Advances in Mathematics 269 (2015) 197–219

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Global existence for some transport equations with nonlocal velocity

Hantaek Bae [a,*], Rafael Granero-Belinchón [b]

[a] Department of Mathematical Sciences, Ulsan National Institute of Science and Technology (UNIST), Republic of Korea
[b] Department of Mathematics, University of California, Davis, USA

ARTICLE INFO

Article history:
Received 21 May 2014
Accepted 22 October 2014
Available online 31 October 2014
Communicated by Charles Fefferman

This paper is dedicated to Professor Eitan Tadmor on the occasion of his 60th birthday

Keywords:
Transport equation
Nonlocal velocity field
Weak solution
Entropy

ABSTRACT

In this paper, we study transport equations with nonlocal velocity fields with rough initial data. We address the global existence of weak solutions of a one dimensional model of the surface quasi-geostrophic equation and the incompressible porous media equation, and one dimensional and n dimensional models of the dissipative quasi-geostrophic equations and the dissipative incompressible porous media equation.

1. Introduction

In this paper, we study several active scalar equations with nonlocal velocity fields. Here, the non-locality means that the velocity field is defined through a singular integral operator that is represented in terms of a Fourier multiplier. For example, in the

* Corresponding author.
E-mail addresses: hantaek@unist.ac.kr (H. Bae), rgranero@math.ucdavis.edu (R. Granero-Belinchón).

http://dx.doi.org/10.1016/j.aim.2014.10.016

two dimensional Euler equation in vorticity form [6], the velocity is recovered from the vorticity ω through

$$u = \nabla^{\perp}(-\Delta)^{-1}\omega \quad \text{or equivalently} \quad \widehat{u}(\xi) = \frac{i\xi^{\perp}}{|\xi|^2}\widehat{\omega}(\xi). \tag{1.1}$$

Other nonlocal and quadratically nonlinear equations appear in many applications. Prototypical examples are the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, magnetogeostrophic equation and their variants. We briefly introduce the equations below.

The surface quasi-geostrophic equations. The surface quasi-geostrophic equation describes the dynamics of the mixture of cold and hot air and the fronts between them in 2 dimensions [23,51]. The equation is of the form

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \tag{1.2}$$

where the scalar function θ is the potential temperature and $\mathcal{R}_j$ is the Riesz transform

$$\mathcal{R}_j f(x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}^2} \frac{(x_j - y_j)f(y)}{|x-y|^3}dy, \quad j = 1, 2.$$

Similar model equations of (1.2) with different types of nonlocal velocities are proposed and analyzed in [2,9,19], respectively (see also [4,10,18,40,44,42,43]):

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \mathcal{R}\theta,$$
$$\theta_t + \nabla\cdot(\theta\mathcal{R}\theta) = 0,$$
$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \nabla^{\perp}\Lambda^{\beta-2}\theta,\ 1 < \beta \le 2.$$

We finally introduce the two dimensional Euler-α model in vorticity form:

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \nabla^{\perp}\Lambda^{-2+\alpha}\theta,\ \alpha \in [0,1] \tag{1.3}$$

which interpolates between (1.1) ($\alpha = 0$) and (1.2) ($\alpha = 1$).

The incompressible porous medium equation. This equation takes the form [17]

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \mathcal{R}^{\perp}\mathcal{R}_1\theta, \tag{1.4}$$

where θ is now the density of the incompressible fluid moving through a homogeneous porous medium. The following version also has been studied with $\beta > 0$ [31]:

$$\theta_t + u\cdot\nabla\theta = 0, \quad u = \Lambda^{\beta}\mathcal{R}^{\perp}\mathcal{R}_1\theta. \tag{1.5}$$

Another important equation related to flow in porous media is the Stokes equation:

$$\theta_t + u \cdot \nabla\theta = 0, \qquad -\Delta u = -\nabla p - (0,\theta)^t, \qquad \nabla \cdot u = 0, \tag{1.6}$$

where the velocity field is given by

$$\begin{aligned} &\text{2D: } u = (-\Delta)^{-1}\mathcal{R}^{\perp}\mathcal{R}_1\theta, \\ &\text{3D: } u = (-\Delta)^{-1}\big(-\mathcal{R}_1\mathcal{R}_3, -\mathcal{R}_2\mathcal{R}_3, \mathcal{R}_1^2 + \mathcal{R}_2^2\big)\theta. \end{aligned}$$

We also define the Stokes-α model

$$\theta_t + u \cdot \nabla\theta = 0, \quad u = \Lambda^{2\alpha-2}\mathcal{R}^{\perp}\mathcal{R}_1\theta \tag{1.7}$$

which interpolates between (1.6) ($\alpha = 0$) and (1.4) ($\alpha = 1$).

Magnetogeostrophic equation. This equation is of the form

$$\theta_t + u \cdot \nabla\theta = 0, \qquad \hat{u}(\xi) = \frac{(\xi_2\xi_3|\xi|^2 - \xi_1\xi_2^2\xi_3, -\xi_1\xi_3|\xi|^2 - \xi_2^3\xi_3, \xi_1^2\xi_2^2 + \xi_2^4)}{|\xi|^2\xi_3^2 + \xi_2^4}\hat{\theta}(\xi), \tag{1.8}$$

where θ represents the buoyancy field. This model is proposed by Moffatt and Loper [49] as a reduced model of the full magnetohydrodynamics to study the geodynamo and turbulence in the Earth's fluid core [32,33,49].

Patch problems. There is a large (and growing) literature in a particular class of weak solution with a *patch* type initial data:

$$\theta_0(x) = \theta^1\mathbf{1}_{\Omega^1} + \theta^2\mathbf{1}_{\mathbb{R}^n\setminus\Omega^1},$$

with θ^i positive constants. In the case of the surface quasi-geostrophic equation, these initial data correspond to a sharp front between two different temperatures and they have obvious interest in meteorology. For the incompressible porous medium equation this situation is known as Muskat problem. This problem is of practical interest because it is a model of the dynamics of the interface between two different fluids in oil wells or geothermal reservoirs. We refer the readers to [5,15,16,21,26,27,34–36,52] and references therein for more details.

All the previous models are posed in two or three spatial dimensions. However, several related one dimensional problems have been studied. The one dimensional reduction idea was initiated by Constatin, Lax and Majda [22]; they proposed the following 1D model

$$\omega_t = \omega\mathcal{H}\omega$$

for the 3D Euler equation in the vorticity form and proved that ω blows up in finite time under certain conditions. Motivated by this work, other similar models were proposed and analyzed in the literature [1,20,25,28,50].

In this paper, we study a one dimensional model of the surface quasi-geostrophic equation and incompressible porous media equation, and one dimensional and n dimensional models of the dissipative quasi-geostrophic equations and the dissipative incompressible porous medium equation in the periodic domain. We begin with the following one dimensional model

$$\theta_t + (\theta \mathcal{H}\theta)_x = 0, \tag{1.9}$$

which is obtained by replacing the Riesz transforms by the Hilbert transform $\mathcal{H}$ in the divergence form of (1.2) and (1.4). We note that (1.9) is also proposed as a model of dislocation dynamics [29,37–39] where θ is related to the density of fractures per length in the material. Eq. (1.9) and related models have been studied by different authors. In a series of papers by A. Castro, D. Chae, A. Córdoba, D. Córdoba and M. Fontelos, the authors addressed the well-posedness and finite time singularities [13,14,20,25]. In particular, A. Castro and D. Córdoba proved global well-posedness for positive $L^2 \cap C^{0,\gamma}$, vanishing at infinity data and finite time singularities for initial data such that $\theta_0(x_0) = 0$ for some x_0. J. Carrillo, L. Ferreira, and J. Precioso in [11] proved the existence of solution corresponding to initial data that are probability measures with finite second moment using gradient flows tools.

We next consider a dissipative model of (1.9):

$$\theta_t + (1-\delta)\mathcal{H}\theta\theta_x + \delta(\theta\mathcal{H}\theta)_x + \nu \Lambda^\gamma \theta = 0, \quad 0 \le \delta \le 1. \tag{1.10}$$

We note that when $\nu = 0$ and $\delta = 1$, we return to (1.9). So, we can understand (1.9) as the limiting case of (1.10). We note that there are several singularity formation results when $\nu = 0$: $0 < \delta < 1/3$ and $\delta = 1$ [50], $0 < \delta \le 1$ [20], and $\delta = 0$ [25,45]. The case $\delta = -1$ is similar to the Kuramoto–Sivashinsky equation and it has been studied in [46].

Finally, we analyze two n-dimensional versions of (1.10). The first model is the equation with $\delta > 0$ and $\nu = 0$:

$$\theta_t + (1-\delta)u \cdot \nabla\theta + \delta \nabla \cdot (\theta \mathcal{R}\theta) = 0, \quad \delta > 0. \tag{1.11}$$

The second model is the equation with $\delta = 0$ and $\nu > 0$ which corresponds to the dissipative quasi-geostrophic equation:

$$\theta_t + u \cdot \nabla\theta + \nu\Lambda^\gamma\theta = 0. \tag{1.12}$$

Here, u in (1.11) and (1.12) is a divergence-free vector field u such that

$$\widehat{u}(k) = m(k)\widehat{\theta}(k), \qquad k \cdot m(k) = 0, \quad m \in L^\infty.$$

Compared with existing results showing global solutions or blowups in finite time with smooth initial data, we establish several global existence of weak solutions with rough

($L^{1+s}, s > 0$) initial data. To the best of our knowledge, the sharpest global existence result for initial data in L^p requires $p > 4/3$ [48]. To this end, we carefully choose dissipative quantities to minimize conditions of initial data. These quantities have the same flavor as the *Shannon entropy*

$$\int \theta \log \theta dx.$$

For more applications of the entropy, see [7,8,12], where the authors apply these ideas jointly with the logarithmic Hardy–Littlewood–Sobolev inequality to the parabolic–elliptic Keller–Segel equation in the plane.

2. Preliminaries

In this paper, all constants will be denoted by C that is a generic constant depending only on the quantities specified in the context.

Hilbert Transform. The Hilbert transform is defined as

$$\mathcal{H}f(x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}} \frac{f(y)}{\tan((x-y)/2)}dy.$$

We list several properties of the Hilbert transform in $\mathbb{T}$.

$$\begin{gathered} \mathcal{H}(\mathcal{H}f) = -f + \langle f \rangle, \quad \langle f \rangle = \int_{\mathbb{T}} f dx, \\ (\mathcal{H}f)_x = \mathcal{H}(f_x), \quad \langle \mathcal{H}f \rangle = 0 \\ \mathcal{H}(f\mathcal{H}g + g\mathcal{H}f) = (\mathcal{H}f)(\mathcal{H}g) - fg - \langle f \rangle \langle g \rangle, \\ \int_{\mathbb{T}} (\mathcal{H}f) g dx = -\int_{\mathbb{T}} f(\mathcal{H}g) dx. \end{gathered} \tag{2.1}$$

Function Spaces. The energy norms in $\mathbb{T}^n$ are defined as follows:

$$\begin{gathered} \|f\|^2_{L^2(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} \big|\widehat{f}(k)\big|^2, \qquad \|f\|^2_{\dot{H}^s(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{2s}\big|\widehat{f}(k)\big|^2 \\ \|f\|^2_{H^s(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} \big(1+|k|^s\big)^2\big|\widehat{f}(k)\big|^2. \end{gathered} \tag{2.2}$$

We also introduce the semi-norm

$$\|f_x\|_{l^1} := \sum_{k\in\mathbb{Z}} |k|\big|\widehat{f}(k)\big|,$$

that is, the derivative is in the Wiener algebra of absolutely convergent Fourier series.

Operator Λ^γ. The differential operator $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$ is defined by the action of the following kernels [24]:

$$\Lambda^\gamma f(x) = c_{\gamma,n}\text{p.v.}\int_{\mathbb{T}^n} \frac{f(x)-f(y)}{|x-y|^{n+\gamma}}dy + c_{\gamma,n}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\int_{\mathbb{T}^n} \frac{f(x)-f(y)}{|x-y+2k\pi|^{n+\gamma}}dy, \tag{2.3}$$

where $c_{\gamma,n} > 0$ is a normalized constant. In particular, in one dimension with $\gamma = 1$,

$$\Lambda f(x) = \mathcal{H}f_x(x) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{T}} \frac{f(x)-f(y)}{\sin^2((x-y)/2)}dy$$

Minimum Principle. In this paper, we assume that $\theta_0 \geq 0$. It is well-known that this property propagates for (1.9), (1.10), (1.11) and (1.12). Let us give a sketch of the proof for (1.12) [20,24]. (The same argument holds for (1.9), (1.10), (1.11).) Let's assume that $\theta(x,t) \in C^1([0,T]\times\mathbb{T}^n)$ and x_t be a point such that $m(t) = \min_x \theta(x,t) = \theta(x_t,t)$. Since $m(t)$ is a continuous Lipschitz function, it is differentiable at almost every point t by Rademacher's theorem. Then, from the definition of Λ^γ and the non-negative assumption, we have

$$\Lambda^\gamma\theta(x_t,t) = c_{\gamma,n}\text{p.v.}\int_{\mathbb{T}^n} \frac{\theta(x_t)-\theta(x_t-y)}{|y|^{n+\gamma}}dy + c_{\gamma,n}\sum_{k\in\mathbb{Z}^n\setminus\{0\}}\int_{\mathbb{T}^n} \frac{\theta(x_t)-\theta(x_t-y)}{|y-2k\pi|^{n+\gamma}}dy \leq 0.$$

This implies that

$$m'(t) = -\nu\Lambda^\gamma\theta(x_t,t) \geq 0.$$

Therefore, we conclude that $\theta(t,x) \geq 0$ for all time. In the paper we will deal with weak solutions that are not continuous in general. However, for the regularized problems, θ^ϵ, the same argument works. Then we construct θ as the limit (in the appropriate space) of θ^ϵ. As the θ will be also the pointwise limit of θ^ϵ, we conclude.

Compactness. Since we look for weak solutions, we use following compactness arguments when we pass to the limit in weak formulations.

Lemma 2.1. *(See [53].) Let X_0, X, X_1 be reflexive Banach spaces such that*

$$X_0 \subset X \subset X_1,$$

where X_0 is compactly embedded in X. Let $T > 0$ be a finite number and let α_0 and α_1 be two finite numbers such that $\alpha_i > 1$. Then,

$$Y = \left\{u \in L^{\alpha_0}(0,T;X_0),\ \partial_t u \in L^{\alpha_1}(0,T;X_1)\right\}$$

is compactly embedded in $L^{\alpha_0}(0,T;X)$.

Lemma 2.2. *(See [47].) Let Ω be a bounded set in $\mathbb{R}^n$. Let (g^ϵ) and (h^ϵ) converge weakly to g and h respectively in $L^{p_1}(0,T;L^{p_2}(\Omega))$ and $L^{q_1}(0,T;L^{q_2}(\Omega))$, with*

$$1 \le p_1, p_2 \le \infty, \qquad \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Suppose that we have the following properties uniformly in $\epsilon > 0$:

$$\begin{aligned} &g_t^\epsilon \text{ is bounded in } L^1\big(0,T;W^{-m,1}(\Omega)\big) \quad \text{for some } m \ge 0, \\ &\big\|h^\epsilon - h^\epsilon(\cdot + y,t)\big\|_{L^{q_1}(0,T;L^{q_2}(\Omega))} \to 0 \quad \text{as } |y| \to 0. \end{aligned} \tag{2.4}$$

Then, $(g^\epsilon h^\epsilon)$ converges to gh in the sense of distributions.

We note that the second condition in (2.4) holds when (h^ϵ) has positive regularity in space.

3. Statements of results

3.1. 1D model of (1.2) and (1.4)

We consider Eq. (1.9) in $\mathbb{T}$ with non-negative initial data. Since (1.9) satisfies the minimum principle, $\theta(t,x) \ge 0$ for all time. We notice that (1.9) is dissipative: the entropy

$$\mathfrak{E}(\theta) = \int_{\mathbb{T}} [\theta \log \theta - \theta + 1] dx \tag{3.1}$$

gains $\dot{H}^{1/2}$ regularity. Therefore, it is natural to assume that θ_0 satisfies the following conditions

$$\theta_0(x) \ge 0, \qquad \theta_0(x) \in L^{1+s}, s > 0, \qquad \mathfrak{E}(\theta_0) < \infty, \tag{3.2}$$

where we need the second condition to obtain a uniform bound of $\mathfrak{E}(\theta_0^\epsilon)$ when we construct an approximate sequence of solutions. We define the function space as follows:

$$\mathcal{A}_T = \left\{ \theta \in L^\infty\big(0,T;L^1(\mathbb{T})\big) : \sup_{0 \le t < T} \mathfrak{E}\big(\theta(t)\big) + \int_0^T \big\|\Lambda^{1/2}\theta(t)\big\|_{L^2}^2 dt < \infty \right\}. \tag{3.3}$$

Definition 3.1. θ is a weak solution of (1.9) if $\theta \in \mathcal{A}_T$ and (1.9) holds in the sense of distributions: for any $\psi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{T})$,

$$\int_0^T \int_{\mathbb{T}} \big[\theta \psi_t + \theta(\mathcal{H}\theta)\psi_x\big] dx dt = \int_{\mathbb{T}} \theta_0(x)\psi(x,0) dx \quad \text{for any } T < \infty.$$

Theorem 3.1. *For any initial datum θ_0 satisfying (3.2), there exists a weak solution of (1.9) in $\mathcal{A}_T$ for all $T > 0$.*

3.2. Dissipative 1D model

We first show the local well-posedness of (1.10) in $H^2(\mathbb{T})$ with $\nu > 0$ and $\delta = 0$. Then, we continue to prove that the solution can be extended beyond $T > 0$ as long as $\|\theta_x(t)\|_{L^\infty}$ is integrable in $[0, T]$. The last condition can be achieved if $\|\theta_{0x}\|_{l^1}$ is sufficiently small. We note that H. Dong and A. Kiselev proved the global existence in the critical case $\gamma = 1$ in [30] and [41], respectively. Therefore, we restrict ourselves to the case $0 < \gamma < 1$ for the local well-posedness.

Theorem 3.2. *Let $\nu > 0$, $\delta = 0$, and $0 < \gamma < 1$. For any initial datum $\theta_0 \in H^2(\mathbb{T})$, there exists $T = T(\theta_0) > 0$ such that there exists a unique solution of (1.10) $\theta \in C(0, T; H^2(\mathbb{T}))$. If $\|\theta_{0x}\|_{l^1} < \nu$, we can take $T = \infty$.*

We recall that the local existence part in this theorem was proved in [30].

The second result is the global existence of a weak solution of (1.10). We note that the additional term on the right-hand side of (1.10) depletes the nonlinear term $(\mathcal{H}\theta)\theta_x$ when $\delta \geq \frac{1}{2}$ with a strictly positive lower bound of θ_0. This enables us to show the existence of a weak solution with $\nu \geq 0$. However, regularity of initial data prescribed below is relatively higher than the usual L^2 regularity in dissipative equations because (1.10) is not in the divergence form. In this paper, we assume that initial data satisfy the following conditions:

$$\theta_0 \in H^{1/2}(\mathbb{T}), \quad m_0 = \operatorname{ess\,inf}_{x\in\mathbb{T}} \theta_0(x) > 0. \tag{3.4}$$

Definition 3.2. θ is a weak solution of (1.10) with $\theta_0 \in H^{1/2}(\mathbb{T})$ if $\theta(t) \in H^{1/2}(\mathbb{T})$ for any $t \leq T$ and (1.10) holds in the sense of distributions: for any $\psi \in \mathcal{C}_c^\infty([0, T) \times \mathbb{T})$,

$$\int_0^T \int_{\mathbb{T}} \left[\theta\psi_t - (1-\delta)\mathcal{H}\theta\theta_x\psi - \nu\theta\Lambda^\gamma\psi + \delta\theta\mathcal{H}\theta\psi_x\right] dxdt = \int_{\mathbb{T}} \theta_0(x)\psi(x,0)dx$$

for any $T < \infty$.

Theorem 3.3. *Let $\nu \geq 0$, $\gamma \geq 0$ and $1/2 \leq \delta < 1$. For any initial datum θ_0 satisfying (3.4), there exists a global weak solution of (1.10) such that*

$$\theta \in L^\infty\big(0, \infty; H^{1/2}(\mathbb{T})\big) \cap L^2\big(0, \infty; H^{\max\{1,(1+\gamma)/2\}}(\mathbb{T})\big).$$

Moreover, such a solution is unique in $L^2(\mathbb{T})$ if $\nu > 0$ and $\gamma \geq 2$.

3.3. High dimensional model

We finally consider Eqs. (1.11) and (1.12) in $\mathbb{T}^n$, $n = 2, 3$, with a divergence-free vector field u satisfying

$$\widehat{u}(k) = m(k)\widehat{\theta}(k), \qquad k \cdot m(k) = 0, \quad m \in L^\infty. \tag{3.5}$$

We begin with Eq. (1.11). We use the n-dimensional version of the entropy (3.1) and functional space (3.3). This is due to the fact that the advection term vanishes in the computation of $\mathfrak{E}(\theta)_t$ by the divergence-free condition of u.

Definition 3.3. θ is a weak solution of (1.11) if $\theta \in \mathcal{A}_T$ and (1.11) holds in the sense of distributions: for any $\psi \in \mathcal{C}_c^\infty([0,T) \times \mathbb{T}^n)$

$$\int_0^T \int_{\mathbb{T}^n} \big[\theta\psi_t + (1-\delta)\theta u \cdot \nabla\psi + \delta\theta\mathcal{R}\theta \cdot \nabla\psi\big]\, dxdt = \int_{\mathbb{T}^n} \theta(x,0)\psi(x,0)dx \quad \text{for every } T < \infty.$$

Theorem 3.4. *For any initial datum θ_0 satisfying (3.2), there exists a weak solution of (1.11) in $\mathcal{A}_T$ for all $T > 0$.*

We note that the same result holds for a smoother velocity field:

$$\widehat{u}(k) = |k|^\beta m(k)\widehat{\theta}(k), \qquad k \cdot m(k) = 0, \quad m(0) = 0 \tag{3.6}$$

with bounded m and $\beta < 0$. We note that u in (3.6) covers (1.3), (1.6) and (1.7).

Corollary 3.5. *For any initial datum θ_0 satisfying (3.2) and u given by (3.6), there exists a weak solution of (1.11) in $\mathcal{A}_T$ for every $T > 0$.*

We finally deal with Eq. (1.12) with a slightly different entropy

$$\mathfrak{E}(\theta) = \int_{\mathbb{T}^n} (\theta + 1)\log(\theta + 1)dx.$$

Due to the lack of the smoothing effect from $\nabla \cdot (\theta\mathcal{R}\theta)$, we only have bounds of θ for $t \geq \tau > 0$. Therefore, we define the function space and the notion of weak solution as follows.

$$\mathcal{B}_T = \bigg\{\theta \in L^\infty\big(0,T; L^1(\mathbb{T}^n)\big) \cap L^\infty\big([\tau,T); L^\infty(\mathbb{T}^n)\big) : \sup_{\tau \leq t < T} \mathfrak{E}\big(\theta(t)\big) + \int_\tau^T \big\|\Lambda^{\gamma/2}\theta(t)\big\|_{L^2}^2 dt < \infty\bigg\}. \tag{3.7}$$

Definition 3.4. θ is a weak solution of (1.12) if $\theta \in \mathcal{B}_T$ and (1.11) holds in the sense of distributions: for any $\psi \in \mathcal{C}_c^\infty([\tau,T)\times\mathbb{T}^n)$

$$\int_\tau^T \int_{\mathbb{T}^n} \left[\theta\psi_t + \theta u\cdot\nabla\psi + \nu\theta\Lambda^\gamma\psi\right]dxdt = 0 \quad \text{for every } 0<\tau<T<\infty.$$

Theorem 3.6. *For any initial datum θ_0 satisfying (3.2), there exists a weak solution of (1.12) in $\mathcal{B}_T$ for every $T>0$. Moreover, $\theta(t)$ converges to θ_0 in $H^{-2}(\mathbb{T}^n)$ as $t\to 0$.*

Remark 1. Actually, following the ideas in the proof of Theorem 3.6, we can prove that the solution θ in Theorems 3.1–3.4, Corollary 3.5 and Theorem 3.6 is in $L^\infty(\tau,T;L^\infty(\mathbb{T}^n))$ for every $0<\tau<T<\infty$.

The proofs of our results are outlined as follows. We first obtain a priori estimates in given function spaces. We then generate approximate sequence of solutions and pass to the limits in weak formulation using Lemma 2.1 or 2.2.

4. Proof of Theorem 3.1

We consider Eq. (1.9)

$$\theta_t + (\theta\mathcal{H}\theta)_x = 0$$

and the entropy

$$\mathfrak{E}(\theta) = \int_{\mathbb{T}} [\theta\log\theta - \theta + 1]dx.$$

Since $\theta(t)\geq 0$, $\mathfrak{E}(\theta)\geq 0$. Moreover, the direct computation yields that

$$\begin{aligned}\frac{d}{dt}\mathfrak{E}(\theta) &= \int_{\mathbb{T}} \left[\theta_t\log\theta(t) + \theta(t)\big(\log\theta(t)\big)_t - \theta_t\right]dx = \int_{\mathbb{T}} \theta_t\log\theta(t)dx \\ &= -\int_{\mathbb{T}} (\theta\mathcal{H}\theta)_x\log\theta dx = \int_{\mathbb{T}} (\mathcal{H}\theta)\theta_x dx = -\int_{\mathbb{T}} \theta\Lambda\theta dx = -\left\|\Lambda^{1/2}\theta\right\|_{L^2}^2. \end{aligned} \tag{4.1}$$

Therefore, we have $\theta\in\mathcal{A}$. We now construct a sequence of solutions (θ^ϵ) by solving

$$\theta_t^\epsilon + \left(\theta^\epsilon\mathcal{H}\theta^\epsilon\right)_x = \epsilon\theta_{xx}^\epsilon, \qquad \theta_0^\epsilon = \rho_\epsilon * \theta_0,$$

where ρ_ϵ is a standard mollifier. Then, θ^ϵ satisfies that

$$\frac{d}{dt}\mathfrak{E}(\theta^\epsilon) + \left\|\Lambda^{1/2}\theta^\epsilon\right\|_{L^2}^2 + 4\epsilon\int_{\mathbb{T}} \left|\left(\sqrt{\theta^\epsilon}\right)_x\right|^2 dx = 0. \tag{4.2}$$

Integrating (4.2) in time, we have

$$\mathfrak{E}\big(\theta^\epsilon(t)\big) + \int_0^t \big\|\Lambda^{1/2}\theta^\epsilon(s)\big\|_{L^2}^2 ds + 4\epsilon \int_0^t \int_{\mathbb{T}} \big|\big(\sqrt{\theta^\epsilon}(s)\big)_x\big|^2 dx ds = \mathfrak{E}\big(\theta_0^\epsilon\big). \tag{4.3}$$

Since $x\log x - x + 1 \le x^{s+1} + 1$ for $x \ge 0$, we can bound the last term in (4.3) as

$$\mathfrak{E}\big(\theta_0^\epsilon\big) \le 2\pi + \big\|\theta_0^\epsilon\big\|_{L^{s+1}}^{s+1} \le 2\pi + \|\theta_0\|_{L^{s+1}}^{s+1}.$$

Therefore, the sequence (θ^ϵ) is uniformly bounded in $\mathcal{A}_T$. By Poincaré's inequality, we obtain uniform bounds of θ^ϵ and $\mathcal{H}\theta^\epsilon$ in $L^2((0,T);H^{1/2}(\mathbb{T}))$. Moreover, by interpolating $L^\infty(0,T;L^1(\mathbb{T}))$ and $L^2((0,T);H^{1/2}(\mathbb{T}))$, we have uniform bounds of θ^ϵ and $\mathcal{H}\theta^\epsilon$ in $L^4(0,T;L^2(\mathbb{T}))$. These estimates and the duality pairing imply that

$$\theta_t^\epsilon = -\big(\theta^\epsilon\mathcal{H}\theta^\epsilon\big)_x + \epsilon\theta_{xx}^\epsilon \in L^2\big(0,T;H^{-2}(\mathbb{T})\big)$$

uniformly in $\epsilon > 0$. Lemma 2.1 with

$$X_0 = L^2\big(0,T;H^{1/2}(\mathbb{T})\big), \qquad X = L^2\big(0,T;L^2(\mathbb{T})\big), \qquad X_1 = L^2\big(0,T;H^{-2}(\mathbb{T})\big),$$

allows to pass to the limit in

$$\int_0^T \int_{\mathbb{T}} \big[\theta^\epsilon\psi_t + \theta^\epsilon\big(\mathcal{H}\theta^\epsilon\big)\psi_x\big] dx dt = \int_{\mathbb{T}} \theta_0^\epsilon(x)\psi(x,0) dx$$

to obtain a weak solution in $\mathcal{A}_T$.

5. Proof of Theorem 3.2

5.1. Local well-posedness

We consider Eq. (1.10) with $\delta = 0$:

$$\theta_t + (\mathcal{H}\theta)\theta_x + \nu\Lambda^\gamma\theta = 0, \quad \nu > 0, \ \gamma > 0. \tag{5.1}$$

We here only provide a priori estimates. We first multiply (5.1) by θ and integrate over $\mathbb{T}$:

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \nu\big\|\Lambda^{\frac{\gamma}{2}}\theta\big\|_{L^2}^2 = -\int_{\mathbb{T}} \big[(\mathcal{H}\theta)\theta_x\theta\big] dx \le \|\theta_x\|_{L^\infty}\|\mathcal{H}\theta\|_{L^2}\|\theta\|_{L^2} \le \|\theta_x\|_{L^\infty}\|\theta\|_{L^2}^2.$$

We next take ∂_x to (5.1) and do the energy estimate.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta_x\|_{L^2}^2+\nu\big\|\Lambda^{\frac{\gamma}{2}}\theta_x\big\|_{L^2}^2&=-\int_{\mathbb{T}}(\mathcal{H}\theta\theta_x)_x\theta_x dx=-\int_{\mathbb{T}}(\mathcal{H}\theta)_x(\theta_x)^2dx-\int_{\mathbb{T}}\mathcal{H}\theta\theta_{xx}\theta_x dx\\&=-\int_{\mathbb{T}}(\mathcal{H}\theta)_x(\theta_x)^2dx-\frac{1}{2}\int_{\mathbb{T}}\mathcal{H}\theta\big[(\theta_x)^2\big]_x dx\\&=-\frac{1}{2}\int_{\mathbb{T}}(\mathcal{H}\theta)_x(\theta_x)^2dx\\&\le\|\theta_x\|_{L^\infty}\|\mathcal{H}\theta_x\|_{L^2}\|\theta_x\|_{L^2}\le\|\theta_x\|_{L^\infty}\|\theta_x\|^2.\end{aligned}$$

Similarly, by taking ∂_{xx} to (5.1), we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_{xx}\|_{L^2}^2+\nu\big\|\Lambda^{\frac{\gamma}{2}}\theta_{xx}\big\|_{L^2}^2\le\big(\|\theta_x\|_{L^\infty}+\|\mathcal{H}\theta_x\|_{L^\infty}\big)\|\theta_{xx}\|_{L^2}^2.$$

Therefore, we obtain that

$$\frac{d}{dt}\|\theta\|_{H^2}^2+\nu\big\|\Lambda^{\frac{\gamma}{2}}\theta\big\|_{H^2}^2\le C\big(\|\theta_x\|_{L^\infty}+\|\mathcal{H}\theta_x\|_{L^\infty}\big)\|\theta\|_{H^2}^2. \tag{5.2}$$

Since $\|\theta_x\|_{L^\infty}+\|\mathcal{H}\theta_x\|_{L^\infty}\le C\|\theta\|_{H^2}$, (5.2) implies the local well-posedness in $H^2(\mathbb{T})$. Moreover, by integrating (5.2) in time, we obtain that

$$\big\|\theta(t)\big\|_{H^2}^2\le C\|\theta_0\|_{H^2}^2\exp\int_0^t C\big(\big\|\theta_x(s)\big\|_{L^\infty}+\big\|\mathcal{H}\theta_x(s)\big\|_{L^\infty}\big)ds.$$

Using the logarithmic bound (similar to the Beale–Kato–Majda criterion [3])

$$\|\mathcal{H}\theta_x\|_{L^\infty}\le C\big(1+\|\theta_x\|_{L^\infty}\log\big(e+\|\theta\|_{H^2}\big)+\|\theta_x\|_{L^2}\big),$$

the solution can be continued as long as we can control $\|\theta_x\|_{L^\infty}$.

5.2. Estimation of $\|\theta_x\|_{l^1}$

We now control $\|\theta_x\|_{L^\infty}$ by $\|\theta_x\|_{l^1}$. For $k\in\mathbb{Z}$,

$$\begin{aligned}|k|\big|\widehat{\theta}(k)\big|_t&=-\nu|k|^{1+\gamma}\big|\widehat{\theta}(k)\big|-\frac{\widehat{\theta}(k)}{|\widehat{\theta}(k)|}|k|\sum_{l\in\mathbb{Z}}\Big[il\widehat{\theta}(l)\frac{i(k-l)}{|k-l|}\widehat{\theta}(k-l)\Big]\\&\le-\nu|k|^{1+\gamma}\big|\widehat{\theta}(k)\big|+|k|\sum_{l\in\mathbb{Z}}\big|\widehat{\theta}(k-l)\big||l|\big|\widehat{\theta}(l)\big|.\end{aligned}$$

By taking the summation over $k\in\mathbb{Z}$,

$$\begin{aligned}\frac{d}{dt}\|\theta_x\|_{l^1} &\leq -\nu\sum_{k\in\mathbb{Z}}|k|^{1+\gamma}\big|\widehat{\theta}(k)\big| + \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|k||l|\big|\widehat{\theta}(l)\big|\big|\widehat{\theta}(k-l)\big| \\ &= -\nu\sum_{k\in\mathbb{Z}}|k|^{1+\gamma}\big|\widehat{\theta}(k)\big| + \sum_{l\in\mathbb{Z}}|l|\big|\widehat{\theta}(l)\big|\sum_{k\in\mathbb{Z}}|k|\big|\widehat{\theta}(k-l)\big| \\ &\leq -\nu\big\|\Lambda^{1+\gamma}\theta\big\|_{l^1} + \|\theta_x\|_{l^1}^2 \leq \|\theta_x\|_{l^1}\big(\|\theta_x\|_{l^1}-\nu\big).\end{aligned}$$

Therefore, $\|\theta_x(t)\|_{l^1} < \nu$ as long as $\|\theta_{0,x}\|_{l^1} < \nu$. This completes the proof.

6. Proof of Theorem 3.3

We consider Eq. (1.10) which is equivalent to

$$\theta_t + (\mathcal{H}\theta)\theta_x + \nu\Lambda^\gamma\theta + \delta\theta\Lambda\theta = 0, \quad \nu\geq 0,\ \gamma\geq 0,\ 1/2\leq\delta<1. \tag{6.1}$$

We begin with a priori estimates. To obtain the L^2 bound, we multiply (6.1) by θ and integrate over $\mathbb{T}$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \nu\big\|\Lambda^{\gamma/2}\theta\big\|_{L^2}^2 &= -\int_{\mathbb{T}}\big[(\mathcal{H}\theta)\theta_x\theta\big]dx - \delta\int_{\mathbb{T}}\big[\theta^2\Lambda\theta\big]dx \\ &= -\frac{1}{2}\int_{\mathbb{T}}\big[(\mathcal{H}\theta)\big(\theta^2\big)_x\big]dx - \delta\int_{\mathbb{T}}\big[\theta^2\Lambda\theta\big]dx \\ &= \left(\frac{1}{2}-\delta\right)\int_{\mathbb{T}}\big[\theta^2\Lambda\theta\big]dx.\end{aligned}$$

Since $\theta\geq 0$, we have

$$\int_{\mathbb{T}}\big[\theta^2\Lambda\theta\big]dx = \int_{\mathbb{T}}\int_{\mathbb{T}}\frac{(\theta(x)-\theta(y))^2}{\sin^2((x-y)/2)}\cdot\frac{\theta(x)+\theta(y)}{2}dxdy\geq 0.$$

Therefore, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \nu\big\|\Lambda^{\gamma/2}\theta\big\|_{L^2}^2\leq 0. \tag{6.2}$$

We next obtain the $\dot{H}^{1/2}$ bound. We multiply (6.1) by $\Lambda\theta$ and integrate over $\mathbb{T}$:

$$\frac{1}{2}\frac{d}{dt}\big\|\Lambda^{1/2}\theta\big\|_{L^2}^2 + \nu\big\|\Lambda^{(1+\gamma)/2}\theta\big\|_{L^2}^2 = -\int_{\mathbb{T}}\big[(\mathcal{H}\theta)\theta_x\Lambda\theta\big]dx - \delta\int_{\mathbb{T}}\big[\theta(\Lambda\theta)^2\big]dx. \tag{6.3}$$

We now compute the first integral on the right-hand side of (6.3). Since

$$-\int_{\mathbb{T}}\big[(\mathcal{H}\theta)\theta_x\Lambda\theta\big]dx = \int_{\mathbb{T}}\big[\theta\mathcal{H}\big(\theta_x(\mathcal{H}\theta_x)\big)\big]dx = \frac{1}{2}\int_{\mathbb{T}}\big[\theta\big((\Lambda\theta)^2-(\theta_x)^2\big)\big]dx,$$

we have

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{1/2}\theta\right\|_{L^2}^2+\nu\left\|\Lambda^{(1+\gamma)/2}\theta\right\|_{L^2}^2=\left(\frac{1}{2}-\delta\right)\int_{\mathbb{T}}\left[\theta(\Lambda\theta)^2\right]dx-\frac{1}{2}\int_{\mathbb{T}}\left[\theta(\theta_x)^2\right]dx,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{1/2}\theta\right\|_{L^2}^2+\nu\left\|\Lambda^{(1+\gamma)/2}\theta\right\|_{L^2}^2+\frac{1}{2}\int_{\mathbb{T}}\left[\theta(\theta_x)^2\right]dx\le 0. \tag{6.4}$$

We note that the minimum principle, with $m_0>0$, implies that $\theta(t,x)\ge m_0>0$ for all time. Therefore, by (6.2), (6.4), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{H^{1/2}}^2+\nu\left\|\Lambda^{\gamma/2}\theta\right\|_{H^{1/2}}^2+\frac{1}{2}m_0\|\theta_x\|_{L^2}^2\le 0. \tag{6.5}$$

Integrating (6.5) in time, we conclude that

$$\theta\in L^\infty\left(0,\infty;H^{1/2}(\mathbb{T})\right)\cap L^2\left(0,\infty;\dot{H}^1(\mathbb{T})\right),\qquad \Lambda^{\gamma/2}\theta\in L^2\left(0,\infty;H^{1/2}(\mathbb{T})\right). \tag{6.6}$$

We now construct an approximate sequence of solutions (θ^ϵ) by solving

$$\theta_t+(\mathcal{H}\theta)\theta_x+\Lambda^\gamma\theta=-\delta\theta\Lambda\theta+\epsilon\theta_{xx},\qquad \theta_0^\epsilon=\rho_\epsilon*\theta_0.$$

Then, (θ^ϵ) is uniformly bounded in the space stated in (6.6). By interpolating $L^\infty(0,T;H^{1/2}(\mathbb{T}))$ and $L^2(0,T;H^1(\mathbb{T}))$, we have uniform bounds of $\mathcal{H}\theta^\epsilon$ and θ^ϵ in $L^4(0,T;H^{3/4}(\mathbb{T}))$ that is embedded in $L^4(0,T;L^2(\mathbb{T}))$. This implies that

$$\left(\mathcal{H}\theta^\epsilon\right)\theta_x^\epsilon\in L^{4/3}\left(0,T;L^1(\mathbb{T})\right),\qquad \theta^\epsilon\Lambda\theta^\epsilon\in L^{4/3}\left(0,T;L^1(\mathbb{T})\right)$$

uniformly in $\epsilon>0$. Therefore, we obtain that

$$\theta_{xt}^\epsilon=-\left(\left(\mathcal{H}\theta^\epsilon\right)\theta_x^\epsilon-\Lambda^\gamma\theta^\epsilon+\delta\theta^\epsilon\Lambda\theta^\epsilon+\epsilon\theta_{xx}^\epsilon\right)_x\in L^{4/3}\left(0,T;W^{-2,1}(\mathbb{T})\right).$$

Similarly,

$$\Lambda\theta_t^\epsilon\in L^{4/3}\left(0,T;W^{-2,1}(\mathbb{T})\right).$$

Moreover, by Sobolev embedding

$$L^2\left(0,T;H^1(\mathbb{T})\right)\subset L^2\left(0,T;C^\alpha(\mathbb{T})\right),\quad 0<\alpha<1/2,$$

we have

$$\left\|\theta^\epsilon(\cdot)-\theta^\epsilon(\cdot+y)\right\|_{L^2(0,T;L^2(\mathbb{T}))}\le C(\theta_0)|y|^\alpha\to 0\quad\text{as } |y|\to 0.$$

Similarly,

$$\left\|\mathcal{H}\theta^\epsilon(\cdot) - \mathcal{H}\theta^\epsilon(\cdot + y)\right\|_{L^2(0,T;L^2(\mathbb{T}))} \to 0 \quad \text{as } |y| \to 0.$$

Therefore, Lemma 2.2 with

$$q_i = p_i = 2, \qquad g^\epsilon = \theta^\epsilon_x, \Lambda\theta^\epsilon, \qquad h^\epsilon = \mathcal{H}\theta^\epsilon, \theta^\epsilon$$

allows to pass to the limit in

$$\int_0^\infty \int_{\mathbb{T}} \left[\theta^\epsilon_t + \left(\mathcal{H}\theta^\epsilon\right)\theta^\epsilon_x + \Lambda^\gamma\theta^\epsilon + \delta\theta^\epsilon\Lambda\theta^\epsilon\right]\psi dx dt = \int_{\mathbb{T}} \theta^\epsilon_0(x)\psi(x,0)dx$$

to obtain a weak solution in the space stated in (6.6).

To show the uniqueness of a solution, let $\theta = \theta_1 - \theta_2$. Then, θ satisfies

$$\theta_t + \nu\Lambda^\gamma\theta = -(\mathcal{H}\theta)\theta_{1x} - (\mathcal{H}\theta_2)\theta_x - \delta\theta\Lambda\theta_1 - \delta\theta_2\Lambda\theta, \qquad \theta(0,x) = 0. \tag{6.7}$$

We multiply θ to (6.7) and integrate over $\mathbb{T}$. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2} + \nu\left\|\Lambda^{\frac{\gamma}{2}}\theta\right\|^2_{L^2} &= \int_{\mathbb{T}} \left[-(\mathcal{H}\theta)\theta_{1x} - (\mathcal{H}\theta_2)\theta_x - \delta\theta\Lambda\theta_1 - \delta\theta_2\Lambda\theta\right]\theta dx \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.\end{aligned}$$

We first estimate $\mathrm{II} + \mathrm{IV}$:

$$\begin{aligned}\mathrm{II} + \mathrm{IV} &\le C\left(\|\theta_2\|_{L^\infty} + \|\mathcal{H}\theta_2\|_{L^\infty}\right)\|\theta\|_{L^2}\|\theta_x\|_{L^2} \\ &\le C(\nu)\left(\|\theta_2\|^2_{L^\infty} + \|\mathcal{H}\theta_2\|^2_{L^\infty}\right)\|\theta\|^2_{L^2} + \frac{\nu}{4}\|\theta_x\|^2_{L^2}.\end{aligned}$$

To estimate I, we do the integration by parts to obtain

$$\mathrm{I} \le C\|\theta_1\|_{L^\infty}\|\theta\|_{L^2}\|\theta_x\|_{L^2} \le \frac{\nu}{8}\|\theta_x\|^2_{L^2} + C(\nu)\|\theta_1\|^2_{L^\infty}\|\theta\|^2_{L^2}.$$

To estimate III, we use $\Lambda\theta_1 = (\mathcal{H}\theta_1)_x$ and do the integration by parts to obtain

$$\mathrm{III} \le C\|\mathcal{H}\theta_1\|_{L^\infty}\|\theta\|_{L^2}\|\theta_x\|_{L^2} \le \frac{\nu}{8}\|\theta_x\|^2_{L^2} + C(\nu)\|\mathcal{H}\theta_1\|^2_{L^\infty}\|\theta\|^2_{L^2}.$$

Since

$$\|\theta_x\|_{L^2} \le \left\|\Lambda^{\gamma/2}\theta\right\|_{L^2}, \qquad \|\theta_i\|_{L^\infty} + \|\mathcal{H}\theta_i\|_{L^\infty} \le C\|\theta_i\|_{H^1}, \quad \text{for } i = 1, 2,$$

we conclude that

$$\frac{d}{dt}\|\theta\|_{L^2}^2 \leq C(\nu)\big(\|\theta_1\|_{H^1}^2 + \|\theta_2\|_{H^1}^2\big)\|\theta\|_{L^2}^2 \tag{6.8}$$

which implies that $\theta = 0$ in L^2.

7. Proof of Theorem 3.4 and Corollary 3.5

Proof of Theorem 3.4. We consider Eq. (1.11):

$$\theta_t + (1-\delta)u\cdot\nabla\theta + \delta\nabla\cdot(\theta\mathcal{R}\theta) = 0.$$

As Eq. (1.9), the entropy (3.1) satisfies that

$$\frac{d}{dt}\mathfrak{E}(\theta) = \int_{\mathbb{T}^n}\theta_t\log\theta dx = \int_{\mathbb{T}^n}\big((1-\delta)u + \delta\mathcal{R}\theta\big)\cdot\nabla\theta dx = -\delta\big\|\Lambda^{1/2}\theta\big\|_{L^2}^2.$$

Therefore, we have $\theta \in \mathcal{A}$. We now construct a sequence of solutions (θ^ϵ) by solving

$$\theta_t^\epsilon + (1-\delta)u^\epsilon\cdot\nabla\theta^\epsilon + \delta\nabla\cdot\big(\theta^\epsilon\mathcal{R}\theta^\epsilon\big) = \epsilon\Delta\theta^\epsilon, \qquad \theta_0^\epsilon = \rho_\epsilon * \theta_0.$$

Then, (θ^ϵ) satisfies that

$$\frac{d}{dt}\mathfrak{E}\big(\theta^\epsilon\big) + \big\|\Lambda^{1/2}\theta^\epsilon\big\|_{L^2}^2 + 4\epsilon\int_{\mathbb{T}^n}\big|\nabla\big(\sqrt{\theta^\epsilon}\big)\big|^2 dx = 0. \tag{7.1}$$

Integrating (7.1) in time,

$$\mathfrak{E}\big(\theta^\epsilon(t)\big) + \int_0^t\big\|\Lambda^{1/2}\theta^\epsilon(s)\big\|_{L^2}^2 ds + 4\epsilon\int_0^t\int_{\mathbb{T}^n}\big|\nabla\big(\sqrt{\theta^\epsilon}(s)\big)\big|^2 dxds \leq \mathfrak{E}\big(\theta_0^\epsilon\big). \tag{7.2}$$

Since $x\log x - x + 1 \leq x^{s+1} + 1$ for $x \geq 0$, we can bound the last term in (7.2) by

$$\mathfrak{E}\big(\theta_0^\epsilon\big) \leq (2\pi)^n + \big\|\theta_0^\epsilon\big\|_{L^{s+1}}^{s+1}.$$

Therefore, the sequence (θ^ϵ) is uniformly bounded in $\mathcal{A}_T$. Using this bound, we first treat the two dimensional case. From Sobolev embedding, we have uniform bounds

$$\mathcal{R}\theta^\epsilon, u^\epsilon, \theta^\epsilon \in L^2\big(0,T;L^4\big(\mathbb{T}^2\big)\big).$$

Moreover, by interpolating $L^\infty(0,T;L^1(\mathbb{T}^2))$ and $L^2(0,T;L^4(\mathbb{T}^2))$, we have uniform bounds

$$\mathcal{R}\theta^\epsilon, u^\epsilon, \theta^\epsilon \in L^6\big(0,T;L^{4/3}\big(\mathbb{T}^2\big)\big).$$

This implies that

$$u^\epsilon\theta^\epsilon, \theta^\epsilon\mathcal{R}\theta^\epsilon \in L^{3/2}\big(0,T;L^1\big(\mathbb{T}^2\big)\big)$$

and thus

$$\theta_t^\epsilon \in L^{3/2}\big(0,T;H^{-2}\big(\mathbb{T}^2\big)\big).$$

Using Lemma 2.1 with

$$X_0 = L^2\big(0,T;H^{1/2}\big(\mathbb{T}^2\big)\big), \qquad X = L^2\big(0,T;L^2\big(\mathbb{T}^2\big)\big), \qquad X_1 = L^{3/2}\big(0,T;H^{-2}\big(\mathbb{T}^2\big)\big)$$

we can pass to the limit in the weak formulation to obtain a weak solution in $\mathcal{A}_T$. Similarly, in three dimensions, we have

$$u^\epsilon\theta^\epsilon, \theta^\epsilon\mathcal{R}\theta^\epsilon \in L^{4/3}\big(0,T;L^1\big(\mathbb{T}^3\big)\big), \qquad \theta_t^\epsilon \in L^{4/3}\big(0,T;H^{-2}\big(\mathbb{T}^3\big)\big).$$

Using Lemma 2.1 with

$$X_0 = L^2\big(0,T;H^{1/2}\big(\mathbb{T}^3\big)\big), \qquad X = L^2\big(0,T;L^2\big(\mathbb{T}^3\big)\big), \qquad X_1 = L^{4/3}2\big(0,T;H^{-2}\big(\mathbb{T}^3\big)\big),$$

we complete the proof of Theorem 3.4. □

Proof of Corollary 3.5. We notice that the hypothesis $m(0) = 0$ together with $\theta^\epsilon \in L^2((0,T);H^{1/2}(\mathbb{T}^n))$ implies $u^\epsilon \in L^2((0,T);H^{1/2+s}(\mathbb{T}^n))$ for some $s > 0$. This is enough to follow the argument in the proof of Theorem 3.4 to complete the proof of Corollary 3.5. □

8. Proof of Theorem 3.6

We finally consider the equation

$$\theta_t + u\cdot\nabla\theta + \nu\Lambda^\gamma\theta = 0$$

with the following entropy

$$\mathfrak{E}(\theta) = \int_{\mathbb{T}^n} (\theta+1)\log(\theta+1)dx.$$

Since $\theta \geq 0$, we have $\mathfrak{E}(\theta) \geq 0$. Let's start with the *a priori* estimates. The direct computation yields that

$$\frac{d}{dt}\mathfrak{E}(\theta) = -\nu\int_{\mathbb{T}^n}\big(\Lambda^\gamma\theta\big)\log(\theta+1)dx. \tag{8.1}$$

To estimate the right hand side of (8.1) we symmetrize the integral using the representation of Λ^γ (2.3):

$$\frac{d}{dt}\mathfrak{E}(\theta) + \frac{c_{\gamma,n}\nu}{2}\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\frac{\theta(x)-\theta(y)}{|x-y+2k\pi|^{n+\gamma}}\log\left[\frac{\theta(x)+1}{\theta(y)+1}\right]dydx = 0.$$

Since $(X-Y)(\log X - \log Y) \geq C(\log X - \log Y)^2$ for $X \geq 1$ and $Y \geq 1$, we obtain

$$\frac{d}{dt}\mathfrak{E}(\theta) + C(\nu,\gamma,n)\left\|\Lambda^{\gamma/2}\log(\theta+1)\right\|_{L^2}^2 \leq 0.$$

To obtain the diffusion, we compute

$$\begin{aligned}&c(\gamma,n,\nu)\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\frac{(\theta(x)-\theta(y))^2}{|x-y+2k\pi|^{n+\gamma}}\frac{(\log(1+\theta(x))-\log(1+\theta(y)))^2}{(\theta(x)-\theta(y))^2}dydx\\ &\geq \frac{c(\gamma,n,\nu)}{1+\|\theta_0\|_{L^\infty}^2}\left\|\Lambda^{\gamma/2}\theta\right\|_{L^2}^2.\end{aligned}$$

This implies that

$$\mathfrak{E}\big(\theta(t)\big) + \frac{c(\gamma,n,\nu)}{1+\|\theta_0\|_{L^\infty}^2}\left\|\Lambda^{\gamma/2}\theta\right\|_{L^2}^2 \leq \mathfrak{E}(\theta_0) \leq (2\pi)^n + \|\theta_0\|_{L^{s+1}}^{s+1}$$

for all $t > 0$ and thus we conclude $\theta \in \mathcal{B}_T$ from (8.1). We now construct a sequence of solutions (θ^ϵ) by solving

$$\theta_t^\epsilon + u^\epsilon\cdot\nabla\theta^\epsilon + \nu\Lambda^\gamma\theta^\epsilon = \epsilon\Delta\theta^\epsilon, \qquad \theta_0^\epsilon = \rho_\epsilon * \theta_0.$$

For such a solution, the following inequality holds:

$$\left\|\theta^\epsilon(t)\right\|_{L^1(\mathbb{T}^n)} = \left\|\theta_0^\epsilon\right\|_{L^1(\mathbb{T}^n)} \leq \|\theta_0\|_{L^1(\mathbb{T}^n)}.$$

Since $\theta^\epsilon(t)$ is smooth (in space and time), the function $\|\theta^\epsilon(t)\|_{L^\infty(\mathbb{T}^n)} = \theta^\epsilon(x_t)$ is Lipschitz. Using Rademacher Theorem, we conclude that $\theta(x_t)$ is differentiable almost everywhere. For each t, let x_t be the point of maximum. Then,

$$\frac{d}{dt}\left\|\theta^\epsilon(t)\right\|_{L^\infty(\mathbb{T}^n)} = \partial_t\theta^\epsilon(x_t).$$

We now estimate nonlocal terms. We take a positive number $0 < r < \pi$ and define

$$\mathcal{U}_1 = \big\{\eta\in[-r,r]^n : \theta^\epsilon(x_t) - \theta^\epsilon(x_t-\eta) > \theta^\epsilon(x_t)/2\big\},$$

and $\mathcal{U}_2 = [-r,r]^n \setminus \mathcal{U}_1$. Then, we have

$$\|\theta_0\|_{L^1(\mathbb{T}^n)} \geq \left\|\theta_0^\epsilon\right\|_{L^1(\mathbb{T}^n)} = \int_{\mathbb{T}^n}\theta^\epsilon(x_t-\eta)d\eta \geq \int_{\mathcal{U}_2}\theta^\epsilon(x_t-\eta)d\eta \geq \frac{\theta^\epsilon(x_t)}{2}|\mathcal{U}_2|$$

or equivalently,

$$(2r)^n - \frac{2\|\theta_0\|_{L^1(\mathbb{T}^n)}}{\theta^\epsilon(x_t)} \leq (2r)^n - |\mathcal{U}_2| = |\mathcal{U}_1|.$$

This implies that

$$\begin{aligned}\frac{1}{c_{\gamma,n}}\Lambda^\gamma\theta^\epsilon(x_t) &\geq \text{p.v.}\int_{\mathbb{T}^n}\frac{\theta^\epsilon(x_t)-\theta^\epsilon(x_t-y)}{|y|^{n+\gamma}}dy \geq \int_{\mathcal{U}_1}\frac{\theta^\epsilon(x_t)-\theta^\epsilon(x_t-y)}{|y|^{n+\gamma}}dy \\ &\geq \frac{\theta^\epsilon(x_t)}{2r^{n+\gamma}}|\mathcal{U}_1| \geq \frac{\theta^\epsilon(x_t)}{r^{n+\gamma}}\left(2^{n-1}r^n - \frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}}{\theta^\epsilon(x_t)}\right).\end{aligned}$$

We now choose r as follows:

$$r = \left(\frac{1}{2^{n-2}}\frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}}{\theta^\epsilon(x_t)}\right)^{1/n}.$$

We assume $r \leq \pi$ for the moment. In this case, we obtain that

$$\frac{1}{c_{\gamma,n}}\Lambda^\gamma\theta^\epsilon(x_t) \geq \frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}}{\left(\frac{1}{2^{n-2}}\frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}}{\theta^\epsilon(x_t)}\right)^{1+\gamma/n}}.$$

This bound implies that

$$\frac{d}{dt}\left\|\theta^\epsilon(t)\right\|_{L^\infty(\mathbb{T}^n)} \leq -\nu c_{\gamma,n}\Lambda^\gamma\theta^\epsilon(x_t) + \epsilon\Delta\theta^\epsilon \leq -\nu c_{\gamma,n}\frac{\|\theta^\epsilon(t)\|_{L^\infty(\mathbb{T}^n)}^{1+\gamma/n}}{\|\theta_0\|_{L^1(\mathbb{T}^n)}^{\gamma/n}},$$

or equivalently,

$$\left\|\theta^\epsilon(t)\right\|_{L^\infty(T^n)} \leq \left(\frac{\nu c_{\gamma,n}}{\|\theta_0\|_{L^1(\mathbb{T}^n)}^{\gamma/n}}\gamma t + \frac{1}{\|\theta_0^\epsilon\|_{L^\infty}^\gamma}\right)^{-1/\gamma} \leq \frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}^{1/n}}{(C(\nu,\gamma,n))^{1/\gamma}}t^{-1/\gamma}.$$

If r is bigger than π,

$$\frac{1}{2^{n-2}\pi^n}\|\theta_0\|_{L^1(\mathbb{T}^n)} > \left\|\theta^\epsilon(t)\right\|_{L^\infty(\mathbb{T}^n)}.$$

As a consequence, for $t > 0$, we have

$$\left\|\theta^\epsilon(t)\right\|_{L^\infty(\mathbb{T}^n)} \leq \max\left\{\frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}^{1/n}}{(C(\nu,\gamma,n))^{1/\gamma}}t^{-1/\gamma}, \frac{1}{2^{n-2}\pi^n}\|\theta_0\|_{L^1(\mathbb{T}^n)}\right\}.$$

In particular, for $t \geq \tau$,

$$\left\|\theta^\epsilon(t)\right\|_{L^\infty(\mathbb{T}^n)} \leq \max\left\{\frac{\|\theta_0\|_{L^1(\mathbb{T}^n)}^{1/n}}{(\nu C_{\gamma,n}\gamma)^{1/\gamma}}\tau^{-1/\gamma}, \frac{1}{2^{n-2}\pi^n}\|\theta_0\|_{L^1(\mathbb{T}^n)}\right\} =: C(\tau,\nu,\gamma,n,\gamma). \quad (8.2)$$

Then, (θ^ϵ) is uniformly bounded in $\mathcal{B}_T$. Following the proof of Theorem 3.1, we obtain a weak solution θ in $\mathcal{B}_T$. Since

$$\theta_t^\epsilon = -u^\epsilon \cdot \nabla\theta^\epsilon - \nu\Lambda^\gamma\theta^\epsilon + \epsilon\Delta\theta^\epsilon \in L^2\big(0,T;H^{-2}\big(\mathbb{T}^n\big)\big)$$

uniformly in $\epsilon > 0$, we have

$$\theta^\epsilon \in C\big(0,T;H^{-2}(\mathbb{T})\big).$$

Therefore, we recover θ_0 in $H^{-2}(\mathbb{T}^n)$.

Acknowledgments

H.B. and R.G.-B. acknowledge Prof. H. Dong for his valuable comments that highly improve the manuscript. The authors also gratefully acknowledge the support by the Department of Mathematics at UC Davis where this research was performed. H.B. is supported by the 2014 Research Fund (Project Number 1.140076.01) of UNIST (Ulsan National Institute of Science and Technology). R.G.-B. is partially supported by the grant MTM2011-26696 from the former Ministerio de Ciencia e Innovación (MICINN, Spain).

Appendix A

A.1. Lyapunov functions

For Eq. (1.9), we have two additional Lyapunov functions defined in terms of

$$-\Lambda^{-1}\theta = \frac{1}{\pi}\int_{\mathbb{T}} \log\left|\sin\left(\frac{x-y}{2}\right)\right|\theta(y)dy. \tag{A.1}$$

When $\theta_0 \geq 0$, by the minimum principle, $-\Lambda^{-1}\theta \leq 0$. The first Lyapunov function is

$$\mathcal{L}_1(\theta) = \int_{\mathbb{T}} \big[\theta\big(\log\big|\Lambda^{-1}\theta\big|\big) + M\big]dx, \quad M = \|\theta_0\|_{L^\infty}\big\|\log\big|\Lambda^{-1}\theta_0\big|\big\|_{L^\infty}, \tag{A.2}$$

where we add M to make $\mathcal{L}_1(\theta)$ to be non-negative. We show that $\mathcal{L}_1(\theta)$ is decreasing in time using the fact $-\Lambda^{-1}f_x = \mathcal{H}f$:

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_1(\theta) &= \int_{\mathbb{T}} \big[\theta_t \log\big|\Lambda^{-1}\theta\big|\big]dx + \int_{\mathbb{T}} \big[\theta\big(\log\big|\Lambda^{-1}\theta\big|\big)_t\big]dx \\ &= \int_{\mathbb{T}} \big[(\theta\mathcal{H}\theta)\big(\log\big|\Lambda^{-1}\theta\big|\big)_x\big]dx + \int_{\mathbb{T}} \left[\frac{\theta\Lambda^{-1}\theta_t}{\Lambda^{-1}\theta}\right]dx = \int_{\mathbb{T}} \left[\frac{-\theta(\mathcal{H}\theta)^2}{\Lambda^{-1}\theta} + \frac{\theta\mathcal{H}(\theta\mathcal{H}\theta)}{\Lambda^{-1}\theta}\right]dx\end{aligned}$$

$$= -\frac{1}{2} \int_{\mathbb{T}} \left[\frac{\theta}{\Lambda^{-1}\theta} \big((\mathcal{H}\theta)^2 + (\theta)^2 + \langle\theta\rangle^2 \big) \right] dx \leq 0.$$

The second Lyapunov function is

$$\mathcal{L}_2(\theta) = \int_{\mathbb{T}} \big[\theta e^{\Lambda^{-1}\theta} \big] dx. \tag{A.3}$$

We show that $\mathcal{L}_2(\theta)$ is exponentially decreasing in time:

$$\begin{aligned} \frac{d}{dt}\mathcal{L}_2(\theta) &= \int_{\mathbb{T}} \big[\theta_t e^{\Lambda^{-1}\theta} \big] dx + \int_{\mathbb{T}} \big[\theta \big(e^{\Lambda^{-1}\theta} \big)_t \big] dx \\ &= \int_{\mathbb{T}} \big[(\theta\mathcal{H}\theta) \big(e^{\Lambda^{-1}\theta} \big)_x \big] dx + \int_{\mathbb{T}} \big[\theta e^{\Lambda^{-1}\theta} \mathcal{H}(\theta\mathcal{H}\theta) \big] dx \\ &= -\int_{\mathbb{T}} \big[\theta(\mathcal{H}\theta)^2 e^{\Lambda^{-1}\theta} \big] dx + \frac{1}{2} \int_{\mathbb{T}} \big[\theta e^{\Lambda^{-1}\theta} \big((\mathcal{H}\theta)^2 - \theta^2 - \langle\theta\rangle^2 \big) \big] dx \\ &= -\frac{1}{2} \int_{\mathbb{T}} \big[\theta e^{\Lambda^{-1}\theta} \big((\mathcal{H}\theta)^2 + \theta^2 \big) \big] dx - \frac{\langle\theta_0\rangle^2}{2} \mathcal{L}, \end{aligned}$$

where we use $\langle\theta\rangle = \langle\theta_0\rangle$. Therefore,

$$\mathcal{L}_2(\theta) \leq e^{-\frac{\langle\theta_0\rangle^2}{2} t} \int_{\mathbb{T}} \big[\theta_0 e^{\Lambda^{-1}\theta_0} \big] dx. \tag{A.4}$$

We note that the bound of $\mathcal{L}_2$ implies the bound of θ in $\dot{H}^{-1/2}(\mathbb{T})$:

$$\big\| \theta(t) \big\|_{\dot{H}^{-1/2}(\mathbb{T})} \leq \int_{\mathbb{T}} \theta \big(1 + \Lambda^{-1}\theta \big) dx \leq \mathcal{L}_2(\theta) < C(\theta_0).$$

References

[1] G.R. Baker, X. Li, A.C. Morlet, Analytic structure of 1D transport equations with nonlocal fluxes, Phys. D 91 (1996) 349–375.

[2] P. Balodis, A. Cordoba, An inequality for Riesz transforms implying blow-up for some nonlinear and nonlocal transport equations, Adv. Math. 214 (1) (2007) 1–39.

[3] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1) (1984) 61–66.

[4] L.C. Berselli, Vanishing viscosity limit and long-time behavior for 2D quasi-geostrophic equations, Indiana Univ. Math. J. 51 (4) (2002) 905–930.

[5] L.C. Berselli, D. Córdoba, R. Granero-Belinchón, Local solvability and turning for the inhomogeneous Muskat problem, Interfaces Free Bound. 16 (2) (2014) 175–213.

[6] A. Bertozzi, A. Majda, Vorticity and Incompressible Flow, Cambridge Univ. Press, 2002.

[7] A. Blanchet, E.A. Carlen, J.A. Carrillo, Functional inequalities, thick tails and asymptotics for the critical mass Patlak–Keller–Segel model, J. Funct. Anal. 262 (5) (2012) 2142–2230.

[8] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller–Segel model: optimal critical mass and qualitative properties of the solutions, Electron. J. Differential Equations 2006 (44) (2006) 1–32.
[9] L. Caffarelli, J.L. Vázquez, Nonlinear porous medium flow with fractional potential pressure, Arch. Ration. Mech. Anal. 202 (2011) 537–565.
[10] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. (2) 171 (3) (2010) 1903–1930.
[11] J.A. Carrillo, L.C.F. Ferreira, J.C. Precioso, A mass-transportation approach to a one dimensional fluid mechanics model with nonlocal velocity, Adv. Math. 231 (1) (2012) 306–327.
[12] J.A. Carrillo, S. Lisini, E. Mainini, Uniqueness for Keller–Segel-type chemotaxis models, Discrete Contin. Dyn. Syst. 34 (4) (2014) 1319–1338.
[13] A. Castro, D. Córdoba, Global existence, singularities and ill-posedness for a nonlocal flux, Adv. Math. 219 (6) (2008) 1916–1936.
[14] A. Castro, D. Córdoba, Self-similar solutions for a transport equation with non-local flux, Chin. Ann. Math. Ser. B 30 (5) (2009) 505–512.
[15] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, Breakdown of smoothness for the Muskat problem, Arch. Ration. Mech. Anal. 208 (3) (2013) 805–909.
[16] A. Castro, D. Córdoba, C. Fefferman, F. Gancedo, M. Lopez-Fernandez, Rayleigh–Taylor breakdown for the Muskat problem with applications to water waves, Ann. of Math. (2) 175 (2) (2012) 909–948.
[17] A. Castro, D. Córdoba, F. Gancedo, R. Orive, Incompressible flow in porous media with fractional diffusion, Nonlinearity 22 (8) (2009) 1791–1815.
[18] D. Chae, On the transport equations with singular/regular nonlocal velocities, SIAM J. Math. Anal. 46 (2) (2014) 1017–1029.
[19] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, J. Wu, Generalized surface quasi-geostrophic equations with singular velocities, Comm. Pure Appl. Math. 65 (8) (2012) 1037–1066.
[20] D. Chae, A. Cordoba, D. Cordoba, M. Fontelos, Finite time singularities in a 1D model of the quasi-geostrophic equation, Adv. Math. 194 (1) (2005) 203–223.
[21] P. Constantin, D. Córdoba, F. Gancedo, R. Strain, On the global existence for the Muskat problem, J. Eur. Math. Soc. (JEMS) 15 (1) (2013) 201–227.
[22] P. Constantin, P. Lax, A. Majda, A simple one-dimensional model for the three dimensional vorticity, Comm. Pure Appl. Math. 38 (1985) 715–724.
[23] P. Constantin, A. Majda, E. Tabak, Formation of strong fronts in the 2D quasi-geostrophic thermal active scalar, Nonlinearity 7 (1994) 1495–1533.
[24] A. Córdoba, D. Córdoba, A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (3) (2004) 511–528.
[25] A. Córdoba, D. Córdoba, M. Fontelos, Formation of singularities for a transport equation with nonlocal velocity, Ann. of Math. (2) 162 (2005) 1–13.
[26] D. Córdoba, F. Gancedo, Contour dynamics of incompressible 3-D fluids in a porous medium with different densities, Comm. Math. Phys. 273 (2) (2007) 445–471.
[27] D. Córdoba, R. Granero-Belinchón, R. Orive, On the confined Muskat problem: differences with the deep water regime, Commun. Math. Sci. 12 (3) (2014) 423–455.
[28] S. De Gregorio, On a one-dimensional model for the 3D vorticity equation, J. Stat. Phys. 59 (1990) 1251–1263.
[29] J. Deslippe, R. Tedstrom, M.S. Daw, D. Chrzan, T. Neeraj, M. Mills, Dynamics scaling in a simple one-dimensional model of dislocation activity, Philos. Mag. 84 (2004) 244–2454.
[30] H. Dong, Well-posedness for a transport equation with nonlocal velocity, J. Funct. Anal. 255 (11) (2008) 3070–3097.
[31] S. Friedlander, F. Gancedo, W. Sun, V. Vicol, On a singular incompressible porous media equation, J. Math. Phys. 53 (11) (2012) 115602, 20 pp.
[32] S. Friedlander, W. Rusin, V. Vicol, The magneto-geostrophic equations: a survey, in: Proceedings of Advances in Mathematical Analysis of PDE, in honor of Olga Ladyzhenskaya, in: Amer. Math. Soc. Trasl., vol. 232, 2014.
[33] S. Friedlander, V. Vicol, Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2) (2011) 283–301.
[34] F. Gancedo, Existence for the α-patch model and the QG sharp front in Sobolev spaces, Adv. Math. 217 (6) (2008) 2569–2598.
[35] J. Gómez-Serrano, R. Granero-Belinchón, On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof, Nonlinearity 27 (6) (2014) 1471–1498.
[36] R. Granero-Belinchón, Global existence for the confined Muskat problem, SIAM J. Math. Anal. 46 (2) (2014) 1651–1680.

[37] A.K. Head, Dislocation group dynamics I. Similarity solutions of the n-body problem, Philos. Mag. 26 (1977) 43–53.
[38] A.K. Head, Dislocation group dynamics II. General solutions of the n-body problem, Philos. Mag. 26 (1977) 55–63.
[39] A.K. Head, Dislocation group dynamics III. Similarity solutions of the continuum approximation, Philos. Mag. 26 (1977) 65–72.
[40] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. 167 (3) (2007) 445–453.
[41] A. Kiselev, Regularity and blow up for active scalars, Math. Model. Nat. Phenom. 5 (2010) 225–255.
[42] O. Lazar, Global existence for the critical dissipative surface quasi-geostrophic equation, Comm. Math. Phys. 322 (1) (2013) 73–93.
[43] O. Lazar, Global and local existence for the dissipative critical SQG equation with small oscillations, arXiv preprint, arXiv:1308.0851 [math.AP].
[44] D. Li, J. Rodrigo, Blow up for the generalized surface quasi-geostrophic equation with supercritical dissipation, Comm. Math. Phys. 286 (1) (2009) 111–124.
[45] D. Li, J. Rodrigo, Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation, Adv. Math. 217 (6) (2008) 2563–2568.
[46] D. Li, J. Rodrigo, On a one-dimensional nonlocal flux with fractional dissipation, SIAM J. Math. Anal. 43 (1) (2011) 507–526.
[47] P.L. Lions, Mathematical Topics in Fluid Dynamics, vol. 2. Compressible Models, Oxford Science Publication, Oxford, 1998.
[48] F. Marchand, Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces L^p or $\dot{H}^{1/2}$, Comm. Math. Phys. 277 (2008) 45–67.
[49] H.K. Moffatt, D.E. Loper, The magnetostrophic rise of a buoyant parcel in the Earth's core, Geophys. J. Int. 117 (2) (1994) 394–402.
[50] A. Morlet, Further properties of a continuum of model equations with globally defined flux, J. Math. Anal. Appl. 221 (1998) 132–160.
[51] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, 1986.
[52] J. Rodrigo, On the evolution of sharp fronts for the quasi-geostrophic equation, Comm. Pure Appl. Math. 58 (6) (2005) 821–866.
[53] R. Temam, Navier–Stokes Equations: Theory and Numerical Analysis, AMS Chelsea Publishing, Providence, 2001.