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The discriminant controls automorphism groups of noncommutative algebras

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We use the discriminant to determine the automorphism groups of some noncommutative algebras, and we prove that a family of noncommutative algebras has tractable automorphism groups.

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0. Introduction

There is a long history and an extensive study of the automorphism groups of algebras. Determining the full automorphism group of an algebra is generally a notoriously difficult problem. For example, the automorphism group of the polynomial ring of three variables is not yet understood, and a remarkable result in this direction is given by Shestakov and Umirbaev [17] which shows the Nagata automorphism is a wild automorphism. Since 1990s, many researchers have been successfully computing the automorphism groups of interesting infinite-dimensional noncommutative algebras, including certain quantum groups, generalized quantum Weyl algebras, skew polynomial rings and many more – see [2–4,6,11,19], which is only a partial list. Recently, by using a rigidity theorem for quantum tori, Yakimov has proved the Andruskiewitsch–Dumas conjecture and the Launois–Lenagan conjecture in [22,21], each of which determines the automorphism group of a family of quantized algebras with parameter q being not a root of unity. A uniform approach to both the Andruskiewitsch–Dumas conjecture and the Launois–Lenagan conjecture is provided in a preprint by Goodearl and Yakimov [12]. These beautiful results, as well as others, motivated us to look into the automorphism groups of noncommutative algebras.

To warm up, let us consider an explicit example. For the rest of the introduction, let k be a field and let $k^\times = k \setminus \{0\}$. For any integer $n \geq 2$, let W_n be the k-algebra generated by $\{x_1, \dots, x_n\}$, subject to the relations $x_i x_j + x_j x_i = 1$ for all $i \neq j$. The action of the symmetric group S_n on the set $\{x_1, \dots, x_n\}$ extends to an action of S_n on the algebra W_n, and the map $x_i \mapsto -x_i$ determines an algebra automorphism of W_n. Therefore $S_n \times \{\pm 1\}$ is a subgroup of the full automorphism group $\operatorname{Aut}(W_n)$ of the k-algebra W_n. We compute $\operatorname{Aut}(W_n)$ when n is even.

Theorem 1. *Assume that* $\operatorname{char} k \neq 2$. *If* $n \geq 4$ *is even, then* $\operatorname{Aut}(W_n) = S_n \times \{\pm 1\}$.

It is well-known that $\operatorname{Aut}(W_2) = S_2 \ltimes k^\times$, see [3]. If n is odd or $\operatorname{char} k = 2$, then $\operatorname{Aut}(W_n)$ is unknown and contains more automorphisms than $S_n \times \{\pm 1\}$: see Example 5.10.

Understanding the automorphism group of an algebra is fundamentally important in general, and for the algebra W_n, is the first step in the study of the invariant theory under group actions [8]. The invariant theory of W_2 was studied in [9], and [9, Theorem 0.4] applies to W_2 as W_2 is filtered Artin–Schelter regular of dimension 2. We have the following for even integers $n \geq 4$.

Theorem 2. *(See [8].) Assume that* $\operatorname{char} k \neq 2$. *Let* n *be an even integer* ≥ 4 *and* G *be any group acting on* W_n. *Then the fixed subring* W_n^G *under the* G*-action is filtered Artin–Schelter Gorenstein.*

By Theorem 2, the W_n's form a class of rings with good homological properties under any group action. The proof of Theorem 2 is heavily dependent on the structure of $\mathrm{Aut}(W_n)$.

As stated in the first sentence of [22], the automorphism group of an algebra is often difficult to describe. For an algebra with many generators, it is usually impossible to compute its automorphism group directly. This leads us to consider the following question.

Question. What invariants of an algebra control its automorphism group?

This question has been implicitly asked by many authors, for example, in the papers mentioned in the first paragraph of the introduction, and different techniques have been used in the study of automorphism groups. In this paper, we use the *discriminant*. When n is even, the discriminant of W_n over its center is a non-unit element of the center, and it is preserved by any algebra automorphism of W_n. This is how we prove Theorem 1. Unfortunately, when n is odd or when the characteristic of k is 2, the discriminant of W_n over its center is (conjecturally) trivial, whence no useful information can be derived from this invariant. This is one reason why the form of $\mathrm{Aut}(W_n)$ is dependent on the parity of n and $\mathrm{char}\, k$.

Our main theorem is an abstract version of Theorem 1. Let A be a filtered algebra with filtration $\{F_i A\}_{i \geq 0}$ such that the associated graded ring $\mathrm{gr}\, A$ is connected graded. We say an automorphism $g \in \mathrm{Aut}(A)$ is *affine* if $g(F_1 A) \subset F_1 A$. Let Af be the category of k-algebras A satisfying the following conditions:

(1) A is a filtered algebra such that the associated graded ring $\mathrm{gr}\, A$ is a domain,
(2) A is a finitely generated free module over its center R, and
(3) the discriminant $d(A/R)$ is dominating (see Definition 2.3).

The morphisms in this category are just isomorphisms of algebras. Conditions (1) and (2) are easy to understand, while the terminology in condition (3) will be defined in Sections 1 and 2. At this point we only mention that the algebras W_n are in Af when n is even and that there are algebras such that (1) and (2) hold and (3) fails [Example 5.7].

Theorem 3. *Let A be in the category* Af*. In parts* (3), (4)*, assume that* $\mathrm{char}\, k = 0$*. Let R be the center of A. Then the following hold.*

(1) *Every automorphism g of A is affine.*
(2) *Every automorphism h of the polynomial extension $A[t]$ is triangular. That is, there is a $g \in \mathrm{Aut}(A)$, $c \in k^{\times}$ and $r \in R$ such that*

$$h(t) = ct + r \quad \textit{and} \quad h(x) = g(x) \in A \quad \textit{for all } x \in A.$$

In other words,

$$\operatorname{Aut}\big(A[t]\big) = \begin{pmatrix} \operatorname{Aut}(A) & R \\ 0 & k^\times \end{pmatrix}.$$

(3) *Every locally nilpotent derivation (defined after Lemma 3.2) of A is zero.*
(4) $\operatorname{Aut}(A)$ *is an algebraic group that fits into the exact sequence*

$$1 \to (k^\times)^r \to \operatorname{Aut}(A) \to S \to 1 \tag{$*$}$$

where $r \geq 0$ and S is a finite group. In other words, $\operatorname{Aut}(A) = S \ltimes (k^\times)^r$.

If $\operatorname{char} k \neq 0$, part (3) of the above could fail, see Example 3.9. Note that parts (3), (4) are consequences of part (2) [Lemmas 3.3(2) and 3.4]. Part (3) suggests that the discriminant controls locally nilpotent derivations too. Part (4) gives a structure theorem for $\operatorname{Aut}(A)$. The integer r is called the *symmetry rank* of A, denoted by $sr(A)$; and the order $|S|$ is called the *symmetry index* of A, denoted by $si(A)$. For example, Theorem 1 says that, when $n \geq 4$ is even, $sr(W_n) = 0$ and $si(W_n) = 2n!$.

Theorem 3(1) provides a uniform approach to the automorphism groups of all algebras in Af. There are many algebras in the category Af [Section 5]. For example, if A is a PI skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$ such that (a) x_i is not in the center of A for all i and (b) A is free over its center, then A is in Af [7]. Here a PI algebra means an algebra satisfying a *polynomial identity* [15, Chapter 13]. The category Af also has the nice property that it is closed under the tensor product [Theorem 5.5].

As we will see below, the discriminant method has limitations. An immediate one is that we need to assume the existence of a "good" trace function, and this does not exist for a general noncommutative algebra – see Example 1.9.

In the sequel [7] we develop other techniques for computing discriminants and automorphism groups. One major goal of that paper is to work with algebras which are not free over their centers. We also deal with algebras B of the following form. First, let A_q be the q-quantum Weyl algebra generated by x and y subject to the relation $yx = qxy+1$ for some $q \in k^\times$ (we assume that $q \neq 1$, but q need not be a root of unity). Consider the tensor product $B := A_{q_1} \otimes \cdots \otimes A_{q_m}$ of quantum Weyl algebras, where $q_i \in k^\times \setminus \{1\}$ for all i. Since we are not assuming that the q_i are roots of unity, B need not be in Af; however, the conclusion of Theorem 3 holds for B:

Theorem 4. *(See [7].) Let $B = A_{q_1} \otimes \cdots \otimes A_{q_m}$ and assume that $q_i \neq 1$ for all $i = 1, \ldots, m$.*

(1) *The automorphism group* $\operatorname{Aut}(B)$ *is an algebraic group that fits into an exact sequence of the form* ($*$).

(2) *The automorphism group of* $B[t]$ *is triangular, namely,*

$$\operatorname{Aut}\bigl(B[t]\bigr) = \begin{pmatrix} \operatorname{Aut}(B) & C(B) \\ 0 & k^\times \end{pmatrix}$$

where $C(B)$ *is the center of* B.

(3) *If* $\operatorname{char} k = 0$, *then every locally nilpotent derivation of* B *is zero.*

Two explicit examples are given in [7]. Let B be as in Theorem 4.

(1) If $q_i \neq \pm 1$ and $q_i \neq q_j^{\pm 1}$ for all $i \neq j$, then $\operatorname{Aut}(B) = (k^\times)^m$.
(2) If $q_i = q \neq \pm 1$ for all i, then $\operatorname{Aut}(B) = S_m \ltimes (k^\times)^m$.

Theorem 4 also holds for the tensor products of A_q's with W_n's (for n even), as well as with many others in Af.

We would like to remark that most results in the literature (including the papers mentioned at the beginning of the introduction) calculate the automorphism group of non-PI algebras, or algebras with a parameter q (or multi-parameters) not being a root of unity. In general it is more difficult to compute the automorphism group in the PI case, or when q is a root of unity. Our method deals with both the PI and non-PI cases. Theorem 3 works for the PI case, and then mod p reduction (to be discussed in the sequel [7]) reduces the non-PI case (with appropriate parameters) to the PI case.

The definition of the discriminant is purely linear algebra, but the computation of the discriminant seems to be very difficult and tedious in general. In this paper we only (partially) compute one nontrivial example that is needed in the proof of Theorem 1. It would be nice to develop basic theory and computational tools for the discriminant in the noncommutative setting.

The paper is laid out as follows. In Section 1, we recall the notion of the discriminant, and we establish some of its basic properties. In Sections 2 and 3, we discuss so-called "affine" and "triangular" automorphisms and prove Theorem 3. The discriminant computation of W_n over its center occupies a major part of Section 4 and Theorem 1 is proved near the end of Section 4. In Section 5 we give comments, remarks, and examples related to the category Af.

1. Discriminant in the noncommutative setting

Throughout let k be a commutative domain. Modules (sometimes called vector spaces), algebras and morphisms are over k.

According to [10], the discriminant for polynomials was introduced by Cayley in 1848. Since then, it has been important in number theory (Galois theory) and algebraic geometry. In this section, we discuss the concept of the discriminant in the noncommutative setting. Let R be a commutative algebra and let B and F be algebras both of which contain R as a subalgebra. In applications, F would be either R or the ring of fractions of R.

Definition 1.1. An R-linear map $\mathrm{tr} : B \to F$ is called a *trace map* if $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in B$.

Here are some examples.

Example 1.2.

(1) Let $B = M_n(R)$. The *internal trace* $\mathrm{tr}_{\mathrm{int}} : B \to R$ is defined to be the usual matrix trace, namely, $\mathrm{tr}_{\mathrm{int}}((r_{ij})) = \sum_{i=1}^{n} r_{ii}$.
(2) Let B be a subalgebra of $M_n(F)$ and R a subalgebra of $F \cap B \subset M_n(F)$. The composition $\mathrm{tr} : B \to M_n(F) \xrightarrow{\mathrm{tr}_{\mathrm{int}}} F$ is a trace map from B to F.
(3) Let B be an R-algebra and F be a commutative R-subalgebra of B such that $B_F := B \otimes_R F$ is finitely generated free over F. Then left multiplication defines a natural embedding of R-algebras $lm : B \to \mathrm{End}_F(B_F) \cong M_n(F)$ where n is the rank $\mathrm{rk}(B/F)$. By part (2), we obtain a trace map, called the *regular trace*, by composing: $\mathrm{tr}_{\mathrm{reg}} : B \xrightarrow{lm} M_n(F) \xrightarrow{\mathrm{tr}_{\mathrm{int}}} F$.

Although we are going to mainly use the regular trace in this paper, the definition of the discriminant works for any trace map. From now on, assume that F is a commutative algebra. Let $R^\times$ be the set of units in R. For any $f, g \in R$, we use the notation $f =_{R^\times} g$ to indicate that $f = cg$ for some $c \in R^\times$. The following definition can be found in Reiner's book [16].

Definition 1.3. Let $\mathrm{tr} : B \to F$ be a trace map and w be a fixed integer. Let $Z := \{z_i\}_{i=1}^{w}$ be a subset of B.

(1) The *discriminant* of Z is defined to be

$$d_w(Z : \mathrm{tr}) = \det\big(\mathrm{tr}(z_i z_j)\big)_{w \times w} \in F.$$

(2) (See [16, Section 10, p. 126].) The *w-discriminant ideal* (or *w-discriminant R-module*) $D_w(B, \mathrm{tr})$ is the R-submodule of F generated by the set of elements $d_w(Z : \mathrm{tr})$ for all $Z = \{z_i\}_{i=1}^{w} \subset B$.
(3) Suppose B is an R-algebra which is finitely generated free over R. If Z is an R-basis of B, the *discriminant* of B is defined to be

$$d(B/R) =_{R^\times} d_w(Z : \mathrm{tr}).$$

(4) We say the discriminant (respectively, discriminant ideal) is *trivial* if it is either 0 or a unit (respectively, it is either the zero ideal or contains 1).

The following well-known proposition establishes some basic properties of the discriminant, including that $d(B/R)$ is independent of the choice of Z.

Proposition 1.4. *Let* $\mathrm{tr} : B \to R$ *be an* R*-linear trace map (so* $F = R$*). Let* $Z := \{z_i\}_{i=1}^w$ *be a set of elements in* B*.*

(1) *(See [16, p. 66, Exer. 4.13].) Suppose that* $Y = \{y_j\}_{j=1}^w$ *is such that* $y_i = \sum_j r_{ij} z_j$ *where* $r_{ij} \in R$*, and denote the matrix* $(r_{ij})_{w \times w}$ *by* $(Y : Z)$*. Then*

$$d_w(Y : \mathrm{tr}) = \det(Y : Z)^2 d_w(Z : \mathrm{tr}).$$

(2) *If both* Y *and* Z *are* R*-linear bases of* B*, then*

$$d_w(Y : \mathrm{tr}) =_{R^\times} d_w(Z : \mathrm{tr}).$$

As a consequence $d(B/R)$ *is well-defined up to a scalar in* $R^\times$*.*

(3) *(See [16, Theorem 10.2].) If* B *is an* R*-algebra which is finitely generated free over* R *with an* R*-basis* Z*, then* $D_w(B, \mathrm{tr})$ *is the principal ideal of* R *generated by* $d_w(Z : \mathrm{tr})$ *or equivalently by* $d(B/R)$*.*

Proof. (2) is an immediate consequence of (1). □

Here are some simple examples. The first two indicate the connection with the classical theory and third one is relevant to Theorem 1.

Example 1.5. If f is a monic polynomial, then its discriminant $\mathrm{Disc}(f)$ is classically defined to be the product of the differences of the roots. If f is the minimal polynomial of an algebraic number α, it is well-known that $d(\mathbb{Z}[\alpha]/\mathbb{Z}) = \mathrm{Disc}(f)$, see [16, pp. 66–67, Exer. 414 and Theorem 4.35], or [1, Theorem 6.4.1], or [18, Definition 6.2.2 and Remark 6.2.3].

Example 1.6. Let $B = M_n(R)$. A word of caution: we are using the regular trace map, not the internal trace map, to compute the discriminant. If we use the basis $Z = \{e_{ij} \mid 1 \le i, j \le n\}$ of matrix units, then we have

$$e_{ij} e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ 0 & \text{else.} \end{cases}$$

So we need to compute the regular trace of the matrix e_{il}: we compute the trace of the matrix giving its action by left multiplication on $M_n(R)$. Diagonal entries in that matrix arise when $e_{il} e_{jk}$ is a scalar multiple of e_{jk}, which can only happen when $i = l = j$; in this case, there are n diagonal entries, each of which is 1, so

$$\mathrm{tr}_{\mathrm{reg}}(e_{ij}e_{kl}) = \begin{cases} n & \text{if } i=l \text{ and } j=k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $d_{n^2}(Z : \mathrm{tr}) = \pm n^{n^2}$.

Example 1.7.

(1) Let $B = W_2 = k\langle x,y\rangle/(xy+yx-1)$ and let $R = k[x^2,y^2] \subset B$. Then it is easy to check that R is the center of B and $B = R \oplus Rx \oplus Ry \oplus Rxy$. Using the regular trace tr, one sees that

$$\mathrm{tr}(1) = 4, \qquad \mathrm{tr}(x) = 0, \qquad \mathrm{tr}(y) = 0, \qquad \mathrm{tr}(xy) = 2.$$

Using these traces and the fact tr is R-linear, we have the matrix

$$\big(\mathrm{tr}(z_iz_j)\big)_{4\times 4} = \begin{pmatrix} 4 & 0 & 0 & 2 \\ 0 & 4x^2 & 2 & 0 \\ 0 & 2 & 4y^2 & 0 \\ 2 & 0 & 0 & 2-4x^2y^2 \end{pmatrix}$$

where $Z = \{z_1,z_2,z_3,z_4\} = \{1,x,y,xy\}$, and therefore the discriminant of $d(B/R)$ is the determinant of the matrix $(\mathrm{tr}(z_iz_j))_{4\times 4}$, which is, by a direct computation, $-2^4(4x^2y^2-1)^2$.

(2) Let C be the skew polynomial ring $k_{-1}[x,y] := k\langle x,y\rangle/(xy+yx)$. A similar computation shows that the discriminant of C over its center $R = k[x^2,y^2]$ is $-2^8x^4y^4$. The details are left to the reader.

Now we consider the case when B contains a central subalgebra R. Assume that F is a localization of R such that $B_F := B \otimes_R F$ is finitely generated free over F. For example, if B_R is free, we may take $F = R$, and if not, we may take F to be the field of fractions of R (assuming R is a domain). We let $\mathrm{tr}_{\mathrm{reg}} : B \to F$ denote the regular trace defined in Example 1.2(3), namely,

$$\mathrm{tr}_{\mathrm{reg}} : B \to B_F \xrightarrow{lm} \mathrm{End}_F(B_F) \xrightarrow{\mathrm{tr}_{\mathrm{int}}} F. \tag{1.7.1}$$

We also simply write tr for $\mathrm{tr}_{\mathrm{reg}}$ since this is used most of the time. For any algebra B, let $\mathrm{Aut}(B)$ denote the full algebra automorphism group of B over the base ring. If C is a central subalgebra of B, the subgroup of automorphisms which fix C is denoted $\mathrm{Aut}_C(B)$. We say that an element $g \in \mathrm{Aut}(B)$ *preserves* a subalgebra A of B if $g(A) \subseteq A$. Note that if g preserves R, then g preserves any localization of R, and in particular, it preserves F. We also note that, in case R is the center of B, any automorphism will preserve it.

Lemma 1.8. *Fix $g \in \mathrm{Aut}(B)$ such that g and g^{-1} preserve R and let $w = \mathrm{rk}(B_F/F)$. Let x be an element in B.*

(1) *For any F-basis $Z = \{z_i\}_{i=1}^w$ of B_F, if $xz_i = \sum_j r_{ij}z_j$ for some $r_{ij} \in F$, then* $\operatorname{tr}(x) = \sum_{i=1}^w r_{ii}$.
(2) $g(\operatorname{tr}(x)) = \operatorname{tr}(g(x))$ *for any $x \in B$.*
(3) $g(d_w(Z : \operatorname{tr})) = d_w(g(Z) : \operatorname{tr})$ *for any set $Z = \{z_i\}_{i=1}^w$.*
(4) *The discriminant R-module $D_w(B, \operatorname{tr})$ is g-invariant.*
(5) *Suppose the image of* tr *is in R and consider the trace map* $\operatorname{tr} : B \to R$. *Then the discriminant ideal $D_w(B, \operatorname{tr})$ is g-invariant.*
(6) *If B is finitely generated free over R, then the discriminant $d(B/R)$ is a g-invariant up to a unit of R.*

Proof. (1) This is the definition of trace, noting that $\operatorname{tr}_{\text{int}}$ is independent of the choices of basis Z.

(2) If $Z = \{z_i\}_{i=1}^w$ is an F-basis, so is $Y = \{g(z_i)\}_{i=1}^w$ by linear algebra. So by part (1), we may use Y to compute tr. Applying g to xz_i we have $g(x)g(z_i) = \sum_j g(r_{ij})g(z_j)$. Since g preserves R, we obtain $\operatorname{tr}(g(x)) = \sum_{i=1}^w g(r_{ii}) = g(\operatorname{tr}(x))$.

(3) This follows from part (2), the definition of $d_w(Z : \operatorname{tr})$ and an easy computation.

(4) It follows from part (3) and the definition that $g(D_w(B, \operatorname{tr})) \subset D_w(B, \operatorname{tr})$. Since g and g^{-1} are automorphisms, we have $g(D_w(B, \operatorname{tr})) = D_w(B, \operatorname{tr})$.

(5) This is a consequence of (4).

(6) By Proposition 1.4(3), $D_w(B, \operatorname{tr})$ is a principal ideal generated by $d(B/R)$. Since g preserves $D_w(B, \operatorname{tr})$, $g(d(B/R)) = cd(B/R)$ for some $c \in R^\times$. □

We conclude this section with a well-known observation.

Example 1.9. Let k be a field. Let A_1 be the first Weyl algebra, the algebra generated by x and y subject to the relation $yx = xy + 1$.

Assume first that $\operatorname{char} k = 0$. Let B be an algebra and let $\operatorname{tr} : A_1 \to B$ be any additive map such that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ for all $a, b \in A_1$. Then $\operatorname{tr}(A_1) = 0$, as every element in A_1 can be written as $ya - ay$ for some $a \in A_1$ – for any $m, n \geq 0$ and any $c \in k$, we have

$$cx^m y^n = y\left(\frac{c}{m+1}x^{m+1}y^n\right) - \left(\frac{c}{m+1}x^{m+1}y^n\right)y.$$

So there is no nontrivial trace map from A_1 to any algebra.

If $\operatorname{char} k = p > 0$, then A_1 is a finitely generated free module over its center $R := k[x^p, y^p]$. A direct computation shows that the regular trace $\operatorname{tr} : A_1 \to R$ is the zero map in this case.

2. Dominating elements and automorphisms

In this section, we establish tools for identifying and constructing certain algebra automorphisms, called "affine" and "triangular" automorphisms. In the situation of

Theorem 1, we can show that every automorphism is affine – see Section 4 – and this allows us to prove the theorem.

The main result in this section is Theorem 3(1). To state and prove it, we need the concept of a "dominating element," which we now develop.

Let A be an algebra over k. We say A is *connected graded* if $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ and A is *locally finite* if each A_i is finitely generated over k. We now consider filtered rings. Let Y be a finitely generated free k-submodule of A. In this case we would also say that Y is finite-dimensional (over k). Suppose $k \cap Y = \{0\}$. Consider the standard filtration $F = \{F_n A := (k \oplus Y)^n \mid n \geq 0\}$ and assume that F is an exhaustive filtration of A and that the associated graded ring $\operatorname{gr} A$ is connected graded. As a consequence of $\operatorname{gr} A$ being connected graded, the unit map $k \to A$ is injective. For each element $f \in F_n A \setminus F_{n-1} A$, the associated element in $\operatorname{gr} A$ is defined to be $\operatorname{gr} f = f + F_{n-1} A \in (\operatorname{gr}_F A)_n$. The degree of an element $f \in A$, denoted by $\deg f$, is defined to be the degree of $\operatorname{gr} f$. By definition, $\deg c = 0$ for all $0 \neq c \in k$.

Using the standard filtration $\{F_n A = (k \oplus Y)^n \mid n \geq 0\}$ makes it easier to talk about affine automorphisms [Definition 2.5]. But the ideas in this section also apply to non-standard filtrations, see Example 5.6.

Note that, if $\operatorname{gr} A$ is a domain, then, for any elements $f_1, f_2 \in A$,

$$\deg(f_1 f_2) = \deg f_1 + \deg f_2. \tag{2.0.1}$$

Let $A^\times$ denote the set of all units of A. If $\operatorname{gr} A$ is a connected graded domain, as we assume in much of what follows, it is easy to see that $A^\times = k^\times$. In this case, if R is any subalgebra of A (for example, if R is the center of A), $R^\times = k^\times$.

One can check that assigning degrees (which could be different from 1) to a set of generators of A is almost equivalent to giving a filtration on A, though not every filtration has the property that $\operatorname{gr} A$ is a domain. See [23, Section 1] for some details.

Definition 2.1. Suppose that $Y = \bigoplus_{i=1}^n kx_i$ generates A as an algebra.

(1) A nonzero element $f := f(x_1, x_2, \ldots, x_n) \in A$ is called *locally dominating* if, for every $g \in \operatorname{Aut}(A)$, one has
  (a) $\deg f(y_1, \ldots, y_n) \geq \deg f$ where $y_i = g(x_i)$ for all i, and
  (b) if, further, $\deg y_{i_0} > 1$ for some i_0, then $\deg f(y_1, \ldots, y_n) > \deg f$.

(2) Assume that $\operatorname{gr} A$ is a connected graded domain. A nonzero element $f \in A$ is called *dominating* if, for every filtered PI algebra T with $\operatorname{gr} T$ a connected graded domain, and for every subset of elements $\{y_1, \ldots, y_n\} \subset T$ that is linearly independent in the quotient k-module T/F_0T, there is a lift of f, say $f(x_1, \ldots, x_n)$, in the free algebra $k\langle x_1, \ldots, x_n \rangle$, such that the following hold: either $f(y_1, \ldots, y_n) = 0$ or
  (a) $\deg f(y_1, \ldots, y_n) \geq \deg f$, and
  (b) if, further, $\deg y_{i_0} > 1$ for some i_0, then $\deg f(y_1, \ldots, y_n) > \deg f$.

We refer to T as a “testing” algebra. To prove our main Theorem 3, we only need one testing algebra, $T = A \otimes k[t] = A[t]$. But it is convenient to include all testing algebras T in order to prove Theorem 5.5. In almost all applications, it is easy to see that $f(y_1, \ldots, y_n) \neq 0$; so we only need to verify (a) and (b) in order to show that f is dominating. If this is the case, we will not mention the subcase of $f(y_1, \ldots, y_n) = 0$.

It is not hard to see that dominating elements are locally dominating. Next we give some examples of dominating elements. A monomial $x_1^{b_1} \cdots x_n^{b_n}$ is said to have degree *component-wise less than* (or, *cwlt*, for short) $x_1^{a_1} \cdots x_n^{a_n}$ if $b_i \leq a_i$ for all i and $b_{i_0} < a_{i_0}$ for some i_0. We write $f = cx_1^{b_1} \cdots x_n^{b_n} + (\text{cwlt})$ if $f - cx_1^{b_1} \cdots x_n^{b_n}$ is a linear combination of monomials with degree component-wise less than $x_1^{b_1} \cdots x_n^{b_n}$. The following is easy.

Lemma 2.2. *Retain the above notation and assume that* $\operatorname{gr} A$ *is a connected graded domain. Fix* $f \in A$.

(1) *If* $f = cx_1^{b_1} \cdots x_n^{b_n} + (\text{cwlt})$ *where* $n > 0$, $b_1 b_2 \cdots b_n > 0$, *and* $0 \neq c \in k$, *then* f *is dominating.*
(2) *For any positive integer* d, f *is dominating* (*respectively, locally dominating*) *if and only if* f^d *is.*

Proof. (2) is clear, using (2.0.1). To prove (1), write

$$f = cx_1^{b_1} \cdots x_n^{b_n} + \sum c_{a_s} x_1^{a_1} \cdots x_n^{a_n}.$$

Let T be any $\mathbb{N}$-filtered PI domain and $\{y_1, \cdots, y_n\}$ be a set of elements in T of degree at least 1. Suppose that $\deg y_{i_0} > 1$ for some i_0. Since each term $x_1^{a_1} \cdots x_n^{a_n}$ is cwlt $x_1^{b_1} \cdots x_n^{b_n}$, we have $\deg y_1^{a_1} \cdots y_n^{a_n} < \deg y_1^{b_1} \cdots y_n^{b_n}$, again by (2.0.1). Hence $f(y_1, \ldots, y_n)$ has leading term $cy_1^{b_1} \cdots y_n^{b_n}$. Thus

$$\deg f(y_1, \ldots, y_n) = \deg y_1^{b_1} \cdots y_n^{b_n} = \sum_{i=1}^{n} b_i \deg y_i > \sum_{i=1}^{n} b_i = \deg f.$$

Therefore part (b) in Definition 2.1(2) is verified. Part (a) can be checked similarly. The assertion follows. □

Definition 2.3. Retain the hypotheses in Definition 2.1. Let $\operatorname{tr} : A \to R = F$ be the regular trace function (1.7.1) and $w = \operatorname{rk}(A_R/R)$. We say the discriminant of A over R is *dominating* (respectively, *locally dominating*) if the discriminant ideal $D_w(A, \operatorname{tr})$ is a principal ideal of R generated by a dominating (respectively, locally dominating) element.

Usually we assume that A is finitely generated free over R; then by Proposition 1.4(3), $D_w(A, \operatorname{tr})$ is generated by $d(A/R)$. In this case we also say that $d(A/R)$ is dominating in Definition 2.3. We now recall a few other definitions given in the introduction.

Definition 2.4. Let Af be the category consisting of all k-flat k-algebras A satisfying the following conditions:

(1) A is a filtered algebra as in Definition 2.1 such that the associated graded ring $\operatorname{gr} A$ is a connected graded domain,
(2) A is a finitely generated free module over its center R, and
(3) the discriminant $d(A/R)$ is dominating.

The morphisms in this category are isomorphisms of algebras.

Definition 2.5. Let (A, Y) be defined as in Definition 2.1.

(1) An algebra automorphism g of A is said to be *affine* if $\deg g(x_i) = 1$ for all i, or equivalently, $g(Y) \subset Y \oplus k$.
(2) If every $g \in \operatorname{Aut}(A)$ is affine, we call $\operatorname{Aut}(A)$ *affine*.

The definition of an affine automorphism (and that of a dominating element) is dependent on Y (or the filtration of A). But in most cases, the filtration (which is not unique in general) is relatively easy to determine. Dominating elements help us to determine the automorphism group in the following way.

Lemma 2.6. *Let A be an algebra generated by Y with a locally dominating element f. If $g \in \operatorname{Aut}(A)$ such that $g(f) = \lambda f$ for some $0 \neq \lambda \in k$, then g is affine.*

Proof. Since g is an automorphism, the elements $g_i := g(x_i)$ are not in k. Thus $\deg g_i \geq 1$. If $\deg g_{i_0} > 1$ for some i_0, then $\deg f(g_1, \ldots, g_n) > \deg f$ as f is locally dominating. Note that $g(f) = f(g_1, \ldots, g_n)$, whence $\deg g(f) > \deg f$, contradicting the hypothesis $g(f) = \lambda f$. Therefore $\deg g(x_i) = 1$ for all i. □

By Lemma 1.8(6), the discriminant $d(B/R)$ is g-invariant for any automorphism g such that g and g^{-1} preserve R. In several situations – see Theorem 4.9(1), Example 5.1, and [7] – we show that the discriminant is dominating, and so any automorphism g is affine by Lemma 2.6. Here is a general statement, which is also Theorem 3(1).

Theorem 2.7. *Let A be a filtered algebra with standard filtration $F_n A = (Y \oplus k)^n$. Assume that the discriminant of A over its center R is locally dominating in A (for example, A is in* Af*). Then every automorphism of A is affine.*

Proof. This follows from Lemmas 1.8(6) and 2.6. □

Remark 2.8. For a filtered algebra A generated by $Y = \bigoplus_{i=1}^{n} k x_i$, here is a general way of determining affine automorphisms of A. For simplicity, let k be a field. Write

$$g(x_i) = \sum_{j=1}^{n} a_{ij}x_j + b_i, \quad \text{for all } i = 1, \dots, n,$$

with $(a_{ij})_{n\times n} \in GL_n(k)$ and $b_i \in k$. Write the inverse of g on the generators as

$$g^{-1}(x_i) = \sum_{j=1}^{n} a'_{ij}x_j + b'_i, \quad \text{for all } i = 1, \dots, n,$$

with $(a'_{ij})_{n\times n} = (a_{ij})^{-1} \in GL_n(k)$ and $b'_i \in k$. List all of the relations of A, say,

$$r_s(x_1, \dots, x_n) = 0$$

for $s = 1, 2, \dots$. Then g is an automorphism of A if and only if

$$r_s\big(g(x_1), \dots, g(x_n)\big) = r_s\big(g^{-1}(x_1), \dots, g^{-1}(x_n)\big) = 0$$

for all s. After we fix a k-basis of A, this is an explicit linear algebra problem and can be solved completely if we have an explicit description of the relations r_s. If A is noetherian, then it is enough to use $r_s(g(x_1), \dots, g(x_n)) = 0$ only. In conclusion, in many situations it is relatively easy to determine all affine automorphisms of A.

Let $\mathrm{Aut}_{\mathrm{af}}(A)$ be the set of affine automorphisms of A. Since k is a field, $\mathrm{Aut}_{\mathrm{af}}(A)$ is a subgroup of $GL(Y \oplus k)$. Since every relation of A gives rise to some closed conditions, $\mathrm{Aut}_{\mathrm{af}}(A)$ is a closed subgroup of $GL(Y \oplus k)$. As a consequence, $\mathrm{Aut}_{\mathrm{af}}(A)$ is an algebraic group and acts on $Y \oplus k$ rationally.

3. Consequences

In the previous section, we proved Theorem 3(1); our goal now is to prove the rest of that theorem. This involves an examination of triangular automorphisms and locally nilpotent derivations.

First we consider the automorphism group of $A[t]$ when A has a dominating discriminant over its center R. For any $g \in \mathrm{Aut}(A)$, $c \in k^\times$ and $r \in R$, the map

$$\sigma : t \to ct + r, \quad x \to g(x), \quad \text{for all } x \in A \tag{3.0.1}$$

determines uniquely a so-called *triangular* automorphism of $A[t]$. The automorphisms given in Example 5.10 can be viewed as triangular automorphisms of the Ore extension $D[x_n; \tau, \delta]$ where D is the subalgebra generated by $\{x_1, \dots, x_{n-1}\}$.

One may associate the triangular automorphism σ (3.0.1) with the upper triangular matrix $\left(\begin{smallmatrix} g & r \\ 0 & c \end{smallmatrix}\right)$. The product of two such automorphisms (or two such matrices) is given by

$$\begin{pmatrix} g_1 & r_1 \\ 0 & c_1 \end{pmatrix} \circ \begin{pmatrix} g_2 & r_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} g_1g_2 & g_1(r_2) + r_1c_2 \\ 0 & c_1c_2 \end{pmatrix}.$$

The inverse is given by

$$\begin{pmatrix} g & r \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} & -c^{-1}g^{-1}(r) \\ 0 & c^{-1} \end{pmatrix}.$$

This shows that all triangular automorphisms form a subgroup of $\mathrm{Aut}(A[t])$, which is denoted by

$$\mathrm{Aut}_{\mathrm{tr}}(A[t]) \quad \text{or} \quad \begin{pmatrix} \mathrm{Aut}(A) & R \\ 0 & k^{\times} \end{pmatrix}.$$

Using the dominating discriminant we can show that $\mathrm{Aut}_{\mathrm{tr}}(A[t]) = \mathrm{Aut}(A[t])$. The following lemma is obvious.

Lemma 3.1. *Suppose A is a finitely generated free module over its center R. Let C be a commutative algebra that is k-flat. Then $d(A \otimes C/R \otimes C) =_{(R\otimes C)^{\times}} d(A/R)$. If, further, $(R \otimes C)^{\times} = R^{\times}$, then $d(A \otimes C/R \otimes C) =_{R^{\times}} d(A/R)$.*

The next lemma says that discriminant of $d(A[t]/R[t])$ is dominating among the elements in $g(A)$, for $g \in \mathrm{Aut}(A[t])$: it controls the degree of $g(x_i)$ for $x_i \in Y$ and for $g \in \mathrm{Aut}(A[t])$. However, it does not control the degree of $g(t)$.

Lemma 3.2. *Let A be in* Af. *Then the following hold.*

(1) *Let C be a k-flat commutative filtered algebra such that $\mathrm{gr}\, A \otimes \mathrm{gr}\, C$ is a connected graded domain. If $g \in \mathrm{Aut}(A \otimes C)$, then $g(Y) \subset Y \oplus k$.*
(2) *Let m be a positive integer. If g is an automorphism of $A[t_1, \ldots, t_m]$, then $g(Y) \subseteq Y \oplus k$.*

Proof. (2) is a consequence of (1). So we only prove (1).

Let T be the corresponding filtered algebra $A \otimes C$ such that $\mathrm{gr}\, T = \mathrm{gr}\, A \otimes \mathrm{gr}\, C$, which is a domain by hypothesis. Hence (2.0.1) holds and $(A \otimes C)^{\times} = k^{\times}$. It is clear that the center of $A \otimes C$ is $R \otimes C$. By Lemma 3.1, $f := d(A \otimes C/R \otimes C) =_{k^{\times}} d(A/R)$. Let $Y = \bigoplus_{i=1}^{n} kx_i$.

Consider a new filtration on the testing algebra $A \otimes C$ with assignment $\deg'(c) = 2\deg(c)$ for all $c \in C$ and $\deg'(x_i) = 1$ for all i. Consequently, $\deg'(c) \geq 2$ for any $c \in C \setminus k$. It is easy to verify that $\mathrm{gr}'(A \otimes C) \cong (\mathrm{gr}\, A) \otimes (\mathrm{gr}'\, C)$, and the latter is isomorphic to $(\mathrm{gr}\, A) \otimes (\mathrm{gr}\, C)$ as ungraded algebras.

Let $g \in \mathrm{Aut}(A \otimes C)$. Since g preserves f (up to a scalar), $\deg' g(f) = \deg' f$. Since $x_i \in Y \setminus \{0\}$ are not in the center, $y_i := g(x_i)$ is not in the center of $A \otimes C$ for all i. Consequently, $\deg y_i \geq 1$ for all i. Since f is dominating, there is a presentations of f, say $f(x_1, \ldots, x_n)$, such that

$$\deg' g(f) = \deg' f(y_1, \ldots, y_n) > \deg' f \,(= \deg f)$$

if $\deg' y_i > 1$ for some i. This yields a contradiction and therefore $\deg' y_i \leq 1$ for all i. This means that $g(x_i) \in Y \oplus k$ for all i as $\deg'(c) \geq 2$ for any $c \in C \setminus k$. □

Derivations are closely related to automorphisms. Recall that a k-linear map $\partial : A \to A$ is called a *derivation* if

$$\partial(xy) = \partial(x)y + x\partial(y)$$

for all $x, y \in A$. We call ∂ *locally nilpotent* if for every $x \in A$, $\partial^n(x) = 0$ for some n. Given a locally nilpotent derivation ∂ (and assuming that $\mathbb{Q} \subseteq k$), the exponential map $\exp(\partial) : A \to A$ is defined by

$$\exp(\partial)(x) = \sum_{i=0}^{\infty} \frac{1}{i!}\partial^i(x), \quad \text{for all } x \in A.$$

Since ∂ is locally nilpotent, $\exp(\partial)$ is an algebra automorphism of A with inverse $\exp(-\partial)$.

Lemma 3.3. *Suppose that $\mathbb{Q} \subseteq k$. Let C be a commutative algebra that is k-flat.*

(1) *If every k-algebra automorphism of $A \otimes C[t]$ restricts to an algebra automorphism of A, then every locally nilpotent derivation of $A \otimes C$ becomes zero when restricted to A.*
(2) *If $\mathrm{Aut}(A[t]) = \mathrm{Aut}_{\mathrm{tr}}(A[t])$, then every locally nilpotent derivation of A is zero.*
(3) *If A is in Af, then every locally nilpotent derivation of $A[t_1, \ldots, t_m]$ becomes zero when restricted to A.*

Proof. (1) Let ∂ be a locally nilpotent derivation of $A \otimes C$. Extend ∂ to $\partial' : A \otimes C[t] \to A \otimes C[t]$ by defining $\partial'(t) = 0$ and $\partial'|_{A\otimes C} = \partial$. Then ∂' is a locally nilpotent derivation of $A \otimes C[t]$. Further, $t\partial'$ is a locally nilpotent derivation of $A \otimes C[t]$. Then the exponential map $\exp(t\partial')$ is a k-algebra automorphism of $A \otimes C[t]$. By hypothesis, the restriction of $\exp(t\partial')$ to A is an automorphism of A. But,

$$\exp\bigl(t\partial'\bigr)(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\partial^i(x), \quad \text{for all } x \in A,$$

which is in A only if $\partial(x) = 0$. The assertion follows.

(2) This is a special case of (1) when $C = k$.

(3) Let $C = k[t_1, \ldots, t_m]$. By Lemma 3.2(1) (for $C = k[t_1, \ldots, t_m, t]$), the hypotheses of part (1) hold. Then the assertion follows from part (1). □

From now until Lemma 3.6 we suppose that k is a field of characteristic zero. We refer to [13] for basic definitions about (affine) algebraic groups. By Remark 2.8, if $\mathrm{Aut}(A)$ is affine, then it is an algebraic subgroup of $GL(Y \oplus k)$. Let $\mathrm{Aut}^1(A)$ denote the identity component of $\mathrm{Aut}(A)$, which is the unique closed, connected, normal subgroup of finite index in $\mathrm{Aut}(A)$. An element $\sigma \in \mathrm{Aut}^1(A)$ or in $\mathrm{Aut}(A)$ is called *unipotent* if $Id - \sigma$, as a linear map of $Y \oplus k$, is nilpotent.

Lemma 3.4. *Let k be a field of characteristic zero. Assume that $\mathrm{Aut}(A)$ is affine (namely, $\mathrm{Aut}(A) \subset GL(Y \oplus k)$) and that every locally nilpotent derivation of A is zero. Then $\mathrm{Aut}^1(A)$ is a torus – it is isomorphic to $(k^\times)^r$ for some $r \geq 0$ – and $\mathrm{Aut}(A)$ is an algebraic group that fits into an exact sequence*

$$1 \to \left(k^\times\right)^r \to \mathrm{Aut}(A) \to S \to 1$$

for some finite group S.

Proof. Let σ be in $\mathrm{Aut}(A)$ such that $Id - \sigma$ is nilpotent on $Y \oplus k$. Then $\log \sigma := \sum_{n=1}^{\infty} \frac{-1}{n}(Id - \sigma)^n$ is a locally nilpotent derivation. By hypothesis, $\log \sigma$ is zero. Then $Id - \sigma$ is zero, so $\sigma = Id$. So every unipotent element in $\mathrm{Aut}(A)$ is the identity. Then $\mathrm{Aut}^1(A)$ is a torus by [13, Exer. 21.4.2]. Since $\mathrm{Aut}^1(A)$ has finite index in $\mathrm{Aut}(A)$, the exact sequence is clear. □

Now we are ready to prove Theorem 3(2)–(4).

Theorem 3.5. *Let k be a field of characteristic zero and A be in Af. Then the following hold.*

(1) $\mathrm{Aut}(A[t]) = \mathrm{Aut}_{\mathrm{tr}}(A[t])$.
(2) *Every locally nilpotent derivation ∂ of $A[t]$ is of the form*

$$\partial(x) = 0 \quad \text{for all } x \in A, \qquad \partial(t) = r \quad \text{for some } r \in R.$$

(3) *Every locally nilpotent derivation of A is zero.*
(4) $\mathrm{Aut}(A)$ *is an algebraic group that fits into an exact sequence*

$$1 \to \left(k^\times\right)^r \to \mathrm{Aut}(A) \to S \to 1$$

for some finite group S.

Proof. (1) Let $Y = \bigoplus_{i=1}^n kx_i$ and $g \in \mathrm{Aut}(A[t])$. By Lemma 3.2(2), $g(x_i) \in Y \oplus k \subset A$, or $g(A) \subset A$. Applying Lemma 3.2(2) to $h := g^{-1}$, we have $h(A) \subset A$. Thus $g|_A$ and $h|_A$ are inverse to each other and hence $g|_A \in \mathrm{Aut}(A)$. Let $g(t) = \sum_{i=0}^n a_i t^i$ with $a_n \neq 0$ and $h(t) = \sum_{j=0}^m b_j t^j$ with $b_m \neq 0$. Then $gh(t) = \sum_{i=0}^{nm} c_i t^i$ with $c_{nm} = a_n (b_m)^n \neq 0$. Since $gh(t) = t$, $nm = 1$ (consequently, $n = m = 1$) and $a_1 b_1 = 1$. Thus $c := a_1 \in R^\times = k^\times$. This shows that $g(t) = ct + a_0$ where $c \in k^\times$ and $a_0 \in A$. Since t is central, $r := a_0 \in R$. The assertion follows.

(2) By Lemma 3.3(3), $\partial(x) = 0$ for all $x \in A$. Let $\partial(t) = \sum_{i=0}^{d} c_i t^i$ for some $c_i \in A$. Suppose $\partial(t) \neq 0$ and it has t-degree d (namely, $c_d \neq 0$). If $n > 0$, the induction shows that $\partial^n(t)$ has t-degree $nd - (n-1)$. Hence ∂ is not locally nilpotent, a contradiction. Thus $\partial(t) = c_0 \in A$. Since $xt = tx$ for all $x \in A$, applying ∂ to the equation, we have $xc_0 = c_0 x$. Thus c_0 is in the center of A and the assertion follows.

(3) Follows from part (1) and Lemma 3.3(2).

(4) Follows from Theorem 2.7, part (3) and Lemma 3.4. □

Next we compute another automorphism group and we assume that k is a commutative domain. For any positive integer m, define $A[\underline{t}_m^{\pm 1}]$ to be the Laurent polynomial extension $A[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_m^{\pm 1}]$. The following lemma is easy and the proof is omitted.

Lemma 3.6. *Let A be any algebra.*

(1) $(A[\underline{t}_m^{\pm 1}])^\times = \bigcup_{(n_s)\in\mathbb{Z}^m} A^\times \cdot t_1^{n_1} t_2^{n_2} \cdots t_m^{n_m}$.
(2) *Suppose $A^\times = k^\times$. Then every automorphism of $A[\underline{t}_m^{\pm 1}]$ preserves $k[\underline{t}_m^{\pm 1}]$.*
(3) $\mathrm{Aut}(k[\underline{t}_m^{\pm 1}]) = (k^\times)^m \rtimes GL_m(\mathbb{Z})$.

Proposition 3.7. *Let m be a positive integer. If $A^\times = k^\times$, then*

$$\mathrm{Aut}\big(A\big[\underline{t}_m^{\pm 1}\big]\big) = \mathrm{Aut}_{k[\underline{t}_m^{\pm 1}]}\big(A\big[\underline{t}_m^{\pm 1}\big]\big) \times \mathrm{Aut}\big(k\big[\underline{t}_m^{\pm 1}\big]\big).$$

Proof. Let $g \in \mathrm{Aut}(A[\underline{t}_m^{\pm 1}])$. By Lemma 3.6(2), $g|_{k[\underline{t}_m^{\pm 1}]} := g_2$ preserves $k[\underline{t}_m^{\pm 1}]$. Thus $g_2 \in \mathrm{Aut}(k[\underline{t}_m^{\pm 1}])$. Then $(1 \otimes g_2)^{-1} g$ is in $\mathrm{Aut}_{k[\underline{t}_m^{\pm 1}]}(A[\underline{t}_m^{\pm 1}])$. The assertion holds. □

If $\mathrm{gr}\, A$ is a connected graded domain, then $A^\times = k^\times$. Therefore Proposition 3.7 applies. Note that $\mathrm{Aut}_{k[\underline{t}_m^{\pm 1}]}(A[\underline{t}_m^{\pm 1}])$ is affine by Lemma 3.2(1), and therefore computable [Remark 2.8]. By using Proposition 3.7, $\mathrm{Aut}(A[\underline{t}_m^{\pm 1}])$ can be described explicitly. In general, it would be interesting to understand the relationship between $\mathrm{Aut}(A \otimes C)$ and the pair $(\mathrm{Aut}(A), \mathrm{Aut}(C))$. Under the situation of Lemma 3.3(1), we have some useful information. On the other hand, this relationship is extremely complicated when A and C are arbitrary.

To conclude this section we give two examples. The first one shows that parts (2), (3), (4) of Theorem 3 do not follow from part (1) of Theorem 3, and the second one shows that Theorem 3(3) fails without the hypothesis that $\mathrm{char}\, k = 0$.

Example 3.8. Let $q \in k^\times$ be not a root of unity. Let A be the skew polynomial ring generated by x_1, x_2, x_3 subject to the relations

$$x_2 x_1 = x_1 x_2, \qquad x_3 x_1 = q x_1 x_3, \qquad x_3 x_2 = q x_2 x_3.$$

Let $Y = kx_1 \oplus kx_2 \oplus kx_3$. Then A is graded with $F_1 A = Y \oplus k$. Using the fact that q is not a root of unity, one can check that every automorphism g of A is affine, namely,

$g(Y) \subset Y$. In fact, $\operatorname{Aut}(A) \cong GL(2,k) \times k^\times$. So it is not of the form in Theorem 3(4). The map $\partial : x_1 \to 0$, $x_2 \to x_1$, $x_3 \to 0$ extends to a nonzero locally nilpotent derivation. Further, there is an automorphism of $A[t]$

$$h : x_1 \to x_1, \quad x_2 \to x_2 + tx_1, \quad x_3 \to x_3, \quad t \to t + a,$$

which is not in $\operatorname{Aut}_{\mathrm{tr}}(A[t])$. Therefore parts (2), (3), (4) of Theorem 3 fail.

Example 3.9. Let A be the skew polynomial ring $k_{-1}[x_1, x_2]$ and $R := k[x_1^2, x_2^2]$ be the center of A. For any $a, b \in k$ and any $h \in R$, define a derivation by

$$\partial : x_1 \to ax_1h, \quad x_2 \to bx_2h.$$

This ∂ extends to a derivation for any commutative base ring k and, by induction, $\partial(x_1^m x_2^n) = (am + bn)x_1^m x_2^n h$ for all non-negative integers m and n.

Now assume that $\operatorname{char} k = p > 2$. Let $a = 1$, $b = 0$ and $h = x_1^2$. Then $\partial(x_2) = 0$ and $\partial(x_1^m) = mx_1^{m+2}$. By induction, $\partial^n(x_1) = 1 \cdot 3 \cdot 5 \cdots (2n-1)\, x_1^{2n+1}$ for all $n \geq 1$. It follows that $\partial^p = 0$. Therefore ∂ is locally nilpotent. By Example 1.7(2), the discriminant of A over its center is $x_1^4 x_2^4$, which is dominating. So Theorem 3(3) fails without the hypothesis that $\operatorname{char} k = 0$. Let d be the discriminant $x_1^4 x_2^4$. Then $\partial(d) = 4x_1^6 x_2^4 = 4dx_1^2 \neq 0$. In this case, d is not an eigenvector of ∂.

4. An example

In this section, we assume that k is a commutative domain and that 2 is invertible in k. Our goal here is to prove Theorem 1 by computing enough information about the discriminant for the algebra W_n to show that this algebra is in Af.

Let $\mathcal{A} := \{a_{ij} \mid 1 \leq i < j \leq n\}$ be a set of scalars in k. Define the (-1)-quantum Weyl algebra $V_n(\mathcal{A})$ to be generated by $\{x_1, x_2, \ldots, x_n\}$ subject to the relations

$$x_i x_j + x_j x_i = a_{ij}$$

for all $i < j$. Example 1.7(1) is a special case with $n = 2$ and $a_{12} = 1$. If $a_{ij} = 0$ for all $i < j$, then this algebra is denoted by $k_{-1}[x_1, \ldots, x_n]$. If $a_{ij} = 1$ for all $i < j$, we get the algebra W_n of the introduction.

We refer to [15] for the definitions of Gelfand–Kirillov dimension (or GK-dimension, for short), and Krull dimension.

Lemma 4.1. *The following hold for $V := V_n(\mathcal{A})$.*

(1) *V is an iterated Ore extension $k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_n; \sigma_n, \delta_n]$ where $\sigma_j : x_i \mapsto -x_i$ and $\delta_j : x_i \mapsto a_{ij}$ for all $i < j$.*
(2) *V is a filtered algebra with associated graded ring $\operatorname{gr} V \cong k_{-1}[x_1, \ldots, x_n]$.*

(3) *The center of* $k_{-1}[x_1, \ldots, x_n]$ *is*

$$\begin{cases} k[x_1^2, \ldots, x_n^2] & \text{if } n \text{ is even,} \\ k\left[x_1^2, \ldots, x_n^2, \prod_i x_i\right] & \text{if } n \text{ is odd.} \end{cases}$$

(4) *If* n *is even, the center of* V *is* $R := k[x_1^2, \ldots, x_n^2]$, *and* V *is finitely generated free over* R *of rank* 2^n.

Proof. (1) It is easy to check that σ_{j+1} is an algebra automorphism of $K_j := k[x_1][x_2; \sigma_2, \delta_2] \cdots [x_j; \sigma_j, \delta_j]$ and δ_{j+1} is a σ_{j+1}-derivation of K_j. The assertion follows.

(2) Let $Y = \sum_{i=1}^n kx_i$. Then $F_n := (k + Y)^n$ defines a filtration of V such that $\operatorname{gr} V$ is generated by $\{x_1, \ldots, x_n\}$ and subject to the relations $x_i x_j + x_j x_i = 0$ for all $i \neq j$. The assertion follows.

(4) Since $k_{-1}[x_1, \ldots, x_n]$ is $\mathbb{Z}^n$-graded and $\mathbb{Z}^n$ is an ordered group, the center of $k_{-1}[x_1, \ldots, x_n]$ is $\mathbb{Z}^n$-graded. So every central element is a linear combination of monomials. It can be checked directly that each central monomial is generated by $x_1^2, \ldots, x_n^2$ when n is even and by $x_1^2, \ldots, x_n^2, \prod_i x_i$ when n is odd.

(5) Let C be the center of V. Since $x_i x_j^2 - x_j^2 x_i = (-x_j x_i + a_{ij})x_j - x_j(-x_i x_j + a_{ij}) = 0$, $x_j^2 \in C$. Thus $k[x_1^2, \ldots, x_n^2] \subset C$. It is clear that $\operatorname{gr} C \subset C(\operatorname{gr} V) = k[x_1^2, \ldots, x_n^2]$. Thus $\operatorname{gr} C = k[x_1^2, \ldots, x_n^2]$. By lifting, $C = k[x_1^2, \ldots, x_n^2]$. □

We are interested in $\operatorname{Aut}(V)$, which is related to the graded algebra automorphism group, denoted by $\operatorname{Aut}_{\mathrm{gr}}$, of $k_{-1}[x_1, \ldots, x_n]$. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and S_n be the symmetric group consisting of all permutations of $[n]$. Recall that W_n is the algebra $V(\{1\}_{i<j})$, namely, $a_{ij} = 1$ for all $1 \le i < j \le n$.

Lemma 4.2. *The following hold.*

(1) *(See [14, Lemma 1.12].)* $\operatorname{Aut}_{\mathrm{gr}}(k_{-1}[x_1, \ldots, x_n]) = S_n \ltimes (k^\times)^n$.
(2) $S_n \times \{\pm 1\} \subseteq \operatorname{Aut}(W_n)$.

Proof. (2) is clear. We only prove (1). This was proved in [14, Lemma 1.12] when k is a field. The assertion in the general case follows by passing from k to the ring of fractions of k. □

Here is an application of Remark 2.8. Recall that $\operatorname{Aut}_{\mathrm{af}}(V)$ denotes the group of affine automorphisms of V. We take $Y = \bigoplus_{i=1}^n kx_i$ for the algebra V.

Lemma 4.3. *Let* g *be an affine automorphism of* V. *Then there is a permutation* $\sigma \in S_n$ *and* $r_i \in k^\times$ *such that* $g(x_i) = r_i x_{\sigma(i)}$ *for all* i. *As a consequence,*

$$\mathrm{Aut_{af}}(W_n) = \begin{cases} S_2 \ltimes k^\times & \textit{if } n = 2, \\ S_n \times \{\pm 1\} & \textit{if } n \geq 3. \end{cases}$$

Proof. Since g preserves the filtration, the associated graded automorphism, denoted by $\bar{g}$, is a graded algebra automorphism of $k_{-1}[x_1, \dots, x_n]$. By Lemma 4.2(1), there is a permutation $\sigma \in S_n$ and $r_i \in k^\times$ such that $\bar{g}(x_i) = r_i x_{\sigma(i)}$ for all i. Thus we have $g(x_i) = r_i x_{\sigma(i)} + a_i$ for some $a_i \in k$. It remains to show that $a_i = 0$ for all i. Applying g to the relations $x_i x_j + x_j x_i = a_{ij}$, we have

$$\begin{aligned} a_{ij} &= g(x_i x_j + x_j x_i) \\ &= (r_i x_{\sigma(i)} + a_i)(r_j x_{\sigma(j)} + a_j) + (r_j x_{\sigma(j)} + a_j)(r_i x_{\sigma(i)} + a_i) \\ &= r_i r_j (x_{\sigma(i)} x_{\sigma(j)} + x_{\sigma(j)} x_{\sigma(i)}) + 2a_i r_j x_{\sigma(j)} + 2a_j r_i x_{\sigma(i)} + 2a_i a_j \\ &= r_i r_j a_{\sigma(i)\sigma(j)} + 2a_i r_j x_{\sigma(j)} + 2a_j r_i x_{\sigma(i)} + 2a_i a_j. \end{aligned}$$

Since $r_i \neq 0$, we have $a_j = 0$ for all j. The consequence follows easily from the fact that in W_n, we have $a_{ij} = 1$ for all $i < j$, and so

$$1 = a_{ij} = r_i r_j a_{\sigma(i)\sigma(j)} = r_i r_j$$

for all $i < j$. $\square$

Let $I = \{i_1, i_2, \dots, i_s\}$ be a set of integers between 1 and n with repetitions. We let $X_I = x_{i'_1} x_{i'_2} \cdots x_{i'_s} \in V_n(\mathcal{A})$ where $\{i'_1, i'_2, \dots, i'_s\}$ is a re-ordering of the elements in I such that $i'_1 \leq i'_2 \leq \cdots \leq i'_s$. Since $V_n(\mathcal{A})$ has a PBW basis, $V_n(\mathcal{A})$ has a k-linear basis consisting of all different monomials X_I. For two sets I and J of integers between 1 and n, let $I + J$ denote the union of I and J with repetitions. Suppose K_1 and K_2 are two sets of integers. We write $K_1 \to K_2$ if there is a presentation $K_1 = \{k_1, \dots, k_w\}$ and $K_2 = \{k'_1, \dots, k'_w\}$ such that $k_\alpha > k'_\alpha$ for all α from 1 to w.

Lemma 4.4. *Let $I = \{i_1, i_2, \dots, i_s\}$ and $J = \{j_1, j_2, \dots, j_u\}$ where the i's and j's are in non-decreasing order. Then*

$$X_I X_J = c X_{I+J} + \sum_{\substack{\emptyset \neq K_1 \subset I \\ \emptyset \neq K_2 \subset J \\ K_1 \to K_2}} c_{K_1, K_2} X_{(I \setminus K_1) + (J \setminus K_2)}$$

where $c \in k^\times$, $c_{K_1,K_2} \in k$.

Proof. First suppose that I has a single element i_1. If $i_1 \leq j_1$, then the assertion is trivial. Now assume $i_1 > j_1$. By induction on u, we have

$$x_{i_1} x_{j_2} \cdots x_{j_u} = c' X_{\{i_1\} + (J \setminus \{j_1\})} + \sum_{\substack{K_2 = \{k_1\} \subset (J \setminus \{j_1\}) \\ i_1 > k_1}} c_{K_2} X_{(J \setminus (K_2 + \{j_1\}))}.$$

Then

$$\begin{aligned}X_I X_J &= x_{i_1} x_{j_1} \cdots x_{j_u} \\ &= (x_{j_1} x_{i_1} + a_{i_1 j_1}) x_{j_2} \cdots x_{j_u} \\ &= x_{j_1} x_{i_1} x_{j_2} \cdots x_{j_u} + a_{i_1 j_1} x_{j_2} \cdots x_{j_u} \\ &= x_{j_1} \Bigg[c' X_{\{i_1\}+(J\setminus\{j_1\})} + \sum_{\substack{K_2=\{k_1\}\subset(J\setminus\{j_1\}) \\ i_1>k_1}} c_{K_2} X_{(J\setminus(K_2+\{j_1\}))} \Bigg] \\ &\quad + a_{i_1 j_1} x_{j_2} \cdots x_{j_u} \\ &= c X_{\{i_1\}+J} + \sum_{\substack{\emptyset\neq K_2\subset J \\ I\to K_2}} c_{K_2} X_{(J\setminus K_2)}.\end{aligned}$$

Now we assume that $|I| > 1$. We write $I = \{i_1\} + I'$ where $|I'| = |I| - 1$. By induction,

$$X_{I'} X_J = b X_{I'+J} + \sum_{\substack{\emptyset\neq K_1\subset I' \\ \emptyset\neq K_2\subset J \\ K_1\to K_2}} b_{K_1,K_2} X_{(I'\setminus K_1)+(J\setminus K_2)}.$$

Then

$$\begin{aligned}X_I X_J = x_{i_1} X_{I'} X_J &= x_{i_1} \Bigg[b X_{I'+J} + \sum_{\substack{\emptyset\neq K_1\subset I' \\ \emptyset\neq K_2\subset J \\ K_1\to K_2}} b_{K_1,K_2} X_{(I'\setminus K_1)+(J\setminus K_2)} \Bigg] \\ &= b x_{i_1} X_{I'+J} + \sum_{\substack{\emptyset\neq K_1\subset I' \\ \emptyset\neq K_2\subset J \\ K_1\to K_2}} b_{K_1,K_2} x_{i_1} X_{(I'\setminus K_1)+(J\setminus K_2)}.\end{aligned}$$

For $x_{i_1} X_{I'+J}$ and $x_{i_1} X_{(I'\setminus K_1)+(J\setminus K_2)}$, we use the case when $|I| = 1$. Note that i_1 is no larger than any element in I'. So

$$\begin{aligned}x_{i_1} X_{I'+J} &= c' X_{I+J} + \sum_{\substack{K_2=\{k_1\}\subset J \\ i_1>k_1}} c_{K_2} X_{(I+J\setminus(K_2+\{i_1\}))} \\ &= c' X_{I+J} + \sum_{\substack{K_1=\{i_1\} \\ K_2=\{k_1\}\subset J \\ K_1\to K_2}} c_{K_2} X_{(I\setminus K_1)+(J\setminus K_2)}.\end{aligned}$$

Similarly, by using the fact that i_1 is no larger than any element in I', one can obtain that the linear combination

$$\sum_{\substack{\emptyset\neq K_1\subset I' \\ \emptyset\neq K_2\subset J \\ K_1\to K_2}} b_{K_1,K_2} x_{i_1} X_{(I'\setminus K_1)+(J\setminus K_2)}$$

is of the form

$$\sum_{\substack{\emptyset \neq K_1 \subset I \\ \emptyset \neq K_2 \subset J \\ K_1 \to K_2}} c_{K_1,K_2} X_{(I \setminus K_1)+(J \setminus K_2)}.$$

The assertion follows. □

For the rest of this section, we work on computing the discriminant of $V_n(\mathcal{A})$ and proving Theorem 1.

Let $B = V = V_n(\mathcal{A})$ and $R = k[x_1^2, \ldots, x_n^2]$. Then B is a finitely generated free module over R of rank 2^n (and R is the center of B if n is even). Let $\operatorname{tr} : B \to R$ be the regular trace map as defined in Example 1.2(3). For any set of elements $X = \{f_1, \ldots, f_w\}$ in V, define

$$\Omega(X) = \Omega(f_1, \ldots, f_n) = \sum_{\sigma \in S_w} (-1)^{|\sigma|} f_{\sigma(1)} \cdots f_{\sigma(w)}. \tag{4.4.1}$$

Let $x_{i_1 i_2 \cdots i_w}$ denote the element $\Omega(x_{i_1}, x_{i_2}, \ldots, x_{i_w})$.

Lemma 4.5. *We work in the algebra $V := V_n(\mathcal{A})$.*

(1) $\operatorname{tr}(1) = 2^n$.
(2) *V is $\mathbb{Z}/(2)$-graded: $V = V_{\text{even}} \oplus V_{\text{odd}}$ with x_i having odd degree.*
(3) *If f has odd degree, then $\operatorname{tr}(f) = 0$. As a consequence, if w is odd, then $\operatorname{tr}(x_{i_1 i_2 \cdots i_w}) = 0$.*
(4) *If w is even, then $\operatorname{tr}(\Omega(f_1, \ldots, f_w)) = 0$. As a consequence, if w is even, then $\operatorname{tr}(x_{i_1 i_2 \cdots i_w}) = 0$.*

Proof. (1)–(3) These are clear.

(4) Using the trace property $\operatorname{tr}(ab) = \operatorname{tr}(ba)$, we have

$$\begin{aligned}
\operatorname{tr}\big(\Omega(f_1, \ldots, f_w)\big) &= \operatorname{tr}\left(\sum_{\sigma \in S_w} (-1)^{|\sigma|} f_{\sigma(1)} \cdots f_{\sigma(w)}\right) = \sum_{\sigma \in S_w} (-1)^{|\sigma|} \operatorname{tr}(f_{\sigma(1)} \cdots f_{\sigma(w)}) \\
&= \sum_{\sigma \in S_w} (-1)^{|\sigma|} \operatorname{tr}(f_{\sigma(w)} f_{\sigma(1)} \cdots f_{\sigma(w-1)}) \\
&= \sum_{\sigma \in S_w} (-1)^{|\sigma(1,2,3,\ldots,w)|} \operatorname{tr}(f_{\sigma(1)} \cdots f_{\sigma(w-1)} f_{\sigma(w)}) \\
&= -\sum_{\sigma \in S_w} (-1)^{|\sigma|} \operatorname{tr}(f_{\sigma(1)} \cdots f_{\sigma(w-1)} f_{\sigma(w)}) = -\operatorname{tr}\big(\Omega(f_1, \ldots, f_w)\big).
\end{aligned}$$

Since 2 is invertible in k, the assertion follows. □

Lemma 4.6. *We continue to work in the algebra V.*

(1) *If $i_1 < i_2 < \cdots < i_s$ and $s > 0$, then $\operatorname{tr}(x_{i_1} \cdots x_{i_s}) \in k$.*
(2) *If $I = \{i_1 < i_2 < \cdots < i_s\}$ and $J = \{j_1 < j_2 < \cdots < j_u\}$, then*

$$\operatorname{tr}(x_{i_1} \cdots x_{i_s} x_{j_1} \cdots x_{j_u}) = b x_{k_1}^2 x_{k_2}^2 \cdots x_{k_n}^2 + (\text{cwlt})$$

where $\{k_1, k_2, \ldots, k_n\} = I \cap J$ and $b \in k$.
(3) *If $i_1 < i_2 < \cdots < i_s$, then*

$$\operatorname{tr}(x_{i_1} \cdots x_{i_s} x_{i_1} \cdots x_{i_s}) = c x_{i_1}^2 x_{i_2}^2 \cdots x_{i_s}^2 + (\text{cwlt})$$

for some $c \in k^\times$.

Proof. (1) We compute the trace using the basis

$$\{x_{j_1} x_{j_2} \cdots x_{j_u} \mid j_1 < j_2 < \cdots < j_u\}.$$

Write $I = \{i_1 < i_2 < \cdots < i_s\}$ and $J = \{j_1 < j_2 < \cdots < j_u\}$. We use Lemma 4.4 to compute:

$$(x_{i_1} \cdots x_{i_s})(x_{j_1} \cdots x_{j_u}) = X_I X_J = c X_{I+J} + \sum_{\substack{\emptyset \neq K_1 \subset I \\ \emptyset \neq K_2 \subset J \\ K_1 \to K_2}} c_{K_1,K_2} X_{(I \setminus K_1)+(J \setminus K_2)}.$$

If $X_{I+J} = r X_J$ for some $r \in R$, then r is a scalar multiple of $x_{k_1}^2 \cdots x_{k_w}^2$ where $\{k_1, \ldots, k_w\} = I \cap J$. As a consequence $J = (I \setminus K) + (J \setminus K)$, which is impossible as $|K| > 0$. If $X_{(I \setminus K_1)+(J \setminus K_2)} = r X_J$ for some $r \in R$, then r is a scalar multiple of $x_{k_1}^2 \cdots x_{k_w}^2$ where $K := \{k_1, \ldots, k_w\} = (I \setminus K_1) \cap (J \setminus K_2)$. As a consequence $J = (I \setminus K + K_1) + (J \setminus K + K_2)$. If $|K| > 0$, then K is not in $(I \setminus K + K_1) + (J \setminus K + K_2)$, a contradiction. Therefore, the only possible case is when K is empty. When K is empty, the coefficient of X_J is in k. The assertion follows.

(2) By Lemma 4.4, we need to compute

$$\operatorname{tr}(X_I X_J) = c \operatorname{tr}(X_{I+J}) + \sum_{\substack{\emptyset \neq K_1 \subset I \\ \emptyset \neq K_2 \subset J \\ K_1 \to K_2}} c_{K_1,K_2} \operatorname{tr}(X_{(I \setminus K_1)+(J \setminus K_2)}).$$

Let $I \cap J = K = \{k_1, \ldots, k_n\}$. Clearly

$$\operatorname{tr}(X_{I+J}) = x_{k_1}^2 \cdots x_{k_n}^2 \operatorname{tr}(X_{(I \setminus K)+(J \setminus K)}) = x_{k_1}^2 \cdots x_{k_n}^2 b$$

for some $b = \operatorname{tr}(X_{(I \setminus K)+(J \setminus K)}) \in k$ by part (1).

Let $(I \setminus K_1) \cap (J \setminus K_2) = K' = \{k'_1, \ldots, k'_m\}$. Clearly

$$\operatorname{tr}(X_{(I\setminus K_1)+(J\setminus K_2)}) = x^2_{k'_1} \cdots x^2_{k'_m} \operatorname{tr}(X_{(I\setminus(K'\cup K_1))+(J\setminus(K'\cup K_2))}) = x^2_{k'_1} \cdots x^2_{k'_m} b'$$

for some $b' \in k$ by part (1). Since K' is a subset of K, $\operatorname{tr}(X_{(I\setminus K_1)+(J\setminus K_2)})$ is either a scalar multiple of $\operatorname{tr}(X_{I+J})$ or a scalar multiple of some monomial in (cwlt). The assertion follows.

(3) For the most part, this is a special case of part (2). To prove c is invertible, we note $\operatorname{tr}(x^2_{i_1} \cdots x^2_{i_s}) = 2^n x^2_{i_1} \cdots x^2_{i_s}$ and that 2 is invertible. □

Remark 4.7. Let $V = W_n$, so the relations are $x_i x_j + x_j x_i = 1$ for all $i \neq j$. Then we have an explicit formula for the trace of each basis element $x_{i_1} \cdots x_{i_s}$, where $1 \leq i_1 < \cdots < i_s \leq n$.

(1) If s is odd, then $\operatorname{tr}(x_{i_1} \cdots x_{i_s}) = 0$, by Lemma 4.5(3).
(2) If $\sigma \in S_n$ is a permutation of $[n]$, then, by Lemmas 1.8(2) and 4.6(1), $\operatorname{tr}(x_{i_1} \cdots x_{i_s}) = \operatorname{tr}(x_{\sigma(i_1)} \cdots x_{\sigma(i_s)})$.
(3) If s is even, then $\operatorname{tr}(x_{i_1} \cdots x_{i_s}) = 2^{n-s/2}$. To see this, we use induction on s. Note that $\operatorname{tr}(x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_s}) = \operatorname{tr}(x_{i_2} x_{i_1} x_{i_3} \cdots x_{i_s})$ by part (2). Using the relation, we have

$$\begin{aligned}\operatorname{tr}(x_{i_3} \cdots x_{i_s}) &= \operatorname{tr}\big((x_{i_1} x_{i_2} + x_{i_2} x_{i_1}) x_{i_3} \cdots x_{i_s}\big) \\ &= \operatorname{tr}(x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_s}) + \operatorname{tr}(x_{i_2} x_{i_1} x_{i_3} \cdots x_{i_s}) \\ &= 2 \operatorname{tr}(x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_s}).\end{aligned}$$

For any nonzero element f in the (graded) polynomial ring $k[x_1^2, x_2^2, \ldots, x_n^2]$, let $\operatorname{pr}(f)$ denote the highest degree component of f, which is called the *principal term* of f or the *leading term* of f.

Using the basis

$$\{X_I = x_{i_1} \cdots x_{i_s} \mid I = \{i_1 < \cdots < i_s\} \subset [n]\}$$

to compute the discriminant, we need to compute the determinant of the matrix

$$M = (m_{IJ})_{2^n \times 2^n},$$

where $m_{IJ} = \operatorname{tr}(x_{i_1} \cdots x_{i_w} x_{j_1} \cdots x_{j_s})$. By Lemma 4.6, we have the following.

- $m_{\emptyset,\emptyset} = 2^n$,
- if $I = \{i_1, \ldots, i_s\}$, then $\operatorname{pr}(m_{II})$ is of the form $c x^2_{i_1} \cdots x^2_{i_s}$ where $c \in k^\times$, and other terms of m_{II} are cwlt $x^2_{i_1} \cdots x^2_{i_s}$,
- for every pair $I \neq J$, m_{IJ} is cwlt both $\operatorname{pr}(m_{II})$ and $\operatorname{pr}(m_{JJ})$.

Therefore we have the following.

Proposition 4.8. *Retain the notation above.*

(1) *The product* $\prod_{I\subset[n]} m_{II}$ *has principal term of the form* $c(\prod_{i=1}^n x_i^2)^{2^{n-1}}$ *for some* $c \in k^\times$.
(2) *Thus* $\prod_{I\subset[n]} m_{II} = c(\prod_{i=1}^n x_i^2)^{2^{n-1}} + (\text{cwlt})$.
(3) *For each non-identity permutation* τ *of* $2^{[n]}$, *each monomial in the product* $\prod_{I\subset[n]} m_{I\tau(I)}$ *is cwlt* $(\prod_{i=1}^n x_i^2)^{2^{n-1}}$.

Recall that 2 is invertible in the commutative domain k.

Theorem 4.9. *Let* $B = V_n(\mathcal{A})$ *and* $R = k[x_1^2, \dots, x_n^2] \subset B$.

(1) *The discriminant satisfies* $d(B/R) = c(\prod_{i=1}^n x_i^2)^{2^{n-1}} + (\text{cwlt})$ *where* $c \in k^\times$. *As a consequence,* $d(B/R)$ *is a dominating element of* B.
(2) *If* $g \in \text{Aut}(B)$ *is an automorphism so that* g *and* g^{-1} *preserve* R, *then* g *is affine.*
(3) *If* n *is even, then* $V_n(\mathcal{A})$ *is in* Af.

Proof. (1) By definition, $d(B/R)$ is the determinant of M, which is equal to

$$\sum_{\tau\in S_{2^n}} (-1)^{|\tau|} \prod_{I\subset[n]} m_{I\tau(I)}.$$

In every summand, by Proposition 3.7(2), (3), $\prod_{I\subset[n]} m_{II}$ has the highest possible degree and it is equal to $c(\prod_{i=1}^n x_i^2)^{2^{n-1}} + (\text{cwlt})$ for some $c \in k^\times$. Any other term $\prod_{I\subset[n]} m_{I\tau(I)}$, for a non-identity permutation τ, is a linear combination of monomials that are cwlt $(\prod_{i=1}^n x_i^2)^{2^{n-1}}$ by Proposition 3.7(3). Therefore

$$\sum_{\tau\in S_{2^n}} (-1)^{|\tau|} \prod_{I\subset[n]} m_{I\tau(I)} = c\left(\prod_{i=1}^n x_i^2\right)^{2^{n-1}} + (\text{cwlt})$$

and the assertion follows.

(2) Assume that g is an automorphism such that g and g^{-1} preserve R. By Lemma 1.8(f), $g(d) = cd$ for some $c \in k^\times$. By part (1), $d(B/R)$ is dominating. By Lemma 2.6, g is affine.

(3) This follows from Lemma 4.1(4) and part (1). □

When n is odd, part (3) no longer holds. See Example 5.10 and Remark 5.12 for more about what happens when n is odd or when $\text{char}\, k = 2$.

Now we are ready to prove Theorem 1, as well as the following.

Theorem 4.10. *Assume that n is a positive even integer. Then $k_{-1}[x_1,\dots,x_n]$ is in* Af *and the following hold.*

(1) $\mathrm{Aut}(k_{-1}[x_1,\dots,x_n]) = S_n \ltimes (k^\times)^n$.
(2) $\mathrm{Aut}(k_{-1}[x_1,\dots,x_n][t]) = \begin{pmatrix} S_n \ltimes (k^\times)^n & k[x_1^2,\cdots,x_n^2] \\ 0 & k^\times \end{pmatrix}$.
(3) *If $\mathbb{Q} \subseteq k$, then every locally nilpotent derivation of $k_{-1}[x_1,\dots,x_n]$ is zero.*

Proof of Theorems 1 and 4.10. Let $B = W_n$ and $R = k[x_1^2,\dots,x_n^2]$. Note that W_n is a special case of $V_n(\mathcal{A})$. By Theorem 4.9(3), W_n is in Af. Theorem 1 follows from Theorem 3 and Lemma 4.3.

Now consider $B = k_{-1}[x_1,\dots,x_n]$. The first part of the proof is the same as for $B = W_n$. By Theorem 4.9(3), B is in Af and every $g \in \mathrm{Aut}(B)$ is affine by Theorem 3. By Lemma 4.3, there is a $\sigma \in S_n$ such that $g(x_i) = r_i x_{\sigma(i)}$, where $r_i \in k^\times$. Thus part (1) follows. Parts (2), (3) follow from Theorem 3. □

We also have the following results, which follow immediately from Theorems 1 and 3 and Proposition 3.7.

Theorem 4.11. *Let n be a positive even integer and m a positive integer.*

(1) $\mathrm{Aut}(W_2[t]) = \begin{pmatrix} S_2 \ltimes k^\times & k[x_1^2,x_2^2] \\ 0 & k^\times \end{pmatrix}$.
(2) $\mathrm{Aut}(W_2[\underline{t}_m^{\pm 1}]) = (S_2 \ltimes (k[\underline{t}_m])^\times) \times ((k^\times)^m \rtimes \{\pm 1\})$.
(3) *If $n \geq 4$,* $\mathrm{Aut}(W_n[t]) = \begin{pmatrix} S_n \times \{\pm 1\} & k[x_1^2,\dots,x_n^2] \\ 0 & k^\times \end{pmatrix}$.
(4) *If $n \geq 4$,* $\mathrm{Aut}(W_n[\underline{t}_m^{\pm 1}]) = (S_n \times \{\pm 1\}) \times ((k^\times)^m \rtimes GL_m(\mathbb{Z}))$.
(5) *If $\mathbb{Q} \subseteq k$, then every locally nilpotent derivation of W_n is zero.*

Further results can be found in [7].

Question 4.12. In the above we don't need the exact computation of the discriminant $d(W_n/R)$, but it would be nice to have. Let $x_{123\cdots n} = \Omega(\{x_1,\dots,x_n\})$ be defined as in (4.4.1), let

$$M = \begin{pmatrix} 2x_1^2 & 1 & \cdots & 1 \\ 1 & 2x_2^2 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 2x_n^2 \end{pmatrix},$$

and let $D = \det M$. We have the following questions (or conjectures).

(1) Is $x_{123\cdots n}^2 =_{k^\times} D$?
(2) Is $d(W_n/k[x_1^2,\dots,x_n^2]) =_{k^\times} D^{2^{n-1}}$?

Both formulas have been verified by computer for even integers $n \leq 6$ (see also Example 1.7(1) for $n = 2$). It also appears that if we use the basis

$$\{\Omega(x_{i_1}, \ldots, x_{i_s}) \mid i_1 < \cdots < i_s\},$$

ordered by s, to compute the discriminant, then the corresponding matrix of traces is block diagonal, and the jth block is the matrix of $j \times j$ minors of M. Verifying this last statement would give the above computation of the discriminant, by the Sylvester–Franke theorem (see [20], for example).

5. Comments and examples

In this section we provide some comments, remarks, examples and questions related to automorphisms. To save space, some details are omitted. By Theorem 3, if A is in the category Af, then we can compute its automorphism group. In this section we would like to show that there are many algebras in Af.

First of all, a dominating discriminant may be in a form different from the one given in Lemma 2.2(1).

Example 5.1. Consider the algebra $S(p) := k\langle x, y\rangle/(y^2x - pxy^2, yx^2 + px^2y)$ where $p \in k^\times$. Suppose k is a field. By [5, (8.11)], $S(p)$ is a noetherian Artin–Schelter regular domain of global dimension 3, which is of type S_2 in the classification given in [5]. Setting $\deg x = \deg y = 1$, $S(p)$ is graded and its Hilbert series is

$$H_{S(p)}(t) = \frac{1}{(1-t)^2(1-t^2)}.$$

It is known that $\operatorname{GKdim} S(p) = \operatorname{Kdim} S(p) = 3$. We are interested in the case when $p = 1$, so we set $A = S(1)$. One can check that the center of A is the commutative polynomial subring $R := k[x^4, y^2, \Omega]$ where $\Omega = (xy)^2 + (yx)^2$. As an R-module, A is free of rank 16. A computation (omitted) shows that

$$d(A/R) =_{k^\times} \left(x^4\right)^8 \left(\Omega^2 + 4x^4y^4\right)^8.$$

We claim that this element is dominating.

Note that, in the algebra A, $d(A/R)$ has different presentations

$$\left(x^4\right)^8 \left(\Omega^2 + 4x^4y^4\right)^8 = \left(x^4\right)^8 (xy + iyx)^{32} = \left(x^4\right)^8 (xy - iyx)^{32}$$

where $i^2 = -1$. Let B be any $\mathbb{N}$-filtered algebra such that $\operatorname{gr} B$ is a domain. Let y_1, y_2 be any elements in B of degree at least 1. If $\deg y_1 > 1$ or $\deg y_2 > 1$, then either $\deg(y_1y_2 - iy_2y_1) > 2$ or $\deg(y_1y_2 + iy_2y_1) > 2$. Assume the former by symmetry. Then $\deg(y_1^4)^8(y_1y_2 - iy_2y_1)^{32} > \deg d(A/R)$. Therefore $d(A/R)$ is dominating. Consequently, A is in Af and Theorem 3 applies. One can then easily check that $\operatorname{Aut}(A) = (k^\times)^2$.

Next we show that Af is closed under tensor products. We start with a few easy lemmas.

Lemma 5.2. *Let A and B be algebras such that their centers $C(A)$ and $C(B)$ are k-flat. Then $C(A \otimes B) = C(A) \otimes C(B)$.*

Lemma 5.3. *Suppose that A is a free module over $C(A)$ of rank m, and B is a free module over $C(B)$ of rank n. Assume that both $C(A)$ and $C(B)$ are flat over k. Then $A \otimes B$ is a free module over $C(A \otimes B)$ of rank mn and*

$$d\big(A \otimes B/C(A \otimes B)\big) = d\big(A/C(A)\big)^n d\big(B/C(B)\big)^m.$$

Proof. Pick a basis $\{x_i\}$ of A over $C(A)$ and basis $\{y_j\}$ of B over $C(B)$. For any $a \in A$, $b \in B$, write $ax_i = \sum_{i'} r_{ii'} x_{i'}$ and $by_j = \sum_{j'} s_{jj'} y_{j'}$. Then $\operatorname{tr}(a) = \sum_i r_{ii}$ and $\operatorname{tr}(b) = \sum_j s_{jj}$. Using $\{x_i \otimes y_j\}$ as a basis of $A \otimes B$ over $C(A) \otimes C(B)$, we have

$$(a \otimes b)(x_i \otimes y_j) = \sum_{i'} \sum_{j'} r_{ii'} s_{jj'} x_{i'} \otimes y_{j'}$$

which implies that $\operatorname{tr}(a \otimes b) = \sum_i \sum_j r_{ii} s_{jj} = \operatorname{tr}(a) \operatorname{tr}(b)$. Now

$$\begin{aligned} d(A \otimes B/C_A \otimes C_B) &= \det\big(\operatorname{tr}\big((x_i \otimes y_j)(x_{i'} \otimes y_{j'})\big)\big) = \det\big(\operatorname{tr}(x_i x_{i'}) \operatorname{tr}(y_j y_{j'})\big) \\ &= \det\big(\operatorname{tr}(x_i x_{i'})\big)^n \det\big(\operatorname{tr}(y_j y_{j'})\big)^m \\ &= d\big(A/C(A)\big)^n d\big(B/C(B)\big)^m. \qquad \square \end{aligned}$$

Lemma 5.4. *Retain the hypotheses of Lemma 5.3. Suppose that $d(A/C(A))$ and $d(B/C(B))$ are dominating. Then so is $d(A \otimes B/C(A \otimes B))$.*

Proof. Since $d(A/C(A))$ is a dominating element, $A \neq C(A)$ (unless $A = k$), so $m := \operatorname{rk}(A/C(A)) > 1$. Similarly, $n := \operatorname{rk}(B/C(B)) > 1$. By Lemma 5.3, the discriminant of $A \otimes B$ over its center is $d(A/C(A))^n d(B/C(B))^m$. By hypothesis, both $d(A/C(A))$ and $d(B/C(B))$ are dominating, and it is routine to check that $d(A/C(A))^n d(B/C(B))^m$ is dominating. $\square$

Theorem 5.5. *Retain the hypotheses of Lemma 5.3. Suppose that $\operatorname{gr} A \otimes \operatorname{gr} B$ is a connected graded domain. If A and B are in Af, so is $A \otimes B$.*

Proof. This follows from Lemmas 5.3 and 5.4. $\square$

Therefore the following algebras are in Af:

(1) All $V_n(\mathcal{A})$ when n is even [Theorem 4.9]. Special cases are $k_{-1}[x_1, \dots, x_n]$ and W_n when n is even.

(2) $A = k\langle x, y\rangle/(y^2x - xy^2, yx^2 + x^2y)$ [Example 5.1].
(3) Any skew polynomial ring $A = k_{p_{ij}}[x_1, \dots, x_n]$ satisfying the properties that (a) x_i are not central for all i and (b) A is a finitely generated free module over its center [7].
(4) Quantum Weyl algebras $A_q := k\langle x, y\rangle/(yx - qxy - 1)$ where $q \neq 1$ and q is a root of unity [7].
(5) Any tensor product of the algebras listed above.

In Section 2 we used standard filtrations in the definitions of dominating elements and affine automorphisms. In practice one might have to use non-standard filtrations in order to determine automorphism groups. Here is an example.

Example 5.6. Suppose 2 is invertible in k. Let D be the fixed subring $k_{-1}[x_1, x_2]^{S_2}$ where the group S_2 is generated by the permutation $\sigma : x_1 \leftrightarrow x_2$. Hence D is a graded PI domain. A presentation of D is given by

$$D \cong k\langle x, y\rangle/\bigl(x^2y - yx^2, xy^2 - y^2x, 2x^6 - 3x^3y - 3yx^3 + 4y^2\bigr)$$

where $x = x_1 + x_2$, $y = x_1^3 + x_2^3$ [14, Example 3.1]. Replacing y by $4y - 3x^3$, D has a better presentation

$$D \cong k\langle x, y\rangle/\bigl(x^2y - yx^2, xy^2 - y^2x, x^6 - y^2\bigr)$$

which we will use for the rest of this example. Then D is a connected graded algebra with $\deg x = 1$ and $\deg y = 3$. If we use a standard filtration for any possible generating set Y, the associated graded ring will not be a domain due to the third relation. Therefore it is not a good idea to use the standard filtration as we need to use (2.0.1) in our argument. A computation shows that the center of D is the polynomial ring generated by x^2 and $z := xy + yx$, and the discriminant of $d(D/C(D))$ is $f := (xy - yx)^4$. Using the relations of D, one has

$$f = \bigl((xy - yx)^2\bigr)^2 = \bigl((xy + yx)^2 - 4x^2y^2\bigr)^2 = \bigl(z^2 - 4x^8\bigr)^2 = \bigl(z - 2x^4\bigr)^2\bigl(z + 2x^4\bigr)^2.$$

Let g be any automorphism of D. By Lemma 1.8(6), $g(f) = cf$ for some $c \in k^\times$. Since the polynomial ring $k[x^2, z]$ is a unique factorization domain, we have

$$\begin{cases} g\bigl(z - 2x^4\bigr) = a\bigl(z - 2x^4\bigr) \\ g\bigl(z + 2x^4\bigr) = b\bigl(z + 2x^4\bigr) \end{cases} \quad \text{or} \quad \begin{cases} g\bigl(z - 2x^4\bigr) = a\bigl(z + 2x^4\bigr) \\ g\bigl(z + 2x^4\bigr) = b\bigl(z - 2x^4\bigr) \end{cases}$$

for some $a, b \in k^\times$. Hence $g(z \pm 2x^4)$ has degree 4. Consequently, $g(x^4)$ has degree (at most) 4, which implies that $g(x)$ has degree 1. By the third relation of D, $g(y)$ has degree 3. From this it is easy to check that

$$\begin{cases} g(x) = ax \\ g(y) = a^3 y \end{cases} \quad \text{or} \quad \begin{cases} g(x) = ax \\ g(y) = -a^3 y \end{cases}$$

for some $a \in k^{\times}$. Therefore $\mathrm{Aut}(D) = k^{\times} \rtimes S_2$.

We could modify the definition of Af so that D is in the category Af, but the definition would be more complicated in order to keep the tensor product property [Theorem 5.5]. At this point we would like to treat D separately. We have checked that all conclusions of Theorem 3 hold for D.

Note that $k_{-1}[x_1, x_2]$ is in Af and $D = k_{-1}[x_1, x_2]^{S_2}$. We may ask the following question: does $k_{-1}[x_1, \ldots, x_{2m}]^{S_{2m}}$ have an "affine" automorphism group for all $m \geq 2$?

Example 5.7. Let $\ell \geq 3$ and q be a primitive ℓth root of unity. Let A be the algebra $(k_q[x_1, x_2])[x_3]$. Then A is a connected graded domain with $\deg(x_i) = 1$ for $i = 1, 2, 3$. Since x_3 is central, it is not hard to check that the center of A is $R = k[x_1^{\ell}, x_2^{\ell}, x_3]$. Hence A is finitely generated free over its center with an R-basis $\{x_1^a x_2^b \mid 0 \leq a, b \leq \ell - 1\}$. Therefore (1) and (2) of Definition 2.4 hold. By a computation, the discriminant $d(A/R)$ is equal to $(x_1 x_2)^{\ell^2(\ell-1)}$, which is not dominating. Therefore (3) of Definition 2.4 fails. With some effort, one can show that every automorphism g of A is of the form

$$g(x_i) = \begin{cases} a_1 x_1 & i = 1, \\ a_2 x_2 & i = 2, \\ a_3 x_3 + f(x_1^{\ell}, x_2^{\ell}) & i = 3, \end{cases}$$

where $a_i \in k^{\times}$ and f is a polynomial of two variables, and every locally nilpotent derivation ∂ of A is of the form

$$\partial(x_i) = \begin{cases} 0 & i = 1, \\ 0 & i = 2, \\ f(x_1^{\ell}, x_2^{\ell}) & i = 3. \end{cases}$$

By Theorem 3(4), if k is a field, then $\mathrm{Aut} : A \mapsto \mathrm{Aut}(A)$ defines a functor from Af to the category of algebraic groups over k. There are some interesting questions about this functor. It is well-known that the symmetry index si (defined after Theorem 3) is neither additive nor multiplicative. For example, if A and $A^{\otimes n}$ are both in Af, then $si(A^{\otimes n}) \geq n!(si(A))^n$. What about the symmetry rank?

Question 5.8. Let k be a field and let A and B be in Af. Is $sr(A \otimes B) = sr(A) + sr(B)$?

Remark 5.9. In [7] we use the discriminant to propose another category Af_{-1} that has the following properties:

(1) If A is in Af, then the polynomial extension $A[t]$ is in Af_{-1} (and there are many other algebras in Af_{-1}).
(2) If B is in Af_{-1}, then $\mathrm{Aut}(B)$ is tame.

Therefore the automorphism groups of the algebras in Af_{-1} can be understood (in theory).

We now consider $W_n = k\langle x_1, \ldots, x_n\rangle/(x_i x_j + x_j x_i - 1, \forall i \neq j)$ again, when n is odd or $\operatorname{char} k = 2$.

Example 5.10. Consider the standard filtration of W_n defined by $Y = \bigoplus_{i=1}^n kx_i$. As stated in Theorem 1, if n is even and $\operatorname{char} k \neq 2$, then every automorphism of W_n is affine. Here are some examples of non-affine automorphisms in other cases.

(1) If $\operatorname{char} k = 2$, then for any nonzero polynomial $f(t_1, \ldots, t_{n-1})$, the following determines a non-affine algebra automorphism of W_n:

$$x_i \mapsto \begin{cases} x_i & \text{if } i < n, \\ x_n + f(x_1^2, \ldots, x_{n-1}^2) & \text{if } i = n. \end{cases}$$

The associated locally nilpotent derivation is determined by

$$x_i \mapsto \begin{cases} 0 & \text{if } i < n, \\ f(x_1^2, \ldots, x_{n-1}^2) & \text{if } i = n. \end{cases}$$

(2) As in (4.4.1), define

$$\Omega(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Then we claim that $x_i \Omega(x_1, \ldots, x_{2m}) = -\Omega(x_1, \ldots, x_{2m}) x_i$ for all $i = 1, 2, \ldots, 2m$: see Lemma 5.11 below. Given this, if n is odd, say $n = 2m + 1$, then for any nonzero polynomial $f(t_1, \ldots, t_{2m})$, the following determines a non-affine algebra automorphism σ of W_n:

$$x_i \mapsto \begin{cases} x_i & \text{if } i < 2m+1, \\ x_{2m+1} + f(x_1^2, \ldots, x_{2m}^2)\Omega(x_1, \ldots, x_{2m}) & \text{if } i = 2m+1. \end{cases}$$

The associated locally nilpotent derivation ∂ is determined by

$$x_i \mapsto \begin{cases} 0 & \text{if } i < 2m+1, \\ f(x_1^2, \ldots, x_{2m}^2)\Omega(x_1, \ldots, x_{2m}) & \text{if } i = 2m+1, \end{cases}$$

and $\sigma = \exp(\partial)$.

The automorphisms in (1) and (2) are examples of *elementary* automorphisms – see [17].

Lemma 5.11. *Let W_n and $\Omega_n := \Omega(x_1, \dots, x_n)$ be defined as in Example 5.10. Then $x_i\Omega_n = (-1)^{n-1}\Omega_n x_i$ for all $i = 1, 2, \dots, n$.*

Proof. It is easy to reduce to the case when $k = \mathbb{Z}$.

We proceed by induction. It is easy to check that the assertion holds when $n = 2$ by using the fact that x_i^2 is central. Now assume the assertion holds for $n - 1 \geq 2$ and we want to show that it holds for n. Note that, for every $\sigma \in S_n$, $\Omega(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = (-1)^{|\sigma|}\Omega(x_1, \dots, x_n)$. By symmetry, it suffices to show that $x_1\Omega_n = (-1)^{n-1}\Omega_n x_1$. The argument below is dependent on the parity of n, and we only give a proof when n is odd. The proof when n is even is very similar, and we omit it. Since n is odd, it suffices to show that $x_1\Omega_n - \Omega_n x_1 = 0$. We compute $x_1\Omega_n - \Omega_n x_1$ in two different ways.

It follows from the definition that $\Omega_n = \sum_{i=1}^{n}(-1)^{i-1}x_i\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)$. Then, by using the induction hypothesis,

$$\begin{aligned}
x_1\Omega_n - \Omega_n x_1 &= x_1\big(x_1\Omega(\widehat{x}_1, x_2, \dots, x_n)\big) - x_1\Omega(\widehat{x}_1, x_2, \dots, x_n)x_1 \\
&\quad + \sum_{i\geq 2}(-1)^{i-1}\big[x_1x_i\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n) - x_i\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_1\big] \\
&= x_1^2\Omega(\widehat{x}_1, x_2, \dots, x_n) - x_1\Omega(\widehat{x}_1, x_2, \dots, x_n)x_1 \\
&\quad + \sum_{i\geq 2}(-1)^{i-1}\big[x_1x_i\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n) + x_ix_1\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)\big] \\
&= x_1^2\Omega(\widehat{x}_1, x_2, \dots, x_n) - x_1\Omega(\widehat{x}_1, x_2, \dots, x_n)x_1 \\
&\quad + \sum_{i\geq 2}(-1)^{i-1}\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Omega_n &= \sum_{i=1}^{n}(-1)^{i-n}\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_i \\
&= \sum_{i=1}^{n}(-1)^{i-1}\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_i
\end{aligned}$$

as n is odd. So we have

$$\begin{aligned}
x_1\Omega_n - \Omega_n x_1 &= x_1\big(\Omega(\widehat{x}_1, x_2, \dots, x_n)x_1\big) - \Omega(\widehat{x}_1, x_2, \dots, x_n)x_1^2 \\
&\quad + \sum_{i\geq 2}(-1)^{i-1}\big[x_1\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_i - \Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_ix_1\big] \\
&= -x_1^2\Omega(\widehat{x}_1, x_2, \dots, x_n) + x_1\Omega(\widehat{x}_1, x_2, \dots, x_n)x_1 \\
&\quad + \sum_{i\geq 2}(-1)^{i-1}\big[-\Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_1x_i - \Omega(x_1, \dots, \widehat{x}_i, \dots, x_n)x_ix_1\big]
\end{aligned}$$

$$
\begin{aligned}
&= -x_1^2\Omega(\widehat{x}_1, x_2, \ldots, x_n) + x_1\Omega(\widehat{x}_1, x_2, \ldots, x_n)x_1 \\
&\quad + \sum_{i\geq 2}(-1)^i\Omega(x_1, \ldots, \widehat{x}_i, \ldots, x_n) \\
&= -(x_1\Omega_n - \Omega_n x_1).
\end{aligned}
$$

Since $2 \neq 0$ in $\mathbb{Z}$, $x_1\Omega_n - \Omega_n x_1 = 0$ as required. □

Remark 5.12. By the previous example, when n is odd or when $\operatorname{char} k = 2$, there are non-affine automorphisms. Thus the automorphism group looks complicated. Also, it appears that the discriminant does not provide useful information in either case: a (non-trivial) computation shows that the discriminant ideal of W_3 over its center contains 1, and hence it is trivial. We conjecture that this holds for any odd integer $n \geq 3$. We also note when n is odd, the center R contains $\Omega(x_1, \ldots, x_n)$, so W_n is not free over R. When $\operatorname{char} k = 2$, Lemma 4.5(1) says that $\operatorname{tr}(1) = 0$ in k, and computer calculations suggest that the discriminant is zero (whence trivial) in general. (For more evidence, see Remark 4.7 – some of these computations remain valid in characteristic 2.) In conclusion, new invariants are needed to understand (or control) $\operatorname{Aut}(W_n)$ when n is odd or when $\operatorname{char} k = 2$.

We conclude this paper with the following question.

Question 5.13. If n is odd and/or $\operatorname{char} k = 2$, what is the group $\operatorname{Aut}(W_n)$?

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