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Splitting families of sets in ZFC

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A B S T R A C T

Miller's 1937 splitting theorem was proved for every finite $n > 0$ for all ρ-uniform families of sets in which ρ is infinite. A simple method for proving Miller-type splitting theorems is presented here and an extension of Miller's theorem is proved in ZFC for every cardinal ν for all ρ-uniform families in which $\rho \geq \beth_\omega(\nu)$. The main ingredient in the method is an asymptotic infinitary Löwenheim–Skolem theorem for anti-monotone set functions.

As corollaries, the use of additional axioms is eliminated from splitting theorems due to Erdős and Hajnal [1], Komjáth [7], Hajnal, Juhász and Shelah [4]; upper bounds are set on conflict-free coloring numbers of families of sets; and a general comparison theorem for ρ-uniform families of sets is proved, which generalizes Komjáth's comparison theorem for $\aleph_0$-uniform families [8].

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1. Introduction

All modern theorems on splitting families of infinite sets follow Miller's seminal 1937 inductive proof [10], in which Miller assumes that an arbitrary ρ-uniform family of sets satisfies that the intersection of any ρ^+ of its members has cardinality smaller than some fixed *finite* number n.

Erdős and Hajnal devoted [1] to relaxing Miller's condition, which they denoted by $C(\rho^+, n)$, to $C(\rho^+, \nu)$ with a given infinite ν for all sufficiently large ρ. With the Generalized Continuum Hypothesis adopted as an additional axiom, they generalized Miller's theorem to all infinite ν and ρ such that $\rho > \nu^+$.

Hajnal, Juhász and Shelah [4] proved a sophisticated general theorem, with a rather difficult proof, that enabled the relaxation of the GCH axiom to a considerably weaker additional axiom, some weak variant of the Singular Cardinals Hypothesis, in proving several Miller-type theorems, including the one by Erdős and Hajnal. These results were proved for an infinite ν and $\rho > 2^\nu$.

Here we prove those and a few other Miller type theorems in ZFC with no additional axioms for an arbitrary ν and $\rho \geq \beth_\omega(\nu)$. We present and use a general method, which extends the method that was used in [6] for infinite graphs colorings. This new method does not require any specialized notions and is not more complicated than Miller's original method. It may be useful in other contexts as well. Let us describe next the set-theoretic development that made this method possible.

Miller's original proof used a closure argument under finitary operations, which amounts to a weak version of the well-known Löwenheim–Skolem theorem, to obtain filtrations of arbitrary large families, from which the inductive argument could be carried out. With $C(\rho^+, \nu)$ replacing $C(\rho^+, n)$, the closure needs to be formed with respect to an infinitary operation. However, there is no general Löwenheim–Skolem theorem for ν-ary operations, for a good reason: the equation $\lambda^n = \lambda$, which entails the Löwenheim–Skolem theorem at λ, holds for every infinite λ and finite $n > 0$, but $\lambda^\nu = \lambda$, which implies the Löwenheim–Skolem theorem at λ for ν-ary operations, fails periodically by König's lemma and, worse still, cannot be generally upper-bounded in terms of λ on the basis of ZFC.

Erdős and Hajnal used in [1] a GCH theorem by Tarski [15] to obtain filtrations with respect to the required closure notion, as did Komjáth in [7]. Hajnal, Juhász and Shelah [4] used a weak version of the SCH and L-like combinatorial principles in their proof, in which filtrations are replaced by a more sophisticated abstract notion. Here we use simple arithmetic relations concerning the density of κ-subsets of λ to prove an asymptotic Löwenheim–Skolem theorem and obtain filtrations in ZFC for order-reversing set operations. The theorem holds at all sufficiently large λ.

What makes this possible is Shelah's spectacular revised GCH theorem [12] (which, of course, was not available at the time [4] was published). This theorem is Shelah's most important achievement in pcf theory. It provides, for the first time since

$\lambda^n = \lambda$ was proved, cardinal arithmetic equations which are true in an end-segment of the cardinals. Simply put, Shelah's new absolute arithmetic can be used in place of the additional axioms in this combinatorial context. Furthermore, no intricacies of pcf theory or L-like principles are used in the proofs. The usefulness of Shelah's RGCH theorem is completely encapsulated into its simple arithmetic statement.

1.1. The results

After developing the required density equations, a general Löwenheim–Skolem theorem and the existence of filtrations are established for order-reversing set operations. This is done in the first two sections. In the third, the existence of filtrations is used almost verbatim to replace the finite ν by an arbitrary infinite ν in Miller's original proof. A similar argument, relying only on elementary properties of density and not involving the RGCH theorem, provides short proofs to the combinatorial splitting theorems from [4] from the additional assumption there.

We prove theorems of three types. First, the classical combinatorial splitting theorems are proved in ZFC for an arbitrary ν and $\rho \geq \beth_\omega(\nu)$. Then, conflict-free colorings are treated. Finally, a comparison theorem by Komjáth is extended to infinite cardinals.

2. Arithmetic of density

In this section we list some properties of the arithmetic density function $\mathcal{D}(\lambda, \kappa_1, \kappa_1)$.

Definition 2.1 *(Density).*

(1) Let $\kappa_1 \leq \kappa_2 \leq \lambda$ be cardinals. A set $\mathcal{D} \subseteq [\lambda]^{\kappa_1}$ is *dense in* $[\lambda]^{\kappa_2}$ if for every $X \in [\lambda]^{\kappa_2}$ there is $Y \in \mathcal{D}$ such that $Y \subseteq X$.

(2) For $\kappa_1 \leq \kappa_2 \leq \lambda$ the (κ_1, κ_2)*-density of* λ, denoted by $\mathcal{D}(\lambda, \kappa_1, \kappa_2)$, is the least cardinality of a set $\mathcal{D} \subseteq [\lambda]^{\kappa_1}$ which is dense in $[\lambda]^{\kappa_2}$.

(3) For $\kappa \leq \lambda$ the κ*-density of* λ, denoted by $\mathcal{D}(\lambda, \kappa)$, is $\mathcal{D}(\lambda, \kappa, \kappa)$.

Claim 2.2 *(Basic properties of density). Suppose $\kappa_1 \leq \kappa_2 \leq \lambda$.*

(1) *If $X \subseteq Y$ and $\mathcal{D} \subseteq [Y]^{\kappa_1}$ is dense in $[Y]^{\kappa_2}$, then $\mathcal{D} \cap [X]^{\kappa_1}$ is dense in $[X]^{\kappa_2}$.*

(2) *(Monotonicity) $\mathcal{D}(\lambda, \kappa_1, \kappa_2)$ is increasing in λ and if $\kappa_1' \leq \kappa_1 \leq \kappa_2 \leq \kappa_2' \leq \lambda$ then $\mathcal{D}(\lambda, \kappa_1', \kappa_2') \leq \mathcal{D}(\lambda, \kappa_1, \kappa_2)$.*

(3) *Suppose $\langle X_i : i < \theta \rangle$ is $\subseteq$-increasing, $X = \bigcup_{i<\theta} X_i$, $\kappa_1 \leq \kappa_2 \leq |X_0|$ and $\mathcal{D}_i \subseteq [X_i]^{\kappa_1}$ is dense in $[X_i]^{\kappa_2}$. Then $\mathcal{D} := \bigcup_{i<\theta} \mathcal{D}_i$ is dense in $[X]^{\kappa_3}$ for all $\kappa_3 > \kappa_2$ such that $\kappa_3 \leq |X|$, and if $\operatorname{cf} \theta \neq \operatorname{cf} \kappa_2$ then $\mathcal{D}$ is dense also in $[X]^{\kappa_2}$.*

(4) *(Continuity at limits) If* $\lambda = \langle \lambda_i : i < \theta \rangle$ *is increasing with limit* λ*, and* $\mathcal{D}(\lambda_i, \kappa_1, \kappa_2) \le \lambda$*, then* $\mathcal{D}(\lambda, \kappa_1, \kappa_3) = \lambda$ *for all* $\kappa_3 > \kappa_2$ *such that* $\kappa_3 \le |X|$*, and if* $\operatorname{cf}\lambda \neq \operatorname{cf}\kappa_2$ *then also* $\mathcal{D}(\lambda, \kappa_1, \kappa_2) = \lambda$.
(5) *If* $\operatorname{cf}\mu = \operatorname{cf}\kappa < \mu$ *then* $\mathcal{D}(\mu, \kappa) > \mu$.
(6) *If* μ *is a strong limit cardinal, then* $\mathcal{D}(\mu, \kappa) = \mu$ *for all* $\kappa < \mu$ *such that* $\operatorname{cf}\kappa \neq \operatorname{cf}\mu$.
(7) $\mathcal{D}(\lambda, \kappa)$ *is* not *increasing in* κ *and the inequality in (2) can be strict.*

Proof. Item (1) follows from the definition of density. The monotonicity in λ in (2) follows from (1).

Assume $\kappa_1' \le \kappa_1 \le \kappa_2 \le \kappa_2' \le \lambda$, let $\mathcal{D} \subseteq [\lambda]^{\kappa_1}$ be an arbitrary dense set in $[\lambda]^{\kappa_2}$ and for each $X \in \mathcal{D}$ fix $X' \in [X]^{\kappa_1'}$. Let $\mathcal{D}' = \{X' : X \in \mathcal{D}\}$. Given any $Y \in [\lambda]^{\kappa_2'}$ there is some $X \in \mathcal{D}$ such that $X \subseteq Y$, hence $X' \subseteq Y$. Clearly, $|\mathcal{D}'| \le |\mathcal{D}|$. Thus, $\mathcal{D}(\lambda, \kappa_1', \kappa_2') \le \mathcal{D}(\kappa_1, \kappa_2)$.

To prove (3) assume first that $\operatorname{cf}\theta \neq \operatorname{cf}\kappa_2$ and let $Z \in [X]^{\kappa_2}$ be arbitrary. Since $\operatorname{cf}\kappa_2 \neq \operatorname{cf}\theta$ there is some $i < \theta$ such that $|Z \cap X_i| = \kappa_2$, and thus there is some $Y \in \mathcal{D}_i \subseteq \mathcal{D}$ such that $Y \subseteq Z$. This establishes the density of $\mathcal{D}$ in $[X]^{\kappa_2}$; as $[X]^{\kappa_2}$ is dense in $[X]^{\kappa_3}$ for all $\lambda \ge \kappa_3 \ge \kappa_2$, the density of $\mathcal{D}$ in $[X]^{\kappa_3}$ follows. Now it remains to prove the density of $\mathcal{D}$ in $[X]^{\kappa_3}$ for $\lambda \ge \kappa_3 > \kappa_2$ when $\operatorname{cf}\theta = \operatorname{cf}\kappa_2$. Given $Z \in [X]^{\kappa_3}$ it suffices to show that $|Z \cap X_i| \ge \kappa_2$ for some $i < \theta$. As $\kappa_3 > \kappa_2$ and $\operatorname{cf}\kappa_2 = \operatorname{cf}\theta$ actually more is true: there is $i < \theta$ such that $|Z \cap X_i| > \kappa_2$ (or else $|Z| \le \operatorname{cf}\kappa_2 \times \kappa_2 = \kappa_2$).

Item (4) follows from (3).

The inequality (5) follows from the standard diagonalization argument in the proof of König's lemma which, in fact, proves that there exists an almost disjoint $\mathcal{F} \subseteq [\mu]^\kappa$ of cardinality $> \mu$. This implies that $\mathcal{D}(\mu, \kappa) > \mu$.

For (6), let μ be a strong limit and let $\kappa < \mu$ be arbitrary. There is an unbounded set of cardinals $\lambda < \mu$ below μ which satisfy $\lambda^\kappa = \lambda$ hence $\mathcal{D}(\mu, \kappa) = \mu$ follows from (4) if $\operatorname{cf}\mu \neq \operatorname{cf}\kappa$.

To see (7) let ν be an arbitrary cardinal. By (5), $\mathcal{D}(\beth_\omega(\nu), \aleph_0) > \beth_\omega(\nu)$ while by (6) $\mathcal{D}(\beth_\omega(\nu), \aleph_1) = \beth_\omega(\nu)$. Similarly, $\mathcal{D}(\beth_\omega(\nu), \aleph_0) > \beth_\omega(\nu) = \mathcal{D}(\beth_\omega(\nu), \aleph_0, \aleph_1) = \mathcal{D}(\beth_\omega(\nu), \aleph_1)$ and $\mathcal{D}(\beth_{\omega_1}(\nu), \aleph_0, \aleph_1) = \beth_{\omega_1}(\nu) < \mathcal{D}(\beth_{\omega_1}(\nu), \aleph_1)$, so (7) is proved. □

The next claim provides a simple sufficient condition for the validity of the equation $\mathcal{D}(\lambda, \kappa_1, \kappa_2) = \lambda$ for all λ in an end-segment of the cardinals.

Claim 2.3. *Suppose* $\theta > \kappa_2 > \kappa_1 \ge \aleph_0$, $\mathcal{D}(\theta, \kappa_1, \kappa_2) = \theta$ *and for every* $\lambda \ge \theta$*, if* $\operatorname{cf}\lambda = \operatorname{cf}\kappa_2$ *there exists some* ν *such that* $\kappa_1 \le \nu < \kappa_2$ *and* $\lambda = \sup\{\delta : \delta < \lambda \wedge \mathcal{D}(\delta, \kappa_1, \nu) \le \lambda\}$. *Then* $\mathcal{D}(\lambda, \kappa_1, \kappa_2) = \lambda$ *for every* $\lambda \ge \theta$.

Proof. By induction on $\lambda \ge \theta$. For $\lambda = \theta$ the equality $\mathcal{D}(\lambda, \kappa_1, \kappa_2) = \lambda$ is assumed, and if $\lambda > \theta$ satisfies $\operatorname{cf}\lambda \neq \operatorname{cf}\kappa_2$ then $\mathcal{D}(\lambda, \kappa_1, \kappa_2) = \lambda$ by (4). Assume then that $\operatorname{cf}\lambda = \operatorname{cf}\kappa_2$. By the assumption, there is $\kappa_1 \le \nu < \kappa_2$ and $\langle \lambda_i : i < \operatorname{cf}\kappa_2 \rangle$, an increasing sequence of cardinals with limit λ such that $\mathcal{D}(\lambda_i, \kappa_1, \nu,) \le \lambda$. Now $\mathcal{D}(\lambda, \kappa_1, \kappa_2) = \lambda$ by (4). □

Corollary 2.4.

(1) *Let ν be an infinite cardinal, $\kappa = \text{cf}\kappa > \nu$ and assume that $\theta \geq \kappa$ satisfies such that $\mathcal{D}(\theta, \nu, \kappa) = \theta$ and for every $\mu > \theta$ all limit cardinals in the interval $[\mu, \mu^\nu)$ are of cofinality strictly smaller than κ. Then $\mathcal{D}(\lambda, \nu, \kappa) = \lambda$ for all $\lambda \geq \theta$.*
(2) *The SCH implies that for every infinite ν it holds that $\mathcal{D}(\lambda, \nu, \nu^+) = \lambda$ for all $\lambda \geq 2^\nu$.*

Proof. If $\lambda \geq \theta$ and $\text{cf}\,\lambda = \kappa$ then $\theta^\mu < \lambda$ for all $\theta < \lambda$, so $\lambda = \sup\{\theta^\nu : \theta < \lambda\}$. If $\delta = \theta^\nu$ then $\delta^\nu = \delta$ and trivially $\mathcal{D}(\delta, \nu) = \delta$. Thus (1) follows from Claim 2.3. Item (2) follows from (1) since by the SCH for every cardinal $\lambda \geq 2^\nu$ it holds that $\lambda^\nu \in \{\lambda, \lambda^+\}$. □

By Corollary 2.4, the existence of a bound on the gaps between μ and μ^κ in some end segment of the cardinals suffices to cover the end segment by a single equation of the form $\mathcal{D}(x, \nu, \kappa) = x$. However, models of set theory with arbitrarily large gaps between μ and $\mu^{\aleph_0}$ can be built from a proper class of suitable large cardinals by the methods in [3].

Lemma 2.7 and its Corollary 2.8 below, hold in every model of ZFC.

Definition 2.5 *(Shelah's revised power).* Let $\theta \leq \lambda$ be cardinals. A set $\mathcal{D} \subseteq [\lambda]^\theta$ is *weakly covering* if for every $X \in [\lambda]^\theta$ there exists $\mathcal{Y} \in [\mathcal{D}]^{<\theta}$ such that $X \subseteq \bigcup \mathcal{Y}$.

Shelah's revised θ-power of λ, denoted $\lambda^{[\theta]}$, is the least cardinality of a weakly covering $\mathcal{D} \subseteq [\lambda]^\theta$.

Theorem 2.6 *(Shelah's revised GCH theorem). (See [12].) For every strong limit cardinal μ, for all $\lambda \geq \mu$, for every sufficiently large regular $\theta < \mu$,*

$$\lambda^{[\theta]} = \lambda.$$

Lemma 2.7. *Let μ be a strong limit cardinal. For every $\lambda \geq \mu$ there is some $\theta(\lambda) < \mu$ such that for every $\theta < \mu$ with $\text{cf}\,\theta > \theta(\lambda)$ it holds that $\mathcal{D}(\lambda, \theta) = \lambda$.*

Proof. Let $\lambda \geq \mu$ be given and let $\theta(\lambda) < \mu$ be fixed by Shelah's revised GCH theorem such that for every regular $\theta \in (\theta(\lambda), \mu)$ it holds that $\lambda^{[\theta]} = \lambda$.

Assume first that $\theta \in (\theta(\lambda), \mu)$ is regular. Let $\mathcal{D} \subseteq [\lambda]^\theta$ witness $\lambda^{[\theta]} = \lambda$ and, as $2^\theta < \lambda$, assume that $X \in \mathcal{D} \Rightarrow [X]^\theta \subseteq \mathcal{D}$. If $Z \in [\lambda]^\theta$ is arbitrary, fix $\mathcal{Y} \in [\mathcal{D}]^{<\theta}$ such that $Z \subseteq \bigcup \mathcal{Y}$. As θ is regular, there is $X \in \mathcal{Y}$ such that $Y := X \cap Z$ has cardinality θ. Since $Y \in \mathcal{D}$ and $Y \subseteq Z$, we have shown that $\mathcal{D}$ is dense in $[\lambda]^\theta$.

Assume now that $\theta \in (\theta(\lambda), \mu)$ is singular and satisfies $\text{cf}\,\theta > \theta(\lambda)$. Write $\theta = \sum_{i<\text{cf}\,\theta} \theta_i$ with $\theta_i = \text{cf}\,\theta_i > \theta(\lambda)$ for each i. Next fix dense $\mathcal{D}_i = \{X^i_\alpha : \alpha < \lambda\} \subseteq [\lambda]^{\theta_i}$ and dense $\mathcal{T}' \subseteq [\text{cf}\,\theta \times \lambda]^{\text{cf}\,\theta}$ with $|\mathcal{T}'| = \lambda$. The set $\mathcal{T} := \{Y \in \mathcal{T} : i < \text{cf}\,\theta \Rightarrow |Y \cap (\{i\} \times \lambda)| \leq 1\}$, namely, all members of $\mathcal{T}'$ which are partial functions from $\text{cf}\,\theta$ to λ with domain cofinal in $\text{cf}\,\theta$, has cardinality λ and for every $f : \text{cf}\,\theta \to \lambda$ there exists $Y \in \mathcal{T}$ such that $Y \subseteq f$.

Let

$$\mathcal{D} = \Big\{ \bigcup \{ X^i_\alpha : \langle i, \alpha \rangle \in Y \} : Y \in \mathcal{T} \Big\}.$$

Clearly, $\mathcal{D} \subseteq [\lambda]^\theta$ and $|\mathcal{D}| = \lambda$.

Given any $Z \in [\lambda]^\theta$, define $f(i) = \min\{\alpha < \lambda : X^i_\alpha \subseteq Z\}$ for $i < \operatorname{cf}\theta$. Now $\bigcup_{i<\operatorname{cf}\theta} X^i_{f(i)} \in [Z]^\theta$. There is some $Y \in \mathcal{T}$ such that $Y \subseteq f$. The set $\bigcup\{X^i_\alpha : \langle i, \alpha\rangle \in Y\}$ therefore belongs to $[Z]^\theta \cap \mathcal{D}$. □

Now the asymptotic equations concerning density follow:

Corollary 2.8. *For every infinite cardinal ν and $\rho \geq \beth_\omega(\nu)$, for all but finitely many $n < \omega$ it holds that*

$$\mathcal{D}\big(\rho, \beth_n(\nu)\big) = \rho, \tag{1}$$

and

$$\mathcal{D}\big(\rho, \nu, \beth_\omega(\nu)\big) = \rho. \tag{2}$$

Proof. For all n, $\operatorname{cf}\beth_{n+1}(\nu) > \beth_n(\nu)$, so $\operatorname{cf}\beth_n(\nu)$ converges to $\beth_\omega(\nu)$, hence (1) follows from Lemma 2.7. To prove (2) fix, for a given $\rho \geq \beth_\omega(\nu)$, some n such that $\mathcal{D}(\rho, \beth_n(\nu)) = \rho$ and as $\nu \leq \beth_n(\nu) < \beth_\omega(\nu)$, by monotonicity $\mathcal{D}(\rho, \nu, \beth_\omega(\nu)) \leq \mathcal{D}(\rho, \beth_n(\nu))$. □

The consistency of $\mathcal{D}(\rho, \nu, \beth_n(\nu)) > \rho$ for $\beth_n(\nu) \leq \rho < \beth_\omega(\nu)$ would imply that $\beth_\omega(\nu)$ cannot be relaxed to $\beth_n(\nu)$ in (2) above. See Section 3.1 below for a discussion of this — at the moment unknown — consistency.

Shelah pointed out to me the next lemma, which will be used in the proof of Theorem 4.14.

Lemma 2.9. *Suppose μ is a strong limit cardinal. Let χ be a sufficiently large regular cardinal and $\delta < \chi$ is a limit ordinal. Suppose a sequence $\langle M_i : i \leq \delta\rangle$ of elementary submodels of $(H(\chi), \in, \ldots)$ satisfies:*

(1) $M_i \subseteq M_{i+1}$ *and* $M_j = \bigcup_{i<j} M_i$ *when* $j \leq \delta$ *is limit;*
(2) $\langle M_j : j \leq i\rangle \in M_{i+1}$ *for all* $i < \delta$*;*
(3) $\mu \subseteq M_0$.

Then there exists $\kappa(*) < \mu$ such that $M_\delta \cap [M_\delta]^\kappa$ is dense in $[M_\delta]^\kappa$ for all κ such that $\kappa(*) \leq \operatorname{cf}\kappa \leq \kappa < \mu$.

Proof. Denote $\lambda = |M_\delta|$ and for $i < \delta$ denote $\lambda_i = |M_i|$. Let $\kappa(*) < \mu$ be such that $\mathcal{D}(\lambda, \kappa) = \lambda$ for all $\kappa < \mu$ with $\operatorname{cf}\kappa \geq \kappa(*)$. By increasing $\kappa(*)$ if necessary, we assume that $\operatorname{cf}\delta \notin [\kappa(*), \mu)$.

Given $i < \delta$ and κ such that $\kappa(*) \le \mathrm{cf}\kappa \le \kappa < \mu$, observe that, by elementarity, there exists a dense $\mathcal{D}_i^\kappa \subseteq [M_i]^\kappa$ of size $\mathcal{D}(\lambda_i, \kappa)$ such that $\mathcal{D}_i^\kappa \in M_{i+1}$. As $\lambda_i \le \lambda$, it holds by Claim 2.2(2) that $|\mathcal{D}_i^\kappa| \le \lambda$ for all $i < \delta$.

Now observe that as $M_i \in M_{i+1}$, also $\lambda_i = |M_i| \in M_{i+1}$ by elementarity, and as $M_i \subseteq M_{i+1}$ it holds that $\lambda_i \subseteq M_{i+1}$. Thus, $\lambda_i \subseteq M_{i+1}$ for all $i < \delta$ and hence $\bigcup_{i<\delta} \lambda_i = \lambda \subseteq M_\delta$ as well.

For each $i < \delta$, since $\mathcal{D}_i^\kappa \in M_\delta$ and $|\mathcal{D}_i^\kappa| \le \lambda \subseteq M_\delta$, it holds that $\mathcal{D}_i^\kappa \subseteq M_\delta$. Now $\mathcal{D}^\kappa := \bigcup_{i<\delta} \mathcal{D}_i^\kappa \subseteq M_\delta$ for all $\kappa < \mu$ with $\mathrm{cf}\,\kappa \ge \kappa(*)$. As $\mathrm{cf}\,\delta \ne \mathrm{cf}\,\kappa$ for such κ, the set $\mathcal{D}^\kappa \subseteq M_\delta$ is dense in $[M_\delta]^\kappa$. □

3. Filtrations with respect to anti-monotone set functions

Every infinite subset of a structure with countably many finite-place operations is contained in a subset of the same cardinality which is closed under all operations by the downward Löwenheim–Skolem theorem. Also, the union of any increasing chain of closed sets is closed. Consequently, every uncountable structure with countably many finitary operations is *filtrable*, that is, presentable as an increasing and continuous union of substructures of smaller cardinality.

If infinitary operations are admitted, both facts above are no longer true since the cardinality of the closure of a subset of cardinality λ under a κ-place operation depends on the valued of the exponent λ^κ which is undecidable and is periodically larger than λ — e.g. for λ with countable cofinality — in every model of set theory, including models of the GCH.

The main theorem of this section asserts that for anti-monotone set-functions a version of the Löwenheim–Skolem theorem and filtrations to closed sets exist without appealing to additional axioms.

Definition 3.1.

(1) A *filtration* of an infinite set V is a sequence of sets $\langle D_\alpha : \alpha < \kappa \rangle$ for some cardinal κ which satisfies:
   (a) $|D_\alpha| < |V|$.
   (b) $\alpha < \beta < \kappa \Rightarrow D_\alpha \subseteq D_\beta$.
   (c) If $\alpha < \kappa$ is limit then $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$.
   (d) $V = \bigcup_{\alpha<\kappa} D_\alpha$.
   Condition (c) in the definition is the *continuity* of $\langle D_\alpha : \alpha < \kappa \rangle$.
(2) Suppose V is an infinite set and $\mathcal{C} \subseteq \mathcal{P}(V)$. A $\mathcal{C}$-filtration of V is a filtration $\langle D_\alpha : \alpha < \kappa \rangle$ such that $D_\alpha \in \mathcal{C}$ for all $\alpha < \kappa$.
(3) We say that V is $\mathcal{C}$-*filtrable*, for $\mathcal{C} \subseteq \mathcal{P}(V)$, if there exists a $\mathcal{C}$-filtration of V.

Theorem 3.2. *Suppose $\mathcal{C} \subseteq \mathcal{P}(V)$, $I \ne \emptyset$ is countable and $\rho < |V|$ a cardinal. Suppose $\mathcal{S}_i \subseteq \mathcal{C}$ for $i \in I$ and:*

(1) *For every $i \in I$, the union of every chain of sets from S_i belongs to $\mathcal{C}$.*
(2) *For every $\rho \le \theta < |V|$ there exists $i \in I$ such that for every set $A \in [V]^\theta$ there is $D \in \mathcal{S}_i$ such that $A \subseteq D \in [V]^\theta$.*

Then V is $\mathcal{C}$-filtrable.

Proof. Let $\lambda := |V|$.

Assume first that $\operatorname{cf} \lambda > \aleph_0$. Fix an increasing and continuous chain of sets $\langle A_\alpha : \alpha < \operatorname{cf} \lambda\rangle$ such that $V = \bigcup_{\alpha<\operatorname{cf}\lambda} A_\alpha$ and $|A_\alpha| < \lambda$ for each $\alpha < \operatorname{cf} \lambda$ and $|A_0| \ge \rho$. Denote $\theta_\alpha := |A_\alpha| < \lambda$.

As $\operatorname{cf} \lambda > \aleph_0$ is regular and I is countable, we may assume, by passing to a subsequence, that for some fixed $i \in I$ it holds that for every $\alpha < \operatorname{cf} \lambda$ and every $B \in [V]^{\theta_{\alpha+1}}$ there is $D \in \mathcal{S}_i$ such that $B \subseteq D$ and $|D| = |B|$.

Define inductively D_α for $0 < \alpha < \operatorname{cf} \lambda$. For limit $\alpha < \operatorname{cf} \lambda$ let $D_\alpha = \bigcup_{\beta<\alpha} D_\beta$. For $\alpha = \beta + 1$ choose $D_\alpha \in \mathcal{S}_i$ which contains $A_\alpha \cup \bigcup_{\beta<\alpha} D_\beta$ and is of cardinality $|A_\alpha \cup \bigcup_{\beta<\alpha} D_\beta|$.

$V = \bigcup_{\alpha<\operatorname{cf}\lambda}$ since $A_\alpha \subseteq D_\alpha$ for all $\alpha < \operatorname{cf} \lambda$, and as $D_{\alpha+1} \in \mathcal{S}_i$, each $D_{\alpha+1}$ belongs to $\mathcal{C}$. By (1) $D_\alpha \in \mathcal{C}$ for limit $\alpha < \operatorname{cf} \lambda$ as well. Thus $\langle D_\alpha : 0 < \alpha < \operatorname{cf} \lambda\rangle$ is a $\mathcal{C}$-filtration.

If $\operatorname{cf} \lambda = \aleph_0$ fix an increasing union $\lambda = \bigcup_{n<\omega} A_n$ with $|A_n| < \lambda$ and $|A_0| \ge \rho$. Using (2), choose inductively sets $D_n \in \mathcal{C}$ such that $A_0 \subseteq D_0$ and for all $n < \omega$, $|D_n| = |A_n|$ and $D_n \cup A_{n+1} \subseteq D_{n+1}$. Now $\langle D_n : n < \omega\rangle$ is a $\mathcal{C}$-filtration. $\square$

Definition 3.3. Let V be a nonempty set.

(1) A *notion of closure* over V is a family of sets $\mathcal{C} \subseteq \mathcal{P}(V)$ that satisfies:
   (a) $\emptyset, V \in \mathcal{C}$.
   (b) $D_1, D_2 \in \mathcal{C} \Rightarrow D_1 \cup D_2 \in \mathcal{C}$.
   (c) $\bigcap X \in \mathcal{C}$ for all $X \subseteq \mathcal{C}$.
(2) A function $K : \mathcal{P}(V) \to \mathcal{P}(V)$ is a *closure operator* if it satisfies Kuratowski's closure axioms:
   (a) $K(\emptyset) = \emptyset$;
   (b) $A \subseteq K(A)$ for $A \subseteq V$;
   (c) $K(K(A)) = K(A)$ for $A \subseteq V$;
   (d) $K(A \cup B) = K(A) \cup K(B)$ for $A, B \subseteq V$.
(3) A *notion of semi-closure* over V is a family $\mathcal{C} \subseteq \mathcal{P}(V)$ which satisfies $V \in \mathcal{C}$ and $\bigcap X \in \mathcal{C}$ for all $X \subseteq \mathcal{C}$.
(4) A function $K : \mathcal{P}(V) \to \mathcal{P}(V)$ is a *semi-closure operator* if it satisfies Kuratowski's axioms (b) and (c) and the following implication of (d):
   (d′) $A_1 \subseteq A_2 \subseteq X \Rightarrow K(A_1) \subseteq K(A_2)$,
(5) Given a notion of semi-closure $\mathcal{C} \subseteq \mathcal{P}(V)$ let $K_{\mathcal{C}}(A) = \bigcap\{D : A \subseteq D \in \mathcal{C}\}$. Conversely, given a semi-closure operator $K : \mathcal{P}(V) \to \mathcal{P}(V)$ let $\mathcal{C}_K = \{D : D = K(D) \subseteq V\}$.

For every semi-closure operator K on a set V it holds that $K = K_{\mathcal{C}_K}$ and for every notion of semi-closure $\mathcal{C}$ it holds that $\mathcal{C} = \mathcal{C}_{K_\mathcal{C}}$. If K is a closure operator then $\mathcal{C}_k$ is a notion of closure and if $\mathcal{C}$ is a notion of closure then $K_\mathcal{C}$ is a closure operator. Thus, notions of [semi]-closure and [semi]-closure operators are interchangeable.

Definition 3.4. Suppose that $F : \mathcal{P}(V) \to \mathcal{P}(V)$ and κ is a cardinal. Denote by F_κ the restriction $F \restriction [X]^\kappa$ to sets of size κ. The subsets which are closed under F_κ form the following notion of semi-closure $\mathcal{C}_{F,\kappa} = \{D : D \subseteq V \wedge (Y \in [D]^\kappa \Rightarrow F(Y) \subseteq D)\}$. Let $K_{F,\kappa}$ denote the semi-closure operator corresponding to $\mathcal{C}_{F,\kappa}$ and let us refer to sets $D \in \mathcal{C}_{F,\kappa}$ as F_κ-*closed* sets.

Definition 3.5. A function $F : \mathcal{P}(X) \to \mathcal{P}(X)$ is *anti-monotone* if $A \subseteq B \subseteq X \Rightarrow F(B) \subseteq F(A)$.

Claim 3.6. *Suppose $F : \mathcal{P}(V) \to \mathcal{P}(V)$ is anti-monotone and κ is an infinite cardinal. Then:*

(1) *$K_{F,\kappa}$ is a closure operator.*
(2) *An increasing union $\bigcup_{\alpha<\theta} D_\alpha$ of $K_{F,\kappa}$-closed sets is $K_{F,\kappa}$-closed if* $\operatorname{cf}\theta \neq \operatorname{cf}\kappa$.

Proof. We verify axiom (4). If $X \in [A \cup B]^\kappa$ then, since κ is infinite, $|X \cap A| = \kappa$ or $|X \cap B| = \kappa$, so by anti-monotonicity $F(X) \subseteq K(A)$ or $F(X) \subseteq K(B)$.

Assume now that $D = \bigcup_{\alpha<\theta} D_i$ is an increasing union of $K_{F,\kappa}$-closed sets and $\operatorname{cf}\theta \neq \operatorname{cf}\kappa$. Let $X \in [D]^\kappa$. There exists some $\alpha < \theta$ such that $X \cap D_\alpha \in [D_\alpha]^\kappa$. Since D_α is $K_{F,\kappa}$-closed, $F(X \cap D_\alpha) \subseteq D_\alpha$ and by anti-monotonicity $F(X) \subseteq F(X \cap D_\alpha)$. □

Lemma 3.7. *Suppose V is a set and $F : \mathcal{P}(V) \to \mathcal{P}(V)$ is anti-monotone. Suppose $\kappa_1 \leq \kappa_2 < \rho$ are cardinals and $|F(X)| \leq \rho$ for all $X \in [V]^{\kappa_1}$. Let $K = K_{F,\kappa_2}$. If $\theta \geq \rho$ satisfies $\mathcal{D}(\theta, \kappa_1, \kappa_2) = \theta$ then $|K(A)| = \theta$ for all $A \in [V]^\theta$.*

Proof. Given $A \in [V]^\theta$ put $A_0 = A$ and for $\zeta \leq \kappa_2^+$ define $A_\zeta \in [V]^\theta$ inductively. For limit $\zeta \leq \kappa_2^+$ let $A_\zeta = \bigcup_{\xi<\zeta} A_\xi$, which is of cardinality θ. To define $A_{\zeta+1}$ for $\zeta < \kappa_2^+$ fix, by the assumption $\mathcal{D}(\theta, \kappa_1, \kappa_2) = \theta$, a set $\mathcal{D}_\zeta \subseteq [A_\zeta]^{\kappa_1}$ of cardinality θ with the property that for all $Y \in [A_\zeta]^{\kappa_2}$ there exists $X \in \mathcal{D}_\zeta$ such that $X \subseteq Y$ and let $A_{\zeta+1} = A_\zeta \cup \bigcup\{F(X) : X \in \mathcal{D}_\zeta\}$, which is of cardinality θ since $\rho \leq \theta$.

For every $X \in [A_{\kappa_2^+}]^{\kappa_2}$ there is some $\zeta < \kappa_2^+$ such that $Y \in [A_\zeta]^{\kappa_2}$ and hence there is some $X \in \mathcal{D}_\zeta$ such that $X \subseteq Y$. As $F(Y) \subseteq F(X) \subseteq A_{\zeta+1}$, it holds that $F(Y) \subseteq A_{\kappa_2^+}$, so $A_{\kappa_2^+}$ is K-closed. □

Theorem 3.8 *(Asymptotic filtrations for anti-monotone set functions). Let ν be an infinite cardinal and denote $\mu := \beth_\omega(\nu)$. Let V be a set of cardinality $|V| > \mu$ and suppose $F : \mathcal{P}(V) \to \mathcal{P}(V)$ is an anti-monotone function. If there exists a cardinal $\rho \geq \mu$ such that $\rho < |V|$ and $|F(Y)| \leq \rho$ for every $Y \in [X]^\nu$ then V is $K_{F,\mu}$-filtrable.*

Proof. Let $\mathcal{C}$ be the family of all $K_{F,\mu}$-closed subsets of V. Let κ_n denote $\beth_n(\nu)$ and let $K_n = K_{F,\kappa_n}$. Let $\mathcal{S}_n$ be the collection of all K_n-closed subsets of V. We check that conditions (1) and (2) in Theorem 3.2 are satisfied for $\mathcal{C}$ and $\{\mathcal{S}_n : n < \omega\}$.

As $\kappa_n < \mu$ and F is anti-monotone, it holds that $\mathcal{S}_n \subseteq \mathcal{S}_{n+1} \subseteq \mathcal{C}$ for all n. The union of any chain of sets from $\mathcal{S}_n$ belongs to $\mathcal{S}_{n+1} \subseteq \mathcal{C}$ by Lemma 3.6(2).

To verify condition (2) let $\theta \geq \rho$ be given. By Lemma 2.7, there exists $m(\theta) < \omega$ such that $D(\theta, \kappa_n) = \theta$ for all $n \geq m(\theta)$. By Lemma 3.7, $|K_n(A)| = \theta$ for $A \in [V]^\theta$ for all $n \geq m(\theta)$.

Now V is $K_{F,\nu}$-filtrable by Theorem 3.2. □

We remark that each $\mathcal{S}_n$ is in fact the collection of closed sets with respect to a closure operator and that $m < n \Rightarrow \mathcal{S}_m \subseteq \mathcal{S}_n$. Neither of these two properties was used in proving Theorem 3.8.

The following lemma relates filtrations to elementary chains of models:

Lemma 3.9. *Suppose F is anti-monotone, $|F(X)| \leq \rho$ for all $X \in [V]^\nu$ and $\rho \geq \mu = \beth_\omega(\nu)$. Then for every chain $\langle M_i : i < \lambda \rangle$ of elementary submodels of a sufficiently large $(H(\chi), \in)$ which satisfies $\mu \subseteq M_0$, $|M_i| \subseteq |M_i|$, $V, F \in M_0$ and $\langle M_j : j \leq i \rangle \in M_{i+1}$ the set $D_i := M_i \cap V$ is F_μ closed for every limit $i < \lambda$. If $|M_i| < \lambda := |V|$ then $\langle D_i : i < \lambda \text{ is limit} \rangle$ is a μ-filtration of V.*

Proof. If $M \prec (H(\chi), \in, \ldots)$ and $V, F \in M$, $\rho \subseteq M$, then for every $X \in M \cap [V]^\kappa$ for $\nu \leq \kappa < \nu$ the set $F(X)$ belongs to M and $|F(X)| \leq \rho$, hence $F(X) \subseteq M$. Thus, if $M \cap [V]^\kappa$ is dense in $[V]^\kappa$, the set $V \cap M$ is κ-closed. By Lemma 2.9 this is the case for M_i for all limits $i < \lambda$.

If indeed $|M_i| < \lambda$ for all i then $|D_i| < \lambda$. The fact that $\bigcup_{i<\lambda} D_i = V$ follows from $V \in M_0$ and $\lambda \subseteq \bigcup_{i<\lambda} M_i$. □

3.1. The need for $\rho \geq \beth_\omega(\nu)$

Is the restriction $\rho \geq \beth_\omega(\nu)$ optimal in Theorem 3.8? This is not known at the moment. As a result of a discussion [13] with Shelah, something can be said about it: improving it will be at least as hard as proving one of the following versions of Shelah's Weak Hypothesis, indexed by $n > 0$:

(SWH$_n$) There are no infinite ν and ρ such that $\beth_n(\nu) < \rho < \beth_\omega(\nu)$ and $\mathcal{F} = \{A_\alpha : \alpha < \rho^+\} \subseteq [\rho]^\rho$ satisfies $|A_\alpha \cap A_\beta| < \nu$ for $\alpha < \beta < \rho^+$.

Shelah's Weak Hypothesis is a dynamic statement whose evolving contents is the weakest unproved pcf-theoretic statement. It is formulated in terms of patterns in Shelah's function pp. See [11] for the translation of SWH from pcf to almost disjoint families. The negations of first two versions of the SWH have been shown consistent.

Gitik [3] showed the consistency of an ω sequence of cardinals with pp of each exceeding the limit of the sequence. SWH was then revised to the statement that there was no such ω_1 sequence. Recently, Gitik [2] proved the consistency of the existence of an ω_1 sequence of cardinals with the pp of each exceeding the limit of the sequence.

Shelah's believes that the negations of the versions above may eventually be proved consistent [13].

Suppose that $\beth_n(\nu) < \rho < \beth_\omega(\nu)$ and $\mathcal{F} = \{A_\alpha : \alpha < \rho^+\} \subseteq [\rho]^\rho$ satisfies $|A_\alpha \cap A_\beta| < \nu$ for $\alpha < \beta < \rho^+$. Let $V = \rho \dot\cup \mathcal{F}$ and let $F(X) = \{\alpha : X \cap \rho \subseteq A_\alpha\} \cup \bigcap_{A \in X \cap \mathcal{F}} A$. This is an anti-monotone function and for every X with $|X| \geq \nu^+$ it holds that $|F(X)| < \nu$. Yet, there is no filtration of V to F_ν-closed sets, as ρ would have to be contained in one of the parts, and every F_ν-closed set which contains ρ is equal to V.

Thus, a ZFC proof of Theorem 3.8 with $\beth_n(\nu)$ in place of $\beth_\omega(\nu)$ for some $n > 0$ will imply the n-th version of the Shelah Weak Hypothesis. If, however, the negations of (SWH_n) are consistent for all $n > 0$, then $\beth_\omega$ is optimal.

4. Splitting families of sets

Following [1] let us say that a family $\mathcal{F}$ of sets satisfies condition $C(\lambda, \kappa)$ if the intersection of every subfamily of $\mathcal{F}$ of size λ is of size strictly less than ν. Miller [10] defined *property B* (after Felix Bernstein's "Bernstein set") of a family of sets $\mathcal{F}$ as: there exists a set B such that $B \cap A \neq \emptyset$ and $A \nsubseteq B$ for all $A \in \mathcal{F}$ and proved that for every infinite ρ, every ρ-uniform family of sets that satisfies $C(\rho^+, n)$ for some natural number n satisfies property B. Erdős and Hajnal [1] used Miller's method and a theorem of Tarski [15] to prove from the assumption GCH that every ρ-uniform family that satisfies $C(\rho^+, \nu)$ for an infinite cardinal ν satisfies property B if $\rho > \nu^+$.

More results with the GCH followed along this line [7,8,4,5]. Komjáth [7] proved from the GCH that for $\nu^+ < \rho$ every ρ-uniform $\mathcal{F}$ which satisfies $C(\rho^+, \nu)$ and is also almost disjoint is *essentially disjoint*, that is, it can be made pairwise disjoint by removing a set of size $< \rho$ from each member of $\mathcal{F}$.

Komjáth [8] investigated further the property of essential disjointness which he introduced in [7] (under the name "sparseness") and proved that the array of cardinalities of pairwise intersections in an $\aleph_0$-uniform a.d. family determines whether the family is essentially disjoint or not.

Hajnal, Juhász and Shelah proved in [4] a general theorem which implied many of the Miller-type theorems known at the time and derived also new combinatorial and topological consequences with it by assuming as an additional axiom a relaxation of the GCH to a weak version of the Singular Cardinal Hypothesis. Soukup [14] proved recently, using Shelah's revised GCH theorem, that for $\nu < \beth_\omega(\aleph_0)$, every ν-almost disjoint $\beth_\omega$-uniform family of sets is essentially disjoint.

Definition 4.1. Let $\mathcal{F}$ be a family of sets.

(1) The *universe of* $\mathcal{F}$ is the set $\bigcup \mathcal{F}$, and is denoted by $V(\mathcal{F})$.
(2) Given $U \subseteq V(\mathcal{F})$ let $\mathcal{F}(U) = \{\mathcal{F} \cap \mathcal{P}(U)\} = \{A : A \in \mathcal{F} \wedge A \subseteq U\}$.
(3) $\mathcal{F}$ is ρ-uniform, for a cardinal ρ, if $|A| = \rho$ for all $A \in \mathcal{F}$.

Definition 4.2. Suppose that $\mathcal{F}$ is a family of sets and $V = V(\mathcal{F})$.

(1) A set $U \subseteq V$ is κ-closed, for a cardinal κ, if $|A \cap U| \geq \kappa \Rightarrow A \subseteq U$ for all $A \in \mathcal{F}$.
(2) Let $F : \mathcal{P}(V) \to \mathcal{P}(V)$ be defined by

$$F(X) = \bigcup \{A : X \subseteq A \in \mathcal{F}\}.$$

Claim 4.3. *For every family $\mathcal{F}$ with $V = V(\mathcal{F})$ and cardinal κ,*

(1) *The collection $\mathcal{C}_\kappa$ of all κ-closed sets is a notion of semi-closure over V and if κ is infinite then $\mathcal{C}_\kappa$ is a notion of closure.*
(2) *If $U \subseteq V$ is κ-closed and $A \in \mathcal{F}$ is not contained in U then $|A \cap U| < \kappa$.*
(3) *The function F defined above is anti-monotone.*
(4) *For every $U \subseteq V$, U is κ-closed iff U is $K_{F,\kappa}$-closed.*
(5) *If $\kappa_1 \leq \kappa_2$ are cardinals then $\mathcal{C}_{\kappa_1} \subseteq \mathcal{C}_{\kappa_2}$.*
(6) *If $\mathcal{F}$ is ρ-uniform and $\kappa \leq \rho$ then a set $U \subseteq V$ is κ-closed iff $\mathcal{F}(U) = \{A : A \in \mathcal{F} \wedge |A \cap U| \geq \kappa\}$.*
(7) *Abusing notation, we say that a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ is κ-closed if $V(\mathcal{F}')$ is κ-closed and we denote by $K_{F,\kappa}(\mathcal{F})$ the family $\mathcal{F}(K_{F,\kappa}(V(\mathcal{F})))$.*

Definition 4.4 *(Disjointness conditions).*

(1) $\mathcal{F}$ is *almost disjoint* (a.d.) if $|A \cap B| < \min\{|A|, |B|\}$ for distinct $A, B \in \mathcal{F}$.
(2) $\mathcal{F}$ is ν-disjoint, for a cardinal ν, if $|A \cap B| < \nu$ for any distinct $A, B \in \mathcal{F}$.
(3) (Miller [10], Erdős and Hajnal [1]) $\mathcal{F}$ satisfies condition $C(\theta, \nu)$ for cardinals θ, ν if $|\bigcap \mathcal{A}| < \nu$ for all $\mathcal{A} \in [\mathcal{F}]^\theta$ (so, $C(2, \nu)$ is ν-disjointness).
(4) (Komjáth [7], Hajnal, Juhász and Shelah [4]) $\mathcal{F}$ is *essentially disjoint* (e.d.) if there exists an assignment of subsets $B(A) \in [A]^{<|A|}$ for all $A \in \mathcal{A}$ such that the family $\{A \setminus B(A) : A \in \mathcal{F}\}$ is pairwise disjoint.
(5) $\mathcal{F}$ is ν-e.d. if for every $A \in \mathcal{F}$ there exists an assignment $B(A) \in [A]^\nu$ such that $\{A \setminus B(A) : A \in \mathcal{F}\}$ is pairwise disjoint. Remark: in [7] the term *sparse* is used instead of "e.d.".

Claim 4.5.

(1) *Suppose $\mathcal{F}$ is ρ-uniform and satisfies $C(\rho^+, \nu)$ for some $\nu < \rho$. If $\nu \leq \kappa \leq \rho \leq \theta$ and $D(\theta, \kappa) = \theta$ then for every $U \in [V]^\theta$ it holds that $|\{A \in \mathcal{F} : |A \cap U| \geq \kappa\}| \leq \theta$.*

(2) *Suppose ν is an infinite cardinal, $\mu = \beth_\omega(\nu)$, $\rho \geq \mu$ and $\mathcal{F}$ is a ρ-uniform family with universe V. If $\mathcal{F}$ satisfies $C(\rho^+, \nu)$ then for every $\theta \geq \rho$ and $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = \theta \iff |V(\mathcal{F}')| = \theta$.*

Proof. To prove (1) fix a dense $\mathcal{D} \subseteq [U]^\kappa$ of cardinality θ. If $|A \cap U| \geq \kappa$ for some $A \in \mathcal{F}$ then there exists $X \in \mathcal{D}$ such that $X \subseteq A$. By $C(\rho^+, \nu)$ and $\nu \leq \kappa$, each $X \in \mathcal{D}$ is contained in no more than ρ members of $\mathcal{F}$, so $|\{A \in \mathcal{F} : (\exists X \in \mathcal{D})(X \subseteq A)\}| \leq \theta \times \rho = \theta$.

For (2) assume first that $|V(\mathcal{F}')| = \theta \geq \rho$. Let n be such that $\mathcal{D}(\theta, \kappa_n) = \theta$ (where $\kappa_n = \beth_n(\nu)$). Clearly $\mathcal{F}(U) \subseteq \{A \in \mathcal{F} : |A \cap U| \geq \kappa_n\}$ which, by (1), has cardinality $\leq \theta$. The converse implication is trivial. □

By the above lemma, for all sufficiently large n it holds that $|K_n(\mathcal{F}')| = |\mathcal{F}'|$ if $|\mathcal{F}'| \geq \rho$.

Lemma 4.6. *Suppose ν is an infinite cardinal, $\mu = \beth_\omega(\nu)$, $\rho \geq \mu$ and $\mathcal{F}$ is a ρ-uniform family with universe V such that $|V| > \rho$. If $\mathcal{F}$ satisfies $C(\rho^+, \nu)$ then V is μ-filtrable.*

Proof. The function F defined above is anti-monotone, for every $X \in [V]^\nu$ it holds that $|F(X)| \leq \rho$ by $C(\rho^+, \nu)$ and $\rho < |V|$. By Theorem 3.8, V is $K_{F,\mu}$-filtrable hence μ-filtrable. □

The next theorem extends Miller's Theorem 2 from [10] when ν is finite, and is proved in ZFC for all cardinals ν.

Theorem 4.7. *Suppose ν is a cardinal and $\rho \geq \mu := \beth_\omega(\nu)$. For every ρ-uniform family $\mathcal{F}$ that satisfies $C(\rho^+, \nu)$ there exists an enumeration $\mathcal{F} = \{A_\alpha : \alpha < |\mathcal{F}|\}$ and a family of sets $\{d_\alpha : \alpha < |\mathcal{F}|\}$ such that*

(*i*) *$d_\alpha \in [A_\alpha]^\rho$ for every $\alpha < |\mathcal{F}|$ and $\beta < \alpha < |\mathcal{F}| \Rightarrow d_\beta \cap d_\alpha = \emptyset$;*
(*ii*) *$|\{\beta < \alpha : A_\beta \cap d_\alpha \neq \emptyset\}| < \rho$ for all $\alpha < |\mathcal{F}|$.*

Proof. Let $\lambda = |\mathcal{F}|$ and we prove the theorem by induction on λ. If $|\mathcal{F}| \leq \rho$ fix any enumeration $\mathcal{F} = \{A_\alpha : \alpha < \lambda\}$. Every ρ-uniform family of cardinality at most ρ has a disjoint refinement, so we can fix $d_\alpha \in [A_\alpha]^\rho$ so that $\alpha < \beta < \lambda \Rightarrow d_\beta \cap d_\alpha = \emptyset$. This settles ($i$). Conclusion ($ii$) holds trivially as $\lambda \leq \rho$.

Assume that $\lambda > \rho$. Since $\mathcal{F}$ satisfies $C(\rho^+, \nu)$ it is μ-filtrable by Lemma 4.6. For finite ν, filtrability is proved in Miller's [10] — or, in modern terms, is an immediate consequence of the Löwenheim–Skolem theorem — and does not require Lemma 4.6.

Fix a strictly increasing and continuous chain of μ-closed families $\{\mathcal{H}_i : i < \mathrm{cf}\,\lambda\}$ such that $|\mathcal{H}_0| \geq \rho$, $|\mathcal{H}_i| < \lambda$ for all $i < \mathrm{cf}\lambda$ and $\mathcal{F} = \bigcup\{\mathcal{H}_i : i < \mathrm{cf}\,\lambda\}$. Put $\mathcal{F}'_i = \mathcal{H}_i \setminus \bigcup_{j<i} \mathcal{H}_j$ and thin out $\langle \mathcal{F}'_i : i < \mathrm{cf}\,\lambda\rangle$ to $\langle \mathcal{F}_i : i < \mathrm{cf}\,\lambda\rangle$ by removing the empty terms.

Now $\{\mathcal{F}_i : i < \mathrm{cf}\,\lambda\}$ is a partition of $\mathcal{F}$, $|\mathcal{F}_i| < \lambda$ for each i and $A \in \mathcal{F}_i$ implies that

$$\Big|A \cap \bigcup_{j<i} \mathcal{F}_j\Big| < \mu. \tag{3}$$

Denote $\lambda_i = |\mathcal{F}_i|$. As $\lambda_i < \lambda$ for every $i < \mathrm{cf}\,\lambda$ and $\mathcal{F}_i$ satisfies $C(\rho^+, \nu)$, the induction hypothesis allows us to fix for each i an enumeration $\mathcal{F}_i = \{A_{\langle i,\gamma\rangle} : \gamma < \lambda_i\}$ and $\{d'_{\langle i,\gamma\rangle} : \gamma < \lambda_i\}$ such that $d'_{\langle i,\gamma\rangle} \in [A_{\langle i,\gamma\rangle}]^\rho$ for each $\gamma < \lambda_i$, $d'_{\langle i,\gamma\rangle} \cap d'_{\langle i,\delta\rangle} = \emptyset$ for all $\gamma < \delta < \lambda_i$ it holds that and for each $\gamma < \lambda_i$:

$$\big|\{\delta < \gamma : A_{\langle i,\delta\rangle} \cap d'_{\langle i,\gamma\rangle} \neq \emptyset\}\big| < \rho. \tag{4}$$

For $i < \mathrm{cf}\,\lambda$ and $\gamma < \lambda_i$ define

$$d_{\langle i,\gamma\rangle} := d'_{\langle i,\gamma\rangle} \setminus \bigcup_{j<i} V(\mathcal{F}_j). \tag{5}$$

By (3) and $d'_{\langle i,\gamma\rangle} \subseteq A_{\langle i,\gamma\rangle}$ it holds that $|d_{\langle i,\gamma\rangle}| = \rho$.

Let $I = \{\langle i,\gamma\rangle : i < \mathrm{cf}\,\lambda \wedge \gamma < \lambda_i\}$ be well-ordered by the lexicographic ordering $<_{lx}$ of pairs of ordinals. As $|I| = \lambda$ and each proper initial segment of I has cardinality $< \lambda$, it holds that $\langle I, <_{lx}\rangle$ is order-isomorphic to λ.

Identifying λ with $\langle I, <_{lx}\rangle$ we now have an enumeration $\mathcal{F} = \{A_{\langle i,\gamma\rangle} : \langle i,\gamma\rangle \in I\}$. The sets $d_{\langle i,g\rangle} \in [A_{\langle i,\gamma\rangle}]^\rho$ have been defined in (5) for each $\langle i,\gamma\rangle \in I$ and satisfy $d_{\langle \delta,j\rangle} \cap d_{\langle i,\gamma\rangle} = \emptyset$ for $\langle j,\delta\rangle \neq \langle i,\gamma\rangle$ in I. This shows that the enumeration of $\mathcal{F}$ and the family $\{d_{\langle i,\gamma\rangle} : \langle i,\gamma\rangle \in I\}$ satisfy conclusion (i) of the theorem.

To show that (ii) also holds, let $\langle i,\gamma\rangle \in I$ be arbitrary and let

$$X = \{\langle j,\delta\rangle <_{lx} \langle i,\gamma\rangle : A_{\langle j,\delta\rangle} \cap d_{\langle i,\gamma\rangle} \neq \emptyset\}. \tag{6}$$

If $j < i$ then $\langle j,\delta\rangle \notin X$ as $A_{\langle j,\delta\rangle} \cap d_{\langle i,\gamma\rangle} = \emptyset$ by (5), hence

$$X = \{\langle i,\delta\rangle <_{lx} \langle i,\gamma\rangle : A_{\langle i,\delta\rangle} \cap d_{\langle i,\gamma\rangle} \neq 0\}. \tag{7}$$

As $d_{\langle i,\gamma\rangle} \subseteq d'_{\langle i,\gamma\rangle}$, it follows from (4) that $|X| < \rho$. □

Theorem 4.7 and Lemma 4.6 have the following corollaries.

Corollary 4.8. *If ν is infinite and $\rho \geq \mu := \beth_\omega(\nu)$ then for every ρ-uniform family $\mathcal{F}$:*

(1) *If $\mathcal{F}$ satisfies $C(\rho^+, \nu)$ then $\mathcal{F}$ has a disjoint refinement.*
(2) *If $\mathcal{F}$ satisfies $C(\rho^+, \nu)$ and for a cardinal $\mu \leq \theta \leq \rho$ every subfamily of $\mathcal{F}$ of cardinality ρ is θ-e.d. then $\mathcal{F}$ is θ-e.d.*
(3) *If $\mathcal{F}$ is ν-a.d. then $\mathcal{F}$ is e.d.*
(4) *If ρ is regular and $\mathcal{F}$ is a.d. and satisfies $C(\rho^+, \nu)$ then $\mathcal{F}$ is e.d.*

Corollary (1) extends to infinite ν the case of ρ-uniform $\mathcal{F}$ in Theorem 3 of [7] and eliminates the GCH from a stronger form of a Theorem 6 in [1]; (3) eliminates the additional axiom $A(\nu, \rho)$ from Theorem 2.4 in [4] for sufficiently large ρ — in the notation there, $\mathrm{ED}(\nu, \rho)$ for all $\rho \geq \beth_\omega(\nu)$ in ZFC. (4) eliminates the GCH from Theorem 5 in [7] for sufficiently large ρ.

Proof. (1) follows directly from the theorem.

To prove (2) let $|\mathcal{F}| = \lambda > \rho$ and assume, by μ-filtrability, that $\mathcal{F}$ is partitioned to $\{\mathcal{H}_\alpha : \alpha < \lambda\}$ such that $|\mathcal{H}_\alpha| < \lambda$ and $|A \cap \bigcup_{\beta<\alpha} V(\mathcal{H}_\beta)| < \mu$ for all $A \in \mathcal{H}_\alpha$. Fix, by the induction hypothesis, $B_\alpha(A) \in [A]^{<\theta}$ for all $A \in \mathcal{H}_\alpha$ such that $\{A \setminus B_\alpha(A) : A \in \mathcal{H}_\alpha\}$ is pairwise disjoint. For $A \in \mathcal{F}$ let $B(A) = B_\alpha(A) \cup (A \cap \bigcup_{\beta<\alpha} V(\mathcal{H}_\beta))$ for the unique α such that $A \in \mathcal{H}_\alpha$. As $|A \cap \bigcup_{\beta<\alpha} V(\mathcal{H}_\beta)| < \mu$ it holds that $|B(A)| < \theta$. Clearly, $\{A \setminus B(A) : A \in \mathcal{F}\}$ is pairwise disjoint.

For (3): every ν-a.d. ρ-uniform family satisfies $C(2, \nu)$ hence $C(\rho^+, \nu)$. If $\mathcal{F}' \subseteq \mathcal{F}$ has cardinality ρ and is well ordered by $\mathcal{F}' = \{A_\alpha : \alpha < \rho\}$ let $B(A_\alpha) = A_\alpha \cap \bigcup_{\beta<\alpha} A_\beta$. The cardinality of B_α is strictly smaller than ρ and $\{A_\alpha \setminus B_\alpha : \alpha < \rho\}$ is disjoint. Now (3) follows from (2).

For (4), as ρ is regular and $\mathcal{F}$ is a.d., every subfamily of $\mathcal{F}$ of cardinality ρ is e.d. and now use (2). □

4.1. Conflict-free colorings

Neither of the corollaries above used conclusion (*ii*) of Theorem 4.7. A generalization of this condition is used for the next theorem on *list conflict-free numbers*.

Definition 4.9.

(1) A coloring c of $V(\mathcal{F})$ for a family of sets $\mathcal{F}$ is *conflict free* if for every $A \in \mathcal{F}$ there is $x \in A$ such that $c(x) \neq c(y)$ for all $y \in A \setminus \{x\}$.
(2) The *conflict free number* $\chi_{CF}(\mathcal{F})$ is the smallest cardinal κ for which there exists a conflict-free coloring $c : V(\mathcal{F}) \to \kappa$.
(3) The *list-conflict-free number* $\chi_{\ell CF}(\mathcal{F})$ is the smallest cardinal κ such that for every assignments L of sets $L(v)$ to every $v \in V$ which satisfies $|L(v)| \geq \kappa$ there exists a conflict-free coloring c on V which satisfies $c(v) \in L(v)$ for all $v \in V$.

Clearly, every ν-e.d. family $\mathcal{F}$ has a coloring by ν-colors such that all but $< \nu$ colors in each $A \in \mathcal{F}$ are unique in A.

Claim 4.10. *Let $\rho \geq \theta > 0$ be cardinals and let $\mathcal{F}$ is a nonempty ρ-uniform family. Suppose there exists an enumeration $\mathcal{F} = \{A_\alpha : \alpha < \lambda\}$ and a family $\{d_\alpha : \alpha < \lambda\}$ such that*

(i) $d_\alpha \in [A_\alpha]^\theta$ *for every* $\alpha < \lambda$ *and* $\beta < \alpha < \lambda \Rightarrow d_\beta \cap d_\alpha = \emptyset$*;*
(ii) $|\{\beta < \alpha : A_\beta \cap d_\alpha \neq \emptyset\}| \leq \rho$ *for all* $\alpha < \lambda$*.*

Then

(1) *For every list-assignment* $L(v)$ *for* $v \in V$ *with* $|L(v)| \geq \rho^+$ *there exists a coloring* $c \in \prod_{v\in V} L(v)$ *such that for every* $\alpha < \lambda$ *and every* $x \in D_\alpha$ *the color* $c(x)$ *is unique in* A_α*.*
(2) $\chi_{\ell CF}(\mathcal{F}) \leq \rho^+$*.*

Proof. Assume that $L(v)$ is given with $|L(v)| \geq \rho^+$ for all $v \in V(\mathcal{F})$. For each $\alpha < \lambda$ enumerate $d_\alpha = \langle x_{\langle\alpha,i\rangle} : i < \theta\rangle$. Define $c \in \prod_{v\in V} L(v)$ arbitrarily on $V \setminus \bigcup_{\alpha<\lambda} d_\alpha$. Next define $c(x_{\langle\alpha,i\rangle})$ by induction on the lexicographic ordering on $\lambda \times \theta$. Suppose $c(x_{\langle\beta,j\rangle})$ is defined for $\langle\beta, j\rangle <_{lx} \langle\alpha, i\rangle$. Define

$$F(x_{\langle\alpha,i\rangle}) = \big\{c(y) : (\exists\beta \leq \alpha)\big[x_{\langle\alpha,i\rangle} \in A_\beta \wedge y \in A_\beta \wedge c(y) \text{ is defined}\big]\big\} \tag{8}$$

and choose

$$c(x_{\langle\alpha,i\rangle}) \in L(x_{\langle\alpha,i\rangle}) \setminus F(x_{\langle\alpha,i\rangle}). \tag{9}$$

By the assumption (3), $|F(x_{\langle\alpha,i\rangle})| \leq \rho$, so as $|L(x_{\langle\alpha,i\rangle})| \geq \rho^+$ the set $L(x_{\langle\alpha,i\rangle}) \setminus F(x_{\langle\alpha,i\rangle})$ is not empty and therefore $c(x_{\langle\alpha,i\rangle})$ can be chosen as required in (9).

To prove that the color $c(x_{\langle\alpha,i\rangle})$ is unique in A_α for every $\alpha < \lambda$ and $i < \theta$ suppose that $\alpha < \lambda$ and $i < \theta$ are given and $z \in A_\alpha \setminus \{x_{\langle\alpha,i\rangle}\}$. If $c(z)$ is defined at stage $\langle\alpha, i\rangle$ of the inductive definition, then $c(z) \in F(x_{\langle\alpha,i\rangle})$ by (8) and hence $c(x_{\langle\alpha,i\rangle}) \neq c(z)$ by (9).

Otherwise, there is some $\beta > \alpha$ and $j < \theta$ such that $z = x_{\langle\beta,j\rangle}$. This means that $x_{\langle\beta,j\rangle} \in A_\alpha$. As $c(x_{\langle\alpha,i\rangle})$ is defined at stage $\langle\beta, j\rangle$, it follows by (8) that $c(x_{\langle\alpha,i\rangle}) \in F(x_{\langle\beta,j\rangle})$ and $c(z) = c(x_{\langle\beta,j\rangle}) \neq c(x_{\langle\alpha,i\rangle})$ by (9). □

By combining Theorem 4.7 with Claim 4.10 we get:

Corollary 4.11. *For every infinite* ν *and* $\rho \geq \beth_\omega(\nu)$*, every* ρ*-uniform family which satisfies* $C(\nu, \rho^+)$ *satisfies* $\chi_{\ell CF}(\mathcal{F}) \leq \rho^+$*.*

Hajnal, Juhász, Soukup and Szentmiklóssy [5] proved that a κ-uniform family which is r-a.d. for finite r has countable conflict-free number. See also Komjáth [9], where the uniformity condition is removed. Soukup [14] proved recently, using Shelah's revised GCH theorem, that every ν-almost disjoint family $\mathcal{F} \subseteq [\lambda]^{\geq\beth_\omega(\aleph_0)}$ has conflict-free coloring number at most $\beth_\omega(\aleph_0)$.

4.2. Comparing almost disjoint families

We conclude this section with a generalization of a theorem by Komjáth on comparing families of sets.

Definition 4.12. Two families of sets $\mathcal{F}$ and $\mathcal{G}$ are *similar* if there is a bijection $f : \mathcal{F} \to \mathcal{G}$ such that $|A \cap B| = |f(A) \cap f(B)|$ for all $A, B \in \mathcal{F}$.

Komjáth [8] addressed the following question: which combinatorial properties of $\aleph_0$-uniform families are invariant under similarity? Property B and the existence of a disjoint refinement are not similarity invariant, as one can replace $A \in \mathcal{F}$ by $A \cup D(A)$ where $D(A) \cap D(B) = \emptyset$ for distinct $A, B \in \mathcal{F}$ and obtain an equivalent family which has a disjoint refinement from a family which does not satisfy property B. However, Komjáth proved:

Theorem 4.13. *(See Komjáth [8].) Suppose $\mathcal{F}$ and $\mathcal{G}$ are $\aleph_0$-uniform, almost disjoint and equivalent. If $\mathcal{F}$ is e.d. then also $\mathcal{G}$ is e.d. More generally: G is e.d. if $\mathcal{F}$ is and there exists a bijection $f : \mathcal{F} \to \mathcal{G}$ such that $|A \cap B| \geq |f(A) \cap f(B)|$ for all $A, B \in \mathcal{F}$.*

The cardinal $\aleph_0$ is strong limit and regular, that is, strongly inaccessible. As a corollary of the next theorem, Komjáth's theorem extends from $\aleph_0$ to all strongly inaccessible cardinals, but actually more is proved.

Theorem 4.14. *Suppose μ is a strong limit cardinal and $\rho \geq \mu$. Suppose $\mathcal{F} = \{A_a : \alpha < \lambda\}$, $\mathcal{G} = \{B_\alpha : \alpha < \lambda\}$ are ρ-uniform and for every $\alpha < \beta < \lambda$ it holds that*

$$|A_\alpha \cap A_\beta| \geq |B_\alpha \cap B_\beta|. \tag{10}$$

Then if $\mathcal{F}$ is $< \mu$-essentially disjoint, so is $\mathcal{G}$, provided that every subfamily of $\mathcal{G}$ of cardinality ρ is μ-e.d.

Proof. Let $C_\alpha \in [A_\alpha]^{<\mu}$ be fixed for all $\alpha < \lambda$ such that $(A_\alpha \setminus C_\alpha) \cap (A_\beta \setminus C_\beta) = \emptyset$ for all $\alpha < \beta < \lambda$. We prove by induction on $\lambda \geq \rho$ that $\mathcal{G}$ is $< \mu$-e.d. If $\lambda = \rho$ then $\mathcal{G}$ is μ-e.d. by the assumption that every subfamily of $\mathcal{G}$ of cardinality ρ is μ-e.d.

For singular $\lambda > \rho$ the conclusion follows from the induction hypothesis by Proposition 5 in [7].

Assume then that $\lambda > \rho$ is regular. We may assume that $V(\mathcal{F}) = V(\mathcal{G}) \subseteq \lambda$. As $\mathcal{F}$ is $< \mu$-a.d. it follows that $|V(\mathcal{F})| = \lambda$, so we assume that actually $V(\mathcal{F}) = \lambda$.

Let $\overline{M} = \langle M_\alpha : \alpha < \lambda \rangle$ be an elementary chain of models $M_\alpha \prec (H(\Omega), \in, \prec)$ for a sufficiently large regular Ω such that $\mathcal{F}, \mathcal{G} \in M_0$ and also the function $A_\alpha \mapsto C_\alpha$ belongs to M_0, such that:

(1) $\rho \subseteq M_0$ and $\alpha < \beta \Rightarrow M_\alpha \subseteq M_\beta$.
(2) $M_\alpha \cap \lambda \in \lambda$.
(3) $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$.

Let $\delta(\alpha) = M_\alpha \cap \lambda$.

For each ordinal $\gamma < \delta(\alpha)$ there is at most one $\beta < \lambda$ such that $\gamma \in A_\beta \setminus C_\beta$. Thus, whenever $\gamma \in A_\beta \setminus C_\beta$ and $\gamma < \delta(\alpha)$ for some $\alpha < \lambda$, by elementarity $A_\beta \in M_\alpha$ and hence $A_\beta \subseteq \delta(\alpha)$. If $\beta \geq \delta(\alpha)$, then, it follows that $A_\beta \cap \delta(\alpha) \subseteq C_\beta$.

For each $\beta < \lambda$ let $\kappa(\beta) = |C_\beta|$. We get:

$$\beta \geq \delta(\alpha) \quad \Rightarrow \quad \left|A_\beta \cap \delta(\alpha)\right| \leq \kappa(\beta) \tag{11}$$

and, since $\gamma < \delta(\alpha) \Rightarrow A_\gamma \subseteq \delta(\alpha)$,

$$\beta \geq \delta(\alpha) > \gamma \quad \Rightarrow \quad |A_\beta \cap A_\gamma| \leq \kappa(\beta). \tag{12}$$

By assumption (10), condition (12) can be copied over to $\mathcal{G}$, that is:

$$\beta \geq \delta(\alpha) > \gamma \quad \Rightarrow \quad |B_\beta \cap B_\gamma| \leq \kappa(\beta). \tag{13}$$

Claim 4.15. *There exists a closed unbounded $E \subseteq \{\delta(\alpha) : \alpha < \lambda\}$ such that for every $\delta \in E$ and $\beta \geq \delta$ it holds that*

$$|B_\beta \cap \delta| < \mu. \tag{14}$$

Proof. Suppose to the contrary that $S \subseteq \{\delta(\alpha) : \alpha < \lambda\}$ is stationary and that $\beta(\delta) \geq \delta$ is chosen for $\delta \in S$ such that $|B_{\beta(\delta)} \cap \delta| \geq \mu$. We may assume that $\delta_1 < \delta_2 \Rightarrow \beta(\delta_1) < \delta_2$ for $\delta_1, \delta_2 \in S$ by intersecting S with a club, and by thinning S out assume that $\kappa(\beta(\delta)) = |C_{\beta(\delta)}| = \kappa(*)$ is fixed for all $\delta \in S$.

The set $S' = S \cap \operatorname{acc} S$ is stationary. If $\delta \in S'$ then for all sufficiently large regular $\kappa < \mu$ it holds that $M_\delta \cap [\delta]^\kappa$ is dense in $[\delta]^\kappa$ by Lemma 2.9, hence, as $|B_{\beta(\delta)} \cap \delta| \geq \mu$, for each sufficiently large regular $\kappa < \mu$ there is $X \in M_\delta$ such that $|X| = \kappa$ and $X \subseteq B_{\beta(\delta)}$.

Define $F(\delta) = X$, for each $\delta \in S'$, such that $X \subseteq B_{\beta(\delta)}$, $X \in M_\delta$ and $|X| > \kappa(*)$. By Fodor's Lemma, we can assume that F is fixed on a stationary $S'' \subseteq S'$. Fix, then, $\delta_1 < \delta_2$ in S'' with $F(\delta_1) = F(\delta_2) = X \in M_{\delta_1}$. But now $|B_{\beta(\delta_1)} \cap B_{\beta(\delta_2)}| \geq |X| > \kappa(*)$, and as $\beta(\delta_1) < \delta_2$ this contradicts Eq. (13) above. □

The conclusion of the theorem now follows from the induction hypothesis. □

In the case that $\rho = \mu = \operatorname{cf} \mu$ the assumption that every subfamily of $\mathcal{G}$ of cardinality ρ is μ-e.d. is, of course, superfluous, as $\mathcal{G}$ is a.d. by the similarity with $\mathcal{F}$ and every ρ-uniform a.d. family of regular size ρ is e.d.

Concluding remarks. After Cohen's and Easton's results showed that the GCH and many of its instances were not provable in ZFC the impression may have been that the GCH or some other additional axiom was required for proving general combinatorial theorems for all infinite cardinals. It now seems that, at least the combinatorics of splitting families of sets, which for a long time was hindered by the "totally independent" aspect of cardinal arithmetic, is, in the end, amenable to investigation on the basis of ZFC. We expect that the method presented here will be useful proving additional absolute combinatorial relations which generalize known combinatorial relations between finite cardinals to infinite ones.

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