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Weil-Petersson Teichmüller space II: Smoothness of flow curves of $H^{\frac{3}{2}}$-vector fields ☆

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A B S T R A C T

Given a continuous vector field $\lambda(t,\cdot)$ of Sobolev class $H^{\frac{3}{2}}$ on the unit circle S^1, the flow maps $\eta = g(t,\cdot)$ of the differential equation

$$\begin{cases} \frac{d\eta}{dt} = \lambda(t,\eta) \\ \eta(0,\zeta) = \zeta \end{cases}$$

are known to be quasisymmetric homeomorphisms. Very recently, Gay-Balmaz-Ratiu [15] conjectured that the flow curve $g(t,\cdot)$ is in the Weil-Petersson class $\mathrm{WP}(S^1)$ and is continuously differentiable with respect to the Hilbert manifold structure of $\mathrm{WP}(S^1)$ introduced by Takhtajan-Teo [40]. The first assertion had already been demonstrated in our previous paper [36]. In this sequel to [36], we will continue to deal with the Weil-Petersson class $\mathrm{WP}(S^1)$ and completely solve this conjecture in the affirmative.

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1. Introduction

This is a continuous work of our previous paper [36], where we presented some recent results on the Weil-Petersson geometry theory of the universal Teichmüller space, a topic which is important in Teichmüller theory and has wide applications to various areas such as mathematical physics (see [4]-[5], [18], [19], [27]-[28]), differential equation and computer vision (see [14], [15], [20]).

A sense-preserving homeomorphism g of the unit circle S^1 onto itself is said to belong to the Weil-Petersson class, which is denoted by $\mathrm{WP}(S^1)$, if it has a quasiconformal extension to the unit disk Δ whose Beltrami coefficient ν is square integrable in the Poincaré metric, namely,

$$\iint_{\Delta} |\nu(z)|^2(1-|z|^2)^{-2}dxdy < +\infty. \tag{1.1}$$

In an important paper [40], Takhtajan-Teo showed how to endow $\mathrm{WP}(S^1)$ and its two close relatives, the Weil-Petersson Teichmüller space $T_0 = \mathrm{WP}(S^1)/\mathrm{Möb}(S^1)$ and the Weil-Petersson Teichmüller curve $\hat{T}_0 = \mathrm{WP}(S^1)/\operatorname{Rot}(S^1)$, with Hilbert manifold structures (see also [15], [36]). Here, $\mathrm{Möb}(S^1)$ denotes the group of all Möbius transformations keeping the unit disk Δ fixed, while $\mathrm{Rot}(S^1)$ denotes the sub-group of all rotations about the circle S^1. In [36], we gave the following intrinsic characterization of a quasisymmetric homeomorphism in the Weil-Petersson class $\mathrm{WP}(S^1)$ without using quasiconformal extensions, which solves a problem proposed by Takhtajan-Teo in 2006. Recall that, for a function f defined on a subset Γ of the complex plane, f' denotes the derivative of f, namely, for $z \in \Gamma$,

$$f'(z) \doteq \lim_{\Gamma \ni \zeta \to z} \frac{f(\zeta)-f(z)}{\zeta - z}$$

provided the limit exists, while $f'(z) \doteq 0$ otherwise.

Theorem 1.1. *([36]) A sense-preserving homeomorphism g on the unit circle S^1 belongs to the Weil-Petersson class $\mathrm{WP}(S^1)$ if and only if g is absolutely continuous (with respect to the arc-length measure) such that $\log g'$ belongs to the Sobolev class $H^{\frac{1}{2}}$ on the unit circle. Moreover, the correspondence $g \mapsto \log|g'|$ induces a homeomorphism from the Weil-Petersson Teichmüller curve $\hat{T}_0$ onto the real Sobolev space $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$.*

It should be pointed out that the second assertion of Theorem 1.1 was stated in a different but an equivalent way in [36] (see Theorem 8.1 in [36]). The definition of Sobolev class will be given in the next section. In this paper, we will continue to deal with the Weil-Petersson class $\mathrm{WP}(S^1)$. Recall that $\mathrm{WP}(S^1)$ is modeled on the Sobolev space $H^{\frac{3}{2}}$, namely, the tangent space to $\mathrm{WP}(S^1)$ at the identity consists of precisely the $H^{\frac{3}{2}}$ vector fields λ on the unit circle (see [25], [40] and also [17]). We will be mainly concerned with the flows of $H^{\frac{3}{2}}$ vector fields on the unit circle. It is known that the Weil-Petersson class $\mathrm{WP}(S^1)$ can be generated by the flows of the $H^{\frac{3}{2}}$ vector fields on the unit circle (see [9], [15]). Here we consider the converse problem and prove the following result, completely solving a conjecture posed by Gay-Balmaz-Ratiu in the recent paper [15] (see page 760 in [15] and also [9]).

Theorem 1.2. *Let $\lambda(t,\cdot) \in C^0([0,M], H^{\frac{3}{2}})$ be a continuous vector field of Sobolev class $H^{\frac{3}{2}}$ on the unit circle S^1. Then the flow curve $\eta = g(t,\cdot)$ of the differential equation*

$$\begin{cases} \frac{d\eta}{dt} = \lambda(t,\eta) \\ \eta(0,\zeta) = \zeta \end{cases} \tag{1.2}$$

is in the Weil-Petersson class $\mathrm{WP}(S^1)$ and is continuously differentiable with respect to the Hilbert manifold structure of $\mathrm{WP}(S^1)$ such that

$$\frac{d}{dt} g(t,\cdot) = \lambda(t, g(t,\cdot)). \tag{1.3}$$

Recall that the first assertion in Theorem 1.2 was already proved by the author in [36]. In fact, we have proved

Theorem 1.3. *([36]) Under the assumption of Theorem 1.2, the flow curve $g(t,\cdot)$ of the differential equation (1.2) satisfies $\log g'(t,\cdot) \in H^{\frac{1}{2}}$, which implies by Theorem 1.1 that the flow curve $\eta = g(t,\cdot)$ is in the Weil-Petersson class $\mathrm{WP}(S^1)$, and the mapping $t \mapsto \log g'(t,\cdot)$ from $[0,M]$ into $H^{\frac{1}{2}}$ is continuously differentiable such that*

$$\frac{d}{dt} \log g'(t,\cdot) = \lambda'(t, g(t,\cdot)). \tag{1.4}$$

Theorem 1.2 has several important consequences on the regularity of the Weil-Petersson class $\mathrm{WP}(S^1)$ and on the flows of the vector fields of Sobolev class $H^{\frac{3}{2}}$ on the unit circle S^1 (see [15] and [36] for more details). It is also assumed to be useful to the further study of the geometry of the Weil-Petersson Teichmüller space T_0. We hope to pursue this in a separated paper.

An open problem (see page 68 in [40]) is to give a geometric characterization of a Weil-Petersson quasi-circle, the image of the unit circle S^1 under a quasiconformal mapping which is conformal outside the unit disk Δ and has Beltrami coefficient in Δ satisfying (1.1). A partial answer to this problem was obtained by Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä [10]. A Weil-Petersson quasi-line is defined in the same way, namely, it is the image of the real line $\mathbb{R}$ under a quasiconformal mapping which is conformal on the lower plane $\mathbb{U}^*$ and has Beltrami coefficient in the upper half plane $\mathbb{U}$ being square integrable in the Poincaré metric, that is, satisfying (2.1) below. In a forth-coming paper [39], we will endow the set of all Weil-Petersson quasi-lines (with certain normalized condition) with a real Hilbert manifold structure from a geometric point of view and show that new manifold structure is topologically equivalent to the standard complex Hilbert manifold structure given by Takhtajan-Teo [40]. Theorem 2.3 in the next section will play an essential role in that work.

2. Weil-Petersson Teichmüller space on the real line

In this section, we give some basic definitions and results on the Weil-Petersson Teichmüller space (see [36] and [40] for more details). As will be seen later, it is convenient to define the Weil-Petersson Teichmüller space and prove Theorem 1.2 in the setting of the real line $\mathbb{R}$ instead of the unit circle S^1. Actually, as stated at the end of the first section, the results in the real line case, e.g. Theorems 2.2 and 2.3 below, turn out to be very useful to the study of geometric characterizations of Weil-Petersson quasi-lines (see [39]).

Let $M(\mathbb{U})$ denote the open unit ball of the Banach space $L^\infty(\mathbb{U})$ of essentially bounded measurable functions on the upper half plane $\mathbb{U}$ in the complex plane $\mathbb{C}$. For $\mu \in M(\mathbb{U})$, let f_μ be the unique quasiconformal mapping on $\mathbb{U}$ onto itself which has complex dilatation μ and keeps the points 0, 1 and ∞ fixed. We say two elements μ and ν in $M(\mathbb{U})$ are equivalent, denoted by $\mu \sim \nu$, if $f_\mu = f_\nu$ on the real line $\mathbb{R}$. Then $T = M(\mathbb{U})/_\sim$ is known as the Bers model of the universal Teichmüller space. We let Φ denote the natural projection from $M(\mathbb{U})$ onto T so that $\Phi(\mu)$ is the equivalence class $[\mu]$. $[0]$ is called the base point of T. It is known that T has a unique complex Banach manifold structure such that the natural projection Φ from $M(\mathbb{U})$ onto T is a holomorphic split submersion (see [11], [21], [22]).

We denote by $\mathcal{L}(\mathbb{U})$ the Banach space of all measurable functions μ with norm

$$\|\mu\|_{\mathrm{WP}} \doteq \|\mu\|_\infty + \left(\frac{1}{\pi}\iint\limits_{\mathbb{U}} \frac{|\mu(z)|^2}{y^2}dxdy\right)^{\frac{1}{2}}, \quad z = x+iy. \tag{2.1}$$

Set $\mathcal{M}(\mathbb{U}) = M(\mathbb{U}) \cap \mathcal{L}(\mathbb{U})$. Then $T_0 = \mathcal{M}(\mathbb{U})/_\sim$ is the complex model of the Weil-Petersson Teichmüller space. It is known that T_0 has a unique complex Hilbert manifold structure such that the natural projection Φ from $\mathcal{M}(\mathbb{U})$ onto T_0 is a holomorphic split submersion (see [40] and also [36]).

It is well known that a quasiconformal self-mapping of $\mathbb{U}$ induces a bi-holomorphic automorphism of the universal Teichmüller space (see [11], [21], [22]). Precisely, let $w : \mathbb{U} \to \mathbb{U}$ be a quasiconformal mapping with complex dilatation μ. Then w induces an bi-holomorphic isomorphism $R_w : M(\mathbb{U}) \to M(\mathbb{U})$ as

$$R_w(\nu) = \left(\frac{\nu-\mu}{1-\bar{\mu}\nu}\frac{\partial w}{\overline{\partial w}}\right) \circ w^{-1}. \tag{2.2}$$

R_w descends down a bi-holomorphic isomorphism $w^* : T \to T$ by $w^* \circ \Phi = \Phi \circ R_w$. w^* is usually called an allowable mapping. When w is quasi-isometric under the Poincaré metric $|dz|/y$ with Beltrami coefficient $\mu \in \mathcal{M}(\mathbb{U})$, R_w maps $\mathcal{M}(\mathbb{U})$ into itself and $w^* : T_0 \to T_0$ is bi-holomorphic.

We denote by $\mathrm{WP}(\mathbb{R})$ the Weil-Petersson class of all increasing homeomorphisms h of $\mathbb{R}$ onto itself which have quasiconformal extensions w to the upper half plane $\mathbb{U}$ whose Beltrami coefficients μ belong to the class $\mathcal{M}(\mathbb{U})$. We also denote by $\mathrm{WP}_0(\mathbb{R})$ the sub-class of $\mathrm{WP}(\mathbb{R})$ of all mappings h with the normalized condition $h(0) = 0$, $h(1) = 1$. Then the correspondence $[\mu] \mapsto f_\mu|_{\mathbb{R}}$ induces a one-to-one map I from T_0 onto the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$, which endows $\mathrm{WP}_0(\mathbb{R})$ with a complex Hilbert manifold structure (under which I is a bi-holomorphic isomorphism).

Recall that the Sobolev class $H^{\frac{1}{2}}$ ($H^{\frac{1}{2}}_{\mathbb{R}}$) on the unit circle S^1 or the real line $\mathbb{R}$ is the set of all locally integrable (real-valued) functions φ with

$$\|\varphi\|^2_{H^{\frac{1}{2}}} = \frac{1}{4\pi^2} \int\limits_{\mathbb{S}} \int\limits_{\mathbb{S}} \frac{|\varphi(u) - \varphi(\tilde{u})|^2}{|u - \tilde{u}|^2} |du||d\tilde{u}| < +\infty, \tag{2.3}$$

where $\mathbb{S}$ denotes the unit circle S^1 or the real line $\mathbb{R}$. We denote by $H^{\frac{3}{2}}$ ($H^{\frac{3}{2}}_{\mathbb{R}}$) the class of all (real-valued) functions φ on the unit circle S^1 or the real line $\mathbb{R}$ which are locally absolutely continuous such that $\varphi' \in H^{\frac{1}{2}}$. As will be seen in section 8 (Theorem 8.1 below), the tangent space to $\mathrm{WP}_0(\mathbb{R})$ at the identity consists of precisely the $H^{\frac{3}{2}}$ real-valued vector fields on the real line vanishing at the points 0 and 1.

We have the following result on the real line parallel to Theorem 1.2.

Theorem 2.1. *Let $\omega(t, \cdot) \in C^0([0, M], H^{\frac{3}{2}}_{\mathbb{R}})$ be a continuous real-valued vector field on the real line $\mathbb{R}$ with the normalized condition $\omega(t, 0) = \omega(t, 1) = 0$. Then the flow curve $u = h(t, \cdot)$ of the differential equation*

$$\begin{cases} \frac{du}{dt} = \omega(t, u) \\ u(0, x) = x \end{cases} \tag{2.4}$$

is in the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$ and is continuously differentiable with respect to the Hilbert manifold structure of $\mathrm{WP}_0(\mathbb{R})$ such that

$$\frac{d}{dt} h(t, \cdot) = \omega(t, h(t, \cdot)). \tag{2.5}$$

The following result plays an essential role in the proof of Theorem 2.1.

Theorem 2.2. *Let h be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\log h'$ belongs to the Sobolev class $H^{\frac{1}{2}}$. Then h belongs to the Weil-Petersson class $\mathrm{WP}(\mathbb{R})$. Moreover, the correspondence*

$$u \mapsto h_u : \quad h_u(x) = \frac{\int_0^x e^{u(t)} dt}{\int_0^1 e^{u(t)} dt}, \quad x \in \mathbb{R} \tag{2.6}$$

induces a real analytic map Ψ *from the real Sobolev space* $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ *into the normalized Weil-Petersson class* $\mathrm{WP}_0(\mathbb{R})(= I(T_0))$.

Since the logarithmic derivative is not invariant under a Möbius transformation, (the first assertion of) Theorem 2.2 can not be deduced from Theorem 1.1 directly. We will prove Theorem 2.2 by means of a construction due to Semmes (see [33]-[34]), which is largely different from the approach in our previous paper [36]. Theorem 2.2 only gives a sufficient condition for an increasing homeomorphism on the real line being in the Weil-Petersson class $\mathrm{WP}(\mathbb{R})$. We have shown in a separated paper (see [38]) that this sufficient condition is also a necessary one. Consequently, Ψ is a one-to-one analytic map from the real Sobolev space $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ onto the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$. We will show that the inverse map Ψ^{-1} is also real analytic.

Theorem 2.3. Ψ *is a one-to-one analytic map from the real Sobolev space* $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ *onto the normalized Weil-Petersson class* $\mathrm{WP}_0(\mathbb{R})$ *whose inverse* Ψ^{-1} *is also real analytic.*

Remark 2.1. Here is an appropriate place to point out why we first prove our main results in the real line case and then come back to the unit circle case. As will be seen later, the main effort of the paper is to prove the real analyticity of the map sending an $H^{\frac{1}{2}}$ function to a Weil-Petersson homeomorphism. The proof is based on an important instruction due to Semmes [34], which is available on the real line but not on the unit circle. On the other hand, we have a program to study the Weil-Petersson Teichmüller space from several points of view. In the forth-coming work [39], we will study how the Riemann mapping depends on a Weil-Petersson quasi-line. In [39] we need the Weil-Petersson theory on the real line parallel to the unit circle case, which will be carried out in the present paper. Theorem 2.3 implies that the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$, the real model of the Weil-Petersson Teichmüller space $\mathcal{T}$, can be endowed with a real Hilbert manifold structure from $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ by the correspondence $h \mapsto \log h'$, which is real analytically equivalent to the standard complex Hilbert manifold structure on $\mathcal{T}$ given by Takhtajan-Teo [40]. This fact will play an important role in the sequel [39].

3. BMO functions

In order to prove Theorem 2.2, we need a construction concerning quasiconformal extensions of strongly quasisymmetric homeomorphisms introduced by Semmes [33]-[34], which relies heavily on BMO estimates (see section 4 below). In this section we recall some basic definitions and results on BMO functions (see [13]).

A locally integrable function $u \in L^1_{loc}(\mathbb{R})$ is said to have bounded mean oscillation and belongs to the space BMO if

$$\|u\|_{\mathrm{BMO}} \doteq \sup \frac{1}{|I|} \int\limits_I |u(t) - u_I| dt < +\infty, \tag{3.1}$$

where the supremum is taken over all finite sub-intervals I of $\mathbb{R}$, while u_I is the average of u on the interval I, namely,

$$u_I = \frac{1}{|I|} \int_I u(t)dt. \tag{3.2}$$

If u also satisfies the condition

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |u(t) - u_I| dt = 0,$$

we say u has vanishing mean oscillation and belongs to the space VMO. We can define BMO functions and VMO functions on the unit circle in a similar way. It is well known that $H^{\frac{1}{2}} \subset \mathrm{VMO}$, and the inclusion map is continuous (see [42]). In the following, we denote by $\mathrm{BMO}_{\mathbb{R}}$ the set of all real-valued BMO functions.

We need some basic results on BMO functions. For simplicity, we fix some notations. $C, C_1, C_2 \cdots$ will denote universal constants that might change from one line to another, while $C(\cdot), C_1(\cdot), C_2(\cdot) \cdots$ will denote constants that depend only on the elements put in the brackets. The notation $A \asymp B$ means that there is a positive constant C independent of A and B such that $A/C \le B \le CA$. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a positive constant C independent of A and B such that $A \le CB$ ($A \ge CB$). By the well-known theorem of John-Nirenberg for BMO functions (see [13]), there exist two universal positive constants C_1 and C_2 such that for any BMO function u, any subinterval I of $\mathbb{R}$ and any $\lambda > 0$, it holds that

$$\frac{|\{t \in I : |u(t) - u_I| \ge \lambda\}|}{|I|} \le C_1 \exp\left(\frac{-C_2\lambda}{\|u\|_{\mathrm{BMO}}}\right). \tag{3.3}$$

By Chebychev's inequality, we obtain that for u with $\|u\|_{\mathrm{BMO}} < C_2$,

$$\begin{aligned} \frac{1}{|I|} \int_I (e^{|u-u_I|} - 1)dt &= \frac{1}{|I|} \int_0^\infty |\{t \in I : |u - u_I| \ge \lambda\}| \, d(e^\lambda - 1) \\ &\le C_1 \int_0^\infty e^\lambda \exp\left(\frac{-C_2\lambda}{\|u\|_{\mathrm{BMO}}}\right) d\lambda \\ &\le \frac{C_1\|u\|_{\mathrm{BMO}}}{C_2 - \|u\|_{\mathrm{BMO}}}. \end{aligned} \tag{3.4}$$

Similarly, for any $p \ge 1$ we have

$$\frac{1}{|I|} \int_I |u - u_I|^p dt \lesssim C(p)\|u\|^p_{\mathrm{BMO}}. \tag{3.5}$$

Lemma 3.1. *Let ϕ be a C^∞ function on the real line which is supported on $[-1,1]$ and satisfies $\int_{\mathbb{R}} \phi(x)dx = 1$. Set $\phi_y(x) = y^{-1}\phi(y^{-1}x)$ for $y > 0$, and*

$$\phi_y * v(x) = \int_{\mathbb{R}} \phi_y(x-t)v(t)dt. \tag{3.6}$$

Then it holds that

$$|\phi_y * e^u| \asymp |e^{\phi_y * u}| \tag{3.7}$$

when $\|u\|_{\mathrm{BMO}}$ is small.

Proof. Lemma 3.1 appears in [34]. For completeness and for convenience of later use, we write down the detailed proof here. Actually, besides Lemma 3.1 itself, the following inequalities (3.8) and (3.9) will also be used in the proof of Theorem 2.2.

For $x \in \mathbb{R}$ and $y > 0$, consider $I = [x-y, x+y]$ so that

$$u_I = \frac{1}{2y}\int_{x-y}^{x+y} u(t)dt.$$

Since $\int_{\mathbb{R}} \phi(x)dx = 1$, which implies that $\int_{\mathbb{R}} \phi_y(x)dx = 1$, we obtain

$$|\phi_y * u(x) - u_I| = |\phi_y * (u - u_I)(x)| \leq C(\phi)\frac{1}{|I|}\int_I |u(t) - u_I|dt \lesssim \|u\|_{\mathrm{BMO}}. \tag{3.8}$$

Since $|e^z - 1| \leq |ze^z| \leq |z|e^{|z|}$, we have

$$\begin{aligned}\frac{1}{|I|}\int_I |e^{u(t)-\phi_y*u(x)} - 1|dt &\leq \frac{1}{|I|}\int_I |e^{u(t)-\phi_y*u(x)}||u(t) - \phi_y * u(x)|dt \\ &\leq \frac{|e^{u_I - \phi_y*u(x)}|}{|I|}\int_I |e^{u(t)-u_I}|(|u(t) - u_I| + |u_I - \phi_y * u(x)|dt.\end{aligned}$$

Using Hölder inequality, we conclude from (3.4), (3.5) and (3.8) that

$$\frac{1}{|I|}\int_I |e^{u(t)-\phi_y*u(x)} - 1|dt \lesssim \|u\|_{\mathrm{BMO}} \tag{3.9}$$

when $\|u\|_{\mathrm{BMO}}$ is small. Noting that

$$\phi_y * e^u(x) - e^{\phi_y*u(x)} = e^{\phi_y*u(x)}\phi_y * (e^{u-\phi_y*u(x)} - 1)(x),$$

we obtain

$$
\begin{aligned}
|\phi_y * e^u(x) - e^{\phi_y * u(x)}| &= |e^{\phi_y * u(x)}||\phi_y * (e^{u-\phi_y * u(x)} - 1)(x)| \\
&\lesssim \frac{|e^{\phi_y * u(x)}|}{|I|} \int_I |e^{u(t)-\phi_y * u(x)} - 1| dt,
\end{aligned}
$$

which implies by (3.9) the required relation (3.7). □

4. Semmes' construction revisited

We begin this section with a basic result of Coifman-Meyer [6]. For $u \in \mathrm{BMO}$ on the real line, set

$$
\gamma_u(x) = \frac{\int_0^x e^{u(t)} dt}{\int_0^1 e^{u(t)} dt}, \quad x \in \mathbb{R}. \tag{4.1}
$$

Coifman-Meyer [6] showed that γ_u is a strongly quasisymmetric homeomorphism from the real line $\mathbb{R}$ onto a chord-arc curve $\Gamma_u = \gamma_u(\mathbb{R})$ when $\|u\|_{\mathrm{BMO}}$ is small. If, in addition, u is real-valued, then γ_u is a strongly quasisymmetric homeomorphism of $\mathbb{R}$ onto itself. Recall that a sense preserving homeomorphism h on $\mathbb{R}$ is strongly quasisymmetric if it is locally absolutely continuous so that $|h'|$ belongs to the class of weights A^∞ introduced by Muckenhoupt (see [13]) and it maps $\mathbb{R}$ onto a chord-arc curve passing through the point at infinity (see [34]).

In an important paper [34], Semmes showed that, when $\|u\|_{\mathrm{BMO}}$ is small, γ_u can be extended to a quasiconformal mapping to the whole plane whose Beltrami coefficient satisfies certain Carleson measure condition. To be precise, let φ and ψ be two C^∞ real-valued function on the real line supported on $[-1, 1]$ such that φ is even, ψ is odd and $\int_{\mathbb{R}} \varphi(x)dx = 1$, $\int_{\mathbb{R}} \psi(x)xdx = 1$. Define

$$
\rho(x, y) = \rho_u(x, y) = \varphi_y * \gamma_u(x) - i\psi_y * \gamma_u(x), \quad z = x + iy \in \mathbb{U}. \tag{4.2}
$$

Semmes proved that ρ is a quasiconformal mapping from the upper half plane $\mathbb{U}$ onto the left domain bounded by Γ_u when $\|u\|_{\mathrm{BMO}}$ is small. Furthermore, when u is real-valued, ρ is a quasiconformal mapping of $\mathbb{U}$ onto itself and is quasi-isometric under the Poincaré metric $|dz|/y$. In fact, there exist two C^∞ functions α and β on the real line which are supported on $[-1, 1]$ and satisfy $\int_{\mathbb{R}} \alpha(x)dx = 0$, $\int_{\mathbb{R}} \beta(x)dx = 1$ such that

$$
\bar\partial\rho(z) = \alpha_y * e^u(x), \, \partial\rho(z) = \beta_y * e^u(x), \quad z = x + iy \in \mathbb{U}. \tag{4.3}
$$

It follows from Lemma 3.1 that the Beltrami coefficient μ of ρ satisfies

$$
\begin{aligned}
|\mu(z)| = \frac{|\bar{\partial}\rho(z)|}{|\partial\rho(z)|} &= \frac{|\alpha_y * e^u(x)|}{|\beta_y * e^u(x)|} \\
&\asymp \frac{|\alpha_y * e^u(x)|}{|e^{\beta_y * u(x)}|} \\
&= |\alpha_y * e^{u-\beta_y * u(x)}(x)| \\
&= |\alpha_y * (e^{u-\beta_y * u(x)} - 1)(x)| \\
&\lesssim \frac{1}{2y}\int_{x-y}^{x+y} |e^{u(t)-\beta_y * u(x)} - 1)|dt \\
&\lesssim \|u\|_{\mathrm{BMO}}
\end{aligned}
\tag{4.4}
$$

if $\|u\|_{\mathrm{BMO}}$ is small, by (3.9).

5. Proof of Theorem 2.2 (first part)

We first prove the following result.

Lemma 5.1. *There exists some universal constant $\delta > 0$ such that, for any $u \in H^{\frac{1}{2}}$ with $\|u\|_{H^{\frac{1}{2}}} < \delta$, the mapping $\rho = \rho_u$ defined by (4.2) is quasiconformal whose Beltrami coefficient μ satisfies $\|\mu\|_{\mathrm{WP}} \lesssim \|u\|_{H^{\frac{1}{2}}}$ and thus belongs to the class $\mathcal{M}(\mathbb{U})$.*

Proof. By the continuity of the inclusion $H^{\frac{1}{2}} \to \mathrm{BMO}$, we conclude that there exists some universal constant $\delta > 0$ such that, for any $u \in H^{\frac{1}{2}}$ with $\|u\|_{\frac{1}{2}} < \delta$, the mapping $\rho = \rho_u$ defined by (4.2) is quasiconformal. It remains to show that $\mu \in \mathcal{M}(\mathbb{U})$.

For $z = x + iy \in \mathbb{U}$, set $I = [x - y, x + y]$ as before so that

$$
u_I = \frac{1}{2y}\int_{x-y}^{x+y} u(t)dt.
$$

Noting that $\int_{\mathbb{R}} \alpha(x)dx = 0$, and $|e^z - 1| \le |ze^z| \le |z|e^{|z|}$, we conclude by (4.3) that

$$
\begin{aligned}
|\bar{\partial}\rho(z)| = |\alpha_y * e^u(x)| &= \left|\int_{\mathbb{R}} \alpha_y(x-t)e^{u(t)}dt\right| \\
&= \left|\int_{\mathbb{R}} \alpha_y(x-t)(e^{u(t)} - e^{u(x)})dt\right| \\
&= \left|\int_{\mathbb{R}} \alpha_y(x-t)(e^{u(t)-u(x)} - 1)e^{u(x)}dt\right|
\end{aligned}
$$

$$\leq \int_{\mathbb{R}} |\alpha_y(x-t)||u(t)-u(x)||e^{u(t)}|dt$$
$$\lesssim \frac{1}{|I|}\int_I |u(t)-u(x)||e^{u(t)}|dt.$$

On the other hand, since $\int_{\mathbb{R}} \beta(x)dx = 1$, we conclude by Lemma 3.1 and (4.3) that

$$|\partial\rho(z)| = |\beta_y * e^u(x)| \asymp |e^{\beta_y * u(x)}|.$$

Thus,

$$|\mu(z)| = \frac{|\bar{\partial}\rho(z)|}{|\partial\rho(z)|} \lesssim \frac{1}{|I|}\int_I |u(t)-u(x)||e^{u(t)-\beta_y*u(x)}|dt.$$

By Hölder inequality, we conclude by (3.9) that

$$\begin{aligned} |\mu(z)|^2 &\lesssim \frac{1}{|I|^2}\int_I |u(t)-u(x)|^2dt \int_I |e^{u(t)-\beta_y*u(x)}|^2dt \\ &\lesssim \frac{1}{|I|}\int_I |u(t)-u(x)|^2dt \qquad (5.1) \\ &\lesssim \frac{1}{y}\int_{-y}^{y} |u(t+x)-u(x)|^2dt. \end{aligned}$$

Consequently,

$$\begin{aligned} \iint_{\mathbb{U}} \frac{|\mu(z)|^2}{y^2}dxdy &\lesssim \iint_{\mathbb{U}} \int_{-y}^{y} \frac{|u(t+x)-u(x)|^2}{y^3}dtdxdy \\ &= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{dy}{y^3} \int_{-y}^{y} |u(t+x)-u(x)|^2dt \\ &= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{dy}{y^3} \int_0^{y} (|u(x+t)-u(x)|^2 + |u(x-t)-u(x)|^2)dt \\ &= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} (|u(x+t)-u(x)|^2 + |u(x-t)-u(x)|^2)dt \int_t^{+\infty} \frac{dy}{y^3} \qquad (5.2) \\ &= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{|u(x+t)-u(x)|^2 + |u(x-t)-u(x)|^2}{2t^2}dt \end{aligned}$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{|u(x+t) - u(x)|^2}{2t^2} dt$$

$$\asymp \|u\|^2_{H^{\frac{1}{2}}}. \quad \square$$

Corollary 5.1. *Let h be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\|\log h'\|_{H^{\frac{1}{2}}} < \delta$. Then h can be extended to a quasiconformal mapping to the upper half plane which is quasi-isometric under the Poincaré metric $|dz|/y$ and has Beltrami coefficient in $\mathcal{M}(\mathbb{U})$. In particular, h belongs the Weil-Petersson class* WP($\mathbb{R}$).

To prove (the first part of) Theorem 2.2, we will decompose a homeomorphism h with finite $\|\log h'\|_{H^{\frac{1}{2}}}$ into homeomorphisms h_j with small norms $\|\log h_j'\|_{H^{\frac{1}{2}}}$. We need some preliminary results. The first is about the pull-back operator induced by a quasisymmetric homeomorphism. Recall that an increasing homeomorphism h from the real line onto itself is said to be quasisymmetric if there exists a (least) positive constant $C(h)$, called the quasisymmetric constant of h, such that $|h(I_1)| \leq C(h)|h(I_2)|$ for all pairs of adjacent intervals I_1 and I_2 on $\mathbb{R}$ with the same length $|I_1| = |I_2|$. A strongly quasisymmetric homeomorphism is obviously quasisymmetric. We have the following well-known result.

Proposition 5.1. *([3], [24]) Let h be an increasing homeomorphism h from the real line onto itself. Then the pull-back operator P_h defined by $P_h u = u \circ h$ is a bounded operator from $H^{\frac{1}{2}}$ into itself if and only if h is quasisymmetric.*

Lemma 5.2. *Let h be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\|\log h'\|_{H^{\frac{1}{2}}} < \infty$. Then h is strongly quasisymmetric.*

Proof. Consider the Cayley transformation $\gamma(z) = \frac{z-i}{z+i}$ from the upper half plane $\mathbb{U}$ onto the unit disk Δ. Since $\log h'$ is in $H^{\frac{1}{2}}$ on the real line, $\log h' \circ \gamma^{-1}$ is in $H^{\frac{1}{2}}$ on the unit circle and consequently in VMO on the unit circle, which implies that $\log h' \circ \gamma^{-1}$ can be approximated by a sequence of bounded functions (u_n) on the unit circle under the BMO norm (see [13]). Thus, $\log h'$ can be approximated by the bounded functions $u_n \circ \gamma$ on the real line under the BMO norm. By Lemma 1.4 in [26] stating that an increasing and locally absolutely continuous homeomorphism g from the real line onto itself is strongly quasisymmetric if $\log g'$ can be approximated by bounded functions on the real line under the BMO norm, we conclude that h is strongly quasisymmetric. $\square$

Proof of Theorem 2.2 (first part). Let h be an increasing and locally absolutely continuous homeomorphism from the real line onto itself such that $\log h'$ belongs to the Sobolev class $H^{\frac{1}{2}}$. Without loss of generality, we assume $h(0) = 0$. For each real number $t \in [0, 1]$, set

$$h_t(x) = \int_0^x (h'(s))^t ds,\ x \in \mathbb{R}. \tag{5.3}$$

Then h_t is an increasing and locally absolutely continuous homeomorphism from the real line onto itself with $h_0 = \mathrm{id}$, $h_1 = h$, and $\log h_t' = t \log h'$, which implies by Lemma 5.2 that h_t is strongly quasisymmetric. Noting that for any fixed $t \in [0,1]$,

$$\|\log(h_s \circ h_t^{-1})'\|_{H^{\frac{1}{2}}} = \|(\log h_s' - \log h_t') \circ h_t^{-1}\|_{H^{\frac{1}{2}}} = |s-t| \|P_{h_t}^{-1} \log h'\|_{H^{\frac{1}{2}}},$$

we conclude by Proposition 5.1 that there exists a neighborhood I_t such that $\|\log(h_s \circ h_t^{-1})'\|_{H^{\frac{1}{2}}} < \delta$ when $s \in I_t$. By compactness, we conclude that there exists a sequence of finite numbers $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$ such that $\|\log(h_{t_j} \circ h_{t_{j+1}}^{-1})'\|_{H^{\frac{1}{2}}} < \delta$ for $j = 0, 1, 2, \cdots, n-1, n$. Since $\mathrm{WP}(\mathbb{R})$ is a group,[1] and

$$h^{-1} = (h_{t_0} \circ h_{t_1}^{-1}) \circ (h_{t_1} \circ h_{t_2}^{-1}) \circ \cdots \circ (h_{t_n} \circ h_{t_{n+1}}^{-1}),$$

we conclude by Corollary 5.1 that $h \in \mathrm{WP}(\mathbb{R})$. □

6. Proof of Theorem 2.2 (second part) and Theorem 2.3

Lemma 6.1. *Let $H_\delta^{\frac{1}{2}} = \{u \in H^{\frac{1}{2}} : \|u\|_{H^{\frac{1}{2}}} < \delta\}$, where δ is the universal constant obtained in Lemma 5.1. For $u \in H_\delta^{\frac{1}{2}}$, let $\Lambda(u)$ denote the Beltrami coefficient for the quasiconformal mapping ρ_u defined by (4.2). Then $\Lambda : H_\delta^{\frac{1}{2}} \to \mathcal{M}(\mathbb{U})$ is holomorphic.*

Proof. Since Λ is bounded in $H_\delta^{\frac{1}{2}}$, it is sufficient to show that, for each fixed pair of (u, v) with $u \in H_\delta^{\frac{1}{2}}$, $v \in H^{\frac{1}{2}}$, $\tilde{\Lambda}(t) \doteq \Lambda(u + tv)$ is holomorphic in a small neighborhood of $t = 0$ in the complex plane. To do so, choose

$$0 < \epsilon < \frac{\delta - \|u\|_{H^{\frac{1}{2}}}}{2\|v\|_{H^{\frac{1}{2}}}}$$

so that $u + tv \in H_\delta^{\frac{1}{2}}$ when $|t| \le 2\epsilon$. We conclude by (4.3) that $\tilde{\Lambda}(t)(z)$ is holomorphic in $|t| \le 2\epsilon$ for fixed $z \in \mathbb{U}$. For $|t_0| < \epsilon$, $|t| < \epsilon$, Cauchy formula yields that

$$\left| \frac{\tilde{\Lambda}(t)(z) - \tilde{\Lambda}(t_0)(z)}{t - t_0} - \frac{d}{dt}\Big|_{t=t_0} \tilde{\Lambda}(t)(z) \right| = \frac{|t - t_0|}{2\pi} \left| \int_{|\zeta|=2\epsilon} \frac{\tilde{\Lambda}(\zeta)(z)}{(\zeta - t)(\zeta - t_0)^2} d\zeta \right|$$

[1] Cui [7] first proved that $\mathrm{WP}(S^1)$, $\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ and $\mathrm{WP}(S^1)/\text{Möb}(S^1)$ are all groups (see also [40]). This can also be seen by means of Theorem 1.1 and Proposition 5.1. Consider the Cayley transformation $\gamma(z) = \frac{z-i}{z+i}$ from the upper half plane $\mathbb{U}$ onto the unit disk Δ. Then the correspondence $g \mapsto h = \gamma^{-1} \circ g \circ \gamma$ induces a one-to-one from $\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ onto $\mathrm{WP}(\mathbb{R})$ when $\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ is considered as the sub-class of $\mathrm{WP}(S^1)$ of all mappings h with $h(1) = 1$. This already implies that $\mathrm{WP}(\mathbb{R})$ is also a group.

$$\leq \frac{|t-t_0|}{2\pi\epsilon^3} \int\limits_{|\zeta|=2\epsilon} |\tilde{\Lambda}(\zeta)(z)||d\zeta|.$$

Thus, by (4.4),

$$\left\| \frac{\tilde{\Lambda}(t)-\tilde{\Lambda}(t_0)}{t-t_0} - \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t) \right\|_\infty \leq \frac{|t-t_0|}{2\pi\epsilon^3} \int\limits_{|\zeta|=2\epsilon} \|\tilde{\Lambda}(\zeta)\|_\infty |d\zeta| \leq C(u,v)|t-t_0|,$$

and by (5.2),

$$\begin{aligned}
&\iint\limits_{\mathbb{U}} \frac{1}{y^2} \left| \frac{\tilde{\Lambda}(t)(z)-\tilde{\Lambda}(t_0)(z)}{t-t_0} - \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t)(z) \right|^2 dxdy \\
&\leq \frac{|t-t_0|^2}{4\pi^2\epsilon^6} \iint\limits_{\mathbb{U}} \frac{1}{y^2} \left(\int\limits_{|\zeta|=2\epsilon} |\tilde{\Lambda}(\zeta)(z)||d\zeta| \right)^2 dxdy \\
&\leq \frac{|t-t_0|^2}{\pi\epsilon^5} \iint\limits_{\mathbb{U}} \int\limits_{|\zeta|=2\epsilon} \frac{|\tilde{\Lambda}(\zeta)(z)|^2}{y^2} |d\zeta| dxdy \\
&= \frac{|t-t_0|^2}{\pi\epsilon^5} \int\limits_{|\zeta|=2\epsilon} \iint\limits_{\mathbb{U}} \frac{|\tilde{\Lambda}(\zeta)(z)|^2}{y^2} dxdy|d\zeta| \\
&\lesssim C(u,v)|t-t_0|^2.
\end{aligned}$$

Consequently, the limit

$$\lim_{t\to t_0} \frac{\tilde{\Lambda}(t)-\tilde{\Lambda}(t_0)}{t-t_0} = \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t)$$

exists in $\mathcal{M}(\mathbb{U})$ and $\Lambda : H^{\frac{1}{2}}_{\delta} \to \mathcal{M}(\mathbb{U})$ is holomorphic. □

To complete the proof of (the second part of) of Theorem 2.2, we need to use the allowable maps introduced in section 2. Let $h_0 \in \mathrm{WP}_0(\mathbb{R})$ be a normalized mapping in the Weil-Petersson class. Then $g_0 = \gamma \circ h_0 \circ \gamma^{-1}$ belongs to the Weil-Petersson class $\mathrm{WP}(S^1)$ on the unit circle, where $\gamma(z) = \frac{z-i}{z+i}$ is the Cayley transformation from the upper half plane $\mathbb{U}$ onto the unit disk Δ. Cui [7] showed that the Douady-Earle [8] extension $DE(g_0)$ of g_0 is a quasiconformal mapping of the unit disk onto itself whose Beltrami coefficient satisfies the condition (1.1). Set $w_0 = \gamma^{-1} \circ DE(g_0) \circ \gamma$. Then w_0 is a quasiconformal extension of h_0 with Beltrami coefficient $\mu_0 \in \mathcal{M}(\mathbb{U})$. Since the Douady-Earle extension $DE(g_0)$ is quasi-isometric under the Poincaré metric $|dz|/(1-|z|^2)$ (see [8]), w_0 is quasi-isometric under the Poincaré metric $|dz|/y$. Thus w_0 induces an allowable map $w_0^*: T_0 \to T_0$ which is a bi-holomorphic isomorphism. Now h_0 induces a one-to-one mapping $R_{h_0}: \mathrm{WP}_0(\mathbb{R}) \to \mathrm{WP}_0(\mathbb{R})$ which sends h to $h \circ h_0^{-1}$. Clearly, it holds that $I \circ w_0^* = R_{h_0} \circ I$, which implies that $R_{h_0}: \mathrm{WP}_0(\mathbb{R}) \to \mathrm{WP}_0(\mathbb{R})$ is bi-holomorphic. On the other hand, when $\log h_0' \in H_{\mathbb{R}}^{\frac{1}{2}}$, h_0 also induces a bi-holomorphic isomorphism $L_{h_0}: H^{\frac{1}{2}}/\mathbb{C} \to H^{\frac{1}{2}}/\mathbb{C}$ defined by

$$L_{h_0} u = (u - \log h_0') \circ h_0^{-1}.$$

When restricted to $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$, both L_{h_0} and its inverse are real analytic. Clearly, $R_{h_0}: \mathrm{WP}_0(\mathbb{R}) \to \mathrm{WP}_0(\mathbb{R})$ and $L_{h_0}: H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R} \to H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$ are related by $R_{h_0} \circ \Psi = \Psi \circ L_{h_0}$. Summarizing these, we obtain

Lemma 6.2. *In order to prove that $\Psi: H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R} \to \mathrm{WP}_0(\mathbb{R})$ is real analytic, it is sufficient to show that Ψ is real analytic near the base point.*

Proof of Theorem 2.2 (second part). Since $\Psi = I \circ \Phi \circ \Lambda$ on $(H_\delta^{\frac{1}{2}} \cap H_{\mathbb{R}}^{\frac{1}{2}})/\mathbb{R}$, we conclude by Lemma 6.1 that Ψ is real analytic near the base point. Combining this with Lemma 6.2 completes the proof of Theorem 2.2. A direct computation shows that the differential of Ψ at a point $u \in H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$ is the linear operator

$$v \mapsto \frac{1}{\left(\int_0^1 e^{u(t)}dt\right)^2}\left(\int_0^1 e^{u(t)}dt \int_0^x e^{u(t)}v(t)dt - \int_0^1 e^{u(t)}v(t)dt \int_0^x e^{u(t)}dt\right) \tag{6.1}$$

for $v \in H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$. □

Proof of Theorem 2.3. As stated in section 2, Ψ is a one-to-one map from the real Sobolev space $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$ onto the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$. Now Ψ is real analytic, and its derivative $d_u\Psi: H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R} \to H_{\mathbb{R}}^{\frac{3}{2}}(0,1) \circ \Psi(u)$ is given by the linear operator (6.1). Here, $H_{\mathbb{R}}^{\frac{3}{2}}(0,1)$ is the set of all real-valued $H^{\frac{3}{2}}$-functions ω with the normalized conditions $\omega(0) = \omega(1) = 0$. Recall that $H_{\mathbb{R}}^{\frac{3}{2}}(0,1)$ is the tangent space at the base point of the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$ (see Theorem 8.1 below), which implies that $H_{\mathbb{R}}^{\frac{3}{2}}(0,1) \circ \Psi(u)$ is the tangent space at the point $\Psi(u)$ of $\mathrm{WP}_0(\mathbb{R})$. Clearly, $d_u\Psi: H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R} \to H_{\mathbb{R}}^{\frac{3}{2}}(0,1) \circ \Psi(u)$ is invertible. In fact, for each $\omega \in H_{\mathbb{R}}^{\frac{3}{2}}(0,1) \circ \Psi(u)$,

$$(d_u\Psi)^{-1}\omega(x) = \left(\int_0^1 e^{u(t)}dt\right)\frac{\omega'(x)}{e^{u(x)}}, \quad x \in \mathbb{R}. \tag{6.2}$$

Now the invertibility of $d_u\Psi$ implies that the inverse mapping Ψ^{-1} is also real analytic by the implicit function theorem. □

Remark 6.1. Theorem 2.3 says that, with the standard real Hilbert manifold structure of $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$, Ψ is a one-to-one analytic map from $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ onto the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$ whose inverse Ψ^{-1} is also real analytic. Therefore, there exists a unique complex Hilbert manifold structure on $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ such that Ψ is a bi-holomorphism from $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ onto $\mathrm{WP}_0(\mathbb{R})$. This complex Hilbert manifold structure on $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ can be assigned as follows. For $u \in H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$, define

$$h_u(x) = \frac{\int_0^x e^{u(t)}dt}{\int_0^1 e^{u(t)}dt}, \quad x \in \mathbb{R}$$

as before. By the well-known conformal sewing principle (see [1], [21], [22], [36], [40]), there exists a pair of quasiconformal mappings f_u, g_u on the whole plane $\mathbb{C}$ which satisfies the following properties:

(1) Both f_u and g_u fix the points 0, 1, and ∞;

(2) $f_u = g_u \circ h_u$ on the real line;

(3) f_u is conformal in the lower half plane $\mathbb{U}^*$, with Beltrami coefficient μ_1 in $\mathbb{U}$ being square integrable in the Poincaré metric, that is, $\mu_1 \in \mathcal{M}(\mathbb{U})$;

(4) g_u is conformal in the upper half plane $\mathbb{U}$, with Beltrami coefficient μ_2 in $\mathbb{U}^*$ being square integrable in the Poincaré metric, that is, $\mu_2 \in \mathcal{M}(\mathbb{U}^*)$.[2]

Let $\mathcal{D}(\mathbb{U}^*)$ denote the Dirichlet space of functions ϕ holomorphic in $\mathbb{U}^*$ with semi-norm

$$\|\phi\|_{\mathcal{D}(\mathbb{U}^*)} \doteq \left(\frac{1}{\pi} \iint\limits_{\mathbb{U}^*} |\phi'(z)|^2 dxdy \right)^{\frac{1}{2}}.$$

Then the correspondence $u \mapsto \log f_u'$ induces a one-to-one map from $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ onto a connected open subset in $\mathcal{D}(\mathbb{U}^*)/\mathbb{C}$, which endows $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ with a complex Hilbert manifold structure. Under this complex Hilbert manifold structure, Ψ is a bi-holomorphism from $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ onto $\mathrm{WP}_0(\mathbb{R})$. For more details, see [40] and also [36]. Theorem 2.3 says that this complex Hilbert manifold structure on $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ is compatible with the standard real Hilbert manifold structure $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$, which is obtained by the standard way under the semi-norm (2.3).

7. Proof of Theorems 1.2 and 2.1

We first point out the following analogous result to Theorem 1.3. A detailed proof can be found in our paper [16].

[2] $\mathcal{M}(\mathbb{U}^*)$ can be defined in the same manner as $\mathcal{M}(\mathbb{U})$.

Theorem 7.1. *Under the assumption of Theorem 2.1, the flow curve $h(t,\cdot)$ of the differential equation (2.3) satisfies $\log h'(t,\cdot) \in H^{\frac{1}{2}}_{\mathbb{R}}$ and the mapping $t \mapsto \log h'(t,\cdot)$ from $[0,M]$ into $H^{\frac{1}{2}}_{\mathbb{R}}$ is continuously differentiable such that*

$$\frac{d}{dt}\log h'(t,\cdot) = \omega'(t, h(t,\cdot)). \tag{7.1}$$

Proof of Theorem 2.1. Since the vector field $\omega(t,\cdot)$ satisfies the normalized condition $\omega(t,0) = \omega(t,1) = 0$, the flow curve $h(t,\cdot)$ of the differential equation (2.4) satisfies the condition $h(t,0) = h(t,1) - 1 = 0$. Then it holds that $h(t,\cdot) = \Psi(\log h'(t,\cdot))$. Consequently, $h(t,\cdot)$ is in the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$ and is continuously differentiable with respect to the Hilbert manifold structure of $\mathrm{WP}_0(\mathbb{R})$ by Theorems 2.2 and 7.1. Now since $h(t,\cdot)$ is a smooth curve in the Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$, we have

$$\left(\frac{d}{dt}h(t,\cdot)\right)(x) = \frac{\partial}{\partial t}(h(t,x)),$$

which implies (2.5) from (2.4). □

Proof of Theorem 1.2. Without loss of generality, we may assume that the vector field $\lambda(t,\cdot)$ satisfies the normalized condition $\lambda(t,1) = \lambda(t,-1) = \lambda(t,-i) = 0$ so that the flow curve $g(t,\cdot)$ of the differential equation (1.2) satisfies the condition $g(t,1) = 1$, $g(t,-1) = -1$, $g(t,-i) = -i$. Consider as above the Cayley transformation $\gamma(z) = \frac{z-i}{z+i}$ from the upper half plane $\mathbb{U}$ onto the unit disk Δ. Set

$$\omega(t,u) = \frac{\lambda(t,\gamma(u))}{\gamma'(u)}, \quad u \in \mathbb{R}, \tag{7.2}$$

and

$$h(t,x) = \gamma^{-1} \circ g(t,\gamma(x)), \quad x \in \mathbb{R}. \tag{7.3}$$

By Corollary 8.1 below, we see that $\omega(t,\cdot) \in C^0([0,M], H^{\frac{3}{2}}_{\mathbb{R}})$ is a continuous real-valued vector field on the real line $\mathbb{R}$ with $\omega(t,0) = \omega(t,1) = 0$. A direct computation yields that $h(t,\cdot)$ is the flow curve of the differential equation

$$\begin{cases} \frac{du}{dt} = \omega(t,u) \\ u(0,x) = x. \end{cases}$$

By Theorem 2.1, $h(t,\cdot)$ is in the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$ and is continuously differentiable with respect to the Hilbert manifold structure of $\mathrm{WP}_0(\mathbb{R})$ such that

$$\frac{d}{dt}h(t,\cdot) = \omega(t, h(t,\cdot)). \tag{7.4}$$

Noting that the correspondence $g \mapsto h = \gamma^{-1} \circ g \circ \gamma$ is a real analytic diffeomorphism from $T_0 = \mathrm{WP}(S^1)/\text{Möb}(S^1)$ onto $\mathrm{WP}_0(\mathbb{R})$, we conclude that $g(t,\cdot)$ is in $T_0 = \mathrm{WP}(S^1)/\text{Möb}(S^1)$ and is continuously differentiable with respect to the Hilbert manifold structure of $T_0 = \mathrm{WP}(S^1)/\text{Möb}(S^1)$. Finally, (1.3) follows from (1.2) immediately. □

8. Appendix: on the tangent space to $\mathbf{WP_0(\mathbb{R})}$

The tangent space to the universal Teichmüller space is well understood (see [11], [12], [23], [32]). In this section, we will deal with the tangent space to the Weil-Petersson Teichmüller space, showing some results which have been used in previous sections and have independent interests of their own.

Recall that a complex-valued function F defined in a domain Ω is called a quasiconformal deformation (abbreviated to q.d.) if it has the generalized derivative $\bar{\partial}F$ such that $\bar{\partial}F \in L^{\infty}(\Omega)$. There are several reasons for being interested in quasiconformal deformations because of their close relation with quasiconformal mappings and Teichmüller spaces (see [1], [11], [12], [22], [32], [41]) and also of their own interests (see [2], [17], [29]-[30], [31], [35], [37]). The notion of quasiconformal deformation is also closely related to the Zygmund class Λ_* in the usual sense (see [43]). Reich-Chen [31] proved that any Zygmund function $g \in \Lambda_*$ on the unit circle has a quasiconformal deformation extension to the unit disk and conversely, any continuous function g on the unit circle which has a quasiconformal deformation extension to the unit disk must belong to the Zygmund class Λ_* if g also satisfies the condition $\Re\bar{\eta}g(\eta) = 0$ for all $\eta \in S^1$. We will need the following result. A proof of Proposition 8.1 may be founded in our paper [17].

Proposition 8.1. *Let g be a continuous function on the unit circle with the normalized condition $\Re\bar{w}g(w) = 0$ on S^1. Then $g \in H^{\frac{3}{2}}$ if and only if g can be extended to a quasiconformal deformation $\tilde{g}$ to the unit disk so that*

$$\iint_{\Delta} |\bar{\partial}\tilde{g}(w)|^2(1-|w|^2)^{-2}dudv < +\infty. \tag{8.1}$$

Proposition 8.1 implies that the tangent space to $\mathrm{WP}(S^1)$ at the identity consists of precisely the $H^{\frac{3}{2}}$ vector fields λ on the unit circle (see [25], [40] and also [17]), a fact which was already pointed out in section 1. In this section, we will prove the following analogous result.

Theorem 8.1. *The tangent space to $\mathrm{WP}_0(\mathbb{R})$ at the identity consists of precisely the $H^{\frac{3}{2}}$ real-valued vector fields f on the real line with the normalized condition $f(0) = f(1) = 0$.*

By the standard Ahlfors-Bers theory of quasiconformal mappings, Theorem 8.1 follows from the following result immediately.

Theorem 8.2. *Let f be a real-valued continuous function on the real line. Then $f \in H^{\frac{3}{2}}$ if and only if f can be extended to a quasiconformal deformation $\tilde{f}$ to the upper half plane so that $\tilde{f}(z) = o(z^2)$ as $z \to \infty$ and*

$$\iint_{\mathbb{U}} |\bar{\partial}\tilde{f}(z)|^2 y^{-2} dxdy < +\infty, \quad z = x + iy. \tag{8.2}$$

We sketch the standard proof how Theorem 8.1 is deduced from Theorem 8.2 (see [12], [41]). Suppose we are given a curve of Weil-Petersson class mappings $h^t(x)$ ($t > 0$ is small) normalized to fix 0 and 1, which is the identity for $t = 0$ and differentiable with respect to t for the Hilbert manifold structure on $\mathrm{WP}_0(\mathbb{R}) = I(T_0)$. Denote

$$h^t(x) = x + tf(x) + o(t), \qquad t \to 0.$$

Since the natural projection $\Phi: \mathcal{M}(\mathbb{U}) \to T_0$ is a holomorphic split submersion, we conclude that there is a differentiable curve of Beltrami coefficients $\nu_t \in \mathcal{M}(\mathbb{U})$ such that h^t is the restriction to the real line of the normalized quasiconformal mapping f_{ν_t}. Now there exists some $\mu \in \mathcal{L}(\mathbb{U})$ such that

$$\nu_t = t\mu + o(t).$$

Consequently,

$$f_{\nu_t}(z) = z + t\dot{f}[\mu](z) + o(t), \qquad t \to 0.$$

Here $\dot{f}[\mu]$ satisfies the normalized conditions $\dot{f}[\mu](0) = \dot{f}[\mu](1) = 0$, $\dot{f}[\mu](z) = o(z^2)$ as $z \to \infty$, and is uniquely determined by the condition $\overline{\partial}\dot{f}[\mu] = \mu$ (see [12]). Noting that $f = \dot{f}[\mu]|_{\mathbb{R}}$, we conclude by Theorem 8.2 that $f \in H^{\frac{3}{2}}$.

Conversely, suppose we are given a function $f \in H^{\frac{3}{2}}$ satisfying the normalized condition $f(0) = f(1) = 0$. By Theorem 8.2 again, we deduce that f can be extended to the upper half plane to a quasiconformal deformation $\tilde{f}$ with $\overline{\partial}-$derivative $\mu = \overline{\partial}\tilde{f} \in \mathcal{L}(\mathbb{U})$ and $\tilde{f}(z) = o(z^2)$ as $z \to \infty$. Set $\mu_t = t\mu$ for small $t > 0$. Then

$$f_{\mu_t}(z) = z + t\dot{f}[\mu](z) + o(t), \qquad t \to 0.$$

Noting that both $\dot{f}[\mu]$ and $\tilde{f}$ satisfy the normalized conditions $\dot{f}[\mu](0) = \dot{f}[\mu](1) = 0$, $\dot{f}[\mu](z) = o(z^2)$ as $z \to \infty$, and have the same $\overline{\partial}$-derivative μ, we conclude that $\dot{f}[\mu] = \tilde{f}$. Then,

$$f_{\mu_t}(z) = z + t\tilde{f}(z) + o(t), \qquad t \to 0.$$

Set $h^t = f_{\mu_t}|_{\mathbb{R}}$. Then it holds that

$$h^t(x) = x + tf(x) + o(t), \qquad t \to 0,$$

which implies that h^t is a differentiable curve in $\mathrm{WP}_0(\mathbb{R}) = I(T_0)$ with the tangent vector f.

Before giving the proof of Theorem 8.2, we point out the following corollary, which was already used in the proof of Theorem 1.2.

Corollary 8.1. *Let g be a continuous function on the unit circle with the normalized conditions $g(1) = 0$, and $\Re \bar{w} g(w) = 0$, and $\gamma(z) = \frac{z-i}{z+i}$ be the Cayley transformation from the upper half plane $\mathbb{U}$ onto the unit disk Δ. Set $f = (g \circ \gamma)/\gamma'$ so that f is a continuous real-valued function on the real line with the normalized condition $f(x) = o(x^2)$ as $x \to \infty$. Then $g \in H^{\frac{3}{2}}$ on S^1 if and only if $f \in H^{\frac{3}{2}}$ on $\mathbb{R}$.*

Proof. For a q.d. extension $\tilde{g}$ of g, $\tilde{f} = (\tilde{g} \circ \gamma)/\gamma'$ is a q.d. extension of f with the normalized condition $\tilde{f}(z) = o(z^2)$ as $z \to \infty$, and vice versa. Moreover, it holds that

$$\bar{\partial} \tilde{f} = (\bar{\partial} \tilde{g} \circ \gamma) \frac{\overline{\gamma'}}{\gamma'}.$$

Since $\bar{\partial}\tilde{g}$ satisfies (8.1) if and only if $\bar{\partial}\tilde{f}$ satisfies (8.2), this corollary follows directly from Proposition 8.1 and Theorem 8.2. □

Now we begin to prove Theorem 8.2. We first recall the following well-known result (see [42]).

Proposition 8.2. *Let ϕ be analytic in the unit disk. Then it holds that*

$$\iint_{\Delta} |\phi(w)|^2 dudv \asymp |\phi(0)|^2 + \iint_{\Delta} |\phi'(w)|^2 (1 - |w|^2)^2 dudv. \tag{8.3}$$

We show that a similar result also holds on the upper half plane.

Proposition 8.3. *Let ψ be analytic in the upper half plane with $\psi(\infty) = 0$. Then it holds that*

$$\iint_{\mathbb{U}} |\psi(z)|^2 dxdy \asymp \iint_{\mathbb{U}} |\psi'(z)|^2 y^2 dxdy, \quad z = x + iy. \tag{8.4}$$

Proof. Suppose first that $\iint_{\mathbb{U}} |\psi(z)|^2 dxdy < +\infty$. Let $\gamma(z) = \frac{z-i}{z+i}$ be the Cayley transformation from the upper half plane $\mathbb{U}$ onto the unit disk Δ as before. Set $\phi = (\psi \circ \gamma^{-1})(\gamma^{-1})'$. Noting that

$$\phi' = (\psi' \circ \gamma^{-1})(\gamma^{-1})'^2 + \psi \circ \gamma^{-1}(\gamma^{-1})'',$$

we obtain by Proposition 8.2 that

$$\begin{aligned}
&\iint_{\mathbb{U}} |\psi'(z)|^2 y^2 dxdy \\
&= \iint_{\Delta} |(\psi' \circ \gamma^{-1})(\gamma^{-1})'^2|^2 (1-|w|^2)^2 dudv \\
&\lesssim \iint_{\Delta} (|\phi'|^2 + |\psi \circ \gamma^{-1}(\gamma^{-1})''|^2)(1-|w|^2)^2 dudv \\
&\lesssim \iint_{\Delta} (|\phi'(w)|^2(1-|w|^2)^2 + |\phi(w)|^2) dudv \\
&\lesssim \iint_{\Delta} |\phi(w)|^2 dudv \\
&= \iint_{\mathbb{U}} |\psi(z)|^2 dxdy.
\end{aligned}$$

Here we have used the relation

$$\frac{(\gamma^{-1})''}{(\gamma^{-1})'}(w) = \frac{2}{1-w}.$$

Conversely, suppose that $\iint_{\mathbb{U}} |\psi'(z)|^2 y^2 dxdy < +\infty$. Then we have the following reproducing formula (see [11]):

$$\psi'(z) = \frac{12}{\pi} \iint_{\mathbb{U}} \frac{v^2 \psi'(w)}{(\bar{w}-z)^4} dudv, \quad w = u + iv,$$

or equivalently,

$$\psi(z) = \frac{4}{\pi} \iint_{\mathbb{U}} \frac{v^2 \psi'(w)}{(\bar{w}-z)^3} dudv, \quad w = u + iv.$$

Now for any holomorphic function φ in the upper half plane with $\iint_{\mathbb{U}} |\varphi(z)|^2 dxdy < +\infty$, we have

$$\iint_{\mathbb{U}} \overline{\psi(z)} \varphi(z) dxdy$$

$$
\begin{aligned}
&= \frac{4}{\pi} \iint\limits_{\mathbb{U}} \iint\limits_{\mathbb{U}} \frac{v^2 \overline{\psi'(w)} \varphi(z)}{(w-\bar{z})^3} du dv dx dy \\
&= -\frac{4}{\pi} \iint\limits_{\mathbb{U}} v^2 \overline{\psi'(w)} du dv \iint\limits_{\mathbb{U}} \frac{\varphi(z)}{(\bar{z}-w)^3} dx dy \\
&= -\frac{4}{\pi} \iint\limits_{\mathbb{U}} v^2 \overline{\psi'(w)} du dv \int\limits_0^{+\infty} dy \int\limits_{-\infty+iy}^{+\infty+iy} \frac{\varphi(z)}{(z-2iy-w)^3} dz \\
&= -4i \iint\limits_{\mathbb{U}} v^2 \overline{\psi'(w)} du dv \int\limits_0^{+\infty} \varphi''(2iy+w) dy \\
&= 2 \iint\limits_{\mathbb{U}} v^2 \overline{\psi'(w)} \varphi'(w) du dv,
\end{aligned}
$$

which implies by what we have proved in the first part that

$$
\begin{aligned}
\left| \iint\limits_{\mathbb{U}} \overline{\psi(z)} \varphi(z) dx dy \right|^2 &\leq 4 \iint\limits_{\mathbb{U}} |\psi'(z)|^2 y^2 dx dy \iint\limits_{\mathbb{U}} |\varphi'(z)|^2 y^2 dx dy \\
&\lesssim \iint\limits_{\mathbb{U}} |\psi'(z)|^2 y^2 dx dy \iint\limits_{\mathbb{U}} |\varphi(z)|^2 dx dy.
\end{aligned}
$$

Consequently,

$$
\iint\limits_{\mathbb{U}} |\psi(z)|^2 dx dy \lesssim \iint\limits_{\mathbb{U}} |\psi'(z)|^2 y^2 dx dy. \quad \square
$$

Now suppose that f is a real-valued continuous function on the real line, and there exists some constant $\alpha < 2$ such that $f(t) = O(|t|^\alpha)$ as $t \to \infty$. Following Reich [29], set

$$
Af(z) = \frac{z^2+1}{i\pi} \int\limits_{-\infty}^{+\infty} \frac{f(t)}{(t-z)(t^2+1)} dt, \quad z \in \mathbb{U}, \tag{8.5}
$$

and

$$
Hf(z) = \frac{(z-\bar{z})^3}{2i\pi} \int\limits_{-\infty}^{+\infty} \frac{f(t)}{(t-z)(t-\bar{z})^3} dt, \quad z \in \mathbb{U}. \tag{8.6}
$$

Clearly, Af is analytic on the upper half plane $\mathbb{U}$, and

$$(Af)'''(z) = \frac{12}{i\pi} \int_{-\infty}^{+\infty} \frac{f(t)}{(t-z)^4} dt, \quad z \in \mathbb{U}. \tag{8.7}$$

Reich [29] showed that Hf is a C^∞ extension of f to $\mathbb{U}$, and

$$\bar{\partial}(Hf)(z) = -y^2 \overline{(Af)'''(z)}, \quad z = x + iy \in \mathbb{U}. \tag{8.8}$$

He also showed that $\Re(Af)$ is continuous extension of f to $\mathbb{U}$ (see also [37]).

Lemma 8.1. *Let f be a real-valued continuous function on the real line with the normalized condition $f(t) = O(|t|^\alpha)$ as $t \to \infty$ for some constant $\alpha < 2$. Then $f \in H^{\frac{3}{2}}$ if and only if*

$$\iint_{\mathbb{U}} |(Af)'''(z)|^2 y^2 dxdy, \quad z = x + iy. \tag{8.9}$$

Proof. Recall that a function ω on the real line belongs to the class $H^{\frac{1}{2}}$ if and only if there exists some harmonic function $\tilde{\omega}$ on the upper half plane with boundary values ω and has finite Dirichlet integral $\iint_{\mathbb{U}}(|\partial\tilde{\omega}|^2 + |\bar{\partial}\tilde{\omega}|^2) < +\infty$. Consequently, under the assumption of the lemma, $f \in H^{\frac{3}{2}}$ if and only if $\iint_{\mathbb{U}} |(Af')' - (Af')'(\infty)|^2 < +\infty$, which is equivalent to (8.9) by Proposition 8.3 due to the fact that $(Af)''' = (Af')''$. $\square$

Proof of Theorem 8.2. Let $f \in H^{\frac{3}{2}}$ be a real-valued continuous function on the real line. Then we conclude by (8.8) and Lemma 8.1 that Hf is the required quasiconformal deformation extension of f to the upper half plane. Conversely, suppose f can be extended to a quasiconformal deformation $\tilde{f}$ to the upper half plane so that $\tilde{f}(z) = o(z^2)$ as $z \to \infty$ and (8.2) holds. Then it holds the following equality (see [29] and also [37]):

$$\overline{(Af)'''(z)} = -\frac{12}{\pi} \iint_{\mathbb{U}} \frac{\bar{\partial}\tilde{f}(w)}{(w-\bar{z})^4} dudv, \quad z = x + iy \in \mathbb{U}. \tag{8.10}$$

A direct computation shows that (8.9) holds by means of (8.2) and (8.10). In fact, by (8.10) we obtain

$$|(Af)'''(z)|^2 \leq \frac{144}{\pi^2} \iint_{\mathbb{U}} \frac{|\bar{\partial}\tilde{f}(w)|^2}{|w-\bar{z}|^4} dudv \iint_{\mathbb{U}} \frac{1}{|w-\bar{z}|^4} dudv = \frac{36}{\pi y^2} \iint_{\mathbb{U}} \frac{|\bar{\partial}\tilde{f}(w)|^2}{|w-\bar{z}|^4} dudv,$$

which implies by (8.2) that

$$\begin{aligned} \iint_{\mathbb{U}} |(Af)'''(z)|^2 y^2 dxdy &\leq \frac{36}{\pi} \iint_{\mathbb{U}} \iint_{\mathbb{U}} \frac{|\bar{\partial}\tilde{f}(w)|^2}{|w-\bar{z}|^4} dudvdxdy \\ &= \frac{36}{\pi} \iint_{\mathbb{U}} |\bar{\partial}\tilde{f}(w)|^2 \iint_{\mathbb{U}} \frac{1}{|w-\bar{z}|^4} dxdydudv \end{aligned}$$

$$= 9 \iint\limits_{\mathbb{U}} |\bar{\partial} \tilde{f}(w)|^2 v^{-2} dudv < +\infty, \; w = u + iv.$$

We conclude that $f \in H^{\frac{3}{2}}$ by Lemma 8.1 again. □

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