The Brunn–Minkowski inequality for the principal eigenvalue of fully nonlinear homogeneous elliptic operators

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ABSTRACT

We prove that the principal eigenvalue of any fully nonlinear homogeneous elliptic operator which fulfills a very simple convexity assumption satisfies a Brunn-Minkowski type inequality on the class of open bounded sets in $\mathbb{R}^n$ satisfying a uniform exterior sphere condition. In particular the result applies to the (possibly normalized) p-Laplacian, and to the minimal Pucci operator. The proof is inspired by the approach introduced by Colesanti for the principal frequency of the Laplacian within the class of convex domains, and relies on a generalization of the convex envelope method by Alvarez-Lasry-Lions. We also deal with the existence and log-concavity of positive viscosity eigenfunctions.

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1. Introduction

In its classical formulation, the Brunn-Minkowski inequality states that the volume functional, raised to the power $1/n$, is concave on the class $\mathcal{K}^n$ of convex bodies in the n-dimensional Euclidean space. Specifically, for every pair K_0, K_1 of nonempty convex compact subsets of $\mathbb{R}^n$ and every $t \in [0, 1]$, denoting by $(1-t)K_0 + tK_1$ the set of points of the form $(1-t)x + ty$ for $x \in K_0$ and $y \in K_1$, and by $V(\cdot)$ the n-dimensional Lebesgue measure, it holds

$$V^{1/n}\big((1-t)K_0 + tK_1\big) \geq (1-t)V^{1/n}(K_0) + tV^{1/n}(K_1), \tag{1}$$

with equality sign if and only if K_0 and K_1 are homothetic.

Named after Brunn, who firstly proved it in dimension 2 and 3 [22,23], and Minkowski, who shortly afterwards gave a full analytic proof in n-dimensions and characterized the equality case [62], in the last century this fundamental inequality has been proved and generalized in many different ways by an impressive list of mathematicians, including Hilbert [44], Bonnesen [17], Kneser-Suss [54], Blaschke [16], Hadwiger [42], Knothe [55], Dinghas [39], MacCann [60], McMullen [61], Ball [4], Klain [53].

It is not conceivable to give here an idea about the impact of the Brunn-Minkowski inequality in both Analysis and Geometry, and in their interplay. We limit ourselves to refer to Chapter 7 in the treatise [68] by Schneider, which includes a lot of historical and bibliographical notes, and to the excellent survey paper [40] by Gardner, from which we quote: "In a sea of mathematics, the Brunn-Minkowski inequality appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams from one area to the next."

Aim of this paper is to reveal a new tentacle of this fascinating creature, which gets as far as the viscosity theory of nonlinear PDEs, by proving the validity of a Brunn-Minkowski type inequality for the principal frequency of fully non-linear homogeneous elliptic operators.

As a starting point to introduce our results, we recall that the Brunn-Minkowski inequality has been generalized, in a suitable form, to several functionals other than volume. They include not only geometric quantities (such as quermassintegrals [68, Section 7.4]), but also some energies from physics and calculus of variations. To be more precise, following [32], we say that a functional Φ which is invariant under rigid motions and homogeneous of degree $\gamma \neq 0$ on $\mathcal{K}^n$ satisfies a Brunn-Minkowski type inequality if, by analogy to (1), it holds

$$\Phi^{1/\gamma}\big((1-t)K_0 + tK_1\big) \geq (1-t)\Phi^{1/\gamma}(K_0) + t\Phi^{1/\gamma}(K_1). \tag{2}$$

The most significant choices of functionals Φ for which the above inequality has been proved are: the principal frequency of the Laplacian (see Brascamp-Lieb [21]), the torsional rigidity (see Borell [20]), the Newtonian capacity (see Borell [18] and Caffarelli-Jerison-Lieb [26]), the logarithmic capacity and a n-dimensional version of it (see Borell

[19] and Colesanti-Cuoghi [31]), the p-capacity (see Colesanti-Salani [33]), the first eigenvalue of the p-Laplacian and the p-torsional rigidity (see Colesanti-Cuoghi-Salani [32]), the first eigenvalue of the Monge-Ampère operator (see Salani [65]), the Bernoulli constant (see Bianchini-Salani [9]), the Hessian eigenvalue in three-dimensional convex domains (see Liu-Ma-Xu [58]), functionals related to Hessian equations (see Salani [66]). For large part of these results, a nice account can be found in the paper [30] by Colesanti.

Regarding this spectrum of extensions of the Brunn-Minkowski inequality, we wish to draw attention on the class of domains where the inequality is known to work. Actually, the validity of inequality (1) for the volume functional goes far beyond the class of convex bodies: it has been extended to all measurable sets; a short and elegant proof due to Hadwiger-Ohmann [43] can be found in the above mentioned survey paper by Gardner. In spite, to our knowledge, for all the functionals Φ mentioned above the validity of inequality (2) has been established only within convex bodies, exception made for the first eigenvalue of the Laplacian and the torsional rigidity, for which the inequality is known to hold for all open bounded domains with sufficiently regular boundary.

It is now time to present the new family of Brunn-Minkowski type inequalities we obtain in this paper. Given an open bounded domain Ω in $\mathbb{R}^n$, we consider the following eigenvalue problem for a fully nonlinear, degenerate elliptic, homogeneous operator:

$$\begin{cases} F(\nabla u, D^2 u) = \lambda |u|^\alpha u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{3}$$

Here $F \colon (\mathbb{R}^n \setminus \{0\}) \times S^n \to \mathbb{R}$ is a continuous function satisfying, for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and X in the space S^n of symmetric real matrices, the following conditions:

(H1) *Homogeneity*: for some $\alpha > -1$ and every $(t, \mu) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$,

$$F(t\xi, \mu X) = |t|^\alpha \mu F(\xi, X);$$

(H2) *Uniform ellipticity*: for some $C \geq c > 0$ and every Y in the space S^n_+ of positive semidefinite symmetric matrices,

$$c|\xi|^\alpha \text{tr}(Y) \leq F(\xi, X) - F(\xi, X + Y) \leq C|\xi|^\alpha \text{tr}(Y).$$

For any operator satisfying (H1)-(H2), inspired by the celebrated work by Berestycki, Nirenberg and Varadhan [7], Birindelli and Demengel introduced in [12] the principal eigenvalue $\overline{\lambda}(\Omega)$ as

$$\overline{\lambda}(\Omega) := \sup \left\{ \lambda \in \mathbb{R} \; : \; \exists u > 0 \text{ in } \Omega \text{ viscosity super-solution to the pde in } (3) \right\};$$

here the notion of viscosity super-solution has to be meant as specified in Section 2.1 below.

The bibliography related to the eigenvalue problem for fully nonlinear second order operators is very wide. With no attempt of completeness, we limit ourselves to quote Birindelli-Demengel for many related works including [11,13–15], Ikoma-Ishii for the computation of eigenvalues on balls [45,46], Armstrong [2], Berestycki-Capuzzo Dolcetta-Porretta-Rossi [6], and Quaas-Sirakov [63] for related maximum principles, Berestycki-Rossi for the case of unbounded domains [8], Kawohl and different coauthors for the case of the game theoretic p-Laplacian [5,49–52] (see also our recent joint work [37]), Juutinen for the case of the normalized infinity Laplacian [48], Busca-Esteban-Quaas for the case of Pucci operators [24].

As far as we know, there is no previous attempt to prove that the Brunn-Minkowski inequality holds true for the principal eigenvalue of a fully nonlinear operator. Our main result states that this is indeed the case as soon as the operator enjoys, besides (H1)-(H2), the following condition

(H3) *Convexity:* for every $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$X \mapsto F(\xi, X) \text{ is convex on } S^n\,,$$

and the involved domains belong to the class

$$\mathcal{A}^n := \left\{ \begin{array}{c} \text{open bounded connected Lipschitz domains of } \mathbb{R}^n \\ \text{satisfying a uniform exterior sphere condition} \end{array} \right\}. \tag{4}$$

We remark that this class is closed with respect to the Minkowski addition of sets.

Theorem 1 *(Brunn-Minkowski inequality).* If F satisfies conditions (H1)-(H2)-(H3), for every pair of domains $\Omega_0, \Omega_1 \in \mathcal{A}^n$, and every $t \in [0,1]$, it holds

$$\overline{\lambda}\big((1-t)\Omega_0 + t\Omega_1\big)^{-1/(\alpha+2)} \geq (1-t)\overline{\lambda}(\Omega_0)^{-1/(\alpha+2)} + t\,\overline{\lambda}(\Omega_1)^{-1/(\alpha+2)}\,. \tag{5}$$

We emphasize that the class $\mathcal{A}^n$ contains all bounded open sets which are convex or of class C^2, but domains in $\mathcal{A}^n$ do not need to be convex, nor of class C^2. In particular, for the first eigenvalue of the p-Laplacian, Theorem 1 extends to domains in $\mathcal{A}^n$ the Brunn-Minkowski inequality proved for C^2 convex bodies by Colesanti-Cuoghi-Salani [32]. Besides the p-Laplacian, a list of further relevant operators fitting the assumptions of Theorem 1 is postponed at the end of this section.

The reason why we work on the class $\mathcal{A}^n$ is that, for such domains, we are able to prove the existence of positive viscosity eigenfunctions, until now known only for C^2 domains (see [12,14]). This side result, which may have its own interest, is given in Section 3 (see Theorem 19): it is derived as a by-product of a global Hölder estimate (see Proposition 17), which in turn is obtained via a barrier argument, adapted from Birindelli-Demengel, involving the distance from the boundary.

Our approach to obtain Theorem 1 can be synthetically defined as a synergy between the method introduced by Colesanti in [30] to obtain the Brunn-Minkowski inequality for the first eigenvalue of the Laplacian for convex domains, and the method introduced by Alvarez, Lasry and Lions in [1] to obtain the convexity of viscosity solutions to second order fully nonlinear elliptic equations with state constraint boundary conditions. We also point out that in the paper [47] parabolic problems are considered under a close perspective, working on possibly non-convex domains, yet still with classical solutions; more specifically, using Lemma 3.1 in [47] it can be realized that the general theory previously developed by Salani in [67] (covering for instance the case of the Pucci operator) can be extended to non-convex domains.

Roughly speaking, the proof of Theorem 1 goes as follows. The key point is that, in order to prove the inequality (2) for $\Phi(\cdot) = \overline{\lambda}(\cdot)$, it is enough to construct a sub-solution to the corresponding eigenvalue problem on the domain $(1 - t)\Omega_0 + t\Omega_1$. In case of the Laplacian, this assertion relies on the variational characterization of the eigenvalue as minimum of the Rayleigh quotient. In our fully nonlinear setting, though there is no variational interpretation of the eigenvalue, the same principle remains true thanks to a maximum principle proved by Birindelli-Demengel (see Theorem 7 below). Then the next step is how to construct a sub-solution. To that aim the idea is to look at the transformed equation satisfied by (minus) the logarithms of the eigenfunctions (which on convex domains are known to be convex functions [21,27]), consider (minus) the infimal convolution between these logarithms, and take its exponential. In case of the Laplacian, the function thus obtained turns out to be a sub-solution essentially because the infimal convolution linearizes the Fenchel transform, and the map $M \mapsto \operatorname{tr}(M^{-1})$ is convex on the family of positive definite matrices. In our fully non-linear setting, we still consider the function constructed in the same way, but in order to show that it is a sub-solution we have to adopt a different procedure. Indeed, since we do not have enough regularity information on the eigenfunctions, we cannot write pointwise Hessians; moreover, since we want to get rid of the convexity assumptions on the domains, we cannot exploit the log-concavity of eigenfunctions. To overcome these difficulties, we set up a generalization of the method introduced by Alvarez-Lasry-Lions in order to show that the convex envelope is a sub-solution, the difference being that we work with a family of *distinct* functions on *distinct, possibly non-convex*, domains (compare Propositions 8 and 14 below respectively with Propositions 1 and 3 in [1]). We remark that similar techniques have been used in the above mentioned paper [67] by Salani, where the author has introduced a very general theory of Brunn-Minkowski inequalities for functionals related to elliptic PDEs, for a very general class of nonlinear operators. Yet, the effective applicability of the results in [67] is limited by the fact that only classical solutions are considered.

Let us point out that at present we are not able to push over our viscosity approach in order to deal with the equality case in Theorem 1. We address such characterization as an interesting open problem, which seems to be quite delicate. Actually, for a lot of Brunn-Minkowski type inequalities, the characterization of the equality case is still open, especially when dealing with non-convex domains. The case of the first eigenvalue of the

Laplacian is emblematic in this respect: since Brascamp-Lieb [21], the inequality (2) is known to hold for all compact, connected domains having sufficiently regular boundary, but the equality case has been settled only forty years later by Colesanti [30], and his approach works just for convex domains.

On the other hand, as a companion result to Theorem 1, we are able to establish the log-concavity of positive viscosity eigenfunctions. As well as in Theorem 1, we need as a key assumption the convexity of F in its second variable. However, for technical reasons which will be explained during the proof, here it is needed in the following stronger form:

(H3)' *Reinforced convexity:* F is of class C^2, and for every $\delta > 0$ there exists a positive constant c_0 such that

$$\nabla_X^2 F(\xi, X) M \cdot M \geq c_0 |M|^2 \qquad \forall M, X \in S^n, \ \forall \xi \in \mathbb{R}^n \text{ with } |\xi| > \delta.$$

Theorem 2 *(log-concavity of eigenfunctions). Assume that F satisfies conditions (H1)-(H2)-(H3)'. Then:*

(i) *if Ω is a strongly convex bounded open set of class $C^{2,\beta}$ for some $\beta \in (0,1)$, then any positive viscosity eigenfunction is log-concave;*

(ii) *if Ω is a convex bounded open set, then there exists a positive viscosity eigenfunction which is log-concave.*

The above theorem can be read as an extension to viscosity solutions of general fully nonlinear operators of the result proved by Sakaguchi in [64] for the p-Laplacian (see also [57]) and by Bianchini and Salani in [10] for a general class of operators including the ones considered here. Part (i) of the statement is obtained essentially via the convex envelope method of Alvarez-Lasry-Lions, whereas, for part (ii), we use our afore mentioned existence result (Theorem 19), which involves an approximation argument with smooth domains. In particular, the fact that an approximation procedure is needed explains why part (ii) of the statement is formulated for *some* (not for *any*) positive viscosity eigenfunction. Clearly, in case the eigenvalue is simple, also for Ω as in (ii) any positive viscosity solution is log-concave. This is for instance the case of the p-Laplacian [64] and of the normalized p-Laplacian [37].

We conclude this Introduction by providing a short list of some relevant operators to which the results stated above apply.

Example 3. The following operators satisfy assumptions (H1)-(H2)-(H3). Moreover, all of them satisfy also assumption (H3)' (the corresponding function F being linear in X), except for the minimal Pucci operator, which however satisfies assumption (H3) (see [25, Lemma 2.10]).

- *The p-Laplacian, for $p > 1$:*

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

$$F(\xi, X) = -|\xi|^{p-2}\text{tr}X - (p-2)|\xi|^{p-4}\langle X\xi, \xi\rangle, \quad \alpha = p - 2$$

- *The normalized p-Laplacian, for $p > 1$:*

$$\Delta_p^N u = \frac{1}{p}|\nabla u|^{2-p}\text{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

$$F(\xi, X) = -\frac{1}{p}\text{tr}X - \frac{p-2}{p}|\xi|^{-2}\langle X\xi, \xi\rangle, \quad \alpha = 0$$

- *The minimal Pucci operator:*

$$\mathcal{M}_{\lambda,\Lambda}(D^2 u) = \lambda \sum_{e_i > 0} e_i(D^2 u) + \Lambda \sum_{e_i < 0} e_i(D^2 u), \quad 0 < \lambda \leq \Lambda,$$

$$F(\xi, X) = \lambda \sum_{e_i > 0} e_i(X) + \Lambda \sum_{e_i < 0} e_i(X), \quad \alpha = 0 \quad (e_i(X) = \text{eigenvalues of } X).$$

The remaining of the paper is organized as follows:

- in Section 2 we provide the intermediate results we need about viscosity solutions and infimal convolutions;
- in Section 3 we prove the existence of eigenfunctions for domains in $\mathcal{A}^n$;
- in Section 4 we give the proofs of Theorems 1 and 2.

2. Preliminary results

2.1. Viscosity solutions and maximum principle

Below we adopt the following standard notation: if u, φ are two real functions on Ω and $x \in \Omega$, by writing $\varphi \prec_x u$ (resp. $u \prec_x \varphi$), we mean that φ *touches u from below (resp. from above) at x*, that is $u(x) = \varphi(x)$ and $\varphi(y) \leq u(y)$ (resp. $u(y) \leq \varphi(y)$) for every $y \in \Omega$. Moreover, we denote by $J_\Omega^{2,-}u(x)$ (resp. $J_\Omega^{2,+}u(x)$) the *second order sub-jet* (resp. *super-jet*) of u at x, which is by definition the set of pairs $(\xi, A) \in \mathbb{R}^n \times S^n$ such that, as $y \to x$, $y \in \overline{\Omega}$, it holds

$$u(y) \geq (\leq) \ u(x) + \langle \xi, y - x\rangle + \frac{1}{2}\langle A(y - x), y - x\rangle + o(|y - x|^2).$$

For any $\lambda > 0$, the notion of viscosity sub- and super-solutions to the pde

$$F(\nabla u, D^2 u) = \lambda |u|^\alpha u$$

can be intended according Crandall-Ishii-Lions [35] or according to Birindelli-Demengel [14], as formulated respectively in Definition 4 and Definition 5. For later use, we give these two definitions for the more general equation

$$F(\nabla u, D^2 u) = g(u), \tag{6}$$

where $g \colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 4. – An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity sub-solution* to (6) if, for every $x \in \Omega$ and for every smooth function φ such that $u \prec_x \varphi$, denoting by F_* the lower semicontinuous envelope of F, it holds

$$F_*(\nabla\varphi(x), D^2\varphi(x)) \le g(\varphi(x))$$

(or equivalently $F_*(\xi, A) \le g(u(x))$ for every $(\xi, A) \in J^{2,+}_\Omega u(x)$).
   – A lower semicontinuous function $u : \Omega \to \mathbb{R}$ is a *viscosity super-solution* to (6) if, for every $x \in \Omega$ and for every smooth function φ such that $\varphi \prec_x u$, denoting by F^* the upper semicontinuous envelope of F, it holds

$$F^*(\nabla\varphi(x), D^2\varphi(x)) \ge g(\varphi(x))$$

(or equivalently $F^*(\xi, A) \ge g(u(x))$ for every $(\xi, A) \in J^{2,-}_\Omega u(x)$).
   – A continuous function $u \colon \Omega \to \mathbb{R}$ is a *viscosity solution* to (6) in Ω if it is both a viscosity super-solution and a viscosity sub-solution.

Definition 5. – An upper semicontinuous function $u \colon \Omega \to \mathbb{R}$ is a *viscosity sub-solution* to (6) if, for every $x \in \Omega$:
   ·) either u is equal to a constant c on an open ball $B_r(x) \subset \Omega$ and $0 \le g(c)$;
   ·) or for every smooth function φ such that $u \prec_x \varphi$ with $\nabla\varphi(x) \ne 0$, it holds

$$F(\nabla\varphi(x), D^2\varphi(x)) \le g(\varphi(x))$$

(or equivalently $F(\xi, A) \le g(u(x))$ for every $(\xi, A) \in J^{2,+}_\Omega u(x)$ with $\xi \ne 0$).
   – A lower semicontinuous function $u \colon \Omega \to \mathbb{R}$ is a *viscosity super-solution* of (6) if, for every $x \in \Omega$:
   ·) either u is equal to a constant c on an open ball $B_r(x) \subset \Omega$ and $0 \ge g(c)$;
   ·) or for every smooth function φ such that $\varphi \prec_x u$ with $\nabla\varphi(x) \ne 0$, it holds

$$F(\nabla\varphi(x), D^2\varphi(x)) \ge g(\varphi(x))$$

(or equivalently $F(\xi, A) \ge g(u(x))$ for every $(\xi, A) \in J^{2,-}_\Omega u(x)$ with $\xi \ne 0$).
   – A continuous function $u : \Omega \to \mathbb{R}$ is a *viscosity solution* to (6) in Ω if it is both a viscosity supersolution and a viscosity subsolution.

The following equivalence lemma is adapted from [38, Lemma 2.1] and [3, Proposition 2.4], and will be very useful in the sequel (*cf.* Remark 15). For this result and the subsequent Theorem 7, the uniform ellipticity condition (H2) can be replaced by the much weaker *degenerate ellipticity condition*:

(H2)' $F(\xi, X) \geq F(\xi, Y)$ for every $\xi \in \mathbb{R}^n \setminus \{0\}$ and for every $X, Y \in S^n$, $X \leq Y$.

Lemma 6. *For any operator F satisfying (H2)' and*

$$F^*(0,0) = F_*(0,0) = 0, \tag{7}$$

and any continuous function g, Definitions 4 and 5 are equivalent.

Proof. Let us show the equivalence for super-solutions, the case of sub-solutions being analogous. Let u be a super-solution according to Definition 4 and let $x \in \Omega$. To show that u is a super-solution according to Definition 5, we have just to consider the case when u is equal to a constant c on a ball $B_r(x)$, and show that $0 \geq g(c)$. Let us fix an arbitrary point $y \in B_r(x)$, and let us consider the test function $\varphi(z) = c - |z - y|^q$, with $q > 2$. We have that φ touches u from below at y, with $\nabla\varphi(y) = 0$ and $D^2\varphi(y) = 0$. Therefore, by assumption

$$F^*(\nabla\varphi(y), D^2\varphi(y)) \geq g(\varphi(y))$$

or equivalently, in view of (7),

$$0 = F^*(0,0) \geq g(c).$$

Conversely, let u be a super-solution according to Definition 5 and let $x \in \Omega$. To show that u is a super-solution according to Definition 4, we have to consider just the situation when φ touches u from below at x with $\nabla\varphi(x) = 0$. We distinguish two cases. First case: u is equal to a constant c on an open ball $B_r(x) \subset \Omega$. Then it holds $0 \geq g(c)$ (because u is assumed to be a super-solution according to Definition 5), and $D^2\varphi(x) \leq 0$ (because φ is touching from below the locally constant function u). Observe that, if $X \leq 0$, by the degenerate ellipticity assumption (H2)' we have that $F(\xi, X + Y) \geq F(\xi, Y)$ for every $\xi \neq 0$ and $Y \in S^n$, so that, from (7),

$$F^*(0, X) \geq 0, \qquad \forall X \leq 0,$$

hence we conclude that

$$F^*(0, D^2\varphi(x)) \geq 0 \geq g(c).$$

Second case: u is not equal to a constant on any open ball $B_r(x) \subset \Omega$. Given $y \in B_\rho(0)$, with $\rho > 0$ small enough, we consider the function

$$\varphi_y(z) = \varphi(y + z) \qquad \forall z \in B_r(x).$$

Since it is not restrictive to assume that x is a strict minimum point of $u - \varphi$ in $\overline{B}_r(x)$, for $|y|$ small enough we have that φ_y touches u from below at some point $x_y \in B_r(x)$. We claim that, with no loss of generality, we may assume that there exists a sequence $y_k \to 0$ such that $\nabla\varphi_{y_k}(x_{y_k}) \neq 0$ for every k. If this is the case, by testing the equation at x_{y_k}, we obtain

$$F(\nabla\varphi_{y_k}(x_{y_k}), D^2\varphi_{y_k}(x_{y_k})) \geq g(\varphi_{y_k}(x_{y_k})),$$

which by passing to the limsup as $k \to +\infty$ yields

$$F^*(0, D^2\varphi(x)) \geq g(\varphi(x)).$$

Finally, it remains to prove the claim. By making r smaller if necessary, we can assume that x is the unique critical point of φ in $B_r(x)$. (This is immediate if $D^2\varphi(x)$ is invertible, and such condition can always be assumed up to replacing $\varphi(z)$ by $\varphi_\varepsilon(z) = \varphi(z) - \frac{\varepsilon}{2}(z - x)^t M(z - x)$, being M a positive definite matrix in S^n such that $D^2\varphi(x) - \varepsilon M$ is invertible for all $\varepsilon > 0$.) Then, arguing by contradiction, and exploiting the fact that x is the unique critical point of φ in $B_r(x)$, one can show that, if the sequence y_k would not exist, u should be constant around x (see [38, Lemma 2.1] or [3, Proposition 2.4] for more details). $\square$

We remark that assumption (7) is fulfilled by every operator F satisfying the homogeneity condition (H1) with $\alpha > -1$. Hence, in view of Lemma 6, in the remaining of the paper we write the words sub- and super-solutions referring indistinctly to Definition 4 or 5.

The following maximum principle will be used as a keystone in our proof of Theorem 1:

Theorem 7. *[12, Thm. 3.3] Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let F satisfy assumptions (H1)-(H2)'. Let $\tau < \overline{\lambda}(\Omega)$, and let u be a viscosity sub-solution to*

$$F(\nabla u, D^2 u) = \tau |u|^\alpha u \qquad in \ \Omega,$$

satisfying $u \leq 0$ on $\partial\Omega$. Then $u \leq 0$ in Ω.

The idea to prove Theorem 1 is to construct a subsolution which, if the inequality (5) would be false, would violate the maximum principle above. To that aim we drive our attention to the operation of infimal convolution.

2.2. Infimal convolutions

For a fixed $k \in \mathbb{N}$, set

$$\Lambda_k^+ := \left\{ \boldsymbol{t} = (t_0, \dots, t_k) : \ t_i \in (0,1], \ \sum_{i=0}^k t_i = 1 \right\},$$

$$\mathcal{O}_k := \left\{ (\Omega_0, \dots, \Omega_k) \ : \ \Omega_i \subset \mathbb{R}^n \text{ open bounded set } \right\}.$$

Given $(\Omega_0, \dots, \Omega_k) \in \mathcal{O}_k$ and $\boldsymbol{t} \in \Lambda_k^+$, we consider the convex Minkowski combination

$$\Omega_{\boldsymbol{t}} := t_0 \, \Omega_0 + \cdots + t_k \, \Omega_k = \left\{ \sum_{i=0}^k t_i x_i \ : \ x_i \in \Omega_i \right\}.$$

Notice that $\Omega_{\boldsymbol{t}}$ is an open set: indeed, if $x = \sum_{i=0}^k t_i x_i$, for any $j \in \{0, \dots, k\}$ it holds

$$B_\delta(x_j) \subset \Omega_j \implies B_{t_j \delta}(x) \subset \Omega_{\boldsymbol{t}}. \tag{8}$$

Let $v_i : \Omega_i \to \mathbb{R}$, $i = 0, \dots, k$, be given functions. We can think v_i as defined on $\mathbb{R}^n$, by extending them to $+\infty$ outside Ω_i.

We call *weighted infimal convolution of the functions* $v_0, \dots, v_k$ (with weight $\boldsymbol{t}$) the function defined on $\mathbb{R}^n$ by

$$(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}(x) := \inf \left\{ \sum_{i=0}^k t_i \, v_i(x_i) : \ x_0, \dots, x_k \in \mathbb{R}^n, \ x = \sum_{i=0}^k t_i \, x_i \right\}, \quad x \in \mathbb{R}^n.$$

Clearly, the weighted infimal convolution $(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}$ has finiteness domain

$$\mathrm{Dom}((v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}) = \Omega_{\boldsymbol{t}}.$$

We say that the infimal convolution $(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}$ is *exact* at a point $x \in \Omega_{\boldsymbol{t}}$, if the above infimum is attained.

The next result is inspired from [1, Propositions 1 and 4]. Given a family of continuous functions bounded from below, it provides a key information on the subjets of their weighted infimal convolution, provided the latter is exact.

Proposition 8. *Let* $(\Omega_0, \dots, \Omega_k) \in \mathcal{O}_k$ *and* $\boldsymbol{t} = (t_0, \dots, t_k) \in \Lambda_k^+$. *Let* $v_i : \Omega_i \to \mathbb{R}$ *be continuous functions bounded from below, and assume that* $(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}$ *is exact at* $x \in \Omega_{\boldsymbol{t}}$, *with*

$$(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}(x) = \sum_{i=0}^k t_i \, v_i(x_i), \quad x = \sum_{i=0}^k t_i \, x_i, \ x_i \in \Omega_i \ \forall i = 0, \dots, k. \tag{9}$$

Then, for a given pair $(\xi, A) \in J^{2-}(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}(x)$, and for every $\varepsilon > 0$, there exist $A_0, \ldots, A_k \in S^n$ such that $(\xi, A_i) \in \overline{J}^{2-} v_i(x_i)$, $i = 0, \ldots, k$, and

$$A - \varepsilon A^2 \le \sum_{i=0}^{k} t_i A_i. \tag{10}$$

If, in addition, $A \ge 0$, and ε is small enough, then $A_i \ge 0$ for every $i = 0, \ldots, k$ and

$$A - \varepsilon A^2 \le \left(t_0 A_0^{-1} + \ldots + t_k A_k^{-1} \right)^{-1}. \tag{11}$$

Remark 9. The above result (and its proof) is quite similar to Proposition 1 in [1]. For completeness, we give the proof in some detail, since we are going to exploit inequality (10), which is not explicitly given in [1].

Proof. To simplify the notation, let us denote $w := (v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}$. Let $\varphi \in C^2(\Omega_{\boldsymbol{t}})$ be a test function such that $\varphi \prec_x w$. Let $(y_0, \ldots, y_k) \in \Omega_0 \times \cdots \times \Omega_k$. By the definition of w, the fact that $(w - \varphi)(y) \ge (w - \varphi)(x)$ for every $y \in \Omega_{\boldsymbol{t}}$, and since $(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}$ is exact at x, we have that

$$
\begin{aligned}
\sum_{i=0}^{k} t_i v_i(y_i) - \varphi \left(\sum_{i=0}^{k} t_i\, y_i \right) &\ge w \left(\sum_{i=0}^{k} t_i\, y_i \right) - \varphi \left(\sum_{i=0}^{k} t_i\, y_i \right) \\
&\ge w \left(\sum_{i=0}^{k} t_i\, x_i \right) - \varphi \left(\sum_{i=0}^{k} t_i\, x_i \right) \\
&= \sum_{i=0}^{k} t_i\, v_i(x_i) - \varphi \left(\sum_{i=0}^{k} t_i\, x_i \right).
\end{aligned}
$$

In other words, the point $(x_0, \ldots, x_k)$ where the infimum in (9) is attained turns out to be a minimum point for the function

$$\Omega_0 \times \cdots \times \Omega_k \ni (y_0, \ldots, y_k) \mapsto \sum_{i=0}^{k} t_i v_i(y_i) - \varphi \left(\sum_{i=0}^{k} t_i\, y_i \right).$$

Then, by [35, Theorem 3.2], for every $\varepsilon > 0$ there exist $A_0, \ldots, A_k \in S^n$ such that $(\xi, A_i) \in \overline{J}^{2-} v_i(x_i)$, $i = 0, \ldots, k$, and

$$
\begin{pmatrix}
t_0 A_0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & t_k A_k
\end{pmatrix}
\ge
\begin{pmatrix}
t_0^2 B & \cdots & t_0 t_k B \\
\vdots & \ddots & \vdots \\
t_0 t_k B & \cdots & t_k^2 B
\end{pmatrix}
\tag{12}
$$

with $B := A - \varepsilon A^2$.

The inequality in (10) follows by testing (12) with a vector of the form $(h, \dots, h) \in (\mathbb{R}^n)^k$.

Moreover, by testing (12) with vectors of the form $(0, \dots, h_i, \dots, 0)$, we get the inequalities

$$t_i(A - \varepsilon A^2) \le A_i, \qquad \forall i = 0, \dots, k, \tag{13}$$

whereas, testing (12) with an arbitrary vector $(h_0, \dots, h_k)$, we see that

$$\langle B\,h,\, h \rangle \le \sum_{i=0}^{k} t_i \, \langle A_i\, h_i,\, h_i \rangle, \qquad \text{with } h := \sum_{i=0}^{k} t_i\, h_i. \tag{14}$$

Assume now that $A \ge 0$, and choose $\varepsilon > 0$ so that $I > \varepsilon A$, and hence $B \ge 0$. From (13), we see that $A_i \ge 0$ for every i. In fact, it is not restrictive to assume that A_i are positive definite, since the case of degenerate matrices can be handled as in [1], p. 273.

Finally, minimizing the right-hand side of (14) under the constraint $\sum_{i=0}^{k} t_i\, h_i = h$ leads to (11). $\square$

In order to be able to apply Proposition 8, we complement it with the following statement, which provides sufficient conditions for the weighted convolution to be exact.

Proposition 10. *Let $(\Omega_0, \dots, \Omega_k) \in \mathcal{O}_k$ and $\boldsymbol{t} = (t_0, \dots, t_k) \in \Lambda_k^+$. Let $v_i : \Omega_i \to \mathbb{R}$ be continuous functions bounded from below, with*

$$v_i \to +\infty \quad \text{as } x \to \partial\Omega_i, \quad \forall i = 1 \dots, k. \tag{15}$$

Then the weighted infimal convolution $(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}$ is continuous and exact at every point $x \in \Omega_{\boldsymbol{t}}$. Moreover, it holds

$$(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}} \to +\infty \quad \text{as } x \to \partial\Omega_{\boldsymbol{t}}. \tag{16}$$

Proof. For the continuity of the weighted infimal convolution and the fact that it is exact, we refer to [69], Theorem 2.5 and Corollary 2.1. In order to prove the last part of the statement, let us consider a sequence of points $x^n \to \partial\Omega_{\boldsymbol{t}}$ as $n \to +\infty$. Since the weighted infimal convolution is exact, there exist sequences x_i^n, $i = 0, \dots, k$, such that

$$(v_0 \sharp \cdots \sharp v_k)_{\boldsymbol{t}}(x^n) = \sum_{i=0}^{k} t_i\, v_i(x_i^n), \quad x^n = \sum_{i=0}^{k} t_i\, x_i^n, \quad x_i^n \in \Omega_i \ \forall i = 0, \dots, k.$$

We claim that $x_i^n \to \partial\Omega_i$ as $n \to +\infty$, $\forall i = 1, \dots, k$. Once proved the claim, the required property (16) follows at once from (15) and the assumptions that the functions v_i's are bounded from below.

To show the claim it is enough to observe that, if $\text{dist}(x, \partial\Omega_{\boldsymbol{t}}) < \delta$, then $\text{dist}(x_i, \partial\Omega_i) < \delta/t_i$ for every $i = 0, \ldots, k$. Indeed, if we assume by contradiction that there exists j such that $\text{dist}(x_j, \partial\Omega_j) \geq \delta/t_j$, then $B_{t_j^{-1}\delta}(x_j) \subset \Omega_j$. By (8), this implies $B_\delta(x) \subset \Omega_{\boldsymbol{t}}$, contradiction. □

2.3. The modified equation

In view of Proposition 10, it is convenient to look at the equation satisfied by minus the logarithm of viscosity eigenfunctions, so to deal with functions which diverge on the boundary. To that aim, let us introduce the operator G associated with F by

$$G(\xi, X) := -F(\xi, \xi \otimes \xi - X), \tag{17}$$

and let us consider the modified equation

$$G(\nabla v, D^2 v) = -\overline{\lambda}(\Omega) \quad \text{in } \Omega. \tag{18}$$

Remark 11. If F satisfies (H2) (resp. (H2)'), the same holds for G. Moreover, if F satisfies (H3), namely F is convex in X, then G is concave in X.

Remark 12. Similarly as in Section 2, also viscosity sub- and super-solutions to (18) can be intended either à la Crandall-Ishii-Lions or à la Birindelli-Demengel, namely according to Definition 4 or to Definition 5. Thanks to Lemma 6, the two notions are equivalent. Note in particular that, since the right–hand side of (18) is negative, for super-solutions the "either" condition in Definition 5 is automatically satisfied.

Lemma 13. *Assume that F satisfies (H1)-(H2)', and let G be defined by (17). Then a function u is a positive viscosity sub-solution to*

$$F(\nabla u, D^2 u) = \overline{\lambda}(\Omega) u^{\alpha+1} \quad \text{in } \Omega$$

if and only if the function $v = -\log u$ is a viscosity super-solution to

$$G(\nabla v, D^2 v) = -\overline{\lambda}(\Omega) \quad \text{in } \Omega.$$

Proof. Let us give the proof working with solutions à la Crandall-Ishii-Lions. We observe that $u \prec_x \varphi$ if and only if $\psi := -\log \varphi \prec_x v$, and that the inequality $F_*(\nabla\varphi, D^2\varphi) \leq \overline{\lambda}(\Omega)\varphi^{\alpha+1}$ can be rewritten as

$$F_*\big(-e^{-\psi}\nabla\psi, e^{-\psi}(\nabla\psi \otimes \nabla\psi - D^2\psi)\big) \leq \overline{\lambda}(\Omega)e^{-(\alpha+1)\psi}.$$

By (H1), this amounts to

$$F_*\big(\nabla\psi, \nabla\psi \otimes \nabla\psi - D^2\psi\big) \leq \overline{\lambda}(\Omega).$$

The required equivalence follows by observing that

$$G^*(\xi, X) = [-F(\xi, \xi \otimes \xi - X)]^* = -F_*(\xi, \xi \otimes \xi - X). \quad \square$$

We are finally in a position to give the main brick for the proof of Theorem 1:

Proposition 14. *Assume that F satisfies $F_*(0,0) = 0$, (H2)' and (H3), and let G be defined by (17). Let $(\Omega_0, \ldots, \Omega_k) \in \mathcal{O}_k$, and $\mathbf{t} = (t_0, \ldots, t_k) \in \Lambda_k^+$. Let $v_0, \ldots, v_k$ be continuous functions bounded from below which are viscosity super-solutions to*

$$\begin{cases} G(\nabla v_i, D^2 v_i) = -\overline{\lambda}(\Omega_i) & in \ \Omega_i, \\ v_i \to +\infty & on \ \partial\Omega_i. \end{cases}$$

Then $w = (v_0 \sharp \cdots \sharp v_k)_{\mathbf{t}}$ is a viscosity super-solution to

$$\begin{cases} G(\nabla w, D^2 w) = -\sum_{i=0}^k t_i \overline{\lambda}(\Omega_i) & in \ \Omega_{\mathbf{t}}, \\ w \to +\infty & on \ \partial\Omega_{\mathbf{t}}. \end{cases}$$

Proof. From Proposition 10, we know that w is continuous, exact, and satisfies $w \to +\infty$ as $x \to \partial\Omega_{\mathbf{t}}$. In order to check that w is a viscosity super-solution to $G(\nabla w, D^2 w) = -\sum_{i=0}^k t_i \overline{\lambda}(\Omega_i)$ in $\Omega_{\mathbf{t}}$, we use the definition à la Birindelli-Demengel. Let $x \in \Omega_{\mathbf{t}}$. If w is constant on a ball centered at x, we have nothing to check. Otherwise, let $(\xi, A) \in J^{2-} w(x)$, with $\xi \neq 0$. Let $x_i \in \Omega_i$ be such that (9) holds. By Proposition 8, there exist $A_0, \ldots, A_k \in S^n$ such that $(\xi, A_i) \in \overline{J}^{2-} v_i(x_i)$, satisfying (10). Hence,

$$G(\xi, A - \varepsilon A^2) \geq G\left(\xi, \sum_{i=0}^k t_i A_i\right) \geq \sum_{i=0}^k t_i G(\xi, A_i) \geq -\sum_{i=0}^k t_i \overline{\lambda}(\Omega_i),$$

where in the first inequality we have used the fact that G is degenerate elliptic, in the second one the fact that it is concave in X (*cf.* Remark 11), and in the third one the fact that the v_i's are super-solutions to $G(\nabla v_i, D^2 v_i) = -\overline{\lambda}(\Omega_i)$.

Passing to the limit as $\varepsilon \to 0$ we conclude that $G(\xi, A) \geq -\sum_{i=0}^k t_i \overline{\lambda}(\Omega_i)$. $\quad \square$

Remark 15. We warn the reader that the above proof cannot be successfully concluded if one adopts the definition of viscosity super-solution à la Crandall-Ishii-Lions. Indeed, in this case, one would need to use the concavity of the upper semicontinuous envelope G^*. But, in general, the concavity of G is not inherited by G^* (for instance, in case of the normalized p-Laplacian, one can easily check that G^* fails to be concave). This sheds some light on the importance of the equivalence Lemma 6.

3. Existence of eigenfunctions for domains in $\mathcal{A}^n$

In this section we prove the existence of eigenfunctions for operators F satisfying assumptions (H1)-(H2) on domains belonging to the class $\mathcal{A}^n$ defined in (4) (see Theorem 19), along with their global Hölder continuity (see Proposition 17). We remark that the restriction $\alpha > -1$ in (H1) is fundamental for the proof of Lemma 16 below, and hence also for the subsequent results. For domains of class C^2, the corresponding results have been proved in [12, Theorem 5.5 and 4.1] (see also [14, Theorem 8 and Proposition 6]).

We recall that, for any Lipschitz domain Ω, denoting by d_Ω the distance function from the boundary

$$d_\Omega(x) := \min_{y \in \partial\Omega} |y - x|, \qquad x \in \mathbb{R}^n,$$

the following properties are equivalent (see e.g. [29,34,36]):

(a) $\Omega \in \mathcal{A}^n$;
(b) there exists $r > 0$ such that the distance function d_Ω is differentiable at any point of the exterior tubular neighborhood

$$\mathcal{N}_r := \{x \in \mathbb{R}^n \setminus \Omega : \ 0 < d_\Omega(x) < r\};$$

(c) Ω is a set of *positive reach*, i.e. there exists $r > 0$ such that every point $x \in \mathcal{N}_r$ admits a unique projection on $\overline{\Omega}$.

These properties are clearly satisfied if Ω is of class C^2 or if Ω is a convex set.

Let us also recall that, if $\Omega \in \mathcal{A}^n$, the distance function d_Ω is semiconcave in $\overline{\Omega}$, i.e. there exists a constant $\kappa > 0$ such that the map $x \mapsto d_\Omega(x) - \frac{\kappa}{2}|x|^2$ is concave in $\overline{\Omega}$ (see [28, Proposition 2.2(iii)]). The constant κ is called a semiconcavity constant for d_Ω, and can be chosen equal to the reciprocal of the radius in the uniform external sphere condition.

As a consequence of the semiconcavity of d_Ω, for any Ω in $\mathcal{A}^n$ and any function f bounded in $\overline{\Omega}$, we are able to construct a barrier for sub-solutions to

$$\begin{cases} F(\nabla u, D^2 u) = f(x), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{19}$$

We prove:

Lemma 16. *Let $\Omega \in \mathcal{A}^n$, let F satisfy (H1)-(H2), and let f be a bounded function in $\overline{\Omega}$. Then, for every upper semicontinuous sub-solution u of* (19) *and every $\gamma \in (0, 1)$, there*

exist constants $M, \delta > 0$, depending only on the semiconcavity constant of d_Ω and on the structural constants of F, such that

$$u(x) \leq M \, d_\Omega(x)^\gamma, \qquad \forall x \in \overline{\Omega} \text{ such that } d_\Omega(x) \leq \delta.$$

Proof. Throughout the proof we write for brevity d in place of d_Ω. If $\kappa > 0$ is a semi-concavity constant for d, since the map $x \mapsto d(x) - \frac{\kappa}{2}|x|^2$ is concave in $\overline{\Omega}$, we have

$$x \in \Omega, \quad (\xi, A) \in J^{2,-}d(x) \qquad \Longrightarrow \qquad \nabla d(x) = \xi, \ A \leq \kappa I. \qquad (20)$$

Let $\gamma \in (0,1)$ be fixed, and let us consider the function $g(x) := M \, d(x)^\gamma$, where $M > 0$ is a constant that will be determined later. For every $x \in \Omega$, by (20) we have that $(\xi, A) \in J^{2,-}d(x)$ if and only if $(\zeta, X) \in J^{2,-}g(x)$, with

$$\zeta = M\gamma d(x)^{\gamma-1}\xi, \quad X = M\gamma d(x)^{\gamma-2} \left[(d(x) A + (\gamma - 1)\xi \otimes \xi \right], \quad A \leq \kappa I, \quad |\xi| = 1,$$

and, in this case, both d and g are differentiable at x, with $\nabla d(x) = \xi$ and $\nabla g(x) = \zeta \neq 0$.

Hence, if $x \in \Omega$ and $(\zeta, X) \in J^{2,-}g(x)$, from (H1)-(H2) it holds

$$\begin{aligned}
F(\zeta, X) &= (M \gamma)^{\alpha+1} d(x)^{(\alpha+1)\gamma-\alpha-2} \, F(\xi, d(x)A + (\gamma - 1)\xi \otimes \xi) \\
&\geq (M \gamma)^{\alpha+1} d(x)^{(\alpha+1)\gamma-\alpha-2} \, F(\xi, \kappa \, d(x)\, I - (1 - \gamma)\xi \otimes \xi) \\
&\geq (M \gamma)^{\alpha+1} d(x)^{(\alpha+1)\gamma-\alpha-2} \left[c(1 - \gamma) - C \, n \, \kappa \, d(x) \right],
\end{aligned}$$

where in the last inequality we have used the fact that $|\xi| = 1$.

Since the exponent $[(\alpha + 1)\gamma - \alpha - 2]$ is negative, if we choose $\delta < c(1 - \gamma)/(C \, n \, \kappa)$, we conclude that there exists $\varepsilon > 0$, depending only on γ and κ (and on the structural constants of F), such that

$$F(\zeta, X) \geq M^{\alpha+1}\varepsilon, \qquad \forall \, (\zeta, X) \in J^{2,-}g(x), \text{ with } x \in \Omega, \ d(x) \leq \delta.$$

In other words, g is a positive supersolution of the equation

$$F(\nabla g, D^2 g) \geq M^{\alpha+1}\varepsilon \qquad \text{in } \Omega_\delta := \{x \in \Omega : \ d(x) < \delta\}.$$

Finally, we can now choose

$$M := \max \left\{ \delta^{-\gamma} \max_{x \in \overline{\Omega}_\delta} u \,, \ \left(\frac{\|f\|_\infty}{\varepsilon} \right)^{\frac{1}{\alpha+1}} + 1 \right\},$$

so that $g \geq u$ on $\partial \Omega_\delta$ and $|f(x)| < M^{\alpha+1}\varepsilon$ for every $x \in \overline{\Omega}_\delta$, hence the claim follows from the comparison result proved in [12, Theorem 3.6]. $\square$

We can now derive a global Hölder estimate:

Proposition 17. *Let $\Omega \in \mathcal{A}^n$, let F satisfy (H1)-(H2), let f be a bounded function in $\overline{\Omega}$, and let u be a non-negative bounded viscosity solution of* (19).

Then, for every $\gamma \in (0,1)$ there exists a constant $H > 0$, depending only on γ, $\|f\|_\infty$ and the semiconcavity constant of d_Ω, such that

$$|u(x) - u(y)| \leq H\,|x - y|^\gamma, \qquad \forall x, y \in \overline{\Omega}.$$

Proof. Thanks to Lemma 16, the result can be obtained following line by line the proof of Proposition 6 in [14] (see also [12, Theorem 4.1]). $\quad\square$

Remark 18. As a consequence of the global Hölder estimate given in Proposition 17, it is possible to obtain also a local Lipschitz regularity result. More precisely, under the hypotheses of Proposition 17, assume in addition that F satisfies the following Hölder continuity assumption with respect to $\xi \neq 0$: there exist $\mu \in (1/2, 1]$ and $K > 0$ such that

$$|F(\xi + \zeta, X) - F(\xi, X)| \leq K|\zeta|^\mu|X|, \qquad \forall\,|\xi| = 1,\ |\zeta| < 1/2,\ X \in S^n\,.$$

Then, by arguing as in Theorem 4.2 of [12], one can see that every non-negative bounded viscosity solution of (19) is locally Lipschitz continuous in Ω.

Finally, thanks to Proposition 17 we are in a position to give

Theorem 19. *Let $\Omega \in \mathcal{A}^n$ and let F satisfy (H1)-(H2). Then for $\lambda = \overline{\lambda}(\Omega)$ the eigenvalue problem* (3) *admits a positive viscosity solution u. Moreover, u can be obtained as the uniform limit of a sequence of positive eigenfunctions $\{u_k\}$, associated with an increasing sequence of smooth domains $\{\Omega_k\}$ such that*

$$\bigcup_k \Omega_k = \Omega, \qquad \lim_{k \to +\infty} \overline{\lambda}(\Omega_k) = \overline{\lambda}(\Omega)\,.$$

Proof. Since Ω satisfies a uniform exterior sphere condition, we can construct a sequence of smooth (C^∞) domains $\{\Omega_k\}$, still satisfying a uniform sphere condition (possibly with a smaller radius r, independent of k), such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\bigcup \Omega_k = \Omega$. This can be achieved by a standard regularization argument, i.e. by mollifying the function whose graph locally defines the boundary of Ω.

For every $k \in \mathbb{N}$, let now u_k be a positive eigenfunction in Ω_k, normalized by $\|u_k\|_\infty = 1$, and let us extend it in $\overline{\Omega}$ by setting $u_k = 0$ in $\overline{\Omega} \setminus \Omega_k$.

Let us fix $\gamma \in (0,1)$. By Proposition 17, there exists a constant $H > 0$, depending only on r, such that

$$|u_k(x) - u_k(y)| \leq H|x - y|^\gamma, \qquad \forall x, y \in \overline{\Omega}, \quad \forall k \in \mathbb{N}.$$

Hence, by the Ascoli–Arzelà theorem, from $\{u_k\}$ we can extract a subsequence that converges uniformly in $\overline{\Omega}$ to some continuous function u. Moreover, by monotonicity, the sequence $\overline{\lambda}(\Omega_k)$ converges decreasingly to some limit L. Thus the function u is a non-negative viscosity solution to the equation $F(\nabla u, D^2 u) = L|u|^\alpha u$ in Ω. Since $u \geq 0$ and $u \not\equiv 0$, by the strict maximum principle proved in [14, Theorem 2] we deduce that $u > 0$ in Ω. By definition of $\overline{\lambda}(\Omega)$, this gives the inequality $\overline{\lambda}(\Omega) \geq L$. On the other hand, since $\Omega_k \subset \Omega$, we have $\overline{\lambda}(\Omega) \leq \overline{\lambda}(\Omega_k)$ and hence in the limit $\overline{\lambda}(\Omega) \leq L$, so that u is a positive eigenfunction associated with $\overline{\lambda}(\Omega)$. $\square$

4. Proofs of Theorems 1 and 2

4.1. Proof of Theorem 1

First of all we observe that it is enough to prove the inequality

$$\overline{\lambda}\big((1-t)\Omega_0 + t\Omega_1\big) \leq (1-t)\overline{\lambda}(\Omega_0) + t\,\overline{\lambda}(\Omega_1), \qquad \forall t \in [0,1]. \tag{21}$$

Indeed, by a standard argument, the Brunn–Minkowski inequality (5) follows from (21) and the fact that

$$\overline{\lambda}(k\,\Omega) = \frac{1}{k^{\alpha+2}}\,\overline{\lambda}(\Omega), \qquad \forall k > 0.$$

Specifically, it is enough to apply (21) with

$$t' = \frac{t\,\overline{\lambda}(\Omega_1)^{-1/(\alpha+2)}}{(1-t)\,\overline{\lambda}(\Omega_0)^{-1/(\alpha+2)} + t\,\overline{\lambda}(\Omega_1)^{-1/(\alpha+2)}}, \qquad \Omega_i' = \overline{\lambda}(\Omega_i)^{1/(\alpha+2)}\,\Omega_i,\ i = 0,1.$$

Let us prove (21). For $i = 0,1$, thanks to Theorem 19, there exists a positive eigenfunction u_i associated with $\overline{\lambda}(\Omega_i)$, i.e. a positive function in $C(\overline{\Omega}_i)$ which is a viscosity solution to

$$\begin{cases} F(\nabla u_i, D^2 u_i) = \overline{\lambda}(\Omega_i)\,u_i^{\alpha+1} & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i. \end{cases}$$

By Lemma 13, for $i = 0,1$, the function $v_i := -\log u_i$ is a viscosity super-solution of

$$\begin{cases} G(\nabla v_i, D^2 v_i) = -\overline{\lambda}(\Omega_i) & \text{in } \Omega_i, \\ v_i \to +\infty & \text{on } \partial\Omega_i, \end{cases}$$

where G is the function defined in (17).

Let $w\colon \Omega_{\boldsymbol{t}} \to \mathbb{R}$ be the infimal convolution of v_0, v_1 with coefficients $\boldsymbol{t} = (1-t,t)$, in $\Omega_{\boldsymbol{t}} = (1-t)\Omega_0 + t\Omega_1$, i.e.,

$$w(x) := \inf\left\{(1-t)v_0(x_0) + t\, v_1(x_1):\ x_0 \in \Omega_0,\ x_1 \in \Omega_1,\ x = (1-t)x_0 + t\, x_1\right\}.$$

By Proposition 14, w is a viscosity super-solution to

$$\begin{cases} G(\nabla w, D^2 w) = -\left[(1-t)\overline{\lambda}(\Omega_0) + t\,\overline{\lambda}(\Omega_1)\right] & \text{in } \Omega_t, \\ w \to +\infty & \text{on } \partial\Omega_t\,. \end{cases}$$

Let us define $\overline{u}\colon \overline{\Omega}_t \to \mathbb{R}$ as $\overline{u}(x) := e^{-w(x)}$ for every $x \in \Omega_t$, $\overline{u}(x) = 0$ for every $x \in \partial\Omega_t$. Clearly, $\overline{u} > 0$ in Ω_t and $\overline{u} \in C(\overline{\Omega}_t)$.

Moreover, applying again Lemma 13, we infer that $\overline{u}$ is a viscosity sub-solution to

$$F(\nabla\overline{u}, D^2\overline{u}) = \left[(1-t)\overline{\lambda}(\Omega_0) + t\,\overline{\lambda}(\Omega_1)\right]\overline{u}^{\alpha+1} \qquad \text{in } \Omega_t.$$

Since $\overline{u} > 0$ in Ω_t and $\overline{u} = 0$ on $\partial\Omega_t$, by Theorem 7 we conclude that (21) holds. □

4.2. Proof of Theorem 2(I)

Let $v := -\log u$. In order to prove that v is a convex function, we exploit the convex envelope method by Alvarez-Lasry-Lions. By definition, the convex envelope v_{**} of v satisfies $v_{**} \leq v$. In order to show the converse inequality, we apply a comparison argument to the modified equation

$$G(\nabla v, D^2 v) = -\overline{\lambda}(\Omega) \tag{22}$$

settled on a suitable level set $\Omega_\varepsilon := \{u > \varepsilon\}$. To be more precise, the comparison principle given in [59, Theorem 1.3] ensures that the inequality $v_{**} \geq v$ in Ω_ε holds true in Ω_ε (and hence in the limit as $\varepsilon \to 0^+$ also in Ω), provided the following two properties hold true:

(a) v_{**} is a viscosity super-solution to (22) in Ω_ε;
(b) $v_{**} = v$ on $\partial\Omega_\varepsilon$.

We point out that we cannot take $\varepsilon = 0$ (namely work directly on Ω) because $v \to +\infty$ on $\partial\Omega$. We also stress that the assumption (H3)' intervenes in the proof of item (b) given below, and this is the reason why the statement cannot be proved under the weaker condition $X \mapsto F(\xi, X^{-1})$ convex appearing in [1].

Proof of (a). Let us show that v_{**} is a viscosity super-solution to (22) in the whole Ω. We observe that

$$v_{**} = \min\left\{(v\sharp\cdots\sharp v)_t\ :\ t \in \bigcup_{k\leq(n+1)} \Lambda_k^+\right\}.$$

Thus, for some $t \in \Lambda_k^+$ (depending on x), we have

$$v_{**}(x) = (v \sharp \cdots \sharp v)_{\boldsymbol{t}}(x),$$

and hence, by applying Proposition 14 (with $\Omega_i = \Omega$ and $v_i = v$ for every i), we conclude that v_{**} is a super-solution to (22). (As well, one could apply here Proposition 3 in [1]).

Proof of (b). By Lemma 4 in [1], the required equality $v_{**} = v$ on $\partial \Omega_\varepsilon$ is satisfied provided the level set Ω_ε is *convex*. We are thus reduced to prove the convexity of Ω_ε for ε small enough.

We start by noticing that, by [15, Proposition 3.5], v belongs to $C^{1,\beta}(\overline{\Omega})$ (for some $\beta \in (0,1)$). Combined with the Hopf boundary point principle given in [14, Corollary 1], this ensures that $|\nabla v| \geq \alpha > 0$ in $\overline{N}$, where N is an inner neighborhood of $\partial \Omega$. This fact, and the strong convexity assumption made on Ω, enable us to apply Lemma 2.4 in [56] (see also [64, Proposition 3.2]) to infer that the required convexity property of Ω_ε is satisfied, for ε sufficiently small, as soon as we know that $u \in C^2(\overline{N})$.

The latter property follows by standard elliptic regularity, in particular thanks to the convexity hypothesis made on F and to the condition $\partial \Omega \in C^{2,\beta}$. So we limit ourselves to give adequate references, along with a few additional comments. By the convexity of F, we can apply the method of continuity as done for instance in the proof of Theorem 9.7 in [25]. There is just one point where we need to be careful when following the proof of Theorem 9.7 in [25]: since F depends also on ξ, we cannot exploit the a priori estimates used therein (which are those given in Theorem 9.5 in [25]). In place, we can invoke the a priori estimates given in [41, Theorem 17.26]. These estimates are stated actually for more regular solutions, but this is not restrictive thanks to classical Schauder estimates, which hold in particular by the $C^{2,\beta}$ regularity of $\partial \Omega$ (see [41, Section 6.4]). The relevant point is that the estimates in [41, Theorem 17.26] continue to hold for $F = F(\xi, X)$, and enable us to conclude along the proof line of [25, Theorem 9.7]. As a drawback, we have to ask the convexity condition in the reinforced form (H3)', which is needed precisely to ensure the validity of condition (17.85) in [41]. $\quad \square$

4.3. Proof of Theorem 2(ii)

Let $\Omega \in \mathcal{A}^n$, and let $\{u_k\}$ be the approximating sequence given by Theorem 19. We remark that the approximating smooth sets $\{\Omega_k\}$ can be chosen to be strongly convex. Since, by Theorem 2(i), every function u_k is log-concave, then also their uniform limit u is a log-concave positive eigenfunction. $\quad \square$

Acknowledgments

References

[1] O. Alvarez, J. Lasry, P. Lions, Convex viscosity solutions and state constraints, J. Math. Pures Appl. (9) 76 (3) (1997) 265–288. MR1441987 (98k:35045).

[2] S. Armstrong, Principal eigenvalues and an anti-maximum principle for homogeneous fully nonlinear elliptic equations, J. Differential Equations 246 (7) (2009) 2958–2987.

[3] A. Attouchi, E. Ruosteenoja, Remarks on regularity for p-Laplacian type equations in non-divergence form, J. Differential Equations 265 (5) (2018) 1922–1961.

[4] K. Ball, An elementary introduction to monotone transportation, in: Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 1850, Springer, Berlin, 2004, pp. 41–52.

[5] A. Banerjee, B. Kawohl, Overdetermined problems for the normalized p-Laplacian, Proc. Amer. Math. Soc. Ser. B 5 (2018) 18–24.

[6] H. Berestycki, I. Capuzzo Dolcetta, A. Porretta, L. Rossi, Maximum principle and generalized principal eigenvalue for degenerate elliptic operators, J. Math. Pures Appl. (9) 103 (5) (2015) 1276–1293.

[7] H. Berestycki, L. Nirenberg, S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Comm. Pure Appl. Math. 47 (1) (1994) 47–92.

[8] H. Berestycki, L. Rossi, Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains, Comm. Pure Appl. Math. 68 (6) (2015) 1014–1065.

[9] C. Bianchini, P. Salani, Concavity properties for elliptic free boundary problems, Nonlinear Anal. 71 (10) (2009) 4461–4470.

[10] M. Bianchini, P. Salani, Power concavity for solutions of nonlinear elliptic problems in convex domains, in: Geometric Properties for Parabolic and Elliptic PDE's, in: Springer INdAM Ser., vol. 2, Springer, Milan, 2013, pp. 35–48.

[11] I. Birindelli, F. Demengel, Comparison principle and Liouville type results for singular fully nonlinear operators, Ann. Fac. Sci. Toulouse Math. (6) 13 (2) (2004) 261–287. MR2126744.

[12] I. Birindelli, F. Demengel, First eigenvalue and maximum principle for fully nonlinear singular operators, Adv. Differential Equations 11 (1) (2006) 91–119.

[13] I. Birindelli, F. Demengel, The Dirichlet problem for singular fully nonlinear operators, Discrete Contin. Dyn. Syst. (2007) 110–121.

[14] I. Birindelli, F. Demengel, Eigenvalue, maximum principle and regularity for fully non linear homogeneous operators, Commun. Pure Appl. Anal. 6 (2) (2007) 335–366. MR2289825.

[15] I. Birindelli, F. Demengel, Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators, J. Differential Equations 249 (5) (2010) 1089–1110. MR2652165.

[16] W. Blaschke, Kreis und Kugel, Walter de Gruyter & Co., Berlin, 1956, 2te Aufl.

[17] T. Bonnesen, Les problèmes des isopérimètres et des iséphiphanes, Gauthiers–Villars, Paris, 1929.

[18] C. Borell, Capacitary inequalities of the Brunn-Minkowski type, Math. Ann. 263 (2) (1983) 179–184.

[19] C. Borell, Hitting probabilities of killed Brownian motion: a study on geometric regularity, Ann. Sci. Éc. Norm. Supér. (4) 17 (3) (1984) 451–467.

[20] C. Borell, Greenian potentials and concavity, Math. Ann. 272 (1) (1985) 155–160.

[21] H. Brascamp, E. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, J. Funct. Anal. 22 (4) (1976) 366–389.

[22] H. Brunn, Über curven ohne wendepunkte, Habilitationsschrift, Munchen, 1887.

[23] H. Brunn, Über ovale und eiflachen, Dissertation, Munchen, 1887.

[24] J. Busca, M. Esteban, A. Quaas, Nonlinear eigenvalues and bifurcation problems for Pucci's operators, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2) (2005) 187–206.

[25] L. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995.

[26] L. Caffarelli, D. Jerison, E. Lieb, On the case of equality in the Brunn-Minkowski inequality for capacity, Adv. Math. 117 (2) (1996) 193–207.

[27] L. Caffarelli, J. Spruck, Convexity properties of solutions to some classical variational problems, Comm. Partial Differential Equations 7 (11) (1982) 1337–1379.

[28] P. Cannarsa, C. Sinestrari, Semiconcave Functions, Hamilton-Jacobi Equations and Optimal Control, Progress in Nonlinear Differential Equations and Their Applications, vol. 58, Birkhäuser, Boston, 2004.

[29] F. Clarke, R. Stern, P. Wolenski, Proximal smoothness and the lower-C^2 property, J. Convex Anal. 2 (1995) 117–144.

[30] A. Colesanti, Brunn-Minkowski inequalities for variational functionals and related problems, Adv. Math. 194 (1) (2005) 105–140. MR2141856.

[31] A. Colesanti, P. Cuoghi, The Brunn-Minkowski inequality for the n-dimensional logarithmic capacity of convex bodies, Potential Anal. 22 (3) (2005) 289–304.

[32] A. Colesanti, P. Cuoghi, P. Salani, Brunn-Minkowski inequalities for two functionals involving the p-Laplace operator, Appl. Anal. 85 (2006).

[33] A. Colesanti, P. Salani, The Brunn-Minkowski inequality for p-capacity of convex bodies, Math. Ann. 327 (3) (2003) 459–479.

[34] G. Colombo, L. Thibault, Prox-regular sets and applications, in: Handbook of Nonconvex Analysis and Applications, Int. Press, Somerville, MA, 2010, pp. 99–182.

[35] M. Crandall, H. Ishii, P. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. (N.S.) 27 (1992) 1–67.

[36] G. Crasta, I. Fragalà, On the characterization of some classes of proximally smooth sets, ESAIM Control Optim. Calc. Var. 22 (3) (2016) 710–727.

[37] G. Crasta, I. Fragalà, B. Kawohl, On the first eigenvalue of the normalized p-Laplacian, Proc. Amer. Math. Soc. (2019), in press, arXiv:1811.10024.

[38] G. Dávila, P. Felmer, A. Quaas, Harnack inequality for singular fully nonlinear operators and some existence results, Calc. Var. Partial Differential Equations 39 (3–4) (2010) 557–578.

[39] A. Dinghas, Über eine klasse superadditiver mengenfunktionale von Brunn–Minkowski–Lusternikschem typus, Math. Z. 69 (1957) 111–125.

[40] R. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. (N.S.) 39 (3) (2002) 355–405.

[41] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1977.

[42] H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer, Berlin, 1957.

[43] H. Hadwiger, D. Ohmann, Brunn-Minkowskischer Satz und Isoperimetrie, Math. Z. 66 (1956) 1–8.

[44] D. Hilbert, Minkowskis theorie von volumen und oberfläche, Nachr. Ges. Wiss. Göttingen, 1910.

[45] N. Ikoma, H. Ishii, Eigenvalue problem for fully nonlinear second-order elliptic PDE on balls, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (5) (2012) 783–812.

[46] N. Ikoma, H. Ishii, Eigenvalue problem for fully nonlinear second-order elliptic PDE on balls, II, Bull. Math. Sci. 5 (3) (2015) 451–510.

[47] K. Ishige, P. Salani, Parabolic Minkowski convolutions of solutions to parabolic boundary value problems, Adv. Math. 287 (2016) 640–673.

[48] P. Juutinen, Principal eigenvalue of a very badly degenerate operator and applications, J. Differential Equations 236 (2) (2007) 532–550.

[49] B. Kawohl, Variational versus PDE-based approaches in mathematical image processing, in: Singularities in PDE and the Calculus of Variations, in: CRM Proc. Lecture Notes, vol. 44, Amer. Math. Soc., Providence, RI, 2008, pp. 113–126. MR2528737.

[50] B. Kawohl, Variations on the p-Laplacian, in: Nonlinear Elliptic Partial Differential Equations, in: Contemp. Math., vol. 540, Amer. Math. Soc., Providence, RI, 2011, pp. 35–46.

[51] B. Kawohl, J. Horák, On the geometry of the p-Laplacian operator, Discrete Contin. Dyn. Syst. Ser. S 10 (4) (2017) 799–813. MR3640538.

[52] B. Kawohl, S. Krömer, J. Kurtz, Radial eigenfunctions for the game-theoretic p-Laplacian on a ball, Differential Integral Equations 27 (7–8) (2014) 659–670.

[53] D. Klain, On the equality conditions of the Brunn-Minkowski theorem, Proc. Amer. Math. Soc. 139 (10) (2011) 3719–3726.

[54] H. Kneser, W. Suss, Die Volumina in Linearen Scharen Konvexer Körper, Mat. Tidsskr., B (1935) 19–25.

[55] H. Knothe, Contributions to the theory of convex bodies, Michigan Math. J. 4 (1957) 39–52.

[56] N. Korevaar, Convex solutions to nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. 32 (4) (1983) 603–614.

[57] M. Kühn, Power- and log-concavity of viscosity solutions to some elliptic Dirichlet problems, Commun. Pure Appl. Anal. 17 (6) (2018) 2773–2788.

[58] P. Liu, X. Ma, X. L., A Brunn-Minkowski inequality for the Hessian eigenvalue in three-dimensional convex domain, Adv. Math. 225 (3) (2010) 1616–1633.

[59] G. Lu, P. Wang, A uniqueness theorem for degenerate elliptic equations, in: Geometric Methods in PDE's, in: Lect. Notes Semin. Interdiscip. Mat., vol. 7, Semin. Interdiscip. Mat. (S.I.M.), Potenza, 2008, pp. 207–222. MR2605157.

[60] R. McCann, A convexity principle for interacting gases, Adv. Math. 128 (1) (1997) 153–179.

[61] P. McMullen, The Brunn–Minkowski theorem for polytopes, Rend. Circ. Mat. Palermo 2 (Suppl. 70, Part II) (2002) 139–150.
[62] H. Minkowski, Geometrie der zahlen, Teubner, Leipzig, 1910.
[63] A. Quaas, B. Sirakov, Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators, Adv. Math. 218 (1) (2008) 105–135.
[64] S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 14 (3) (1987) 403–421 (1988). MR951227 (89h:35133).
[65] P. Salani, A Brunn-Minkowski inequality for the Monge-Ampère eigenvalue, Adv. Math. 194 (1) (2005) 67–86.
[66] P. Salani, Convexity of solutions and Brunn-Minkowski inequalities for Hessian equations in $\mathbb{R}^3$, Adv. Math. 229 (3) (2012) 1924–1948.
[67] P. Salani, Combination and mean width rearrangements of solutions to elliptic equations in convex sets, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (4) (2015) 763–783. MR3390083.
[68] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge Univ. Press, Cambridge, 1993.
[69] T. Strömberg, The operation of infimal convolution, Dissertationes Math. (Rozprawy Mat.) 352 (1996) 58. MR1387951.