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Extreme differences between weakly open subsets and convex combinations of slices in Banach spaces ☆

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We show that every Banach space containing isomorphic copies of c_0 can be equivalently renormed so that every nonempty relatively weakly open subset of its unit ball has diameter 2 and, however, its unit ball still contains convex combinations of slices with diameter arbitrarily small, which improves in an optimal way the known results about the size of this kind of subsets in Banach spaces.

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1. Introduction

The study of the size of slices, relatively weakly open subsets or convex combinations of slices in the unit ball of a Banach space is a relatively recent topic which has received intensive attention in the last years. For example, in [14] it is proved that the unit ball of every uniform algebra has all its slices with diameter 2 and in [6] it is shown that the unit ball of every non-hilbertizable real JB^*-triple has all its relatively weakly open subsets with diameter 2. Many other results in this direction have appeared [5,2,13] giving new geometrical properties in Banach spaces, extremely opposite to the well-known Radon–Nikodym property. See also [1]. We pass now to present these properties joint to its w^*-versions.

Given a Banach space X, X is said to have the slice diameter 2 property (slice-D2P) if every slice in the unit ball of X has diameter 2. If every nonempty relatively weakly open subset, respectively every convex combinations of slices, of the unit ball of X has diameter 2, we say that X has the diameter 2 property (D2P), respectively the strong diameter 2 property (strong-D2P). Also we define the weak-star versions of the above properties, the w^*-slice-S2P, w^*-D2P and w^*-strong-D2P property, respectively, asking for the above conditions for w^*-slices, nonempty relatively w^*-weakly open subsets and convex combinations of w^*-slices of B_{X^*}, respectively.

It is clear that (w^*)-strong-D2P ⇒ (w^*)-D2P ⇒ (w^*)-slice-D2P. In [7], examples of Banach spaces X are exhibited satisfying the slice-D2P and failing in an extreme way the D2P, in the sense that there are nonempty relatively weakly open subsets in the unit ball with arbitrarily small diameter. Then the biduals of these spaces, X^{**}, are examples of dual Banach spaces satisfying the w^*-slice-D2P such that its unit ball contains nonempty relatively weak-star open subsets with diameter arbitrarily small.

On the other hand there is a Banach space X such that X^* satisfies the w^*-strong-D2P, but its unit ball contains convex combinations of slices with diameter arbitrarily small. Indeed, take $X = C([0,1])$, the classical Banach space of continuous functions on $[0,1]$ with the sup norm. Now, it is known that $X^* = L_1[0,1] \oplus_1 Z$, for some subspace Z of X^* with RNP [4]. Then the unit ball of Z contains slices with arbitrarily small diameter and so, X^* also contains slices with arbitrarily small diameter. On the other hand, X has Daugavet property, which implies that X^* has w^*-strong-D2P [8, Lemma 2.3]. Observe that now we have trivially that X^* has the w^*-slice-D2P and its unit ball contains slices with diameter arbitrarily small and also X^* has w^*-D2P and its unit ball contains nonempty relatively weakly open subsets with diameter arbitrarily small. Then the general situation is shown in the following diagram

$$\begin{tikzcd}
\text{Strong-D2P} \arrow[r, Rightarrow, "(1)"] \arrow[d, Rightarrow] & \text{D2P} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{slice-D2P} \arrow[d, Rightarrow] \\
w^*\text{-Strong-D2P} \arrow[r, Rightarrow, "(2)"] & w^*\text{-D2P} \arrow[r, Rightarrow] & w^*\text{-slice-D2P}
\end{tikzcd}$$

Following the above comments, we observe that all converse implications, unless (1) and (2) are false in an extreme way, that is, one can get diameter 2 for one of the properties in every above pair and diameter arbitrarily small in the other one.

The aim of this note is to prove that (w^*)-D2P and (w^*)-strong-D2P are also extremely different in the above sense, and so the converse implications (1) and (2) in the above diagram are again false in an extreme way. Indeed, we show in Theorem 2.5 that there are Banach spaces X with D2P such that its unit ball contains convex combinations of slices with diameter arbitrarily small. In fact every Banach space X containing isomorphic copies of c_0 works. Then X^{**} will be an example of the extreme difference between w^*-D2P and w^*-strong-D2P. Note that in [2], it is proved that $c_0 \oplus_2 c_0$ is a Banach space with D2P and failing the strong-D2P, but as we will see in Proposition 2.1 every convex combination of slices in the unit ball of $c_0 \oplus_p c_0$ has diameter, at least, 1 for every $p \geq 1$.

We pass now to introduce some notation. For a Banach space X, X^* denotes the topological dual of X, B_X and S_X stand for the closed unit ball and unit sphere of X, respectively, and w, respectively w^*, denotes the weak and weak-star topology in X, respectively X^*. $[A]$ stands for the closed linear span of the subset A of X. We consider only real Banach spaces. A slice of a set C in X is a set of X given by

$$S = \big\{x \in C : x^*(x) > \sup x^*(C) - \alpha\big\}$$

where $x^* \in X^*$ and $0 < \alpha$. A w^*-slice of a set C of X^* is a slice of C determined by elements of X, seen in X^{**}.

Recall that a slice of B_X is a nonempty relatively weakly open subset of B_X and the family

$$\big\{\big\{x \in B_X : \big|x_i^*(x - x_0)\big| < \varepsilon,\ 1 \leq i \leq n\big\} : n \in \mathbb{N},\ x_1^*, \cdots, x_n^* \in X^*\big\}$$

is a basis of relatively weakly open neighborhoods of $x_0 \in B_X$. So every relatively weakly open subset of B_X has nonempty intersection with S_X, whenever X has infinite dimension.

Finally recall some connections between diameter 2 properties and another well-known geometrical properties in Banach spaces. Given a Banach space X, X is said to have the Daugavet property if the equality $\|I + T\| = 1 + \|T\|$ holds for every finite rank operator T on X, where I denotes the identity operator on X. The norm of X is said to be octahedral if for every finite-dimensional subspace F of X and for every $\varepsilon > 0$ there is $x \in S_X$ satisfying

$$\|y + \alpha x\| \geq (1 - \varepsilon)\big(\|y\| + |\alpha|\big) \quad \forall (y \in F,\ \alpha \in \mathbb{R}).$$

The norm of X is called extremely rough if

$$\limsup_{\|h\|\to 0} \frac{\|u+h\|+\|u-h\|-2}{\|h\|} = 2$$

for every $u \in S_X$.

The Daugavet property implies the strong-D2P [16], the dual of a Banach space with octahedral norm satisfies the w^*-strong-D2P (see [10]) and the dual (or predual, if it exists) of a Banach space with D2P has an extremely rough norm [10, Proposition I.1.11].

2. Main results

The following proposition shows that the space $c_0 \oplus_p c_0$, which has slice-D2P and fails the strong D2P [2], is far to satisfy that its unit ball contains convex combination of slices with arbitrarily small diameter.

Proposition 2.1. *If $p \geq 1$, every convex combination of slices in $B_{c_0 \oplus_p c_0}$ has diameter at least 1.*

Proof. Put $X = c_0 \oplus_p c_0$ and consider $\sum_{i=1}^n \lambda_i S(B_X, (x_i^*, y_i^*), \alpha_i)$ a convex combination of slices in B_X, where $n \in \mathbb{N}$, $0 < \alpha_i < 1$ for every i, $(x_i^*, y_i^*) \in S_{X^*}$ and $\lambda_i > 0$ for every i with $\sum_{i=1}^n \lambda_i = 1$. If $\alpha = \min_i \alpha_i$, then $S_i \subset S(B_X, (x_i^*, y_i^*), \alpha_i)$, where $S_i = S(B_X, (x_i^*, y_i^*), \alpha)$ for every i. Now, given $\varepsilon > 0$ arbitrary, for every $1 \leq i \leq n$ we choose $(x_i, y_i) \in S_i$ such that $\|(x_i, y_i)\|_X > 1 - \varepsilon$ with $A_i := supp(x_i)$ finite and $B_i := supp(y_i)$ finite, where $supp(z) = \{n \in \mathbb{N} : z(n) \neq 0\}$ for every $z \in c_0$. Pick $k_0 \geq \max \bigcup_{i=1}^n A_i \cup \bigcup_{i=1}^n B_i$ and $k > k_0$ such that $x_i \pm \|x_i\|_\infty e_k$, $y_i \pm \|y_i\|_\infty e_k \in S_i$ for every i. From here we have that

$$\begin{aligned} diam\left(\sum_{i=1}^n \lambda_i S\big(B_X, (x_i^*, y_i^*), \alpha_i\big)\right) &\geq diam\left(\sum_{i=1}^n \lambda_i S_i\right) \\ &\geq 2\left\|\sum_{i=1}^n \lambda_i\big(\|x_i\|_\infty e_k, \|y_i\|_\infty e_k\big)\right\|. \end{aligned}$$

As $\|x_i\|_\infty^p + \|y_i\|_\infty^p > 1 - \varepsilon$ one has that for every i either $\|x_i\|_\infty \geq (\frac{1-\varepsilon}{2})^{1/p}$ or $\|y_i\|_\infty \geq (\frac{1-\varepsilon}{2})^{1/p}$. Put $I = \{i : \|x_i\|_\infty \geq (\frac{1-\varepsilon}{2})^{1/p}\}$ and $t = \sum_{i \in I} \lambda_i$ ($t = 0$ if $I = \emptyset$). Then $t \in [0,1]$ and $1 - t = \sum_{i \notin I} \lambda_i$. Now we have that

$$\begin{aligned} diam\left(\sum_{i=1}^n \lambda_i S\big(B_X, (x_i^*, y_i^*), \alpha_i\big)\right) &\geq diam\left(\sum_{i=1}^n \lambda_i S_i\right) \\ &\geq 2\left\|\sum_{i=1}^n \lambda_i\big(\|x_i\|_\infty e_k, \|y_i\|_\infty e_k\big)\right\| \end{aligned}$$

$$\begin{aligned}
&\geq 2\left(\left(\frac{t(1-\varepsilon)^{1/p}}{2^{1/p}}\right)^p + \left(\frac{(1-t)(1-\varepsilon)^{1/p}}{2^{1/p}}\right)^p\right)^{1/p} \\
&= \frac{2(1-\varepsilon)^{1/p}}{2^{1/p}}\left(t^p + (1-t)^p\right)^{1/p} \\
&\geq \frac{2(1-\varepsilon)^{1/p}}{2^{1/p}}\left(\frac{1}{2^p} + \frac{1}{2^p}\right)^{1/p} = (1-\varepsilon)^{1/p}.
\end{aligned}$$

Since ε is arbitrary we get that $diam(\sum_{i=1}^n \lambda_i S(B_X, (x_i^*, y_i^*), \alpha_i)) \geq 1$ and we are done. $\square$

Our first goal in order constructing a Banach space with D2P so that its unit ball contains convex combinations of slices with diameter arbitrarily small should be find out a closed, bounded and absolutely convex subset with diameter 2 so that every nonempty relatively weakly open subset has diameter 2 and containing convex combinations of slices with diameter arbitrarily small. We pass now to describe a family of closed, bounded and convex subsets in c_0 with diameter 1 satisfying that every nonempty relatively weakly open subset has diameter 1 and containing convex combinations of slices with diameter arbitrarily small.

Pick $\{\varepsilon_n\}$ a nonincreasing null scalars sequence. We construct an increasing sequence of closed, bounded and convex subsets $\{K_n\}$ in c_0 and a sequence $\{g_n\}$ in c_0 as follows: Let $K_1 = \{e_1\}$, $g_1 = e_1$ and $K_2 = co(e_1, e_1 + e_2)$. Choose $l_2 > 1$ and $g_2, \ldots, g_{l_2} \in K_2$ an ε_2-net in K_2. Assume that $n \geq 2$ and m_n, l_n, K_n and $\{g_1, \ldots, g_{l_n}\}$ have been constructed, with $K_n \subset B_{[e_1, \ldots, e_{m_n}]}$ and $g_i \in K_n$ for every $1 \leq i \leq l_n$. Define K_{n+1} as

$$K_{n+1} = co\big(K_n \cup \{g_i + e_{m_n+i} : 1 \leq i \leq l_n\}\big).$$

Let $l_{n+1} = m_n + l_n$ and choose $\{g_{l_n+1}, \ldots, g_{l_{n+1}}\} \subset K_{n+1}$ so that $\{g_1, \ldots, g_{l_{n+1}}\}$ is an ε_{n+1}-net in K_{n+1}. Finally we define $K_0 = \overline{\bigcup_n K_n}$. Then it follows that K_0 is a nonempty closed, bounded and convex subset of c_0 such that $x(n) \geq 0$ for every $n \in \mathbb{N}$ and $\|x\|_\infty = 1$ for every $x \in K_0$ and so $diam(K_0) \leq 1$.

Now, if i is fixed, we have from the construction that $\{g_i + e_{m_n+i}\}_n$ is a sequence in K_0 weakly convergent to g_i and $\|(g_i - e_{m_n+i}) - g_i\| = \|e_{m_n+i}\| = 1$ for every n. Then $diam(K_0) = 1$. We will use freely below the subset K_0 and the above construction. Observe that, from the above construction, one has that

$$K_0 = \overline{\{g_i : i \in \mathbb{N}\}}^w = \overline{\{g_i : i \in \mathbb{N}\}}.$$

Mention that the construction of K_0 follows word for word the definition of Poulsen simplex in ℓ_2 [15], that is, the unique, unless homeomorphism, Choquet simplex with a dense subset of extreme points [12]. In fact, it is known [3] that the weak-star closure of K_0 in ℓ_∞ is affinely weak-star homeomorphic to the Poulsen simplex. However K_0 is not a Choquet simplex, because it is not weakly compact, K_0 is a simplex in a more general definition than Choquet simplex.

Let us see that K_0 satisfies the requirements we are looking for.

Proposition 2.2. *K_0 is a closed, bounded and convex subset of c_0 with $diam(K_0) = 1$ satisfying that every nonempty relatively weakly open subset of K_0 has diameter 1 and K_0 contains convex combinations of slices with diameter arbitrarily small.*

Proof. The fact that K_0 is a closed, bounded and convex subset of c_0 with $diam(K_0) = 1$ has been proved after the construction of K_0. From [3, Theorem 1.2], we deduce that K_0 has convex combinations of slices with diameter arbitrarily small. Now pick U a nonempty relatively weakly open subset of K_0. From the construction of K_0 we noted that $K_0 = \overline{\{g_i : i \in \mathbb{N}\}}^w$ and so there is $i \in \mathbb{N}$ such that $g_i \in U$. Now, again from the construction of K_0, $g_i + e_{m_n+i} \in K_0$ for every n. Thus, $g_i + e_{m_n+i} \in U$ for every n greater than some n_0, since $\{g_i + e_{m_n+i}\}_n$ is weakly convergent to g_i. Therefore, $diam(U) \geq \|e_{m_n+i}\| = 1$. $\square$

Our next goal should be to get from K_0 a closed, absolutely convex, bounded subset with diameter 2, containing convex combinations of slices with diameter arbitrarily small and so that every nonempty relatively weakly open subset has diameter 2. For this, we see K_0 as a subset of c, the space of scalars convergent sequence with the sup norm and define

$$K = 2\overline{co}\left(\left(K_0 - \frac{\mathbf{1}}{2}\right) \cup \left(-K_0 + \frac{\mathbf{1}}{2}\right)\right),$$

where $\mathbf{1}$ is the sequence of c with every coordinate equal 1. Now, it is clear that K is a closed, absolutely convex and bounded subset of c with $diam(K) = 2$.

Our next point is constructing a Banach space with D2P and so that its unit ball contains convex combinations of slices with diameter arbitrarily small. It is natural to think that this Banach space is some renorming of c, which would be in fact a renorming of c_0. For this we need the following lemmas.

Lemma 2.3. *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm $|||\cdot|||$ in X satisfying that $(X, |||\cdot|||)$ contains an isometric copy of c and for every $x \in B_{(X,|||\cdot|||)}$ there are sequences $\{x_n\}, \{y_n\} \in B_{(X,|||\cdot|||)}$ weakly convergent to x such that $|||x_n - y_n||| = 2$ for every $n \in \mathbb{N}$. In fact, $x_n = x + (1 - \alpha_n)e_n$ and $y_n = x - (1 + \alpha_n)e_n$ for some scalar sequence $\{\alpha_n\}$ with $|\alpha_n| \leq 1$ for every n.*

Proof. As X contains isomorphic copies of c, we can assume that c is, in fact, an isometric subspace of X. Then for every Y separable subspace of X containing c, there is a linear and continuous projection $P_Y : Y \longrightarrow c$ with $\|P_Y\| \leq 8$. Indeed, let us consider the onto linear isomorphism $T : c \longrightarrow c_0$ given by $T(x)(1) = \frac{1}{2}\lim_n x(n)$ and $T(x)(n) = \frac{1}{2}(x(n) - \lim_n x(n))$ for every $n > 1$. Note that $\|T\| = 1$ and $\|T^{-1}\| = 4$. On the other hand, by Sobczyk's Theorem, there exists a linear projection $\pi : Y \to c_0$ such that $\|\pi\| \leq 2$. Now $P_Y = T^{-1} \circ \pi$ satisfies $\|P_Y\| \leq 8$ and is the required projection from Y onto c.

Let Υ be the family of subspaces Y of X containing c such that c has finite codimension in Y. Consider the filter basis Υ given by $\{Y \in \Upsilon : Y_0 \subset Y\}$, where $Y_0 \in \Upsilon$ and call $\mathcal{U}$ the ultrafilter containing the generated filter by the above filter basis.

For every $Y \in \Upsilon$, we define a new norm in X given by

$$\|x\|_Y := \max\big\{\big\|P_Y(x)\big\|, \big\|x - P_Y(x)\big\|\big\}.$$

Finally, we define the norm on X given by $|||x||| := \lim_{\mathcal{U}} \|x\|_Y$. Observe that $\frac{1}{8}\|x\| \leq |||x||| \leq 3\|x\|$ for every $x \in X$ and so $|||\cdot|||$ is an equivalent norm in X such that $|||x||| = \|x\|_\infty$ for every $x \in c$, where $\|\cdot\|_\infty$ is the sup norm in c. Hence $(X, |||\cdot|||)$ contains an isometric copy of c.

Pick $x_0 \in B_{(X,|||\cdot|||)}$. In order to prove the remaining statement let $\{e_n\}$ and $\{e_n^*\}$ be the usual basis of c_0 and the biorthogonal functionals sequence, respectively.

Choose $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. For every $Y \in \Upsilon$ with $x_0 \in Y$ we have that

$$\begin{aligned}\|x_0 + \lambda e_n\|_Y &= \max\big\{\big\|P_Y(x_0) + \lambda e_n\big\|, \big\|x_0 - P_Y(x_0)\big\|\big\} \\ &= \max\big\{\big|\lambda + e_n^*\big(P_Y(x_0)\big)\big|, \big\|P_Y(x_0) - e_n^*\big(P_Y(x_0)\big)e_n\big\|, \big\|x_0 - P_Y(x_0)\big\|\big\}.\end{aligned}$$

Define $\beta_n = \lim_{\mathcal{U}} \max\{\|P_Y(x_0) - e_n^*(P_Y(x_0))e_n\|, \|x_0 - P_Y(x_0)\|\}$ and $\alpha_n = \lim_{\mathcal{U}} e_n^*(P_Y(x_0))$. Then $|||x_0 + \lambda e_n||| = \max\{|\lambda + \alpha_n|, \beta_n\}$. Note that $|\alpha_n| \leq 1$ and $\beta_n \leq 1$ since $|||x_0||| \leq 1$.

Doing $x_n := x_0 + (1 - \alpha_n)e_n$ and $y_n := x_0 - (1 + \alpha_n)e_n$ for every n, we get that $x_n, y_n \in B_{(X,|||\cdot|||)}$. Finally, it is clear that $\{x_n\}$ and $\{y_n\}$ are weakly convergent sequences to x_0 and $|||x_n - y_n||| = 2$ for every $n \in \mathbb{N}$. □

Lemma 2.4. *Let X be a vector space and A, B convex subsets of X such that $\frac{A-A}{2} \subset B$. Then*

$$co(A \cup -A \cup B) = co(A \cup B) \cup co(-A \cup B).$$

Proof. It is enough to prove that

$$co(A \cup -A \cup B) \subset co(A \cup B) \cup co(-A \cup B).$$

For this, take $x \in co(A \cup -A \cup B)$. As A and B are convex subsets we get that $x = \lambda_1 a_1 + \lambda_2(-a_2) + \lambda_3 b$, where $a_1, a_2 \in A$, $b \in B$ and $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Assuming that $\lambda_1 \geq \lambda_2$, one has that

$$x = (\lambda_1 - \lambda_2)a_1 + 2\lambda_2 \frac{a_1 - a_2}{2} + \lambda_3 b.$$

Then x is a convex combination of elements in $A \cup B$, since from hypotheses $\frac{a_1 - a_2}{2} \in B$, and so $x \in co(A \cup B)$.

If $\lambda_1 \leq \lambda_2$, one has similarly that $x \in co(-A \cup B)$.

In any case, $x \in co(A \cup B) \cup co(-A \cup B)$ and we are done. □

It would be natural to think that some renorming of c_0 gives us our goal space. The following result shows that this is true for every Banach space containing c_0.

Theorem 2.5. *Let X be a Banach space containing isomorphic copies of c_0. Then there is an equivalent norm $|||\cdot|||$ in X such that every nonempty relatively weakly open subset of $B_{(X,|||\cdot|||)}$ has diameter 2 and $B_{(X,|||\cdot|||)}$ contains convex combinations of slices with diameter arbitrarily small.*

Proof. From Lemma 2.3, we can assume that X contains an isometric copy of c and for every $x \in B_X$ there are sequences $\{x_n\}, \{y_n\} \in B_X$ weakly convergent to x such that $\|x_n - y_n\| = 2$ for every $n \in \mathbb{N}$.

Fix $0 < \varepsilon < 1$ and consider in X the equivalent norm $\|\cdot\|_\varepsilon$ whose unit ball is $B_\varepsilon = \overline{co}(2(K_0 - \frac{\mathbf{1}}{2}) \cup 2(-K_0 + \frac{\mathbf{1}}{2}) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$. Then we have $\|x\| \leq \|x\|_\varepsilon \leq \frac{1}{1-\varepsilon}\|x\|$ for every $x \in X$ and $\|x\| = \|x\|_\infty$ for every $x \in c$.

Fix $\gamma > 0$. From Proposition 2.2, there exist $S_1, \cdots, S_n$ slices of K_0 such that

$$\dim\left(\frac{1}{n}\sum_{i=1}^{n} S_i\right) < \frac{1}{4}(1-\varepsilon)\gamma.$$

We can assume that $S_i = \{x \in K : x_i^*(x) > 1 - \widetilde{\delta}\}$ where $x_i^* \in c^*$ and $\sup x_i^*(K_0) = 1$ for every $i = 1, \ldots, n$ and $0 < \widetilde{\delta} < 1$. Denote by $\mathbf{1}$ the sequence in c with all its coordinates equal 1. It is clear that $\sup x_i^*(2(K_0 - \frac{\mathbf{1}}{2})) = 2(1 - x_i^*(\frac{\mathbf{1}}{2}))$, for all $i = 1, \cdots, n$. We put $\rho, \delta > 0$ such that $\frac{1}{2}\rho\|x_i^*\| + \delta < \widetilde{\delta}$, $2\rho < \varepsilon$, $\rho\|x_i^*\| < 4\delta$, and $\frac{(7-2\varepsilon)\rho}{(1-\varepsilon)} < \gamma$, for all $i = 1, \ldots, n$. We consider the relatively weakly open set of B_ε given by

$$U_i := \left\{x \in B_\varepsilon : x_i^*(x) > 2\left(1 - \delta - x_i^*\left(\frac{\mathbf{1}}{2}\right)\right) + \frac{1}{2}\rho\|x_i^*\|,\ \lim_k x(k) < -1 + \rho^2\right\}$$

for every $i = 1, \ldots, n$, where x_i^* and $\lim_n$ denote the Hahn–Banach extensions to X of the corresponding functionals on c. It is clear that $\|x_i^*\|_\varepsilon = \|x_i^*\|$ for every $i = 1, \ldots, n$ and $\|\lim_n\|_\varepsilon = \|\lim_n\| = 1$.

Since $\rho\|x_i^*\| < 4\delta$, we have that $2(1 - x_i^*(\frac{\mathbf{1}}{2})) > 2(1 - \delta - x_i^*(\frac{\mathbf{1}}{2})) + \frac{1}{2}\rho\|x_i^*\|$. Now, we have that $\sup x_i^*(2(K_0 - \frac{\mathbf{1}}{2})) = 2(1 - x_i^*(\frac{\mathbf{1}}{2}))$, then there exists $x \in K_0$ such that $x_i^*(2(x - \frac{\mathbf{1}}{2})) > 2(1 - \delta - x_i^*(\frac{\mathbf{1}}{2})) + \frac{1}{2}\rho\|x_i^*\|$ and $\lim_k 2(x(k) - \frac{1}{2}) = -1 < -1 + \rho^2$. This implies that $U_i \neq \emptyset$ for every $i = 1, \ldots, n$. In order to estimate the diameter of $\frac{1}{n}\sum_{i=1}^{n} U_i$, it is enough to compute the diameter of

$$\frac{1}{n}\sum_{i=1}^{n} U_i \cap co\left(2\left(K_0 - \frac{\mathbf{1}}{2}\right) \cup -2\left(K_0 - \frac{\mathbf{1}}{2}\right) \cup \left[(1-\varepsilon)B_X + \varepsilon B_{c_0}\right]\right).$$

Since $2(K_0 - \mathbf{\frac{1}{2}})$ and $(1-\varepsilon)B_X + \varepsilon B_{c_0}$ are convex subsets of B_ε, given $x \in B_\varepsilon$, we can assume that $x = \lambda_1 2(a - \mathbf{\frac{1}{2}}) + \lambda_2 2(-b + \mathbf{\frac{1}{2}}) + \lambda_3[(1-\varepsilon)x_0 + \varepsilon y_0]$, where $\lambda_i \in [0,1]$ with $\sum_{i=1}^3 \lambda_i = 1$ and $a, b \in K_0$, $x_0 \in B_X$, and $y_0 \in B_{c_0}$.

Given $x, y \in \frac{1}{n}\sum_{i=1}^n U_i$, for $i = 1, \cdots, n$, there exist $a_i, a_i', b_i, b_i' \in K_0$, $\lambda_{(i,j)}, \lambda'_{(i,j)} \in [0,1]$ with $j = 1,2,3$ and, $x_i, x_i' \in B_X$, and $y_i, y_i' \in B_{c_0}$, such that

$$2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right]$$

$$2\lambda'_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda'_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda'_{(i,3)}\left[(1-\varepsilon)x_i' + \varepsilon y_i'\right]$$

belong to U_i and

$$x = \frac{1}{n}\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right]$$

and

$$y = \frac{1}{n}\sum_{i=1}^n 2\lambda'_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda'_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda'_{(i,3)}\left[(1-\varepsilon)x_i' + \varepsilon y_i'\right].$$

For $i = 1, \ldots, n$, we have that

$$2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right] \in U_i,$$

then

$$\lim_k\left(2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right]\right) < -1 + \rho^2.$$

This implies that

$$2\lambda_{(i,2)} + \lambda_{(i,3)}\varepsilon - 1 = -\lambda_{(i,1)} + \lambda_{(i,2)} - \lambda_{(i,3)}(1-\varepsilon) < -1 + \rho^2.$$

Since $2\rho < \varepsilon$, we deduce that $\lambda_{(i,2)} + \lambda_{(i,3)} < \frac{1}{2}\rho$. As a consequence we get that

$$\lambda_{(i,1)} > 1 - \frac{1}{2}\rho, \tag{2.1}$$

and similarly we get that

$$\lambda'_{(i,1)} > 1 - \frac{1}{2}\rho, \tag{2.2}$$

for every $i = 1, \dots, n$. Now, applying (2.1), and (2.2), we have that

$$\begin{aligned}
\|x-y\|_\varepsilon \leq{}& \frac{1}{n}\left\|\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) - 2\lambda'_{(i,1)}\left(a'_i - \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon \\
&+ \frac{1}{n}\sum_{i=1}^n \left\|2\lambda_{(i,2)}\left(-b_i + \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon + \frac{1}{n}\sum_{i=1}^n \left\|2\lambda'_{(i,2)}\left(-b'_i + \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon \\
&+ \frac{1}{n}\sum_{i=1}^n \left\|\lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right]\right\|_\varepsilon + \frac{1}{n}\sum_{i=1}^n \left\|\lambda'_{(i,3)}\left[(1-\varepsilon)x'_i + \varepsilon y'_i\right]\right\|_\varepsilon \\
\leq{}& \frac{1}{n}\left\|\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) - 2\lambda'_{(i,1)}\left(a'_i - \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon \\
&+ \frac{1}{n}\sum_{i=1}^n (\lambda_{(i,2)} + \lambda_{(i,3)}) + \frac{1}{n}\sum_{i=1}^n (\lambda'_{(i,2)} + \lambda'_{(i,3)}) \\
\leq{}& \frac{1}{n}\left\|\sum_{i=1}^n 2\lambda_{(i,1)}\left(a_i - \frac{\mathbf{1}}{2}\right) - 2\lambda'_{(i,1)}\left(a'_i - \frac{\mathbf{1}}{2}\right)\right\|_\varepsilon + \rho \\
\leq{}& \frac{2}{n}\left\|\sum_{i=1}^n \lambda_{(i,1)}a_i - \lambda'_{(i,1)}a'_i\right\|_\varepsilon + \frac{1}{n}\sum_{i=1}^n \left|\lambda_{(i,1)} - \lambda'_{(i,1)}\right| \|\mathbf{1}\|_\varepsilon + \rho \\
\leq{}& \frac{2}{n}\left\|\sum_{i=1}^n \lambda_{(i,1)}a_i - \lambda'_{(i,1)}a'_i\right\|_\varepsilon + \frac{(3-2\varepsilon)}{2(1-\varepsilon)}\rho.
\end{aligned}$$

Now

$$\begin{aligned}
&\left\|\sum_{i=1}^n \lambda_{(i,1)}a_i - \lambda'_{(i,1)}a'_i\right\|_\varepsilon \\
&\quad\leq \left\|\sum_{i=1}^n (\lambda_{(i,1)} - 1)a_i\right\|_\varepsilon + \left\|\sum_{i=1}^n a_i - a'_i\right\|_\varepsilon + \left\|\sum_{i=1}^n (\lambda'_{(i,1)} - 1)a'_i\right\|_\varepsilon \\
&\quad\leq \frac{1}{1-\varepsilon}\left\|\sum_{i=1}^n a_i - a'_i\right\| + \sum_{i=1}^n \frac{1}{1-\varepsilon}\left|\lambda_{(i,1)} - 1\right| \|a_i\| + \sum_{i=1}^n \frac{1}{1-\varepsilon}\left|\lambda'_{(i,1)} - 1\right| \|a'_i\| \\
&\quad\leq \frac{1}{1-\varepsilon}\left\|\sum_{i=1}^n a_i - a'_i\right\| + \frac{1}{1-\varepsilon} n\rho.
\end{aligned}$$

We deduce that

$$\|x-y\|_\varepsilon \leq \frac{2}{1-\varepsilon}\left\|\frac{1}{n}\sum_{i=1}^n a_i - a'_i\right\| + \frac{(7-2\varepsilon)}{2(1-\varepsilon)}\rho. \tag{2.3}$$

On the other hand, we have that, for every $i = 1, \dots, n$,

$$\begin{aligned} &x_i^*\left(2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right]\right) \\ &\quad > 2\left(1 - \delta - x_i^*\left(\mathbf{\frac{1}{2}}\right)\right) + \rho\|x_i^*\|, \end{aligned}$$

then

$$\begin{aligned} &x_i^*\left(2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right)\right) + \frac{1}{2}\rho\|x_i^*\| \\ &\quad \geq x_i^*\left(2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right)\right) + \lambda_{(i,2)}\|x_i^*\|_\varepsilon + \lambda_{(i,3)}\|x_i^*\|_\varepsilon \\ &\quad \geq x_i^*\left(2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right) + 2\lambda_{(i,2)}\left(-b_i + \mathbf{\frac{1}{2}}\right) + \lambda_{(i,3)}\left[(1-\varepsilon)x_i + \varepsilon y_i\right]\right). \end{aligned}$$

We have that

$$x_i^*\left(2\lambda_{(i,1)}\left(a_i - \mathbf{\frac{1}{2}}\right)\right) > 2\left(1 - \delta - x_i^*\left(\mathbf{\frac{1}{2}}\right)\right),$$

and hence

$$x_i^*(\lambda_{(i,1)}a_i) > 1 - \delta - (1 - \lambda_{(i,1)})x_i^*\left(\mathbf{\frac{1}{2}}\right) \geq 1 - \delta - \frac{1}{2}\rho\|x_i^*\|.$$

We recall that $\delta + \frac{1}{2}\rho\|x_i^*\| < \widetilde{\delta}$, then $x_i^*(\lambda_{(i,1)}a_i) > 1 - \widetilde{\delta}$. It follows that $x_i^*(a_i) > 1 - \widetilde{\delta}$. Now $a_i \in K_0 \cap S_i$, and similarly we get that $a_i' \in K_0 \cap S_i$, for every $i = 1, \dots, n$, and $\frac{1}{n}\sum_{i=1}^n a_i$, $\frac{1}{n}\sum_{i=1}^n a_i' \in \frac{1}{n}\sum_{i=1}^n S_i$. Since the diameter of $\frac{1}{n}\sum_{i=1}^n S_i$ is less than $\frac{1}{4}(1-\varepsilon)\gamma$, we deduce that $\frac{1}{n}\|\sum_{i=1}^n a_i - a_i'\| < \frac{1}{4}(1-\varepsilon)\gamma$. Finally, we conclude from (2.3) and the above estimation that

$$\|x - y\|_\varepsilon \leq \gamma.$$

Hence the set $\frac{1}{n}\sum_{i=1}^n U_i$ has diameter, at most γ, for the norm $\|\cdot\|_\varepsilon$. We recall now that every relatively weakly open subset of B_ε contains a convex combination of slices [9, Lemme 5.3]. So we conclude that B_ε has convex combinations of slices with diameter arbitrarily small.

In order to prove that every nonempty relatively weakly open subset of B_ε has diameter 2, we recall that $K_0 = \overline{\{g_i : i \in \mathbb{N}\}}$.

Recall that $B_\varepsilon = \overline{co}(2(K_0 - \mathbf{\frac{1}{2}}) \cup 2(-K_0 + \mathbf{\frac{1}{2}}) \cup [(1-\varepsilon)B_X + \varepsilon B_{c_0}])$. Call $A = 2(K_0 - \mathbf{\frac{1}{2}})$ and $B = (1-\varepsilon)B_X + \varepsilon B_{c_0}$. Now A and B are convex subsets of X and $B_\varepsilon = co(A \cup -A \cup B)$. Observe that $\frac{A-A}{2} = K_0 - K_0$ and so $\frac{A-A}{2} \subset B_{c_0} \subset B$, from the definition of K_0.

Thus, in order to prove that every nonempty relatively weakly open subset of B_ε has $\|\cdot\|_\varepsilon$-diameter 2 it is enough to prove, from Lemma 2.4, that every nonempty relatively weakly open subset of $\overline{co}((2K_0-\mathbf{1})\cup[(1-\varepsilon)B_X+\varepsilon B_{c_0}])$ has $\|\cdot\|_\varepsilon$-diameter 2.

Pick U a weakly open subset of X such that

$$U\cap\overline{co}\big((2K_0-\mathbf{1})\cup\big[(1-\varepsilon)B_X+\varepsilon B_{c_0}\big]\big)\neq\emptyset,$$

then there are $g_i\in K_0$, $x_0\in B_X$, $y_0\in B_{c_0}$ and $\lambda\in[0,1]$ such that $\lambda(2g_i-\mathbf{1})+(1-\lambda)[(1-\varepsilon)x_0+\varepsilon y_0]$ belong to U.

As U is a norm open set, we can assume that y_0 has finite support. From Lemma 2.3, there is a scalar sequence $\{t_j\}$ with $|t_j|\le 1$ for every j such that, putting $x_j=x_0+(1-t_j)e_j$ and $y_j=x_0-(1+t_j)e_j$ for every j, we have that $\{x_j\}$ and $\{y_j\}$ are weakly convergent sequences in B_X to x_0. We put j_0 such that $e_j^*(y_0)=0$ for every $j\ge j_0$, then $y_0+e_j, y_0-e_j\in B_{c_0}$ for every $j\ge j_0$. Now, again from the construction of K_0, $g_i+e_{m_n+i}\in K_0$ for every n, and hence, $\{g_i+e_{m_n+i}\}_n$ is weakly convergent to g_i.

Therefore we get for n conveniently big that

$$x:=\lambda\big(2(g_i+e_{m_n+i})-\mathbf{1}\big)+(1-\lambda)\big[(1-\varepsilon)x_{m_n+i}+\varepsilon(y_0+e_{m_n+i})\big]$$

and

$$y:=\lambda(2g_i-\mathbf{1})+(1-\lambda)\big[(1-\varepsilon)y_{m_n+i}+\varepsilon(y_0-e_{m_n+i})\big]$$

belong to U. Therefore

$$\begin{aligned}diam_{\|\cdot\|_\varepsilon}(U)&\ge\|x-y\|_\varepsilon\\&=\big\|2\lambda e_{m_n+i}+(1-\lambda)\big[2(1-\varepsilon)e_{m_n+i}+2\varepsilon e_{m_n+i}\big]\big\|_\varepsilon\\&=2\|e_{m_n+i}\|_\varepsilon\ge 2\|e_{m_n+i}\|=2\|e_{m_n+i}\|_\infty=2.\end{aligned}$$

We conclude that $diam_{\|\cdot\|_\varepsilon}(U)=2$. □

The following consequence shows that there are many spaces satisfying D2P and failing strong-D2P.

Corollary 2.6. *Every Banach space containing isomorphic copies of c_0 can be equivalently renormed satisfying D2P and failing strong-D2P.*

Finally, we get a stability property for Banach spaces with D2P and failing strong-D2P.

Corollary 2.7. *The Banach spaces with D2P and failing strong-D2P are stable for l_1-sums.*

The proof of the above corollary follows from the following general proposition, which gives the stability under ℓ_1-sums of the D2P and small convex combinations of slices. In fact this stability property holds for $1\le p<\infty$.

Proposition 2.8. *Let $\{X_n\}$ be a sequence of Banach spaces satisfying the D2P and put $Z := \ell_1 - \bigoplus_n X_n$. Assume that $\{\varepsilon_n\}$ is a null scalars sequence such that for every $n \in \mathbb{N}$ there is a convex combination of slices in X_n with diameter, at most, ε_n. Then Z satisfies the D2P and*

$$\inf\big\{\mathit{diam}(T) : T \textit{ convex combination of slices in } B_Z\big\} = 0.$$

Proof. In order to prove that

$$\inf\big\{\mathit{diam}(T) : T \text{ convex combination of slices in } B_Z\big\} = 0,$$

fix $n \in \mathbb{N}$ and let us see that for every slice of B_{X_n} we can define a slice of B_Z with similar diameter. Consider $Z = X_n \oplus_1 Y_n$, being $Y_n = \ell_1 - \bigoplus_{k \neq n} X_k$. Let $S_n = S(B_{X_n}, x_n^*, \alpha)$ be a slice of B_{X_n} and fix $0 < \mu < \alpha$. We can assume that $x_n^* \in S_{X_n^*}$. If $(x_n, y_n) \in S(B_Z, (x_n^*, 0), \mu)$, then $x_n^*(x_n) > 1 - \mu > 1 - \alpha$ and so $\|x_n\| > 1 - \mu$. Thus $\|y_n\| < \mu$. As a consequence, $\|(x_n, y_n) - (x_n, 0)\| < \mu$. Then we have that

$$S\big(B_Z, (x_n^*, 0), \mu\big) \subset S\big(B_{X_n}, x_n^*, \alpha\big) \times \mu B_{Y_n}. \tag{2.4}$$

Now, if T_n is a convex combination of slices of B_{X_n}, for $\mu > 0$ small enough we get that

$$\begin{aligned} &\inf\big\{\mathit{diam}(T) : T \text{ is a convex combination of slices of } B_Z\big\} \\ &\quad \leq \mathit{diam}(T_n) + 2\mu \leq \varepsilon_n + 2\mu. \end{aligned}$$

We conclude that

$$\inf\big\{\mathit{diam}(T) : T \text{ convex combination of slices in } B_Z\big\} = 0,$$

since $\lim_n \varepsilon_n = 0$.

We pass now to prove that Z has D2P. As every nonempty relatively weakly open subset of B_Z contains a nonempty intersection of slices in B_Z [9, Lemme 5.3], take $f_1, \ldots, f_N \in S_Z$, $0 < \alpha_1, \ldots, \alpha_N < 1$ and consider a nonempty intersections of slices in B_Z

$$S = \big\{z \in B_Z : f_i(z) > 1 - \alpha_i,\ 1 \leq i \leq N\big\}.$$

Pick $z_0 \in S_Z \cap S$, then choose $0 < \varepsilon < \alpha_i$ for every i so that $f_i(z_0) > 1 - \alpha_i + \varepsilon$ for every i.

We denote by P_n the projection of Z onto $\ell_1 - \bigoplus_{i=1}^n X_i$, which is a norm one projection for every $n \in \mathbb{N}$. As $f_i(z_0) > 1 - \alpha_i + \varepsilon$, there is $k \in \mathbb{N}$ such that $P_k^*(f_i)(P_k(z_0)) > 1 - \alpha_i + \varepsilon$, where P_k^* denotes the transposed projection of P_k.

Consider the intersections of slices in the unit ball of $Y = \ell_1 - \bigoplus_{i=1}^k X_i$ given by $T = \{y \in B_Y : P_k^*(f_i)(y) > 1 - \alpha_i + \varepsilon,\ 1 \leq i \leq N\}$. Observe that $T \neq \emptyset$, since

$P_K(z_0) \in T$. In order to prove that $diam(S) = 2$, fix $\rho > 0$ and take $y_1, y_2 \in B_Y \cap T$ such that $\|y_1 - y_2\| > 2 - \rho$. This is possible, because it is known that the finite ℓ_1-sum of Banach spaces with D2P has too D2P [2]. Now we see y_1, y_2 as elements in Z, via the natural isometric embedding of Y into Z, and we have that $y_1, y_2 \in S$ with $\|y_1 - y_2\|_Z > 2 - \rho$, hence $diam(S) \geq 2 - \rho$. As ρ was arbitrary, we conclude that $diam(S) = 2$. □

Finally, we would like to pose the following questions:

(1) We don't know if L_1 can be equivalently renormed satisfying D2P so that every convex combination of slices of its unit ball has diameter arbitrarily small.
(2) What Banach spaces can be equivalently renormed to satisfy slice-D2P, D2P or strong-D2P?
(3) Is there some strongly regular Banach space with D2P?

About the third question, recall that a Banach space X is said to be strongly regular (SR) if every closed, convex and bounded subset of X has convex combination of slices with diameter arbitrarily small (we refer to [11] for background about this topic). It is well known that every Banach space containing isomorphic copies of c_0 fails to be SR. As SR is an isomorphic property, that is independent on the equivalent norm considered in the space, every renorming of c_0 fails to be SR. Also it is known that there are SR Banach spaces so that every relatively weakly open subset of its unit ball has diameter, at least, some $\delta > 0$, but with $\delta < 2$.

About the second question, it seems natural to think that every Banach space failing to be strongly regular can be equivalently renormed with the strong-D2P, but we don't know if this is true. In [8] it is proved that every Banach space X, whose dual X^* fails to be strongly regular can be equivalently renormed so that every convex combination of w^*-slices in the unit ball of X^* has diameter 2. Moreover, if X is separable, also it is shown there that for every $\varepsilon > 0$, X can be equivalently renormed so that every convex combination of slices in the unit ball of X^* has diameter, at least, $2 - \varepsilon$.

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