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Product type actions of G_q

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ABSTRACT

We will study a faithful product type action of G_q, the q-deformation of a connected semisimple compact Lie group G, and prove that such an action is induced from a minimal action of the maximal torus T of G_q. This enables us to classify product type actions of $SU_q(2)$ up to conjugacy. We also compute the intrinsic group of $G_{q,\Omega}$, the 2-cocycle deformation of G_q that is naturally associated with the quantum flag manifold $L^\infty(T\backslash G_q)$.

1. Introduction

In this paper, we will study a product type action of a q-deformed compact quantum group. Theory of a quantum group was initiated by Drinfel'd and Jimbo [13,21]. They have introduced the quantum group $U_q(\mathfrak{g})$, the q-deformation of the enveloping algebra of a Kac–Moody Lie algebra $\mathfrak{g}$. In the operator algebraic approach, Woronowicz has

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defined $SU_q(N)$ and introduced the concept of a compact quantum group [51–53,55] by deforming a function algebra. We can construct the q-deformed compact quantum group G_q for a connected semisimple compact Lie group G using $U_q(\mathfrak{g})$ for a finite dimensional simple $\mathfrak{g}$ (see [27,15]).

Let us consider a product type action of G_q on a uniformly hyperfinite C*-algebra. Such an action has been studied by Konishi, Nagisa, and Watatani [26]. They have shown that the fixed point algebra of the product type action of $SU_q(2)$ with respect to the spin-1/2 irreducible representation is generated by certain Jones projections. In particular, if we take its weak closure, then the product type action is never minimal.

In [17], Izumi has elucidated this interesting phenomenon by introducing the concept of a (non-commutative) Poisson boundary. Namely, he has constructed a von Neumann algebra from a random walk on the dual discrete quantum group of a compact quantum group, and shown that is isomorphic to the relative commutant of a fixed point algebra inside an infinite tensor product factor of matrix algebras. Since this pioneering work, the program of realization of Poisson boundaries has been carried out in several papers [17,20,44,49,50]. In particular, it is known that the Poisson boundary of G_q is isomorphic to the quantum flag manifold $T\backslash G_q$ [17,20,44].

On the center of a Poisson boundary as a von Neumann algebra, it has been conjectured in [46] that the center could coincide with the classical part of the Poisson boundary, that is, the center could come exactly from the random walk on the von Neumann algebraic center of the dual discrete quantum group. Indeed, it is well-known for experts that this is the case to $SU_q(2)$. Also for universal quantum groups $A_o(F)$ and $A_u(F)$, the conjecture has been affirmatively solved [47,49,50]. We will show the following result which states that the conjecture also holds for every G_q (Theorem 3.1).

Theorem 1. *The von Neumann algebra $L^\infty(T\backslash G_q)$ is a factor of type I.*

Let us again consider a product type action of G_q on a factor $\mathcal{M}$. From Izumi's result and Theorem 1, it turns out that the relative commutant $\mathcal{Q} := (\mathcal{M}^{G_q})' \cap \mathcal{M}$ is the infinite dimensional type I factor, where $\mathcal{M}^{G_q}$ denotes the fixed point algebra. Thus we obtain the tensor product splitting $\mathcal{M} \cong \mathcal{R} \otimes \mathcal{Q}$, where $\mathcal{R} := \mathcal{Q}' \cap \mathcal{M}$. It is then shown that the inclusion $\mathcal{M}^{G_q} \subset \mathcal{R}$ is irreducible and of depth 2 (Lemma 4.1). Hence it arises from a minimal action of a unique compact quantum group on $\mathcal{R}$. Actually, we will show that the compact quantum group is nothing but the maximal torus T (Theorem 4.6). Hence we have a T-equivariant copy of $L^\infty(T)$ inside $\mathcal{R}$. Then it is natural to ask whether this copy and $\mathcal{Q}$ generate a von Neumann algebra that is G_q-isomorphic to $L^\infty(G_q)$.

To solve this question, we need to show the triviality of the G_q-equivariant automorphism on $L^\infty(T\backslash G_q)$. In [37], Soibel'man has classified irreducible representations of $C(G_q)$ as a C*-algebra. Namely, it has become clear that irreducible representations of $C(G_q)$ are parametrized by the maximal torus T and the Weyl group W of $\mathfrak{g}$ as $\{\pi_{t,w}\}_{t\in T, w\in W}$. Then Dijkhuizen–Stokman's result [12, Theorem 5.9] states that any irreducible representation of $C(T\backslash G_q)$ actually comes from that of $C(G_q)$. This, in particular, implies that the counit gives a unique character on $C(T\backslash G_q)$, and we obtain the triviality of the G_q-equivariant automorphism group of $C(T\backslash G_q)$ (Corollary 3.5).

By using these results, we study an $L^\infty(T\backslash G_q)$-valued invariant cocycle. Actually, it is shown that any such cocycle is a coboundary with a unique solution in $Z(L^\infty(G_q))$ up to a scalar multiple (Theorem 3.6). As an application, we can show the following main result of this paper (Theorem 4.14).

Theorem 2. *A faithful product type action of G_q is induced from a minimal action of T on a type III factor. Moreover, such a minimal action is unique.*

Another application of Theorem 1 concerns theory of 2-cocycle deformation of locally compact quantum groups. In [9], De Commer has shown that a 2-cocycle twisted von Neumann bi-algebra of a locally compact quantum group again has a locally compact quantum group structure. In our setting, we will encounter with a 2-cocycle Ω that is canonically associated with an irreducible projective unitary representation of G_q coming from $L^\infty(T\backslash G_q)$. We can determine the intrinsic group of $G_{q,\Omega}$, the deformation of G_q by Ω (Theorem 3.10) as follows.

Theorem 3. *The intrinsic group of $G_{q,\Omega}$ is isomorphic to $\widehat{T}$.*

When $G_q = SU_q(2)$, it has been proved by De Commer that $G_{q,\Omega}$ is isomorphic to $\widetilde{E}_q(2)$, Woronowicz's quantum $E(2)$ group [11]. The above theorem generalizes a partial result of [54].

This paper is organized as follows.

In Section 2, we will give a brief summary of theory of a compact quantum group and a q-deformed Lie group G_q.

In Section 3, the notion of an invariant cocycle is introduced. We will prove that all invariant cocycles evaluated in $L^\infty(T\backslash G_q)$ come from the canonical generators of $Z(L^\infty(G_q))$. As an application, we compute the intrinsic group of a 2-cocycle deformation $G_{q,\Omega}$.

In Section 4, we will discuss a product type action. First, we will deduce from sector theory that the canonical inclusion of factors stated before corresponds to a minimal action of the maximal torus T. Next, by using invariant cocycles and the triviality of G_q-equivariant automorphism of $L^\infty(T\backslash G_q)$, we will show that a faithful product type action is actually induced from a minimal action of T. Then the classification of product type actions is studied. Especially, we will present a complete classification of product type actions of $SU_q(2)$. Uncountably many non-product type and mutually non-cocycle conjugate actions of $SU_q(2)$ on the injective type III$_1$ factors are also constructed.

In the last section, we will pose a problem concerning the main results in a more general situation.

2. Preliminary

2.1. Notation and terminology

In this paper, $\mathbb{Z}_+$ denotes the set of non-negative integers, that is, $\mathbb{Z}_+ = \{0, 1, \dots\}$.

The tensor symbol $\otimes$ denotes the minimal tensor product for C*-algebras and the von Neumann algebra tensor product for von Neumann algebras.

We denote by $\operatorname{span} S$ and $\overline{\operatorname{span}}^{\mathrm{w}} S$, the linear span of a set S and the weak closure of $\operatorname{span} S$, respectively.

For a von Neumann algebra $\mathcal{M}$, we will denote by $Z(\mathcal{M})$ its center. By $\operatorname{End}(\mathcal{M})$, we will denote the set of normal endomorphisms on $\mathcal{M}$. For $\rho, \sigma \in \operatorname{End}(\mathcal{M})$, (ρ, σ) denotes the set of intertwiners. Namely, an element $a \in (\rho, \sigma)$ satisfies $a\rho(x) = \sigma(x)a$ for all $x \in \mathcal{M}$. If $(\rho, \rho) = \mathbb{C}$, then we will say that ρ is irreducible. Two endomorphisms ρ, σ on $\mathcal{M}$ are said to be equivalent if there exists a unitary $u \in \mathcal{M}$ such that $\rho = \operatorname{Ad} u \circ \sigma$. By $\operatorname{Sect}(\mathcal{M})$, we denote the quotient space of $\operatorname{End}(\mathcal{M})$. The equivalence class of ρ is denoted by $[\rho]$ which is called a *sector*. For sector theory, reader's are referred to [16,29,30].

Recall the notion of a Hilbert space in a von Neumann algebra [36]. A weakly closed linear space $\mathscr{H}$ in a von Neumann algebra $\mathcal{M}$ is called a *Hilbert space in* $\mathcal{M}$ if $W^*V \in \mathbb{C}$ for all $V, W \in \mathscr{H}$. Then $\mathscr{H}$ is a Hilbert space with the inner product $\langle V, W\rangle := W^*V$. The support of $\mathscr{H}$, which we denote by $s(\mathscr{H})$, is the infimum of projections $p \in \mathcal{M}$ such that $pV = V$ for all $V \in \mathscr{H}$. If $\{V_i\}_{i\in I}$ is an orthonormal base of $\mathscr{H}$, then we have $s(\mathscr{H}) = \sum_{i\in I} V_i V_i^*$.

If $\rho, \sigma \in \operatorname{End}(\mathcal{M})$ and ρ is irreducible, then (ρ, σ) is a Hilbert space in $\mathcal{M}$ by the inner product $\langle V, W\rangle := W^*V$ for $V, W \in (\rho, \sigma)$.

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of properly infinite von Neumann algebras. Then $L^2(\mathcal{M})$ also has the structure of the standard form for $\mathcal{N}$. Let $J_\mathcal{M}$ and $J_\mathcal{N}$ be the modular conjugations of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then $\gamma_\mathcal{N}^\mathcal{M}(x) := J_\mathcal{N} J_\mathcal{M} x J_\mathcal{M} J_\mathcal{N}$, $x \in \mathcal{M}$, is called the *canonical endomorphism* from $\mathcal{M}$ into $\mathcal{N}$. It is known that the sector $[\gamma_\mathcal{N}^\mathcal{M}]$ in $\operatorname{Sect}(\mathcal{N})$ does not depend on the choice of the structure of the standard forms of $\mathcal{N}$ and $\mathcal{M}$.

2.2. Compact quantum group

We will freely use the notions and the notation introduced in [44].

Let $\mathbb{G} := (C(\mathbb{G}), \delta)$ be a (C*-algebraic) compact quantum group. We always assume that the Haar state h is faithful. Let $\{L^2(\mathbb{G}), 1_h\}$ be the GNS representation with respect to h. We will regard $C(\mathbb{G})$ as a C*-subalgebra of $B(L^2(\mathbb{G}))$ from now on. By $L^\infty(\mathbb{G})$, we denote the weak closure of $C(\mathbb{G})$.

The *multiplicative unitary* V is a unitary on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ such that

$$V(x1_h \otimes \xi) = \delta(x)(1_h \otimes \xi) \quad \text{for } x \in C(\mathbb{G}),\ \xi \in L^2(\mathbb{G}).$$

Then V satisfies the pentagon equation $V_{12}V_{13}V_{23} = V_{23}V_{12}$. The coproduct δ extends to the normal coproduct $\delta\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \otimes L^\infty(\mathbb{G})$ by putting $\delta(x) = V(x \otimes 1)V^*$

for $x \in L^\infty(\mathbb{G})$. Note V belongs to $B(L^2(\mathbb{G})) \otimes L^\infty(\mathbb{G})$. The Haar state h also extends to a faithful normal invariant state on $L^\infty(\mathbb{G})$.

Let H be a Hilbert space. A unitary $v \in B(H) \otimes L^\infty(\mathbb{G})$ is called a *unitary representation* on H when $(\mathrm{id} \otimes \delta)(v) = v_{12}v_{13}$. By $\mathrm{Rep}_f(\mathbb{G})$, we denote the set of finite dimensional unitary representations. We set

$$A(\mathbb{G}) := \mathrm{span}\bigl\{(\omega \otimes \mathrm{id})(v) \bigm| \omega \in B(H)_*,\ v \in \mathrm{Rep}_f(\mathbb{G})\bigr\}.$$

Then $A(\mathbb{G})$ is a dense unital $*$-subalgebra of $C(\mathbb{G})$. We can define an anti-multiplicative linear map κ on $A(\mathbb{G})$, which is called the *antipode*, such that $(\mathrm{id} \otimes \kappa)(v) = v^*$ for $v \in \mathrm{Rep}_f(\mathbb{G})$. The *counit* is the character $\varepsilon : A(\mathbb{G}) \to \mathbb{C}$ such that $(\mathrm{id} \otimes \varepsilon)(v) = 1$ for $v \in \mathrm{Rep}_f(\mathbb{G})$. We only treat a co-amenable $\mathbb{G}$ in this paper, and ε extends to the character on $C(\mathbb{G})$. See [3–5,43] for details of the amenability.

By using the Woronowicz characters $\{f_z\}_{z\in\mathbb{C}}$, the modular automorphism group σ^h and the *scaling automorphism group* τ are given by

$$\sigma_t^h(x) = (f_{it} \otimes \mathrm{id} \otimes f_{it})\bigl(\delta^{(2)}(x)\bigr),$$
$$\tau_t(x) = (f_{it} \otimes \mathrm{id} \otimes f_{-it})\bigl(\delta^{(2)}(x)\bigr) \quad \text{for all } t \in \mathbb{R},\ x \in A(\mathbb{G}),$$

where $\delta^{(2)} := (\delta \otimes \mathrm{id}) \circ \delta$. In general, for $k \in \mathbb{N}$, we let $\delta^{(k)} := (\delta^{(k-1)} \otimes \mathrm{id}) \circ \delta$. Then $\delta^{(k+\ell)} = (\delta^{(k)} \otimes \delta^{(\ell-1)}) \circ \delta$ for $k, \ell \in \mathbb{N}$.

Let $v \in B(H) \otimes L^\infty(\mathbb{G})$ be a finite dimensional unitary representation. We let $F_v := (\mathrm{id} \otimes f_1)(v)$, which is non-singular and positive. Then $F_v^z = (\mathrm{id} \otimes f_z)(v)$ for all $z \in \mathbb{C}$. It is known that $\mathrm{Tr}(F_v) = \mathrm{Tr}(F_v^{-1})$, which is called the *quantum dimension* of v, and denoted by $\dim_q(v)$ or $\dim_q H$.

Let $\mathrm{Irr}(\mathbb{G})$ be the complete set of unitary equivalence classes of irreducible unitary representations. For $s \in \mathrm{Irr}(\mathbb{G})$, we fix a section $v(s) = (v(s)_{ij})_{i,j\in I_s}$. Then we have the following orthogonal equalities: for all $s, t \in \mathrm{Irr}(\mathbb{G})$, $i, j \in I_s$ and $k, \ell \in I_t$,

$$h\bigl(v(s)_{ij}v(t)_{k\ell}^*\bigr) = \dim_q\bigl(v(s)\bigr)^{-1}(F_{v_s})_{\ell,j}\delta_{s,t}\delta_{i,k},$$
$$h\bigl(v(s)_{ij}^*v(t)_{k\ell}\bigr) = \dim_q\bigl(v(s)\bigr)^{-1}\bigl(F_{v_s}^{-1}\bigr)_{k,i}\delta_{s,t}\delta_{j,\ell}. \tag{2.1}$$

2.3. Action

Let A be a unital C*-algebra. We will say that a faithful unital $*$-homomorphism $\alpha : A \to A \otimes C(\mathbb{G})$ is a *(right) action* of $\mathbb{G}$ on A if $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \delta) \circ \alpha$, and $\alpha(A)(\mathbb{C} \otimes C(\mathbb{G}))$ is a dense subspace of $A \otimes C(\mathbb{G})$. Similarly, we can define a left action.

By A^α, we denote the fixed point algebra $\{x \in A \mid \alpha(x) = x \otimes 1\}$. If $A^\alpha = \mathbb{C}$, then α is said to be *ergodic*.

For a von Neumann algebra $\mathcal{M}$, a (right) action means a faithful normal unital $*$-homomorphism $\alpha : \mathcal{M} \to \mathcal{M} \otimes L^\infty(\mathbb{G})$ satisfying $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \delta) \circ \alpha$. The fixed point algebra $\mathcal{M}^\alpha$ is similarly defined.

We will say that a state $\varphi \in \mathcal{M}_*$ is *invariant* when $(\varphi \otimes \mathrm{id})(\alpha(x)) = \varphi(x)1$ for all $x \in \mathcal{M}$.

The crossed product is defined by

$$\mathcal{M} \rtimes_\alpha \mathbb{G} := \overline{\mathrm{span}}^{\mathrm{w}}\big\{\alpha(\mathcal{M})\big(\mathbb{C} \otimes R(\mathbb{G})\big)\big\} \subset \mathcal{M} \otimes B\big(L^2(\mathbb{G})\big),$$

where $R(\mathbb{G})$ denotes the right quantum group algebra, that is,

$$R(\mathbb{G}) := \overline{\mathrm{span}}^{\mathrm{w}}\big\{(\mathrm{id} \otimes \omega)(V) \bigm| \omega \in L^\infty(\mathbb{G})_*\big\}.$$

Let $e_1 := (\mathrm{id} \otimes h)(V) \in R(\mathbb{G})$. Then e_1 is a minimal projection of $B(L^2(\mathbb{G}))$, and $(1 \otimes e_1)(\mathcal{M} \rtimes_\alpha \mathbb{G})(1 \otimes e_1) = \mathcal{M}^\alpha \otimes \mathbb{C}e_1$. Thus $\mathcal{M}^\alpha$ is a corner of $\mathcal{M} \rtimes_\alpha \mathbb{G}$. In particular, if $\mathcal{M} \rtimes_\alpha \mathbb{G}$ is a factor, then so is $\mathcal{M}^\alpha$.

2.4. Quantum subgroup

Let $\mathbb{H}$ and $\mathbb{G}$ be compact quantum groups. We will say that $\mathbb{H}$ is a *quantum subgroup* of $\mathbb{G}$ when there exists a unital surjective $*$-homomorphism $r_\mathbb{H}: C(\mathbb{G}) \to C(\mathbb{H})$, which we will call a *restriction map*, such that $\delta_\mathbb{H} \circ r_\mathbb{H} = (r_\mathbb{H} \otimes r_\mathbb{H}) \circ \delta_\mathbb{G}$, where $\delta_\mathbb{H}$ and $\delta_\mathbb{G}$ denote the coproducts of $\mathbb{H}$ and $\mathbb{G}$, respectively.

Then $\mathbb{H}$ acts on $C(\mathbb{G})$ from the both sides. Namely, $\gamma^\ell_\mathbb{H} := (r_\mathbb{H} \otimes \mathrm{id}) \circ \delta_\mathbb{G}$ and $\gamma^r_\mathbb{H} := (\mathrm{id} \otimes r_\mathbb{H}) \circ \delta_\mathbb{G}$ define left and right actions of $\mathbb{H}$, respectively. Moreover, they are commuting, that is, $(\mathrm{id} \otimes \gamma^r_\mathbb{H}) \circ \gamma^\ell_\mathbb{H} = (\gamma^\ell_\mathbb{H} \otimes \mathrm{id}) \circ \gamma^r_\mathbb{H}$.

Let us introduce the function algebras on the homogeneous spaces as follows:

$$C(\mathbb{H}\backslash\mathbb{G}) := \big\{x \in C(\mathbb{G}) \bigm| \gamma^\ell_\mathbb{H}(x) = 1 \otimes x\big\},$$
$$C(\mathbb{G}/\mathbb{H}) := \big\{x \in C(\mathbb{G}) \bigm| \gamma^r_\mathbb{H}(x) = x \otimes 1\big\}.$$

Then the restrictions of $\delta_\mathbb{G}$ on $C(\mathbb{H}\backslash\mathbb{G})$ and $C(\mathbb{G}/\mathbb{H})$ yield actions from the right and left, respectively. The weak closures of $C(\mathbb{H}\backslash\mathbb{G})$ and $C(\mathbb{G}/\mathbb{H})$ are denoted by $L^\infty(\mathbb{H}\backslash\mathbb{G})$ and $L^\infty(\mathbb{G}/\mathbb{H})$, respectively.

Let $\varepsilon_\mathbb{H}$ and $\varepsilon_\mathbb{G}$ be the counits of $\mathbb{H}$ and $\mathbb{G}$, respectively. Then $\varepsilon_\mathbb{H} \circ r_\mathbb{H} = \varepsilon_\mathbb{G}$. So we will denote simply by ε the counits of $\mathbb{H}$ and $\mathbb{G}$.

Lemma 2.1. *Let α be an action of $\mathbb{G}$ on a von Neumann algebra $\mathcal{M}$. Then there uniquely exists a unital* C*-*subalgebra A *of* $\mathcal{M}$ *such that*

- $\alpha(A) \subset A \otimes C(\mathbb{G})$;
- *if a* C*-*subalgebra* $B \subset \mathcal{M}$ *satisfies* $\alpha(B) \subset B \otimes C(\mathbb{G})$, *then* $B \subset A$;
- A *is weakly dense in* $\mathcal{M}$.

Proof. Let A be a C*-algebra generated by all C*-subalgebras $B \subset \mathcal{M}$ which satisfy $\alpha(B) \subset B \otimes C(\mathbb{G})$. Then it is clear that (1) and (2) hold. We will check (3) as follows.

Let $s \in \mathrm{Irr}(\mathbb{G})$ and H_s the corresponding irreducible module. Let $\mathrm{Hom}_{\mathbb{G}}(H_s, \mathcal{M})$ be the set of $\mathbb{G}$-equivariant linear maps. Then $\mathcal{M}$ is weakly spanned by $T(H_s)$ for $T \in \mathrm{Hom}_{\mathbb{G}}(H_s, \mathcal{M})$ and $s \in \mathrm{Irr}(\mathbb{G})$. It is clear that $T(H_s) \subset A$, and we are done. Note that A is in fact generated by such $T(H_s)$'s. □

Let α and A be as above and $\mathbb{H}$ a quantum subgroup of $\mathbb{G}$ with a restriction map $r_{\mathbb{H}}$. Then we can restrict α on $\mathbb{H}$, that is, $\alpha_{\mathbb{H}} := (\mathrm{id} \otimes r_{\mathbb{H}}) \circ \alpha$ gives an action of $\mathbb{H}$ on A. It is clear that $\alpha_{\mathbb{H}}$ preserves any α-invariant normal state on $\mathcal{M}$. Thus $\alpha_{\mathbb{H}}$ extends to $\mathcal{M}$ as the action of $\mathbb{H}$. We will call $\alpha_{\mathbb{H}}$ the restriction of α by $\mathbb{H}$.

2.5. $U_q(\mathfrak{g})$

We will quickly review the definition of $U_q(\mathfrak{g})$ introduced by Drinfel'd and Jimbo [13, 21], and the highest weight theory. Our references are [6,22,23,25,27]. Let $A = (a_{ij})_{i,j \in I}$ be an irreducible Cartan matrix of finite type ($I := \{1, \dots, n\}$), and $(\mathfrak{h}, \{h_i\}_{i \in I}, \{\alpha_i\}_{i \in I})$ the root data (the realization of A), that is,

- $\mathfrak{h}$ is an n-dimensional vector space over $\mathbb{C}$, and $\{h_i\}_i$ is a base of $\mathfrak{h}$;
- $\{\alpha_i\}_i$ is a base of $\mathfrak{h}^*$, the space of linear functionals on $\mathfrak{h}$;
- $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$.

Each α_i is called a simple root. The simple reflection $s_i \colon \mathfrak{h}^* \to \mathfrak{h}^*$ is defined by $s_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$. Note that $s_i^2 = 1$. The Weyl group W is the finite group generated by s_i's. The word length of $w \in W$ with respect to $\{s_1, \dots, s_n\}$ is denoted by $\ell(w)$. We denote by w_0 an element of maximal length. It is known that any $w \in W$ is contained in w_0, that is, $\ell(w_0) = \ell(w) + \ell(w^{-1}w_0)$. It follows from this equality that w_0 is unique and $w_0^2 = 1$.

Take positive integers $\{d_i\}_{i \in I}$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in I$. A standard form is an inner product $(\cdot,\cdot)$ on $\mathfrak{h}^*$ such that $(\alpha_i, \alpha_j) = d_i a_{ij}$ for $i, j \in I$. It is known that $(\alpha_i, \alpha_i) = 2d_i$ can attain at most two values. We normalize $\{d_i\}_{i \in I}$ so that the smallest value of (α_i, α_i) is equal to 2. Then the normalized standard form satisfies the following properties:

- W-invariance, that is, $(w\lambda, w\mu) = (\lambda, \mu)$ for $w \in W$, $\lambda, \mu \in \mathfrak{h}^*$;
- $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}_+$ for $i \in I$;
- $\lambda(h_i) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ for $i \in I$.

We associate A with a finite dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ whose Cartan subalgebra is $\mathfrak{h}$.

Definition 2.2 *(Drinfel'd, Jimbo).* Let $0 < q < 1$. The *quantum universal enveloping algebra* $U_q(\mathfrak{g})$ is the unital $\mathbb{C}$-algebra generated by $\{K_i, X_i^+, X_i^-\}_{i\in I}$ such that K_i is invertible for all $i \in I$, and the following relations hold for all $i, j \in I$:

$$K_iK_j = K_jK_i, \qquad K_iX_j^\pm = q^{\pm(\alpha_i,\alpha_j)/2}X_j^\pm K_i; \tag{2.2}$$

$$X_i^+X_j^- - X_j^-X_i^+ = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}; \tag{2.3}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i^2} q_i^{-k(1-a_{ij}-k)}\left(X_i^\pm\right)^k X_j^\pm \left(X_i^\pm\right)^{1-a_{ij}-k} = 0 \quad \text{if } i \neq j, \tag{2.4}$$

where $q_i := q^{(\alpha_i,\alpha_i)/2}$, and the t-binomial is defined by

$$\binom{m}{n}_t := \frac{(t;t)_m}{(t;t)_n(t;t)_{m-n}}, \qquad (a;t)_m = (1-a)(1-at)\cdots\left(1-at^{m-1}\right)$$

for $m, n \in \mathbb{Z}_+$ and $a \in \mathbb{C}$.

Then $U_q(\mathfrak{g})$ has the Hopf$*$-algebra structure defined as follows: For $i \in I$,

- (Coproduct)

$$\Delta\left(K_i^\pm\right) = K_i^\pm \otimes K_i^\pm, \qquad \Delta\left(X_i^\pm\right) = X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm,$$

- (Antipode)

$$S\left(K_i^\pm\right) = K_i^\mp, \qquad S\left(X_i^\pm\right) = -q_i^{\pm 1}X_i^\pm,$$

- (Counit)

$$\varepsilon(K_i) = 1, \qquad \varepsilon\left(X_i^\pm\right) = 0,$$

- (Involution)

$$K_i^* = K_i, \qquad \left(X_i^\pm\right)^* = X_i^\mp. \tag{2.5}$$

2.6. Representation theory of $U_q(\mathfrak{g})$

For a finite dimensional Hilbert space H, we call a $*$-homomorphism π from $U_q(\mathfrak{g})$ into $B(H)$ a *representation*. If the commutant of $\operatorname{Im}\pi$ is trivial, π is said to be *irreducible*. In general, $\operatorname{Im}\pi$ is a finite dimensional C*-algebra, and π is the direct sum of irreducibles.

Let $\pi: U_q(\mathfrak{g}) \to B(H)$ be an irreducible representation. From (2.5), we obtain non-singular self-adjoint operators $\pi(K_i)$. Let $p_i^\pm$ be the projections onto the positive and negative spectral subspaces of $\pi(K_i)$, respectively. Then the relations (2.2) and (2.5) imply $p_i^{\pm}H$ are $U_q(\mathfrak{g})$-invariant. Hence each K_i has either only positive eigenvalues or only negative ones. We will say that π is *admissible* when $\pi(K_i)$ is a positive operator for all $i \in I$.

For $g = (g_k)_k \in \mathbb{Z}_2^n = \prod_{k=1}^n \{1, -1\}$, we will define an automorphism σ_g on $U_q(\mathfrak{g})$ as a $*$-algebra (not as a Hopf$*$-algebra) by

$$\sigma_g\bigl(K_i^{\pm}\bigr) = g_i K_i^{\pm}, \qquad \sigma_g\bigl(X_i^{\pm}\bigr) = X_i^{\pm} \quad \text{for } i \in I.$$

Then for any irreducible π, we can find $g \in \mathbb{Z}_2^n$ so that $\pi \circ \sigma_g$ is admissible. Hence the classification of irreducible admissible representations is essential.

Let $\pi: U_q(\mathfrak{g}) \to B(H)$ be a finite dimensional admissible representation. Then $\pi(K_i)$'s are generating a commutative C*-subalgebra. Hence we obtain the decomposition of H into the common eigenspaces. For $\lambda \in \mathfrak{h}^*$, we define

$$H_\lambda := \bigl\{\xi \in H \bigm| \pi(K_i)\xi = q^{(\lambda,\alpha_i)/2}\xi\bigr\}.$$

Note that $q^{(\lambda,\alpha_i)/2} = q_i^{\lambda(h_i)/2}$. Then the positivity of each $\pi(K_i)$ implies the direct sum decomposition of H:

$$H = \bigoplus_{\lambda \in \mathfrak{h}^*} H_\lambda.$$

If $H_\lambda \neq \{0\}$, then λ and H_λ are called a *weight* and a *weight space* of π (or of the $U_q(\mathfrak{g})$-module H), respectively. By $\mathrm{Wt}(H)$, we denote the collection of the weights of π. It is known that $\dim H_\lambda = \dim H_{w\lambda}$ for all $\lambda \in \mathrm{Wt}(H)$ and $w \in W$.

If a non-zero vector $\xi \in H_\lambda$ is cyclic and $\pi(X_i^+)\xi = 0$ for all $i \in I$, then λ and ξ are called a *highest weight* and a *highest weight vector* of H, respectively. A lowest weight and a lowest vector are similarly introduced by using X_i^- instead of X_i^+. It is known that if λ is a highest weight, then $w_0\lambda$ is a lowest weight.

By $\mathrm{Irr}^+(U_q(\mathfrak{g}))$, we denote the set of the unitary equivalence classes of irreducible admissible representations of finite dimension.

Let us introduce the root lattice $Q := \sum_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$, and set $Q_+ := \sum_{i\in I} \mathbb{Z}_+\alpha_i$. We will equip $\mathfrak{h}^*$ with the partial order $\le$ such that $\lambda \le \mu$ if and only if $\mu - \lambda \in Q_+$. Then it turns out that a finite dimensional irreducible admissible representation has a unique maximal weight that is in fact the highest weight. Moreover, the weight space of the highest weight is one-dimensional.

Let

$$P := \bigl\{\lambda \in \mathfrak{h}^* \bigm| \lambda(h_i) \in \mathbb{Z} \text{ for all } i \in I\bigr\},$$

and

$$P_+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_+ \text{ for all } i \in I\}.$$

We will call an element of P and P_+ an *integral weight* and a *dominant integral weight*, respectively. When $\lambda \in P_+$ satisfies $\lambda(h_i) > 0$ for all $i \in I$, λ is said to be *regular*. We will denote by P_{++} the set of regular dominant weights. Note that $Q \subset P \subset \sum_{i\in I} \mathbb{Q}\alpha_i$ since A is an invertible matrix.

From $\lambda \in P_+$, we can construct an irreducible module $L(\lambda)$ that is the irreducible quotient of the Verma module $M(\lambda)$. The highest weight theory says that there exists a one-to-one correspondence $P_+ \ni \lambda \longleftrightarrow L(\lambda) \in \mathrm{Irr}^+(U_q(\mathfrak{g}))$. Let $\pi_\lambda : U_q(\mathfrak{g}) \to B(L(\lambda))$ be the corresponding representation.

Let $\omega_i \in P_+$ be the *fundamental weight* that is defined by $\omega_i(h_j) = \delta_{i,j}$ for $i, j \in I$. Then $P = \sum_{i\in I} \mathbb{Z}\omega_i$ and $P_+ = \sum_{i\in I} \mathbb{Z}_+\omega_i$.

The Weyl vector is defined by $\rho := (1/2)\sum_{\alpha\in\Delta_+} \alpha$, where $\Delta := \{w\alpha_i \mid w \in W,\ i \in I\}$ and $\Delta_+ := \Delta \cap Q_+$. It is known that $\rho(h_i) = 1$ for all $i \in I$, that is, $\rho = \sum_{i\in I} \omega_i$. In particular, ρ is a dominant integral weight.

2.7. $C(G_q)$

Let $\pi : U_q(\mathfrak{g}) \to B(H_\pi)$ be a finite dimensional representation. For vectors $\xi, \eta \in H_\pi$, we will define a linear functional $C^\pi_{\xi,\eta}$ on $U_q(\mathfrak{g})$ by

$$C^\pi_{\xi,\eta}(x) := \langle \pi(x)\eta, \xi\rangle \quad \text{for } x \in U_q(\mathfrak{g}).$$

For $\lambda \in P_+$ and $\mu \in \mathrm{Wt}(L(\lambda))$, we will fix an orthonormal base $\{\xi^i_\mu \mid i \in I^\lambda_\mu\}$ of $L(\lambda)_\mu$, where $I^\lambda_\mu := \{1, \dots, \dim L(\lambda)_\mu\}$. We often write $C^\lambda_{\xi^i_\mu,\xi^j_\nu}$ or, more simply, $C^\lambda_{\mu_i,\nu_j}$ for $C^{\pi_\lambda}_{\xi^i_\mu,\xi^j_\nu}$.

Since $\dim L(\lambda)_{w\lambda} = \dim L(\lambda)_\lambda = 1$ for $w \in W$, we simply denote by $C^\lambda_{w\lambda,w'\lambda}$ for $C^\lambda_{\xi_{w\lambda},\xi_{w'\lambda}}$, where $\xi_{w\lambda} \in L(\lambda)_{w\lambda}$ and $\xi_{w'\lambda} \in L(\lambda)_{w'\lambda}$ are fixed unit vectors for $w, w' \in W$. Note that there exists an ambiguity of a constant factor of modulus one about this expression.

Then we will introduce the following subspace of $U_q(\mathfrak{g})^*$:

$$\begin{aligned} A(G_q) &:= \mathrm{span}\{C^\pi_{\xi,\eta} \mid \xi, \eta \in H_\pi,\ \pi \text{ is an admissible representation}\} \\ &= \mathrm{span}\{C^\lambda_{\mu_i,\nu_j} \mid \mu, \nu \in \mathrm{Wt}(L(\lambda)),\ i \in I^\lambda_\mu,\ j \in I^\lambda_\nu,\ \lambda \in P_+\}. \end{aligned}$$

Then $A(G_q)$ inherits the Hopf$*$-algebra structure from $U_q(\mathfrak{g})$. (See [27, Chapter 3].) The following equality is frequently used:

$$\delta(C^\lambda_{\mu_i,\nu_j}) = \sum_{\zeta\in\mathrm{Wt}(\lambda),\ k\in I^\lambda_\zeta} C^\lambda_{\mu_i,\zeta_k} \otimes C^\lambda_{\zeta_k,\nu_j},$$

where $\mathrm{Wt}(\lambda)$ denotes $\mathrm{Wt}(L(\lambda))$. The C*-completion of $A(G_q)$ is denoted by $C(G_q)$. It is known that G_q is co-amenable. Hence the Haar state h is faithful, and the counit ε is norm bounded on $C(G_q)$ (see, for example, [2, Corollary 5.1]). The following result is well-known for experts (see [25, Example 9, p. 425]).

Lemma 2.3. *For $z \in \mathbb{C}$, the Woronowicz character $f_z : A(G_q) \to \mathbb{C}$ is given by*

$$f_z\big(C^\lambda_{\xi_\mu,\xi_\nu}\big) = \langle \xi_\nu, \xi_\mu \rangle q^{2(\mu,\rho)z} \quad \textit{for all } \xi_\mu \in L(\lambda)_\mu,\ \xi_\nu \in L(\lambda)_\nu.$$

Hence for $t \in \mathbb{R}$, $\lambda \in P_+$ and $\mu, \nu \in \mathrm{Wt}(\lambda)$, we have

$$\sigma^h_t\big(C^\lambda_{\xi_\mu,\xi_\nu}\big) = q^{2(\mu+\nu,\rho)it} C^\lambda_{\xi_\mu,\xi_\nu} \quad \text{for all } \xi_\mu \in L(\lambda)_\mu,\ \xi_\nu \in L(\lambda)_\nu, \tag{2.6}$$

and the quantum dimension $\dim_q L(\lambda)$ is given by

$$\dim_q L(\lambda) = \sum_{\mu \in \mathrm{Wt}(\lambda)} q^{2(\mu,\rho)} \dim L(\lambda)_\mu. \tag{2.7}$$

2.8. The maximal torus and the quantum flag manifold

Let us denote by T the n-torus, that is, $T = \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid |t_i| = 1,\ i = 1, \dots, n\}$. Recall that $n = |I|$. By the coupling $\langle \cdot, \cdot \rangle : T \times P \to \mathbb{C}$ defined by

$$\big\langle (t_1, \dots, t_n), \mu \big\rangle := t_1^{\mu(h_1)} \cdots t_n^{\mu(h_n)} \quad \text{for } (t_1, \dots, t_n) \in T,\ \mu \in P,$$

we will regard P as the dual group of T. Let $\mathrm{ev}_t : C(T) \to \mathbb{C}$ be the evaluation map at $t \in T$. Then the restriction map $r_T : C(G_q) \to C(T)$ is defined as follows:

$$\mathrm{ev}_t \circ r_T\big(C^\lambda_{\xi_\mu,\xi_\nu}\big) = \langle t, \mu \rangle \langle \xi_\nu, \xi_\mu \rangle \quad \text{for all } t \in T,\ \xi_\mu \in L(\lambda)_\mu,\ \xi_\nu \in L(\lambda)_\nu. \tag{2.8}$$

With this map, T is a quantum subgroup of G_q and called the *maximal torus*. Note that r_T actually comes from the inclusion map $U_q(\mathfrak{h})$ into $U_q(\mathfrak{g})$.

Let $\gamma_t := (\mathrm{ev}_t \circ r_T \otimes \mathrm{id}) \circ \delta$ be the left action of T on $C(G_q)$. Then we have

$$\gamma_t\big(C^\lambda_{\xi_\mu,\xi_\nu}\big) = \langle t, \mu \rangle C^\lambda_{\xi_\mu,\xi_\nu} \quad \text{for all } t \in T,\ \xi_\mu \in L(\lambda)_\mu,\ \xi_\nu \in L(\lambda)_\nu. \tag{2.9}$$

Let us denote by $C(T\backslash G_q)$ the fixed point algebra of the T-action γ. The weak closure of $C(T\backslash G_q)$ is denoted by $L^\infty(T\backslash G_q)$ that is called the *quantum flag manifold*.

3. Invariant cocycles and 2-cocycle deformations

In this section, we will introduce the notion of invariant cocycles evaluated in $L^\infty(T\backslash G_q)$ and determine them. As an application, we will describe the intrinsic group of a twisted quantum group $G_{q,\Omega}$.

3.1. Factoriality of $L^\infty(T\backslash G_q)$

Let $\mathbb{G}$ be a compact quantum group. In [46], it is conjectured that the classical Poisson boundary $H^\infty_{\text{class}}(\widehat{\mathbb{G}}, \mu)$ could coincide with the center of the Poisson boundary $Z(H^\infty(\widehat{\mathbb{G}}, \mu))$. (See [17,20,44] for a detail of theory of a Poisson boundary.) In particular, if the fusion rule of $\mathbb{G}$ is commutative, the conjecture asks the factoriality of the Poisson boundary. It has been verified that the conjecture is true for $SU_q(2)$, $A_o(F)$ [47,49] and $A_u(F)$ [50]. To these examples, we can show that a stronger property that the canonical $\mathbb{G}$-action on a Poisson boundary is "approximately inner." We will show the factoriality of $L^\infty(T\backslash G_q)$ as follows.

For $\lambda \in P_+$, we let $a_\lambda := C^\lambda_{\lambda, w_0\lambda}$. Then they generate a commutative C*-subalgebra of $C(G_q)$. See [27, Corollary 2.1.5, Propositions 2.2.4, 2.3.2, Chapter 3] for its proof. Readers should note that several typographical errors concerning signs are found in [27, Proposition 2.1.4, Corollary 2.1.5, Proposition 2.3.2, Chapter 3].

Theorem 3.1. *The following statements hold:*

(1) *γ is a faithful action of T on the center $Z(L^\infty(G_q))$;*
(2) *$L^\infty(T\backslash G_q)$ is the type I_∞ factor.*

Proof. (1). Let $\lambda, \Lambda \in P_+$, $\mu, \nu \in \mathrm{Wt}(\lambda)$, $\xi_\mu \in L(\lambda)_\mu$ and $\xi_\nu \in L(\lambda)_\nu$. Then by [27, Corollary 2.1.5, Proposition 2.3.2, Chapter 3], we have

$$C^\lambda_{\xi_\mu,\xi_\nu} a_\Lambda = q^{(\Lambda, -\mu+w_0\nu)} a_\Lambda C^\lambda_{\xi_\mu,\xi_\nu}, \qquad C^\lambda_{\xi_\mu,\xi_\nu} a^*_\Lambda = q^{(\Lambda, -\mu+w_0\nu)} a^*_\Lambda C^\lambda_{\xi_\mu,\xi_\nu}.$$

Thus

$$C^\lambda_{\xi_\mu,\xi_\nu} |a_\Lambda| = q^{(\Lambda, -\mu+w_0\nu)} |a_\Lambda| C^\lambda_{\xi_\mu,\xi_\nu}. \tag{3.1}$$

Let $a_\Lambda = v_\Lambda |a_\Lambda|$ be the polar decomposition in $L^\infty(G_q)$. Then $v_\Lambda C^\lambda_{\xi_\mu,\xi_\nu} = C^\lambda_{\xi_\mu,\xi_\nu} v_\Lambda$. Hence v_Λ belongs to the center of $L^\infty(G_q)$. Then $\gamma_t(v_\Lambda) = \langle t, \Lambda\rangle v_\Lambda$ for $t \in T$. Thus γ is faithful on $Z(L^\infty(G_q))$.

(2). This is a direct consequence of [46, Theorem 4.7]. We will sketch the proof for readers' convenience. Let $H^\infty(\widehat{G}_q)$ be the Poisson boundary of $\widehat{G}_q$ and $\Theta: L^\infty(G_q) \to R(G_q)$ the Poisson integral defined by $\Theta(x) := (\mathrm{id} \otimes h)(V^*(1 \otimes x)V)$ for $x \in L^\infty(G_q)$. Then Θ is a $\widehat{G}_q$–G_q-equivariant isomorphism from $L^\infty(T\backslash G_q)$ onto $H^\infty(\widehat{G}_q)$. See [17, Theorem 5.10], [20, Theorems A, B] and [44, Corollary 4.11] for the proof.

Let $x \in Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q)$. Then $\Theta(x)$ is fixed by the coproduct of $R(G_q)$, and $\Theta(x)$ is a scalar. Hence $Z(L^\infty(G_q))^\gamma = Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q) = \mathbb{C}$ which shows the central ergodicity of γ.

By (1), $Z(L^\infty(G_q))$ is generated by unitaries v_λ, $\lambda \in P_+$, such that $\gamma_t(v_\lambda) = \langle t, \lambda\rangle v_\lambda$ for $t \in T$. From (2.9), it turns out that $L^\infty(G_q) = Z(L^\infty(G_q)) \vee L^\infty(T\backslash G_q)$. Hence $Z(L^\infty(T\backslash G_q)) \subset Z(L^\infty(G_q)) \cap L^\infty(T\backslash G_q) = \mathbb{C}$. $\square$

Note that by ergodicity, v_Λ in the above proof must be a unitary. As a corollary, we have the following.

Corollary 3.2. *The Poisson boundary $H^\infty(\widehat{G}_q)$ is the type I_∞ factor.*

Recall Dijkhuizen–Stokman's result [12, Theorem 5.9] which states that any C*-irreducible representation of $C(T\backslash G_q)$ is obtained by restricting a C*-irreducible representation of $C(G_q)$. Among them, a one-dimensional representation of $C(G_q)$ factors through $C(T)$ via the canonical restriction map and its restriction of $C(T\backslash G_q)$ coincides with the counit ε.

Lemma 3.3 *(Dijkhuizen–Stokman). The counit is the unique character of $C(T\backslash G_q)$.*

For an action α of G_q on a unital C*-algebra (or von Neumann algebra) A, we denote by $\mathrm{Aut}_{G_q}(A)$ the set of G_q-equivariant automorphisms on A, that is,

$$\mathrm{Aut}_{G_q}(A) := \{\theta \in \mathrm{Aut}(A) \mid \alpha \circ \theta = (\theta \otimes \mathrm{id}) \circ \alpha\}.$$

Theorem 3.4. *The embedding of $C(T\backslash G_q)$ into $C(G_q)$ is unique.*

Proof. Suppose that $\psi: C(T\backslash G_q) \to C(G_q)$ is a G_q-equivariant embedding. By Lemma 3.3, we have $\varepsilon \circ \psi = \varepsilon$ on $C(T\backslash G_q)$. Then for $x \in C(T\backslash G_q)$,

$$x = (\varepsilon \otimes \mathrm{id})(\delta(x)) = (\varepsilon \circ \psi \otimes \mathrm{id})(\delta(x)) = (\varepsilon \otimes \mathrm{id})(\delta(\psi(x))) = \psi(x). \qquad \square$$

From the previous result, we have the following.

Corollary 3.5. *One has $\mathrm{Aut}_{G_q}(C(T\backslash G_q)) = \{\mathrm{id}\}$.*

This also implies $\mathrm{Aut}_{G_q}(L^\infty(T\backslash G_q)) = \{\mathrm{id}\}$ because an equivariant map on $L^\infty(T\backslash G_q)$ preserves each finite dimensional spectral subspace and also $C(T\backslash G_q)$.

3.2. Invariant cocycles evaluated in $L^\infty(T\backslash G_q)$

Recall that the left T-action γ is faithful and ergodic on $Z(L^\infty(G_q))$. For each fundamental weight $\omega_i \in P$, $i \in I$, we take a unitary v_{ω_i} in $Z(L^\infty(G_q))$ such that $\gamma_t(v_{\omega_i}) = \langle t, \omega_i \rangle v_{\omega_i}$ for $t \in T$. Next, for $\lambda = \sum_{i=1}^n a_i\omega_i \in P$, $a_i \in \mathbb{Z}$, we set $v_\lambda := v_{\omega_1}^{a_1} \cdots v_{\omega_n}^{a_n}$. Then we have

$$\gamma_t(v_\lambda) = \langle t, \lambda \rangle v_\lambda, \qquad v_\lambda v_\mu = v_{\lambda+\mu} \quad \text{for } t \in T,\ \lambda, \mu \in P. \tag{3.2}$$

Note that v_λ's are generating $Z(L^\infty(G_q))$ as a von Neumann algebra, and $L^\infty(G_q) = Z(L^\infty(G_q)) \vee L^\infty(T\backslash G_q)$.

Now for each $\lambda \in P$, we introduce the unitary w_λ defined by:

$$\delta(v_\lambda) = (v_\lambda \otimes 1)w_\lambda. \tag{3.3}$$

Since δ and γ are commuting, w_λ belongs to $L^\infty(T\backslash G_q) \otimes L^\infty(G_q)$. From the above equation, the cocycle relation $(w_\lambda \otimes 1)(\delta \otimes \mathrm{id})(w_\lambda) = (\mathrm{id} \otimes \delta)(w_\lambda)$ holds. Moreover, setting $\delta^{w_\lambda} := \mathrm{Ad}\, w_\lambda \circ \delta$, we have $\delta^{w_\lambda} = \delta$ on $L^\infty(G_q)$ because v_λ is a central unitary.

Let us denote by $Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$ the collection of $L^\infty(T\backslash G_q)$-valued cocycles w such that $\delta^w = \delta$ on $L^\infty(T\backslash G_q)$. In general, when α is an action of G_q on a von Neumann algebra $\mathcal{N}$, we denote by $Z^1_{\mathrm{inv}}(\alpha, \mathcal{N})$ the set of α-cocycles w with $\alpha^w = \alpha$. We call such w an *invariant cocycle*.

We know w_λ's are invariant cocycles of the action of δ on $L^\infty(T\backslash G_q)$, but in fact, they are all.

Theorem 3.6. *In the above setting, the following statements hold:*

(1) $Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q)) = \{w_\lambda \mid \lambda \in P\}$;
(2) *the map* $P \ni \lambda \mapsto w_\lambda \in Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$ *is a group isomorphism.*

Proof. (1). Let $w \in Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$. First, we will observe that the twisted right action δ^w is ergodic on $L^\infty(G_q)$. Let $x \in L^\infty(G_q)^{\delta^w}$. Since $w \in L^\infty(T\backslash G_q) \otimes L^\infty(G_q)$, the actions γ and δ^w are commuting. Thus we may and do assume that $\gamma_t(x) = \langle t, \mu\rangle x$ for some $\mu \in P$ and all $t \in T$. Then γ fixes $y := v_\mu^* x$, and y belongs to $L^\infty(T\backslash G_q)$. Using $w\delta(x)w^* = \delta^w(x) = x \otimes 1$ and $\delta^w = \delta$ on $L^\infty(T\backslash G_q)$, we have

$$ww_\mu w^*\delta(y) = w\bigl(v_\mu^* \otimes 1\bigr)\delta(v_\mu) \cdot \delta(y)w^* = \bigl(v_\mu^* \otimes 1\bigr)w\delta(x)w^* = y \otimes 1. \tag{3.4}$$

In particular, y^*y and yy^* are fixed by δ since $ww_\mu w^*$ commutes with $\delta(yy^*)$. By ergodicity of δ on $L^\infty(T\backslash G_q)$, we may and do assume that y is a unitary.

Set the inner automorphism $\psi := \mathrm{Ad}\, y^*$ on $L^\infty(T\backslash G_q)$. Then for $z \in L^\infty(T\backslash G_q)$,

$$\begin{aligned}\delta\bigl(\psi(z)\bigr) &= \delta\bigl(y^*\bigr)\delta(z)\delta(y) = \bigl(y^* \otimes 1\bigr)ww_\mu w^* \cdot \delta(z) \cdot ww_\mu^* w^*(y \otimes 1) \quad \text{by (3.4)} \\ &= \bigl(y^* \otimes 1\bigr)\delta(z)(y \otimes 1) = (\psi \otimes \mathrm{id})\bigl(\delta(z)\bigr).\end{aligned}$$

Namely, $\psi \in \mathrm{Aut}_{G_q}(L^\infty(T\backslash G_q))$. However, we must have $\psi = \mathrm{id}$ by Corollary 3.5. It turns out from Theorem 3.1 that $y \in \mathbb{C}$. From (3.4), we have $w_\mu = 1$. This shows $\mu = 0$. (See the proof of the second statement.) Hence x is a scalar, and δ^w is ergodic on $L^\infty(G_q)$.

Second, we will use Connes' 2×2 matrix trick. Let $\mathcal{N} := M_2(\mathbb{C}) \otimes L^\infty(G_q)$ and $\{e_{ij}\}_{i,j=1}^2$ be a system of matrix units of $M_2(\mathbb{C})$. We set $\alpha := \mathrm{id} \otimes \delta$ and $\overline{w} := e_{11} \otimes 1 + e_{22} \otimes w$. Then $\overline{w}$ is an α-cocycle. We will show that the projections $p_1 := e_{11} \otimes 1$ and $p_2 := e_{22} \otimes 1$ are Murray–von Neumann equivalent in $\mathcal{N}^\alpha$.

Consider the crossed product $\mathcal{N} \rtimes_\alpha G_q$. Since α is a cocycle perturbation of $\mathrm{id} \otimes \delta$, $\mathcal{N} \rtimes_\alpha G_q$ is canonically isomorphic to $M_2(\mathbb{C}) \otimes L^\infty(G_q) \rtimes_\delta G_q$, and also to $M_2(\mathbb{C}) \otimes B(L^2(G_q))$.

Thus $\mathcal{N} \rtimes_\alpha G_q$ is the type I_∞ factor. The fixed point algebra $\mathcal{N}^\alpha$ is a corner of $\mathcal{N} \rtimes_\alpha G_q$, and $\mathcal{N}^\alpha$ is a type I factor. Since $p_1\mathcal{N}^\alpha p_1 = L^\infty(G_q)^\delta = \mathbb{C}$ and $p_2\mathcal{N}^\alpha p_2 = L^\infty(G_q)^{\delta^w} = \mathbb{C}$, p_1 and p_2 are minimal projections in $\mathcal{N}^\alpha$. Hence they are equivalent.

Let us take a unitary $v \in L^\infty(G_q)$ such that $e_{12} \otimes v \in \mathcal{N}^\alpha$, that is, $\delta(v) = (v \otimes 1)w$. The ergodicity of δ shows that v is the unique solution of this equation up to a scalar multiple. Indeed, if v' is a (not necessarily unitary) another solution, then $\delta(v'v^*) = (v' \otimes 1)w \cdot w^*(v^* \otimes 1) = v'v^* \otimes 1$, and $v'v^* \in \mathbb{C}$. Then by the commutativity of γ and δ and also the equality $(\gamma_t \otimes \mathrm{id})(w) = w$, we can find a unique $\lambda \in P$ such that $v = v_\lambda v_1$ for some unitary $v_1 \in L^\infty(T\backslash G_q)$. However, the invariance $\delta^w = \delta$ on $L^\infty(T\backslash G_q)$ deduces $\mathrm{Ad}\, v_1 \in \mathrm{Aut}_{G_q}(L^\infty(T\backslash G_q))$. Hence v_1 is a scalar as before, and $w = w_\lambda$.

(2). Let $\lambda, \mu \in P$. Since v_λ is central and $v_\lambda v_\mu = v_{\lambda+\mu}$, we have

$$w_\lambda w_\mu = (v_\lambda^* \otimes 1)\delta(v_\lambda) \cdot (v_\mu^* \otimes 1)\delta(v_\mu) = (v_\lambda^* \otimes 1)(v_\mu^* \otimes 1)\delta(v_\lambda) \cdot \delta(v_\mu) = w_{\lambda+\mu}.$$

To show the injectivity, let $w_\lambda = 1$. Then v_λ is fixed by δ, and v_λ is a scalar. Hence $\langle t, \lambda\rangle = 1$ for all $t \in T$, and $\lambda = 0$. □

3.3. 2-Cocycle deformations

As we have shown, the quantum flag manifold $L^\infty(T\backslash G_q)$ is a type I factor. So, we would like to find a unitary which implements the right action δ on $L^\infty(T\backslash G_q)$.

Lemma 3.7. *There exists a unitary $U \in L^\infty(T\backslash G_q) \otimes L^\infty(G_q)$ such that $\delta(x) = U(x \otimes 1)U^*$ for all $x \in L^\infty(T\backslash G_q)$.*

Proof. Let p_0 be the distinguished minimal projection of $C(T\backslash G_q)$. (See [27, Chapter 3.5].) Namely, p_0 satisfies $|a_\rho|p_0 = p_0$, where $a_\rho = C^\rho_{\rho,w_0\rho}$. Actually, p_0 is also contained in $C(G_q/T)$, and $\delta(p_0) \in C(T\backslash G_q) \otimes C(G_q/T)$. Recall that $L^\infty(T\backslash G_q) \otimes L^\infty(G_q/T)$ is a type I factor. We will show that $\delta(p_0)$ is an infinite projection.

Suppose that $\delta(p_0)$ is of finite rank. Then $\delta(p_0)(C(T\backslash G_q) \otimes C(G_q/T))\delta(p_0)$ is finite dimensional. Recall π_{w_0} that is a C*-irreducible representation of $C(G_q)$ associated by $w_0 \in W$ (see [37, Theorems 3.4, 5.7] or [27, Theorems 5.3.3, 6.2.7, Chapter 3]). Thanks to [12, Theorem 5.9], the restrictions of π_{w_0} on $C(T\backslash G_q)$ and $C(G_q/T)$ are irreducible, $e := (\pi_{w_0} \otimes \pi_{w_0})(\delta(p_0))$ is a finite rank projection of $B(H_{w_0}) \otimes B(H_{w_0})$. In particular, e is a compact operator on $H_{w_0} \otimes H_{w_0}$.

Recall the fact that the counit of $C(G_q)$ factors through π_{w_0} and the quotient map $B(H_{w_0}) \to B(H_{w_0})/K(H_{w_0})$. (See [44, p. 294].) Thus we have a $*$-homomorphism $\eta_{w_0}\colon \mathrm{Im}\,\pi_{w_0} \to \mathbb{C}$ such that $\eta_{w_0} \circ \pi_{w_0} = \varepsilon$ and $\eta_{w_0} = 0$ on $\mathrm{Im}\,\pi_{w_0} \cap K(H_{w_0})$. Then we obtain $(\mathrm{id}_{K(H_{w_0})} \otimes \eta_{w_0})(e) = 0$. This implies that $\pi_{w_0}(p_0) = 0$. It follows from the faithfulness of π_{w_0} that p_0 equals 0, a contradiction. Hence $\delta(p_0)$ is an infinite projection of $L^\infty(T\backslash G_q) \otimes L^\infty(G_q/T)$.

Take a partial isometry $v \in L^\infty(T\backslash G_q) \otimes L^\infty(G_q/T)$ such that $v^*v = p_0 \otimes 1$ and $vv^* = \delta(p_0)$. Let $\{e_{ij}\}_{i,j=0}^\infty$ be a system of matrix units of $L^\infty(T\backslash G_q)$ such that $e_{00} = p_0$. Then a unitary $U := \sum_{i=0}^\infty \delta(e_{i0})v(e_{0i} \otimes 1)$ implements δ. □

Remark 3.8. By the same proof as the above, we can show that $(\pi_{w_0} \otimes \pi_w)(\delta(p_0))$ is a projection of infinite rank on $H_{w_0} \otimes H_w$ for any $w \in W \neq \{e\}$.

The coassociativity of δ, implies that $(\mathrm{id} \otimes \delta)(U)^* U_{12} U_{13}$ commutes with $x \otimes 1 \otimes 1$ for all $x \in L^\infty(T\backslash G_q)$. By factoriality of $L^\infty(T\backslash G_q)$, we obtain a unitary $\Omega \in L^\infty(G_q) \otimes L^\infty(G_q)$ such that

$$U_{12}U_{13} = (\mathrm{id} \otimes \delta)(U)\big(1 \otimes \Omega^*\big).$$

Then Ω satisfies the following 2-cocycle relation:

$$(\Omega \otimes 1)(\delta \otimes \mathrm{id})(\Omega) = (1 \otimes \Omega)(\mathrm{id} \otimes \delta)(\Omega).$$

Denote by δ_Ω the twisted coproduct $\mathrm{Ad}\,\Omega \circ \delta$. Then thanks to [9, Theorem 6.2], the pair $(L^\infty(G_q), \delta_\Omega)$ becomes a new, in general non-compact, locally compact quantum group in the sense of [28], which we will denote by $G_{q,\Omega}$. Set $L^\infty(G_{q,\Omega}) := L^\infty(G_q)$. Readers are referred to [10] for a general treatment of projective representations.

We will not study a concrete description of $G_{q,\Omega}$, but describe group-like elements. A unitary $u \in L^\infty(G_{q,\Omega})$ is said to be *group-like* when $\delta_\Omega(u) = u \otimes u$. Denote by $\mathscr{G}(G_{q,\Omega})$ the collection of all group-like elements of $G_{q,\Omega}$, which is called the *intrinsic group* of $G_{q,\Omega}$.

For a unitary $u \in L^\infty(G_{q,\Omega})$, we will put

$$w^u := U(1 \otimes u)U^* \in L^\infty(T\backslash G_q) \otimes L^\infty(G_q). \tag{3.5}$$

Lemma 3.9. *The unitary w^u is a δ-cocycle if and only if $u \in \mathscr{G}(G_{q,\Omega})$.*

Proof. This is shown by the following straightforward computations:

$$\begin{aligned}\big(w^u \otimes 1\big)(\delta \otimes \mathrm{id})\big(w^u\big) &= U_{12}u_2U_{12}^* \cdot U_{12}\big(w^u\big)_{13}U_{12}^* \\ &= U_{12}u_2U_{13}u_3U_{13}^*U_{12}^* \\ &= (\mathrm{id} \otimes \delta)(U)\Omega_{23}^*(1 \otimes u \otimes u)\Omega_{23}(\mathrm{id} \otimes \delta)\big(U^*\big),\end{aligned}$$

and

$$(\mathrm{id} \otimes \delta)\big(w^u\big) = (\mathrm{id} \otimes \delta)(U)\big(1 \otimes \delta(u)\big)(\mathrm{id} \otimes \delta)\big(U^*\big). \qquad \square$$

By Lemma 3.7 and (3.5), $w^u\delta(x) = \delta(x)w^u$ for all $x \in L^\infty(T\backslash G_q)$. Thus w^u belongs to $Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$ if $u \in \mathscr{G}(G_{q,\Omega})$.

Theorem 3.10. *The map $\mathscr{G}(G_{q,\Omega}) \ni u \mapsto w^u \in Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$ is a group isomorphism. In particular, $\mathscr{G}(G_{q,\Omega})$ is isomorphic to $P = \widehat{T}$.*

Proof. It is trivial that this map is an injective group homomorphism. To show the surjectivity, let $w \in Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$. Then for $x \in L^\infty(T\backslash G_q)$, we have $w\delta(x)w^* = \delta(x)$, and so U^*wU commutes with $x \otimes 1$. By factoriality of $L^\infty(T\backslash G_q)$, there exists a unitary $u \in L^\infty(G_q)$ such that $w = U(1\otimes u)U^*$. Since w is a δ-cocycle, u is group-like with respect to δ_Ω by the previous lemma. The remaining statement follows from Theorem 3.6. □

This result shows that we have a Hopf algebra embedding of $L^\infty(T)$ into $L^\infty(G_{q,\Omega})$, that is, the n-dimensional torus T is a "quotient quantum group" of $G_{q,\Omega}$.

When $G_q = SU_q(2)$, it has been proved that $G_{q,\Omega}$ is isomorphic to $\widetilde{E}_q(2)$, Woronowicz's quantum $E(2)$ group in [11, Theorem 4.5]. In [54, Theorem 2.1], Woronowicz has classified unitary representations of $\widetilde{E}_q(2)$. In particular, the intrinsic group of $\widetilde{E}_q(2)$ is generated by the canonical unitary representation v (see [54, p. 254, (1)]), and is indeed isomorphic to $\mathbb{Z} \cong \widehat{T}$.

4. Product type actions

In this section, we will study a product type action of G_q. We will fix our notation.

Let v be a unitary representation of G_q on a Hilbert space H_v with $2 \le \dim H_v \le \infty$. Take a faithful invariant state $\phi \in B(H_v)_*$, that is, ϕ satisfies $(\phi\otimes\mathrm{id})(v(x\otimes 1)v^*) = \phi(x)1$ for all $x \in B(H_v)$. Note that ϕ is never tracial since $(\mathrm{id} \otimes f_1)(C^\lambda) \ne 1$ for any non-zero $\lambda \in P_+$.

Consider the infinite tensor product $(\mathcal{M}, \varphi) := \bigotimes_{m=1}^\infty (B(H_v), \phi)''$ that is a factor of type III_λ with $\lambda \ne 0$. So, Connes' S-invariant is computed from the period of ϕ. Let $\alpha: \mathcal{M} \to \mathcal{M} \otimes L^\infty(G_q)$ be the product type action with respect to v [17,20,26]. Let $E_\alpha: \mathcal{M} \to \mathcal{M}^\alpha$ be the conditional expectation defined by $E_\alpha := (\mathrm{id} \otimes h) \circ \alpha$. Then $\varphi \circ E_\alpha = \varphi$.

4.1. Depth 2 inclusions

In what follows, we always assume that α is faithful, that is, the subspace $\alpha(\mathcal{M})(\mathcal{M}\otimes\mathbb{C})$ is dense in $\mathcal{M} \otimes L^\infty(G_q)$. This is the case when each irreducible representation of G_q is contained in a product unitary representations $(v \otimes \bar{v})^{\otimes m}$ for some $m \in \mathbb{N}$. Then as remarked in [20, p. 509], each irreducible is contained in $v^{\otimes m}$ for some $m \in \mathbb{N}$. Therefore, the faithfulness of α implies the generating property of the corresponding probability measure on $\mathrm{Irr}(G_q)$.

Thanks to [17, Corollary 3.9], the relative commutant $\mathcal{Q} := (\mathcal{M}^\alpha)' \cap \mathcal{M}$ is non-trivial. Moreover, we know by [17, Theorem 5.10], [20, Theorem A] and [44, Corollary 4.11] that there exists a G_q-equivariant isomorphism from $L^\infty(T\backslash G_q)$ onto $\mathcal{Q}$. In particular, $\mathcal{Q}$ is a type I factor by Theorem 3.1. Then we have the following tensor product decomposition:

$$\mathcal{M} = \mathcal{R} \vee \mathcal{Q} \cong \mathcal{R} \otimes \mathcal{Q}, \tag{4.1}$$

where $\mathcal{R} := \mathcal{Q}' \cap \mathcal{M}$. Note that the invariant state φ is of the form $\varphi|_{\mathcal{R}} \otimes \varphi|_{\mathcal{Q}}$. Indeed, the modular automorphism group σ^φ preserves $\mathcal{M}^\alpha$, and it does $\mathcal{Q}$. Hence, by Takesaki's theorem [41, p. 309], there exists a unique conditional expectation $F: \mathcal{M} \to \mathcal{Q}$ with $\varphi \circ F = \varphi$. Then F maps $\mathcal{R}$ into the center $Z(\mathcal{Q}) = \mathbb{C}$, and

$$\varphi(xy) = \varphi\big(F(x)y\big) = \varphi\big(F(x)\big)\varphi(y) = \varphi(x)\varphi(y) \quad \text{for } x \in \mathcal{R},\ y \in \mathcal{Q}.$$

We will study the inclusion $\mathcal{M}^\alpha \subset \mathcal{R}$.

Lemma 4.1. *The inclusion $\mathcal{M}^\alpha \subset \mathcal{R}$ is irreducible and of depth 2.*

Proof. First we will show the irreducibility. Let $x \in (\mathcal{M}^\alpha)' \cap \mathcal{R}$. Then $x \in (\mathcal{M}^\alpha)' \cap \mathcal{M} = \mathcal{Q}$, but $x \in \mathcal{R} = \mathcal{Q}' \cap \mathcal{M}$. Hence $x \in Z(\mathcal{Q}) = \mathbb{C}$.

Next we let $\mathcal{M}^\alpha \subset \mathcal{R} \subset \mathcal{R}_1 \subset \mathcal{R}_2$ be the Jones tower. We will show that $(\mathcal{M}^\alpha)' \cap \mathcal{R}_2$ is a type I factor. Let $\mathcal{M}^\alpha \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2$ be the Jones tower. By (4.1), this is isomorphic to the following:

$$\mathcal{M}^\alpha \otimes \mathbb{C} \subset \mathcal{R} \otimes \mathcal{Q} \subset \mathcal{R}_1 \otimes \mathcal{Q}_1 \subset \mathcal{R}_2 \otimes \mathcal{Q}_2,$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are type I factors. Thus it suffices to show that $(\mathcal{M}^\alpha)' \cap \mathcal{M}_2$ is a type I factor.

Claim. *The Jones tower $\mathcal{M}^\alpha \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2$ is isomorphic to $\mathcal{M}^\alpha \otimes \mathbb{C} \subset \alpha(\mathcal{M}) \subset \mathcal{M} \rtimes_\alpha G_q \subset \mathcal{M} \otimes B(L^2(G_q))$.*

Proof of Claim. Since α is integrable, there exists a canonical surjection from $\mathcal{M} \rtimes_\alpha G_q$ onto $\mathcal{M}_1$ (see, for example, [20, p. 510] or [48, Theorem 5.3]). Recall that α is faithful. Thanks to [20, Corollary 1.5], we have an isomorphism from $\alpha(\mathcal{M})' \cap (\mathcal{M} \rtimes_\alpha G_q)$ onto $\mathcal{M}' \cap \mathcal{M}_1$. In particular, the canonical surjection from $\mathcal{M} \rtimes_\alpha G_q$ onto $\mathcal{M}_1$ is an isomorphism. Hence the tower $\mathcal{M}^\alpha \subset \mathcal{M} \subset \mathcal{M}_1$ is isomorphic to $\mathcal{M}^\alpha \otimes \mathbb{C} \subset \alpha(\mathcal{M}) \subset \mathcal{M} \rtimes_\alpha G_q$. The basic extension of $\alpha(\mathcal{M}) \subset \mathcal{M} \rtimes_\alpha G_q$ is realized as $\mathcal{M} \otimes B(L^2(G_q))$ through the computation of the modular conjugation of $\mathcal{M} \rtimes_\alpha G_q$. See [48, Proof of Proposition 5.9] or [18, Lemma 5.7] for its proof. □

Hence $(\mathcal{M}^\alpha)' \cap \mathcal{M}_2$ is isomorphic to $((\mathcal{M}^\alpha)' \cap \mathcal{M}) \otimes B(L^2(G_q)) = \mathcal{Q} \otimes B(L^2(G_q))$, which is a type I factor. □

The restriction of E_α on $\mathcal{R}$ gives a conditional expectation from $\mathcal{R}$ onto $\mathcal{M}^\alpha$. Therefore, there exists a compact quantum group and its minimal action β on $\mathcal{R}$ with the fixed point algebra $\mathcal{M}^\alpha$. (See [14,31,40]. It is worth mentioning that Longo's sector-approach still works in our case.) We will show that the quantum group is nothing but the maximal torus T.

Note that $\mathcal{R}^\beta$ can be of type II_1 though $\mathcal{R}$ is of type III. Thus β is not dual in general (see [18, Proposition 5.2 (5)]). However, the action is semidual (see, for example, [34, Theorem 4.4], [48, Proposition 6.4] and [56, Theorem 2.2]). So let us take the tensor product by $B(\ell^2)$ as follows:

$$\overline{\mathcal{M}} := B\big(\ell^2\big) \otimes \mathcal{M}, \qquad \overline{\alpha} := \mathrm{id} \otimes \alpha.$$

Also set $\overline{\mathcal{R}} := B(\ell^2) \otimes \mathcal{R}$ and $\overline{\mathcal{Q}} := \mathbb{C} \otimes \mathcal{Q}$. Then we have

$$\overline{\mathcal{M}} = \overline{\mathcal{R}} \vee \overline{\mathcal{Q}} \cong \overline{\mathcal{R}} \otimes \overline{\mathcal{Q}}, \qquad \overline{\mathcal{M}}^{\overline{\alpha}} = B\big(\ell^2\big) \otimes \mathcal{M}^\alpha, \qquad \big(\overline{\mathcal{M}}^{\overline{\alpha}}\big)' \cap \overline{\mathcal{M}} = \overline{\mathcal{Q}}.$$

Let π be a G_q-equivariant isomorphism from $L^\infty(T\backslash G_q)$ onto $\overline{\mathcal{Q}}$, which is unique by Corollary 3.5.

Recall $w_\lambda \in Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$ introduced in (3.3). Set an $\overline{\alpha}$-cocycle w^o_λ defined by

$$w^o_\lambda := (\pi \otimes \mathrm{id})(w_\lambda). \tag{4.2}$$

Then $w^o_\lambda \in Z^1_{\mathrm{inv}}(\overline{\alpha}, \overline{\mathcal{Q}})$ for all $\lambda \in P$.

Note that $\overline{\mathcal{M}} \rtimes_{\overline{\alpha}} G_q$ is an infinite factor. Since $\overline{\mathcal{M}}^{\overline{\alpha}}$ and $\overline{\mathcal{M}}^{\overline{\alpha}^{w^o_\lambda}}$ contain $B(\ell^2) \otimes \mathbb{C}$, they are also infinite factors. Using the 2×2 matrix trick as before, we obtain a unitary $u_\lambda \in \overline{\mathcal{M}}$ for $\lambda \in P$ such that $\overline{\alpha}(u_\lambda) = (u_\lambda \otimes 1)w^o_\lambda$. For $x \in \overline{\mathcal{M}}^{\overline{\alpha}}$, we have

$$\overline{\alpha}\big(u_\lambda x u^*_\lambda\big) = (u_\lambda \otimes 1)w^o_\lambda(x \otimes 1)\big(w^o_\lambda\big)^*\big(u^*_\lambda \otimes 1\big) = u_\lambda x u^*_\lambda \otimes 1.$$

Hence $\theta_\lambda := \mathrm{Ad}\, u_\lambda$ gives an endomorphism on $\overline{\mathcal{M}}^{\overline{\alpha}}$. We will show that θ_λ is in fact an automorphism.

Lemma 4.2. *For any $\lambda \in P$, u_λ belongs to $\overline{\mathcal{R}}$.*

Proof. Let $x \in \overline{\mathcal{Q}}$ and $y \in \overline{\mathcal{M}}^{\overline{\alpha}}$. Then

$$u^*_\lambda x u_\lambda y = u^*_\lambda x \theta_\lambda(y) u_\lambda = u^*_\lambda \theta_\lambda(y) x u_\lambda = y u^*_\lambda x u_\lambda.$$

Hence $\rho_\lambda := \mathrm{Ad}\, u^*_\lambda$ defines an endomorphism on $\overline{\mathcal{Q}}$. Moreover for $x \in \overline{\mathcal{Q}}$, we have

$$\begin{aligned}\overline{\alpha}\big(\rho_\lambda(x)\big) &= \overline{\alpha}\big(u^*_\lambda\big)\overline{\alpha}(x)\overline{\alpha}(u_\lambda) = \big(w^o_\lambda\big)^*\big(u^*_\lambda \otimes 1\big)\overline{\alpha}(x)(u_\lambda \otimes 1)w^o_\lambda \\ &= \big(w^o_\lambda\big)^*(\rho_\lambda \otimes \mathrm{id})\big(\overline{\alpha}(x)\big)w^o_\lambda.\end{aligned}$$

Thus

$$\begin{aligned}\overline{\alpha}\big(\rho_\lambda(x)\big) &= w_\lambda^o \overline{\alpha}\big(\rho_\lambda(x)\big)\big(w_\lambda^o\big)^* \quad \text{because } w_\lambda^o \in Z^1_{\mathrm{inv}}(\overline{\alpha}, \overline{\mathcal{Q}}) \\ &= w_\lambda^o \cdot \big(w_\lambda^o\big)^* (\rho_\lambda \otimes \mathrm{id})\big(\overline{\alpha}(x)\big) w_\lambda^o \cdot \big(w_\lambda^o\big)^* \\ &= (\rho_\lambda \otimes \mathrm{id})\big(\overline{\alpha}(x)\big).\end{aligned}$$

Namely, ρ_λ is a G_q-equivariant embedding of $\overline{\mathcal{Q}}$ into itself. The injectivity of ρ_λ implies that spectral multiplicities of $\rho_\lambda(\overline{\mathcal{Q}})$ and $\overline{\mathcal{Q}}$ must coincide, and $\rho_\lambda(\overline{\mathcal{Q}}) = \overline{\mathcal{Q}}$. Hence ρ_λ is a G_q-equivariant automorphism on $\overline{\mathcal{Q}} \cong L^\infty(T\backslash G_q)$. By Corollary 3.5, we obtain $\rho_\lambda = \mathrm{id}$, that is, $u_\lambda \in \overline{\mathcal{Q}}' \cap \overline{\mathcal{M}} = \overline{\mathcal{R}}$. $\square$

Let $\lambda, \mu \in P$. Since u_μ belongs to $\overline{\mathcal{R}}$, we see $w_\lambda^o(u_\mu \otimes 1) = (u_\mu \otimes 1)w_\lambda^o$, and

$$\begin{aligned}\overline{\alpha}(u_\lambda u_\mu) &= (u_\lambda \otimes 1)w_\lambda^o(u_\mu \otimes 1)w_\mu^o = (u_\lambda u_\mu \otimes 1)w_\lambda^o w_\mu^o \\ &= (u_\lambda u_\mu \otimes 1)w_{\lambda+\mu}^o = \big(u_\lambda u_\mu u_{\lambda+\mu}^* \otimes 1\big)\overline{\alpha}(u_{\lambda+\mu}).\end{aligned}$$

It follows that $c_{\lambda,\mu} := u_\lambda u_\mu u_{\lambda+\mu}^*$ is contained in $\overline{\mathcal{M}}^{\overline{\alpha}}$, and

$$\theta_\lambda \circ \theta_\mu = \mathrm{Ad}\, c_{\lambda,\mu} \circ \theta_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in P. \tag{4.3}$$

We will show that (θ, c) is a cocycle action of P on $\overline{\mathcal{M}}^{\overline{\alpha}}$ as below. (See [35] for the definition of a cocycle action.) If we put $\mu = -\lambda$, we have $u_{\lambda+\mu} = u_0 = 1$, and $\theta_\lambda \circ \theta_{-\lambda} = \mathrm{Ad}\, c_{\lambda,-\lambda}$. This shows the surjectivity of θ_λ, that is, $\theta_\lambda \in \mathrm{Aut}(\overline{\mathcal{M}}^{\overline{\alpha}})$.

The 2-cocycle identity of c is verified as follows: for $\lambda, \mu, \nu \in P$,

$$\begin{aligned}c_{\lambda,\mu} c_{\lambda+\mu,\nu} &= u_\lambda u_\mu u_{\lambda+\mu}^* \cdot u_{\lambda+\mu} u_\nu u_{\lambda+\mu+\nu}^* \\ &= u_\lambda u_\mu u_\nu u_{\lambda+\mu+\nu}^*,\end{aligned}$$

and

$$\begin{aligned}\theta_\lambda(c_{\mu,\nu}) u_{\lambda,\mu+\nu} &= u_\lambda \cdot u_\mu u_\nu u_{\mu+\nu}^* \cdot u_\lambda^* \cdot u_\lambda u_{\mu+\nu} u_{\lambda+\mu+\nu}^* \\ &= u_\lambda u_\mu u_\nu u_{\lambda+\mu+\nu}^*.\end{aligned}$$

From (4.3), it turns out that (θ, c) gives a cocycle action on an infinite factor $\overline{\mathcal{M}}^{\overline{\alpha}}$. Then c is in fact a coboundary by [39, Proposition 2.1.3]. Take unitaries u_λ' in $\overline{\mathcal{M}}^{\overline{\alpha}}$ for $\lambda \in P$ so that $u_\lambda' \theta_\lambda(u_\mu') c_{\lambda,\mu} (u_{\lambda+\mu}')^* = 1$ for $\lambda, \mu \in P$. By replacing u_λ with $u_\lambda' u_\lambda$ if necessary, we may and do assume that our u_λ's are satisfying

$$u_\lambda \in \overline{\mathcal{R}}, \qquad u_\lambda u_\mu = u_{\lambda+\mu}, \qquad \overline{\alpha}(u_\lambda) = (u_\lambda \otimes 1) w_\lambda^o \quad \text{for all } \lambda, \mu \in P. \tag{4.4}$$

Then we have an outer action θ of P on $\overline{\mathcal{M}}^{\overline{\alpha}}$. Indeed, if for some $\lambda \in P$, $a \in \overline{\mathcal{M}}^{\overline{\alpha}}$ satisfies $ax = \theta_\lambda(x)a$ for all $x \in \overline{\mathcal{M}}^{\overline{\alpha}}$, then $u_\lambda^* a \in (\overline{\mathcal{M}}^{\overline{\alpha}})' \cap \overline{\mathcal{R}} = \mathbb{C}$. This, however, implies that $u_\lambda \in \overline{\mathcal{M}}^{\overline{\alpha}}$ and $w_\lambda^o = 1$, that is, $\lambda = 0$. We will prove that $\overline{\mathcal{R}}$ is actually generated by $\overline{\mathcal{M}}^{\overline{\alpha}}$ and $\{u_\lambda\}_{\lambda \in P}$ by sector technique developed in [19].

Since the inclusion $\mathcal{N} := \overline{\mathcal{M}}^{\overline{\alpha}} \subset \overline{\mathcal{R}}$ comes from a compact quantum group action, $\overline{\mathcal{R}}$ is the crossed product of $\mathcal{N}$ by the dual discrete quantum group action. We would like to determine this action. For the sake of this, we should study the sector $[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}]$ in $\mathrm{Sect}(\mathcal{N})$, where $\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}$ denotes the canonical endomorphism from $\overline{\mathcal{R}}$ into $\mathcal{N}$. Since there exists a conditional expectation from $\overline{\mathcal{M}}$ onto $\overline{\mathcal{R}}$, we have a canonical embedding ${}_{\mathcal{N}}L^2(\overline{\mathcal{R}})_{\mathcal{N}} \subset {}_{\mathcal{N}}L^2(\overline{\mathcal{M}})_{\mathcal{N}}$. Hence $[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}]$ is contained in $[\gamma_{\mathcal{N}}^{\overline{\mathcal{M}}}|_{\mathcal{N}}]$. So let us study ${}_{\mathcal{N}}L^2(\overline{\mathcal{M}})_{\mathcal{N}}$ first. The author thanks Izumi for suggesting this method.

Lemma 4.3. *For any $\lambda \in P_+$, there exists a Hilbert space $\mathscr{H}_\lambda$ with $s(\mathscr{H}_\lambda) = 1$ in $\overline{\mathcal{M}}$ which admits an isometric G_q-isomorphism from $L(\lambda)$ onto $\mathscr{H}_\lambda$.*

Proof. Recall that $\mathcal{M}^\alpha$ is not a type I factor. Otherwise, it follows that $\mathcal{M} = \mathcal{M}^\alpha \vee \mathcal{Q}$, and $\mathcal{M}$ would be of type I. Thus we can take a von Neumann algebra embedding ψ of $B(L(\lambda))$ into $\mathbb{C}1_{\ell^2} \otimes \mathcal{M}^\alpha \subset \overline{\mathcal{M}}^{\overline{\alpha}}$. Let $w := (\psi \otimes \mathrm{id})(C^\lambda)$. Then w is an $\overline{\alpha}$-cocycle, and $B(\ell^2) \otimes \mathbb{C}$ is contained in the fixed point algebra of $\overline{\mathcal{M}}$ by $\overline{\alpha}^w$. Hence we can employ the 2×2-matrix trick as usual, we obtain a unitary $u \in \overline{\mathcal{M}}$ such that $\overline{\alpha}(u) = (u \otimes 1)w$.

Let $\{e_{ij}\}_{i,j \in I}$ be a system of matrix units of $\operatorname{Im} \psi$ such that each e_{ii} is minimal. Fix an element i_0 in I. Since $1_{\ell^2} \otimes e_{i_0 i_0}$ is infinite projection, there exists an isometry in $V_{i_0} \in B(\ell^2) \otimes \mathcal{M}^\alpha$ such that $V_{i_0} V_{i_0}^* = 1_{\ell^2} \otimes e_{i_0 i_0}$. For $i \in I$, we set $V_i := (1 \otimes e_{i i_0}) V_{i_0}$. Then it turns out that $V_i^* V_j = \delta_{ij} 1$ and $V_i V_j^* = 1 \otimes e_{ij}$ for $i, j \in I$. We let $W_i := u V_i$. Then

$$\begin{aligned}\overline{\alpha}(W_i) &= \overline{\alpha}(u)\overline{\alpha}(V_i) = (u \otimes 1) w (V_i \otimes 1) \\ &= \sum_{j \in I} (W_j \otimes 1)\bigl(V_j^* \otimes 1\bigr) w (V_i \otimes 1).\end{aligned}$$

Therefore, the statement follows by setting $\mathscr{H}_\lambda := \mathrm{span}\{W_i \mid i \in I\}$. □

For $\lambda \in P_+$, let T_λ be an isometric G_q-isomorphism from $L(\lambda)$ onto a Hilbert space $\mathscr{H}_\lambda$ in $\overline{\mathcal{M}}$. We let $V_{\mu_i}^\lambda := T_\lambda(\xi_{\mu_i})$ for $\mu \in \mathrm{Wt}(\lambda)$ and $i \in I_\mu^\lambda$. Then we obtain

$$\overline{\alpha}\bigl(V_{\mu_i}^\lambda\bigr) = \sum_{\nu \in \mathrm{Wt}(\lambda),\, j \in I_\nu^\lambda} V_{\nu_j}^\lambda \otimes C_{\nu_j,\mu_i}^\lambda.$$

For $\lambda, \Lambda \in P_+$, $\mu \in \mathrm{Wt}(\lambda)$, $i \in I_\mu^\lambda$, $\nu \in \mathrm{Wt}(\Lambda)$ and $j \in I_\nu^\Lambda$, we have

$$\begin{aligned}E_{\overline{\alpha}}\bigl(V_{\mu_i}^\lambda \bigl(V_{\nu_j}^\Lambda\bigr)^*\bigr) &= (\mathrm{id} \otimes h)\bigl(\overline{\alpha}\bigl(V_{\mu_i}^\lambda \bigl(V_{\nu_j}^\Lambda\bigr)^*\bigr)\bigr) \\ &= \sum_{\eta,\zeta,k,\ell} V_{\eta_k}^\lambda \bigl(V_{\zeta_\ell}^\Lambda\bigr)^* \cdot h\bigl(C_{\eta_k,\mu_i}^\lambda \bigl(C_{\zeta_\ell,\nu_j}^\Lambda\bigr)^*\bigr) \\ &= \delta_{\lambda,\Lambda}\delta_{\mu,\nu}\delta_{i,j}\bigl(\dim_q L(\lambda)\bigr)^{-1} F_{\mu_i,\mu_i}^\lambda \sum_{\eta,\zeta,k,\ell} \delta_{\eta,\zeta}\delta_{k,\ell} V_{\eta_k}^\lambda \bigl(V_{\zeta_\ell}^\Lambda\bigr)^* \quad \text{by (2.1)}\end{aligned}$$

$$= \delta_{\lambda,\Lambda}\delta_{\mu,\nu}\delta_{i,j}\big(\dim_q L(\lambda)\big)^{-1} F^\lambda_{\mu_i,\mu_i} \sum_{\eta,k} V^\lambda_{\eta_k}\big(V^\Lambda_{\eta_k}\big)^*$$
$$= \delta_{\lambda,\Lambda}\delta_{\mu,\nu}\delta_{i,j}\big(\dim_q L(\lambda)\big)^{-1} F^\lambda_{\mu_i,\mu_i}$$
$$= \delta_{\lambda,\Lambda}\delta_{\mu,\nu}\delta_{i,j}\big(\dim_q L(\lambda)\big)^{-1} q^{2(\mu,\rho)} \quad \text{by Lemma 2.3.}$$

So if we put $W^\lambda_{\mu_i} := (\dim_q L(\lambda))^{1/2} q^{-(\mu,\rho)} V^\lambda_{\mu_i}$, then $\{W^\lambda_{\mu_i}\}_{\mu,i}$ is a linear base of $\mathscr{H}_\lambda$ such that

$$E_{\bar\alpha}\big(W^\lambda_{\mu_i}\big(W^\Lambda_{\nu_j}\big)^*\big) = \delta_{\lambda,\Lambda}\delta_{\mu,\nu}\delta_{i,j}. \tag{4.5}$$

Now we let

$$\sigma_\lambda(x) := \sum_{\mu\in \mathrm{Wt}(\lambda),\, i\in I^\lambda_\mu} V^\lambda_{\mu_i} x \big(V^\lambda_{\mu_i}\big)^* \quad \text{for } x \in \mathcal{N}.$$

Then σ_λ is an endomorphism on $\mathcal{N}$. We will determine the intertwiner space $(\sigma_\lambda, \sigma_\lambda)$. Recall $v_\mu \in Z(L^\infty(G_q))$ and $w_\mu \in Z^1_{\mathrm{inv}}(\delta, L^\infty(T\backslash G_q))$ introduced in (3.2) and (3.3). Then we set

$$a^\lambda_{\mu_i} := \sum_{\nu\in \mathrm{Wt}(\lambda),\, j\in I^\lambda_\nu} V^\lambda_{\nu_j} \pi\big(\big(C^\lambda_{\mu_i,\nu_j}\big)^* v_\mu\big) u^*_\mu \quad \text{for } \lambda \in P_+,\ \mu \in \mathrm{Wt}(\lambda),\ i \in I^\lambda_\mu,$$

where we recall that u_μ is satisfying (4.4). Note that $(C^\lambda_{\mu_i,\nu_j})^* v_\mu$ is contained in $L^\infty(T\backslash G_q)$, and $\pi((C^\lambda_{\mu_i,\nu_j})^* v_\mu)$ is well-defined.

Lemma 4.4. *Let $\lambda \in P_+$. Then the following statements hold:*

(1) *For all $\mu \in \mathrm{Wt}(\lambda)$, $\{a^\lambda_{\mu_i}\}_{i\in I^\lambda_\mu}$ is an orthonormal base of $(\theta_\mu, \sigma_\lambda)$.*
(2) *$(\sigma_\lambda, \sigma_\lambda)$ is linearly spanned by $a^\lambda_{\mu_i}(a^\lambda_{\mu_j})^*$ for $\mu \in \mathrm{Wt}(\lambda)$ and $i, j \in I^\lambda_\mu$.*

Proof. (1). We will check that $a^\lambda_{\mu_i}$ is contained in $\overline{\mathcal{M}}^{\bar\alpha}$. Indeed,

$$\bar\alpha\big(a^\lambda_{\mu_i}\big) = \sum_{\nu,j} \bar\alpha\big(V^\lambda_{\nu_j}\big) \cdot (\pi\otimes \mathrm{id})\big(\delta\big(\big(C^\lambda_{\mu_i,\nu_j}\big)^* v_\mu\big)\big) \cdot \bar\alpha\big(u^*_\mu\big)$$
$$= \sum_{\nu,\eta,\zeta,j,k,\ell} \big(V^\lambda_{\eta_k} \otimes C^\lambda_{\eta_k,\nu_j}\big) \cdot (\pi\otimes \mathrm{id})\big(\big(\big(C^\lambda_{\mu_i,\zeta_\ell}\big)^* \otimes \big(C^\lambda_{\zeta_\ell,\nu_j}\big)^*\big)(v_\mu\otimes 1)w_\mu\big)$$
$$\cdot \big(w^o_\mu\big)^*\big(u^*_\mu \otimes 1\big)$$
$$= \sum_{\nu,\eta,\zeta,j,k,\ell} \delta_{\eta,\zeta}\delta_{k,\ell}\big(V^\lambda_{\eta_k}\otimes 1\big) \cdot (\pi\otimes \mathrm{id})\big(\big(\big(C^\lambda_{\mu_i,\zeta_\ell}\big)^* \otimes 1\big)(v_\mu\otimes 1)\big) \cdot \big(u^*_\mu \otimes 1\big)$$
$$= a^\lambda_{\mu_i} \otimes 1.$$

Next we have the following for $x \in \overline{\mathcal{M}}^{\bar{\alpha}}$:

$$\begin{aligned} a^\lambda_{\mu_i}\theta_\mu(x) &= \sum_{\nu,j} V^\lambda_{\nu_j}\pi\big(\big(C^\lambda_{\mu_i,\nu_j}\big)^* v_\mu\big)u_\mu^*\theta_\mu(x) \\ &= \sum_{\nu,j} V^\lambda_{\nu_j}\pi\big(\big(C^\lambda_{\mu_i,\nu_j}\big)^* v_\mu\big)x u_\mu^* \\ &= \sum_{\nu,j} V^\lambda_{\nu_j}x\pi\big(\big(C^\lambda_{\mu_i,\nu_j}\big)^* v_\mu\big)u_\mu^* = \sigma_\lambda(x)a^\lambda_{\mu_i}. \end{aligned}$$

Thus $a^\lambda_{\mu_i} \in (\theta_\mu, \sigma_\lambda)$. We have

$$\begin{aligned} \big(a^\lambda_{\mu_i}\big)^* a^\lambda_{\nu_j} &= \sum_{\eta,\zeta,k,\ell} u_\mu\pi\big(v_\mu^* C^\lambda_{\mu_i,\eta_k}\big)\big(V^\lambda_{\eta_k}\big)^* \cdot V^\lambda_{\zeta_\ell}\pi\big(\big(C^\lambda_{\nu_j,\zeta_\ell}\big)^* v_\nu\big)u_\nu^* \\ &= \sum_{\eta,k} u_\mu\pi\big(v_\mu^* C^\lambda_{\mu_i,\eta_k}\big(C^\lambda_{\nu_j,\eta_k}\big)^* v_\nu\big)u_\nu^* \\ &= \delta_{\mu,\nu}\delta_{i,j}, \end{aligned}$$

and

$$\begin{aligned} \sum_{\mu,i} a^\lambda_{\mu_i}\big(a^\lambda_{\mu_i}\big)^* &= \sum_{\mu,\eta,\zeta,i,k,\ell} V^\lambda_{\eta_k}\pi\big(\big(C^\lambda_{\mu_i,\eta_k}\big)^* v_\mu\big)u_\mu^* \cdot u_\mu\pi\big(v_\mu^* C^\lambda_{\mu_i,\zeta_\ell}\big)\big(V^\lambda_{\zeta_\ell}\big)^* \\ &= \sum_{\eta,\zeta,k,\ell} \delta_{\eta,\zeta}\delta_{k,\ell}V^\lambda_{\eta_k}\big(V^\lambda_{\zeta_\ell}\big)^* = 1. \end{aligned}$$

Hence for $x \in \overline{\mathcal{M}}^{\bar{\alpha}}$, we obtain

$$\sigma_\lambda(x) = \sum_{\mu,i} a^\lambda_{\mu_i}\theta_\mu(x)\big(a^\lambda_{\mu_i}\big)^*,$$

and we are done. □

By the previous lemma, we get the following equality in $\mathrm{Sect}(\mathcal{N})$:

$$[\sigma_\lambda] = \bigoplus_{\mu\in \mathrm{Wt}(\lambda)} \dim L(\lambda)_\mu[\theta_\mu]. \tag{4.6}$$

Lemma 4.5. *In* $\mathrm{Sect}(\mathcal{N})$*, one has*

$$\big[\gamma^{\overline{\mathcal{M}}}_{\mathcal{N}}\big|_{\mathcal{N}}\big] = \bigoplus_{\lambda\in P_+} \dim L(\lambda)[\sigma_\lambda].$$

Proof. We will decompose the $\mathcal{N}$–$\mathcal{N}$ bimodule ${}_{\mathcal{N}}L^2(\overline{\mathcal{M}})_{\mathcal{N}}$ as follows. First we observe that the linear span of $\mathscr{H}^*_\lambda\mathcal{N}$, $\lambda \in P_+$ is weakly dense in $\overline{\mathcal{M}}$. Indeed, for any $\lambda \in P$ and any equivariant map $T: L(\lambda) \to \overline{\mathcal{M}}$, it turns out that $a := \sum_{\mu,i} V_{\mu_i}^{\lambda} T(\xi_{\mu_i}^{\lambda})^*$ belongs to $\overline{\mathcal{M}}^{\bar{\alpha}}$. It follows that $T(\xi_{\mu_i}^{\lambda})^* = (V_{\mu_i}^{\lambda})^* a \in \mathscr{H}_{\lambda}^* \mathcal{N}$. Since the linear span of $T(\xi_{\mu_i}^{\lambda})^*$'s for T and λ, μ, i is weakly dense in $\overline{\mathcal{M}}$, we are done.

Next recall the α-invariant state φ on $\mathcal{M}$. Take a faithful normal state ψ on $B(\ell^2)$ and put $\overline{\varphi} := \psi \otimes \varphi$. Then $\overline{\varphi}$ is an $\bar{\alpha}$-invariant state on $\overline{\mathcal{M}}$. For $\lambda \in P_+$, we let $e_N: L^2(\mathcal{M}) \to \overline{\mathcal{N}1_{\overline{\varphi}}}$ be the Jones projection. Then from (4.5), $z_\lambda := \sum_{\mu,i} (W_{\mu_i}^{\lambda})^* e_N W_{\mu_i}^{\lambda}$ is a projection onto the subspace $X_\lambda := \mathscr{H}_{\lambda}^* \overline{\mathcal{N}1_{\overline{\varphi}}}$. Since each $(W_{\mu_i}^{\lambda})^* e_N W_{\mu_i}^{\lambda}$ belongs to $\mathcal{N}' \cap J_{\overline{\varphi}} \mathcal{N}' J_{\overline{\varphi}}$, the subspace $(W_{\mu_i}^{\lambda})^* \overline{\mathcal{N}1_{\overline{\varphi}}}$ is an $\mathcal{N}$–$\mathcal{N}$-bimodule. Thus we have the following decomposition as the $\mathcal{N}$–$\mathcal{N}$-bimodules:

$$ {}_{\mathcal{N}}L^2(\mathcal{M})_{\mathcal{N}} = \bigoplus_{\lambda \in P_+} \bigoplus_{\mu \in \mathrm{Wt}(\lambda),\, i \in I_\mu^\lambda} \left(W_{\mu_i}^{\lambda}\right)^* \overline{\mathcal{N}1_{\overline{\varphi}}}. \tag{4.7} $$

Let us consider the map $\mathcal{N}1_{\overline{\varphi}} \ni x1_{\overline{\varphi}} \mapsto (W_{\mu_i}^{\lambda})^* x1_{\overline{\varphi}}$. Again by (4.5), it turns out that this map extends to the unitary map U from $\overline{\mathcal{N}1_{\overline{\varphi}}}$ onto $(W_{\mu_i}^{\lambda})^* \overline{\mathcal{N}1_{\overline{\varphi}}}$. Then for $a, b \in \mathcal{N}$ and $\xi \in \overline{\mathcal{N}1_{\overline{\varphi}}}$, we have

$$ \begin{aligned} U\big(\sigma_\lambda(a)\xi b\big) &= \left(W_{\mu_i}^{\lambda}\right)^* \cdot \big(\sigma_\lambda(a)\xi b\big) \\ &= \left(W_{\mu_i}^{\lambda}\right)^* \sigma_\lambda(a)\xi b \\ &= a\big(\left(W_{\mu_i}^{\lambda}\right)^* \xi\big) b = a(U\xi)b. \end{aligned} $$

Hence as the $\mathcal{N}$–$\mathcal{N}$-bimodules, $(W_{\mu_i}^{\lambda})^* \overline{\mathcal{N}1_{\overline{\varphi}}}$ and ${}_{\mathcal{N}\sigma_\lambda}L^2(\mathcal{N})_{\mathcal{N}}$ are isomorphic. Then the statement follows from (4.7). □

By (4.6) and the previous lemma, we obtain

$$ \left[\gamma_{\mathcal{N}}^{\overline{\mathcal{M}}}|_{\mathcal{N}}\right] = \bigoplus_{\mu \in P} \infty[\theta_\mu]. $$

Since $[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}]$ is contained in $[\gamma_{\mathcal{N}}^{\overline{\mathcal{M}}}|_{\mathcal{N}}]$, $[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}]$ is a direct sum of multiples of $[\theta_\mu]$'s. Now we know $\mathcal{N} \subset \overline{\mathcal{R}}$ comes from a minimal action of a compact quantum group. Since every θ_μ is an automorphism, each irreducible representation of the compact quantum group is one-dimensional. Thanks to [19, p. 49] or [45, Lemma 3.5], we have

$$ \left[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}\right] = \bigoplus_{\mu \in S} [\theta_\mu] \quad \text{for some } S \subset P. $$

However, each u_μ is actually an element of $\overline{\mathcal{R}}$ which implements θ_μ, $[\theta_\mu]$ must be contained in $[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}]$. Thus we obtain $S = P$, that is,

$$ \left[\gamma_{\mathcal{N}}^{\overline{\mathcal{R}}}|_{\mathcal{N}}\right] = \bigoplus_{\mu \in P} [\theta_\mu]. $$

Then by [19, Theorem 3.9], it turns out that $\overline{\mathcal{R}}$ is generated by $\mathcal{N} = \overline{\mathcal{M}}^{\bar{\alpha}}$ and u_μ, $\mu \in P$. For $\mu \in P$, $E_{\bar{\alpha}}(u_\mu)$ is an element in $(\mathrm{id}, \theta_\mu)$ The outerness of θ implies that $E_{\bar{\alpha}}(u_\mu) = 0$ for $\mu \neq 0$. Hence we obtain the following result.

Theorem 4.6. *The inclusion $\overline{\mathcal{M}}^{\bar{\alpha}} \subset \overline{\mathcal{R}}$ is isomorphic to $\overline{\mathcal{M}}^{\bar{\alpha}} \subset \overline{\mathcal{M}}^{\bar{\alpha}} \rtimes_\theta \widehat{T}$, where $\widehat{T} = P$ as usual.*

Remark 4.7. Recall the unitary U introduced in Lemma 3.7. Let $\Gamma(x) := (\pi \otimes \mathrm{id})(U^*) \times \alpha(x)(\pi \otimes \mathrm{id})(U)$ for $x \in \mathcal{M}$. Then $(\Gamma \otimes \mathrm{id}) \circ \Gamma = (\mathrm{id} \otimes \delta_\Omega) \circ \Gamma$, that is, Γ is an action of $G_{q,\Omega}$ on $\mathcal{R}$. However, Γ is not faithful. Indeed, $\mathcal{R}^\Gamma = \mathcal{M}^\alpha$ and $\Gamma(u_\lambda) = u_\lambda \otimes \nu_\lambda$ for $\lambda \in P$, where ν_λ is a group-like unitary of $G_{q,\Omega}$ such that $U^* w_\lambda U = 1 \otimes \nu_\lambda$. Hence Γ is nothing but the dual action of θ.

4.2. Induced actions

Let us introduce the action β on $\overline{\mathcal{R}}$, that is,

$$\beta_t(u_\mu) = \langle t, \mu \rangle u_\mu \quad \text{for all } t \in T,\ \mu \in P.$$

By definition, $\beta_t = \hat{\theta}_{t^{-1}}$, where $\hat{\theta}$ denotes the dual action of θ. Then β extends to $\overline{\mathcal{M}}$ by putting $\beta = \mathrm{id}$ on $\overline{\mathcal{Q}}$. Then $\bar{\varphi} \circ \beta_t = \bar{\varphi}$ for all $t \in T$ since $E_{\bar{\alpha}}(u_\mu) = 0$ if $\mu \neq 0$. We will show that $W^*(u_\mu \mid \mu \in P) \vee \mathcal{Q}$ is naturally isomorphic to $L^\infty(G_q)$.

Lemma 4.8. *There exists a von Neumann algebra isomorphism $\pi: L^\infty(G_q) \to W^*(u_\mu \mid \mu \in P) \vee \mathcal{Q}$ such that*

- $\bar{\alpha} \circ \pi = (\pi \otimes \mathrm{id}) \circ \delta$;
- $\beta_t \circ \pi = \pi \circ \gamma_t$ *for all* $t \in T$;
- $\pi(v_\mu) = u_\mu$ *for* $\mu \in P$;
- $\pi(L^\infty(T \backslash G_q)) = \mathcal{Q}$.

Proof. Let $\pi: L^\infty(T \backslash G_q) \to \mathcal{Q}$ be a G_q-equivariant isomorphism as before. Let w_μ, w_μ^o be the invariant cocycles defined in (3.3) and (4.2). They are satisfying the following equalities:

$$\delta(v_\mu) = (v_\mu \otimes 1) w_\mu, \qquad \alpha(u_\mu) = (u_\mu \otimes 1) w_\mu^o \quad \text{for } \mu \in P.$$

Put $\mathcal{P} := W^*(u_\mu \mid \mu \in P) \vee \mathcal{Q}$. Let us introduce a unitary map $U: L^2(G_q) \to L^2(\mathcal{P})$ such that $U(v_\mu a 1_h) = u_\mu \pi(a) 1_{\bar{\varphi}}$ for $\mu \in P$ and $a \in L^\infty(T \backslash G_q)$. Then we have $U v_\mu U^* = u_\mu$ and $U a U^* = \pi(a)$ for $\mu \in P$ and $a \in L^\infty(T \backslash G_q)$. The map π extends to a map, which we also denote by π, from $L^\infty(G_q)$ into $\mathcal{M}$. The G_q-equivariance of π is verified as

$$\begin{aligned}\bar{\alpha}\big(\operatorname{Ad} U(v_\mu)\big) &= \bar{\alpha}(u_\mu) = (u_\mu \otimes 1)w_\mu^o \\ &= (u_\mu \otimes 1)(\pi \otimes \mathrm{id})(w_\mu) \\ &= (\operatorname{Ad} U \otimes \mathrm{id})\big((v_\mu \otimes 1)w_\mu\big) \\ &= (\operatorname{Ad} U \otimes \mathrm{id})\big(\delta(v_\mu)\big). \qquad \square\end{aligned}$$

Remark 4.9. It turns out from the previous lemma that α is semidual. Hence there exists an action σ of $\widehat{G}_q$ on $\mathcal{N} = B(\ell^2) \otimes \mathcal{M}^\alpha$ such that $\overline{\mathcal{M}} = \mathcal{N} \rtimes_\sigma \widehat{G}_q$.

Recall the restriction of an action by a quantum subgroup (see Section 2.4). In the following lemma, we will show that the minimal action β actually comes from the restriction of α by the maximal torus T though it seems not so clear at first.

Let α_T be the restriction of α on T. We denote by α_t the restriction of α on $t \in T$, that is, $\alpha_t := (\mathrm{id} \otimes \mathrm{ev}_t) \circ \alpha_T$ for $t \in T$. Let $w_0 t$ be the element satisfying $\langle w_0 t, \mu\rangle = \langle t, w_0\mu\rangle$ for all $\mu \in P$.

Lemma 4.10. *The minimal action β_t of T on $\mathcal{R}$ is given by $\alpha_{w_0 t}$ on $\mathcal{R}$.*

Proof. To see this, we may assume that $\mathcal{M}^\alpha$ is infinite. Then $\mathcal{R}$ is generated by $\mathcal{M}^\alpha$ and u_λ's as before.

By the above equivariant embedding π, α_T on $\{u_\lambda\}_\lambda'' \vee \mathcal{Q}$ is conjugate to the right torus action γ^R on $L^\infty(G_q)$, where $\gamma_t^R := (\mathrm{id} \otimes \mathrm{ev}_t \circ r_T) \circ \delta$ for $t \in T$. Using $\pi(v_\lambda) = u_\lambda$ and the polar decomposition of $C^\Lambda_{\Lambda, w_0\Lambda}$ with $\Lambda \in P_+$, we have $\alpha_t(u_\lambda) = \pi(\gamma_t^R(v_\lambda)) = \langle t, w_0\lambda\rangle u_\lambda = \beta_{w_0 t}(u_\lambda)$. $\square$

Remark 4.11. Let $x \in \mathbb{R}^n$ and $y := A^{-1}x$, where A denotes the Cartan matrix. Then we put $t = (t_j)_j$ with $t_j = q_j^{iy_j}$ for $j = 1, \ldots, n$, and we get $(w_0 t, \nu) = \prod_j q^{i(w_0\omega_j, \nu)x_j}$. By the commutation relation in the proof of Theorem 3.1, we obtain

$$\gamma^R_{w_0 t} = \operatorname{Ad} |a_{\omega_1}|^{ix_1} \cdots |a_{\omega_n}|^{ix_n} \quad \text{on } L^\infty(T \backslash G_q).$$

This shows the right action γ^R on $L^\infty(T\backslash G_q)$ is implemented by a unitary representation.

Lemma 4.12. *The map*

$$\Xi \colon \big(\overline{\mathcal{M}}^{\bar{\alpha}} \otimes \mathbb{C}\big) \vee W^*(u_\mu \otimes v_\mu \mid \mu \in P) \vee \big(\mathbb{C} \otimes L^\infty(T\backslash G_q)\big) \to \overline{\mathcal{M}}$$

with $\Xi((a \otimes 1)(u_\mu \otimes v_\mu)(1 \otimes b)) = a u_\mu \pi(b)$ for $a \in \overline{\mathcal{M}}^{\bar{\alpha}}$, $\lambda \in P$ and $b \in \mathcal{Q}$ is a well-defined G_q-equivariant isomorphism.

Proof. Let $\mathcal{L} := (\overline{\mathcal{M}}^{\bar{\alpha}} \otimes \mathbb{C}) \vee W^*(u_\mu \otimes v_\mu \mid \lambda \in P) \vee (\mathbb{C} \otimes L^\infty(T\backslash G_q))$. Then $\mathcal{L} \subset \overline{\mathcal{R}} \otimes L^\infty(G_q)$.

Claim. *The following map* $U: L^2(\mathcal{L}) \to L^2(\overline{\mathcal{M}})$ *is a well-defined unitary:*

$$U\big((a \otimes 1)(u_\mu \otimes v_\mu)(1 \otimes b)(1_{\overline{\varphi}} \otimes 1_h)\big) := a u_\mu \pi(b) 1_{\overline{\varphi}}$$

for $a \in \overline{\mathcal{M}}^{\overline{\alpha}}$, $\mu \in P$ *and* $b \in L^\infty(T \backslash G_q)$, *where* π *is the one defined in the previous lemma.*

Proof of Claim. Recall that $\overline{\mathcal{M}} \cong \overline{\mathcal{R}} \otimes \mathcal{Q}$ and $\overline{\varphi}$ is splitted to $\overline{\varphi}|_{\overline{\mathcal{R}}} \otimes \varphi_{\mathcal{Q}}$. Then the well-definedness follows from $\overline{\varphi}(a u_\mu) = 0$ for $a \in \overline{\mathcal{M}}^{\overline{\alpha}}$ and a non-zero $\mu \in P$. □

Using this map, we obtain an isomorphism $\Xi: \mathcal{L} \to \overline{\mathcal{M}}$ as in the statement. We will check the G_q-equivariance. Let $a \in \overline{\mathcal{M}}^{\overline{\alpha}}$, $\mu \in P$ and $b \in L^\infty(T \backslash G_q)$. Then $\Xi(a) = a$ and $\Xi(1 \otimes b) = \pi(b)$. Next,

$$\begin{aligned}(\Xi \otimes \mathrm{id}_{L^\infty(G_q)})\big((\mathrm{id}_{\overline{\mathcal{R}}} \otimes \delta)(u_\mu \otimes v_\mu)\big) &= (\Xi \otimes \mathrm{id}_{L^\infty(G_q)})\big((u_\mu \otimes v_\mu \otimes 1)(1 \otimes w_\mu)\big) \\ &= (u_\mu \otimes 1)(\pi \otimes \mathrm{id})(w_\mu) \\ &= (u_\mu \otimes 1) w_\mu^o = \overline{\alpha}(u_\mu) \\ &= \overline{\alpha}\big(\Xi(u_\mu \otimes v_\mu)\big).\end{aligned}$$

Therefore, Ξ is G_q-equivariant. □

We will recall the notion of the induction of actions.

Definition 4.13. Let $\mathbb{H}$ be a quantum subgroup of $\mathbb{G}$ and $\Gamma: \mathcal{A} \to \mathcal{A} \otimes L^\infty(\mathbb{H})$ an action of $\mathbb{H}$ on a von Neumann algebra $\mathcal{A}$. Let $\gamma_{\mathbb{H}} := (r_{\mathbb{H}} \otimes \mathrm{id}) \circ \delta$ be the left action of $\mathbb{H}$ on $L^\infty(\mathbb{G})$. Set

$$\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}} \mathcal{A} := \mathcal{A} \otimes_{\mathbb{H}} L^\infty(\mathbb{G}) = \big\{ x \in \mathcal{A} \otimes L^\infty(\mathbb{G}) \bigm| (\Gamma \otimes \mathrm{id})(x) = (\mathrm{id} \otimes \gamma_{\mathbb{H}})(x) \big\}.$$

Then the restriction of $\mathrm{id} \otimes \delta$ on $\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}} \mathcal{A}$, which we will denote by $\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}} \Gamma$, gives an action of $\mathbb{G}$, and we will call it the *induction* of Γ from $\mathbb{H}$ to $\mathbb{G}$.

Note that the fixed point algebra of $\mathrm{Ind}_{\mathbb{H}}^{\mathbb{G}} \Gamma$ is equal to $\mathcal{A}^\Gamma$. Now we will prove the following main result of this paper.

Theorem 4.14. *A faithful product type action of* G_q *is induced from a minimal action of* T *on a type III factor. Moreover, such minimal action is unique in the following sense: If there exists a minimal action* χ *of* T *on a factor* $\mathcal{N}$ *such that* $\mathrm{Ind}_T^{G_q} \mathcal{N}$ *is* G_q*-equivariantly isomorphic to* $\mathcal{M}$, *then there exist a* $*$*-isomorphism* ζ *from* $\mathcal{R}$ *onto* $\mathcal{N}$ *and a topological group isomorphism* f *on* T *such that* $\chi_t = \zeta \circ \beta_{f(t)} \circ \zeta^{-1}$.

Proof. We let $\mathcal{A} := W^*(u_\mu \mid \lambda \in P) \subset \overline{\mathcal{R}}$. Since $\beta_t(u_\mu) = \langle t, \mu \rangle u_\mu$ and $\gamma_t(v_\mu) = \langle t, \mu \rangle v_\mu$, we have

$$\mathcal{A} \otimes_T Z\big(L^\infty(T\backslash G_q)\big) = W^*(u_\mu \otimes v_\mu \mid \lambda \in P).$$

Therefore,

$$\begin{aligned} B\big(\ell^2\big) \otimes \operatorname{Ind}_T^{G_q} \mathcal{R} = \overline{\mathcal{R}} \otimes_T L^\infty(G_q) &= \big(\overline{\mathcal{M}}^{\bar\alpha} \vee \mathcal{A}\big) \otimes_T \big(Z\big(L^\infty(G_q)\big) \vee L^\infty(T\backslash G_q)\big) \\ &= \big(\overline{\mathcal{M}}^{\bar\alpha} \otimes \mathbb{C}\big) \vee W^*(u_\mu \otimes v_\mu \mid \lambda \in P) \vee \big(\mathbb{C} \otimes L^\infty(T\backslash G_q)\big), \end{aligned}$$

which is isomorphic to $B(\ell^2)\otimes\mathcal{M}$ through Ξ, the map constructed in the previous lemma. By definition, Ξ maps the fixed point algebra $(\overline{\mathcal{R}} \otimes_T L^\infty(G_q))^{G_q} = \overline{\mathcal{R}}^{\bar\beta} = \overline{\mathcal{M}}^{\bar\alpha}$ onto $\overline{\mathcal{M}}^{\bar\alpha}$ identically. Thus we can remove the contribution of $B(\ell^2)$.

Next suppose that we have a minimal action χ of T on a factor $\mathcal{N}$ such that $\mathcal{P} := \operatorname{Ind}_T^{G_q} \mathcal{N}$ is G_q-equivariantly isomorphic to $\mathcal{M}$. Then the inclusion $\mathcal{N}^\chi \subset (\mathbb{C} \otimes L^\infty(T\backslash G_q))' \cap \mathcal{P}$ is isomorphic to $\mathcal{M}^\alpha \subset \mathcal{R}$. Thus there exist a $*$-isomorphism ζ from $\mathcal{R}$ onto $\mathcal{N}$ and a topological group isomorphism f on T such that $\chi_t = \zeta \circ \beta_{f(t)} \circ \zeta^{-1}$ since every automorphism ψ on $\mathcal{R}$ which fixes $\mathcal{M}^\alpha$ is of the form β_t for some $t \in T$. Indeed, $u_\lambda^* \psi(u_\lambda)$ commutes with $\mathcal{M}^\alpha$, and it is a scalar (see [1, p. 131] for a more general situation). □

4.3. Classification of product type actions

As mentioned in Lemma 4.10, the minimal action β on $\mathcal{R}$ comes from the restriction of α on T, which we denote by α_T as usual. Let $\alpha_t := (\mathrm{id} \otimes \mathrm{ev}_t) \circ \alpha_T$ for $t \in T$. Readers are referred to [33,35] for the notion of *conjugacy* and *cocycle conjugacy*. We will say that a (quantum) group action is *stable* when every cocycle is a coboundary.

Recall that α_T on $\mathcal{Q}$ is implemented by a unitary representation (see Remark 4.11). Then we have

$$\begin{aligned} \alpha_{w_0 t} &\approx \alpha_{w_0 t}|_{\mathcal{R}} \otimes \alpha_{w_0 t}|_{\mathcal{Q}} = \beta_t \otimes \alpha_{w_0 t}|_{\mathcal{Q}} \\ &\sim \beta_t \otimes \mathrm{id}|_{\mathcal{Q}} \\ &\sim \beta_t \quad \text{for } t \in T, \end{aligned}$$

where we have used the infiniteness of $\mathcal{R}$ at the last cocycle conjugacy. The notation $\approx$ and $\sim$ denote the conjugacy and the cocycle conjugacy, respectively. We will summarize this observation in the following (cf. Lemma 4.10). For the notion of *invariant approximate innerness*, readers are referred to [33, Definition 4.5, Lemma 4.7].

Theorem 4.15. *The minimal action β_t of the maximal torus T on $\mathcal{R}$ is cocycle conjugate to $\alpha_{w_0 t}$. In particular, β is invariantly approximately inner.*

Note that $\mathcal{M}$ is the completion of the infinite tensor product of $B(H)$ by a product state, $\mathcal{M}$ is of type III_λ with $0 < \lambda \le 1$. To compute the type of $\mathcal{M}^\alpha$, the following result is useful. Note that $\mathcal{M}^\alpha$ is not of type I as remarked in the proof of Lemma 4.3.

Corollary 4.16. *The following statements hold:*

(1) *The fixed point algebra $\mathcal{M}^{\alpha_T}$ is not of type III$_0$.*
(2) *If $\mathcal{M}^{\alpha_T}$ is of type III$_\lambda$ with $0 < \lambda \leq 1$, then so is $\mathcal{M}^\alpha$. In this case, α is stable.*
(3) *If $\mathcal{M}^{\alpha_T}$ is of type II, then so is $\mathcal{M}^\alpha$.*

Proof. (1). It is clear that the canonical action of the infinite symmetric group $\mathfrak{S}_\infty$ is commuting not only α_T but σ^φ, where φ is the product state with respect to ϕ. Therefore, $(\mathcal{M}^{\alpha_T})'_\varphi \cap \mathcal{M}^{\alpha_T} = \mathbb{C}$, and $\Gamma(\sigma^\varphi|_{\mathcal{M}^{\alpha_T}}) = \mathrm{Sp}(\sigma^\varphi|_{\mathcal{M}^{\alpha_T}})$. This shows that $\mathcal{M}^{\alpha_T}$ is not of type III$_0$.

(2). By [18, Proposition 5.2 (4)], α_T is stable. This implies that α_t is conjugate to $\beta_{w_0 t}$, and $\mathcal{M}^{\alpha_T} \cong \mathcal{R}^\beta = \mathcal{M}^\alpha$. The stability of α is shown by using 2×2-matrix trick.

(2). If $\mathcal{M}^\alpha = \mathcal{R}^\beta$ were of type III, then so would $\mathcal{M}^{\alpha_T}$ since there exists a normal conditional expectation from $\mathcal{M}^{\alpha_T}$ onto $\mathcal{M}^\alpha$. This is a contradiction. □

Theorem 4.15 enables us to classify some product type actions of G_q.

Corollary 4.17. *A product type action α is unique up to conjugacy if $\mathcal{M}^\alpha$ is of type III$_1$. More precisely, such α is conjugate to $\mathrm{Ind}_T^{G_q}(\mathrm{id}_{\mathcal{R}_\infty} \otimes m)$, where $\mathcal{R}_\infty$ denotes the injective type III$_1$ factor and m the minimal action of T on the type II$_1$ injective factor $\mathcal{R}_0$.*

Proof. Let β be the associated minimal action on $\mathcal{R}$. Then $\mathcal{R}^\beta = \mathcal{M}^\alpha$ is of type III$_1$. It follows that β is a dual action of an outer action θ^{-1} on $\mathcal{R}^\beta$. Then θ_μ for each $\mu \in \widehat{T}$ is approximately inner since $\mathrm{Aut}(\mathcal{R}_\infty) = \overline{\mathrm{Int}}(\mathcal{R}_\infty)$ [24, Theorem 1]. By [33, Theorem 4.11], θ has the Rohlin property, that is, the central freeness. Thus θ is unique up to cocycle conjugacy [35, Theorem 1.4, p. 7]. This implies the uniqueness of β up to conjugacy. □

Example 4.18. We will construct a model of a product type action whose fixed point algebra is of type III$_1$. As a result, it turns out that $\mathrm{Ind}_T^{G_q}(\mathrm{id}_{\mathcal{R}_\infty} \otimes m)$ is indeed of product type.

Take an n-dimensional unitary representation v of G_q such that the matrix elements v_{ij} generate $C(G_q)$. Then we set the $(n+3)$-dimensional representation $w := 1^{\oplus 3} \oplus v$. Let $\lambda, \mu > 0$ such that $\lambda/\mu \notin \mathbb{Q}$. We introduce $\mathrm{Ad}\, w$-invariant state ϕ defined by the normalization of Tr_k, where k denotes the diagonal matrix $\mathrm{diag}(1, \lambda, \mu, F_v)$.

By Corollary 4.16, it suffices to show that $\mathcal{M}^{\alpha_T}$ is of type III$_1$. It follows from the proof of Corollary 4.16 that $\Gamma(\sigma^\varphi|_{\mathcal{M}^{\alpha_T}}) = \mathrm{Sp}(\sigma^\varphi|_{\mathcal{M}^{\alpha_T}})$. By construction of w, it turns out that $\log\lambda, \log\mu \in \mathrm{Sp}(\sigma^\varphi|_{\mathcal{M}^{\alpha_T}})$. Thus $\mathcal{M}^{\alpha_T}$ is of type III$_1$.

When the fixed point algebra is of another type, it seems that the general classification is complicated. So, let us treat $SU_q(2)$ in what follows. Our main ingredient is the complete invariant treated in [33, Theorem 6.28]. Note that two actions of the torus $\mathbb{R}/2\pi\mathbb{Z}$ are cocycle conjugate if and only if so are they as $\mathbb{R}$-actions.

We now suppose that α is a product type action of $SU_q(2)$ and v a finite dimensional representation. To compute the invariant, we give a parametrization of v and ϕ as follows. The irreducible representations of $SU_q(2)$ are parametrized by $\mathbb{Z}_+\omega_1$, or equivalently, the half spins $(1/2)\mathbb{Z}_+$. Let us decompose v into the direct sum of irreducible representations as follows:

$$v = \bigoplus_{\nu\in(1/2)\mathbb{Z}_+} \bigoplus_{k=1}^{m_\nu} C^\nu,$$

where m_ν denotes the multiplicity of C^ν in v. Under identification of $T = \mathbb{R}/2\pi\mathbb{Z}$, we have

$$v_t = \bigoplus_{\nu\in(1/2)\mathbb{Z}_+} \bigoplus_{k=1}^{m_\nu} \operatorname{diag}\bigl(e^{2\nu it}, e^{(2\nu-2)it}, \ldots, e^{-2\nu it}\bigr) \quad \text{for } t \in \mathbb{R}.$$

Changing the orthonormal base of each intertwiner space if necessary, we may and do assume that ϕ is the normalization of Tr_{k_ϕ}, where k_ϕ is defined as

$$k_\phi = \bigoplus_{\nu\in(1/2)\mathbb{Z}_+} \bigoplus_{k=1}^{m_\nu} c_k^\nu \operatorname{diag}\bigl(q^{2\nu}, q^{2\nu-2}, \ldots, q^{-2\nu}\bigr), \quad \text{for some } c_k^\nu > 0.$$

From the faithfulness of α, v has at least one non-integer-spin representation and at least one integer. Thus we may assume that $c_k^\nu = 1$ for a fixed even ν and k. Note that the density matrix $\operatorname{diag}(q^{2\nu}, q^{2\nu-2}, \ldots, q^{-2\nu})$ contains 1 as its spectrum for any integer-spin ν.

Then the invariant $G_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ stated in [33, Theorem 6.28] is computed as follows:

$$G_{\alpha_T} := \bigl\langle \bigl(\log\bigl(c_k^\nu q^\ell\bigr), \ell\bigr) \bigm| \ell = 2\nu, 2\nu-2, \ldots, -2\nu,\ k = 1, \ldots, m_\nu,\ \nu \in (1/2)\mathbb{Z}_+ \bigr\rangle,$$

which is a closed subgroup of $\mathbb{R}^2$. Since there exists $\nu \in (1/2)+\mathbb{Z}_+$ with $m_\nu > 0$, G_{α_T} can be written as the following form:

$$G_{\alpha_T} = \bigl\langle \bigl(\log c_k^{\nu_e}, 0\bigr), \bigl(2\log c_k^{\nu_o}, 0\bigr), \bigl(\log\bigl(c_k^{\nu_o} q\bigr), 1\bigr) \bigm| k,\ \nu_e \in \mathbb{Z}_+,\ \nu_o \in 1/2 + \mathbb{Z}_+ \bigr\rangle \tag{4.8}$$

Theorem 4.19. *If $G_q = SU_q(2)$, and $\mathcal{M}^\alpha$ is of type II, then $\mathcal{M}^\alpha$ and $\mathcal{M}$ must be of type II_1 and III_q, respectively. Moreover, α is conjugate to the induction of the torus action $\sigma^{\varphi_q}_{t/\log q}$, where φ_q denotes the Powers state on the Powers factor $\mathcal{R}_q$ of type III_q.*

Proof. Let tr be the tracial weight on $\mathcal{R}^\beta$. Then by [18, Proposition 5.2 (5)], $\{\sigma_t^{\tau\circ E_\alpha|_{\mathcal{R}}}\}_{t\in\mathbb{R}}$ is contained in $\{\beta_t\}_{t\in\mathbb{R}/2\pi\mathbb{Z}}$. In particular, $\sigma^{\tau\circ E_\alpha|_{\mathcal{R}}}$ is periodic, and $\mathcal{R}$ is of type III_λ for some $0 < \lambda < 1$. We will show that λ must be equal to q.

By [33, Proposition 6.34], β_t is cocycle conjugate to $\sigma^{\psi_\lambda}_{t/\log\lambda}$ or $\sigma^{\psi_\lambda}_{-t/\log\lambda}$, where ψ_λ denotes the Powers state on the Powers factor $\mathcal{R}_\lambda$. From Theorem 4.15, we have

$\alpha_t \sim \beta_{-t} \sim \sigma^{\psi_\lambda}_{\mp t/\log\lambda}$, and their invariants introduced in [33, Section 6.5] coincide. The invariant of $\sigma^{\psi_\lambda}_{\mp t/\log\lambda}$ equals $G_\lambda = \mathbb{Z}(\log\lambda, \mp 1)$. It follows immediately from (4.8) that $c_k^\nu = 1$ for all ν and k, and $\lambda = q$. Hence we have $\beta_t \sim \sigma^{\psi_\lambda}_{-t/\log\lambda}$ and $\alpha_t = \sigma^{\varphi}_{t/\log q}$ for $t \in \mathbb{R}$. So, $\mathcal{M}^{\alpha_T} = \mathcal{M}_\varphi$ is of type II_1.

We will show that β_t is in fact conjugate to $\sigma^{\psi_q}_{-t/\log q}$. Employing Lemma 4.10, we have $\beta_t = \alpha_{-t} = \sigma^{\varphi}_{-t/\log q} = \sigma^{\varphi|_{\mathcal{R}}}_{-t/\log q}$ on $\mathcal{R}$. Note that $\mathcal{R} \cong \mathcal{R}_q$ and φ and ψ_q are periodic states. Then by adjusting a Connes–Takesaki module, there exists an isomorphism $\zeta: \mathcal{R} \to \mathcal{R}_q$ such that $\varphi|_{\mathcal{R}} = \psi_q \circ \zeta$. Thus $\beta_t \approx \sigma^{\psi_q}_{-t/\log q}$. It is not so difficult to show that the induction of an action is stable with respect to an automorphism of T, and we have $\alpha \approx \mathrm{Ind}_T^{G_q} \sigma^{\psi_q}_{-t/\log q} \approx \mathrm{Ind}_T^{G_q} \sigma^{\psi_q}_{t/\log q}$. $\square$

Example 4.20. Let v be the direct sum of the spin-0 and 1/2 irreducible representations. Namely, a unitary v has the following form:

$$v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & u \\ 0 & v & y \end{pmatrix} \in M_3(\mathbb{C}) \otimes C\big(SU_q(2)\big) = M_3\big(C\big(SU_q(2)\big)\big),$$

where x, u, v and y are the canonical generators of $C(SU_q(2))$ as a C*-algebra (see [32]).

Now we set the following density matrix:

$$k_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix},$$

where Tr denotes the canonical non-normalized trace of $M_3(\mathbb{C})$. Let φ be the product state of ϕ as usual. Then $\alpha_t = \sigma^{\varphi}_{-t/\log q}$ for $t \in \mathbb{R}/2\pi\mathbb{Z}$, and $\mathcal{M}^{\alpha_T}$ is of type II_1. Thus so is $\mathcal{M}^\alpha$.

The remaining case is when $\mathcal{M}^\alpha$ is of type III_λ with $0 < \lambda < 1$. The infiniteness of $\mathcal{R}^\beta$ implies that the crossed product decomposition of $\mathcal{R}$, that is, $\mathcal{R} = \mathcal{R}^\beta \rtimes_\theta \mathbb{Z}$. Recall that $\beta_t = \hat{\theta}_{e^{-it}}$ for $t \in \mathbb{R}/2\pi\mathbb{Z}$. Since β is invariantly approximately inner, θ is centrally free.

Let $\mathrm{mod}(\theta)$ be the Connes–Takesaki module of θ [8]. Identifying the flow space of $\mathcal{R}^\beta$ with $(\lambda, 1] = \mathbb{R}_{>0}/\lambda^{\mathbb{Z}}$, we may assume that $\lambda \le \mathrm{mod}(\theta) < 1$. Let $\mu := \mathrm{mod}(\theta)$. Thanks to the classification of $\mathbb{Z}$-actions, (see [7, Theorem 1, Corollary 6, p. 385] or [42, Theorem 1.13, p. 311]), θ is cocycle conjugate to $\mathrm{id}_{\mathcal{R}_\lambda} \otimes \theta^\mu$, where θ^μ denotes the automorphism on the injective type II_∞ factor $\mathcal{R}_{0,1}$ with $\mathrm{tr} \circ \theta^\mu = \mu\,\mathrm{tr}$. Thus $\mathcal{R} \cong \mathcal{R}_\lambda \otimes \mathcal{R}_\mu$ and $\beta_t \approx \mathrm{id}_{\mathcal{R}_\lambda} \otimes \sigma^{\varphi_\mu}_{t/\log\mu}$. So, the invariant of β is computed as follows:

$$G_{\lambda,\mu} := \mathbb{Z}(\log\lambda, 0) + \mathbb{Z}(\log\mu, 1). \tag{4.9}$$

Note that we can replace μ with $\lambda\mu$, that is, $G_{\lambda,\lambda\mu} = G_{\lambda,\mu}$. This shows that $\mathrm{id}_{\mathcal{R}_\lambda} \otimes \sigma^{\varphi_\mu}_{t/\log\mu}$ is (cocycle) conjugate to $\mathrm{id}_{\mathcal{R}_\lambda} \otimes \sigma^{\varphi_{\lambda\mu}}_{t/\log(\lambda\mu)}$.

Theorem 4.21. *If $G_q = SU_q(2)$, and $\mathcal{M}^\alpha$ is of type III_λ with $0 < \lambda < 1$, then $\operatorname{mod}(\theta) = q$ or $\lambda^{1/2}q$ in $\mathbb{R}_{>0}/\lambda^{\mathbb{Z}}$. In each case, α is unique up to conjugacy.*

Proof. In this case, we have $G_{\alpha_T} = G_{\lambda,\mu}$. By (4.8) and (4.9), we $c_k^{\nu_o} \in \lambda^{m_k}$ with $m_k \in (1/2)\mathbb{Z}_+$ and $\mu \in qc_k^{\nu_o}\lambda^{\mathbb{Z}}$. Hence $\mu = q\lambda^n$ for some $n \in (1/2)\mathbb{Z}_+$, and $\operatorname{mod}(\theta) = q$ or $\lambda^{1/2}q$ in $\mathbb{R}_{>0}/\lambda^{\mathbb{Z}}$. □

Example 4.22. Let $0 < \lambda < 1$ and $\varepsilon \in \{0, 1/2\}$. Set v and k_ϕ as follows:

$$v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & u \\ 0 & 0 & v & y \end{pmatrix}, \qquad k_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^\varepsilon q & 0 \\ 0 & 0 & 0 & \lambda^\varepsilon q^{-1} \end{pmatrix}.$$

Then we can see $G_{\alpha_T} = G_{\lambda,\lambda^\varepsilon q}$ by direct calculation.

So, if $\mu = \lambda^k q < 1$ with k a half integer, then $\operatorname{Ind}_T^{G_q}(\mathrm{id}_{\mathcal{R}_\lambda} \otimes \sigma_{t/\log\mu}^{\varphi_\mu})$ is of product type and falls into two categories.

The following result is a direct consequence of the previous theorem.

Corollary 4.23. *Let $G_q = SU_q(2)$ and $0 < \lambda < 1$. Suppose that μ satisfies $0 < \mu < 1$ and $\mu/q \notin (\lambda^{1/2})^{\mathbb{Z}_+}$. Then the induced action $\operatorname{Ind}_T^{G_q}(\mathrm{id}_{\mathcal{R}_\lambda} \otimes \sigma_{t/\log\mu}^{\varphi_\mu})$ is not of product type. In particular, for any $0 < \lambda < 1$, there exist uncountably many, non-product type, mutually non-cocycle conjugate actions of $SU_q(2)$ on the injective type III_1 factor with fixed point factor of type III_λ.*

5. Related problems

Let $\mathbb{G}$ be a compact quantum group and $\mathbb{K}$ the maximal quantum subgroup of Kac type introduced in [38, App. A] and [44, Definition 4.6]. We would like to generalize Dijkhuizen–Stokman's result.

Problem 5.1. Does the following equality hold (up to multiplicity)?

$$\operatorname{Irr}\bigl(C(\mathbb{K}\backslash\mathbb{G})\bigr) = \bigl\{\pi|_{C(\mathbb{K}\backslash\mathbb{G})} \bigm| \pi \in \operatorname{Irr}\bigl(C(\mathbb{G})\bigr)\bigr\},$$

where $\operatorname{Irr}(A)$ denotes the equivalence classes of irreducible representation of a C*-algebra A.

Problem 5.2. Is the counit a unique character on $C(\mathbb{K}\backslash\mathbb{G})$?

We will remark on this problem. Let Γ be the set of characters on $C(\mathbb{G})$. Then it is probably well-known for experts that Γ is a compact group that is regarded as a quantum subgroup of $\mathbb{G}$. The maximality of $\mathbb{K}$ implies that $C(\mathbb{K}\backslash\mathbb{G}) \subset C(\Gamma\backslash\mathbb{G})$. In particular, the restriction of every element of Γ on $C(\mathbb{K}\backslash\mathbb{G})$ gives a counit. Thus if Problem 5.1 is solved, this problem holds.

Problem 5.3. $\mathrm{Aut}_{\mathbb{G}}(C(\mathbb{K}\backslash\mathbb{G})) = \{\mathrm{id}\}$?

If $\mathbb{G}$ is a compact group, then $\mathbb{K} = \mathbb{G}$. So these problems are trivial. We will explain why the last problem seems plausible. Let G be a compact group and H a closed subgroup of G. Then $\mathrm{Aut}_G(C(H\backslash G))$ is isomorphic to $N_G(H)/H$, where $N_G(H)$ denotes the normalizer group of H. If there exists a non-trivial $g \in N_G(H)$, then H and g generate a closed subgroup larger than H. Hence the maximality of $\mathbb{K}$ would imply the triviality of $\mathrm{Aut}_{\mathbb{G}}(C(\mathbb{K}\backslash\mathbb{G}))$.

In the last section, in order to show that a faithful product type action of G_q is induced from a minimal action of the maximal torus T, we have exploited the representation theory of G_q and $C(G_q)$ to a full. We would like to obtain this in a more conceptual way.

Problem 5.4. Let $\mathbb{G}$ be a co-amenable compact quantum group with commutative fusion rules and $\mathbb{K}$ the maximal quantum subgroup of Kac type. Then is any faithful product type action of $\mathbb{G}$ induced from a minimal action of $\mathbb{K}$?

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