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A Geometric Localization Theorem*

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The purpose of this paper is to provide a proof of Theorem 3.11 in the previous paper of Jerison and Kenig [1]. A connected domain $\mathscr{D} \subset \mathbb{R}^n$ is said to be an (ε, δ) domain if for all $x, y \in \mathscr{D}$, $|x-y| < \delta$, there is a rectifiable arc $\gamma \subset \mathscr{D}$ joining x to y and satisfying

$$l(\gamma) \leqslant \frac{1}{\varepsilon} |x-y| \tag{1}$$

and

$$\operatorname{dist}(z, \partial\mathscr{D}) \equiv d(z) \geqslant \varepsilon \frac{|x-z|\,|y-z|}{|x-y|} \qquad \text{for all } z \text{ on } \gamma. \tag{2}$$

Here $l(\gamma)$ denotes the Euclidean arclength of a rectifiable arc γ and $|x-y|$ denotes the Euclidean distance between $x, y \in \mathbb{R}^n$. A domain $\mathscr{D}$ is said to be an (ε, δ) nontangentially accessible domain (N.T.A.) if $\mathscr{D}$ is an (ε, δ) domain and for every point $q \in \partial\mathscr{D}$ and $r \in (0, \delta)$, there is a point z such that $B(z, r) \subset B(q, r/\varepsilon) \cap \mathscr{D}^c$. Here $B(x, r)$ denotes the Euclidean ball centered at x and of radius r. The following localization theorem is used by Jerison and Kenig in their paper [1].

THEOREM. *Suppose $\mathscr{D}$ is an (ε, δ) N.T.A. There is a positive constant A depending only on the values of ε and the dimension n such that whenever $q \in \partial\mathscr{D}$ and $r \in (0, \delta/A)$, there is a domain $\mathscr{D}_{q,r}$ such that $\mathscr{D}_{q,r}$ is a $(1/A, 1/A)$ N.T.A. and*

$$\mathscr{D} \cap B(q, r) \subset \mathscr{D}_{q,r} \subset \mathscr{D} \cap B(q, Ar). \tag{3}$$

Jerison and Kenig call $\mathscr{D}_{q,r}$ a *cap*. Before proving the theorem we make two observations. Firstly, the example $\mathscr{D} = \{(x, y, z) \in \mathbb{R}^3 : z < (x^2 + y^2)^{1/4}\}$ shows that not every (ε, δ) domain is an N.T.A. Secondly, the theorem is not hard to prove when $n = 2$. We say no more than this can be accomplished by cutting $\mathscr{D}$ into two components by constructing a suitable arc. Thus our only

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problem is to prove the theorem in the case where $n \geqslant 3$. The proof we give, however, is valid for all dimensions greater than one. Our figures will be drawn only for the planar case to ease comprehension, though their higher dimensional analogues are not too much harder to visualize.

Fix $q \in \partial\mathscr{D}$ and $r > 0$; we may assume $r = 2^{-m}$ for some integer m. By a translation, we may also assume that q is the center of a dyadic cube Q with sidelength $l(Q) = 2^{-m+1}$. Let $\mathscr{D}_0 = \mathscr{D} \cap Q^0$, where E^0 denotes the interior of a set E. Then $B(q, r) \cap \mathscr{D} \subset \mathscr{D}_0$, but $\mathscr{D}_0$ may not even be connected. To rectify this we will add onto $\mathscr{D}_0$ a certain portion of $\mathscr{D}$. Before doing this we pause to collect some information on $\mathscr{D}_0$'s local behavior.

Let $W = \{Q_j\}$ be the dyadic Whitney decomposition of $\mathscr{D}$. (See [2] for a construction of W.) Then each Q_j is a closed dyadic cube and $\bigcup_j Q_j = \mathscr{D}$. Furthermore

$$Q_j^0 \cap Q_k^0 = \varnothing, \qquad j \neq k, \tag{4}$$

$$1 \leqslant \frac{\operatorname{dist}(Q_j, \partial\mathscr{D})}{l(Q_j)} \leqslant 4\sqrt{n} \tag{5}$$

and

$$1/4 \leqslant \frac{l(Q_j)}{l(Q_k)} \leqslant 4 \qquad \text{if} \quad Q_j \cap Q_k \neq \varnothing. \tag{6}$$

Because all cubes in W are dyadic, each $Q_j \in W$ satisfies either $Q_j \subset Q$ or $Q_j^0 \cap Q^0 = \varnothing$. In what follows $A_1, A_2, \ldots,$ denote certain constants whose values depend only upon ε and the dimension n.

LEMMA 1. *Suppose $Q_j, Q_k \in W$ and $Q_j \cap Q_k \cap \mathscr{D}_0 \neq \varnothing$. If $x \in Q_j^0$ and $y \in Q_k^0$, then there is an arc γ joining x to y, with length $l(\gamma) \leqslant A_1 |x - y|$, and such that*

$$\operatorname{dist}(z, \mathscr{D}_0^c) \geqslant \frac{1}{A_1} \cdot \frac{|x - z|\,|y - z|}{|x - y|}$$

for all z on γ.

Proof. We first observe that the line segment $\overline{xy}$ is contained in $\mathscr{D}$. This follows from properties (5) and (6) of the Whitney decomposition. Since Q^0 is convex, $\overline{xy} \in Q^0$. Let z_0 be the midpoint of $\overline{xy}$. There is a point $z_1 \in Q^0$ such that $|z_0 - z_1| \leqslant |x - y|/10^n$ and $B(z_1, |x - y|/100n^2) \subset Q$. Let γ be the union of the line segments $\overline{xz_1}$ and $\overline{z_1 y}$. Then $\gamma \in Q^0$ and $l(\gamma) \leqslant 2|x - y|$. By properties (5) and (6),

$$d(z) \geqslant \frac{1}{10n} \min(d(x), d(y))$$

for all z on γ. Since every face of Q is a hyperplane, the construction of γ shows that

$$\operatorname{dist}(z, Q^c) \geqslant \frac{1}{200n^2} \cdot \frac{|x-z|\,|y-z|}{|x-y|}$$

for all z on γ. The proof of the lemma is complete.

An application of conditions (1) and (2) shows there is a cube $Q_0 \in W$, $Q_0 \subset Q$, with length $l(Q) \geqslant \varepsilon^2 r/64n$. (If $\operatorname{rad}(\mathscr{D}) < 2r$, the lemma is trivial.) We fix now and hereafter one such cube Q_0. For $Q_j \in W$, let z_j denote the center of Q_j. To make $\mathscr{D}_0$ connected, we would like to do something like the following. For each $z_j \in Q$, take an arc γ_j, connecting z_j to z_0 and satisfying (1) and (2). Let $F_j = \{Q_k \in W: Q_k \cap \gamma_j \neq \varnothing\}$. Then put $\mathscr{D}_{q,r} = \mathscr{D}_0 \cup \bigcup_{z_j \in Q} F_j$. Unfortunately, this idea does not quite work. This is because we could have added to $\mathscr{D}_0$ two cubes Q_j, Q_k, which intersect at only one point x, while all other Whitney cubes containing x were not added to $\mathscr{D}_0$. To rectify this we could chop off certain "bad" pieces of each F_j, forming new sets $\tilde{F}_j$, and then put $\mathscr{D}_1 = \mathscr{D}_0 \cup \bigcup_{z_j \in Q} \tilde{F}_j$. It is not hard to convince oneself that even then $\mathscr{D}_1$ need not be an (η, ∞) domain for any value of $\eta > 0$. As a last attempt we could connect each F_j to every $\tilde{F}_k$ which comes near to it, proceeding in the above manner. We could obtain new sets G_{jk} and hope that $\mathscr{D}_{q,r} = \mathscr{D}_1 \cup \bigcup_{j,k} G_{j,k}$ satisfies the conclusions of the theorem. It is exactly this strategy that we adopt in the following paragraphs. The technicalities involved are a bit arduous, but the ideas behind our construction are rather simple.

The building blocks for our construction will be certain "good" pieces of Whitney cubes, which we call pipe segments and tees. Let us fix our attention on an (arbitrary) cube $Q_j \in W$. Let $Q_{j,1}, Q_{j,2}, \ldots, Q_{j,M}$ be all those cubes $Q_k \in W$ such that $Q_j \cap Q_k$ is an $n-1$ dimensional cube. By (6) there are at most $4^{n-1} \cdot 2n$ such cubes. Let $z_{j,k}$ be the center of $Q_j \cap Q_{j,k}$, and let $L_{j,k}$ be the line segment $\overline{z_j z_{j,k}}$, $1 \leqslant k \leqslant M$. For each such $L_{j,k}$ let $S_{j,k} = \{x \in Q_j: \operatorname{dist}(x, L_{j,k}) < 10^{-3n} \min(l(Q_j), l(Q_k))\}$. Each $S_{j,k}$ is a *pipe segment* connecting z_j to a neighborhood on ∂Q_j of $z_{j,k}$; it is also clearly an (η, ∞) domain for some value of η depending only on the dimension n. Let $T_j = B(z_j, 10^{-1} l(Q_j))$. Then the pipe segments $S_{j,k}$ are joined together by T_j. In the terminology of plumbing, T_j is a *tee.* An *admissible set* $S \subset Q_j$ is a nonvoid collection of pipe segments $S_{j,k}$ together with T_j. The null set is also defined to be an admissible set. The following lemma can be easily verified by the reader.

LEMMA 2. *Suppose* $Q_j, Q_k \in W$ *and* $Q_j \cap Q_k \neq \varnothing$. *If* $S_j \subset Q_j$ *and* $S_k \subset Q_k$ *are admissible sets and* $S_j \cap S_k \neq \varnothing$, *then* $(S_j \cup S_k)^0$ *is a* $(10^{-4n}, \infty)$ *domain. If* $S_j \subset Q_j$ *is an admissible set and* $S_j \cap Q_k \neq \varnothing$, *then* $(S_j \cup Q_k)^0$ *is a* $(10^{-4n}, \infty)$ *domain.*

An arc $\gamma \subset \mathscr{D}$ connecting z_1 to z_2 is said to be a *pipeline* if it satisfies the following conditions:

$$\gamma \text{ is the union of a collection of centers } L_{j,k} \text{ of pipe segments } S_{j,k}. \tag{7}$$

$$d(z) \geqslant \frac{1}{A} \frac{|z_1 - z| \, |z_2 - z|}{|z_1 - z_2|} \qquad \text{for all } z \text{ on } \gamma. \tag{8}$$

and

$$\text{If } x, y \in \gamma \text{ and } \gamma(x,y) \text{ is the subarc of } \gamma \text{ having endpoints } x \text{ and } y, \text{ then } l(\gamma(x,y)) \leqslant A\,|x-y|. \tag{9}$$

The number A appearing in (8) and (9) is called the pipeline constant of γ, and we say that γ is an A pipeline.

LEMMA 3. *Suppose $z_1, z_2 \in \mathscr{D}$ and $|z_1 - z_2| < \delta/2$. Then there is an A_2 pipeline joining z_1 to z_2.*

Proof. We may assume $|z_1 - z_2| > 2\min(d(z_1), d(z_2))$, for otherwise the lemma is trivial. Let γ_0 be an arc joining z_1 to z_2 and satisfying (1) and (2), and let $s_0 \in Z$ satisfy $\frac{1}{4}|z_1 - z_2| < 2^{s_0} d(z_1) \leqslant \frac{1}{2}|z_1 - z_2|$. For $s \in Z$, $-1 \leqslant s \leqslant s_0$, pick a point x_s in $\gamma_0 \cap \{x : |x - z_1| = 2^s d(z_1)\}$. Then

$$\frac{\varepsilon}{2} 2^s d(z_1) \leqslant d(x_s) \leqslant 2^{s+1} d(z_1), \qquad -1 \leqslant s \leqslant s_0. \tag{10}$$

Let $x_{-2} = z_1$. By (10), whenever $-2 \leqslant s \leqslant s_0 - 1$, there is an arc Γ_s joining x_s to x_{s+1} satisfying

$$l(\Gamma_s) \leqslant \frac{4}{\varepsilon} 2^s d(z_1) \tag{11}$$

and

$$d(z) \geqslant \frac{\varepsilon^2}{8} 2^s d(z_1) \qquad \text{for all } z \text{ on } \Gamma_s. \tag{12}$$

Let $F_{z_1,s} = \{Q_{s,t}\} = \{Q_j \in W : Q_j \cap \Gamma_s \neq \varnothing\}$, $-2 \leqslant s \leqslant s_0 - 1$. By (11) and (12) each $F_{z_1,s}$ contains at most $\varepsilon^{-3} 10^{2n}$ cubes and

$$\bigcup_t Q_{s,t} \subset B\left(z_1, \frac{8}{\varepsilon} 2^s d(z_1)\right).$$

By (12), $\{\bigcup_t Q_{s,t}\} \cap B(z_1, (\varepsilon^2/32)\, 2^s d(z_1)) = \varnothing$, $64/\varepsilon^2 \leqslant s \leqslant s_0 - 1$. For each $Q_{s,t} \in F_{z_1,s}$, let $\{L_{s,t,k}\}$ be the collection of all centers of pipe segments

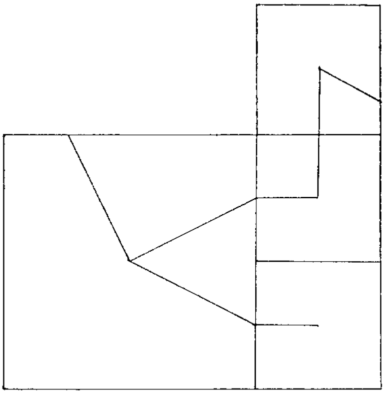

FIGURE 1

$S_{s,t,k} \subset Q_{s,t}$. Then by the above properties of the cubes $Q_{s,t}$, there is a pipeline Γ_{z_1} connecting z_1 to a center z_j of a cube $Q_j \in W$ such that $Q_j \cap \{x: |x - z_1| = 2^{s_0} d(z_1)\} \neq \varnothing$ and such that $l(Q_j) \geqslant (\varepsilon/16n)|z_1 - z_2|$. Furthermore, Γ_{z_1} is an $n10^{4n}\varepsilon^{-5}$ pipeline. Form in exactly the same manner a pipeline Γ_{z_2} connecting z_2 to z_j and with pipeline constant $n10^{4n}\varepsilon^{-5}$. Since $l(Q_j) \geqslant (\varepsilon/16n)|z_1 - z_2|$, a suitable subset of $\Gamma_{z_1} \cup \Gamma_{z_2}$ provides us with an A_2 pipeline joining z_1 to z_2.

We now start to connect up $\mathscr{S}_0$. For each $z_j \in \mathscr{S}_0$, let p_j be a pipeline joining z_j to z_0. We now form the associated *pipe* P_j by taking P_j to be the smallest set containing p_j such that $P_j \cap Q_k$ is an admissible set for all $Q_k \in W$. Let $\{P_j\}$ be the collection of all such pipes; these pipes are called *primary pipes*. See Figs. 1 and 2. If we put $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\bigcup_j P_j\}$, then by (9),

$$\mathscr{S}_1 \subset B(q, 2A_2 r). \tag{13}$$

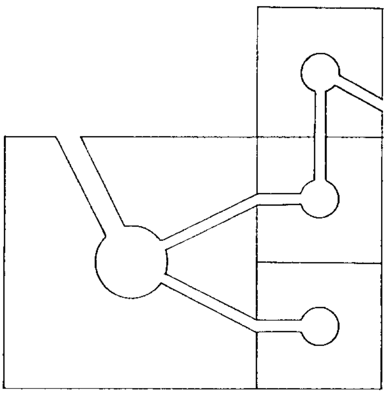

FIGURE 2

In Fig. 1 there are four Whitney cubes and two pipelines. In Fig. 2 the pipelines have been replaced by their associated pipes.

$\mathscr{D}_1$ is connected, but there is little else we can say about it. We fix this by adding more pipes onto $\mathscr{D}_1$. For all pairs $z_j, z_k \in \mathscr{D}_1$ satisfying $\frac{1}{4} \leqslant l(Q_j)/l(Q_k) \leqslant 4$ and $|z_1 - z_2| \leqslant 8n^2 A_2 l(Q_j)$, construct a pipeline $p_{j,k}$ connecting z_j to z_k and let $P_{j,k}$ be the associated pipe. Let $\{P_{j,k}\}$ be the collection of all such pipes; these pipes are called *secondary pipes*. By Lemma 3,

$$l(p_{j,k}) \leqslant A_3 |z_j - z_k| \tag{14}$$

and

$$d(z) \geqslant \frac{1}{A_3} l(Q_j) \qquad \text{for all } z \text{ on } p_{j,k}, \tag{15}$$

for all secondary pipelines $p_{j,k}$. Now put $\mathscr{D}_{q,r} = \mathscr{D}_1 \cup \{\bigcup_{j,k} P_{j,k}\}$. By (13) and (14), $\mathscr{D}_{q,r} \subset B(q, A_3 r)$, so we need only verify that $\mathscr{D}_{q,r}$ is an N.T.A. By Lemmas 1, 2 and the construction of the pipes we obtain

LEMMA 4. *Suppose $Q_j, Q_k \in W$ and $Q_j \cap Q_k \cap \mathscr{D}_{q,r} \neq \varnothing$. Then if $x \in Q_j \cap \mathscr{D}_{q,r}$ and $y \in Q_k \cap \mathscr{D}_{q,r}$, there is an arc $\gamma \subset \mathscr{D}_{q,r}$ joining x to y, with length $l(\gamma) \leqslant A_3 |x-y|$, and such that*

$$\operatorname{dist}(z, \mathscr{D}^c_{q,r}) \geqslant \frac{1}{A_3} \frac{|x-z|\,|y-z|}{|x-y|}$$

for all z on γ.

Lemma 4 says we need only be able to find "good" arcs connecting centers z_j, z_k of Whitney cubes Q_j, Q_k. For if

$$Q_j \cap Q_k,\ Q_j \cap \mathscr{D}_{q,r},\ Q_k \cap \mathscr{D}_{q,r} \neq \varnothing$$

but $Q_j \cap Q_k \cap \mathscr{D}_{q,r} = \varnothing$, then either Q_j or Q_k lies outside of $\mathscr{D}_0$, and by the construction of the pipes,

$$\operatorname{dist}(Q_j \cap \mathscr{D}_{q,r}, Q_k \cap \mathscr{D}_{q,r}) \geqslant \tfrac{1}{16} \min(l(Q_j), l(Q_k)).$$

Furthermore, if $Q_j \cap \mathscr{D}_{q,r} \neq \varnothing$, then either $Q_j^0 \subset \mathscr{D}_{q,r}$ or $Q_j \cap \mathscr{D}_{q,r}$ is an admissible set, and Lemma 2 applies. With these observations in mind, we now verify the (ε, δ) condition by examining pairs z_j, z_k of points in $\mathscr{D}_{q,r}$. Notice that *every* $z_j \in \mathscr{D}_{q,r}$ lies on some pipeline p_s or $p_{s,t}$; hence z_j lies in the center of some pipe P_s or $P_{s,t}$. Let $P = \{\bigcup_j p_j\}\{\bigcup_{j,k} p_{j,k}\}$ be the union of all primary and secondary pipelines. By the construction of the pipes

$\operatorname{dist}(z, \mathscr{D}^c_{q,r})$ is proportional to $d(z)$ for all $z \in P$. Thus if $z_j, z_k \in \mathscr{D}_{q,r}$, we need only find an arc $\gamma \subset P$ joining z_j to z_k with length $l(\gamma) \leqslant A\,|z_j - z_k|$ and such that

$$d(z) \geqslant \frac{|z_j - z|\,|z_k - z|}{A\,|z_j - z_k|}$$

for all z on γ.

Case I. $z_j \in P_s$, $z_k \in P_t$ (two primary pipes). Travel down the pipeline p_s from z_j to z_0 until a point $z_1 \in p_s$ is reached which satisfies

$$|z_j - z_k| \leqslant l(Q_1) < 4\,|z_j - z_k|.$$

If there is no such point $z_1 \in p_s$, set $z_1 = z_0$. Let γ_1 be the subarc of p_s between z_j and z_1. By Lemma 3

$$l(\gamma_1) \leqslant A_3\,|z_j - z_k|. \tag{16}$$

Since $l(Q_0) \geqslant \varepsilon^2 r/64n$, Lemma 3 and (13) yield the estimate

$$d(z) \geqslant \frac{1}{A_3}\,|z_j - z| \qquad \text{for all } z \text{ on } \gamma_1. \tag{17}$$

In exactly the same manner find a point z_2 on p_t and let γ_2 be the subarc of p_t between z_k and z_2. Then

$$l(\gamma_2) \leqslant A_3\,|z_j - z_k| \tag{18}$$

and

$$d(z) \geqslant \frac{1}{A_3}\,|z_k - z| \qquad \text{for all } z \text{ on } \gamma_2. \tag{19}$$

We first treat the subcase where either $z_1 = z_0$ or $z_2 = z_0$; by symmetry we may assume that the first of these conditions holds. Let γ_3 be the subarc of p_t between z_2 and z_0. By (16) and Lemma 3, $l(\gamma_3) \leqslant A_2(A_3 + 1)\,|z_j - z_k|$.

Put $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$. Then by (16), (18), and the above remark, $l(\gamma) \leqslant A_4\,|z_j - z_k|$. We also have the estimate $d(z) \geqslant (1/A_3)\,|z_k - z|$ for all z on γ_3. Thus by (17) and (19),

$$d(z) \geqslant \frac{1}{A_4}\,\frac{|z_j - z|\,|z_k - z|}{|z_j - z_k|} \qquad \text{for all } z \text{ on } \gamma.$$

We now treat the subcase where $z_1 \neq z_0$ and $z_2 \neq z_0$. By Lemma 3

$$|z_j - z_1|,\ |z_k - z_1| \leqslant 3n^2 A_2\,|z_j - z_k|,$$

so there is a secondary pipeline $p_{1,2}$ connecting z_1 to z_2. By (14) and (15),

$$l(p_{1,2}) \leqslant A_4 |z_j - z_k| \tag{20}$$

and

$$d(z) \geqslant \frac{1}{A_4} \cdot |z_j - z_k| \qquad \text{for all } z \text{ on } p_{1,2}. \tag{21}$$

Let $\gamma = \gamma_1 \cup \gamma_2 \cup p_{1,2}$. Then by (16), (18), and (20), $l(\gamma) \leqslant A_5 |z_j - z_k|$, and by (17), (19), and (21),

$$d(z) \geqslant \frac{1}{A_5} \frac{|z_j - z||z_k - z|}{|z_j - z_k|} \qquad \text{for all } z \text{ on } \gamma.$$

Case II. $z_j \in P_s$, $z_k \in P_{t,u}$ (one primary pipe and one secondary pipe). Let γ_1 be the subarc of $p_{t,u}$ between z_k and z_t. By (14) and (15), $l(\gamma_1) \leqslant A_4 |z_j - z_k|$ and $d(z) \geqslant (1/A_4)|z_k - z|$ for all z on γ_1. Now let $\gamma_2 \subset P$ be the arc joining z_j to z_t constructed as in case I, and let $\gamma = \gamma_1 \cup \gamma_2$. Then the estimates in case I for γ_2 along with our estimates for γ_1 yield

$$l(\gamma) \leqslant A_5 |z_j - z_k| \qquad \text{and} \qquad d(z) \geqslant \frac{1}{A_5} \frac{|z_j - z||z_k - z|}{|z_j - z_k|} \qquad \text{for all } z \text{ on } \gamma.$$

Case III. $z_j \in P_{s,t}$, $z_k \in P_{u,v}$ (two secondary pipes). Let γ_1 be the subarc of $p_{s,t}$ between z_t and z_s. Then by (14) and (15),

$$l(\gamma_1) \leqslant A_r |z_j - z_k| \qquad \text{and} \qquad d(z) \geqslant \frac{1}{A_4} |z_j - z| \qquad \text{for all } z \text{ on } \gamma_1.$$

Now let $\gamma_2 \subset P$ be the arc joining z_k to z_s constructed as in case II, and let $\gamma = \gamma_1 \cup \gamma_2$. Then the above estimates for γ_1 along with the estimates from cases I and II for γ_2 yield

$$l(\gamma) \leqslant A_6 |z_j - z_k| \qquad \text{and} \qquad d(z) \geqslant \frac{1}{A_6} \frac{|z_j - z||z_k - z|}{|z_j - z_k|} \qquad \text{for all } z \text{ on } \gamma.$$

Therefore $\mathscr{D}_{q,r}$ is a $(1/A, \infty)$ domain.

We now verify N.T.Aness of $\mathscr{D}$. Fix a point $p \in \partial\mathscr{D}_{q,r}$ and a value of $s > 0$. First suppose that $p \in \mathscr{D}^c$. Then since $\mathscr{D}$ is nontangentially accessible there is a point z such that $B(x, s) \subset B(p, As) \cap \mathscr{D}^c$, as long as s is small enough. Now suppose that $p \notin \mathscr{D}^c$; by (3) we may assume $s < r$. Then either

$p \in \partial Q$ or p is in the boundary of some primary or secondary pipe. In either case there is a dyadic cube S containing p such that

$$Q^0 \cap S^0 = \varnothing \qquad \text{and} \qquad \frac{10^{2n}}{\varepsilon} s \leqslant l(S) \leqslant \frac{2 \cdot 10^{2n}}{\varepsilon} s.$$

Let x be the center of S; by our previous remark it is sufficient to handle the case where $x \in \mathscr{D}$. The argument which produced the "large" cube $Q_0 \subset Q$ shows there is $Q_j \in W$, $Q_j \subset S$, with length $l(Q_j) \geqslant (10^{2n}/n)s$. Since $Q_j \cap \mathscr{D}_{q,r}$ is an admissible set, there is a point $z \in Q_j$ such that $B(z, s) \subset \mathscr{D}^c_{q,r}$. This completes the proof of the theorem.

References

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