^{-1} \in E^{0,0}(-2) \subset \mathcal{E}^2$ coming from the cohomology ring \mathcal{A} , cf. (9), induce inverse isomorphisms of bigraded \mathcal{E}^0 -modules $\tau : \mathcal{E}^k \rightarrow \mathcal{E}^{k-2}$, for all $k \in \mathbb{Z}$.

Define $\mathcal{E}_0 = \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_0^k$, where $\mathcal{E}_0^k = \bigoplus_{j \geq 0} E^{0,2j}(j - k)$. In other words, \mathcal{E}_0 is the graded subring of \mathcal{E} formed by the first columns of the spectral sequences $E_*^{*,*}(*).$ It follows that $\mathcal{E}_0 \cong \{\mathcal{CH}^*(X_{\mathbb{C}}) \otimes \mathbb{Z}[\xi, \xi^{-1}]\}^{\mathfrak{S}}$, and hence \mathcal{E}_0 is a free abelian group. In particular, $\mathcal{E}_0^0 \cong \bigoplus_j \mathcal{CH}^j(X_{\mathbb{C}})^{\mathfrak{S}}$ consists of the *Galois invariants* of the Chow ring $CH^*(X_{\mathbb{C}})$, and $\mathcal{E}_0^1 = \bigoplus_j \mathcal{CH}^j(X_{\mathbb{C}})^{-}$, consists of the *anti-invariants*, i.e. those classes $\alpha \in CH^*(X_{\mathbb{C}})$ for which $\sigma^* \alpha = -\alpha$. Multiplication by $\tau = \xi^2$ induces isomorphisms $\tau : \mathcal{E}_0^k \rightarrow \mathcal{E}_0^{k-2}$.

Proposition 3.2. *Let X be a geometrically cellular real variety.*

- (i) $2 \cdot H_{\text{Br}}^{*,*}(X(\mathbb{C}); \mathbb{Z})_{\text{tor}} = 0$.
- (ii) *One has isomorphisms of abelian groups*

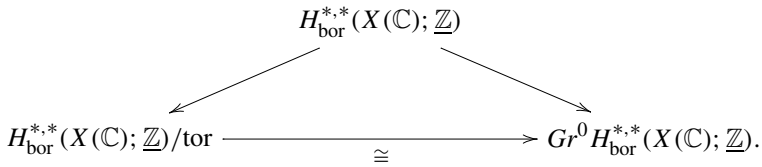
$$H_{\text{bor}}^{p,q}(X(\mathbb{C}); \mathbb{Z}) \cong \bigoplus_{r+s=p} E_{\infty}^{r,s}(q).$$

In particular, one has an isomorphism of abelian groups

$$H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z}) \cong \bigoplus_{j \geq 0} Gr^j H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z}) = F^j / F^{j+1},$$

where $H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z}) = F^0 \supseteq F^1 \supseteq \dots$ denotes the filtration associated to the spectral sequence. Note that the resulting ring structure on $E_{\infty}^{*,*}$ does not determine the ring structure on $H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z})$.

- (iii) *In the filtration above, each F^j is an ideal in F^0 and $Gr^0 := F^0/F^1$ is a free abelian group. Therefore, the natural projection $F^0 \rightarrow F^0/F^1$ factors through F^0/F^0_{tor} , giving the following commutative diagram of ring epimorphisms:*



- (iv) *The forgetful map $\varphi_{\text{bor}} : F^0 = H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z}) \rightarrow H_{\text{sing}}^*(X(\mathbb{C}); \mathbb{Z})$ factors as*

$$F^0 \rightarrow F^0/F^1 \subseteq \{\mathcal{C}\mathcal{H}^*(X_{\mathbb{C}}) \otimes \mathbb{Z}[\xi, \xi^{-1}]\}^{\mathfrak{S}} \subseteq H_{\text{sing}}^{2*}(X(\mathbb{C}); \mathbb{Z}(*)).$$

Proof. (i) This follows immediately from Property 2.3(ii) and the fact that $H_{\text{sing}}^*(X(\mathbb{C}); \mathbb{Z})$ is torsion free.

(ii) Since $H_{\text{sing}}^*(X(\mathbb{C}); \mathbb{Z})$ is free, it follows from Eq. (14) and basic computations in group cohomology that $2 \cdot E_2^{r,s} = 0$ for all $r > 0$. That is, all columns of the E_2 -term but the first are 2-torsion. Hence the same is true for E_{∞} and so $F^1 \cong \bigoplus_{j \geq 1} F^j / F^{j+1}$ because F^j / F^{j+1} is an \mathbb{F}_2 -vector space, for $j \geq 1$. As explained above, the first column of $E_2^{*,*}$ is free. Hence the same is true for E_{∞} . This implies that $F^0 \cong F^1 \oplus F^0/F^1$.

(iii) The first statement is just a consequence of basic properties of multiplicative spectral sequences. The second statement is a consequence of the remarks preceding this proposition. Now, $H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z}) \cong F^0/F^1 \oplus F^1$ and F^0/F^1 is free. As mentioned in the proof of (ii), F^1 is torsion, hence $F^1 = H_{\text{bor}}^{*,*}(X(\mathbb{C}); \mathbb{Z})_{\text{tor}}$. This implies that the diagram commutes and the bottom arrow is an isomorphism.

(iv) Let $i : \mathfrak{S} \rightarrow E\mathfrak{S}$ denote the inclusion as the zero skeleton. By the definition of the filtration F^{\bullet} , F^1 is the kernel of $(\text{id}_X \times i)^* : H_{\text{Br}}^{*,*}(X \times E\mathfrak{S}; \mathbb{Z}) \rightarrow H_{\text{Br}}^{*,*}(X \times \mathfrak{S}; \mathbb{Z})$. The forgetful map φ_{bor} (cf. Property 2.7(iii)), is given by $\varphi_{\text{bor}} = (pr_{13}^*)^{-1} \circ pr_{12}^*$, where pr_{ij} denotes the projection onto

the ij factor of $X \times E\mathfrak{S} \times \mathfrak{S}$. It is easy to see that $(\text{id}_X \times i) \circ \text{pr}_{13} \simeq \text{pr}_{12}$ hence $(\text{id}_X \times i)^* = \varphi_{\text{bor}}$, and φ_{bor} factors as $F^0 \rightarrow F^0/F^1 \subset H_{\text{sing}}^*(X(\mathbb{C}); \mathbb{Z}(\ast))$. Since X is geometrically cellular $\mathcal{H}_{\text{sing}}^{2\ast}(X(\mathbb{C}); \mathbb{Z}(\ast))$ is identified with $\mathcal{C}\mathcal{H}^*(X_{\mathbb{C}}) \otimes \mathbb{Z}[\xi, \xi^{-1}]$. Finally, by Property 2.3(i) we have $\text{im } \varphi_{\text{bor}} \subset \mathcal{H}_{\text{sing}}^*(X(\mathbb{C}); \mathbb{Z}(\ast))^{\mathfrak{S}}$. Together these facts give a factorization of φ_{bor} as stated in the proposition. \square

Corollary 3.3. *Given a geometrically cellular real variety X then a cohomology class $\alpha \in H_{\text{bor}}^{p,q}(X(\mathbb{C}); \mathbb{Z})$ is completely determined by its image under the forgetful map*

$$\varphi : H_{\text{bor}}^{p,q}(X(\mathbb{C}); \mathbb{Z}) \rightarrow H_{\text{sing}}^p(X(\mathbb{C}); \mathbb{Z}(q))$$

and the reduction of coefficients map

$$\rho : H_{\text{bor}}^{p,q}(X(\mathbb{C}); \mathbb{Z}) \rightarrow H_{\text{bor}}^{p,q}(X(\mathbb{C}); \mathbb{F}_2).$$

Corollary 3.4. *The restriction of $\rho : H_{\text{Br}}^{\ast,\ast}(X(\mathbb{C}); \mathbb{Z}) \rightarrow H_{\text{Br}}^{\ast,\ast}(X(\mathbb{C}); \mathbb{F}_2)$ to the torsion subring $H_{\text{Br}}^{\ast,\ast}(X(\mathbb{C}); \mathbb{Z})_{\text{tor}}$ is injective.*

Remark 3.5. If X is a cellular variety, then the spectral sequence (14) degenerates at the E_2 -term. Examples include projective spaces \mathbb{P}^n , Grassmannians $Gr_{r,n}$ and split quadrics $\mathcal{Q}_{2n,n}$.

4. Algebraic preliminaries

This section contains basic algebraic results, which stem from the following presentation of the singular cohomology ring of the real Grassmannian $Gr_{2,n+2}(\mathbb{R})$ of 2-planes in \mathbb{R}^{n+2} . Consider the polynomial ring $\mathbb{F}_2[w_1, w_2]$ on indeterminates w_1, w_2 (Stiefel–Whitney classes) of degrees 1 and 2, respectively. Then

$$H_{\text{sing}}^*(Gr_{2,n+2}(\mathbb{R}); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2]/\bar{J}_n, \tag{17}$$

where $\bar{J}_n = \langle \bar{f}_{n+1}, w_2 \bar{f}_n \rangle$ is the ideal generated by polynomials defined recursively as $\bar{f}_0 = 1$, $\bar{f}_1 = w_1$, and $\bar{f}_{n+1} = w_1 \bar{f}_n + w_2 \bar{f}_{n-1}$. See [11].

Recall from Section 2.1 that ε is an element of degree $(1, 1)$ in the ring \mathcal{A} and that $\mathfrak{K} = \mathbb{Z}[\varepsilon]$. Consider indeterminates ξ and h of degrees $(0, 1)$ and $(2, 1)$, respectively, and define polynomials $F_m \in \mathfrak{K}[\xi, h]$ in a similar fashion:

$$F_0 = 1, \quad F_1 = \varepsilon, \quad \text{and} \quad F_{m+1} = \varepsilon F_m + (\xi h) F_{m-1}. \tag{18}$$

Note that F_m is homogeneous of degree (m, m) . One can assemble these polynomials into generating functions

$$G := \sum_{m \geq 0} F_m Y^m \quad \text{and} \quad H := \sum_{k \geq 0} F_k^2 Y^{2k} \in \mathfrak{K}[\xi, h][[Y]] \tag{19}$$

and verify that

$$G = \frac{1}{1 - p(Y)} \quad \text{and} \quad H = \frac{1}{1 - p(Y)^2},$$

where $p(Y) = Y\{\varepsilon + (\xi h)Y\}$. In particular $G = \{1 + p(Y)\}H$, and these identities give the following.

Lemma 4.1. *For each non-negative integer m one has:*

- (1) $F_m = \sum_{a+2b=m} \binom{a+b}{b} \varepsilon^a \xi^b h^b$.
- (2) $F_{2m+1} = \varepsilon F_m^2$ and $F_{2m} = F_m^2 + (\xi h)F_{m-1}^2$. In particular, one can write

$$F_{2m+1} = \sum_{a+2b=m} \binom{a+b}{b} \varepsilon^{2a+1} \xi^{2b} h^{2b}.$$

Definition 4.2. Let x_n be a variable of degree $(n, -1)$ and let \mathcal{R}_n denote the bigraded ring $\mathcal{R}_n := \mathfrak{N}[\xi, \xi^{-1}, h, x_n]$. Note that sending τ to ξ^2 induces an inclusion of bigraded rings $\mathcal{A} := \mathfrak{N}[\tau, \tau^{-1}] \hookrightarrow \mathfrak{N}[\xi, \xi^{-1}]$, cf. Definition 2.1 and (9), which in turn induces an inclusion

$$\iota : \mathcal{A}[h, x_n] \hookrightarrow \mathcal{R}_n.$$

We now introduce three homomorphisms of bigraded rings. The first one is a *reduction of coefficients* epimorphism

$$\pi : \mathcal{R}_n \rightarrow \mathbb{F}_2[\varepsilon, \xi, \xi^{-1}, x_n, h] \tag{20}$$

induced by the quotient map $\mathfrak{N} \rightarrow \mathfrak{N}/2\mathfrak{N} \cong \mathbb{F}_2[\varepsilon]$. The second one is an isomorphism of bigraded \mathbb{F}_2 -algebras

$$W : \mathbb{F}_2[\varepsilon, \xi, \xi^{-1}, h, x_n] \xrightarrow{\cong} \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2, \bar{w}_n] \tag{21}$$

defined by $\varepsilon \mapsto \xi w_1$, $h \mapsto \xi w_2$ and $x_n \mapsto \xi^{-1} \bar{w}_n$, where w_1, w_2 and \bar{w}_n are given bidegrees $(1, 0)$, $(2, 0)$ and $(n, 0)$, respectively. The third one is an epimorphism of bigraded \mathbb{F}_2 -algebras

$$q : \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2, \bar{w}_n] \rightarrow \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2] \tag{22}$$

defined by $\xi \mapsto \xi$, $w_i \mapsto w_i$, $i = 1, 2$ and $\bar{w}_n \mapsto \bar{f}_n$; where \bar{f}_n is introduced in (17).

Remark 4.3. For all n and m one can consider the polynomial F_m as an element in \mathcal{R}_n . It follows directly from Lemma 4.1(2) and basic definitions that:

- (i) F_{2m+1} is the image under $\mathcal{A}[h, x_n] \hookrightarrow \mathcal{R}_n$ of the polynomial $\mathbf{f}_m \in \mathcal{A}[h, x_n]$, introduced in (8).
- (ii) $W \circ \pi(F_n) = \xi^n \bar{f}_n$; see (17).

Definition 4.4. We introduce the following three ideals:

- (i) Let \bar{J}_n denote the ideal $\langle \bar{f}_{n+1}, w_2 \bar{f}_n \rangle \subset \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2]$; see (17).
- (ii) Write $n = 2m - \delta$, $\delta \in \{0, 1\}$, and define $\widehat{J}_n = \langle \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4 \rangle \subset \mathcal{R}_n$ as the homogeneous ideal generated by

$$\begin{aligned} \mathbf{g}_1 &:= \varepsilon^{1-\delta} \xi^{2m} X_n - h^{1-\delta} F_{2m-1}, \\ \mathbf{g}_2 &:= F_{2m+1}, \\ \mathbf{g}_3 &:= hX_n, \\ \mathbf{g}_4 &:= (\xi h)^{2m} - (-1)^m (\xi h)^\delta (\xi^{2m+1-\delta} X_n)^2. \end{aligned}$$

- (iii) Using the inclusion $\iota : \mathcal{A}[X_n, h] \hookrightarrow \mathcal{R}_n$ define the ideal $J_n := \mathcal{A}[X_n, h] \cap \widehat{J}_n \subset \mathcal{A}[X_n, h]$.

Remark 4.5. Abbreviate X_n to X for simplicity.

(1) It is easy to see, from Remark 4.3(i) and simple degree considerations, that one can present the ideal $J_n \subset \mathcal{A}[X_n, h]$ as:

$$J_n = \begin{cases} \langle \tau^m X - \mathbf{f}_{m-1}, \mathbf{f}_m, hX, h^{2m} \rangle, & \text{if } n = 2m - 1, \\ \langle \varepsilon \tau^m X - h \mathbf{f}_{m-1}, \mathbf{f}_m, hX, h^{2m} - (-1)^m \tau^{m+1} X^2 \rangle, & \text{if } n = 2m. \end{cases}$$

- (2) Note that $\mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2]/\bar{J}_n$ is a presentation of the ring

$$H_{\text{sing}}^*(Gr_{2,n+2}(\mathbb{R}); \mathbb{F}_2(*)) \cong H_{\text{sing}}^*(Gr_{2,n+2}(\mathbb{R}); \mathbb{F}_2) \otimes \mathbb{Z}[\xi, \xi^{-1}].$$

The commutative diagram below summarizes the relations between the ideals and maps defined above. The lower vertical maps are the natural projections to the corresponding quotients, and the existence of the maps represented by dotted arrows is established in Proposition 4.6. The composite $q \circ W \circ \pi$ is denoted by Ψ .

$$\begin{array}{ccccc} \varepsilon \mathcal{A}[h, X_n] \hookrightarrow & \mathbb{F}_2[\xi, \xi^{-1}, \varepsilon, h, X_n] & \xrightarrow[\cong]{W} & \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2, \bar{w}_n] & \\ \downarrow & \uparrow \pi & \searrow \bar{\Psi} & \downarrow q & \\ \mathcal{A}[h, X_n] \hookrightarrow & \mathcal{R}_n & \xrightarrow{\Psi} & \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2] & \\ \downarrow \gamma & \downarrow p & & \downarrow & \\ \mathcal{A}[h, X_n]/J_n \hookrightarrow & \mathcal{R}_n/\widehat{J}_n & \dashrightarrow^r & \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2]/\bar{J}_n & \\ & \dashrightarrow^{\bar{\rho}} & & & \end{array} \tag{23}$$

In the next section we show that $\mathcal{A}[h, X_n]/J_n$ is a presentation for the equivariant cohomology ring of $X_{\mathbf{q}}(\mathbb{C})$, where \mathbf{q} is anisotropic of rank $n + 2$. The main properties of $\mathcal{A}[h, X_n]/J_n$ and J_n needed for the proof of this result are stated in the next proposition.

Proposition 4.6. *Using the notation above:*

- (i) *The composition $\Psi := q \circ W \circ \pi$ sends \widehat{J}_n into \bar{J}_n . In particular, it descends to a map $r : \mathcal{R}_n/\widehat{J}_n \rightarrow \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2]/\bar{J}_n$ making the diagram above commute.*
- (ii) *Writing $n = 2m - \delta$, with $\delta \in \{0, 1\}$ one has*

$$J_{2m-\delta} + \langle \varepsilon \rangle = \langle \varepsilon, h^{1-\delta}x, h^{2m} - (-1)^m \tau^{m+1}x^2 \rangle.$$

- (iii) *The torsion subgroup $(\mathcal{A}[X_n, h]/J_n)_{\text{tor}}$ is precisely the ideal $\langle \varepsilon \rangle$ generated by the class of ε .*
- (iv) *The restriction of $\bar{\rho} := r \circ \bar{t} : \mathcal{A}[X_n, h]/J_n \rightarrow \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2]/\bar{J}_n$ to the torsion subgroup $(\mathcal{A}[X_n, h]/J_n)_{\text{tor}}$ is injective.*

Proof. Using same arguments as in Lemma 4.1, one can prove the following facts concerning the ideal \bar{J}_n :

Fact 4.7.

- (a) $w_2^k \bar{J}_n \subset \bar{J}_{n+k} \subset \bar{J}_n$;
- (b) $\bar{f}_{2n+1} = w_1 \bar{f}_n^2$;
- (c) $\bar{f}_{2n} = \bar{f}_n^2 + w_2 \bar{f}_{n-1}^2$.

Write $n = 2m - \delta$, where $\delta = s(n) \in \{0, 1\}$ is the sign of n , and denote X_n by X .

(i) One just needs to verify the assertion on the generators of \widehat{J}_n .

The generator \mathbf{g}_1 is sent to $\xi^{2m-\delta} \{w_1^{1-\delta} \bar{f}_{2m-\delta} + w_2^{1-\delta} \bar{f}_{2m-1}\}$. This expression is 0 if $\delta = 1$ and equal to $\xi^{2m} \bar{f}_{2m+1} \in \bar{J}_n$ when $\delta = 0$. The generator \mathbf{g}_2 is sent to $\xi^{2m+1} \bar{f}_{2m+1} \in \bar{J}_{2m} \subset \bar{J}_n$; cf. Fact 4.7(a). The generator \mathbf{g}_3 is sent to $w_2 \bar{f}_n \in \bar{J}_n$. Finally, the generator \mathbf{g}_4 is sent to $\xi^{4m} \beta_n$ where $\beta_n := w_2^{s(n)} (\bar{f}_n^2 + w_2^n)$. To show that $\beta_n \in \bar{J}_n$ we use induction. First, observe that $\beta_1 = w_2 \bar{f}_2 \in \bar{J}_1$. Also, $\beta_2 = w_1^4 = w_1 \bar{f}_2 \in \bar{J}_2$. Assuming $\beta_k \in \bar{J}_k$ for $k \leq n$, we can use Fact 4.7 above to write

$$\begin{aligned} \beta_{n+1} &= w_2^{s(n+1)} \{ (w_1 \bar{f}_n + w_2 \bar{f}_{n-1})^2 + w_2^{n+1} \} \\ &= w_2^{s(n+1)} (w_1^2 \bar{f}_n^2 + w_2^2 \bar{f}_{n-1}^2 + w_2^{n+1}) \\ &= w_2^{s(n+1)} (w_1 \bar{f}_{2n+1} + w_2^2 \bar{f}_{n-1}^2 + w_2^{n+1}) \\ &= w_2^{s(n+1)} w_1 \bar{f}_{2n+1} + w_2^2 \beta_{n-1} \in \bar{J}_{2n} + w_2^2 \bar{J}_{n-1} \subset \bar{J}_{n+1}. \end{aligned}$$

This completes the proof of (i).

(ii) This follows directly from inspection of the generators of J_n noting that $F_k \in \langle \varepsilon \rangle$ for k odd; cf. Lemma 4.1(b).

(iii) It follows directly from (ii) that $\mathcal{A}[X, h]/\langle \varepsilon \rangle + J_n$ is torsion free, hence the torsion ideal of $\mathcal{A}[X, h]/J_n$ is $\langle \varepsilon \rangle$.

(iv) Let $\psi, \bar{\psi}$ denote the composites $q \circ W \circ \pi$ and $q \circ W$, respectively. By (ii) it suffices to show that

$$\psi^{-1}(\bar{J}_n \cap \langle w_1 \rangle) \cap \langle \varepsilon \rangle \subset \widehat{J}_n,$$

because $\psi(\varepsilon) = \xi w_1$. Using Fact 4.7(b) and the inductive definition of \bar{f}_n one obtains

$$\bar{f}_{2m+1} \in \langle w_1 \rangle \quad \text{and} \quad \bar{J}_{2m-\delta} \cap \langle w_1 \rangle = \langle \bar{f}_{2m+1}, w_2^{2-\delta} \bar{f}_{2m-1} \rangle.$$

A direct computation gives

$$\begin{aligned} \psi(F_{2m+1}) &= \xi^{2m+1} \bar{f}_{2m+1}, \\ \psi(h^{2-\delta} F_{2m-1}) &= \xi^{2m-\delta+1} w_2^{2-\delta} \bar{f}_{2m-1}, \\ W(\xi^{2m-\delta+1} X - F_{2m-\delta}) &= \xi^{2m-\delta} (\bar{w}_n - \bar{f}_n), \end{aligned}$$

where we denote the image $\pi(F_n) \in \mathbb{F}_2[\varepsilon, \xi, \xi^{-1}, h, X_n]$ by F_n as well. Since q is onto and $\ker q = \langle \bar{w}_n - \bar{f}_n \rangle$ it follows that

$$\bar{\psi}^{-1}(\bar{J}_n \cap \langle w_1 \rangle) = \langle F_{2m+1}, h^{2-\delta} F_{2m-1}, \xi^{2m-\delta+1} X - F_n \rangle.$$

Now, since $F_{2m\pm 1} \in \langle \varepsilon \rangle$ we conclude that

$$\bar{\psi}^{-1}(\bar{J}_n \cap \langle w_1 \rangle) \cap \langle \varepsilon \rangle = \langle F_{2m+1}, h^{2-\delta} F_{2m-1}, \varepsilon(\xi^{2m-\delta+1} X - F_n) \rangle.$$

Finally, noting that $\ker \pi = 2\mathcal{R}_n$ and $2\varepsilon = 0$ we obtain

$$\psi^{-1}(\bar{J}_n \cap \langle w_1 \rangle) \cap \langle \varepsilon \rangle = \langle F_{2m+1}, h^{2-\delta} F_{2m-1}, \varepsilon(\xi^{n+1} X - F_n) \rangle \subset \widehat{J}_n. \quad \square$$

It is interesting to note that when n is odd the ring $\mathcal{A}[t, X_n]/J_n$ is considerably simpler.

Proposition 4.8. *If $n = 2m - 1$ is odd, define $I_{2m-1} = \mathcal{A}[h] \cap \widehat{J}_{2m-1}$. Then $I_{2m-1} = \langle \mathbf{f}_m, h\mathbf{f}_{m-1}, h^{2m} \rangle$ and the inclusion $\mathcal{A}[h] \hookrightarrow \mathcal{A}[X_n, h]$ induces an isomorphism*

$$\bar{i}: \mathcal{A}[h]/I_{2m-1} \rightarrow \mathcal{A}[h, X_{2m-1}]/J_{2m-1}.$$

Proof. This follows directly from Remark 4.5 and the fact that τ is invertible in \mathcal{A} . \square

5. Anisotropic quadrics

A non-degenerate quadratic form \mathbf{q} on a real vector space V of dimension $n + 2$ determines a real smooth n -dimensional quadric hypersurface $X_{\mathbf{q}} \subset \mathbb{P}(V)$. Any such hypersurface is isomorphic to the quadric $\mathcal{Q}_{n,s} \subset \mathbb{P}_{\mathbb{R}}^{n+1}$ given by the equation

$$\mathbf{q}_{n,s}(z_0, \dots, z_{n+1}) = z_0^2 + \dots + z_{n-s+1}^2 - z_{n-s+2}^2 - \dots - z_{n+1}^2 = 0,$$

for some s satisfying $0 \leq 2s \leq n + 2$. We denote $\mathcal{Q}_n := \mathcal{Q}_{n,0}$.

The main theorem in this section computes the ring $H_{\text{Bf}}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z})$, showing that is generated by two elements over the Borel cohomology of a point. These elements come from the following geometric constructions.

Let S_a^n be the unit sphere in \mathbb{R}^{n+1} endowed with the antipodal \mathcal{G} -action $a: S^n \rightarrow S^n$. One has a classical equivariant embedding $j: (S_a^n, a) \rightarrow (\mathcal{Q}_n(\mathbb{C}), \sigma)$ defined by $j(y_0, \dots, y_{n+1}) =$

$[iy_0 : \dots : iy_{n+1} : 1]$. It is easy to see that S_a^n is $H\mathbb{Z}$ -orientable, with fundamental class $[S_a^n] \in H_{n,n+1}^{Br}(S_a^n; \mathbb{Z})$, and hence we obtain Gysin maps

$$j_!^b : H_{Br}^{r,s}(S_a^n; \mathbb{Z}) \rightarrow H_{Br}^{r+n,s-1}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}), \tag{24}$$

defined using Poincaré duality by

$$j_!^b(c) \cap [\mathcal{Q}_n(\mathbb{C})] = j_*(c \cap [S_a^n]),$$

since the fundamental class $[\mathcal{Q}_n(\mathbb{C})]$ lives in $H_{2n,n}^{Br}(\mathcal{Q}_n(\mathbb{C}), \mathbb{Z})$. One can obtain Gysin maps $j_!$ in singular cohomology in a similar fashion and, using the notation in Property 2.3, one can see that the Gysin maps are compatible with forgetful functors. In other words, the following diagram commutes

$$\begin{CD} H_{Br}^{r,s}(S_a^{2m}; \mathbb{Z}) @>j_!^b>> H_{Br}^{r+2m,s-1}(\mathcal{Q}_{2m}(\mathbb{C}); \mathbb{Z}) \\ @V\varphi VV @VV\varphi V \\ H_{sing}^r(S_a^{2m}; \mathbb{Z}(s)) @>j_!>> H_{sing}^{r+2m}(\mathcal{Q}_{2m}(\mathbb{C}); \mathbb{Z}(s-1)) \end{CD} \tag{25}$$

Let $\mathcal{O}(1)$ denote the hyperplane bundle $\mathcal{O}_{\mathcal{Q}_n}(1)$, under the embedding $\mathcal{Q}_n \hookrightarrow \mathbb{P}^{n+1}$. The space of complex points $\mathcal{O}(1)(\mathbb{C})$ becomes a Real bundle over $\mathcal{Q}_n(\mathbb{C})$, in the sense of [1]. It follows that $\mathcal{O}(1)(\mathbb{C})$ has Chern classes

$$\underline{c}_i(\mathcal{O}(1)(\mathbb{C})) \in H_{Br}^{2i,i}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}) \tag{26}$$

with values in Bredon cohomology with coefficients in \mathbb{Z} ; cf. [5].

Since the action of \mathfrak{S} on $\mathcal{Q}_n(\mathbb{C})$ is free one can identify $H_{Br}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z})$ with $H_{bor}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z})$, cf. Property 2.7(i), and use its structure of algebra over $\mathcal{A} := H_{bor}^{*,*}(pt; \mathbb{Z})$ in the following construction.

Definition 5.1. Let $\Psi_n : \mathcal{A}[h, x_n] \rightarrow H_{Br}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z})$ denote the homomorphism of bigraded \mathcal{A} -algebras that sends

$$\begin{aligned} x_n &\mapsto j_!^b \mathbf{1} \in H_{Br}^{n,-1}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}) \quad \text{and} \\ h &\mapsto \underline{c}_1(\mathcal{O}(1)(\mathbb{C})) \in H_{Br}^{2,1}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}). \end{aligned}$$

The classes $j_!^b \mathbf{1}$ and $\underline{c}_1(\mathcal{O}(1)(\mathbb{C}))$ are completely determined by their images under the forgetful map φ into singular cohomology, and the reduction of coefficients map ρ into Bredon cohomology with coefficients in \mathbb{F}_2 ; cf. Corollary 3.3. Hence, it is useful to identify the compositions

$$\varphi \circ \Psi_n : \mathcal{A}[h, x_n] \xrightarrow{\Psi_n} H_{Br}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}) \xrightarrow{\varphi} H_{sing}^*(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}(*)) \tag{27}$$

and

$$\rho \circ \Psi_n : \mathcal{A}[h, x_n] \xrightarrow{\Psi_n} H_{\text{Br}}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}) \xrightarrow{\rho} H_{\text{Br}}^{*,*}(\mathcal{Q}_n(\mathbb{C}); \mathbb{F}_2). \tag{28}$$

Since quadrics are geometrically cellular, the cycle map (13) induces a natural isomorphism

$$H_{\text{sing}}^*(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}(*)) := \bigoplus_{p,q} H_{\text{sing}}^p(\mathcal{Q}_n(\mathbb{C}); \mathbb{Z}(q)) \cong CH^*(\mathcal{Q}_{n,\mathbb{C}}) \otimes \mathbb{Z}[\xi, \xi^{-1}], \tag{29}$$

where the elements of $CH^j(\mathcal{Q}_{n,\mathbb{C}})$ are given bidegree $(2j, j)$. Furthermore, the Chow groups of complex quadrics are well-known classical objects, and have the following presentation.

Fact 5.2. Write $n = 2m - \delta$, with $\delta \in \{0, 1\}$. Then, the Chow ring $CH^*(\mathcal{Q}_{2m-\delta,\mathbb{C}})$ is isomorphic to the graded ring $\mathbb{Z}[h, \phi]/\mathcal{C}_{2m-\delta}$, where $\deg h = 1$, $\deg \phi = m$, and $\mathcal{C}_{2m-\delta}$ is the ideal

$$\mathcal{C}_{2m-\delta} := \left\langle h^{1-\delta}(h^m - 2\phi), \phi^2 - \frac{1 + (-1)^m}{2} h^m \phi \right\rangle.$$

(1) The action of $Gal(\mathbb{C}/\mathbb{R})$ on $CH^*(\mathcal{Q}_{n,\mathbb{C}})$ is determined by

$$\sigma_*(h) = h \quad \text{and} \quad \sigma_*(\phi) = \frac{1 + (-1)^m}{2} h^m - (-1)^m \phi.$$

(2) If $i : \mathcal{Q}_n \hookrightarrow \mathcal{Q}_{n+1}$ is the canonical inclusion, then

$$i^* : CH^*(\mathcal{Q}_{n+1,\mathbb{C}}) \rightarrow CH^*(\mathcal{Q}_{n,\mathbb{C}})$$

is given by

$$i^*(\phi) = h \frac{1+(-1)^n}{2} \phi \quad \text{and} \quad i^*(h) = h.$$

Remark 5.3.

(i) Denote the homogeneous coordinates in $\mathcal{Q}_{2m-\delta}(\mathbb{C})$ by $[\mathbf{z}] = [\mathbf{x} : x_{2m-\delta} : x_{2m-\delta+1}]$, where $\mathbf{x} = (x_0, \dots, x_{2m-\delta-1}) \in \mathbb{C}^{2m-\delta}$. Let $L^+, L^- \subset \mathcal{Q}_{2m-\delta,\mathbb{C}}$ be the (maximal) linear subspaces given by

$$L^+ = \{[\mathbf{z}] \mid x_0 = (-1)^\delta i x_1, x_{2j} = i x_{2j+1}, \delta \cdot x_{2m-\delta+1} = 0, 1 \leq j \leq m - \delta\},$$

$$L^- = \{[\mathbf{z}] \mid x_0 = -i x_1, x_{2j} = -i x_{2j+1}, \delta \cdot x_{2m-\delta+1} = 0, 1 \leq j \leq m - \delta\}.$$

It follows from [10] that when $\delta = 0$, the cohomology duals of $[L^+]$ and $[L^-]$, respectively, are distinct classes generating the cohomology group in middle dimension.

(ii) In the choices of generators in the description of the singular cohomology of $\mathcal{Q}_{2m}(\mathbb{C})$, given in Fact 5.2 and (29), ϕ is the Poincaré dual to $L^{\epsilon(m)}$, where

$$\epsilon(m) = \begin{cases} +, & \text{if } m \text{ is odd,} \\ -, & \text{if } m \text{ is even.} \end{cases}$$

(iii) It follows from the above description that when $n \not\equiv 0 \pmod 4$, one has $\mathcal{C}\mathcal{H}^*(\mathbb{Q}_n, \mathbb{C})^{\mathfrak{S}} = \mathcal{C}\mathcal{H}^*(\mathbb{Q}_n, \mathbb{C})$, and when $n = 4m$ one has

$$\mathcal{C}\mathcal{H}^*(\mathbb{Q}_{4m, \mathbb{C}})^{\mathfrak{S}} = \mathbb{Z}[h, h\phi] / \langle h^{4m+1} - 2h\phi, (h\phi)^2 \rangle \tag{30}$$

and

$$\mathcal{C}\mathcal{H}^*(\mathbb{Q}_{4m, \mathbb{C}})^{-} = \mathbb{Z} \cdot \{h^{2m} - 2\phi\}. \tag{31}$$

Note that $h \cdot (h^{2m} - 2\phi) = 0$, and this determines the structure of $\mathcal{C}\mathcal{H}^*(\mathbb{Q}_{4m, \mathbb{C}})^{-}$ as a $\mathcal{C}\mathcal{H}^*(\mathbb{Q}_{4m, \mathbb{C}})^{\mathfrak{S}}$ -module. Therefore, the E_2 -term of the spectral sequence (14) is given by

$$\mathcal{E} = \begin{cases} \mathcal{A} \otimes \mathcal{C}\mathcal{H}^*(\mathbb{Q}_n) & \text{if } n \not\equiv 0 \pmod 4, \\ \mathcal{A}[h, h\phi] / \langle h^{4m+1} - 2h\phi, (h\phi)^2 \rangle \oplus \mathcal{A} \cdot \{h^{2m} - 2\phi\} & \text{if } n = 4m. \end{cases}$$

Eq. (29) and Fact 5.2 provide an identification

$$H_{\text{sing}}^*(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z}(\ast)) \cong \mathbb{Z}[\xi, \xi^{-1}][h, \phi] / \mathcal{C}_n. \tag{32}$$

On the other hand, it is well known that one has a homeomorphism

$$i_n : \mathbb{Q}_n(\mathbb{C}) / \mathfrak{S} \rightarrow \text{Gr}_{2, n+2}(\mathbb{R}),$$

where the latter denotes the Grassmanian of two-planes in \mathbb{R}^{n+2} with its classical topology. In particular, it follows from (17) and Properties 2.7(ii) and (vi) that one has isomorphisms

$$\begin{aligned} H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{F}_2) &\cong H_{\text{sing}}^*(\text{Gr}_{2, n+2}(\mathbb{R}); \mathbb{F}_2) \otimes \mathbb{Z}[\xi, \xi^{-1}] \\ &\cong \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2] / \bar{J}_n \end{aligned} \tag{33}$$

where $\bar{J}_n = \langle \bar{f}_{n+1}, w_2 \bar{f}_n \rangle$ is introduced in Definition 4.4(i).

Lemma 5.4. *Under the identifications (32) and (33), one has*

- (i) $\rho(\underline{\mathbf{c}}_1(\mathcal{O}(1)(\mathbb{C}))) = \xi w_2, \quad \rho(\varepsilon) = \xi w_1 \quad \text{and} \quad \varphi(\underline{\mathbf{c}}_1(\mathcal{O}(1)(\mathbb{C}))) = h$
- (ii) $\rho(j_!^b \mathbf{1}) = \xi^{-1} \bar{f}_n \quad \text{and} \quad \varphi(j_!^b \mathbf{1}) = \begin{cases} 0, & n = 2m - 1, \\ \xi^{-m-1} (h^m - 2\phi), & n = 2m. \end{cases}$

Proof. The proof of the assertions in (i) is clear.

Now, first assume that $n = 2m$. In order to prove the second assertion in (ii) observe that

$$\begin{aligned} j(S_a^{2m}) \cap L^+ &= [0 : \dots : 0 : i : 1] := p^+, \\ j(S_a^{2m}) \cap L^- &= [0 : \dots : 0 : -i : 1] := p^-. \end{aligned}$$

Furthermore, a simple calculation in local coordinates shows that

- (i) $j(S_a^{2m})$ intersects both L^+ and L^- transversally.
- (ii) The intersection multiplicities $\epsilon(S_a^{2m}, L^+, p^+)$ and $\epsilon(S_a^{2m}, L^-, p^-)$ are precisely $(-1)^m$ and $(-1)^{m-1}$, respectively.

Using the identification (32) and Poincaré duality one sees that $\xi^{-m-1} \cap [p^+] = \xi^{-m-1} \cap [p^-]$ represents the canonical generator $[*]_{m+1}$ for the homology group $H_0^{\text{sing}}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z}(m+1))$. Therefore, the observations above show that the Gysin map $j_!$ in singular cohomology gives

$$(j_!1) \cap [L^+] = (-1)^m [*]_{m+1} \quad \text{and} \quad (j_!1) \cap [L^-] = (-1)^{m-1} [*]_{m+1}.$$

In terms of generators, one can write $j_!1 = a \cdot (\xi^{-m-1}h^m) + b \cdot (\xi^{-m-1}\phi)$, with $a, b \in \mathbb{Z}$, and hence

$$\begin{aligned} (j_!1) \cap [L^{\epsilon(m)}] &= a \cdot (\xi^{-m-1}h^m) \cap [L^{\epsilon(m)}] + b \cdot (\xi^{-m-1}\phi) \cap [L^{\epsilon(m)}] \\ &= (a + b)[*]_{m+1}, \end{aligned}$$

and

$$(j_!1) \cap [L^{\epsilon(m+1)}] = a[*]_{m+1},$$

cf. Remark 5.3(ii).

It follows that, when m is even one gets $a + b = (-1)^{m-1} = -1$ and $a = (-1)^m = 1$ so that $b = -2$. Similarly, if m is odd, $a + b = (-1)^m = -1$ and $a = (-1)^{m-1} = 1$ so that, once again, $b = -2$. Therefore

$$j_!1 = \xi^{-m-1}(h^m - 2\phi),$$

in singular cohomology.

Now, (25) gives

$$\varphi(j_!^b(1)) = j_!(\varphi(1)) = j_!1 = \xi^{-m-1}(h^m - 2\phi).$$

The case $n = 2m - 1$ follows from similar arguments and relations in the singular cohomology of quadrics.

It remains to identify $\rho(j_!^b(1))$. Consider the pull-back square

$$\begin{array}{ccc} S_a^{2m} & \xrightarrow{j} & \mathbb{Q}_{2m}(\mathbb{C}) \\ \downarrow q & & \downarrow \pi \\ Gr_1\mathbb{R}^{2m+1} & \xrightarrow{\iota} & Gr_2\mathbb{R}^{2m+2} = \mathbb{Q}_{2m}/\mathfrak{S} \end{array}$$

where the vertical arrows denote quotient maps.

Let $\Theta_{r,k} \rightarrow Gr_k \mathbb{R}^{r+k}$ denote the universal bundle (of rank r). Then $\iota^* \Theta_{2m,2} = \Theta_{2m,1}$. Now, it follows from Property 2.3(v) that one has a natural isomorphism

$$\hat{\pi} : H_{\text{sing}}^r(\mathbb{Q}_{2m}(\mathbb{C})/\mathfrak{S}; \mathbb{F}_2(s)) \rightarrow H_{\text{Br}}^{r,s}(\mathbb{Q}_{2m}(\mathbb{C}); \underline{\mathbb{F}}_2).$$

Similarly, one has an isomorphism

$$\hat{q} : H_{\text{sing}}^r(S_a^{2m}/\mathfrak{S}; \mathbb{F}_2(s)) \rightarrow H_{\text{Br}}^{r,s}(S_a^{2m}; \underline{\mathbb{F}}_2).$$

We now proceed to identify $\rho(X_{2m})$:

$$\begin{aligned} \rho(X_{2m}) &:= \rho(j_!^b(\mathbf{1})) = \rho(j_!^b(\hat{q}\mathbf{1})) \\ &= \rho(\hat{\pi}(\iota_!(\mathbf{1}))) \quad (\text{pull-back square property}) \\ &= \underline{\hat{\pi}}(\rho(\iota_!(\mathbf{1}))) \quad (\text{naturality of change of coefficients functor}) \\ &= \underline{\hat{\pi}}(\underline{\iota}_!(\rho(\mathbf{1}))) \quad (\text{compatibility with Poincaré duality}) \\ &= \underline{\hat{\pi}}(\underline{\iota}_!(\mathbf{1})), \end{aligned}$$

where the underlined symbols denote homomorphisms in the singular cohomology with \mathbb{F}_2 -coefficients. Now observe that

$$\underline{\iota}_!(\mathbf{1}) \cap [Gr_2 \mathbb{R}^{m+1}] = \underline{\iota}_* [Gr_1 \mathbb{R}^{m+1}].$$

Since $\iota(Gr_1 \mathbb{R}^{2m+1})$ is the zero set of a section of $\Theta_{2m,2}$ then

$$\underline{\iota}_* [Gr_1 \mathbb{R}^{2m+1}] = \omega_{2m}(\Theta_{2m,2}) \cap [Gr_2 \mathbb{R}^{2m+2}],$$

where $\omega_{2m}(\Theta_{2m,2})$ is the top Stiefel–Whitney class of $\Theta_{2m,2}$.

Using (17) together with a simple induction argument and the definition of the universal quotient bundle, one concludes that

$$\underline{\hat{\pi}}(\underline{\iota}_!(\mathbf{1})) = \xi^{-1} \bar{f}_n. \quad \square$$

Proposition 5.5. *The map $\Psi_n : \mathcal{A}[h, X_n] \rightarrow H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z})$ factors through a map $\bar{\Psi}_n : \mathcal{A}[h, X_n]/J_n \rightarrow H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z})$, where J_n is introduced in Definition 4.4(ii).*

Proof. It follows from Corollary 3.3 that it suffices to prove that $J_n \subseteq \ker \rho \circ \Psi_n$ and $J_n \subseteq \ker \varphi \circ \Psi_n$. Lemma 5.4 shows that $\rho \circ \Psi_n$ can be factored as the upper triangle in the following diagram:

$$\begin{array}{ccc} \mathcal{A}[h, X_n] & \xrightarrow{\iota} & \mathcal{R}_n = \mathbb{Z}[\xi, \xi^{-1}, \varepsilon, h, X_n] \xrightarrow{q \circ W \circ \pi} \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2] \\ \downarrow & \searrow^{\rho \circ \Psi_n} & \downarrow \\ \mathcal{A}[h, X_n]/J_n & \xrightarrow{\bar{\rho}} & \mathbb{F}_2[\xi, \xi^{-1}, w_1, w_2]/\bar{J}_n, \end{array} \tag{34}$$

see diagram (23). It follows from Proposition 4.6(ii) that $\rho \circ \Psi_n$ descends to $\bar{\rho}$, making the diagram above commute and proving the first inclusion.

Since $CH^*(\mathbb{Q}_n, \mathbb{C})$ is torsion-free, Lemma 5.4 shows that the composition $\varphi \circ \Psi_n$ factors as

$$\mathcal{A}[h, x_n] \xrightarrow{\widehat{\Psi}_n} \mathbb{Z}[\xi, \xi^{-1}][h, \phi] \rightarrow \mathbb{Z}[\xi, \xi^{-1}][h, \phi]/\mathcal{C}_n, \tag{35}$$

where the last arrow is the quotient map, and $\widehat{\Psi}_n$ is the map induced by the composition $\mathcal{A} \hookrightarrow \mathfrak{N}[\xi, \xi^{-1}] \rightarrow \mathbb{Z}[\xi, \xi^{-1}]$ and by sending

$$h \mapsto h \quad \text{and} \quad x_n \mapsto \begin{cases} 0, & n = 2m - 1, \\ \xi^{-m-1}(h^m - 2\phi), & n = 2m. \end{cases}$$

In particular $\langle \varepsilon \rangle \subseteq \ker(\varphi \circ \Psi_n)$. As before, write $n = 2m - \delta$ with $\delta \in \{0, 1\}$ and abbreviate x_n to x . From the presentation of J_n in Remark 4.5, it now follows that it suffices to show that $h^{1-\delta}x$ and $h^{2m} - (-1)^m \tau^{m+1}x^2$ are sent to $\mathcal{C}_{2m-\delta}$ by the correspondence above.

Computing we get $(\varphi \circ \Psi)(h^{1-\delta}x) = \xi^{-m-1}\{h^{1-\delta}(h^m - 2\phi)\}$ and, using $\tau = \xi^2$, one can easily verify that

$$\begin{aligned} (\varphi \circ \Psi)(h^{2m} - (-1)^m \tau^{m+1}x^2) &= [1 - (-1)^m]h^m(h^m - 2\phi) \\ &\quad - 4(-1)^m \left\{ \phi^2 - \frac{1 + (-1)^m}{2} h^m \phi \right\}. \end{aligned}$$

Since $m \geq 1$ one concludes that this elements lies in $\mathcal{C}_{2m-\delta}$.

This completes the proof. \square

Corollary 5.6. *The restriction*

$$\bar{\Psi}_{n|\text{tor}} : (\mathcal{A}[h, x_n]/J_n)_{\text{tor}} \rightarrow H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z})$$

of $\bar{\Psi}_n$ to the torsion subgroup is injective.

Proof. It follows from Corollary 3.4 that

$$\rho : H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z}) \rightarrow H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{F}_2)$$

is injective on torsion. Hence, $\bar{\Psi}_{n|\text{tor}}$ is injective if and only if so is $\rho \circ \bar{\Psi}_{n|\text{tor}}$, but the latter statement follows immediately from diagram (34) and Proposition 4.6(iv). \square

The following result along with Proposition 4.8, yield the proof of Theorem B in the introduction.

Theorem 5.7. *The map $\bar{\Psi}_n : \mathcal{A}[h, x_n]/J_n \rightarrow H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z})$ is an isomorphism of bigraded algebras over \mathcal{A} .*

Proof. The injectivity of $\bar{\Psi}_n$ follows from Corollaries 5.6, 3.3 and the following claim.

Claim 5.8. $\ker(\varphi \circ \Psi_n) = \langle \varepsilon \rangle + J_n$.

Proof. It suffices to prove that $\ker(\varphi \circ \Psi_n) \subseteq \langle \varepsilon \rangle + J_n$.

Let us first consider the case $n = 2m - 1$. It is clear that any $p(h, x_{2m-1}) \in \mathcal{A}[h, x_{2m-1}]$ is congruent (mod $\langle \varepsilon \rangle + J_{2m-1}$) to a polynomial of the form $p_0(h)$, where $p_0(h) \in \mathbb{Z}[\tau, \tau^{-1}][h] \subset \mathbb{Z}[\xi, \xi^{-1}][h]$. Suppose that $p(h, x_{2m-1}) \in \ker(\varphi \circ \Psi_{2m-1})$. It follows that $\widehat{\Psi}_{2m-1}(p_0(h)) = p_0(h) \in \mathcal{C}_{2m-1}$ (see (35)) and we can write

$$p_0(h) = \alpha(h, \phi)(h^m - 2\phi) + \beta(h, \phi)\phi^2,$$

where $\alpha(h, \phi), \beta(h, \phi) \in \mathbb{Z}[\xi, \xi^{-1}][h, \phi]$. Passing to $\mathbb{Q}(\xi, \xi^{-1})[h, \phi]$ and evaluating at $\phi = \frac{1}{2}h^m$ one obtains the following equality in $\mathbb{Q}(\xi, \xi^{-1})[h]$:

$$p_0(h) = \beta\left(h, \frac{1}{2}h^m\right)\frac{1}{4}h^{2m}.$$

One concludes that $h^{2m} | p_0(h)$. This shows that $p_0(h) \in \langle \varepsilon \rangle + J_{2m-1}$ and hence, so does $p(h, x_{2m-1})$.

The case $n = 2m$ is similar. \square

In order to prove the surjectivity of $\overline{\Psi}_n$ we resort to the spectral sequence whose E_2 -term in described explicitly in Remark 5.3.

By construction, the elements of \mathcal{A} are infinite cycles. Under the description of Remark 5.3, the class $h \in CH^1(\mathbb{Q}_n, \mathbb{C})$ represents an element in $E_2^{0,2}(0)$. Since h is the image of $c_1(\mathcal{O}(1)) \in H_{\text{Br}}^{2,1}(\mathbb{Q}_n, \mathbb{C}; \mathbb{Z})$ under the forgetful map φ , it follows by Proposition 3.2(iv) that h is an infinite cycle.

The discussion above leads us to define the following subsets of the E_2 -term \mathcal{E} :

$$\begin{aligned} \mathcal{S}_n &:= \{\varepsilon^a \tau^b h^c \phi \mid a > 0, c \geq 0\}, & \text{if } n \not\equiv 0 \pmod{4}, \\ \mathcal{S}_n &:= \{\varepsilon^a \tau^b h^c \phi \mid a > 0, c > 0\}, & \text{if } n \equiv 0 \pmod{4}. \end{aligned}$$

Lemma 5.9.

- (i) For each n , the set \mathcal{S}_n contains no non-zero infinite cycles.
- (ii) If $n = 2m$, then $\chi := h^{2m} - 2\phi$ is an infinite cycle and $E_\infty^{*,*}(\ast)$ is generated by h and χ as an \mathcal{A} -algebra.
- (iii) If $n = 2m - 1$, then $E_\infty^{*,*}(\ast)$ is generated by h as an \mathcal{A} -algebra.

Proof. (i) Suppose that $\varepsilon^a \tau^b h^c \phi \in \mathcal{S}_n$ is an infinite cycle. Since h and ε are infinite cycles, this would imply that $\varepsilon^k \tau^b h^n \phi$ is an infinite cycle for all $k \geq a$. These elements live in the top row of the spectral sequence and hence they would survive to E_∞ , contradicting the fact that $H_{\text{br}}^{r,s}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z}) = 0$ for all $r \geq 2n$; cf. Properties 2.7(ii).

(ii) Suppose now $n = 2m$. Under the isomorphism (32) the forgetful map φ sends the element $j_!^b \mathbf{1} \in H_{\text{Br}}^{n,-1}(\mathbb{Q}_{2m}(\mathbb{C}); \mathbb{Z})$ to $\xi^{-1-m}(h^m - 2\phi)$. It follows by Proposition 3.2(iv) that the corresponding element $\xi^{-1-m} \chi \in \mathcal{E}$ (under the description of Remark 5.3) survives to E_∞ .

(iii) Combining the first assertion in this lemma with Remark 5.3 completes the proof. \square

It follows from the lemma that the map Ψ_n is onto. Indeed, consider first the case n odd. By the lemma, $E_{\infty}^{*,*}(\ast)$ is generated as an \mathcal{A} -algebra by the image h of $c_1(\mathcal{O}(1))$ under the projection

$$H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z}) \rightarrow Gr^0(H_{\text{Br}}^{*,*}(\mathbb{Q}_{2m}(\mathbb{C}); \mathbb{Z})).$$

Since $c_1(\mathcal{O}(1)) = \Psi_n(h)$ this gives

$$\frac{\text{Im}(\Psi_n) \cap F^j}{\text{Im}(\Psi_n) \cap F^{j-1}} = Gr^j H_{\text{Br}}^{*,*}(\mathbb{Q}_n(\mathbb{C}); \mathbb{Z}),$$

hence Ψ_n is onto. If $n = 2m$ is even, then $E^{*,*}(\ast)$ is generated as an \mathcal{A} -algebra by h and the image x of $j_1^b \mathbf{1}$ under the projection $H_{\text{Br}}^{2m,-1}(\mathbb{Q}_{2m}(\mathbb{C}); \mathbb{Z}) \rightarrow Gr^0(H_{\text{Br}}^{*,*}(\mathbb{Q}_{2m}(\mathbb{C}); \mathbb{Z}))$. Since $j_1^b \mathbf{1}$ and $c_1(\mathcal{O}(1)) \in \text{Im}(\Psi_{2m})$ it follows, as in the previous case, that Ψ_{2m} is onto.

The theorem is now proven. \square

Corollary 5.10. *The map Ψ_{2m-1} induces an isomorphism*

$$H_{\text{Br}}^{*,*}(\mathbb{Q}_{2m-1}(\mathbb{C}); \mathbb{Z})/\text{tor} \cong \mathbb{Z}[\tau, \tau^{-1}, h]/\langle h^{2m} \rangle.$$

Similarly, Ψ_{2m} induces an isomorphism

$$H_{\text{Br}}^{*,*}(\mathbb{Q}_{2m}(\mathbb{C}); \mathbb{Z})/\text{tor} \cong \mathbb{Z}[\tau, \tau^{-1}, h, \chi]/\langle h^{2m+1}, h\chi, \tau^{m+1}\chi^2 - (-1)^m h^{2m} \rangle.$$

Proof. Left to the reader. \square

6. Isotropic quadrics

Consider a real quadratic form \mathbf{q} of rank $n + 2$ that can be written as $\mathbf{q} = \mathbf{q}' + \mathbf{h}$, where \mathbf{h} is a hyperbolic factor. Let $P \in X_{\mathbf{q}}(\mathbb{R})$ be a real point in $X_{\mathbf{q}}$ and denote by $T_P X_{\mathbf{q}}$ the projective tangent space to $X_{\mathbf{q}}$ at P . The intersection $X_{\mathbf{q}} \cap T_P X_{\mathbf{q}} = \mathcal{Z}X_{\mathbf{q}'}$ is the ruled join of the quadric $X_{\mathbf{q}'}$ and the point P . In particular, the set of complex points $\mathcal{Z}X_{\mathbf{q}'}(\mathbb{C})$ in the analytic topology is the Thom space of the Real bundle $\mathcal{O}_{X_{\mathbf{q}'}}(1)(\mathbb{C})$ over $X_{\mathbf{q}'}(\mathbb{C})$, and $X_{\mathbf{q}} - \mathcal{Z}X_{\mathbf{q}'}$ is isomorphic to affine space \mathbb{A}^n . Let $P_0 \in X_{\mathbf{q}} - \mathcal{Z}X_{\mathbf{q}'}$ correspond to $0 \in \mathbb{A}^n$ under this isomorphism.

Definition 6.1. Denote

$$\begin{aligned} Y_2 &= \{P_0\}, & Y_1 &= X_{\mathbf{q}'}, & Y_0 &= \{P\}, & \text{and} \\ X_2 &= X_{\mathbf{q}}, & X_1 &= \mathcal{Z}X_{\mathbf{q}'}, & X_0 &= \{P\}. \end{aligned}$$

Note that the filtration $X_{\mathbf{q}} = X_2 \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$ comes with projections $\pi_i : X_i - X_{i-1} \rightarrow Y_i$ that are real algebraic vector bundles, and satisfies $a_2 := \text{codim } X_2 = 0$, $a_1 := \text{codim } X_1 = 1$ and $a_0 := \text{codim } X_0 = n$, respectively.

Notation 6.2. Given a bigraded \mathcal{B} -module $N = \bigoplus_{r,s} N^{r,s}$ and integer a , denote by $N(a)$ the bigraded \mathcal{B} -module whose summand of degree (r, s) is $N^{r+2a, s+a}$.

Proposition 6.3. *Given a quadratic form $\mathbf{q} = \mathbf{q}' + \mathbf{h}$ as above, denote $\mathbb{H} := H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z})$ and $\mathbb{H}' := H_{\text{Br}}^{*,*}(X_{\mathbf{q}'}(\mathbb{C}); \mathbb{Z})$. Then the Bredon cohomology of $X_{\mathbf{q}}(\mathbb{C})$ has a decomposition*

$$\mathbb{H} \cong \mathcal{B} \oplus \mathbb{H}'(-1) \oplus \mathcal{B}(-n),$$

as a sum of bigraded \mathcal{B} -modules.

Proof. Denote $H^{(p)}(X) := \bigoplus_{r \in \mathbb{Z}} H_{\text{Br}}^{r,p}(X(\mathbb{C}); \mathbb{Z})$, whenever X is a real variety and $p \in \mathbb{Z}$. The cycle map (13) $CH^p(X) \rightarrow H^{(p)}(X)$ gives $H^{(*)}(X) = \bigoplus_p H^{(p)}(X)$ the structure of a $CH^*(X)$ -algebra, whenever X is smooth. It is easy to see that, in the category $\mathcal{V}_{\mathbb{R}}$ of smooth proper real varieties, the functor $H^{(*)}$ becomes a graded geometric cohomology theory in the sense of [13]. The proposition now follows from Definition 6.1, Theorem 6.5 and Corollary 6.11 [13]. \square

In order to understand the product structure in the Bredon cohomology, we provide a brief description of the isomorphism in the proposition above.

Define $U_i \subset Y_i \times X_{\mathbf{q}}$ as $U_i = \{(y, x) \mid x \in X_i - X_{i-1} \text{ and } \pi_i(x) = y\}$, cf. Definition 6.1, and let $\Gamma_i = \overline{U_i}$ denote its closure. Then, Γ_i defines a correspondence $\Gamma_i \in CH^{\dim Y_i + a_i}(Y_i \times X_{\mathbf{q}})$. Each such correspondence defines a map of (bigraded) \mathcal{B} -modules

$$\gamma_i : H_{\text{Br}}^{*,*}(Y_i(\mathbb{C}); \mathbb{Z})(-a_i) \rightarrow H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z})$$

and the isomorphism above is simply given by the sum $\gamma_2 + \gamma_1 + \gamma_0$. Recall that γ_i is explicitly defined on a cohomology class $\alpha \in H_{\text{Br}}^{*,*}(Y_i(\mathbb{C}); \mathbb{Z})$ as follows. Let $Y_i \xleftarrow{pr_1} Y_i \times X_{\mathbf{q}} \xrightarrow{pr_2} X_{\mathbf{q}}$ be the projections. Then, invoking Poincaré duality, γ_i is uniquely determined by

$$\gamma_i(\alpha) \cap [X_{\mathbf{q}}(\mathbb{C})] = pr_{2*}(pr_1^* \alpha \cap [\Gamma_i(\mathbb{C})]),$$

where $[-]$ denotes the fundamental class of a subvariety in Bredon homology.

These constructions hold in great generality, whenever the variety $X_{\mathbf{q}}$ has a filtration satisfying the conditions in Definition 6.1. In our case, $X_2 - X_1 \cong \mathbb{A}^n$ and the projection $\pi_2 : X_2 - X_1 \rightarrow \{P_0\}$ is identified with the “retraction” from \mathbb{A}^n to $0 \in \mathbb{A}^n$. Hence, $\Gamma_2 = \{pt\} \times X_{\mathbf{q}}$ and the map $\gamma_2 : \mathcal{B} \rightarrow H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z})$ splits off the cohomology of a point via the real point $P_0 \in X_{\mathbf{q}}(\mathbb{R})$. Similarly, $X_0 - X_{-1} = \{P\}$ and $\Gamma_0 = \{(P, P)\} \subset \{P\} \times X_{\mathbf{q}}$, and the map $\gamma_0 : \mathcal{B}(-n) \rightarrow H_{\text{Br}}^{*,*}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z})$ is the Gysin map associated to the inclusion $\{P\} \hookrightarrow X_{\mathbf{q}}$.

Finally, $X_1 - X_0$ is the total space of the hyperplane bundle $\mathcal{O}_{X_{\mathbf{q}'}}(1)$ over $X_{\mathbf{q}'}$. It is easy to see that $\Gamma_1 \subset X_{\mathbf{q}'} \times X_{\mathbf{q}}$ is the projective closure $\mathbb{P}(\mathcal{O}_{X_{\mathbf{q}'}}(1) \oplus \mathbf{1})$ of $\mathcal{O}_{X_{\mathbf{q}'}}(1)$. The restriction of the projection pr_2 to $\Gamma_1(\mathbb{C})$ is the composition $\Gamma_1(\mathbb{C}) \rightarrow \text{Thom}(\mathcal{O}(1)(\mathbb{C})) = X_1(\mathbb{C}) \hookrightarrow X_{\mathbf{q}}(\mathbb{C})$, where the first map corresponds to collapsing the section at infinity and the last one is just the inclusion $i_1 : X_1(\mathbb{C}) \hookrightarrow X_{\mathbf{q}}(\mathbb{C})$.

Remark 6.4. It is easy to see that $\gamma_1(\alpha) \cap [X_{\mathbf{q}}] = i_{1*}(\mathcal{J}(\alpha \cap [X_{\mathbf{q}'}]))$, where $\mathcal{J} : H_{*,*}^{\text{Br}}(X_{\mathbf{q}'}(\mathbb{C}); \mathbb{Z}) \rightarrow H_{*+2, *+1}^{\text{Br}}(\text{Thom}(\mathcal{O}(1)(\mathbb{C})); \mathbb{Z})$ is the Thom isomorphism in Bredon homology.

Proposition 6.5. *Using the notation above, one has*

$$\gamma_1(\mathbf{1}) = \mathbf{c}_1(\mathcal{O}_{X_{\mathbf{q}}}(1)(\mathbb{C})) \in H_{\text{Br}}^{2,1}(X_{\mathbf{q}}(\mathbb{C}); \mathbb{Z}).$$

Proof. By definition $\gamma_1(\mathbf{1}) \cap [X_{\mathbf{q}}] = pr_{2*}(pr_1^*\mathbf{1} \cap [\Gamma_1]) = pr_{2*}[\Gamma_1]$. The latter element is precisely $[pr_2(\Gamma_1)] = [X_1]$, a hyperplane section of $X_{\mathbf{q}}$. \square

Consider the real quadric $\mathcal{Q}_{n,s}$, and assume $2s \leq n$. In order to simplify notation, denote the Bredon cohomology ring of $\mathcal{Q}_{n,s}(\mathbb{C})$ by

$$\mathbb{H}_{n,s} := H_{\text{Br}}^{*,*}(\mathcal{Q}_{n,s}(\mathbb{C}); \mathbb{Z}). \tag{36}$$

Using Propositions 6.3 and 6.5, together with an induction argument one easily obtains the additive structure of $\mathbb{H}_{n,s}$.

Proposition 6.6. Let $\mathbf{h} \in H_{\text{Br}}^{2,1}(\mathcal{Q}_{r,s}(\mathbb{C}); \mathbb{Z})$ denote the first Chern class $\mathbf{c}_1(\mathcal{O}(1))$ of $\mathcal{O}_{\mathcal{Q}_{n,s}}(1)$ and let $\eta \in H_{\text{Br}}^{2(n+1-s), n+1-s}(\mathcal{Q}_{n,s}(\mathbb{C}); \mathbb{Z})$ be the Poincaré dual to a $\mathbb{P}_{\mathbb{R}}^{s-1} \subset \mathcal{Q}_{n,s}$. Then one has an inclusion of \mathcal{B} -modules

$$j_* : \mathbb{H}_{n-2s,0} \rightarrow \mathbb{H}_{n,s}(s) \tag{37}$$

which induces an identification of bigraded \mathcal{B} -modules:

$$\begin{aligned} \mathbb{H}_{n,s} = & \mathcal{B} \cdot \mathbf{1} \oplus \mathcal{B} \cdot \mathbf{h} \oplus \dots \oplus \mathcal{B} \cdot \mathbf{h}^{s-1} \oplus \mathbb{H}_{n-2s,0}(-s) \\ & \oplus \mathcal{B} \cdot \eta \oplus \mathcal{B} \cdot \mathbf{h}\eta \oplus \dots \oplus \mathcal{B} \cdot \mathbf{h}^{s-1}\eta. \end{aligned}$$

6.1. The multiplicative structure

We proceed to determine the multiplicative structure of the cohomology ring of $\mathcal{Q}_{n,s}$, with $n \geq 2s$. The terms *vector bundles* and *spaces* henceforth mean *Real vector bundles* and *Real spaces*; cf. [1].

Given $0 \leq s \in \mathbb{Z}$, define a subring of the polynomial ring $\mathcal{B}[h, x, y]$ by

$$\mathcal{B}_s[h, x, y] := \mathcal{B}[h] + \langle h^s \rangle, \tag{38}$$

where $\langle h^s \rangle$ is the ideal generated by h^s . Every $P \in \mathcal{B}_s[h, x, y]$ can be written uniquely as

$$P = P_0 + h^s \cdot P_1, \tag{39}$$

with $P_0 = a_0 \cdot \mathbf{1} + a_1 \cdot h + \dots + a_{s-1} h^{s-1}$, $a_i \in \mathcal{B}$, $i = 1, \dots, s - 1$, and $P_1 \in \mathcal{B}[h, x, y]$.

In what follows, we identify $\mathbb{H}_{n-2s,0}$ with $\mathcal{A}[h, x]/J_{n-2s}$ and use the surjection of \mathcal{B} -algebras $\mathcal{B}[y] \rightarrow \mathcal{A}$ of \mathcal{B} -algebras; cf. Theorem 5.7 and Remark 2.9(ii), respectively. Let $\Lambda(\eta)$ be the exterior algebra over \mathbb{Z} on one generator η . It follows directly from Proposition 6.6 and (39) that the map of \mathcal{B} -modules defined by

$$\begin{aligned} \Psi : \mathcal{B}_s[h, x, y] \otimes \Lambda(\eta) & \longrightarrow \mathbb{H}_{n,s}, \\ h^j \otimes \eta^\epsilon & \longmapsto \mathbf{h}^j \eta^\epsilon, \quad j = 0, \dots, s - 1, \quad \epsilon = 0, 1, \\ h^s P \otimes \eta & \longmapsto 0, \quad \text{for all } P \in \mathcal{B}[h, x, y], \\ h^{s+r} x^k y^a \otimes 1 & \longmapsto 0, \quad \text{if either } kr \neq 0, \text{ or } k > 1, \end{aligned}$$

$$\begin{aligned}
 h^{s+r}y^a \otimes 1 &\longmapsto \mathbf{h}^r j_{\dagger}(\tau^{-a}), \quad \text{for all } a \geq 0, \\
 h^s X y^b \otimes 1 &\longmapsto j_{\dagger}(\tau^{-b} X), \quad \text{for all } b \geq 0
 \end{aligned}
 \tag{40}$$

is surjective. In what follows, we show that Ψ is indeed a \mathcal{B} -algebra homomorphism and identify its kernel.

Let L_1 and L_2 be line bundles over an $H\mathbb{Z}$ -oriented compact manifold X with first Chern classes $t_1, t_2 \in H_{\text{Br}}^{2,1}(X; \mathbb{Z})$, respectively. Let $\pi : \mathbb{P} \rightarrow X$ denote the projection from $\mathbb{P} := \mathbb{P}(L_1 \oplus L_2)$ onto X and let $s_{\infty} : X \rightarrow \mathbb{P}$ be the “section at infinity,” where X is identified with $\mathbb{P}(0 \oplus L_2)$. The following identities on characteristic classes are well known.

Lemma 6.7. *Using the notation above, let $\zeta := c_1(\mathcal{O}_{L_1 \oplus L_2}(1)) \in H_{\text{Br}}^{2,1}(\mathbb{P}, \mathbb{Z})$. Then*

$$s_{\infty}^* \mathcal{O}_{L_1 \oplus L_2}(1) = L_2^{\vee}, \tag{41}$$

where L_2^{\vee} is the dual of L_2 , and

$$\zeta \cap [\mathbb{P}] = -\pi^* t_1 \cap [\mathbb{P}] + s_{\infty*}[X]. \tag{42}$$

Furthermore, for all $r \geq 0$ one has

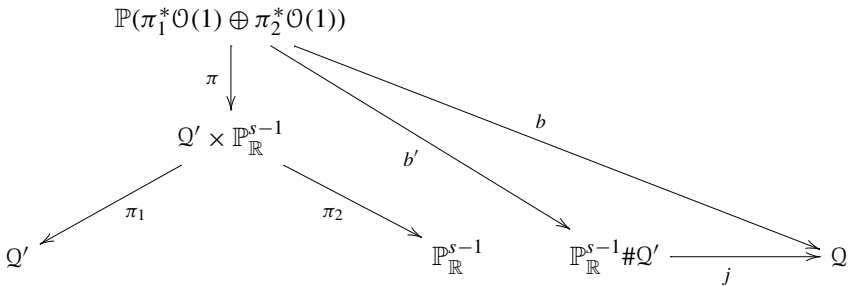
$$\zeta^{r+1} \cap [\mathbb{P}] = (-1)^{r+1} \pi^* t_1^{r+1} \cap [\mathbb{P}] + (-1)^r \sum_{i+j=r} s_{\infty*}(t_1^i t_2^j \cap [X]). \tag{43}$$

Proof. The first two identities are standard facts about characteristic classes; cf. [9].

The last assertion is true for $r = 0$; cf. (42). Assume true for $r_0 - 1$, then

$$\begin{aligned}
 \zeta^{r_0+1} \cap [\mathbb{P}] &= (-1)^{r_0} \pi^* t_1^{r_0} \cap \zeta \cap [\mathbb{P}] \\
 &\quad + (-1)^{r_0-1} \sum_{i+j=r_0-1} \zeta \cap s_{\infty*}(t_1^i t_2^j \cap [X]) \\
 &= (-1)^{r_0} \pi^* t_1^{r_0} \cap \{-\pi^*(t_1) \cap [\mathbb{P}] + s_{\infty*}[X]\} \\
 &\quad + (-1)^{r_0-1} \sum_{i+j=r_0-1} s_{\infty*}(s_{\infty}^* \zeta \cap (t_1^i t_2^j \cap [X])) \\
 &\stackrel{(41)}{=} (-1)^{r_0+1} \pi^*(t_1^{r_0+1}) \cap [\mathbb{P}] \\
 &\quad + (-1)^{r_0} s_{\infty*}(t_1^{r_0} \cap [X]) + (-1)^{r_0} \sum_{i+j=r_0-1} s_{\infty*} t_1^i t_2^{j+1} \cap [X] \\
 &= (-1)^{r_0+1} \pi^* t_1^{r_0+1} \cap [\mathbb{P}] + (-1)^{r_0} \sum_{i+j=r_0} s_{\infty*}(t_1^i t_2^j \cap [X]). \quad \square
 \end{aligned}$$

Denote $\mathcal{Q} = \mathcal{Q}_{n,s}$ and $\mathcal{Q}' = \mathcal{Q}_{n-2s,0}$. The notation in the following diagram will be used in the proofs of all remaining results in this paper.



Here $\mathbb{P}_{\mathbb{R}}^{s-1} \# \mathcal{Q}'$ is the ruled join of $\mathbb{P}_{\mathbb{R}}^{s-1}$ and \mathcal{Q}' , j is the natural inclusion, b' is the blow-up map of $\mathbb{P}_{\mathbb{R}}^{s-1} \# \mathcal{Q}'$ along $\mathbb{P}_{\mathbb{R}}^{s-1} \sqcup \mathcal{Q}'$ and $b = j \circ b'$.

Proposition 6.8. *The map*

$$j_{\dagger} : \mathbb{H}_{n-2s,0} \longrightarrow \mathbb{H}_{n,s}(s)$$

introduced in (37) sends $\alpha \in H_{\text{Br}}^{p,q}(\mathcal{Q}')$ to $j_{\dagger}(\alpha)$ satisfying

$$j_{\dagger}(\alpha) \cap [\mathcal{Q}] := (-1)^q b_* (\pi^*(\alpha \times \mathbf{1}) \cap [\mathbb{P}]) \in H_{\text{Br}}^{p+2s,q+s}(\mathcal{Q}),$$

where $\mathbb{P} := \mathbb{P}(\pi_1^* \mathcal{O}(1) \oplus \pi_2^* \mathcal{O}(1))$. In particular, $j_{\dagger}(1) = \mathbf{h}^s$.

Proof. Left to the reader. \square

Notation 6.9. For $X = \mathcal{Q}' \times \mathbb{P}_{\mathbb{R}}^{s-1}$, let $L_1 = \pi_1^* \mathcal{O}(1)$ and $L_2 = \pi_2^* \mathcal{O}(1)$, and denote $c_1 L_1 = t_1 = h \times \mathbf{1}$ and $c_1 L_2 = t_2 = \mathbf{1} \times t$. Let $i : \mathbb{P}_{\mathbb{R}}^{s-1} \hookrightarrow \mathcal{Q}$ be the inclusion.

Lemma 6.10. *Given $\alpha \in H_{\text{Br}}^{p,q}(\mathcal{Q}'(\mathbb{C}); \mathbb{Z})$ and $r \geq 0$ one has:*

$$\begin{aligned}
 & \mathbf{h}^{r+1} \cap j_{\dagger}(\alpha) \cap [\mathcal{Q}(\mathbb{C})] \\
 &= j_{\dagger}(h^{r+1} \alpha) \cap [\mathcal{Q}(\mathbb{C})] + (-1)^{r+q} i_* \pi_{2*} \sum_{i+j=r} (\alpha h^i \cap [\mathcal{Q}'(\mathbb{C})]) \times (t^j \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})]).
 \end{aligned}$$

Proof. The fact that $b^* \mathbf{h} = \zeta$ and Lemma 6.7 give:

$$\begin{aligned}
 & \mathbf{h}^{r+1} \cap j_{\dagger}(\alpha) \cap [\mathcal{Q}(\mathbb{C})] \\
 &= (-1)^q \mathbf{h}^{r+1} \cap b_* (\pi^*(\alpha \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\
 &= (-1)^q b_* (\pi^*(\alpha \times \mathbf{1}) \cap \{b^* \mathbf{h}^{r+1} \cap [\mathbb{P}(\mathbb{C})]\}) \\
 &= (-1)^q b_* (\pi^*(\alpha \times \mathbf{1}) \cap \{(-1)^{r+1} \pi^*(h^{r+1} \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]\})
 \end{aligned}$$

$$\begin{aligned}
 &+ (-1)^r s_{\infty*} \sum_{i+j=r} (h^i \times t^j) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})]) \\
 = &(-1)^{q+r+1} b_*(\pi^*(h^{r+1}\alpha \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\
 &+ (-1)^{r+q} b_*(\pi^*(\alpha \times \mathbf{1}) \cap s_{\infty*} \sum_{i+j=r} (h^i \times t^j) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})]) \\
 = &j_{\dagger}(h^{r+1}\alpha) \cap [Q] + (-1)^{r+q} b_* s_{\infty*} \left(\sum_{i+j=r} (\alpha h^i \times t^j) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})] \right) \\
 = &j_{\dagger}(h^{r+1}\alpha) \cap [Q(\mathbb{C})] + (-1)^{r+q} i_* \pi_{2*} \left(\sum_{i+j=r} (\alpha h^i \times t^j) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})] \right). \quad \square
 \end{aligned}$$

Corollary 6.11. For all $a \geq 0$ one has:

$$\mathbf{h}^r j_{\dagger}(\tau^{-a} h^{r'}) = \begin{cases} j_{\dagger}(\tau^{-a} h^{r+r'}), & \text{if } 0 \leq r+r' \leq n-2s, \\ 2\tau^{-a} \eta \mathbf{h}^{r+r'-1-n+2s}, & \text{if } n-2s+1 \leq r+r' \leq n-s, \\ 0, & \text{if } n-s < r+r'. \end{cases}$$

Proof. One has

$$\begin{aligned}
 &\mathbf{h}^r j_{\dagger}(\tau^{-a} h^{r'}) \cap [Q(\mathbb{C})] \\
 &= j_{\dagger}(\tau^{-a} h^{r+r'}) \cap [Q(\mathbb{C})] \\
 &\quad + (-1)^{r+r'-1} \sum_{i+j=r-1} i_* \pi_{2*} (\tau^{-a} h^{i+r'} \cap [Q'(\mathbb{C})] \times t^j \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})]).
 \end{aligned}$$

For $r+r'-1 < n-2s$ the terms in the summation above vanish for dimensional reasons, hence $\mathbf{h}^r j_{\dagger}(\tau^{-a} h^{r'}) = j_{\dagger}(\tau^{-a} h^{r+r'})$ in this case. For $r+r'-1 \geq n-2s$ one has $h^{r+r'} = 0$ in $\mathbb{H}_{n-2s,0}$, thus giving

$$\mathbf{h}^r j_{\dagger}(\tau^{-a} h^{r'}) = (-1)^{r-1} i_* \pi_{2*} (\tau^{-a} h^{n-2s} \cap [Q'(\mathbb{C})] \times t^{r+r'-1-n+2s} \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})]). \quad (44)$$

Since $\dim \mathbb{P}_{\mathbb{R}}^{s-1} = s-1$, this immediately shows that $\mathbf{h}^r j_{\dagger}(\tau^{-a} h^{r'}) = 0$, if $r+r' > n-s$.

Now, if $n-2s+1 \leq r+r' \leq n-s$, i.e. $0 \leq r+r'-1-n-2s \leq s-1$, one has

$$i_*(t^{r+r'-1-n+2s} \cap \mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})) = \eta \mathbf{h}^{r+r'-1-n+2s} \cap [Q(\mathbb{C})].$$

On the other hand, for all $m \geq 0$ one has

$$\pi_{2*}(\tau^{-a} h^{n-2s} \cap [Q'(\mathbb{C})] \times t^m \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})]) = (2\tau^{-a}) t^m \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})].$$

Therefore,

$$\mathbf{h}^r j_{\dagger}(\tau^{-a} h^{r'}) = (2\tau^{-a}) \eta \mathbf{h}^{r+r'-1-n+2s},$$

for $n-2s+1 \leq r+r' \leq n-s$. \square

Remark 6.12. Note that $2\tau^{-a}$ is an element in \mathcal{B} for all $a \in \mathbb{Z}$ while \mathcal{B} does not have an inverse τ^{-a} for τ^a , for $a > 0$.

Lemma 6.13. For all $\alpha \in \mathbb{H}_{n-2s,0}$ and $k \in \mathbb{Z}$ one has

- (a) $j_{\dagger}(\alpha) \cdot \eta = 0$;
- (b) $b^*(j_{\dagger}(\tau^k \alpha)) = \tau^k b^*(j_{\dagger}(\alpha)) \in H_{\text{Br}}^{*,*}(\mathbb{P}(\mathbb{C}), \mathbb{Z})$.

Proof. One has

$$\eta j_{\dagger}(\alpha) \cap [\mathcal{Q}(\mathbb{C})] = \pm \eta \cap b_*(\pi^*(\alpha \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) = \pm b_*(\pi^*(\alpha \times \mathbf{1}) \cap b^* \eta \cap [\mathbb{P}(\mathbb{C})]). \tag{45}$$

However, one has $\text{deg } \eta = (2(n - 2 + 1), n - 2 + 1)$, while $\dim \mathbb{P} = n - s$, and hence $b^* \eta = 0$, thus proving the first assertion of the lemma.

Now, observe that the second assertion holds for $k \geq 0$, since j_{\dagger} is a homomorphism of \mathcal{B} -modules. If $k = -a$, with $a > 0$, one has $\tau^a j_{\dagger}(\tau^{-a} \alpha) = j_{\dagger}(\alpha)$, and hence $b^*(\tau^a j_{\dagger}(\tau^{-a} \alpha)) = b^* j_{\dagger}(\alpha)$. This is equivalent to saying that $\tau^a b^*(j_{\dagger}(\tau^{-a} \alpha)) = b^* j_{\dagger}(\alpha)$. Since \mathfrak{S} acts freely on $\mathbb{P}(\mathbb{C})$, τ is invertible in the cohomology of $\mathbb{P}(\mathbb{C})$, and one has $b^*(j_{\dagger}(\tau^{-a} \alpha)) = \tau^{-a} b^*(j_{\dagger} \alpha)$. \square

Corollary 6.14. Denote $X_a := j_{\dagger}(\tau^{-a} X)$ and $T_a = j_{\dagger}(\tau^{-a})$. Then:

- (a) $T_a T_b = T_{a+b} \mathbf{h}^s$,
- (b) $\mathbf{h}^{r+1} X_a = 0$ for all $r \geq 0$,
- (c) $T_a X_b = 0$ for all $a, b \geq 0$,
- (d) $X_a X_b = 0$ for all $a, b \geq 0$.

Proof. To prove (a) one observes that

$$\begin{aligned} T_a T_b \cap [\mathcal{Q}(\mathbb{C})] &= T_a \cap b_*(\pi^*(\tau^{-b} \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= b_*(\tau^{-a} b^*(j_{\dagger}(\mathbf{1})) \pi^*(\tau^{-b} \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= b_*(b^* \mathbf{h}^s \cdot \pi^*(\tau^{-a-b} \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= \mathbf{h}^s T_{a+b} \cap [\mathcal{Q}(\mathbb{C})]. \end{aligned}$$

To prove assertion (b) one has:

$$\begin{aligned} \mathbf{h}^{r+1} X_a \cap [\mathcal{Q}(\mathbb{C})] &= \pm b_*(\pi^*(\tau^{-a} X \times \mathbf{1}) \cap \{b^* \mathbf{h}^{r+1} \cap [\mathbb{P}(\mathbb{C})]\}) \\ &= \pm b_*(\pi^*(\tau^{-a} X \times \mathbf{1}) \cap \{(-1)^{r+1} \pi^*(h^{r+1} \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})] \\ &\quad + (-1)^r s_{\infty*} \sum_{i+j=r} (h^i \times t^j) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})]\}) \\ &= \pm b_*(\pi^*(\tau^{-a} h^{r+1} X \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &\quad \pm s_{\infty*} \sum_{i+j=r} (\tau^{-a} X h^i \times t^j) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})]) \\ &= \pm i_* \pi_{2*} \{(\tau^{-a} X \times t^r) \cap [(Q' \times \mathbb{P}_{\mathbb{R}}^{s-1})(\mathbb{C})]\} \\ &= \pm i_* \pi_{2*} \{ \tau^{-a} \cap [S_a^{n-2s}] \times t^r \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})] \}. \end{aligned}$$

The latter expression is clearly zero if $n > 2s$ (Künneth formula for $\mathcal{Q}' \times \mathbb{P}_{\mathbb{R}}^{s-1}$), and when $n = 2s$ one has $S_a^0 \cong \mathfrak{S} = \{0, 1\}$ and $[S_a^0] = [1] - [0]$. It follows that $\pi_{2*}(\tau^{-a} \cap [S_a^0] \times t^r \cap [\mathbb{P}_{\mathbb{R}}^{s-1}]) = 0$, as well.

To prove assertion (c) one has

$$\begin{aligned} \tau_a X_b \cap [\mathcal{Q}(\mathbb{C})] &= \pm \tau_a \cap b_* (\pi^*(\tau^{-b} X \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= \pm b_* (b^* j_{\dagger}(\tau^{-a}) \cap \pi^*(\tau^{-b} X \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= \pm b_* (\tau^{-a} b^* h^s \cap \pi^*(\tau^{-b} X \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= \pm b_* (\pi^*(\tau^{-a-b} h^s X \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &\quad \pm i_* \pi_{2*} \left(\sum_{i+j=s-1} \tau^{-a-b} X h^i \cap [\mathcal{Q}'(\mathbb{C})] \times t^j \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})] \right) \\ &= \pm i_* \pi_{2*} (\tau^{-a-b} X \cap [\mathcal{Q}'(\mathbb{C})] \times t^{s-1} \cap [\mathbb{P}_{\mathbb{R}}^{s-1}(\mathbb{C})]) = 0, \end{aligned}$$

using the arguments of the previous lemma.

Observe that \mathfrak{S} acts freely on $\mathbb{P}(\mathbb{C})$ and that $\dim \mathbb{P} = n - s$. Hence, whenever $\alpha \in H_{\text{Br}}^{i,j}(\mathbb{P}(\mathbb{C}); \mathbb{Z})$ and $i > 2(n - s)$ one has $\alpha = 0$; cf. Property 2.7(ii). Since

$$\begin{aligned} X_a \cap (X_b \cap [\mathcal{Q}(\mathbb{C})]) &= \pm X_a \cap b_* (\pi^*(\tau^{-b} X \times \mathbf{1}) \cap [\mathbb{P}(\mathbb{C})]) \\ &= \pm b_* (\{b^* X_a \cdot \pi^*(\tau^{-b} X \times \mathbf{1})\} \cap [\mathbb{P}(\mathbb{C})]) \end{aligned}$$

and $\deg b^* X_a \cdot \pi^*(\tau^{-b} X \times \mathbf{1}) = (2n, 2(s - 1) - a - b)$, it follows that $X_a \cdot X_b = 0$, thus showing the last assertion of the lemma. \square

Proposition 6.15. Write $n = 2m - \delta$. Then

- (i) The map $\Psi : \mathcal{B}_s[h, x, y] \otimes \Lambda(\eta) \rightarrow H_{\text{Br}}^{*,*}(\mathcal{Q}_{n,s}(\mathbb{C}); \mathbb{Z})$ (40) is a surjective homomorphism of \mathcal{B} -algebras.
- (ii) The kernel of Ψ is the ideal

$$\mathcal{J}_{n,s} = [h^s] \cdot \tilde{\mathcal{J}}_{n-2s} \otimes \Lambda(\eta) + [h^s] \otimes \langle \eta \rangle + \langle h^{n-s+1} \otimes 1 - 2(1 \otimes \eta) \rangle,$$

where $\tilde{\mathcal{J}}_{n-2s} = [g_1, g_2, g_3, g_4, g_5] \subset \mathcal{B}[h, X, y]$ is the ideal generated by the elements

$$\begin{aligned} g_1 &= \mathbf{f}_{m-s}, & g_2 &= e^{1-\delta} \tau^{m-s} X - h^{1-\delta} \mathbf{f}_{m-s-1}, & g_3 &= hX, \\ g_4 &= h^{2(m-s)} - \delta \{(-1)^{m-s} \tau^{m-s+1} X^2\}, & \text{and } g_5 &= \tau y - 1. \end{aligned}$$

Proof. The fact that Ψ is a ring homomorphism follows directly from Corollaries 6.11, 6.14 and Eq. (39), together with the obvious fact that $\eta^2 = 0$ in $\mathbb{H}_{n,s}$. This concludes the proof of (i), since Ψ was shown to be surjective in (40).

Let $\varrho : \mathcal{B}[h, X, y] \rightarrow \mathbb{H}_{n-2s,0}$ denote the surjection induced by $\mathcal{B}[y] \rightarrow \mathcal{A}$ and the presentation $\mathbb{H}_{n-2s,0} \cong \mathcal{A}[h, X]/\mathcal{J}_{n-2s}$. It follows from Theorem 5.7 and Remark 2.9(ii) that the ideal $\tilde{\mathcal{J}}_{n-2s}$ in the statement of the proposition is precisely the kernel of ϱ .

It follows from the definition of Ψ and Corollary 6.11 that whenever the highest power of h in $P \in \mathcal{B}[h, x, y]$ (denoted $\text{deg}_h P$) is less or equal than $n - 2s$ then

$$\Psi(h^s P \otimes 1) = j_{\dagger}(\varrho(P)). \tag{46}$$

Now, the fact that $\text{deg}_h \mathbf{f}_r \leq r$, for all r , together with (46) and the definition of Ψ shows that $[h^s] \cdot \tilde{J}_{n-2s} \otimes \Lambda(\eta) \subset \ker \Psi$. Also, Corollary 6.11 gives $\mathbf{h}^{n-s+1} = \mathbf{h}^{n-2s+1} j_{\dagger}(1) = 2\eta$ and hence, the element $h^{n-s+1} \otimes 1 - 2(1 \otimes \eta)$ belongs to the kernel of Ψ . Since $[h^s] \otimes \langle \eta \rangle \subset \ker \Psi$, by definition, one concludes that $\mathcal{J}_{n,s} \subset \ker \Psi$.

Consider $\mathbf{u} \in \ker \Psi$. Since $[h^s] \otimes \langle \eta \rangle \subset \mathcal{J}_{n,s}$ then

$$\mathbf{u} \equiv A_0 \otimes 1 + B_0 \otimes \eta + h^s P \otimes 1 \pmod{\mathcal{J}_{n,s}}$$

where $A_0 = a_0 + a_1 h + \dots + a_{s-1} h^{s-1}$, $B_0 = b_0 + b_1 h + \dots + b_{s-1} h^{s-1}$, and P is an arbitrary element of $\mathcal{B}[h, x, y]$.

Let us now write $P = P_0 + h^{n-2s+1} P_1 + h^{n-s+1} P_2$, where $\text{deg}_h P_0 \leq n - 2s$ and $\text{deg}_h P_1 \leq s - 1$. A simple inspection shows that $h^{n-s+1} \in \tilde{J}_{n-2s}$. Therefore, $h^s P \otimes 1 \equiv h^s P_0 \otimes 1 + h^{n-s+1} P_1 \otimes 1 \pmod{\mathcal{J}_{n,s}}$. On the other hand, $h^{n-s+1} P_1 \otimes 1 = (h^{n-s+1} \otimes 1)(P_1 \otimes 1) \equiv 2(1 \otimes \eta)(P_1 \otimes 1) \pmod{\mathcal{J}_{n,s}}$. It follows that one can finally write

$$\mathbf{u} \equiv A_0 \otimes 1 + B'_0 \otimes \eta + h^s P_0 \otimes 1 \pmod{\mathcal{J}_{n,s}}, \tag{47}$$

where $B'_0 = B_0 + 2P_1$ and $\text{deg}_h B'_0 \leq s - 1$.

Finally, it follows from (46) that

$$0 = \Psi(\mathbf{u}) = \Psi(A_0 \otimes 1) + \Psi(B'_0 \otimes \eta) + j_{\dagger}(\varrho(P_0)).$$

Since the decomposition in Proposition 6.6 is a direct sum, one concludes that $A_0 = B'_0 = 0$ and $j_{\dagger}(\varrho(P_0)) = 0$. Since j_{\dagger} is injective, one concludes that $P_0 \in \ker \varrho$, in other words $P_0 \in \tilde{J}_{n-2s}$. This shows that $\mathbf{u} \in \mathcal{J}_{n,s}$ and hence $\ker \Psi = \mathcal{J}_{n,s}$. \square

With this proposition, we conclude the proof of Theorem A, stated in Section 1.

References

- [1] M.F. Atiyah, *K*-theory and reality, *Quart. J. Math. Oxford Ser. (2)* 17 (1966) 367–386.
- [2] A.K. Bousfield, D.M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin, 1972.
- [3] Glen E. Bredon, *Equivariant cohomology theories*, *Bull. Amer. Math. Soc.* 73 (1967) 266–268.
- [4] S.R. Costenoble, S. Waner, *Equivariant Poincaré duality*, *Michigan Math. J.* 39 (2) (1992) 325–351.
- [5] Pedro F. dos Santos, *Algebraic cycles on real varieties and $\mathbb{Z}/2$ -equivariant homotopy theory*, *Proc. London Math. Soc.* (3) 86 (2) (2003) 513–544.
- [6] Pedro F. dos Santos, Paulo Lima-Filho, *Quaternionic algebraic cycles and reality*, *Trans. Amer. Math. Soc.* 356 (12) (2004) 4701–4736 (electronic).
- [7] Daniel Dugger, Daniel C. Isaksen, *Topological hypercovers and \mathbb{A}^1 -realizations*, *Math. Z.* 246 (4) (2004) 667–689.
- [8] Leonard Evens, *The Cohomology of Groups*, Oxford Math. Monogr., Clarendon/Oxford Univ. Press, New York, 1991.
- [9] William Fulton, *Intersection Theory*, second ed., *Ergeb. Math. Grenzgeb.* (3), vol. 2, Springer-Verlag, Berlin, 1998.
- [10] Phillip Griffiths, Joseph Harris, *Principles of Algebraic Geometry*, Wiley Classics Lib., Wiley, New York, 1994, reprint of the 1978 original.

- [11] Howard L. Hiller, On the cohomology of real Grassmanians, *Trans. Amer. Math. Soc.* 257 (2) (1980) 521–533.
- [12] Bruno Kahn, Motivic cohomology of smooth geometrically cellular varieties, in: *Algebraic K-Theory*, Seattle, WA, 1997, in: *Proc. Sympos. Pure Math.*, vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 149–174.
- [13] N.A. Karpenko, Cohomology of relative cellular spaces and of isotropic flag varieties, *Algebra i Analiz* 12 (1) (2000) 3–69.
- [14] V.A. Krasnov, Harnack–Thom inequalities for mappings of real algebraic varieties, *Izv. Akad. Nauk SSSR Ser. Mat.* 47 (2) (1983) 268–297.
- [15] V.A. Krasnov, On the equivariant Grothendieck cohomology of a real algebraic variety and its application, *Izv. Ross. Akad. Nauk Ser. Mat.* 58 (3) (1994) 36–52.
- [16] H. Blaine Lawson, Paulo Lima-Filho, Marie-Louise Michelsohn, Algebraic cycles and the classical groups. I. Real cycles, *Topology* 42 (2) (2003) 467–506.
- [17] G. Lewis, J.P. May, J. McClure, Ordinary $RO(G)$ -graded cohomology, *Bull. Amer. Math. Soc. (N.S.)* 4 (2) (1981) 208–212.
- [18] L.G. Lewis Jr., J.P. May, M. Steinberger, J.E. McClure, *Equivariant Stable Homotopy Theory*, *Lecture Notes in Math.*, vol. 1213, Springer-Verlag, Berlin, 1986, with contributions by J.E. McClure.
- [19] J.P. May, *Equivariant Homotopy and Cohomology Theory*, *CBMS Reg. Conf. Ser. Math.*, vol. 91, Amer. Math. Soc., Providence, RI, 1996, with contributions by M. Cole, G. Comezana, S. Costenoble, A.D. Elmendorf, J.P.C. Greenlees, L.G. Lewis, Jr., R.J. Piacenza, G. Triantafillou, and S. Waner.
- [20] Markus Rost, *The motive of a Pfister form*, preprint, 1998.
- [21] Robert Silhol, *Real Algebraic Surfaces*, *Lecture Notes in Math.*, vol. 1392, Springer-Verlag, Berlin, 1989.
- [22] J.A. van Hamel, *Algebraic Cycles and Topology of Real Algebraic Varieties*, *CWI Tract*, vol. 129, Centrum Wiskund. Inform., Amsterdam, 2000, dissertation, Vrije Universiteit Amsterdam, Amsterdam.
- [23] Nobuaki Yagita, *Coniveau spectral sequence and motivic cohomology of quadrics and classifying spaces*, preprint, September, 2004.