Dominant K-theory and integrable highest weight representations of Kac–Moody groups

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To Haynes Miller on his 60th birthday

Abstract

We give a topological interpretation of the highest weight representations of Kac–Moody groups. Given the unitary form $\mathcal{G}$ of a Kac–Moody group (over $\mathbb{C}$), we define a version of equivariant K-theory, $\mathbb{K}_{\mathcal{G}}$ on the category of proper $\mathcal{G}$-CW complexes. We then study Kac–Moody groups of compact type in detail (see Section 2 for definitions). In particular, we show that the Grothendieck group of integrable highest weight representations of a Kac–Moody group $\mathcal{G}$ of compact type, maps isomorphically onto $\tilde{\mathbb{K}}_{\mathcal{G}}^*(E\mathcal{G})$, where $E\mathcal{G}$ is the classifying space of proper $\mathcal{G}$-actions. For the affine case, this agrees very well with recent results of Freed–Hopkins–Teleman. We also explicitly compute $\mathbb{K}_{\mathcal{G}}^*(E\mathcal{G})$ for Kac–Moody groups of extended compact type, which includes the Kac–Moody group E_{10}.
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1. Introduction

Given a compact Lie group G and a G-space X, the equivariant K-theory of X, $K_G^*(X)$ can be described geometrically in terms of equivariant vector bundles on X. If one tries to relax the condition of G being compact, one immediately runs into technical problems in the definition of equivariant K-theory. One way around this problem is to impose conditions on the action of the group G on the space X. For example, one may restrict their attention to the category of $\mathcal{G}$-CW complexes. Recall that for a topological group $\mathcal{G}$, a $\mathcal{G}$-CW complex X is a $\mathcal{G}$-space built inductively by attaching equivariant discs of the form $\mathcal{G}/H \times D^n$ along their boundary $\mathcal{G}/H \times S^n$, where H is a closed subgroup of $\mathcal{G}$. Notice that in this definition, we do not require X itself to have the structure of a CW-complex. We may further ask for the action of $\mathcal{G}$ on X to be proper. By a proper action, we shall mean that all the isotropy groups H are compact subgroups of $\mathcal{G}$.

There exists a universal space with a proper $\mathcal{G}$-action, known as the classifying space for proper $\mathcal{G}$-actions, which is a terminal object (up to $\mathcal{G}$-equivariant homotopy) in the category of proper $\mathcal{G}$-spaces [23,29]. One may give an alternate identification of this classifying space as:

Definition 1.1. For a topological group $\mathcal{G}$, the classifying space for proper actions $E\mathcal{G}$ is a $\mathcal{G}$-CW complex with the property that all the isotropy subgroups are compact, and given a compact subgroup $H \subseteq \mathcal{G}$, the fixed point space $E\mathcal{G}^H$ is weakly contractible.

Notice that if $\mathcal{G}$ is a compact Lie group, then $E\mathcal{G}$ is simply equivalent to a point, and so $K_{\mathcal{G}}^*(E\mathcal{G})$ is isomorphic to the representation ring of $\mathcal{G}$. For a general non-compact group $\mathcal{G}$, the definition or geometric meaning of $K_{\mathcal{G}}^*(E\mathcal{G})$ remains unclear.

In this paper, we deal with a class of topological groups known as Kac–Moody groups [13,14]. By a Kac–Moody group, we shall mean the unitary form of a split Kac–Moody group over $\mathbb{C}$. We refer the reader to [21] for a beautiful treatment of the subject. These groups form a natural extension of the class of compact Lie groups, and share many of their properties. They are known to contain the class of (polynomial) loop groups, which go by the name of affine Kac–Moody groups. With the exception of compact Lie groups, Kac–Moody groups over $\mathbb{C}$ are not even locally compact (local compactness holds for Kac–Moody groups defined over finite fields). However, the theory of integrable highest weight representations does extend to the world of Kac–Moody groups. For a loop group, these integrable highest weight representations form the well studied class of positive energy representations [26].

It is therefore natural to ask if some version of equivariant K-theory for Kac–Moody groups encodes the integrable highest weight representations. The object of this paper is two fold. Firstly, we define a version of equivariant K-theory $\mathbb{K}_{\mathcal{G}}$ as a functor on the category of proper $\mathcal{G}$-CW

complexes, where $\mathcal{G}$ is a Kac–Moody group. For reasons that will become clear later, we shall call this functor "Dominant K-theory." For a compact Lie group $\mathcal{G}$, this functor is usual equivariant K-theory. Next, we build an explicit model for the classifying space of proper $\mathcal{G}$-actions, $E\mathcal{G}$, and calculate the groups $\mathbb{K}^*_{\mathcal{G}}(E\mathcal{G})$ for Kac–Moody groups $\mathcal{G}$ of compact, and extended compact type (see Section 2). Indeed, we construct a map from the Grothendieck group of highest weight representations of $\mathcal{G}$, to $\mathbb{K}^*_{\mathcal{G}}(E\mathcal{G})$ for a Kac–Moody group of compact type, and show that this map is an isomorphism.

This document is directly inspired by the following recent result of Freed, Hopkins and Teleman [10] (see also [24]): let G denote a compact Lie group and let $\tilde{L}G$ denote the universal central extension of the group of smooth free loops on G. In [10], the authors calculate the twisted equivariant K-theory of the conjugation action of G on itself and describe it as the Grothendieck group of positive energy representations of $\tilde{L}G$. The positive energy representations form an important class of (infinite-dimensional) representations of $\tilde{L}G$ [26] that are indexed by an integer known as 'level.' If G is simply connected, then the loop group $\tilde{L}G$ admits a Kac–Moody form known as the affine Kac–Moody group. We will show that the Dominant K-theory of the classifying space of proper actions of the corresponding affine Kac–Moody group is simply the graded sum (under the action of the central circle) of the twisted equivariant K-theory groups of G. Hence, we recover the theorem of Freed–Hopkins–Teleman in the special case of the loop group of a connected, simply connected, simple, compact Lie group.

As a final remark, we mention that Dominant K-theory is a multiplicative theory by construction. However, the multiplication appears to not be interesting. For example, in the case of the Loop group, our multiplication is trivial. However, there is an interesting (fusion) product on the Grothendieck group of positive energy representations making it an algebra known as the Verlinde algebra. This appears to be special to affine Kac–Moody groups (loop groups). A topological interpretation of this product in terms of Dominant K-theory is something we leave for future discussion.

Organization of the paper. We begin in Section 2 with background on Kac–Moody groups and describe the main definitions, constructions and results of this paper. Section 3 describes the finite type topological Tits building as a model for the classifying space for proper actions of a Kac–Moody group, and in Section 4 we study the properties of Dominant K-theory as an equivariant cohomology theory. In Sections 5 we compute the Dominant K-theory of the Tits building for a Kac–Moody group of compact type, and in Section 6 we give our computation a geometric interpretation in terms of an equivariant family of cubic Dirac operators. Section 7 explores the Dominant K-theory of the building for the extended compact type. Section 8 is a discussion on various related topics, including the relationship of our work with the work of Freed–Hopkins–Teleman. Included also in this section are remarks concerning real forms of Kac–Moody groups, and the group E_{10}. Finally, in Section 9, we define a corresponding Dominant K-homology theory, by introducing the equivariant dual of proper complexes. We also compute the Dominant K-homology of the building in the compact type. Appendix A is devoted to the construction of a compatible family of metrics on the Tits building, which is required in Section 6. Appendix B extends the definition of Dominant K-theory to a more general situation.

2. Background and statement of results

Kac–Moody groups have been around for more than twenty years. They have been extensively studied and much is known about their general structure, representation theory and topology [14–18,21,28] (see [21] for a modern perspective). One begins with a finite integral matrix

$A = (a_{i,j})_{i,j \in I}$ with the properties that $a_{i,i} = 2$ and $a_{i,j} \leqslant 0$ for $i \neq j$. Moreover, we demand that $a_{i,j} = 0$ if and only if $a_{j,i} = 0$. These conditions define a *Generalized Cartan Matrix*. A generalized Cartan matrix is said to be symmetrizable if it becomes symmetric after multiplication with a suitable rational diagonal matrix.

Given a generalized Cartan matrix A, one may construct a complex Lie algebra $\mathfrak{g}(A)$ using the Harish–Chandra–Serre relations. This Lie algebra contains a finite-dimensional Cartan subalgebra $\mathfrak{h}$ that admits an integral form $\mathfrak{h}_{\mathbb{Z}}$ and a real form $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{Z}} \otimes \mathbb{R}$. The lattice $\mathfrak{h}_{\mathbb{Z}}$ contains a finite set of primitive elements h_i, $i \in I$, called "simple coroots." Similarly, the dual lattice $\mathfrak{h}_{\mathbb{Z}}^*$ contains a special set of elements called "simple roots" α_i, $i \in I$. One may decompose $\mathfrak{g}(A)$ under the adjoint action of $\mathfrak{h}$ to obtain a triangular form as in the classical theory of semisimple Lie algebras. Let $\eta_{\pm}$ denote the positive and negative "nilpotent" subalgebras respectively, and let $\mathfrak{b}_{\pm} = \mathfrak{h} \oplus \eta_{\pm}$ denote the corresponding "Borel" subalgebras. The structure theory of $\mathfrak{g}(A)$ leads to a construction (in much the same way that Chevalley groups are constructed), of a topological group $G(A)$ called the (minimal, split) Kac–Moody group over the complex numbers. The group $G(A)$ supports a canonical anti-linear involution ω, and one defines the unitary form $K(A)$ as the fixed group $G(A)^{\omega}$. It is the group $K(A)$ that we study in this article. We refer the reader to [21] for details on the subject.

Given a subset $J \subseteq I$, one may define a parabolic subalgebra $\mathfrak{g}_J(A) \subseteq \mathfrak{g}(A)$ generated by $\mathfrak{b}_+$ and the root spaces corresponding to the set J. For example, $\mathfrak{g}_{\varnothing}(A) = \mathfrak{b}_+$. One may exponentiate these subalgebras to parabolic subgroups $G_J(A) \subset G(A)$. We then define the unitary Levi factors $K_J(A)$ to be the groups $K(A) \cap G_J(A)$. Hence $K_{\varnothing}(A) = T$ is a torus of rank $2|I| - \mathrm{rk}(A)$, called the maximal torus of $K(A)$. The normalizer $N(T)$ of T in $K(A)$, is an extension of a discrete group $W(A)$ by T. The Weyl group $W(A)$ has the structure of a crystallographic Coxeter group generated by reflections r_i, $i \in I$. For $J \subseteq I$, let $W_J(A)$ denote the subgroup generated by the corresponding reflections r_j, $j \in J$. The group $W_J(A)$ is a crystallographic Coxeter group in its own right that can be identified with the Weyl group of $K_J(A)$.

We will identify the type of a Kac–Moody group $K(A)$, by that of its generalized Cartan matrix. For example, a generalized Cartan matrix A is called of *Finite Type* if the Lie algebra $\mathfrak{g}(A)$ is a finite-dimensional semisimple Lie algebra. In this case, the groups $G(A)$ are the corresponding simply connected semisimple complex Lie groups. Another sub-class of groups $G(A)$ correspond to Cartan matrices that are of *Affine Type*. These are non-finite type matrices A which are diagonalizable with nonnegative real eigenvalues. The group of polynomial loops on a complex simply-connected semisimple Lie group can be seen as Kac–Moody group of affine type. Kac–Moody groups that are not of finite type or of affine type, are said to be of *Indefinite Type*. We will say that a generalized Cartan matrix A is of *Compact Type*, if for every proper subset $J \subset I$, the sub-matrix $(a_{i,j})_{i,j \in J}$ is of finite type (see [8] (Section 6.9) for a classification). It is known that indecomposable generalized Cartan matrices of affine type are automatically of compact type. Finally, for the purposes of this paper, we introduce the *Extended Compact Type* defined as a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$, for which there is a (unique) decomposition $I = I_0 \sqcup J_0$, with the property that the sub-Cartan matrix $(a_{i,j})_{i,j \in J}$ is of non-finite type if and only if $I_0 \subseteq J$.

Given a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$, define a category $\mathcal{S}(A)$ to be the poset category (under inclusion) of subsets $J \subseteq I$ such that $K_J(A)$ is a compact Lie group. This is equivalent to demanding that $W_J(A)$ is a finite group. Notice that $\mathcal{S}(A)$ contains all subsets of I of cardinality less than two. In particular, $\mathcal{S}(A)$ is nonempty and has an initial object given by the empty set. However, $\mathcal{S}(A)$ need not have a terminal object. In fact, it has a terminal object exactly if A is of finite type. The category $\mathcal{S}(A)$ is also known as the poset of spherical subsets [9].

Remark 2.1. The topology on the group $K(A)$ is the strong topology generated by the compact subgroups $K_J(A)$ for $J \in \mathcal{S}(A)$ [14,21]. More precisely, $K(A)$ is the amalgamated product of the compact Lie groups $K_J(A)$, in the category of topological groups. For a arbitrary subset $L \subseteq I$, the topology induced on homogeneous space of the form $K(A)/K_L(A)$ makes it into a CW-complex, with only even cells, indexed by the set of cosets $W(A)/W_L(A)$.

We now introduce the topological Tits building, which is a space on which most of our constructions rest. Assume that the set I has cardinality $n + 1$, and let us fix an ordering of the elements of I. Notice that the geometric n simplex $\Delta(n)$ has faces that can be canonically identified with proper subsets of I. Hence, the faces of codimension k correspond to subset of cardinality k. Let $\Delta_J(n)$ be the face of $\Delta(n)$ corresponding to the subset J. If we let $\mathcal{B}\Delta(n)$ denote the Barycentric subdivision of $\Delta(n)$, then it follows that the faces of dimension k in $\mathcal{B}\Delta(n)$ are indexed on chains of length k consisting of proper inclusions $\varnothing \subseteq J_1 \subset J_2 \subset \cdots \subset J_k \subset I$. Let $|\mathcal{S}(A)|$ denote the subcomplex of $\mathcal{B}\Delta(n)$ consisting of those faces for which the corresponding chain is contained entirely in $\mathcal{S}(A)$. Henceforth, we identify $|\mathcal{S}(A)|$ as a subspace of $\Delta(n)$. The terminology is suggestive of the fact that $|\mathcal{S}(A)|$ is canonically homeomorphic to the geometric realization of the nerve of the category $\mathcal{S}(A)$.

Definition 2.2. Define the (finite-type) Topological Tits building $X(A)$ as the $K(A)$-space:

$$X(A) = \frac{K(A)/T \times |\mathcal{S}(A)|}{\sim},$$

where we identify (gT, x) with (hT, y) iff $x = y \in \Delta_J(n)$, and $g = h \mod K_J(A)$.

In the case of the loop group $\mathbb{T} \ltimes \tilde{L}G$, the space $X(A)$ is homeomorphic to the affine space of principal G-connections for the trivial bundle over S^1 [19].

Remark 2.3. An alternate definition of $X(A)$ is as the homotopy colimit [2]:

$$X(A) = \operatorname{hocolim}_{J \in \mathcal{S}(A)} F(J),$$

where F is the functor from $\mathcal{S}(A)$ to $K(A)$-spaces, such that $F(J) = K(A)/K_J(A)$.

Notice that by construction, $X(A)$ is a $K(A)$-CW complex such that all the isotropy subgroups are compact Lie groups. In fact, we prove the following theorem in Section 3.

Theorem 2.4. *The space $X(A)$ is equivalent to the classifying space $EK(A)$ for proper $K(A)$-actions.*

The Weyl chamber C, its faces C_J, and the space Y, are defined as subspaces of $\mathfrak{h}_{\mathbb{R}}^*$:

$$C = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \lambda(h_i) \geqslant 0, \ i \in I\}, \qquad C_J = \operatorname{Interior}\{\lambda \in C \mid \lambda(h_j) = 0, \ \forall j \in J\},$$
$$Y = W(A)C.$$

The space Y has the structure of a cone. Indeed it is called the Tits cone. The Weyl chamber is the fundamental domain for the $W(A)$-action on the Tits cone. Moreover, the stabilizer of any point in C_J is $W_J(A)$.

The subset of $\mathfrak{h}_{\mathbb{Z}}^{*}$ contained in C is called the set of dominant weights D, and those in the interior of C are called the regular dominant weights D_{+}:

$$\mathrm{D} = \big\{\lambda \in \mathfrak{h}_{\mathbb{Z}}^{*} \,\big|\, \lambda(h_i) \geqslant 0,\ i \in I\big\}, \quad \mathrm{D}_{+} = \big\{\lambda \in \mathfrak{h}_{\mathbb{Z}}^{*} \,\big|\, \lambda(h_i) > 0,\ i \in I\big\}.$$

As is the case classically, there is a bijective correspondence between the set of irreducible highest weight representations, and the set D. This bijection identifies an irreducible representation L_{μ} of highest weight μ, with the element $\mu \in \mathrm{D}$. It is more convenient for our purposes to introduce a *Weyl element* $\rho \in \mathfrak{h}_{\mathbb{Z}}^{*}$ with the property that $\rho(h_i) = 1$ for all $i \in I$. The Weyl element may not be unique, but we fix a choice. We may then index the set of irreducible highest weight representations on D_{+} by identifying an irreducible representation L_{μ} of highest weight μ, with the element $\mu + \rho \in \mathrm{D}_{+}$. The relevance of the element ρ will be made clear in Section 5.

We work for the moment with the assumption that A is a symmetrizable generalized Cartan matrix. In this case, any irreducible $K(A)$ representation L_{μ} is unitary. Let us now define a dominant representation of $K(A)$ in a Hilbert space:

Definition 2.5. We say that a $K(A)$-representation in a separable Hilbert space is dominant if it contains a dense subspace given by a countable sum of irreducible highest weight representations. In particular, we obtain a maximal dominant $K(A)$-representation $\mathcal{H}$ (in the sense that any other dominant representation is a summand) by taking a completed sum of countably many copies of all irreducible highest weight representations.

The Dominant K-theory spectrum. We are now ready to define Dominant K-theory as a cohomology theory on the category of proper $K(A)$-CW complexes. As is standard, we will do so by defining a representing object for Dominant K-theory. Recall that usual 2-periodic K-theory is represented by homotopy classes of maps into the infinite grassmannian $\mathbb{Z} \times \mathrm{BU}$ in even parity, and into the infinite unitary group U in odd parity. The theorem of Bott periodicity relates these spaces via $\Omega\mathrm{U} = \mathbb{Z} \times \mathrm{BU}$, ensuring that this defines a cohomology theory.

This structure described above can be formalized using the notion of a spectrum. In particular, a spectrum consists of a family of pointed spaces $\mathbb{E}_n$ indexed over the integers, endowed with homeomorphisms $\mathbb{E}_{n-1} \to \Omega\mathbb{E}_n$. For a topological group G, the objects that represent a G-equivariant cohomology theory are known as G-equivariant spectra. In analogy to a spectrum, an equivariant spectrum $\mathbb{E}$ consists of a collection of pointed G-spaces $\mathbb{E}(V)$, indexed on finite-dimensional sub-representations V of an infinite-dimensional unitary representation of G in a separable Hilbert space (known as a "universe"). In addition, these spaces are related on taking suitable loop spaces. For a comprehensive reference on equivariant spectra, see [22].

Henceforth, all our $K(A)$-equivariant spectra are to be understood as naive equivariant spectra, i.e. indexed on a trivial $K(A)$-universe, or equivalently, indexed over the integers. Indeed, there are no interesting finite-dimensional representations of $K(A)$, and so it is even unclear what a nontrivial universe would mean.

Let $\mathbb{K}\mathrm{U}$ denote the $K(A)$-equivariant periodic K-theory spectrum represented by a suitable model $\mathcal{F}(\mathcal{H})$, for the space of Fredholm operators on $\mathcal{H}$ [1,27]. The space $\mathcal{F}(\mathcal{H})$ is chosen so that the projective unitary group $\mathbb{P}\mathrm{U}(\mathcal{H})$ (with the compact open topology) acts continuously on $\mathcal{F}(\mathcal{H})$ [1] (Proposition 3.1). By maximality, notice that we have an equivalence: $\mathcal{H} \otimes \mathcal{H} = \mathcal{H}$. Hence $\mathbb{K}\mathrm{U}$ is naturally a two periodic, equivariant ring spectrum.

Definition 2.6. Given a proper $K(A)$-CW complex Y, let $\mathcal{H}$ be as in 2.5. Then the Dominant K-theory of Y is defined as the group of equivariant homotopy classes of maps:

$$\mathbb{K}^{2k}_{K(A)}(Y) = \big[Y, \mathcal{F}(\mathcal{H})\big]_{K(A)}, \quad \text{and} \quad \mathbb{K}^{2k+1}_{K(A)}(Y) = \big[Y, \Omega \mathcal{F}(\mathcal{H})\big]_{K(A)}.$$

There is nothing special about spaces. In fact, we may define the Dominant K-cohomology groups of a proper $K(A)$-CW spectrum Y as the group of equivariant homotopy classes of stable maps:

$$\mathbb{K}^{k}_{K(A)}(Y) = \big[Y, \Sigma^k \mathbb{KU}\big]_{K(A)}.$$

Remark 2.7. Given a closed subgroup $G \subseteq K(A)$, we introduce the notation $^A\mathbb{K}^*_G(X)$, when necessary, to mean the restriction of Dominant K-theory from $K(A)$-spectra to G-spectra. Hence

$$^A\mathbb{K}^k_G(X) = \mathbb{K}^k_{K(A)}\big(K(A)_+ \wedge_G X\big) = \big[X, \Sigma^k \mathbb{KU}\big]_G.$$

Conventions. With the exception of Section 3, we will always work under the assumption that the Kac–Moody group being considered corresponds to a symmetrizable generalized Cartan matrix. In Appendix B, we will describe how to define Dominant K-theory even in the case when the generalized Cartan matrix is not symmetrizable. To avoid redundancy, we will also assume that it is not of finite type.

Let $\operatorname{Aut}(A)$ denote the automorphisms of $K(A)$ induced from automorphisms of $\mathfrak{g}(A)$. The outer automorphism group of $K(A)$ is essentially a finite-dimensional torus, extended by a finite group of automorphisms of the Dynkin diagram of A [17]. Given a torus $T' \subseteq \operatorname{Aut}(A)$, we call groups of the form $T' \ltimes K(A)$, *Reductive Kac–Moody Groups*. The unitary Levi factors $K_J(A)$ are examples of such groups.

All arguments in this paper are based on two general notions in the theory of Kac–Moody groups: theory of Highest Weight representations [21], and the theory of BN-pairs and Buildings [9,14,28]. Both these notions extend naturally to reductive Kac–Moody groups, and consequently the results of our paper also extend in an obvious way to reductive Kac–Moody groups. This will be assumed implicitly throughout.

In Section 5 we plan to prove the following theorem:

Theorem 2.8. *Assume that the generalized Cartan matrix A has size $n + 1$, and that $K(A)$ is of compact type, but not of finite type. Let $T \subset K(A)$ be the maximal torus, and let $R^{\varnothing}_T$ denote the regular dominant character ring of $K(A)$ (i.e. characters of T generated by the weights in D_+).*

For the space $X(A)$, define the reduced Dominant K-theory $\tilde{\mathbb{K}}^*_{K(A)}(X(A))$, to be the kernel of the restriction map along any orbit $K(A)/T$ in $X(A)$. We have an isomorphism of graded groups:

$$\tilde{\mathbb{K}}^*_{K(A)}\big(X(A)\big) = R^{\varnothing}_T\big[\beta^{\pm 1}\big],$$

where β is the Bott class in degree 2, and $R^{\varnothing}_T$ is graded so as to belong entirely in degree n.

Remark 2.9. Note that we may identify $R_T^{\varnothing}$ with the Grothendieck group of irreducible highest weight representations of $K(A)$, as described earlier. Under this identification, we will geometrically interpret the isomorphism in the above theorem using an equivariant family of cubic Dirac operators.

In Section 7, we extend the above theorem to show that:

Theorem 2.10. *Let $K(A)$ be a Kac–Moody group of extended compact type, with $|I_0| = n+1$ and $n > 1$. Let $A_0 = (a_{i,j})_{i,j \in I_0}$ denote the sub-Cartan matrix of compact type. Then $X(A_0)$ is also a classifying space of proper $K_{I_0}(A)$-actions, and the following restriction map is an injection:*

$$r : \tilde{\mathbb{K}}_{K(A)}^*\big(X(A)\big) \longrightarrow {}^A\tilde{\mathbb{K}}_{K_{I_0}(A)}^*\big(X(A_0)\big).$$

A dominant $K(A)$-representation restricts to a dominant $K_{I_0}(A)$-representation, inducing a map

$$\mathrm{St} : {}^A\tilde{\mathbb{K}}_{K_{I_0}(A)}^*\big(X(A_0)\big) \longrightarrow \tilde{\mathbb{K}}_{K_{I_0}(A)}^*\big(X(A_0)\big).$$

Since $K_{I_0}(A)$ is a (reductive) Kac–Moody group of compact type, the previous theorem identifies $\tilde{\mathbb{K}}_{K_{I_0}(A)}^*(X(A_0))$ with the ideal generated by regular dominant characters of $K_{I_0}(A)$. Furthermore, the map St is injective, and under the above identification, has image generated by those regular dominant characters of $K_{I_0}(A)$ which are in the $W(A)$-orbit of dominant characters of $K(A)$. The image of $\tilde{\mathbb{K}}_{K(A)}^*(X(A))$ inside $\tilde{\mathbb{K}}_{K_{I_0}(A)}^*(X(A_0))$ may then be identified with characters in the image of St which are also antidominant characters of $K_{J_0}(A)$.

3. Structure of the space $X(A)$

In this section we will not require that A is symmetrizable. Before we begin a detailed study of the space $X(A)$, let us review the structure of the Coxeter group $W(A)$. Details can be found in [11]. The group $W(A)$ is defined as follows:

$$W(A) = \big\langle r_i, \; i \in I \;\big|\; (r_i r_j)^{m_{i,j}} = 1 \big\rangle,$$

where the integers $m_{i,j}$ depend on the product of the entries $a_{i,j} a_{j,i}$ in the generalized Cartan matrix $A = (a_{i,j})$. In particular, $m_{i,i} = 1$. The word length with respect to the generators r_i defines a length function $l(w)$ on $W(A)$. Moreover, we may define a partial order on $W(A)$ known as the Bruhat order. Under the Bruhat order, we say $v \leqslant w$ if v may be obtained from some (in fact any!) reduced expression for w by deleting some generators.

Given any two subsets $J, K \subseteq I$, let $W_J(A)$ and $W_K(A)$ denote the subgroups generated by the elements r_j for $j \in J$ (resp. K). Let $w \in W(A)$ be an arbitrary element. Consider the double coset $W_J(A) w W_K(A) \subseteq W(A)$. This double coset contains a unique element w_0 of minimal length. Moreover, any other element $u \in W_J(A) w W_K(A)$ can be written in reduced form: $u = \alpha w_0 \beta$, where $\alpha \in W_J(A)$ and $\beta \in W_K(A)$. It should be pointed out that the expression $\alpha w_0 \beta$ above, is not unique. We have $\alpha w_0 = w_0 \beta$ if and only if $\alpha \in W_L(A)$, where $W_L(A) = W_J(A) \cap w_0 W_K(A) w_0^{-1}$ (any such intersection can be shown to be generated by a subset $L \subseteq J$).

We denote the set of minimal $W_J(A)$–$W_K(A)$ double coset representatives by ${}^J W^K$. If $K = \varnothing$, then we denote the set of minimal left $W_J(A)$-coset representatives by ${}^J W$. The following claim follows from the above discussion:

Claim 3.1. *Let J, K be any subsets of I. Let w be any element of $^J W$. Then w belongs to the set $^J W^K$ if and only if $l(wr_k) > l(w)$ for all $k \in K$.*

Let us now recall the generalized Bruhat decomposition for the split Kac–Moody group $G(A)$. Given any two subsets $J, K \subseteq I$. Let $G_J(A)$ and $G_K(A)$ denote the corresponding parabolic subgroups. Then the group $G(A)$ admits a Bruhat decomposition into double cosets:

$$G(A) = \coprod_{w \in ^J W^K} G_J(A) \tilde{w} G_K(A),$$

where $\tilde{w}$ denotes any element of $N(T)$ lifting $w \in W(A)$. The closure of a double coset $G_J(A) \tilde{w} G_K(A)$ is given by

$$\overline{G_J(A) \tilde{w} G_K(A)} = \coprod_{v \in ^J W^K,\, v \leqslant w} G_J(A) \tilde{v} G_K(A).$$

One may decompose the subspace $G_J(A) \tilde{w} G_K(A)$ further as

$$G_J(A) \tilde{w} G_K(A) = \coprod_{v \in W_J(A) w W_K(A)} B \tilde{v} B,$$

where $B = G_\varnothing(A)$ denotes the positive Borel subgroup. The subspace $B \tilde{v} B$ has the structure of the right B space $\mathbb{C}^{l(v)} \times B$ and one has an isomorphism:

$$B \tilde{v} B = \prod_{i=1}^{s} B \tilde{r}_s B,$$

where $v = r_1 \ldots r_s$ is a reduced decomposition. The above structure in fact gives the homogeneous space $G(A)/G_K(A)$ the structure of a CW complex, and as such it is homeomorphic to the corresponding homogeneous space for the unitary forms: $K(A)/K_K(A)$.

Now recall the proper $K(A)$-CW complex $X(A)$ defined in the previous section:

$$X(A) = \frac{K(A)/T \times |\mathcal{S}(A)|}{\sim},$$

where we identify (gT, x) with (hT, y) iff $x = y \in \Delta_J(n)$, and $g = h$ mod $K_J(A)$. The space $|\mathcal{S}(A)|$ was the subcomplex of the barycentric subdivision of the n-simplex consisting of those faces indexed on chains $\varnothing \subseteq J_0 \subset J_1 \subset \cdots \subset J_m \subseteq I$ for which $J_k \in \mathcal{S}(A)$ for all k.

Theorem 3.2. *The space $X(A)$ is equivalent to the classifying space $EK(A)$ for proper $K(A)$-actions.*

Proof. To prove the theorem it is sufficient to show that $X(A)$ is equivariantly contractible with respect to any compact subgroup of $K(A)$. By the general theory of Buildings [9,15], any compact subgroup of $K(A)$ is conjugate to a subgroup of $K_J(A)$ for some $J \in \mathcal{S}(A)$. Hence, it is sufficient to show that $X(A)$ is $K_J(A)$-equivariantly contractible for any $J \in \mathcal{S}(A)$. We proceed as follows:

Define a $K_J(A)$ equivariant filtration of $G(A)/B$ by finite complexes:

$$\big(G(A)/B\big)_k = \coprod_{w \in {}^J W, \, l(w) \leqslant k} G_J(A) \tilde{w} B / B, \quad k \geqslant 0.$$

Identifying $K(A)/T$ with $G(A)/B$, we get a filtration of $X(A)$ by $K_J(A)$ invariant finite-dimensional sub-complexes:

$$X_k(A) = \frac{(G(A)/B)_k \times |\mathcal{S}(A)|}{\sim}, \quad k > 0, \qquad X_0(A) = \frac{G_J(A)/B \times |\mathcal{S}_J(A)|}{\sim},$$

where $|\mathcal{S}_J(A)|$ denotes the subspace $|\mathcal{S}(A)| \cap \Delta_J(n)$, which is the realization of the nerve of the subcategory under the object $J \in \mathcal{S}(A)$ (and is hence contractible). Notice that $X_0(A)$ is in fact a trivial $G_J(A)$-space homeomorphic to $|\mathcal{S}_J(A)|$. We now proceed to show that $X_k(A)$ equivariantly deformation retracts onto $X_{k-1}(A)$ thereby showing equivariant contractibility.

An element $(gB, x) \in X_k(A)$ belongs to $X_{k-1}(A)$ if either $gT \in (G(A)/B)_{k-1}$ or if $x \in \Delta_K(n)$ and $gG_K(A) = hG_K(A)$ for some h such that $hB \in (G(A)/B)_{k-1}$. This observation may be expressed as

$$X_k(A)/X_{k-1}(A) = \bigvee_{w \in {}^J W, \, l(w) = k} \big(G_J(A)\tilde{w}B/B\big)_+ \wedge \big(|\mathcal{S}(A)|/|\mathcal{S}_w(A)|\big),$$

where $(G_J(A)\tilde{w}B/B)_+$ indicates the one point compactification of the space, and $|\mathcal{S}_w(A)|$ is a subcomplex of $|\mathcal{S}(A)|$ consisting of faces indexed on chains $\varnothing \subseteq J_0 \subset J_1 \subset \cdots \subset J_m \subseteq I$ for which w does not belong to ${}^J W^{J_0}$. Hence, to show that $X_k(A)$ equivariantly retracts onto $X_{k-1}(A)$ it is sufficient to show that there is a deformation retraction from $|\mathcal{S}(A)|$ to $|\mathcal{S}_w(A)|$ for all $w \in {}^J W$. It is enough to show that the subspace $|\mathcal{S}_w(A)|$ is contractible.

Let $I_w \subseteq I$ be the subset defined as $I_w = \{i \in I \mid l(wr_i) < l(w)\}$. By studying the coset $w W_{I_w}(A)$, it is easy to see that $I_w \in \mathcal{S}(A)$ (see [8, (4.7.2)]). Using Claim 3.1 we see that $|\mathcal{S}_w(A)|$ is the geometric realization of the nerve of the subcategory

$$\mathcal{S}_w(A) = \big\{J \in \mathcal{S}(A) \mid J \cap I_w \neq \varnothing\big\}.$$

Note that $\mathcal{S}_w(A)$ is clearly equivalent to the subcategory consisting of subsets that are contained in I_w, which has a terminal object (namely I_w). Hence the nerve of $\mathcal{S}_w(A)$ is contractible. $\square$

Remark 3.3. One may consider the T-fixed point subspace of $X(A)$, where T is the maximal torus of $K(A)$. It follows that the $X(A)^T$ is a proper $W(A)$-space. Moreover, it is the classifying space for proper $W(A)$-actions. This space has been studied in great detail by M.W. Davis and coauthors [6–8], and is sometimes called the Davis complex Σ, for the Coxeter group $W(A)$. It is not hard to see that

$$\Sigma = X(A)^T = \frac{W(A) \times |\mathcal{S}(A)|}{\sim},$$

where we identify (w, x) with (v, y) iff $x = y \in \Delta_J(n)$, and $w = v \bmod W_J(A)$. As before, this may be expressed as a homotopy colimit of a suitable functor defined on the category $\mathcal{S}(A)$ taking values in the category of $W(A)$-spaces.

4. Dominant K-theory

In this section we study Dominant K-theory in detail. We resume the assumption that the generalized Cartan matrix A is symmetrizable. Under this assumption, any irreducible highest weight representation of $K(A)$ is unitary. This assumption is not strictly necessary, and it is possible to extend our methods to arbitrary generalized Cartan matrices (see Appendix B).

Now recall some definitions introduced in Section 2. We say that a unitary representation of $K(A)$ is dominant if it decomposes as a completed sum of irreducible highest weight representations. Hence one has a maximal dominant representation $\mathcal{H}$ obtained by taking a completed sum of countably many copies of all highest weight representations. Since $K(A)$ is the amalgamated product (in the category of topological groups) of compact Lie groups, we obtain a continuous map from $K(A)$ to the group $\mathbb{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, with the compact open topology.

Now given any highest weight representation L_μ, one may consider the set of weights in $\mathfrak{h}_{\mathbb{Z}}^*$ for L_μ. It is known [21] that this set is contained in the convex hull of the orbit: $W(A)\mu$. Hence we see that the set of weights of any proper highest weight representation L_μ belongs to the set of weights contained in the Tits cone Y. Conversely, any weight in Y is the $W(A)$ translate of some weight in the Weyl chamber C, and so we have:

Claim 4.1. *The set of distinct weights that belong to the maximal dominant representation $\mathcal{H}$ is exactly the set of weights in the Tits cone Y. Moreover, this set is closed under addition.*

Definition 4.2. For $G \subseteq K(A)$, a compact Lie subgroup, let the dominant representation ring of G denoted by DR_G, be the free abelian group on the set of isomorphism classes of irreducible G representations belonging to $\mathcal{H}$.[2] This group admits the structure of a subring of the representation ring R_G of G.

Recall from Section 2 that $\mathbb{KU}$ denotes the periodic K-theory spectrum represented by a model $\mathcal{F}(\mathcal{H})$ for the space of Fredholm operators on $\mathcal{H}$.

In the next few claims, we explore the structure of Dominant K-theory. We begin with a comparison between Dominant K-theory and standard equivariant K-theory with respect to compact subgroups.

Let $G \subseteq K(A)$ be a compact Lie group. Let K_G denote usual G-equivariant K-theory modeled on a the space of Fredholm operators on a G-stable Hilbert space $\mathcal{L}(G)$ [27]. Since, by definition $\mathcal{L}(G)$ contains every G-representation infinitely often, we may fix an equivariant isometry $\mathcal{H} \subseteq \mathcal{L}(G)$ (notice that any two isometries are equivariantly isotopic). Let $\mathcal{M}(G)$ be a closed G-invariant subspace of $\mathcal{L}(G)$, so that $\mathcal{H} \oplus \mathcal{M}(G)$ is a G-stable Hilbert space inside $\mathcal{L}(G)$. We may assume that $\mathcal{M}(G)$ and $\mathcal{H}$ share no nonzero G-representations. Then, if G is a subgroup of the form $K_J(A)$ for $J \in \mathcal{S}(A)$, we have:

Lemma 4.3. *If $G \subseteq K(A)$ is a subgroup of the form $K_J(A)$ for $J \in \mathcal{S}(A)$, then the representations $\mathcal{H}$ and $\mathcal{M}(G)$ share no common characters of the maximal torus T.*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G. One has the triangular decomposition of the complexification: $\mathfrak{g} \otimes \mathbb{C} = \eta_+ \oplus \eta_- \oplus \mathfrak{h}$, where $\eta_\pm$ denote the nilpotent subalgebras. Recall that the Borel

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[2] The Kac–Moody group is implicit in the definition, we hope that the context will prevent any confusion.

subalgebra $\mathfrak{h}$ contains the lattice $\mathfrak{h}_{\mathbb{Z}}$ containing the coroots h_i for $i \in I$. Fix a dual set $h_i^* \in \mathfrak{h}_{\mathbb{Z}}^*$. We may decompose $\eta_{\pm}$ further into root spaces indexed on the roots generated by the simple roots in the set J:

$$\eta_{\pm} = \sum_{\alpha \in \Delta_{\pm}} \mathfrak{g}_{\alpha},$$

where $\Delta_{\pm}$ denotes the positive (resp. negative) roots for G. Now fix a Weyl element ρ_J, defined by

$$\rho_J = \sum_{j \in J} h_j^*.$$

Let L_{ρ_J} denote the irreducible representation of G with highest weight ρ_J. Since ρ_J is a dominant weight, L_{ρ_J} belongs to $\mathcal{H}$. It is easy to see using character theory that L_{ρ_J} has character

$$\mathrm{ch}\, L_{\rho_J} = e^{\rho_J} \prod_{\alpha \in \Delta_+} \left(1 + e^{-\alpha}\right) = e^{\rho_J}\, \mathrm{ch}\, \Lambda^*(\eta_-),$$

where $\Lambda^*(\eta_-)$ denotes the exterior algebra on η_-. In particular, the characters occuring in the formal expression A_J belong to $\mathcal{H}$, where

$$A_J = e^{\rho_J} \prod_{\alpha \in \Delta_+} \left(1 - e^{-\alpha}\right).$$

Now let μ be a dominant weight of G, and let L_{μ} denote the irreducible highest weight representation of G with weight μ. To prove the lemma, it is sufficient to show that if any character of L_{μ} belongs to $\mathcal{H}$, then L_{μ} itself belongs to $\mathcal{H}$. Therefore, let us assume that L_{μ} contains a character in common with $\mathcal{H}$. Using the character formula, we have

$$A_J\, \mathrm{ch}\, L_{\mu} = \sum_{w \in W_J(A)} (-1)^w e^{w(\rho_J + \mu)}.$$

It follows from the $W(A)$-invariance of characters in $\mathcal{H}$, that all characters on the right-hand side belong to $\mathcal{H}$. In particular, $\mu + \rho_J$ is a dominant weight for $K(A)$. Which implies that μ is a dominant weight of $K(A)$, and that L_{μ} belongs to $\mathcal{H}$.

A useful consequence of the above lemma is to help us identify Dominant K-theory as a summand of usual equivariant K-theory, under certain conditions.

Definition 4.4. Define the stabilization map St to be the induced map of Fredholm operators:

$$\mathrm{St} : \mathcal{F}(\mathcal{H}) \times \mathcal{F}\big(\mathcal{M}(G)\big) \longrightarrow \mathcal{F}\big(\mathcal{L}(G)\big).$$

For a G-CW spectrum X, define $\mathbb{M}_G^*(X)$ to be the cohomology theory represented by the infinite loop space $\mathcal{F}(\mathcal{M}(G))$. Hence, the stabilization map St induces a natural transformation of cohomology theories

$$\mathrm{St} : {}^A\mathbb{K}_G^*(X) \oplus \mathbb{M}_G^*(X) \longrightarrow \mathrm{K}_G^*(X).$$

Claim 4.5. *Let G be a subgroup of the form $K_J(A) \subseteq K(A)$ for $J \in \mathcal{S}(A)$, and let X be a G-CW spectrum so that each isotropy group is connected, and has maximal rank. Then the natural transformation* St *defined above, is an isomorphism.*

Proof. It is enough to prove it for a single orbit of the form $X = G/H_+ \wedge S^m$, with H being a connected subgroup containing the maximal torus T. Since the characters of $\mathcal{H}$ and $\mathcal{M}(G)$ are distinct, the stabilization map St induces a homotopy equivalence

$$\text{St} : \mathcal{F}^H(\mathcal{H}) \times \mathcal{F}^H(\mathcal{M}(G)) = \mathcal{F}^H(\mathcal{H} \oplus \mathcal{M}(G)) \longrightarrow \mathcal{F}^H(\mathcal{L}(G)).$$

Applying homotopy to this equivalence, proves the claim for the orbit $G/H_+ \wedge S^m$. $\quad\square$

In the case of a general compact subgroup $G \subseteq K(A)$, we may prove the weaker:

Claim 4.6. *Given a proper orbit $Y = K(A)_+ \wedge_G S^0$ for some compact Lie subgroup $G \subseteq K(A)$, the stabilization map induces an isomorphism*

$$\text{St} : \mathbb{K}^*_{K(A)}(Y) \oplus \mathbb{M}^*_G(S^0) = {}^A\mathbb{K}^*_G(S^0) \oplus \mathbb{M}^*_G(S^0) \longrightarrow K^*_G(S^0) = R_G[\beta^\pm].$$

Furthermore, the image of $\mathbb{K}^*_{K(A)}(Y)$ is $\text{DR}_G[\beta^{\pm 1}]$, where DR_G is the Dominant representation ring of G graded in degree 0, and β is the Bott class in degree 2. In particular, the odd Dominant K-cohomology of single proper $K(A)$-orbits is trivial.

Proof. We repeat the argument from the above claim. Taking G-fixed points of the stabilization map, we get a homotopy equivalence:

$$\text{St} : \mathcal{F}^G(\mathcal{H}) \times \mathcal{F}^G(\mathcal{M}(G)) = \mathcal{F}^G(\mathcal{H} \oplus \mathcal{M}(G)) \longrightarrow \mathcal{F}^G(\mathcal{L}(G)).$$

The result follows on applying homotopy. It is also clear that the image of $\mathbb{K}^*_{K(A)}(Y)$ is $\text{DR}_G[\beta^{\pm 1}] \subseteq \text{R}_G[\beta^{\pm 1}]$. $\quad\square$

The following theorem may be seen as a Thom isomorphism theorem for Dominant K-theory. It would be interesting to know the most general conditions on an equivariant vector bundle that ensure the existence of a Thom class.

Theorem 4.7. *Let $G \subseteq K(A)$ be a subgroup of the form $K_J(A)$ for some $J \in \mathcal{S}(A)$, and let r be the rank of $K(A)$. Let $\mathfrak{g}$ denote the Adjoint representation of $K_J(A)$. Then there exists a fundamental irreducible representation λ of the Clifford algebra $\text{Cliff}(\mathfrak{g} \otimes \mathbb{C})$ that serves as a Thom class in ${}^A\mathbb{K}^r_G(S^\mathfrak{g})$ (a generator for ${}^A\mathbb{K}^*_G(S^\mathfrak{g})$ as a free module of rank one over DR_G), where $S^\mathfrak{g}$ denotes the one point compactification of $\mathfrak{g}$.*

Proof. We begin by giving an explicit description of λ. Fix an invariant inner product B on $\mathfrak{g}$, and let $\text{Cliff}(\mathfrak{g} \otimes \mathbb{C})$ denote the corresponding complex Clifford algebra. Recall the triangular decomposition: $\mathfrak{g} \otimes \mathbb{C} = \eta_+ \oplus \eta_- \oplus \mathfrak{h}$, where $\eta_\pm$ denote the nilpotent subalgebras. The inner product extends to a Hermitian inner product on $\mathfrak{g} \otimes \mathbb{C}$, which we also denote B, and for which the triangular decomposition is orthogonal.

Also recall that the highest weight representation L_{ρ_J} of G has character

$$\operatorname{ch} L_{\rho_J} = e^{\rho_J} \prod_{\alpha \in \Delta_+} \left(1 + e^{-\alpha}\right) = e^{\rho_J} \operatorname{ch} \Lambda^*(\eta_-).$$

In particular, the vector space L_{ρ_J} is naturally $\mathbb{Z}/2$-graded and belongs to $\mathcal{H}$. The exterior algebra $\Lambda^*(\eta_-)$ can naturally be identified with the fundamental Clifford module for the Clifford algebra $\operatorname{Cliff}(\eta_+ \oplus \eta_-)$. Let $\mathbb{S}(\mathfrak{h})$ denote an irreducible Clifford module for $\operatorname{Cliff}(\mathfrak{h})$. It is easy to see that the action of $\operatorname{Cliff}(\mathfrak{g} \otimes \mathbb{C})$ on $\mathbb{S}(\mathfrak{h}) \otimes \Lambda^*(\eta_-)$ extends uniquely to an action of $G \ltimes \operatorname{Cliff}(\mathfrak{g} \otimes \mathbb{C})$ on λ, with highest weight ρ_J, where

$$\lambda = \mathbb{C}_{\rho_J} \otimes \mathbb{S}(\mathfrak{h}) \otimes \Lambda^*(\eta_-) = \mathbb{S}(\mathfrak{h}) \otimes L_{\rho_J}.$$

The Clifford multiplication parametrized by the base space $\mathfrak{g}$, naturally describes λ as an element in ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$.

We now show that λ is a free generator of rank one for ${}^A\mathbb{K}^r_G(S^{\mathfrak{g}})$, as a DR_G-module. Since the G action on $S^{\mathfrak{g}}$ has connected isotropy subgroups of maximal rank, the stabilization map restricted to the Dominant K-theory is injective. Consider the element $\operatorname{St}(\lambda) \in K^r_G(S^{\mathfrak{g}})$ given by the image of λ under the stabilization map. It is clear that $\operatorname{St}(\lambda)$ is a Thom class in equivariant K-theory. Let α be any other class in the image of the stabilization map. Hence α is of the form $\alpha = \operatorname{St}(\lambda) \otimes \beta$, where $\beta \in R_G$. The proof will be complete if we can show that β actually belongs to DR_G. We may assume that β is a virtual sum of irreducible G-representations of the form L_μ, for dominant weights μ of G. Assume for the moment that β has only one summand L_μ.

Consider the restriction of $\operatorname{St}(\lambda) \otimes L_\mu$ to the group $K^r_T(S^{\mathfrak{h}_{\mathbb{R}}})$, along the action map

$$\varphi : G_+ \wedge_T S^{\mathfrak{h}_{\mathbb{R}}} \longrightarrow S^{\mathfrak{g}},$$

where $T \subseteq G$ is the maximal torus. Identifying $K^r_T(S^{\mathfrak{h}_{\mathbb{R}}})$, with R_T, and using the character formula, we see that the restriction of $\operatorname{St}(\lambda) \otimes L_\mu$ has virtual character given by

$$\sum_{w \in W_J(A)} (-1)^w e^{w(\rho_J + \mu)}.$$

Repeating an earlier argument, we notice that for this virtual character to appear in $\mathcal{H}$, μ must be a dominant weight for $K(A)$. The general case for β follows by induction. $\square$

Remark 4.8. The above Thom isomorphism allows us to define the pushforward map in Dominant K-theory. For example, let $\iota : T \to K_J(A)$ be the canonical inclusion of the maximal torus. Then we have the (surjective) pushforward map given by Dirac Induction:

$$\iota_* : \operatorname{DR}_T \longrightarrow \operatorname{DR}_J,$$

where DR_J denotes the dominant representation ring of $K_J(A)$. Moreover, we also recover the character formula as the composite:

$$\iota^* \iota_*(e^\mu) = \frac{\sum_{w \in W_J(A)} (-1)^w e^{w(\mu)}}{e^{\rho_J} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})}.$$

5. The dominant K-theory for the compact type

Fix a Kac–Moody group $K(A)$ of compact type, with maximal torus T. In this section we describe the Dominant K-cohomology groups of the space $X(A)$.

Recall that $X(A)$ can be written as a proper $K(A)$-CW complex given by

$$X(A) = \frac{K(A)/T \times \Delta(n)}{\sim},$$

where we identify (gT, x) with (hT, y) iff $x = y \in \Delta_J(n)$, and $g = h \bmod K_J(A)$.

Define the pinch map induced by the projection: $\Delta(n) \to \Delta(n)/\partial \Delta(n) = S^n$:

$$\pi : X(A) \longrightarrow K(A)_+ \wedge_T S^n.$$

Assume that the generalized Cartan matrix A has size $n + 1$, and that $K(A)$ is not of finite type. Let $R_T^{\varnothing}$ denote the ideal within the character ring of T generated by the regular dominant weights, D_+. In particular, we may identify $R_T^{\varnothing}$ with the Grothendieck group of irreducible highest weight representations of $K(A)$.

For the space $X(A)$, define the reduced Dominant K-theory $\tilde{\mathbb{K}}^*_{K(A)}(X(A))$, to be the kernel of the restriction map along any orbit of the form $K(A)/T$ in $X(A)$.

Theorem 5.1. *We have an isomorphism of graded groups:*

$$\tilde{\mathbb{K}}^*_{K(A)}\big(X(A)\big) = R_T^{\varnothing}\big[\beta^{\pm 1}\big],$$

where β is the Bott class in degree 2, and $R_T^{\varnothing}$ is graded so as to belong entirely in degree n. Moreover, the identification of $\tilde{\mathbb{K}}^n_{K(A)}(X(A))$ with $R_T^{\varnothing}$ is induced by the pinch map π.

Proof. Since $K(A)$ is of compact type (but not of finite type), the category $\mathcal{S}(A)$ is the category of all proper subsets of I. We may see $X(A)$ as a homotopy colimit of a diagram over $\mathcal{S}(A)$ taking the value $K(A)/K_J(A)$ for $J \in \mathcal{S}(A)$.

Filtering $X(A)$ by the equivariant skeleta, we get a convergent spectral sequence (E_n, d_n), $|d_n| = (n, n-1)$, and with E_2 term given by

$$E_2^{p,*} = \varprojlim{}^p \mathbb{K}^*_{K(A)}\big(K(A)/K_\bullet(A)\big) = \varprojlim{}^p \mathrm{DR}_\bullet\big[\beta^{\pm 1}\big] \Rightarrow \mathbb{K}^p_{K(A)}\big(X(A)\big)\big[\beta^{\pm 1}\big],$$

where $\mathrm{DR}_\bullet$ denotes the functor $J \mapsto \mathrm{DR}_{K_J(A)}$, and β denotes the invertible Bott class of degree two. Now using Remark 4.8, it is easy to see that DR_J is isomorphic to the $W_J(A)$-invariant characters: $\mathrm{DR}_T^{W_J(A)}$. Notice also that

$$\mathrm{DR}_T^{W_J(A)} = \mathrm{Hom}_{W(A)}\big(W(A)/W_J(A), \mathrm{DR}_T\big).$$

It follows that $\varprojlim{}^p \mathrm{DR}_\bullet$ is canonically isomorphic to the equivariant cohomology (as defined in [7]) of the Davis complex Σ (3.3), with values in the ring DR_T:

$$\varprojlim{}^p \mathrm{DR}_\bullet = \mathrm{H}^p_{W(A)}(\Sigma, \mathrm{DR}_T).$$

Now recall that the set of dominant weights D has a decomposition indexed by subsets $F \subseteq I$:

$$D = \coprod D_F, \quad \text{where } D_F = \big\{\lambda \in D \mid \lambda(h_k) = 0 \Leftrightarrow k \in F\big\}.$$

Let R_T^F denote the ideal in DR_T generated by the weights belonging to the subset D_F. We get a corresponding decomposition of DR_T as a $W(A)$-module:

$$DR_T = \bigoplus DR_T^F, \quad \text{where } DR_T^F \cong \mathbb{Z}\big[W(A)/W_F(A)\big] \otimes R_T^F.$$

We therefore have an induced decomposition of the functor $DR_\bullet = \bigoplus DR_\bullet^F$ indexed by $F \subseteq I$. On taking derived functors we have:

$$\varprojlim{}^p DR_\bullet = \bigoplus \varprojlim{}^p DR_\bullet^F = \bigoplus H_{W(A)}^p\big(\Sigma, \mathbb{Z}[W(A)/W_F(A)]\big) \otimes R_T^F.$$

We now proceed by considering three cases: $F = I, \varnothing$, and the remaining cases.

Case $F = I$. Observe that $W_I(A) = W(A)$, and that R_T^I is the subring of DR_T generated by the $W(A)$-invariant weights D_I. Moreover, we have:

$$H_{W(A)}^p(\Sigma, \mathbb{Z}) = H^p\big(\Delta(n), \mathbb{Z}\big) = \begin{cases} 0, & \text{if } p > 0, \\ \mathbb{Z} & \text{if } p = 0. \end{cases}$$

Hence $\varprojlim{}^0 DR_\bullet^I = R_T^I$, and all higher inverse limits vanish. Now every weight in D_I represents an element of $\mathbb{K}_{K(A)}^0(X(A))$ given by a one-dimensional highest weight representation of $K(A)$, hence they represent permanent cycles. Moreover, these weights are detected under any orbit $K(A)/T$ in $X(A)$.

Case $F = \varnothing$. Notice that $W_\varnothing = \{1\}$, and that $R_T^\varnothing$ is the ideal in DR_T generated by the weights in D_+. We have by [6,7]:

$$H_{W(A)}^p\big(\Sigma, \mathbb{Z}[W(A)]\big) = H_c^p(\Sigma, \mathbb{Z}) = \begin{cases} 0, & \text{if } p \neq n, \\ \mathbb{Z} & \text{if } p = n, \end{cases}$$

where $H_c^p(\Sigma, \mathbb{Z})$ denotes the cohomology of Σ with compact supports. Hence we deduce that $\varprojlim{}^n DR_\bullet^\varnothing = R_T^\varnothing$, and all other derived inverse limits vanish.

Case $F \neq \{I, \varnothing\}$. If $F \subseteq I$ is a nontrivial proper subset of I, then we know that $W_F(A)$ is a finite subgroup of $W(A)$. The argument in [7] also shows that

$$H_{W(A)}^p\big(\Sigma, \mathbb{Z}[W(A)/W_F(A)]\big) = H_c^p\big(\Sigma/W_F(A), \mathbb{Z}\big).$$

By [6, Section 3], $\Sigma/W_F(A)$ is a proper retract of Σ. Hence the group: $H_c^p(\Sigma/W_F(A), \mathbb{Z})$ is a summand within $H_c^p(\Sigma, \mathbb{Z})$. From the previous case, recall that the latter group is trivial if $p \neq n$, and is free cyclic for $p = n$. Moreover, given any $i \in I$, the reflection $r_i \in W(A)$ acts by reversing the sign of the generator of $H_c^n(\Sigma, \mathbb{Z})$. By picking $i \in F$, we see that the groups $H_c^p(\Sigma/W_F(A), \mathbb{Z})$ must be trivial for all p.

From the above three cases we notice that the spectral sequence must collapse at E_2, showing that the groups $\tilde{\mathbb{K}}_{K(A)}^{*+n}(X(A))$ are isomorphic to $\mathrm{R}_T^\varnothing[\beta^{\pm 1}]$. Since the case $J = \varnothing$ corresponds to the lowest filtration in the E_∞ term, it can be identified with the image of the map given by pinching off the top cell: $\pi : X(A) \to K(A)_+ \wedge_T S^n$. $\quad\square$

6. The geometric identification

For a compact type Kac–Moody group, we would like to give a geometric meaning to the image of π in terms of Fredholm families. One may see this as a global Thom isomorphism theorem for Dominant K-theory (in the spirit of the local Thom isomorphism theorem developed in Section 4). That such a result could hold globally, was first shown in the affine case in [10] using the cubic Dirac operator. We will adapt their argument to our context. It is important to point out that earlier work of Mickelsson [25] on the super-symmetric WZW-model points to the relevance of this operator in physics.

Let A be a symmetrizable generalized Cartan matrix, and let $\mathfrak{g}(A)$ be the corresponding complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. Recall the triangular decomposition:

$$\mathfrak{g}(A) = \eta_+ \oplus \eta_- \oplus \mathfrak{h}.$$

Let $\mathfrak{g}(A)^*$ denote the restricted dual of $\mathfrak{g}(A)$. So $\mathfrak{g}(A)^*$ is a direct sum of the duals of the individual (finite-dimensional) root spaces.

Let $K_J(A)$ be a compact subgroup of $K(A)$ for some $J \in \mathcal{S}(A)$, and let $\mathrm{B}(J)$ be a $K_J(A)$-invariant Hermitian inner product on $\mathfrak{g}(A)$. Let $\mathrm{S} = \mathrm{B}(J)^\omega$ denote the non-degenerate, symmetric, bilinear form on $\mathfrak{g}(A)$ defined in terms of the involution ω, as $\mathrm{S}(x, y) = -\mathrm{B}(J)(x, \omega(y))$. We identify $\mathfrak{g}(A)^*$ with $\mathfrak{g}(A)$ via S, and endow it with the induced form. The subspaces $\eta_\pm^*$ are S-isotropic, and can be seen as a polarization of $\mathfrak{g}(A)^*$ [26]. Let $\mathrm{Cliff}(\mathfrak{g}(A)^*)$ denote the complex Clifford algebra generated by $\mathfrak{g}(A)^*$, modulo the relation $x^2 = \frac{1}{2}\mathrm{S}(x, x)$ for $x \in \mathfrak{g}(A)^*$. Let $\mathbb{S}(A)$ denote the fundamental irreducible representation of $\mathrm{Cliff}(\mathfrak{g}(A)^*)$ endowed with the canonical Hermitian inner product, which we also denote by $\mathrm{B}(J)$. We have a decomposition of algebras:

$$\mathrm{Cliff}\big(\mathfrak{g}(A)^*\big) = \mathrm{Cliff}(\mathfrak{h}^*) \otimes \mathrm{Cliff}(\eta_+^* \oplus \eta_-^*).$$

Let $\mathbb{S}(\mathfrak{h}^*)$ denote the fundamental irreducible $\mathrm{Cliff}(\mathfrak{h}^*)$-module. The above decomposition of $\mathrm{Cliff}(\mathfrak{g}(A)^*)$ induces a decomposition $\mathbb{S}(A) = \mathbb{S}(\mathfrak{h}^*) \otimes \Lambda^*(\eta_-)$. We have identified $\Lambda^*(\eta_-)$ with the fundamental $\mathrm{Cliff}(\eta_+^* \oplus \eta_-^*)$-module, with η_-^* acting by contraction, and η_+^* acting by exterior multiplication (once we identify η_+^* with η_-).

As in the local case, let L_ρ denote the irreducible representation of $K(A)$ with highest weight ρ. From the character formula, we see that the character of L_ρ is given by

$$\mathrm{ch}\,\mathrm{L}_\rho = e^\rho\,\mathrm{ch}\,\Lambda^*(\eta_-).$$

It is easy to see that the $\mathrm{Cliff}(\mathfrak{g}(A)^*)$ action on $\mathbb{S}(A)$ extends uniquely to an irreducible $K_J(A) \ltimes \mathrm{Cliff}(\mathfrak{g}(A)^*)$-module $\hat{\mathbb{S}}(A)$, with highest weight ρ, given by

$$\hat{\mathbb{S}}(A) = \mathbb{S}(A) \otimes \mathbb{C}_\rho = \mathbb{S}(\mathfrak{h}^*) \otimes \mathrm{L}_\rho.$$

Furthermore, $\hat{\mathbb{S}}(A)$ is a unitary $K_J(A)$-representation. We now proceed to construct an elliptic operator as described in [10, Section 14], by twisting the Koszul–Chevalley differential with the cubic Dirac operator:

Given a dominant weight $\mu \in D$, let L_μ be the corresponding irreducible unitary $K(A)$-representation with highest weight μ. Let $\slashed{D}$ denote the self-adjoint operator acting on $\mathbb{H}_\mu$, where

$$\mathbb{H}_\mu = \hat{\mathbb{S}}(A) \otimes L_\mu = \mathbb{S}(\mathfrak{h}^*) \otimes \mathbb{C}_\rho \otimes \Lambda^*(\eta_-) \otimes L_\mu, \qquad \slashed{D} = K + \partial + \partial^\dagger,$$

here K denotes the Dirac operator on $\mathfrak{h}_\mathbb{R}$ with coefficients in the T-representation $\mathbb{C}_\rho \otimes \Lambda^*(\eta_-) \otimes L_\mu$. In particular, K is given by Clifford multiplication with θ on the isotypical summand of weight θ in $\mathbb{C}_\rho \otimes \Lambda^*(\eta_-) \otimes L_\mu$. The operator ∂ denotes the Koszul–Chevalley differential $\partial : \Lambda^k(\eta_-) \otimes L_\mu \to \Lambda^{k-1}(\eta_-) \otimes L_\mu$, and $\partial^\dagger$ its adjoint with respect to the Hermitian inner product on $\Lambda^*(\eta_-) \otimes L_\mu$, induced from L_μ, and the Hermitian inner-product on η_- induced by $B(J)$.

By endowing $\Lambda^*(\eta_-)$ with the canonical $\mathbb{Z}/2$-grading, and trivially grading $\mathbb{C}_\rho$ and L_μ, we see that $\mathbb{H}_\mu$ is $\mathbb{Z}/2$ graded over $\mathbb{S}(\mathfrak{h}^*)$. If the dimension of $\mathfrak{h}$ is even, then $\mathbb{S}(\mathfrak{h}^*)$ is itself $\mathbb{Z}/2$-graded, and $\slashed{D}$ is a graded self-adjoint operator.

We may give an alternate description of $\slashed{D}$ which will be helpful later. Let a_i denote a basis of root-vectors for η_-, and let b_i denote the dual basis for η_+ such that $S(a_m, b_n) = \delta_{m,n}$. Let h_j denote the coroot-basis for $\mathfrak{h}_\mathbb{R}$. It is easy to see that we may write the map ∂ acting on $\hat{\mathbb{S}}(A) \otimes L_\mu$ as the formal expression (using the Einstein summation notation):

$$\partial = \psi(a_i^*) \otimes a_i + \frac{1}{2} \mathrm{ad}_{a_i} \psi(a_i^*) \otimes \mathrm{Id}$$

where $\psi(a_i^*)$ indicates Clifford action, and ad_{a_i} denotes the adjoint action of η_- extended to be a derivation on $\Lambda^*(\eta_-)$. It follows that we may write the elements $\partial^\dagger$ and K as

$$\partial^\dagger = \psi(b_i^*) \otimes b_i + \frac{1}{2} \psi(b_i^*) \mathrm{ad}_{a_i}^\dagger \otimes \mathrm{Id}, \qquad K = \psi(h_j^*) \otimes h_j + \psi(h_j^*) \mathrm{ad}_{h_j} \otimes \mathrm{Id} + \psi(\rho) \otimes \mathrm{Id}.$$

With this in mind, consider the following formal expressions:

$$\slashed{D}_1 = \psi(a_i^*) \otimes a_i + \psi(b_i^*) \otimes b_i + \psi(h_j^*) \otimes h_j \quad \text{and}$$

$$\slashed{D}_2 = \frac{1}{2} \mathrm{ad}_{a_i} \psi(a_i^*) + \frac{1}{2} \psi(b_i^*) \mathrm{ad}_{a_i}^\dagger + \psi(h_j^*) \mathrm{ad}_{h_j} + \psi(\rho).$$

Claim 6.1. *Let $\slashed{D}_1$ and $\slashed{D}_2$ be the formal expressions as given above. Then, seen as operators on $\mathbb{H}_\mu$, they are well defined and $K_J(A)$-invariant. In particular, $\slashed{D} = \slashed{D}_1 + \slashed{D}_2$ is $K_J(A)$-invariant.*

Proof. Working with an explicit basis of elementary tensors in $\Lambda^*(\eta_-)$, it is easy to see that $\slashed{D}_1$, and $\slashed{D}_2$ are well defined. Invariance for $\slashed{D}_1$ is also obvious. It remains to show that $\slashed{D}_2$ is $K_J(A)$-invariant, seen as an operator acting on $\hat{\mathbb{S}}(A)$.

Define an operator E on $\mathfrak{g}(A)^*$, with values in $\operatorname{End}(\hat{\mathbb{S}}(A))$ as a graded commutator:

$$E : \mathfrak{g}(A)^* \longrightarrow \operatorname{End}(\hat{\mathbb{S}}(A)), \quad E(\tau) = [\not{D}_2, \psi(\tau)] = \not{D}_2 \psi(\tau) + \psi(\tau) \not{D}_2,$$

where we take the graded commutator since both operators are odd.

Now let σ denote the action of the Lie algebra of $K_J(A)$ on $\hat{\mathbb{S}}(A)$, and let a be any element of this Lie algebra. We proceed to show the equality (compare [10]):

$$[\sigma(a), E(\tau)] = E(\operatorname{ad}_a^*(\tau)).$$

Before we prove this equality, notice that it is equivalent to $[[\not{D}_2, \sigma(a)], \psi(\tau)] = 0$ for all $\tau \in \mathfrak{g}(A)^*$. Hence $[\not{D}_2, \sigma(a)]$ is a scalar operator. By checking on a highest weight vector, we see that $[\not{D}_2, \sigma(a)] = 0$, or that $\not{D}_2$ is $K_J(A)$-invariant. Hence, all we need to do is to prove the above equality, which we shall prove in the next lemma. $\quad\square$

Lemma 6.2. *Let E be defined as the function:*

$$E : \mathfrak{g}(A)^* \longrightarrow \operatorname{End}(\hat{\mathbb{S}}(A)), \quad E(\tau) = [\not{D}_2, \psi(\tau)] = \not{D}_2 \psi(\tau) + \psi(\tau) \not{D}_2,$$

and let a be an element in the Lie algebra of $K_J(A)$. Then $[\sigma(a), E(\tau)] = E(\operatorname{ad}_a^*(\tau))$, where σ denotes the action of $K_J(A)$ on $\hat{\mathbb{S}}(A)$.

Proof. Let us begin by expressing some structure constants using the summation convention:

$$[a_i, a_s] = A_{i,s}^k a_k, \qquad [h_j, a_t] = -\alpha_t(h_j) a_t,$$

where α_t is the positive root corresponding to a_t. It follows that ad_{a_i}, $\operatorname{ad}_{a_i}^\dagger$ and ad_{h_j} may be written as:

$$\operatorname{ad}_{a_i} = A_{i,s}^k \psi(b_k^*) \psi(a_s^*), \qquad \operatorname{ad}_{a_i}^\dagger = A_{i,s}^k \psi(b_s^*) \psi(a_k^*), \qquad \operatorname{ad}_{h_j} = -\alpha_t(h_j) \psi(b_t^*) \psi(a_t^*).$$

We may therefore express $\not{D}_2$ in terms of the Clifford generators as

$$\not{D}_2 = \frac{1}{2} A_{i,s}^k \{ \psi(b_k^*) \psi(a_s^*) \psi(a_i^*) + \psi(b_i^*) \psi(b_s^*) \psi(a_k^*) \} - \alpha_t(h_j) \psi(h_j^*) \psi(b_t^*) \psi(a_t^*) + \psi(\rho).$$

From this description, it is easy to calculate the value of E for the generators of $\mathfrak{g}(A)^*$

$$E(b_i^*) = A_{i,s}^k \psi(b_k^*) \psi(a_s^*) + \psi(b_i^*) \psi(\alpha_i), \qquad E(h^*) = -S(\alpha_i, h^*) \psi(b_i^*) \psi(a_i^*) + \rho(\bar{h}),$$

where $\bar{h} \in \mathfrak{h}$ is the element dual to h^* via S.

Now let α_0 be a simple root in the subset of roots corresponding to the Lie group $K_J(A)$. Let h_0 be the corresponding co-root. Notice that we may express the elements $[b_0, a_i]$, and $[b_0, h]$ in the form:

$$[b_0, a_i] = \delta_{i,0} A h_0 + B_{i,0}^k a_k, \qquad [b_0, h] = -\alpha_0(h) b_0, \quad \text{where } A = 2/S(h_0, h_0).$$

Similarly, we may write $\mathrm{ad}^*_{b_0}(b_i^*)$ and $\mathrm{ad}^*_{b_0}(h^*)$ in the form:

$$\mathrm{ad}^*_{b_0}(b_i^*) = \delta_{i,0}\alpha_0 + \mathrm{B}^k_{i,0}b_k^*, \qquad \mathrm{ad}^*_{b_0}(h^*) = -h^*(h_0)\mathrm{A}a_0^*.$$

Let us identify $\mathfrak{b}^*_- \,\hat{\otimes}\, \mathfrak{b}^*_-$, with $\mathfrak{b}_- \,\hat{\otimes}\, \mathfrak{b}^*_-$ by applying duality on the first factor. We express the quadratic homogeneous part in the expressions for $\mathrm{E}(b_i^*)$ and $\mathrm{E}(h^*)$ under this identification. These expressions transforms into the formal expressions in $\mathrm{End}(\mathfrak{b}_-) = \mathfrak{b}_- \,\hat{\otimes}\, \mathfrak{b}^*_-$:

$$\tilde{\mathrm{E}}(b_i^*) = \mathrm{A}^k_{i,s}a_k \otimes a_s^* + a_i \otimes \alpha_i, \qquad \tilde{\mathrm{E}}(h^*) = -\mathrm{S}(\alpha_i, h^*)a_i \otimes a_i^*.$$

Notice that $\tilde{\mathrm{E}}(b_i^*)$ represents the formal expression for ad_{a_i} acting on $\mathfrak{b}_-$. Similarly, $\tilde{\mathrm{E}}(h^*)$ represents $\mathrm{ad}_{\tilde{h}}$. Taking the commutator with $\sigma(b_0)$, and isolating the part in $\mathfrak{b}^*_- \,\hat{\otimes}\, \mathfrak{b}_-$:

$$\big[\sigma(b_0), \tilde{\mathrm{E}}(b_i^*)\big] = \tilde{\mathrm{E}}\big([b_0, a_i]^*\big) + \delta_{i,0}\mathrm{A}h_0 \otimes \alpha_0.$$

Reversing the identification and reducing the above equality to an equality of operators on $\hat{\mathbb{S}}(A)$, we see that the expression $[\sigma(b_0), \tilde{\mathrm{E}}(b_i^*)]$ reduces to $[\sigma(b_0), \mathrm{E}(b_i^*)]$, while the expression $\tilde{\mathrm{E}}([b_0, a_i]^*)$ reduces to $\mathrm{E}(\mathrm{ad}^*_{b_0}(a_i^*)) - \rho(\bar{\alpha}_0)\,\mathrm{Id}$. Finally, the term $\delta_{i,0}\mathrm{A}h_0 \otimes \alpha_0$ reduces to the operator $\frac{1}{2}\delta_{i,0}\mathrm{S}(\alpha_0, \alpha_0)\,\mathrm{Id}$. On the other hand $\bar{\alpha}_0$ is $\frac{1}{2}\mathrm{S}(\alpha_0, \alpha_0)h_0$, and so $\rho(\bar{\alpha}_0) = \frac{1}{2}\delta_{i,0}\mathrm{S}(\alpha_0, \alpha_0)$.

Putting this together, we have shown that $[\sigma(b_0), \mathrm{E}(b_i^*)] = \mathrm{E}(\mathrm{ad}^*_{b_0}(b_i^*))$. To complete the lemma, one needs to consider the cases with b_0 replaced by h_0, a_0, and b_i^* replaced by h_i^* (the remaining cases follow on taking adjoints). These cases are much easier, and the computation is left to the reader. □

A parametrized version of the cubic Dirac operator. To apply this to Dominant K-theory, we need a parametrized version of the above operator. We begin with a few preliminary observations. Let $\mathfrak{g}'(A)$ denoted the derived Lie algebra $\mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$. There is a (split) short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{g}'(A) \overset{i}{\longrightarrow} \mathfrak{g}(A) \longrightarrow \mathrm{t} \longrightarrow 0,$$

where t is the quotient of the abelian Lie algebra $\mathfrak{h}$ by the subalgebra $\tilde{\mathfrak{h}}$ generated by the coroots: $\tilde{\mathfrak{h}} = \bigoplus_i \mathbb{C}h_i$. Let $\mathrm{t}_\mathbb{R}$ denote the real form inside t.

Since we are working with a generalized Cartan matrix of compact type, the space $|\mathcal{S}(A)|$ is equivalent to the Barycentric subdivision of the n-simplex $\mathcal{B}\Delta(n)$. Define an affine map from $\mathcal{B}\Delta(n)$ to $\mathfrak{g}(A)^*$ by sending the barycenter corresponding to a proper subset $J \subset I$ to the element $\rho_{I-J} \in \mathfrak{h}^*$, where we recall that for $K \subseteq I$, the elements ρ_K are defined as

$$\rho_K = \sum_{i \in K} h_i^*.$$

This restricts to an affine map $\mathrm{f}: |\mathcal{S}(A)| \to \mathfrak{g}(A)^*$. Since the stabilizer of the element ρ_{I-J} is exactly the subgroup $K_J(A)$, we see that the map f extends $K(A)$ equivariantly to a proper embedding:

$$\mathrm{f}: X(A) \longrightarrow \mathfrak{g}(A)^*.$$

Furthermore, we may extend the map f to a proper equivariant embedding:

$$F : t_{\mathbb{R}}^* \times \mathbb{R}_+ \times X(A) \longrightarrow \mathfrak{g}(A)^*, \quad (t, r, x) \longmapsto t + r f(x),$$

where $\mathbb{R}_+$ denotes the positive reals. Let $Y(A)$ denote the space $t_{\mathbb{R}}^* \times \mathbb{R}_+ \times X(A)$. Since $\tilde{\mathbb{K}}_{K(A)}^r(Y(A))$ is canonically isomorphic to $\tilde{\mathbb{K}}_{K(A)}^n(X(A))$, where r is the rank of $K(A)$, we may as well give a geometric description of the former, which is slightly more convenient.

Notice that the fundamental domain for the $K(A)$-action on $Y(A)$ is given by the subspace $t_{\mathbb{R}}^* \times \mathbb{R}_+ \times |\mathcal{S}(A)|$. Indeed, the image of $t_{\mathbb{R}}^* \times \mathbb{R}_+ \times |\mathcal{S}(A)|$ under the map F belongs to the Weyl chamber C.

Now let $\mu + \rho \in D_+$ be dominant regular weight. We construct a Dominant Fredholm family over $Y(A)$, with the underlying vector space given by the product $\mathbb{H}_\mu \times Y(A)$. Choose a continuous family B_y, of Hermitian inner-products on $\mathfrak{g}(A)$, parametrized by points y in the space $t_{\mathbb{R}}^* \times \mathbb{R}_+ \times |\mathcal{S}(A)|$. We require that B_y is invariant under the stabilizer (in $K(A)$), of the point y. Since all stabilizers are compact Lie groups, it is easy to see that we may choose such a family, and that any two choices are equivariantly homotopic (see the appendix for details). The ω-conjugates $S_y = B_y^\omega$ form a continuous family of non-degenerate, symmetric, bilinear forms on $\mathfrak{g}(A)$.

For a point y in the fundamental domain, let $\lambda \in C$ be the element $F(y)$. Define the operator at y, to be $\slashed{D}_\lambda = \slashed{D} - \psi(\lambda)$, where $\psi(\lambda)$ denotes Clifford multiplication, with respect to the Clifford algebra generated under the relation $x^2 = \frac{1}{2} S_y(x, x)$. Here $\slashed{D}$ refers to the operator with symbol $\slashed{D}_1 + \slashed{D}_2$ defined earlier. By 6.1, we see that the operator $\slashed{D}$ is invariant under the stabilizer of y. Hence $\slashed{D}_\lambda$ is also fixed by stabilizer of the point y, and so we may therefore use the $K(A)$-action to extend the Fredholm family to all of $Y(A)$.

Claim 6.3. *Let $\mu \in D$ be a dominant weight. Then the family $(\mathbb{H}_\mu, \slashed{D}_\lambda)$ is equivalent to the pullback of the character $\mu + \rho$ under the pinch map*

$$\pi^* : DR_T = \tilde{\mathbb{K}}_{K(A)}^r \big(K(A)_+ \wedge_T S^r\big) \longrightarrow \tilde{\mathbb{K}}_{K(A)}^r\big(Y(A)\big).$$

Proof. Since the operators $\partial + \partial^\dagger$ and $K_\lambda = K - \psi(\lambda)$ commute, we may calculate the support of this family in two steps. Firstly, we calculate the kernel of $\partial + \partial^\dagger$ which is [21]

$$\sum_{w \in W(A)} (-1)^w \mathbb{S}(\mathfrak{h}^*) \otimes \mathbb{C}_{w(\mu+\rho)},$$

where the sign of a vector space indicates its parity with respect to the $\mathbb{Z}/2$ grading of $\Lambda^*(\eta_-)$. It is now clear that the operator K_λ acting on this complex is supported at the W-orbit of $\mu + \rho$. Since $\mathbb{S}(\mathfrak{h}^*)$ represents the Thom class in $^A\tilde{\mathbb{K}}_T^r(S^r) = \tilde{\mathbb{K}}_{K(A)}^r(K(A)_+ \wedge_T S^r)$, this operator is induced by the image of the character $\mu + \rho$. $\square$

Remark 6.4. The operator $\slashed{D}_\lambda$ is not a bounded operator. We rectify this as follows.

First notice that by construction, $\slashed{D}_\lambda$ commutes with the T-action, and therefore it has a weight space decomposition as a direct sum of operators acting on the individual finite-dimensional isotypical summands in $\mathbb{H}_\mu$.

Next, taking a regular element $h^* \in \mathfrak{h}_{\mathbb{R}}^*$, the equality $[\slashed{D}, \psi(h^*)] = E(h^*)$, and the formula for $E(h^*)$, implies that $\slashed{D}_\lambda$ has a discrete spectrum. Furthermore, $\slashed{D}_\lambda^2$ is a scalar on each isotypical

summand of the T-action, and one can explicitly compute it (see [21, Section 3.4]). It follows easily that $(1 + \not{D}_\lambda^2)^{-1}$ is a well defined, continuously varying family of compact operators, in the norm topology.

Similarly, one can show that $\not{D}_\lambda (1 + \not{D}_\lambda^2)^{-1/2}$ is a continuously varying family of bounded operators, in the compact open topology. It now follows, as in [1], that $(\mathbb{H}_\mu, \not{D}_\lambda)$ may be suitable normalized to define a continuous family of operators with values in $\mathcal{F}(\mathcal{H})$.

7. Dominant K-theory for the extended compact type

Thus far, we have restricted our attention to Kac–Moody groups of compact type because this assumption greatly simplified the Dominant K-theory of $X(A)$. Let us now explore the behaviour of Dominant K-theory for generalized Cartan matrices that are one level higher in terms of complication.

Recall that a generalized Cartan matrix A is of extended compact type if there exists a decomposition $I = I_0 \sqcup J_0$, such that given any $J \subseteq I$, the sub-generalized Cartan matrix $(a_{i,j})_{j,k \in J}$ is of non-finite type if and only if $I_0 \subseteq J$. In the following section, we shall make a few remarks about the example of the Kac–Moody group E_{10} of extended compact type with J_0 being the unique extended node so that $(a_{i,j})_{i,j \in I_0}$ is the affine type generalized Cartan matrix for E_9.

Let A_0 be the generalized Cartan matrix of compact type given by $A_0 = (a_{i,j})_{j,k \in I_0}$. Then the category $\mathcal{S}(A)$ of spherical subsets of I can easily seen to be equivalent to the category $\mathcal{S}(A_0)$ via the functor sending a subset $J \subset I_0$ to $J \cup J_0 \subset I$. It follows that $X(A)$ is equivariantly equivalent to a subspace $X_0(A)$:

$$X_0(A) = \frac{K(A)/T \times |\mathcal{S}(A_0)|}{\sim},$$

where we identify (gT, x) with (hT, y) iff $x = y \in \Delta_J(n)$, and $g = h \mod K_{J \cup J_0}(A)$. We therefore take $X_0(A)$ to be a model for the classifying space of $K(A)$-actions.

A similar situation holds for the Davis complex: we may take the space Σ_0 as a model for the classifying space of proper $W(A)$-actions, where

$$\Sigma_0 = \frac{W(A) \times |\mathcal{S}(A_0)|}{\sim};$$

we identify (v, x) with (w, y) iff $x = y \in \Delta_J(n)$, and $v = w \mod W_{J \cup J_0}(A)$.

Before we begin the analysis of the extended compact type, it is important to calculate the cohomology of spaces related to the Davis complex. We begin with a definition relevant to this calculation:

Definition 7.1. Let W be an arbitrary Coxeter group on a generating set I. Let $J, K \subseteq I$ be any subsets. Define the set of pure double coset representatives ${}^K \overline{W}{}^J$:

$${}^K \overline{W}{}^J = \big\{ w \in {}^K W^J \mid w W_J(A) w^{-1} \cap W_K(A) = 0 \big\}.$$

Similarly, define the set of (right) maximally pure elements ${}^K \overline{W}{}^J_s$:

$${}^K \overline{W}{}^J_s = \big\{ w \in {}^K \overline{W}{}^J \mid w \notin {}^K \overline{W}{}^L \text{ for all proper inclusions } J \subset L \big\}.$$

Let us now consider the following proper $W(A)$-CW complex:

$$\hat{\Sigma}_0 = W(A) \times_{W(A_0)} \Sigma(A_0).$$

Claim 7.2. *Assume that A is a generalized Cartan matrix of extended compact type as defined above, with $|I_0| = n + 1$. Let $K \subseteq I$ be any subset, then the groups $H_c^p(\hat{\Sigma}_0/W_K(A), \mathbb{Z})$ are trivial if $p \neq \{0, n\}$, and*

$$H_c^0(\hat{\Sigma}_0/W_K(A), \mathbb{Z}) = \mathbb{Z}[{}_K \overline{W}_{I_0}], \qquad H_c^n(\hat{\Sigma}_0/W_K(A), \mathbb{Z}) = \mathbb{Z}[{}^K \overline{W}^{I_0}],$$

where ${}_K \overline{W}_{I_0}$ denotes the fixed point set of the $W(A_0)$ action on $W(A)/W_K(A)$.

Proof. Notice that $W(A)/W_K(A)$ may be decomposed under the $W(A_0)$ action as

$$W(A)/W_K(A) = \coprod_{w \in {}^K W^{I_0}} W(A_0)/W_{K_w}(A), \quad \text{where } W_{K_w}(A) = W(A_0) \cap w^{-1} W_K(A) w.$$

This induces a decomposition of the cohomology:

$$H_c^p(\hat{\Sigma}_0/W_K(A), \mathbb{Z}) = \bigoplus_{w \in {}^K W^{I_0}} H_c^p(\Sigma(A_0)/W_{K_w}(A), \mathbb{Z}).$$

Recall from Section 5, that the summands on the right-hand side are nontrivial exactly if $K_w = \{\varnothing, I_0\}$, and in those cases contribute a copy of $\mathbb{Z}$ in degree $\{n, 0\}$, respectively. It is easy to see that this is equivalent to the statement of the claim. $\quad\square$

A similar claim holds for the space Σ_0.

Claim 7.3. *Assume that A is a generalized Cartan matrix of extended compact type as defined above with $|I_0| = n + 1$, and $n > 1$. Let $K \subseteq I$ be any subset, then the groups $H_c^p(\Sigma_0/W_K(A), \mathbb{Z})$ are trivial if $p \neq \{0, n\}$, and*

$$H_c^n(\Sigma_0/W_K(A), \mathbb{Z}) = \mathbb{Z}[{}^K \overline{W}_s^{I_0}].$$

Moreover, $H_c^0(\Sigma_0/W_K(A), \mathbb{Z})$ is nonzero if and only if $\Sigma_0/W_K(A)$ is compact and contractible, in particular for this case ${}^K \overline{W}_s^{I_0}$ is empty.

Proof. The proof is essentially the argument given in [8, (8.3.1), (8.3.3)]. First notice that $\Sigma_0/W_K(A)$ may be identified with a subspace $A_K \subseteq \Sigma_0$ given by the "K-Sector":

$$A_K = \bigcup_{w \in {}^K W^{J_0}} w|\mathcal{S}(A_0)| \subseteq \Sigma_0.$$

This identification provides a retraction of $\Sigma_0/W_K(A)$ from Σ^0. Let us now order elements of ${}^K W^{J_0}$ in a sequence $\{w_1, w_2, \ldots\}$ such that $l(w_k) \leqslant l(w_{k+1})$. Consider an exhaustive filtration

of A_K by subspaces U_k given by the union of the first k translates $w_i|\mathcal{S}(A_0)|$, $i \leqslant k$. And let $\overline{U}_k$ denote its complement. Consider the following long exact sequence:

$$\cdots \rightarrow H^*(A_K, \overline{U}_{k-1}) \rightarrow H^*(A_K, \overline{U}_k) \rightarrow H^*(\overline{U}_{k-1}, \overline{U}_k) \rightarrow \cdots. \tag{1}$$

We prove by induction on k, that $H^*(A_K, \overline{U}_k)$ has cohomology only in degrees $\{0, n\}$, then we shall identify the generators in degree n.

Setting $\overline{U}_0 = A_K$, notice [8, (8.3.1)], that we may identify the groups $H^*(\overline{U}_{k-1}, \overline{U}_k)$ with $H^*(|\mathcal{S}(A_0)|, |\mathcal{S}^k(A_0)|)$, where $|\mathcal{S}^k(A_0)|$ denotes the realization of the nerve of the subcategory:

$$\mathcal{S}^k(A_0) = \left\{ J \in \mathcal{S}(A_0) \,\big|\, l(w_k r_j) > l(w_k) \text{ and } w_k r_j \in {}^K W \text{ for some } j \in J \cup J_0 \right\}.$$

Assume by induction that the groups $H^*(A_K, \overline{U}_{k-1})$ are concentrated in degrees $\{0, n\}$. Since $|\mathcal{S}(A_0)|$ is homeomorphic to the n-simplex, it is easy to see that $H^*(|\mathcal{S}(A_0)|, |\mathcal{S}^k(A_0)|)$ is nontrivial if and only if $\mathcal{S}^k(A_0)$ is empty, or is the full subcategory $\{J \in \mathcal{S}(A_0) \mid J \neq \varnothing\}$.

In the case when $\mathcal{S}^k(A_0)$ is empty, the long exact sequence (1), along with the induction assumption, forces $H^0(A_K, \overline{U}_k)$ to have a free summand, which implies that $\overline{U}_k = \varnothing$ i.e. $U_k = A_K$ or that A_K is compact. Since A_K is contractible (being a retract of Σ_0), we have shown in this case that $H_c^*(\Sigma^0/W_K(A), \mathbb{Z}) = \mathbb{Z}$, concentrated in degree zero.

For $\mathcal{S}^k(A_0)$ being all nonempty subsets, it is clear that $H^p(|\mathcal{S}(A_0)|, |\mathcal{S}^k(A_0)|)$ is nontrivial only if $p = n$, and is isomorphic to $\mathbb{Z}$ in that degree. It is easy to construct a splitting to the sequence (1), and so we pick up a free generator in degree n. The result follows by induction.

It is straightforward to check that the latter case is equivalent to the condition $w_k \in {}^K \overline{W}_s^{I_0}$. Hence we get one generator in degree n for each maximally pure element. $\quad\square$

Remark 7.4. Consider the canonical map given by projection

$$r : \hat{\Sigma}_0 \longrightarrow \Sigma_0.$$

It follows from the proof of the previous claim that the induced map in cohomology

$$r : H_c^p\big(\Sigma_0/W_K(A), \mathbb{Z}\big) \longrightarrow H_c^p\big(\hat{\Sigma}_0/W_K(A), \mathbb{Z}\big),$$

is the diagonal map in degree 0, if $\Sigma_0/W_K(A)$ is compact, and is the canonical inclusion map of ${}^K \overline{W}_s^{I_0}$ in ${}^K \overline{W}^{I_0}$ in degree n.

We may now attempt to calculate $\mathbb{K}^*_{K(A)}(X(A))$ for groups $K(A)$ of extended compact type with $|I_0| = n + 1$, and $n > 1$. It is sufficient to replace $X(A)$ by $X_0(A)$. Now consider the topological version of the map r above:

$$r : K(A) \times_{K_{I_0}(A)} X(A_0) \longrightarrow X_0(A),$$

where the $K_{I_0}(A)$-action on $X(A_0)$ extends the $K(A_0)$-action. Indeed, we have a homeomorphism between each orbit $K(A_0)/K_J(A_0) = K_{I_0}(A)/K_J(A_0)$ for $J \subseteq I_0$. It follow from the general properties of Buildings [9] that $X(A_0)$ is the classifying space of proper $K_{I_0}(A)$-actions.

In Dominant K-theory we have the induced maps

$$r : \mathbb{K}^*_{K(A)}\big(X_0(A)\big) \longrightarrow {}^A\mathbb{K}^*_{K_{I_0}(A)}\big(X(A_0)\big),$$

$$\mathrm{St} : {}^A\mathbb{K}^*_{K_{I_0}(A)}\big(X(A_0)\big) \longrightarrow \mathbb{K}^*_{K_{I_0}(A)}\big(X(A_0)\big),$$

where the second map St is induced from the inclusion of the maximal Dominant $K(A)$-representation into the maximal Dominant $K_{I_0}(A)$-representation by restriction.

To calculate $\mathbb{K}^*_{K(A)}(X_0(A))$, we begin with the E_2 term of the spectral sequence:

$$E_2^{p,*} = \varprojlim {}^p\mathbb{K}^*_{K(A)}\big(K(A)/K_\bullet(A)\big) = \varprojlim {}^p\mathrm{DR}_\bullet\big[\beta^{\pm 1}\big] \;\Rightarrow\; \mathbb{K}^p_{K(A)}\big(X_0(A)\big)\big[\beta^{\pm 1}\big],$$

where we recall that $\varprojlim {}^p\mathrm{DR}_\bullet$ is canonically isomorphic to the equivariant cohomology of the Davis complex Σ_0, with values in the ring DR_T:

$$\varprojlim {}^p\mathrm{DR}_\bullet = \mathrm{H}^p_{W(A)}(\Sigma_0, \mathrm{DR}_T).$$

Next, we decompose the functor $\mathrm{DR}_\bullet$ as $\bigoplus \mathrm{DR}^K_\bullet$ indexed by $K \subseteq I$:

$$\varprojlim {}^p\mathrm{DR}_\bullet = \bigoplus \varprojlim {}^p\mathrm{DR}^K_\bullet = \bigoplus \mathrm{H}^p_{W(A)}\big(\Sigma_0, \mathbb{Z}[W(A)/W_K(A)]\big) \otimes \mathrm{R}^K_T.$$

It is easy to see that $\mathrm{H}^p_{W(A)}(\Sigma_0, \mathbb{Z}[W(A)/W_K(A)]) = \mathrm{H}^p_c(\Sigma_0/W_K(A), \mathbb{Z})$. Hence using the previous claim we get that $\varprojlim {}^i\mathrm{DR}_\bullet = 0$ if $i \neq \{0, n\}$, and in those degrees, we have:

$$\varprojlim {}^0\mathrm{DR}_\bullet = \bigoplus_{|{}^K W| < \infty} \mathrm{R}^K_T, \qquad \varprojlim {}^n\mathrm{DR}_\bullet = \bigoplus_{K \subset I} \mathbb{Z}\big[{}^K\overline{W}^{I_0}_s\big] \otimes \mathrm{R}^K_T. \tag{2}$$

We may set up the same spectral sequence to compute ${}^A\mathbb{K}^*_{K_{I_0}(A)}(X(A_0))$, with E_2 term:

$$ {}^A E_2^{p,*} = \mathrm{H}^p_{W(A)}(\hat{\Sigma}_0, \mathrm{DR}_T)\big[\beta^{\pm 1}\big] \;\Rightarrow\; {}^A\mathbb{K}^p_{K_{I_0}(A)}\big(X(A_0)\big)\big[\beta^{\pm 1}\big]. $$

It follows from the previous claims that the map r induces a map of spectral sequences which is injective on E_2.

Now, the (reductive) Kac–Moody group $K_{I_0}(A)$ is of compact type, and hence we know from Section 5 that the spectral sequence computing $\mathbb{K}^*_{K_{I_0}(A)}(X(A_0))$ collapses at E_2. The Dominant regular ring DR_T for $K(A)$ is a summand within the Dominant regular ring for $K_{I_0}(A)$, and hence the map St also induces a map of spectral sequences which is injective on E_2.

This argument shows that all spectral sequences collapse at E_2 and we may identify the generators:

Theorem 7.5. *Let $K(A)$ be a Kac–Moody group of extended compact type with $|I_0| = n + 1$, and $n > 1$. Then we have an isomorphism of graded groups:*

$$\mathbb{K}^*_{K(A)}\big(X(A)\big) = \bigoplus_{|{}^K W| < \infty} R^K_T\big[\beta^{\pm 1}\big] \bigoplus_{K \subset I} \mathbb{Z}\big[{}^K\overline{W}^{I_0}_s\big] \otimes R^K_T\big[\beta^{\pm 1}\big].$$

The classes $\mathbb{Z}[{}^K\overline{W}^{I_0}_s] \otimes R^K_T$ appear naturally in degree n, and the others in degree 0. Furthermore, the following map is injective:

$$r : \mathbb{K}^*_{K(A)}\big(X(A)\big) \longrightarrow {}^A\mathbb{K}^*_{K_{I_0}(A)}\big(X(A_0)\big), \quad with$$

$${}^A\mathbb{K}^*_{K_{I_0}(A)}\big(X(A_0)\big) = \bigoplus \mathbb{Z}[{}_K\overline{W}_{I_0}] \otimes R^K_T[\beta^{\pm 1}] \bigoplus_{K \subset I} \mathbb{Z}[{}^K\overline{W}^{I_0}] \otimes R^K_T[\beta^{\pm 1}],$$

where ${}_K\overline{W}_{I_0}$ denotes the set of elements fixed under the $W(A_0)$-action on $W(A)/W_K(A)$.

Remark 7.6. Under the map St, we may identify these classes within the regular dominant characters of $K_{I_0}(A)$. The classes $\bigoplus_{K \subset I} \mathbb{Z}[{}^K\overline{W}^{I_0}] \otimes R^K_T$ map to characters in the $W(A)$-orbit of dominant characters of $K(A)$, that are regular dominant characters of $K_{I_0}(A)$. The classes $\bigoplus_{K \subset I} \mathbb{Z}[{}^K\overline{W}^{I_0}_s] \otimes R^K_T$ are then identified with those characters of the form described above, which are also antidominant for $K_{J_0}(A)$.

8. Some related remarks

The affine case. We would like to make some brief comments relating our result with results of Freed–Hopkins–Teleman. For simplicity, we deal only with the untwisted affine case. The verification of details in this section are left to the interested reader.

Let $K(A)$ denote the affine Kac–Moody group corresponding to the group of loops on a simply connected, simple compact Lie group G. Let $\tilde{L}G$ denote the central extension of the group of polynomial loops on G, and let $\mathbb{T}$ denote the rotational circle acting by reparametrization of loops. Then we may identify $K(A)$ with the group $\mathbb{T} \ltimes \tilde{L}G$.

Let T denote the maximal torus of G, and let K denote the central circle of $K(A)$. The maximal torus T of $K(A)$ may be decomposed as $T = T \times \mathbb{T} \times K$. Let k and d denote the weights dual to K and $\mathbb{T}$ respectively, under the above decomposition. The nonzero vectors in the Tits cone Y are given by the upper-half plane consisting of vectors that have a positive k-coefficient. Hence, the Dominant representation ring of T may be identified as:

$$\mathrm{DR}_T = \mathbb{Z}[e^{\pm \mathrm{d}}] + e^{\mathrm{k}} \mathrm{R}_T[e^{\mathrm{k}}, e^{\pm \mathrm{d}}],$$

where R_T denotes the representation ring of T. The central weight k is known as "level," and we may decompose our maximal dominant Hilbert space $\mathcal{H}$ into a (completed) sum of eigen-spaces corresponding to the level:

$$\mathcal{H} = \hat{\bigoplus}_{m \geqslant 0} \mathcal{H}_m.$$

Now assume that X is a finite, proper $K(A)$-CW complex on which the center K acts trivially. Hence the action of $K(A)$ on X factors through the central quotient: $\mathbb{T} \ltimes LG$. One has a corresponding decomposition of Dominant K-theory:

$$\mathbb{K}^*_{K(A)}(X) = \bigoplus_{m \geqslant 0} {}^m\mathbb{K}^*_{K(A)}(X),$$

where ${}^m\mathbb{K}^0_{K(A)}(X)$ is defined as homotopy classes of $\mathbb{T} \ltimes LG$ equivariant maps from X to the space $\mathcal{F}(\mathcal{H}_m)$, of Fredholm operators on $\mathcal{H}_m$. There is a similar decomposition for homology. We now proceed to relate this decomposition to twisted K-theory.

Recall that X is a proper $K(A)$-CW complex, hence all its isotropy subgroups of X are compact. It is well known that the subgroup of pointed polynomial loops $\Omega G \subset \mathbb{T} \ltimes LG$ contains no compact subgroup, and hence ΩG acts freely on X. It follows that ${}^m \mathbb{K}^0_{K(A)}(X)$ may be seen as homotopy classes of $\mathbb{T} \ltimes LG$-equivariant sections of a fiber bundle over $X/\Omega G$, with fiber $\mathcal{F}(\mathcal{H}_m)$, and structure group $\mathbb{PU}(\mathcal{H}_m)$ (with the compact open topology). For $m > 0$, the Hilbert space $\mathcal{H}_m$ is universal in the sense of [10], and hence the above groups can be identified with the energy equivariant version of twisted G-equivariant K-theory of G.

Remark 8.1. The reader should bear in mind that Dominant K-theory and twisted equivariant K-theory are cohomology theories on two very different categories. The former is defined on the category of $\mathbb{T} \ltimes \tilde{L}G$-CW complexes, and satisfies the Eilenberg–Steenrod axiom for cofiber sequences. The latter is defined on a suitably enriched category of stacks, and satisfies the Mayer–Vietoris axiom for open covers. The above discussion establishes an isomorphism between the $\mathbb{T} \ltimes \tilde{L}G$-equivariant Dominant K-theory of the universal space of proper actions, and the twisted K-theory of the stack G/G given by the conjugation action of G on itself.

Recall that projective Hilbert bundles on $X/\Omega G$ as above, are classified by a "twist" belonging to $\mathrm{H}^3_{\mathbb{T} \ltimes LG}(X, \mathbb{Z})$ [1]. This twist is simply the pullback of the universal twist $[m]$ in $\mathrm{H}^3_{\mathbb{T} \ltimes LG}(X(A), \mathbb{Z}) = \mathbb{Z}$, where $X(A)$ is the classifying space of proper $K(A)$-actions. We therefore conclude that the Dominant K-theory groups ${}^m \mathbb{K}^*_{K(A)}(X)$ represent the (energy equivariant) m-twisted equivariant K-theory groups of the space $X/\Omega G$.

The space $X(A)$ is homeomorphic to the affine space of principal G-connections for the trivial bundle over S^1 [19], and the space $X(A)/\Omega G$ is homeomorphic to G, via the holonomy map. Recall that the geometric isomorphism we constructed in Section 5, identified the irreducible representation L_μ with highest weight μ, with the character $e^{\mu+\rho}$ considered as an element in $\mathbb{K}^n_{K(A)}(X(A))$ (here n may be identified with the rank of the compact group G). The Weyl element ρ has level given by the dual Coxeter number, $h^\vee$, and so we recover the energy equivariant version of the theorem of Freed–Hopkins–Teleman [10, Section 15], that relates the $(k + h^\vee)$-twisted equivariant K-theory groups, with highest weight (or positive energy) level k representations of the loop group.

Real forms of Kac–Moody groups. So far we have been considering the unitary form $K(A) \subset G(A)$, where $G(A)$ denotes the complex points of the Kac–Moody group. There is a functorial construction of the Kac–Moody group over an arbitrary field [28]. In particular, we may consider the real form $G^{\mathbb{R}}(A)$. One may identify the real form as the fixed subgroup: $G^{\mathbb{R}}(A) = G(A)^\tau$, where τ denotes complex conjugation. This action preserves the subgroups $G_J(A)$, we may define the real form of the parabolic subgroups: $G^{\mathbb{R}}_J(A) = G_J(A)^\tau = G_J(A) \cap G^{\mathbb{R}}(A)$. Notice also that the action by complex conjugation descends to the homogeneous spaces: $G(A)/G_J(A)$.

Let $K^{\mathbb{R}}(A)$ denote the "orthogonal form" inside $G^{\mathbb{R}}(A)$. Hence, $K^{\mathbb{R}}(A)$ is given by the fixed points of the involution ω acting on $G^{\mathbb{R}}(A)$. In other words, the orthogonal form is given by: $K^{\mathbb{R}}(A) = G^{\mathbb{R}}(A)^\omega = K(A) \cap G^{\mathbb{R}}(A)$. And along the same lines, define subgroups of $K^{\mathbb{R}}(A)$ by $K^{\mathbb{R}}_J(A) := G^{\mathbb{R}}_J(A)^\omega = K_J(A) \cap G^{\mathbb{R}}(A)$.

Claim 8.2. *The following canonical inclusions are homeomorphisms:*

$$K^{\mathbb{R}}(A)/K^{\mathbb{R}}_J(A) \subseteq G^{\mathbb{R}}(A)/G^{\mathbb{R}}_J(A) \subseteq \left\{ G(A)/G_J(A) \right\}^\tau.$$

Proof. We will prove these equalities by comparing the Bruhat decomposition over the fields $\mathbb{R}$ and $\mathbb{C}$. First recall the Bruhat decomposition for the space $G(A)/G_J(A)$:

$$G(A)/G_J(A) = \coprod_{w \in W^J} B\tilde{w}B/B, \tag{3}$$

where $\tilde{w}$ denotes any element of $N(T)$ lifting $w \in W(A)$. In fact, the general theory of Kac–Moody groups allows us to pick the elements $\tilde{w}$ to belong to $G^{\mathbb{R}}(A)$. Moreover, the subspace $B\tilde{w}B/B$ has the structure of the complex affine space $\mathbb{C}^{l(w)}$. In addition, the set of right coset representatives $\mathbb{C}^{l(w)}$ may be chosen to belong to $K(A)$.

Working over $\mathbb{R}$ we have the Bruhat decomposition

$$G^{\mathbb{R}}(A)/G_J^{\mathbb{R}}(A) = \coprod_{w \in W^J} B^{\mathbb{R}}\tilde{w}B^{\mathbb{R}}/B^{\mathbb{R}}, \tag{4}$$

where the subspace $B^{\mathbb{R}}\tilde{w}B^{\mathbb{R}}/B^{\mathbb{R}}$ has the structure of the real affine space $\mathbb{R}^{l(w)}$, with the representatives $\mathbb{R}^{l(w)}$ belong to $K^{\mathbb{R}}(A)$. By comparing Eqs. (3) and (4) it follows that

$$\mathbb{R}^{l(w)} = B^{\mathbb{R}}\tilde{w}B^{\mathbb{R}}/B^{\mathbb{R}} = \{B\tilde{w}B/B\}^{\tau},$$

which assembles to the statement of the theorem as w ranges over W^J. □

Definition 8.3. Define the real (finite-type) Topological Tits building $X^{\mathbb{R}}(A)$ as the $K^{\mathbb{R}}(A)$-space:

$$X^{\mathbb{R}}(A) = \frac{K^{\mathbb{R}}(A)/T^{\mathbb{R}} \times |\mathcal{S}(A)|}{\sim},$$

where we identify $(gT^{\mathbb{R}}, x)$ with $(hT^{\mathbb{R}}, y)$ iff $x = y \in \Delta_J(n)$, and $g = h \mod K_J^{\mathbb{R}}(A)$. We note that the group $T^{\mathbb{R}} = K_{\varnothing}^{\mathbb{R}}(A)$, is the subgroup of two torsion points in T.

As before, we may identify this space as the homotopy colimit of a suitable functor on the category $\mathcal{S}(A)$. The complex conjugation action τ induces and action on $X(A)$, and we may relate $X^{\mathbb{R}}(A)$ to the space $X(A)$ studied earlier via:

Theorem 8.4. $X^{\mathbb{R}}(A)$ *is equivariantly homeomorphic to the fixed point space* $X(A)^{\tau}$. *Moreover, it is a finite, proper,* $K^{\mathbb{R}}(A)$- *CW space which is* $K_J^{\mathbb{R}}(A)$-*equivariantly contractible for* $J \in \mathcal{S}(A)$.

Proof. The space $X^{\mathbb{R}}(A)$ is clearly a finite, proper, $K^{\mathbb{R}}(A)$-CW complex. Also, the previous claim shows that $X^{\mathbb{R}}(A)$ is equivariantly homeomorphic to $X(A)^{\tau}$. It remains to show that it is $K_J^{\mathbb{R}}(A)$-equivariantly contractible for all $J \in \mathcal{S}(A)$. This follows immediately once we notice that the proof of contractibility given in Theorem 3.2, is τ equivariant. □

Remark 8.5. The space $X^{\mathbb{R}}(A)$ is in fact equivalent to the classifying space of proper $K^{\mathbb{R}}(A)$-actions: it suffices to show that any compact subgroup of $K^{\mathbb{R}}(A)$ is conjugate to a subgroup of $K_J^{\mathbb{R}}(A)$ for some $J \in \mathcal{S}(A)$. This follows from the general properties of Buildings [9].[3]

The group E_{10}. There has been recent interest in the real form of the Kac–Moody group E_{10}. This group is of extended compact type , with J_0 being the unique extended node so that $(a_{i,j})_{i,j \in I_0}$ is the affine type generalized Cartan matrix for E_9.

It appears that the real form of E_{10} encodes the symmetries of the dynamics ensuing from 11-dimensional supergravity [4,5].[4] The dynamics may be (conjecturally) expressed in terms of null geodesics in the Lorentzian space $G^{\mathbb{R}}(\mathrm{E}_{10})/K^{\mathbb{R}}(\mathrm{E}_{10})$. The group E_{10} also appears in a different fashion when 11-dimensional supergravity is compactified on a 10-dimensional torus, [3,12]. There is a conjectural description of the latter in terms of "billiards" in the Weyl chamber of E_{10} [3–5].

Since $X^{\mathbb{R}}(\mathrm{E}_{10})$ is the universal proper $K^{\mathbb{R}}(\mathrm{E}_{10})$-space, it seems reasonable to guess that the configuration space of the dynamics indicated above (perhaps away from singularities), may be related to the space $X^{\mathbb{R}}(\mathrm{E}_{10})$.

The affine type that was just discussed is relevant in field theory. The Dominant K-theory of the affine space $X(A)$ can be seen as the receptacle for the D-brane charge in the G/G-gauged WZW model. It can also be identified with the Verlinde algebra of a modular functor that corresponds to the categorification of Chern–Simons theory.

Invoking Theorem 7.5 for the group E_{10}, we have:

Theorem 8.6. *Let $A = \mathrm{E}_{10}$ denote the Kac–Moody group as above, then the group $K_{I_0}(\mathrm{E}_{10})$ is the (energy extended) affine group E_9, and the following restriction map, is an injection:*

$$r : \tilde{\mathbb{K}}^*_{\mathrm{E}_{10}}\big(X(\mathrm{E}_{10})\big) \longrightarrow {}^A\tilde{\mathbb{K}}^*_{\mathrm{E}_9}\big(X(\mathrm{E}_9)\big),$$

where reduced Dominant K-theory denote the kernel along any orbit E_{10}/T. Moreover, the maximal dominant E_{10}-representation restricts to a dominant E_9-representation, inducing a map

$$\mathrm{St} : {}^A\tilde{\mathbb{K}}^*_{\mathrm{E}_9}\big(X(\mathrm{E}_9)\big) \longrightarrow \tilde{\mathbb{K}}^*_{\mathrm{E}_9}\big(X(\mathrm{E}_9)\big).$$

The map St is injective, and has image given by the (energy extended) regular dominant characters of E_9, which are in the Weyl-orbit of dominant characters of E_{10}.

The image of $\tilde{\mathbb{K}}^*_{\mathrm{E}_{10}}(X(\mathrm{E}_{10}))$ under St is given by those characters of the form described above that are also antidominant characters for $K_{J_0}(\mathrm{E}_{10})$.

Remark 8.7. Consider the action map $K(\mathrm{E}_{10}) \times_{K^{\mathbb{R}}(\mathrm{E}_{10})} X^{\mathbb{R}}(\mathrm{E}_{10}) \longrightarrow X(\mathrm{E}_{10})$. This induces

$$\mathbb{K}^*_{K(\mathrm{E}_{10})}\big(X(\mathrm{E}_{10})\big) \longrightarrow {}^A\mathbb{K}^*_{K^{\mathbb{R}}(\mathrm{E}_{10})}\big(X^{\mathbb{R}}(\mathrm{E}_{10})\big),$$

which allows us to construct elements in ${}^A\tilde{\mathbb{K}}_{K^{\mathbb{R}}(\mathrm{E}_{10})}(X^{\mathbb{R}}(\mathrm{E}_{10}))$ via the previous theorem. The behaviour of this map is still unclear to us.

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[3] We thank P.E. Caprace for pointing this out to us.

[4] We thank Hisham Sati for explaining this to us.

9. Dominant K-homology

To give a natural definition of the Dominant K-homology groups, we need the definition of the equivariant dual of a finite, proper $K(A)$-CW spectrum. Our definition is motivated by [20]:

Definition 9.1. For a finite, proper $K(A)$-CW spectrum X, define the equivariant dual of X as the homotopy fixed point spectrum:

$$DX = \mathrm{Map}\big(X, K(A)_+\big)^{hK(A)},$$

where $K(A)_+$ denotes the suspension spectrum $\Sigma^\infty K(A)_+$ with a right $K(A)$-action.

For a proper $K(A)$-CW spectrum X, we show below that DX has the equivariant homotopy type of a finite, proper $K(A)$-CW spectrum via the residual left $K(A)$-action on $K(A)_+$. For example, let X be a single proper orbit $K(A)_+ \wedge_G S^0$ for some compact Lie subgroup $G \subseteq K(A)$. In this case, we will show below that DX is equivalent to the proper $K(A)$-spectrum $K(A)_+ \wedge_G S^{\mathfrak{g}}$, where $S^{\mathfrak{g}}$ denotes the one point compactification of the adjoint representation of G on its Lie algebra.

Definition 9.2. The Dominant K-homology groups of a finite, proper $K(A)$-CW spectrum X are defined as

$$\mathbb{K}_k^{K(A)}(X) = \mathbb{K}_{K(A)}^{-k}(DX).$$

We may extend this definition to arbitrary proper $K(A)$-CW complexes by taking direct limits over sub-skeleta.

Claim 9.3. *Let $G \subseteq K(A)$ be a compact Lie subgroup, let $S^{\mathfrak{g}}$ denote the one point compactification of the adjoint representation of G on its Lie algebra. Then the equivariant dual of a proper orbit: $X = K(A)_+ \wedge_G S^0$ is given by $DX \simeq K(A)_+ \wedge_G S^{\mathfrak{g}}$.*

Proof. Consider the sequence of equalities:

$$DX = \mathrm{Map}\big(K(A)_+ \wedge_G S^0, K(A)_+\big)^{hK(A)} = \mathrm{Map}\big(S^0, K(A)_+\big)^{hG} = K(A)_+^{hG}.$$

Now recall [20], that there is a $G \times G$-equivalence of spectra: $\mathrm{Map}(G_+, S^{\mathfrak{g}}) \simeq G_+$. Since G_+ is a compact Lie group, and $K(A)$ is a free G-space, we get an $K(A) \times G$-equivalence:

$$\mathrm{Map}\big(G_+, K(A)_+ \wedge_G S^{\mathfrak{g}}\big) = K(A)_+ \wedge_G \mathrm{Map}\big(G_+, S^{\mathfrak{g}}\big) \simeq K(A)_+.$$

Taking homotopy fixed points, we get the required $K(A)$-equivalence:

$$DX = K(A)_+^{hG} \simeq \mathrm{Map}\big(G_+, K(A)_+ \wedge_G S^{\mathfrak{g}}\big)^{hG} \simeq K(A)_+ \wedge_G S^{\mathfrak{g}},$$

which completes the proof. $\square$

Remark 9.4. For loop groups, there are geometrically meaningful definitions of K-homology. In [24], Meinrenken gives a description of the twisted G-equivariant K-homology of G in terms of K-theory with coefficients in the equivariant Dixmier–Douady bundle of algebras over G. This bundle is constructed from the space of bounded operators on a Hilbert space. For a general Kac–Moody group $K(A)$, our algebraic definition of Dominant K-homology above suggests a geometric description in exactly the same way: one simply replaces the Dixmier–Douady bundle of algebras by the $K(A)$-equivariant bundle of algebras over the $X(A)$, whose fiber is given by the bounded operators on the maximal Dominant representation $\mathcal{H}$.

Now recall that $X(A)$ was a finite homotopy colimit of proper homogeneous spaces of the form $K(A)/K_J(A)$. Therefore, we may write the equivariant dual of $X(A)$ as the following homotopy inverse limit in the category of proper $K(A)$-CW spectra:

$$DX(A) = \text{holim}_{J \in \mathcal{S}(A)} K(A)_+ \wedge_{K_J(A)} S^{\mathfrak{g}(J)},$$

where $\mathfrak{g}(J)$ denotes the Lie algebra of $K_J(A)$. This is a finite $K(A)$-CW spectrum, and its Dominant K-cohomology is defined as the Dominant K-homology of $X(A)$. We have:

Theorem 9.5. *Let $K(A)$ be a Kac–Moody group of compact type, where A is a generalized Cartan matrix of size $n + 1$. Assume that $K(A)$ is not of finite type, and let r be the rank of $K(A)$. Define the reduced Dominant K-homology $\tilde{\mathbb{K}}_*^{K(A)}(X(A))$, to be the kernel of the pinch map π. Then we have an isomorphism of graded groups:*

$$\tilde{\mathbb{K}}_*^{K(A)}(X(A)) = R_T^{\varnothing}[\beta^{\pm 1}],$$

where $R_T^{\varnothing}$ is graded so as to belong entirely in degree $n - r$, and β is the Bott class in degree 2. Moreover, the identification of $\tilde{\mathbb{K}}_{n-r}^{K(A)}(X(A))$ with $R_T^{\varnothing}$ is induced by the inclusion of any orbit of the form $K(A)/T$.

Proof. The description of the equivariant dual of $X(A)$ as a homotopy inverse limit, yields a spectral sequence:

$$E_2^{p,*} = \varinjlim {}^p \mathbb{K}_{K(A)}^*\big(K(A)_+ \wedge_{K_\bullet(A)} S^{\mathfrak{g}(\bullet)}\big) = \varinjlim {}^p \mathbb{K}_{K_\bullet(A)}^*\big(S^{\mathfrak{g}(\bullet)}\big) \Rightarrow \mathbb{K}_{p-*}^{K(A)}\big(X(A)\big).$$

To simplify the notation, we will use the Thom isomorphism Theorem 4.7 to identify the (co-variant) functor $J \mapsto {}^A \mathbb{K}_{K_J(A)}^r(S^{\mathfrak{g}(J)})$ with $\text{DR}_\bullet$. Hence, we may write our spectral sequence as

$$E_2^{p,*} = \varinjlim {}^p \text{DR}_\bullet[\beta^{\pm}] \Rightarrow \mathbb{K}_{p-r}^{K(A)}\big(X(A)\big)[\beta^{\pm 1}].$$

The strategy in proving the above theorem is not new. We decompose the functor $\text{DR}_\bullet$ into summands $\tilde{\text{DR}}_\bullet^K$ indexed over $K \subseteq I$, and then consider two separate cases.

Recall the decomposition of the ring DR_T indexed by subsets $K \subseteq I$: $\text{DR}_T = \bigoplus \text{DR}_T^K$. This induces a decomposition of the functor $\text{DR}_\bullet$ using the push forward map 4.8:

$$\text{DR}_\bullet = \bigoplus \tilde{\text{DR}}_\bullet^K, \quad \text{where } \tilde{\text{DR}}_J^K = \iota_J\big(\text{DR}_T^K\big).$$

We now proceed as before by considering two cases:

Case $K = I$. It is clear that the functor $\tilde{D}R^I_\bullet$ takes the value zero on all nonempty proper subsets J, and that $\tilde{D}R^I_\varnothing = R^I_T$. Hence we get

$$\varinjlim {}^p\tilde{D}R^I_\bullet = \begin{cases} 0, & \text{if } p \neq n, \\ R^I_T, & \text{if } p = n. \end{cases}$$

Since R^I_T represents the highest filtration in the spectral sequence, it admits a map to DR_T via π. At the end of this argument, we shall show that the elements R^I_T are permanent cycles representing elements in $\mathbb{K}^{K(A)}_{n-r}(X(A))$.

Case $K \subset I$. To calculate higher derived colimits of the remaining summands: $\tilde{D}R^K_\bullet$, we begin by first splitting the functor $\tilde{D}R^K_\bullet$ out of another functor $E^K_\bullet$, where $K \subset I$ is any proper subset. The functor $E^K_\bullet$ is defined by

$$J \longmapsto E^K_J = R^K_T \otimes \tilde{\mathbb{Z}} \otimes_{\mathbb{Z}[W_K(A)]} \mathbb{Z}\big[W(A)/W_J(A)\big],$$

where $\tilde{\mathbb{Z}}$ stands for the sign representation of $W_K(A)$. One has a map from the functor $\tilde{D}R^K_\bullet$ to $E^K_\bullet$ defined as follows.

Let L_μ be an irreducible representation in $\tilde{D}R^K_J$ with highest weight μ. Assume that we have the equality: $\mu + \rho_J = w^{-1}(\tau)$, for some $\tau \in D_K$, and $w \in W(A)$. Define a map

$$\tilde{D}R^K_J \longrightarrow E^K_J, \quad L_\mu \longmapsto (-1)^w e^\tau \otimes 1 \otimes w,$$

where $(-1)^w$ denotes the sign of the element w, and e^τ stands for a character in R^K_T. It is easy to see that this map well defined and functorial. Moreover, one may check that the retraction is given by

$$E^K_J \longrightarrow \tilde{D}R^K_J, \quad e^\tau \otimes 1 \otimes w \longmapsto (-1)^w \iota_J\big(w^{-1}e^\tau\big).$$

Hence, $\varinjlim {}^p\tilde{D}R^K_\bullet$ splits from the homology of the chain complex $R^K_T \otimes \tilde{\mathbb{Z}} \otimes_{\mathbb{Z}[W_K(A)]} C_*$, where C_* is the simplicial chain complex of the Davis complex Σ. Since $W_K(A)$ is a finite subgroup, this simplicial complex is $W_K(A)$-equivariantly contractible. Thus C_* is $W_K(A)$-equivariantly equivalent to the constant complex $\mathbb{Z}$ in dimension zero.

It follows that $\varinjlim {}^p\tilde{D}R^K_\bullet = 0$ if $p > 0$, and $\varinjlim \tilde{D}R^K_\bullet$ splits from the group

$$R^K_T \otimes \tilde{\mathbb{Z}} \otimes_{\mathbb{Z}[W_K(A)]} \mathbb{Z}.$$

Furthermore, if K is nonempty, then notice that the group $R^K_T \otimes \tilde{\mathbb{Z}} \otimes_{\mathbb{Z}[W_K(A)]} \mathbb{Z}$ is two torsion, hence the map to $\tilde{D}R^K_\bullet$ is trivial. This shows that $\varinjlim \tilde{D}R^K_\bullet = 0$ if K is not the empty set. Finally, for $K = \varnothing$, one simply observes that the above retraction is an isomorphism, and hence $\varinjlim \tilde{D}R^\varnothing_\bullet = R^\varnothing_T$.

To complete the proof, it remains to show that all elements in R_T^I discussed in the first case are permanent cycles. Unfortunately, we cannot invoke a geometric argument as before. Instead we consider the $K(A)$-equivariant action map

$$\varphi : K(A) \times_{N(T)} \Sigma \longrightarrow X(A),$$

where $N(T)$ stands for the normalizer of T in $K(A)$. To show that R_T^I consists of permanent cycles, it is sufficient to show that it belongs to the image of φ_*:

$$\varphi_* : {}^A\mathbb{K}_{n-r}^{N(T)}(\Sigma) = \mathbb{K}_{n-r}^{K(A)}\big(K(A) \times_{N(T)} \Sigma\big) \longrightarrow \mathbb{K}_{n-r}^{K(A)}\big(X(A)\big).$$

Consider now the $N(T)$-equivariant pinch map $\pi : \Sigma \longrightarrow W(A)_+ \wedge S^n$. Taking the $N(T)$-equivariant dual of π yields $D\pi : W(A)_+ \wedge S^{r-n} \longrightarrow D\Sigma$ where r is the rank of the maximal torus T. We have a description of $D\Sigma$ as a homotopy limit in the category of proper $N(T)$-CW spectra:

$$D\Sigma = \mathrm{holim}_{J \in \mathcal{S}(A)} W(A)_+ \wedge_{W_J(A)} S^{\mathfrak{h}_{\mathbb{R}}}.$$

Taking $N(T)$ orbits of $D\Sigma$ yields a non-equivariant CW spectrum:

$$S^0 \wedge_{N(T)} D\Sigma = \mathrm{holim}_{J \in \mathcal{S}(A)} S^0 \wedge_{W_J(A)} S^{\mathfrak{h}_{\mathbb{R}}}.$$

Notice that the orbit space $\mathfrak{h}_{\mathbb{R}}/W_J(A)$ can be identified with a cone within $\mathfrak{h}_{\mathbb{R}}$, and hence, the spectra $S^0 \wedge_{W_J(A)} S^{\mathfrak{h}_{\mathbb{R}}}$ are contractible for all $J \neq \varnothing$. It is now easy to see that the dual pinch map induces an equivalence of non-equivariant spectra:

$$D\pi : S^{r-n} = S^0 \wedge_{N(T)} W(A)_+ \wedge S^{r-n} \longrightarrow S^0 \wedge_{N(T)} D\Sigma.$$

Now given any weight $\mu \in D_I$, let L_μ be the one dimension highest weight representation corresponding to μ. These represent canonical elements of ${}^A\mathbb{K}_T^{r-n}(S^{r-n})$ which extend to corresponding elements in ${}^A\mathbb{K}_{N(T)}^{r-n}(D\Sigma)$ using the above equivalence.

This shows that elements of R_T^I lift to elements in ${}^A\mathbb{K}_{n-r}^{N(T)}(\Sigma)$, thus completing the proof. $\quad\square$

Acknowledgments

Appendix A. A compatible family of metrics

Given an indecomposable, symmetrizable generalized Cartan matrix A, let us fix a non-degenerate, invariant, Hermitian form B on the Lie algebra $\mathfrak{g}(A)$ [21]. This form, however, is not positive definite in general. In this section we will explore the ways in which one can perturb B into positive definite bilinear forms in an equivariantly compatible way (to be made precise below).

Let $J \in \mathcal{S}(A)$ be any subset, and let $K_J(A)$ be the corresponding compact subgroup of $K(A)$. We may decompose the Lie algebra $\mathfrak{g}(A)$ as

$$\mathfrak{g}(A) = \mathfrak{z}_J \oplus \mathfrak{h}_J \oplus \eta_- \oplus \eta_+,$$

where $\mathfrak{h}_J$ is the subspace of co-roots generated by h_j for $j \in J$, and $\mathfrak{z}_J$ is the centralizer of $K_J(A)$ given by

$$\mathfrak{z}_J = \{ h \in \mathfrak{h} \mid \alpha_j(h) = 0, \ \forall j \in J \}.$$

It is easy to see that B restricts to a positive form on the $K_J(A)$-invariant subspace $\mathfrak{g}_J$:

$$\mathfrak{g}_J = \mathfrak{h}_J \oplus \eta_- \oplus \eta_+.$$

Notice also that the action of $K_J(A)$ on $\mathfrak{z}_J$ is trivial, and that $\mathfrak{z}_J$ is orthogonal to $\mathfrak{g}_J$. Let B(J) denote the restriction of B to $\mathfrak{g}_J$. From the above observation, it is clear that extending B(J) to a $K_J(A)$-invariant Hermitian inner-product on $\mathfrak{g}(A)$, is equivalent to constructing a (non-equivariant) Hermitian inner-product on $\mathfrak{z}_J$. Let us therefore define the space of extensions.

Definition A.1. Let **Met**(J) denote the space of $K_J(A)$-invariant Hermitian inner-products on $\mathfrak{g}(A)$, extending B(J). Hence, **Met**(J) is equivalent to the space of Hermitian inner-products on $\mathfrak{z}_J$, and in particular, it is contractible.

Notice that there is an obvious restriction map from **Met**(K) to **Met**(J) if $J \subseteq K$. Thus we obtain an inverse system of spaces, **Met**($\bullet$), on the poset $\mathcal{S}(A)$.

Let **Met** be the homotopy inverse limit of this system [2]. In simple terms, **Met** is nothing other than the space of Hermitian inner-products on $\mathfrak{g}(A)$, which are equivariantly parametrized over the building $X(A)$, and such that the metric over any point in the fundamental domain in $X(A)$ belongs to some space **Met**(J).

One may set up a convergent spectral sequence to compute the homotopy of the space **Met** [2]. The E_2-term of this spectral sequence calculates the derived functors of inverse limit of the system $\pi_*(\mathbf{Met}(J))$. Since **Met**(J) is contractible for all J, it follows from the spectral sequence that **Met** is weakly contractible (and so non-empty). We conclude:

Claim A.2. *The topological space consisting of families of Hermitian inner-products on $\mathfrak{g}(A)$, equivariantly parametrized over $X(A)$ and extending B, is weakly contractible. In particular, it is non-empty.*

Appendix B. Dominant K-theory for the non-symmetrizable case

Let A be a non-symmetrizable generalized Cartan matrix. Let $K(A)$ be the corresponding Kac–Moody group. In this section we will give a description of Dominant K-theory for $K(A)$. The main obstruction to extending our definition is the fact that a highest weight representation of $K(A)$ need not be unitary. These representations however are still well defined, and we may construct the vector space $\check{\mathcal{H}}$ given by the direct sum of each irreducible highest weight representation of $K(A)$ repeated countably many times.

Definition B.1. Let $Met(\check{\mathcal{H}})$ be the space of all Hermitian inner products on $\check{\mathcal{H}}$. Define $\tilde{X}(A)$ to be the space

$$\tilde{X}(A) = \text{hocolim}_{J \in \mathcal{S}(A)} \, K(A) \times_{K_J(A)} Met(\check{\mathcal{H}}).$$

Then $\tilde{X}(A)$ is a $K(A)$-equivariant fiber bundle, denoted by ξ, over $X(A)$, and with fiber $Met(\check{\mathcal{H}})$. Let $s(\xi)$ be an equivariant section of this bundle.

Notice that the space of equivariant sections of ξ is a contractible space given by the homotopy inverse limit of the functor defined on $\mathcal{S}(A)$ given by $J \mapsto Met(\check{\mathcal{H}})^{K_J(A)}$.

It is easy to now construct a $K(A)$-equivariant bundle over $X(A)$ with fiber over a point $x \in X(A)$ being the space $\mathcal{F}(\mathcal{H})_x$ of Fredholm operators on the Hilbert completion $\mathcal{H}$ of $\check{\mathcal{H}}$, with respect to the Hermitian inner product $s(\xi)$ evaluated at x. Let $\mathcal{F}(A)_s$ denote this bundle, and let $\Omega\mathcal{F}(A)_s$ denote the bundle with fiber at $x \in X(A)$ being the loop space $\Omega\mathcal{F}(\mathcal{H})_x$.

Definition B.2. Given a proper $K(A)$-space X, let $f : X \to X(A)$ be the classifying map. Define the Dominant K-theory of X to be:

$$\mathbb{K}_{K(A)}^{2k}(X) = \pi_0\big(\Gamma f^* \mathcal{F}(\mathcal{H})_s\big), \quad \text{and} \quad \mathbb{K}_{K(A)}^{2k+1}(X) = \pi_0\big(\Gamma f^* \Omega\mathcal{F}(\mathcal{H})_s\big),$$

where Γ denotes the space of equivariant sections of the relevant bundle. It is easy to see that the definition is well defined and independent of the choice of section $s(\xi)$.

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