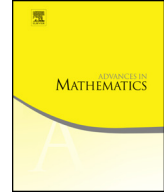




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Cyclic symmetries of A_n -quiver representations



David Nadler

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, United States

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ABSTRACT

This note contains a combinatorial construction of symmetries arising in symplectic geometry (partially wrapped or infinitesimal Fukaya categories), algebraic geometry (derived categories of singularities), and K -theory (Waldhausen’s S -construction). Our specific motivation (in the spirit of expectations of Kontsevich) is a combinatorial construction of categorical quantizations of Lagrangian skeleta of symplectic manifolds. The main result of this paper gives an immediate solution in the one-dimensional case of ribbon graphs.

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E-mail address: nadler@math.berkeley.edu.

1. Introduction

This note contains a combinatorial construction of symmetries arising in several distinct but related areas. In symplectic geometry, it is connected to categorical quantizations in the form of microlocal sheaves [8] or partially wrapped or infinitesimal Fukaya categories [1,18], in particular of Riemann surfaces in the form of ribbon graphs [7,10,11,16] as explored in [19]. In algebraic geometry, via mirror symmetry, it is connected to derived categories of singularities [6,15], in particular of functions on curves and more general Brieskorn singularities via duality and the Thom–Sebastiani theorem [4,9,17]. In K -theory, it is connected to Waldhausen’s S-construction [20], specifically with its corepresentability and S^1 -equivariance [5].¹ Our primary motivation (in the spirit of [12], and to be taken up in general elsewhere [14]) lies in the first direction, with the aim of constructing a combinatorial quantization of Lagrangian skeleta, equivalent to microlocal sheaves in their many guises. We explain below the one-dimensional case of ribbon graphs where the main result of this note gives an immediate solution.

To state the main result of this note (in its simplest two-periodic form), we briefly recall Connes’ cyclic category [2,3,13]. Let \underline{A} denote the big cyclic category of finite cyclically ordered nonempty sets. Objects are finite nonempty subsets of the circle $S \subset S^1$, and morphisms $S \rightarrow S'$ are homotopy classes of degree 1 maps $\varphi : S^1 \rightarrow S^1$ such that $\varphi(S) \subset S'$. The traditional cyclic category A is the full subcategory of \underline{A} on the objects $[n] = \mathbf{Z}/(n + 1)\mathbf{Z} \subset S^1$ embedded as $(n + 1)$ st roots of unity, for $n = 0, 1, 2, \dots$. The inclusion $A \subset \underline{A}$ is an equivalence.

For each finite cyclically ordered nonempty set S , we introduce a triangulated two-periodic A_∞ -category $\mathcal{C}_{S, st}$. (It is the twisted complexes in a particularly simple two-periodic A_∞ -category \mathcal{C}_S with objects given by the set S .) It provides a symmetric presentation of the two-periodic differential graded derived category of finite-dimensional representations of the A_{n-1} -quiver with $n = |S|$. For example, for $n = 0$, it is the zero category, but presented as one object s_0 with a degree one map

$$s_0 \longrightarrow s_0$$

whose differential is the (two-periodic) identity. For $n = 1$, it is generated by two objects s_0, s_1 with degree one maps

$$s_0 \longrightarrow s_1 \quad s_1 \longrightarrow s_0$$

whose compositions are the respective (two-periodic) identities. For $n = 2$, it provides a symmetric presentation of a universal exact triangle (as explained in [18], and attributed to Kontsevich). For general n , it provides a symmetric presentation of n -step filtrations,

¹ I am grateful to J. Lurie for discussions about how this note contributes to an understanding of the S^1 -equivariance of the S-construction as appears in the comprehensive work of Dyckerhoff and Kapranov (see the discussion of [5, Introduction, p. 9]).

which allows for an easy verification of the following. (From a concrete perspective, the primary content is in the precise combinatorial form of the construction rather than the abstract statement itself.)

Theorem 1.1. (See [Theorem 5.1](#) below.) *The assignment of the two-periodic A_∞ -category $\mathcal{C}_{S,st}$ to a finite cyclically ordered nonempty set S naturally extends to a functor from the opposite of the cyclic category to the strict category of two-periodic A_∞ -categories*

$$\mathcal{C}_{st} : \underline{\mathcal{A}}^{op} \longrightarrow \mathcal{A}_{per}^{str}$$

Remark 1.2. We also describe a graded version in [Theorem 6.8](#) where the target is the category \mathcal{A}_∞ of plain (no longer two-periodic) A_∞ -categories. Instead of the cyclic category, the domain becomes a cover of the cyclic category with objects comprising pairs of a finite nonempty subset $S \subset S^1$ and a point $c \in \text{Sym}^2(S^1)$.

Remark 1.3. Thanks to the natural duality equivalence $\underline{\mathcal{A}} \simeq \underline{\mathcal{A}}^{op}$, the theorem also provides an alternatively variant functor.

Because of its basic nature as a universal sequence of composable morphisms, the A_n -quiver appears wherever categories appear. Similarly, its derived category of representations appears wherever triangulated categories appear. What is less immediately evident in some contexts (though certainly of primary focus in others) is the importance of the natural functors (as appear in [Theorem 1.1](#)) between representations of the A_n -quivers for varying n . Let us informally mention three settings where they play a significant role.

1.1. Symplectic geometry

Representations of the A_{n-1} -quiver arise in symplectic geometry as the infinitesimal (or dual partially wrapped) Fukaya category of the exact symplectic manifold $M = \mathbb{C}$ with support Lagrangian $A_n \subset \mathbb{C}$ the union of n rays emanating from the origin. The functoriality of [Theorem 1.1](#) provides the gluing one needs to extend this description to the general case of oriented surfaces as captured by ribbon graphs [\[7,10,11,16\]](#) as explored in [\[19\]](#).

To spell this out, by a ribbon graph we will mean a locally finite graph Γ with an embedding into the germ of an oriented surface Σ . We can allow Γ to have half-infinite edges incident to a single vertex, as well as infinite or circular edges incident to no vertices. We can also allow Γ to have vertices of any finite non-zero valency.

There is a natural cosheaf of sets over Γ given by the local components of the complement $\Sigma \setminus \Gamma$. Moreover, the cosheaf naturally takes values in the cyclic category $\underline{\mathcal{A}}$ thanks to the orientation of Σ . Composing with the functor of [Theorem 1.1](#), we obtain a sheaf of A_∞ -categories (or alternatively, via the natural equivalence $\underline{\mathcal{A}} \simeq \underline{\mathcal{A}}^{op}$, a cosheaf

of A_∞ -categories) over Γ . Taking its global sections in the form of a homotopy limit (or alternatively, homotopy colimit) over Γ , we obtain a combinatorial model for the two-periodic infinitesimal (or alternatively, partially wrapped) Fukaya category of Σ . (Note that one could go further and add a bicanonical trivialization and then apply the graded functor of [Theorem 6.8](#).) This is the first instance of general constructions to be further developed in [\[14\]](#).

1.2. Algebraic geometry

Under mirror symmetry, the above symplectic realization of representations of the A_{n-1} -quiver arises in algebraic geometry as \mathbb{G}_m -equivariant matrix factorizations for the singularity $x^n = 0$. Here the functoriality of [Theorem 1.1](#) is less evident and could be profitably organized in terms of integral transforms [\[4,9,17\]](#) living on Brieskorn singularities. (In informal discussions, J. Lurie has also explained a simple formulation of it in a flexible homotopical setting).

1.3. K -theory

Given a suitable stable category, Waldhausen’s S-construction records the structure of filtered objects and their relationships under restriction and induction of filtrations. By keeping track of the natural symmetries of such operations, it produces a paracyclic space, and in particular, from a two-periodic category, a cyclic space [\[5\]](#). Within the A_∞ -setting, one can interpret a graded version of [Theorem 1.1](#) as the calculation of the object corepresenting the S-construction. In fact, the graded version we present in [Theorem 6.8](#) records slightly more symmetry than the S-construction: it only depends on a point $c \in \text{Sym}^2(S^1)$ rather than a single point $c \in S^1$ as would appear in a paracyclic realization.

2. Two-periodic A_∞ -categories

Fix a base field k . Introduce the commutative graded k -algebra $k[u, u^{-1}]$ where $\deg u = 2$. All of our constructions will be two-periodic in the sense that they will be $k[u, u^{-1}]$ -linear.

Remark 2.1. Our constructions make sense with k replaced by the integers or any base commutative ring. Moreover, J. Lurie has sketched an ∞ -categorical version that takes place over the sphere spectrum.

We will follow the conventions on A_∞ -categories from Seidel’s book [\[18\]](#). By a two-periodic A_∞ -category \mathcal{C} , we will mean the following data:

- (1) A set of objects $\text{Ob } \mathcal{C}$.
- (2) For pairs of objects $c_0, c_1 \in \text{Ob } \mathcal{C}$, a compatibly graded $k[u, u^{-1}]$ -module $\text{hom}_{\mathcal{C}}(c_0, c_1)$.

(3) For every $d \geq 1$, and objects $c_0, \dots, c_d \in \text{Ob } \mathcal{C}$, a $k[u, u^{-1}]$ -linear composition map

$$\mu_d : \text{hom}_{\mathcal{C}}(c_{d-1}, c_d) \otimes \cdots \otimes \text{hom}_{\mathcal{C}}(c_0, c_1) \longrightarrow \text{hom}_{\mathcal{C}}(c_0, c_d)[2 - d]$$

The composition maps must satisfy the quadratic A_∞ -equations

$$\sum_{m=0}^d \sum_{n=0}^{d-m} (-1)^{\dagger_n} \mu_{d-m+1}(a_d, \dots, a_{n+m+1}, \mu_m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) = 0$$

where the sign is determined by $\dagger_n = |a_1| + \cdots + |a_n| - n$.

Remark 2.2. We could equivalently assume our morphisms form $\mathbf{Z}/2\mathbf{Z}$ -graded k -modules. Then the composition map μ_d would be a morphism of degree $2 - d = d$ modulo 2.

Remark 2.3. All of our A_∞ -categories will be strictly unital: for each object $c \in \text{Ob } \mathcal{C}$, there is a degree zero element $\text{id}_c \in \text{hom}_{\mathcal{C}}^0(c, c)$, called the identity morphism, such that

(1) for any $a \in \text{hom}_{\mathcal{C}}(c_0, c_1)$, we have

$$(-1)^{|a|} \mu_2(\text{id}_{c_1}, a) = a = \mu_2(a, \text{id}_{c_0}),$$

(2) for $d \neq 2$, the composition map μ_d vanishes if any entry is an identity morphism.

Definition 2.4. Let $S = [s_0, s_1, \dots, s_n] \subset S^1$ be a finite cyclically ordered nonempty set. We will understand the indices modulo $n + 1$ so that s_{i+n+1} also denotes s_i .

We define the unital $k[u, u^{-1}]$ -linear A_∞ -category \mathcal{C}_S as follows.

(1) Objects: $\text{Ob } \mathcal{C}_S = S$.

(2) Morphisms: the free $k[u, u^{-1}]$ -modules generated by the identity morphisms and additional degree one morphisms

$$v_i \in \text{hom}_{\mathcal{C}_S}(s_i, s_{i+1}), \quad |v_i| = 1, \quad \text{for } i = 0, \dots, n.$$

(3) Compositions: all are zero except for the $k[u, u^{-1}]$ -linear maps given by the identity compositions and the additional compositions

$$\mu_{n+1}(v_{i+n}, \dots, v_i) = u \text{id}_{s_i}, \quad \text{for } i = 0, \dots, n.$$

To check that \mathcal{C}_S satisfies the quadratic A_∞ -equations, it suffices to consider composable sequences of the (non-identity) degree one generating morphisms. By construction, each term of the A_∞ -equations individually vanishes on such sequences, except for sequences of the form

$$s_i \xrightarrow{v_i} s_{i+1} \xrightarrow{v_{i+1}} \cdots \xrightarrow{v_{i+n}} s_{i+n+1} \xrightarrow{v_i} s_{i+1}, \quad \text{for } i = 0, \dots, n.$$

For such sequences, two terms will not vanish, but indeed cancel each other

$$\begin{aligned} &\mu_2(v_i, \mu_{n+1}(v_{i+n}, \dots, v_{i+1}, v_i)) + (-1)^{|v_i|-1} \mu_2(\mu_{n+1}(v_i, v_{i+n}, \dots, v_{i+1}), v_i) \\ &= \mu_2(v_i, u \operatorname{id}_{s_i}) + \mu_2(u \operatorname{id}_{s_{i+1}}, v_i) = u(v_i + (-1)^{|v_i|} v_i) = 0 \end{aligned}$$

Remark 2.5. Note that the (non-identity) degree one generating morphisms of \mathcal{C}_S are in natural bijection with the components of the complement $S^1 \setminus S$.

Example 2.6 ($n = 0$). When $S = [s_0]$, we see that \mathcal{C}_S consists of one object s_0 with endomorphism complex

$$\operatorname{hom}_{\mathcal{C}_S}(s_0, s_0) = k[u, u^{-1}] \operatorname{id}_{s_0} \oplus k[u, u^{-1}] v_0 \quad \mu_1(v_0) = u \operatorname{id}_{s_0}$$

Thus $\operatorname{hom}_{\mathcal{C}_S}(s_0, s_0)$ is acyclic, and hence \mathcal{C}_S is quasi-equivalent to the zero A_∞ -category.

Example 2.7 ($n = 1$). When $S = [s_0, s_1]$, we see that \mathcal{C}_S consists of two objects s_0, s_1 with morphism complexes

$$\begin{aligned} \operatorname{hom}_{\mathcal{C}_S}(s_0, s_0) &= k[u, u^{-1}] \operatorname{id}_{s_0} & \operatorname{hom}_{\mathcal{C}_S}(s_1, s_1) &= k[u, u^{-1}] \operatorname{id}_{s_1} \\ \operatorname{hom}_{\mathcal{C}_S}(s_0, s_1) &= k[u, u^{-1}] v_0 & \operatorname{hom}_{\mathcal{C}_S}(s_1, s_0) &= k[u, u^{-1}] v_1 \end{aligned}$$

with the only non-zero or non-identity compositions the $k[u, u^{-1}]$ -linear maps given by

$$\mu_2(v_1, v_0) = u \operatorname{id}_{s_0} \quad \mu_2(v_0, v_1) = u \operatorname{id}_{s_1}$$

Thus \mathcal{C}_S is non-canonically equivalent to the full A_∞ -subcategory of graded $k[u, u^{-1}]$ -modules given by the free module $k[u, u^{-1}]$ and its odd shift $k[u, u^{-1}][1]$. To give such an equivalence, either object s_0 or s_1 can be sent to $k[u, u^{-1}]$, and then the other will be sent to $k[u, u^{-1}][1]$.

3. Pointed version: injective cyclic functoriality

We continue to follow the conventions on A_∞ -categories from Seidel’s book [18]. By an A_∞ -functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ between two-periodic A_∞ -categories, we will mean the following data:

- (1) A map of sets $\mathcal{F} : \operatorname{Ob} \mathcal{C} \rightarrow \operatorname{Ob} \mathcal{D}$.
- (2) For every $d \geq 1$, and objects $c_0, \dots, c_d \in \operatorname{Ob} \mathcal{C}$, a $k[u, u^{-1}]$ -linear map

$$\mathcal{F}_d : \operatorname{hom}_{\mathcal{C}}(c_{d-1}, c_d) \otimes \cdots \otimes \operatorname{hom}_{\mathcal{C}}(c_0, c_1) \longrightarrow \operatorname{hom}_{\mathcal{D}}(\mathcal{F}c_0, \mathcal{F}c_d)[1 - d]$$

The maps must satisfy the polynomial equations

$$\begin{aligned} &\sum_r \sum_{k_1, \dots, k_r} \mu_r(\mathcal{F}_{k_r}(a_d, \dots, a_{d-k_r+1}), \dots, \mathcal{F}_{k_1}(a_{k_1}, \dots, a_1)) \\ &= \sum_{m=0}^d \sum_{n=0}^{d-m} (-1)^{\dagger_n} \mathcal{F}_{d-m+1}(a_d, \dots, a_{n+m+1}, \mu_m(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1) \end{aligned}$$

where the sums of the left hand side run over all $r \geq 1$ and partitions $k_1 + \dots + k_r = d$, and the sign of the right hand side is determined by $\dagger_n = |a_1| + \dots + |a_n| - n$.

The composition of A_∞ -functors $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$, $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{E}$ is the A_∞ -functor defined by:

- (1) The composite map of sets $\mathcal{G} \circ \mathcal{F} : \text{Ob } \mathcal{C} \rightarrow \text{Ob } \mathcal{E}$.
- (2) The composite $k[u, u^{-1}]$ -linear maps

$$(\mathcal{G} \circ \mathcal{F})_d(a_d, \dots, a_1) = \sum_r \sum_{k_1, \dots, k_r} \mathcal{G}_r(\mathcal{F}_{k_r}(a_d, \dots, a_{d-k_r+1}), \dots, \mathcal{F}_{k_1}(a_{k_1}, \dots, a_1))$$

where the sums run over all $r \geq 1$ and partitions $k_1 + \dots + k_r = d$.

Composition is strictly associative with unit the identity functor $\text{id}_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ given by the identity on objects, $(\text{id}_{\mathcal{C}})_1$ the identity on morphisms, and $(\text{id}_{\mathcal{C}})_d = 0$, for all $d \geq 2$.

Remark 3.1. All of our A_∞ -functors will be strictly unital:

- (1) for each object $c \in \text{Ob } \mathcal{C}$, we have $\mathcal{F}_1(\text{id}_c) = \text{id}_{\mathcal{F}(c)}$,
- (2) for $d \geq 2$, the map \mathcal{F}_d vanishes if any entry is an identity morphism.

Definition 3.2. Let \mathcal{A}_{per}^{str} denote the strict category with objects $k[u, u^{-1}]$ -linear A_∞ -categories and morphisms $k[u, u^{-1}]$ -linear A_∞ -functors. (We refer to it as strict since we will not consider any natural transformations between functors which are not the identity transformation.)

Let $\underline{A}_{inj} \subset \underline{A}$ denote the non-full subcategory with objects finite cyclically ordered nonempty sets and morphisms cyclic maps that are injective as set maps.

Remark 3.3. It is worth noting that the forgetful functor $\underline{A} \rightarrow \underline{Fin}$ is not faithful, but its restriction $\underline{A}_{inj} \rightarrow \underline{Fin}$ is in fact faithful.

Definition 3.4. Let $S \subset S^1$ be a finite cyclically ordered nonempty set.

We define the unital $k[u, u^{-1}]$ -linear A_∞ -category $\mathcal{C}_{S,0}$ to be the A_∞ -category \mathcal{C}_S with a zero object 0 adjoined. In other words, $\text{Ob } \mathcal{C}_{S,0} = \text{Ob } \mathcal{C}_S \sqcup \{0\}$, and the morphism spaces in $\mathcal{C}_{S,0}$ with domain or target 0 are all zero.

Proposition 3.5. *The assignment of the A_∞ -category $\mathcal{C}_{S,0}$ to a finite cyclically ordered nonempty set S naturally extends to a functor on the injective cyclic category*

$$\underline{\mathcal{A}}_{inj}^{op} \longrightarrow \mathcal{A}_{per}^{str}$$

Proof. Let $f : S \rightarrow T$ be a cyclic injection. We will define a unital A_∞ -functor

$$f^* : \mathcal{C}_{T,0} \longrightarrow \mathcal{C}_{S,0}$$

For the map of objects $f^* : T \sqcup \{0\} \rightarrow S \sqcup \{0\}$, we set $f^*(0) = 0$, and for $t \in T$, we set

$$f^*(t) = \begin{cases} s & \text{when } t = f(s) \\ 0 & \text{when } t \notin f(S) \end{cases}$$

For the maps of morphisms, let us introduce the term *f-interval of length k*, to refer to any interval of elements $[t_0, \dots, t_k] \subset T$ such that $t_0, t_k \in f(S)$ and $t_1, \dots, t_{k-1} \notin f(S)$, for some $k \geq 1$. (We allow the possibility that $t_0 = t_k$.)

For $k \geq 1$, we define the map f_k^* on composable sequences of the (non-identity) degree one generating morphisms

$$t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \dots \xrightarrow{v_{k-1}} t_k$$

to be zero, except when $[t_0, \dots, t_k]$ is an *f*-interval of length *k*, in which case we set

$$f_k^*(v_{k-1}, \dots, v_0) = w_0$$

where $w_0 \in \text{hom}_{\mathcal{C}_S}(s_0, s_1)$ denotes the degree one generating morphism for $s_0, s_1 \in S$ the unique elements with $f(s_0) = t_0, f(s_1) = t_k$.

To check that f^* defines a unital A_∞ -functor, it suffices to consider composable sequences of the (non-identity) degree one generating morphisms. By construction, each term of the equations individually vanishes on such sequences, except possibly for complete cycles

$$t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \dots \xrightarrow{v_{n-1}} t_n \xrightarrow{v_n} t_0$$

Furthermore, for some term not to vanish, there must be a (necessarily unique) element $s_0 \in S$ with $f(s_0) = t_0$. For such sequences, two terms will not vanish, but indeed equal each other

$$\mu_r(f_{k_r}^*(v_n, \dots, v_{n-k_r+1}), \dots, f_{k_1}^*(v_{k_1}, \dots, v_0)) = u \text{id}_{s_0} = f_1^*(\mu_{n+1}(v_n, \dots, v_0))$$

where the partition of the left hand side is the unique partition of the sequence into *f*-intervals.

Finally, we check that such functors compose as asserted. Let $f : S \rightarrow T, g : T \rightarrow U$ be cyclic injections. On objects, we clearly have $(g \circ f)^* = f^* \circ g^*$. On morphisms, it suffices to consider composable sequences of the (non-identity) degree one generating morphisms. By construction, each term of the equations individually vanishes on such sequences, except for $(g \circ f)$ -intervals

$$u_0 \xrightarrow{v_0} u_1 \xrightarrow{v_1} \cdots \xrightarrow{v_{k-1}} u_k$$

For such sequences, two terms will not vanish, but indeed equal each other

$$(g \circ f)_k^*(v_{k-1}, \dots, v_0) = w_0 = g_r^*(f_{k_r}(v_k, \dots, v_{k-k_r+1}), \dots, f_{k_1}(v_{k_1}, \dots, v_0))$$

where $w_0 \in \text{hom}_{\mathcal{C}_S}(s_0, s_1)$ denotes the degree one generating morphism for $s_0, s_1 \in S$ the unique elements with $f(s_0) = t_0, f(s_1) = t_k$, and the partition of the right hand side is the unique partition of the sequence into g -intervals. \square

Remark 3.6. It is worth informally noting that the functors of the proposition are the natural quotients by the objects not in the image of the cyclic set map.

4. Twisted complexes: surjective cyclic functoriality

We continue to follow the conventions on A_∞ -categories from Seidel’s book [18].

Definition 4.1. Let $S \subset S^1$ be a finite cyclically ordered nonempty set.

We define the unital $k[u, u^{-1}]$ -linear A_∞ -category $\mathcal{C}_{S, st}$ to be the A_∞ -category of twisted complexes of \mathcal{C}_S , or equivalently of $\mathcal{C}_{S, 0}$.

Let \mathcal{Q}_n denote the $k[u, u^{-1}]$ -linear triangulated differential graded derived category of finite-dimensional left modules over the A_n -quiver

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \cdots \longrightarrow \bullet_n$$

For $i = 1, \dots, n$, denote by $m_i \in \mathcal{Q}_n$ the free rank one module supported at the i th vertex and zero elsewhere. Note that among these objects, there are degree one generating morphisms $w_i : m_i \rightarrow m_{i+1}$, for $i = 1, \dots, n - 1$, and no other linearly independent non-identity morphisms.

Proposition 4.2. Let $S = [s_0, s_1, \dots, s_n] \subset S^1$ be a finite cyclically ordered nonempty set.

Fix the element $s_0 \in S$ so that the remaining elements inherit the linear order s_1, \dots, s_n .

Then there is a quasi-equivalence of A_∞ -categories

$$F : \mathcal{Q}_n \xrightarrow{\sim} \mathcal{C}_{S,st}$$

such that $F(m_i) = s_i$, for $i = 1, \dots, n$, and $F(w_i) = v_i$, for $i = 1, \dots, n - 1$.

Proof. Clearly the functor F is well-defined and a quasi-equivalence on the full subcategory of $\mathcal{C}_{S,st}$ generated by s_1, \dots, s_n . (Note there are no non-identity compositions among those objects.) To see it is essentially surjective, observe that s_0 is isomorphic to the shift by one of the twisted complexes

$$s'_0 = (s_1 \xrightarrow{v_1} s_2 \xrightarrow{v_2} \dots \xrightarrow{v_{n-1}} s_n)$$

Namely the degree one morphisms $v_0 \in \text{hom}_{\mathcal{C}_S}(v_0, v_1)$ and $v_n \in \text{hom}_{\mathcal{C}_S}(v_n, v_0)$ induce degree zero isomorphisms $v'_0 \in \text{hom}_{\mathcal{C}_{S,st}}(s_0, s'_0[-1])$ and $v'_n \in \text{hom}_{\mathcal{C}_{S,st}}(s'_0[1], s_0)$ which are inverse (up to a two-periodic shift). \square

Let $\underline{\mathcal{A}}_{surj} \subset \underline{\mathcal{A}}$ denote the non-full subcategory with objects finite cyclically ordered nonempty sets and morphisms cyclic maps that are surjective as set maps.

Remark 4.3. It is worth noting that the forgetful functor $\underline{\mathcal{A}}_{surj} \rightarrow \underline{Fin}$ is not faithful: for example, there are two distinct maps of cyclic objects $\{1, -1\} \rightarrow \{1\}$.

Proposition 4.4. *The assignment of the A_∞ -category $\mathcal{C}_{S,st}$ to a finite cyclically ordered nonempty set S naturally extends to a functor on the surjective cyclic category*

$$\underline{\mathcal{A}}_{surj}^{op} \longrightarrow \mathcal{A}_{per}^{str}$$

Proof. Let $f : S \rightarrow T$ be a cyclic surjection. We will define a unital A_∞ -functor

$$f^* : \mathcal{C}_{T,st} \longrightarrow \mathcal{C}_{S,st}$$

For this, it suffices to define f^* on the full A_∞ -subcategory $\mathcal{C}_T \subset \mathcal{C}_{T,st}$.

For $t \in T$, consider the fiber $f^{-1}(t) = [s_0(t), \dots, s_k(t)] \subset S$ as an interval equipped with its natural linear ordering. For the map of objects, we set $f^*(t)$ to be the twisted complex

$$(s_0(t) \xrightarrow{w_0} s_1(t) \xrightarrow{w_1} \dots \xrightarrow{w_{k-1}} s_k(t))$$

where $w_i \in \text{hom}_{\mathcal{C}_S}(s_i(t), s_{i+1}(t))$ is the degree one generating morphism. Note each morphism appearing has degree one and thus no shifts are needed. To confirm the twisted complex satisfies the generalized Maurer–Cartan equation, note that the fiber is an interval and so by construction, the composition maps on all subcollections of the morphisms appearing in the twisted complex vanish.

For the maps of morphisms, on the (non-identity) degree one generating morphisms $v_0 \in \text{hom}_{\mathcal{C}_T}(t_0, t_1)$, we define the first component $f_1^*(v_0) \in \text{hom}_{\mathcal{C}_S}(f^*(t_0), f^*(t_1))$ to be the composition of three morphisms: projection to the last term

$$(s_0(t_0) \xrightarrow{w_{0,0}} s_1(t_0) \xrightarrow{w_{0,1}} \cdots \xrightarrow{w_{0,k-1}} s_k(t_0)) \longrightarrow s_k$$

followed by the degree one generating morphism

$$s_k(t_0) \xrightarrow{w_k} s_0(t_1)$$

followed by inclusion to the first term

$$s_0(t_1) \longrightarrow (s_0(t_1) \xrightarrow{w_{1,0}} \sigma_1(t_1) \xrightarrow{w_{1,1}} \cdots \xrightarrow{w_{1,\ell-1}} s_\ell(t_1))$$

Here $w_{0,i} \in \text{hom}_{\mathcal{C}_S}(s_i(t_0), s_{i+1}(t_0))$, $w_{1,j} \in \text{hom}_{\mathcal{C}_S}(s_j(t_1), s_{j+1}(t_1))$ denote the degree one generating morphisms.

For $k > 1$, we set the higher components f_k^* to be identically zero.

To check that f^* defines a unital A_∞ -functor, it suffices to consider composable sequences of the (non-identity) degree one generating morphisms. By construction, each term of the equations individually vanishes on such sequences, except possibly for complete cycles

$$t_0 \xrightarrow{v_0} t_1 \xrightarrow{v_1} \cdots \xrightarrow{v_{n-1}} t_n \xrightarrow{v_n} t_0$$

For such sequences, two terms will not vanish, but indeed equal each other

$$\mu_{n+1}(f_1^*(v_n), \dots, f_1^*(v_0)) = u \text{ id}_{s_0(t_0)} = f_1^*(\mu_{n+1}(v_n, \dots, v_0))$$

where as above $s_0(t_0) \in S$ denotes the initial endpoint of the fiber $f^{-1}(t_0) \subset S$. Note that the evaluation of the left hand side follows from the constructions and the definition of component maps between twisted complexes

$$\mu_{n+1}(f_1^*(v_n), \dots, f_1^*(v_0)) = \mu_{m+1}(w_m, \dots, w_0)$$

where the right hand side is evaluated on a complete cycle

$$s_0 \xrightarrow{w_0} s_1 \xrightarrow{w_1} \cdots \xrightarrow{w_{n-1}} s_n \xrightarrow{w_n} s_0$$

where we start at $s_0 = s_0(t_0) \in S$.

Finally, we check that such functors compose as asserted. Let $f : S \rightarrow T$, $g : T \rightarrow U$ be cyclic surjections. On objects, it is clear we have $(g \circ f)^* = f^* \circ g^*$. On morphisms, it suffices to consider composable sequences of the (non-identity) degree one generating morphisms. By construction, each term of the equations individually vanishes on such

sequences, except for single terms $v_0 \in \text{hom}_{\mathcal{C}_U}(u_0, u_1)$, where it is clear we have $(g \circ f)_1^* = f_1^* \circ g_1^*$. \square

Remark 4.5. It is worth informally noting that the functors of the proposition are fully faithful.

5. Full cyclic functoriality

Theorem 5.1. *The assignment of the A_∞ -category $\mathcal{C}_{S,st}$ to a finite cyclically ordered nonempty set S naturally extends to a functor on the full cyclic category*

$$\mathcal{C}_{st} : \underline{A}^{op} \longrightarrow \mathcal{A}_{per}^{str}$$

Proof. Given a cyclic map $f : S \rightarrow T$, consider the object $Q = f(S) \subset S^1$. We have a canonical cyclic surjection $f_{surj} : S \rightarrow Q$ (given by f), and a canonical cyclic injection $f_{inj} : Q \rightarrow T$ (given by the identity), along with the evident equality $f = f_{inj} \circ f_{surj}$.

Let us define the functor $f^* : \mathcal{C}_{T,st} \rightarrow \mathcal{C}_{S,st}$ to be the composition $f^* = f_{surj}^* \circ f_{inj}^*$ of our previously defined functors. By our previous results, to see that this extends functorially, it suffices to show that if a cyclic map $f : S \rightarrow U$ happens to be a composition $f = g_{surj} \circ h_{inj}$ of an injection $h_{inj} : S \rightarrow T$ followed by a surjection $g_{surj} : T \rightarrow U$, then we have an equality

$$f^* = f_{surj}^* \circ f_{inj}^* = h_{inj}^* \circ g_{surj}^* : \mathcal{C}_{U,st} \longrightarrow \mathcal{C}_{S,st}$$

Furthermore, it suffices to assume the sizes of the sets satisfy $|S| = |T| - 1 = |U|$. It is convenient to consider two cases: (1) the composition f is an isomorphism; (2) the composition f is not an isomorphism.

Case (1): It suffices to consider $S = U = [s_0, \dots, s_n]$, $T = [s_0, \dots, s_n, t]$, with $h_{inj} : S \rightarrow T$ the evident injection, and $g_{surj} : T \rightarrow U$ the surjection such that $g_{surj}(t) = s_n$. (There is also the alternative case where $g_{surj}(t) = s_0$, but it follows from a completely parallel argument.)

We must check that $h_{inj}^* \circ g_{surj}^* : \mathcal{C}_{S,st} \rightarrow \mathcal{C}_{S,st}$ is the identity. On objects, this is clear, with the following observation: first forming the twisted complex $(s_n \rightarrow t)$ along the degree one generating morphism $v_{n,t} \in \text{hom}_{\mathcal{C}_S}(s_n, t)$, then sending t to the zero object results in the object s_n again.

On morphisms, it is also clear, with the following observation: by the definition of functors on twisted complexes, we have

$$(h_{inj}^*)_1 (g_{surj}^*)_1 (v_n) = (h_{inj}^*)_2 (v_{t,0}, v_{n,t}) = v_n \in \text{hom}_{\mathcal{C}_S}(s_n, s_0)$$

for the degree one generating morphisms

$$v_n \in \text{hom}_{\mathcal{C}_S}(s_n, s_0) \quad v_{n,t} \in \text{hom}_{\mathcal{C}_S}(s_n, t) \quad v_{t,0} \in \text{hom}_{\mathcal{C}_S}(t, s_0)$$

Case (2): We will consider two subcases.

(a) The first subcase is particularly simple. Fix $1 < i_0 < n$, and consider $S = [s_0, \dots, s_n]$, $T = [s_0, \dots, s_n, t]$, $U = [s_0, \dots, \hat{s}_{i_0}, \dots, s_n, t]$, with $h_{inj} : S \rightarrow T$ the evident injection, and $g_{surj} : T \rightarrow U$ the surjection such that $g_{surj}(t) = t$, $g_{surj}(s_j) = s_j$, for $j \neq i_0$, and $g_{surj}(s_{i_0}) = s_{i_0-1}$. In this situation, the asserted identity is evident, since the injection and surjection do not interact with each other.

(b) Finally, it suffices to consider $S = [s_0, \dots, s_n]$, $T = [s_0, \dots, s_n, t]$, $U = [s_0, \dots, s_{n-1}, t]$, with $h_{inj} : S \rightarrow T$ the evident injection, and $g_{surj} : T \rightarrow U$ the surjection such that $g_{surj}(s_n) = s_{n-1}$. (There is also the alternative case where $S = [s_0, \dots, s_n]$, $T = [s_0, \dots, s_n, t]$, $U = [s_1, \dots, s_n, t]$, with $h_{inj} : S \rightarrow T$ the evident injection, and $g_{surj} : T \rightarrow U$ the surjection such that $g_{surj}(s_0) = s_1$, but it follows from a completely parallel argument.)

We must check that $f_{surj}^* \circ f_{inj}^* = h_{inj}^* \circ g_{surj}^* : \mathcal{C}_{U,st} \rightarrow \mathcal{C}_{S,st}$ where $Q = f(S) = [s_0, \dots, s_{n-1}]$, $f_{inj} : Q \rightarrow U$ is the evident injection, and $f_{surj} : S \rightarrow Q$ is the surjection such that $f(s_n) = s_{n-1}$.

On objects, this is immediate from the definitions: under both functors, the objects s_0, \dots, s_{n-2} are sent to themselves, t is sent to 0, and s_{n-1} is sent to the twisted complex $(s_{n-1} \rightarrow s_n)$ along the degree one generating morphism $v_{n-1} \in \text{hom}_{\mathcal{C}_S}(s_{n-1}, s_n)$.

On morphisms, it is also clear, with the following observation: by the definition of functors on twisted complexes, we have

$$f_2^*(v_{t,0}, v_{n-1,t}) = (h_{inj}^*)_2((g_{surj}^*)_1 v_{t,0}, (g_{surj}^*)_1 v_{n-1,t}) = v_n \in \text{hom}_{\mathcal{C}_S}((s_{n-1} \rightarrow s_n), s_0)$$

for the degree one generating morphisms

$$v_{n-1,t} \in \text{hom}_{\mathcal{C}_S}(s_{n-1}, t) \quad v_{t,0} \in \text{hom}_{\mathcal{C}_S}(t, s_0) \quad \square$$

Remark 5.2. Let $T = [t_0, t_1, \dots, t_n] \subset S^1$ be a finite cyclically ordered nonempty set regarded as an object of the cyclic category $\underline{\mathcal{A}}$. As usual, we will understand the indices modulo $n+1$ so that t_{i+n+1} also denotes t_i . There is a canonical rotational automorphism $\tau : T \rightarrow T$ that sends $\tau(t_i) = t_{i-1}$ and hence satisfies $\tau^{n+1} = \text{id}_T$. One can check that the functor \mathcal{C}_{st} takes the morphism τ to the shifted Serre functor $S[-1]$ of the category $\mathcal{C}_{T,st}$. (Here we are in a two-periodic setting so only the odd nature of the shift is meaningful, but it is more natural to shift by -1 with the graded version in mind.) Note that this is compatible with the equivalence $\mathcal{C}_{T,st} \simeq \mathcal{Q}_n$ of Proposition 4.2 and the fact that the Serre functor on the category of modules over the A_n -quiver satisfies $S^{n+1} = [n-1]$. See Remark 6.11 for a graded lift of this observation.

6. Graded version

In this section, we indicate how to lift our prior constructions to the traditional \mathbf{Z} -graded (rather than two-periodic) setting.

Definition 6.1. Let \mathcal{A}^{str} denote the strict category with objects k -linear A_∞ -categories and morphisms k -linear A_∞ -functors.

Remark 6.2. There is the evident forgetful functor $Per : \mathcal{A}^{str} \rightarrow \mathcal{A}_{per}^{str}$ where we tensor up from k to the graded k -algebra $k[u, u^{-1}]$ with $\deg u = 2$.

Next we introduce the natural structure needed to lift our prior constructions. Let $\text{Sym}^2(S^1)$ denote the second symmetric product of S^1 . Recall that it is homeomorphic to the Möbius strip, in particular it is homotopy equivalent to a circle.

Definition 6.3. Let $\underline{\mathcal{A}}_{gr}$ denote the category defined as follows.

Objects are pairs (S, c) of a finite nonempty subset of the circle $S \subset S^1$ and a point $c \in \text{Sym}^2(S^1)$.

Morphisms $(S, c) \rightarrow (S', c')$ are homotopy classes of pairs (φ, γ) of a degree 1 map $\varphi : S^1 \rightarrow S^1$ such that $\varphi(S) \subset S'$ and a path $\gamma : [0, 1] \rightarrow \text{Sym}^2(S^1)$ such that $\gamma(0) = \varphi(c)$, $\gamma(1) = c'$.

Composition of morphisms $(S, c) \rightarrow (S', c') \rightarrow (S'', c'')$ is given by the homotopy class of the composition

$$(\varphi', \gamma') \circ (\varphi, \gamma) = (\varphi' \circ \varphi, \gamma' \#_{\varphi'(c')} (\varphi' \circ \gamma))$$

where we write $\#_{\varphi'(c')}$ for the concatenation of paths at $\varphi'(c') \in \text{Sym}^2(S^1)$.

Remark 6.4. Note that every morphism $(\varphi, \gamma) : (S, c) \rightarrow (S', c')$ can be factored

$$(\varphi, \gamma) : (S, c) \xrightarrow{(\varphi, \gamma_{const})} (\varphi(S) = S', \varphi(c)) \xrightarrow{(\text{id}_{S^1}, \gamma)} (S', c')$$

where γ_{const} denotes the constant path.

Remark 6.5. There is the full and essentially surjective forgetful functor $For : \underline{\mathcal{A}}_{gr} \rightarrow \underline{\mathcal{A}}$ where we forget the points $c \in \text{Sym}^2(S^1)$ and the paths $\gamma : [0, 1] \rightarrow \text{Sym}^2(S^1)$.

Next we present a graded version of [Definition 2.4](#).

Let $S = [s_0, s_1, \dots, s_n] \subset S^1$ be a finite cyclically ordered nonempty set. As usual, we will understand the indices modulo $n + 1$ so that s_{i+n+1} also denotes s_i .

Given a point $c \in \text{Sym}^2(S^1)$, we can assign a weight $\alpha_i \in \{0, 1, 2\}$, for all $i = 0, \dots, n$, by taking the multiplicity of c in the closed-open interval $[s_i, s_{i+1})$. Note in particular the total weight satisfies $\sum_{i=0}^{n+1} \alpha_i = 2$.

Definition 6.6. Let $S = [s_0, s_1, \dots, s_n] \subset S^1$ be a finite cyclically ordered nonempty set.

Let $c \in \text{Sym}^2(S^1)$ be a point of the second symmetric power of S^1 .

We define the unital k -linear A_∞ -category $\mathcal{C}_{S,c,gr}$ as follows.

(1) Objects: $\text{Ob } \mathcal{C}_{S,c,gr} = S$.

(2) Morphisms: the free k -modules generated by the identity morphisms and the additional morphisms

$$v_i \in \text{hom}_{\mathcal{C}_S}(s_i, s_{i+1}), \quad |v_i| = 1 - \alpha_i, \quad \text{for } i = 0, \dots, n.$$

(3) Compositions: all are zero except for the k -linear maps given by the identity compositions and the additional compositions

$$\mu_{n+1}(v_{i+n}, \dots, v_i) = \text{id}_{s_i}, \quad \text{for } i = 0, \dots, n.$$

We define the unital k -linear A_∞ -category $\mathcal{C}_{S,c,st,gr}$ to be the A_∞ -category of twisted complexes of $\mathcal{C}_{S,c,gr}$.

We have the following graded version of [Proposition 4.2](#). Let $\mathcal{Q}_{n,gr}$ denote the k -linear triangulated differential graded derived category of finite-dimensional left modules over the A_n -quiver

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_n$$

For $i = 1, \dots, n$, denote by $m_i \in \mathcal{Q}_n$ the free rank one module supported at the i th vertex and zero elsewhere. Note that among these objects, there are degree one generating morphisms $w_i : m_i \rightarrow m_{i+1}$, for $i = 1, \dots, n - 1$, and no other linearly independent non-identity morphisms.

The proof of the following is immediate given that of [Proposition 4.2](#).

Proposition 6.7. *Let $S = [s_0, s_1, \dots, s_n] \subset S^1$ be a finite cyclically ordered nonempty set.*

Let $c \in \text{Sym}^2(S^1)$ be a point of the second symmetric power of S^1 .

Fix the element $s_0 \in S$ so that the remaining elements inherit the linear order s_1, \dots, s_n .

Then there is a quasi-equivalence of A_∞ -categories

$$F : \mathcal{Q}_{n,gr} \xrightarrow{\sim} \mathcal{C}_{S,c,st,gr}$$

such that $F(m_i) = s_i[\eta_i]$, for $i = 1, \dots, n$, and $F(w_i) = v_i$, for $i = 1, \dots, n - 1$, where the shift $\eta_i \in \{0, 1, 2\}$ is the multiplicity of c in the closed-open interval $[s_0, s_i)$.

Now we have arrived at the graded version of [Theorem 5.1](#). The proof is largely a routine check of gradings given our prior constructions.

Theorem 6.8. *The assignment of the A_∞ -category $\mathcal{C}_{S,c,st,gr}$ to a pair of a finite cyclically ordered nonempty set $S \subset S^1$ and a point $c \in \text{Sym}^2(S^1)$ naturally extends to a functor $\mathcal{C}_{st,gr} : \underline{\mathcal{A}}_{gr}^{op} \rightarrow \mathcal{A}^{str}$ fitting into a commutative diagram*

$$\begin{array}{ccc}
 \underline{\mathcal{A}}_{gr}^{op} & \xrightarrow{\mathcal{C}_{st,gr}} & \mathcal{A}^{str} \\
 \text{For} \downarrow & & \downarrow \text{Per} \\
 \underline{\mathcal{A}}^{op} & \xrightarrow{\mathcal{C}_{st}} & \mathcal{A}_{per}^{str}
 \end{array}$$

Proof. Recall that every morphism $(\varphi, \gamma) : (S, c) \rightarrow (S', c')$ can be factored

$$(\varphi, \gamma) : (S, c) \xrightarrow{(\varphi, \gamma_{const})} (\varphi(S) = S', \varphi(c)) \xrightarrow{(\text{id}_{S^1}, \gamma)} (S', c')$$

where γ_{const} denotes the constant path.

Let us explain the functoriality of the assignment $\mathcal{C}_{S,c,st,gr}$ under morphisms of the form

$$\gamma = (\text{id}_{S^1}, \gamma) : (S, c) \longrightarrow (S, c')$$

We will then leave the remaining compatibility check to the reader.

Given a path $\gamma : [0, 1] \rightarrow \text{Sym}^2(S^1)$, we can assign a weight $\beta_i \in \mathbf{Z}$, for all $i = 0, \dots, n$, by the signed count of the number of times $\gamma(t)$ crosses $s_i \in S$ from the closed–open interval $[s_{i-1}, s_i)$ to the open–closed interval $(s_i, s_{i+1}]$.

On objects, we define the induced functor to be given by shifts

$$\gamma^* : \mathcal{C}_{S,c,st,gr} \rightarrow \mathcal{C}_{S,c',st,gr} \quad \gamma^* s_i = s_i[-\beta_i]$$

On morphisms and compositions, we set it to be the identity.

We leave the compatibility check of our prior constructions to the reader. \square

Remark 6.9. The compatibility check can be made particularly easy by the following observation: with the construction of γ^* in hand, we can move any $c \in \text{Sym}^2(S^1)$ to consist of two distinct points, and also move any $c \in \text{Sym}^2(S^1)$ to arrange that a given morphism $(\varphi, \gamma) : (S, c) \rightarrow (S', c')$ has the property that φ is injective on c .

Remark 6.10. Let $T \subset S^1$ be a finite cyclically ordered nonempty set. Let $c \in \text{Sym}^2(S^1)$ be a point of the second symmetric power of S^1 . Consider the pair (T, c) as an object of the category $\underline{\mathcal{A}}_{gr}$. There is a partial sweep automorphism $\sigma = (\text{id}_{S^1}, \gamma) : (T, c) \rightarrow (T, c)$ where $\gamma : [0, 1] \rightarrow \text{Sym}^2(S^1)$ fixes one point of c and wraps (with reverse orientation) the other around S^1 back to itself. One can check that the functor $\mathcal{C}_{st,gr}$ takes the morphism σ to the shift functor $[-1]$ on the category $\mathcal{C}_{T,c,st,gr}$. Similarly, the total sweep automorphism σ^2 that wraps both points of c around S^1 back to themselves is taken to the squared shift functor $[-2]$.

Remark 6.11. Let $T = [t_0, t_1, \dots, t_n] \subset S^1$ be a finite cyclically ordered nonempty set. As usual, we will understand the indices modulo $n + 1$ so that t_{i+n+1} also denotes t_i .

Let $c \in \text{Sym}^2(S^1)$ be a point of the second symmetric power of S^1 . Consider the pair (T, c) as an object of the category \underline{A}_{gr} . There is a canonical rotation automorphism $\tau = (\varphi, \gamma) : (T, c) \rightarrow (T, c)$ where $\varphi(t_i) = t_{i-1}$ and $\gamma : [0, 1] \rightarrow \text{Sym}^2(S^1)$ takes c minimally (with reverse orientation) to $\varphi(c)$. It satisfies $\tau^{n+1} = \sigma^2$ where σ^2 is the total sweep of Remark 6.10. One can check that the functor $\mathcal{C}_{st,gr}$ takes the morphism τ to the shifted Serre functor $S[-1]$ of the category $\mathcal{C}_{T,c,st,gr}$. Note that this is compatible with the equivalence $\mathcal{C}_{T,c,st,gr} \simeq \mathcal{Q}_{n,gr}$ of Proposition 6.7 and the fact that the Serre functor on the category of modules over the A_n -quiver satisfies $S^{n+1} = [n - 1]$.

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