



ELSEVIER

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim



Finite generators for countable group actions in the Borel and Baire category settings

Anush Tserunyan¹

Department of Mathematics, University of Illinois at Urbana-Champaign, IL,
61801, USA

ARTICLE INFO

Article history:

Received 12 April 2012

Accepted 18 October 2014

Available online 11 November 2014

Communicated by H. Jerome Keisler

Keywords:

Finite generator

Generating partition

Group action

Entropy

Compressibility

Shift action

Weakly wandering set

ABSTRACT

For a continuous action of a countable discrete group G on a Polish space X , a countable Borel partition \mathcal{P} of X is called a *generator* if $G\mathcal{P} := \{gP : g \in G, P \in \mathcal{P}\}$ generates the Borel σ -algebra of X . For $G = \mathbb{Z}$, the Kolmogorov–Sinai theorem gives a measure-theoretic obstruction to the existence of finite generators: they do not exist in the presence of an invariant probability measure with infinite entropy. It was asked by Benjamin Weiss in the late 1980s whether the nonexistence of any invariant probability measure guarantees the existence of a finite generator. We show that the answer is positive (in fact, there is a 32-generator) for an arbitrary countable group G and σ -compact X (in particular, for locally compact X). We also show that for aperiodic actions there is always a 4-generator on a comeager set, thus giving a positive answer to a question of Alexander Kechris asked the mid-1990s.

Furthermore, assuming a positive answer to Weiss's question for arbitrary Polish spaces and $G = \mathbb{Z}$, we prove the following dichotomy: every aperiodic Borel action of \mathbb{Z} on a Polish space X admits either an invariant probability measure of infinite entropy or a finite generator. As an auxiliary lemma, we prove the following statement, which may be of independent interest: every aperiodic Borel action of a countable group G on a Polish space X admits a G -equivariant Borel map to the aperiodic part of the shift action $G \curvearrowright 2^G$.

We also obtain a number of other related results, among which is a criterion for the nonexistence of non-meager weakly

E-mail address: anush@illinois.edu.

¹ This work is part of the author's Ph.D. thesis completed at University of California at Los Angeles in 2013.

wandering sets for continuous actions of \mathbb{Z} . A consequence of this is a negative answer to a question asked by Eigen–Hajian–Nadkarni, which was also independently answered by Benjamin Miller.

Published by Elsevier Inc.

Contents

1.	Introduction	587
1.1.	Definition and examples	587
1.2.	Countable generators	589
1.3.	Entropy and finite generators	589
1.4.	Weiss’s question and potential dichotomy theorems	590
1.5.	Weiss’s question for an arbitrary group and the main result of the paper	591
1.6.	Finite generators in the measure-theoretic setting	593
1.7.	Finite generators in the Baire category setting	593
1.8.	Connections with weakly wandering and traveling sets	594
1.9.	Separating smooth-many sets and maps to the aperiodic part of the shift action	596
1.10.	Nadkarni’s theorem	596
1.11.	Outline of the proof of Theorem 5.7	597
1.12.	Open questions	598
2.	The theory of i -compressibility	599
2.1.	The notion of \mathcal{I} -equidecomposability	599
2.2.	The notion of i -compressibility	603
3.	Traveling sets and finite generators	606
3.1.	Traveling and i -traveling sets	606
3.2.	Constructing finite generators using i -traveling sets	610
4.	Finitely additive invariant measures and i -compressibility	612
4.1.	Measuring the size of a set relative to another	614
4.2.	Properties of $[A/B]$	616
4.3.	Fundamental sequence	617
4.4.	Definition and properties of $m(A, x)$	618
4.5.	Proof of Theorem 4.1	620
5.	Finite generators in the case of σ -compact spaces	620
6.	Finitely traveling sets	623
7.	Locally weakly wandering sets and other special cases	625
7.1.	Weakly wandering sets and finite generators	625
7.2.	Localization	626
7.3.	Countable unions of weakly wandering sets	627
7.4.	Further special cases	629
8.	Separating partitions	630
8.1.	Aperiodic separation and G -equivariant maps to 2^G	630
8.2.	Separating smooth-many invariant sets	634
9.	Potential dichotomy theorems	635
10.	Finite generators on comeager sets	639
11.	The nonexistence of non-meager weakly wandering sets	642
11.1.	An arithmetic criterion	642
11.2.	A negative answer to the Eigen–Hajian–Nadkarni question	644
	Acknowledgments	645
	References	645

1. Introduction

1.1. Definition and examples

Throughout the paper let G denote a countably infinite discrete group and let 1_G denote the identity of G . Let X be a *Borel G -space*, i.e. a standard Borel space (see [13, 12.5]) equipped with a Borel action of G . If X is a Polish topological space and G acts continuously, we call it a *Polish G -space*. For $g \in G$ and $x \in X$, we denote the result of the action of g on x by gx (instead of $g \cdot x$).

Consider the following game: Player I chooses a finite or countable Borel partition² $\mathcal{I} = \{A_n\}_{n < k}$ of X ($k \leq \infty$), then Player II chooses $x \in X$ and Player I tries to guess x by asking questions to Player II regarding which piece of the partition x lands in when moved by a certain group element. More precisely, for every $g \in G$, Player I asks “To which A_n does gx belong?” and Player II gives an index $n_g < k$ as an answer. Whether or not Player I can uniquely determine x from the sequence $(n_g)_{g \in G}$ of responses depends on how cleverly he chose the partition \mathcal{I} . We call \mathcal{I} a *generator* (or a *generating partition*) if it guarantees that Player I will determine x correctly no matter which x Player II chooses. Here is the precise definition, which also explains the terminology.

Definition 1.1 (*Generator*). For a Borel G -space X , a countable Borel partition $\mathcal{I} = \{A_n\}_{n < k}$ of X ($k \leq \infty$) is called a generator if $G\mathcal{I} := \{gA_n : g \in G, n < k\}$ generates the Borel σ -algebra of X . We also call \mathcal{I} a k -generator, and, if $k < \infty$, a finite generator.

Example 1.2.

- (a) For $k \leq \infty$, let $X = k^G$ (with the understanding that $X = \mathbb{N}^G$ when $k = \infty$) be equipped with the product topology and the shift action of G , i.e. $gx(h) := x(g^{-1}h)$, for $g, h \in G$, $x \in X$. For each $n < k$, put $V_n = \{x \in X : x(1_G) = n\}$. The partition $\mathcal{I} = \{V_n\}_{n < k}$ of X is clearly a generator because $G\mathcal{I}$ is a sub-basis for the product topology on k^G .
- (b) Let $X = S^1$ be the unit circle in the complex plane, $\alpha \in \mathbb{R}$ be irrational relative to π (i.e. $\alpha/\pi \notin \mathbb{Q}$), and $T_\alpha : X \rightarrow X$ be the rotation by angle α , i.e. $T_\alpha(z) = e^{i\alpha}z$, for $z \in X$. This induces a continuous action $\mathbb{Z} \curvearrowright X$ and we claim that the partition $\mathcal{I} = \{C_u, C_l\}$ of X into upper and lower half circles is a generator. This is not hard to verify directly, using the fact that every orbit is dense, but it is easier to check the equivalent condition to being a generator given in (2) of Proposition 1.3 below.
- (c) If a Borel G -space X admits a Borel weakly wandering (see Definition 7.1) complete section³ W , e.g. the translation action of \mathbb{Z} on \mathbb{R} with $W = [0, 2]$, then it is not

² We call a countable partition Borel if each piece of it is a Borel set.

³ For an action $G \curvearrowright X$, a subset of X is called a *complete section* if it non-trivially intersects every orbit.

hard to prove that it admits a 3-generator (see [Proposition 7.2](#) for a short and self-contained proof).

- (d) The translation action of \mathbb{Z} on \mathbb{R} actually admits a 2-generator because the exponentiation function 2^x makes it Borel isomorphic to the Borel \mathbb{Z} -space $\mathbb{R}^{>0}$, where the action of $1_{\mathbb{Z}}$ is given by multiplication by 2; the latter action in its turn is Borel isomorphic to an invariant subset of the shift action $\mathbb{Z} \curvearrowright 2^{\mathbb{Z}}$ via the binary representation, which, by (a) above, has a 2-generator. The fact that the translation action $\mathbb{Z} \curvearrowright \mathbb{R}$ has a 2-generator is true more generally ([Proposition 7.10](#)) for smooth ([Definition 3.9](#)) free actions.

For a Borel G -space X and a Borel partition $\mathcal{I} = \{A_n\}_{n < k}$ of X , $k \leq \infty$, let $f_{\mathcal{I}} : X \rightarrow k^G$ be defined by $x \mapsto (n_g)_{g \in G}$, where n_g is such that $gx \in A_{n_g}$. This is often called the *symbolic representation* or *coding map* for the process (X, G, \mathcal{I}) . Clearly $f_{\mathcal{I}}$ is a G -equivariant Borel map and, for every $x \in X$, $f_{\mathcal{I}}(x)$ is the sequence of responses of Player I in the above game.

Proposition 1.3. *For a Borel G -space X and a Borel partition $\mathcal{I} = \{A_n\}_{n < k}$, $k \leq \infty$, the following are equivalent:*

- (1) \mathcal{I} is a generator.
- (2) $G\mathcal{I}$ separates points.⁴
- (3) The coding map $f_{\mathcal{I}} : X \rightarrow k^G$ is one-to-one.

Proof. The only part worth proving is (3) \Rightarrow (1), so suppose $f_{\mathcal{I}}$ is one-to-one. Then, by the Luzin–Souslin theorem (see [\[13, 15.1\]](#)), $f_{\mathcal{I}}$ is a G -equivariant Borel isomorphism between X and $Y := f_{\mathcal{I}}(X)$, so it is enough to show that the partition $f_{\mathcal{I}}(\mathcal{I}) := \{f_{\mathcal{I}}(A_n)\}_{n < k}$ of Y is a generator for the shift action of G on Y . But, in the notation of [Example 1.2\(a\)](#), $f_{\mathcal{I}}(A_n) = V_n \cap Y$, for each $n < k$, so $f_{\mathcal{I}}(\mathcal{I})$ is indeed a generator for $G \curvearrowright Y$. \square

Conversely, given a G -equivariant Borel map $f : X \rightarrow k^G$ for some $k \leq \infty$, define a partition $\mathcal{I}_f = \{A_n\}_{n < k}$ by $A_n = f^{-1}(V_n)$, where V_n is as in [Example 1.2\(a\)](#). Note that $f_{\mathcal{I}_f} = f$. This and [Proposition 1.3](#) imply the following.

Corollary 1.4. *For $k \leq \infty$, X admits a k -generator if and only if there is a G -equivariant Borel embedding of X into k^G .*

In all arguments to follow in this paper, we use these equivalent descriptions of a generator without comment.

⁴ A collection \mathcal{F} of subsets of X separates points if for all distinct $x, y \in X$ there is $A \in \mathcal{F}$ such that $x \in A \not\curvearrowright y \in A$.

1.2. Countable generators

In [24], it was shown that every aperiodic (i.e. having no finite orbits) \mathbb{Z} -space admits a countable generator. This was later generalized to any countable group in [12].

Theorem 1.5 (Weiss, Jackson–Kechris–Louveau). *Every aperiodic Borel G -space X admits a countable generator. In particular, there is a Borel G -equivariant embedding of X into \mathbb{N}^G .*

This is sharp in the sense that we could not hope to obtain a finite generator solely from the aperiodicity assumption because of the measure-theoretic obstruction provided by the Kolmogorov–Sinai theorem (see Theorem 1.9) as explained below. For example, the action of \mathbb{Z} on $[0, 1]^{\mathbb{Z}} \setminus A$ by shift, where A is the set of periodic points, is aperiodic, but it does not admit a finite generator since it admits an invariant probability measure of infinite entropy, namely, the Lebesgue measure.

Thus, the question of the existence of countable generators is completely resolved and the current paper studies the existence of finite generators.

1.3. Entropy and finite generators

Generators arose in the study of entropy in ergodic theory. Let (X, μ, T) be a dynamical system, i.e. (X, μ) is a standard probability space and T is a Borel measure preserving automorphism of X . We can interpret the above game as follows:

- X is the set of possible pictures of the world,
- \mathcal{I} is an experiment that Player I conducts,
- the point $x \in X$ that Player II chooses is the true picture of the world,
- T is the unit of time.

Assume that \mathcal{I} is finite (indeed, we want our experiment to have finitely many possible outcomes). Player I repeats the experiment every day (also going back in time) and Player II tells its outcome. The goal is to find the true picture of the world x with probability 1. This happens exactly when \mathcal{I} is a generator μ -a.e.

The *static entropy* of the experiment $\mathcal{I} = \{P_n\}_{n < k}$ is defined by

$$h_\mu(\mathcal{I}) = - \sum_{n < k} \mu(A_n) \log_2 \mu(A_n), \quad (1.6)$$

and intuitively, it measures our probabilistic uncertainty about the outcome of the experiment. For example, if for some $n < k$, A_n had measure 1, then we would be probabilistically certain that the outcome is going to be in A_n and $h_\mu(\mathcal{I}) = 0$. On the other hand, if all of A_n had probability $\frac{1}{k}$, then our uncertainty would be the highest, namely, $h_\mu(\mathcal{I}) = \log_2 k$. Equivalently, according to Shannon's interpretation, $h_\mu(\mathcal{I})$ measures how much information we gain from learning the outcome of the experiment.

We now define the *time average* or *dynamic* entropy of \mathcal{I} by

$$h_\mu(\mathcal{I}, T) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} h_\mu \left(\bigvee_{i=-n}^n T^i \mathcal{I} \right), \quad (1.7)$$

where \bigvee denotes the join (the least common refinement) of the partitions. The sequence in the limit is decreasing and hence the limit always exists and is finite (see [8] or [21]).

Finally the *entropy of the dynamical system* (X, μ, T) is defined as the supremum over all (finite) experiments:

$$h_\mu(T) = \sup_{\mathcal{I}} h_\mu(\mathcal{I}, T), \quad (1.8)$$

and it could be finite or infinite. Now it is plausible that if \mathcal{I} is a finite generator (and hence Player I wins the above game), then $h_\mu(\mathcal{I}, T)$ should be all the information there is to obtain about X and hence \mathcal{I} achieves the supremum above. This is indeed the case as the following theorem [8, Theorem 14.33] shows.

Theorem 1.9 (*Kolmogorov–Sinai, 1958–1959*). *If \mathcal{I} is a finite generator modulo μ -NULL, then $h_\mu(T) = h_\mu(\mathcal{I}, T)$. In particular, $h_\mu(T) \leq \log_2(|\mathcal{I}|) < \infty$.*

Here μ -NULL denotes the σ -ideal of μ -null sets and, by definition, a statement holds modulo a σ -ideal \mathfrak{J} if it holds on $X \setminus Z$, for some $Z \in \mathfrak{J}$. We will also use this for $\mathfrak{J} = \text{MEAGER}$ (the σ -ideal of meager sets in a Polish space).

In case of ergodic systems, i.e. dynamical systems where every measurable invariant set is either null or conull, the converse of Kolmogorov–Sinai theorem is true (see [15]):

Theorem 1.10 (*Krieger, 1970*). *Suppose (X, μ, T) is ergodic. If $h_\mu(T) < \log_2 k$, for some $k \geq 2$, then there is a k -generator modulo μ -NULL.*

1.4. Weiss’s question and potential dichotomy theorems

Now let X be just a Borel \mathbb{Z} -space with no measure specified. By the Kolmogorov–Sinai theorem, if there exists an invariant Borel probability measure on X with infinite entropy, then X does not admit a finite generator. Is this the only obstruction to having a finite generator? More precisely:

Question 1.11. If a Borel \mathbb{Z} -space X does not admit any invariant Borel probability measure of infinite entropy, does it admit a finite generator?

The following seemingly simpler question was first asked by Weiss in [24]:

Question 1.12 (*Weiss, 1987*). If a Borel \mathbb{Z} -space X does not admit any invariant Borel probability measure, does it admit a finite generator?

It is shown below in Section 9 that these two questions are actually equivalent, and thus, a positive answer to Weiss’s question would imply the following dichotomy theorem:

Theorem 9.5. *Suppose the answer to Question 1.12 is positive and let X be an aperiodic Borel \mathbb{Z} -space. Then exactly one of the following holds:*

- (1) *there exists an invariant Borel probability measure with infinite entropy;*
- (2) *X admits a finite generator.*

We remark that the nonexistence of an invariant ergodic probability measure of infinite entropy does not guarantee the existence of a finite generator. For example, let X be a direct sum of uniquely ergodic actions $\mathbb{Z} \curvearrowright X_n$ such that the entropy h_n of each X_n is finite but $h_n \rightarrow \infty$. Then X does not admit an invariant ergodic probability measure with infinite entropy since otherwise it would have to be supported on one of the X_n , contradicting the unique ergodicity. Neither does X admit a finite generator since that would contradict the Kolmogorov–Sinai theorem applied to X_n for large enough n .

However, assuming again that the answer to Question 1.12 is positive, we prove the following dichotomy suggested by Kechris:

Theorem 9.3. *Suppose the answer to Question 1.12 is positive and let X be an aperiodic Borel \mathbb{Z} -space. Then exactly one of the following holds:*

- (1) *there exists an invariant ergodic Borel probability measure with infinite entropy,*
- (2) *there exists a partition $\{Y_n\}_{n \in \mathbb{N}}$ of X into invariant Borel sets such that each Y_n admits a finite generator.*

The proofs of these dichotomies (presented in Section 9) use the Ergodic Decomposition Theorem and a uniform version of Krieger’s theorem in tandem with Theorem 8.12, which provides a finite partition that separates the equivalence classes of an invariant smooth equivalence relation (Definition 8.11).

1.5. Weiss’s question for an arbitrary group and the main result of the paper

Because Questions 1.11 and 1.12 are equivalent, we may focus on answering the latter. Moreover, since the statement of Question 1.12 does not use the notion of entropy, one may as well state it for an arbitrary countable group G as it is done in [12]:

Question 1.13 (Weiss 1987, Jackson–Kechris–Louveau 2002). *Let G be a countable group and X be a Borel G -space. If X does not admit any invariant Borel probability measure, does it admit a finite generator?*

In order to state our answer, we need the following:

Definition 1.14. Let X be a Borel G -space and denote its Borel σ -algebra by $\mathfrak{B}(X)$. For a topological property P (e.g. Polish, σ -compact, etc.), we say that X admits a P topological realization, if there exists a Hausdorff second countable topology on X satisfying P such that it makes the G -action continuous and its induced Borel σ -algebra is equal to $\mathfrak{B}(X)$.

We remark that every Borel G -space admits a Polish topological realization,⁵ see, [13, 13.12.ii]. The main result of this paper is a positive answer to Question 1.13 in case X has a σ -compact realization:

Theorem 5.7. *Let X be a Borel G -space that admits a σ -compact realization. If there is no G -invariant Borel probability measure on X , then X admits a Borel 32-generator.*

For example, Question 1.13 has a positive answer when G acts continuously on a locally compact or even just σ -compact Polish space.

Remark 1.15. The number 32 in the above theorem comes from the fact that the generator is constructed as the partition generated by 5 Borel sets.

Remark 1.16. The fact that a concrete numerical bound of 32 is obtained in the conclusion of the above theorem is somewhat surprising. However, Robin Tucker-Drob pointed out that if Question 1.13 had a positive answer, then automatically there would be a uniform finite bound on the number generators; indeed, otherwise, there is an unbounded sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that for each $n \in \mathbb{N}$, there is a Borel G -space X_n that

- (i) does not admit an invariant probability measure,
- (ii) admits a k_n -generator,
- (iii) does not admit a k -generator for $k < k_n$.

Then, letting X be the disjoint union of the X_n , $n \in \mathbb{N}$, we see that X still does not admit an invariant probability measure, but neither does it admit a finite generator, contradicting the assumption that the answer to Question 1.13 is positive.

Addendum 1.17. In the original version of the current paper, it was also asked whether every Borel G -space admits a σ -compact realization. However, this was later answered negatively by Conley, Kechris and Miller in [3].

Before explaining the idea of the proof of the above theorem, we present previously known results as well as other related results obtained in this paper.

⁵ This is actually true also for an uncountable Polish group G , but it is a highly non-trivial result of Becker and Kechris, see [1, 5.2].

1.6. Finite generators in the measure-theoretic setting

The following result gives a positive answer to a version of [Question 1.13](#) in the measure-theoretic context.

Theorem 1.18 (Krengel–Kuntz [\[16,17\]](#)). *Let (X, μ) be a standard probability space equipped with a nonsingular⁶ Borel action of G . If there is no invariant Borel probability measure absolutely continuous with respect to μ , then X admits a 2-generator modulo μ -NULL.*

The proof of this uses a version of the Hajian–Kakutani–Itô theorem (see [Theorem 1.21](#) below), which states that the hypothesis of the Krengel–Kuntz theorem is equivalent to the existence of a weakly wandering Borel set (see [Definition 7.1](#)) of arbitrarily large μ -measure.⁷ However, as explained below in Subsection 1.8, the analogues of the Hajian–Kakutani–Itô theorem fail in the Borel and Baire category settings. Therefore, other means needed to be used to obtain the results of the current paper.

1.7. Finite generators in the Baire category setting

In the mid-1990s, Kechris asked whether an analogue of the Krengel–Kuntz theorem holds in the context of Baire category (see [\[12, 6.6.\(B\)\]](#) for a more particular version of this question), more precisely:

Question 1.19 (Kechris, mid-1990s). *Does every aperiodic Polish G -space admit a finite generator on an invariant comeager set?*

The nonexistence of invariant measures is not mentioned in the hypothesis of the question because it is automatic in the context of Baire category due to the following:

Theorem 1.20 (Kechris–Miller [\[14, 13.1\]](#)). *For any aperiodic Polish G -space, there is an invariant comeager set that does not admit any invariant probability measure.*

Thus, a positive answer to [Question 1.13](#) for *all* Borel G -spaces would imply a positive answer to [Question 1.19](#), but the latter does not follow from [Theorem 5.7](#). However, using entirely different techniques, we still give an affirmative answer to [Question 1.19](#):

Theorem 10.1. *Any aperiodic Polish G -space admits a 4-generator on an invariant comeager set.*

⁶ An action is *nonsingular* if it preserves the μ -null sets.

⁷ Here, by arbitrarily large we mean arbitrarily close to 1.

The proof of this is entirely contained in Section 10 and can be read independently from the rest of the paper. It uses the Kuratowski–Ulam method introduced in the proofs of Theorems 12.1 and 13.1 in [14]. This method was inspired by product forcing and its idea is as follows. Suppose we want to prove the existence of an object A that satisfies a certain condition on a comeager set (in our case a finite partition). We give a parametrized construction of such objects A_α , where the parameter α ranges over $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ (or any other Polish space), and then try to show that for comeager many values of the parameter α , the object A_α has the desired property Φ on a comeager set. In other words, we want to prove

$$\forall^* \alpha \forall^* x \Phi(\alpha, x),$$

where \forall^* means “for comeager many”. Now the key point is that the Kuratowski–Ulam theorem allows us to switch the order of the quantifiers and prove

$$\forall^* x \forall^* \alpha \Phi(\alpha, x)$$

instead. The latter is often an easier task since it allows us to work locally with a fixed $x \in X$.

1.8. Connections with weakly wandering and traveling sets

Going back to the Borel setting and Question 1.13, we now explore the hypothesis of the question, namely, the nonexistence of an invariant probability measure. In the measure-theoretic setting, the latter is tightly connected to weakly wandering sets (Definition 7.1):

Theorem 1.21 (Hajian–Kakutani–Itô [11,10]). *Let (X, μ) be a standard probability space equipped with a nonsingular Borel action of G . There is no invariant Borel probability measure equivalent to μ if and only if there is a μ -positive weakly wandering Borel set.*

In the Borel setting, the following statement would be an analogue of Theorem 1.21: *a Borel G -space does not admit any invariant Borel probability measure if and only if it admits a weakly wandering Borel complete section*. However, in [6] the authors show that it does not hold. Nevertheless, their counterexample does not rule out local versions of the statement, leaving the following question unsettled [6, question (ii) on page 9]:

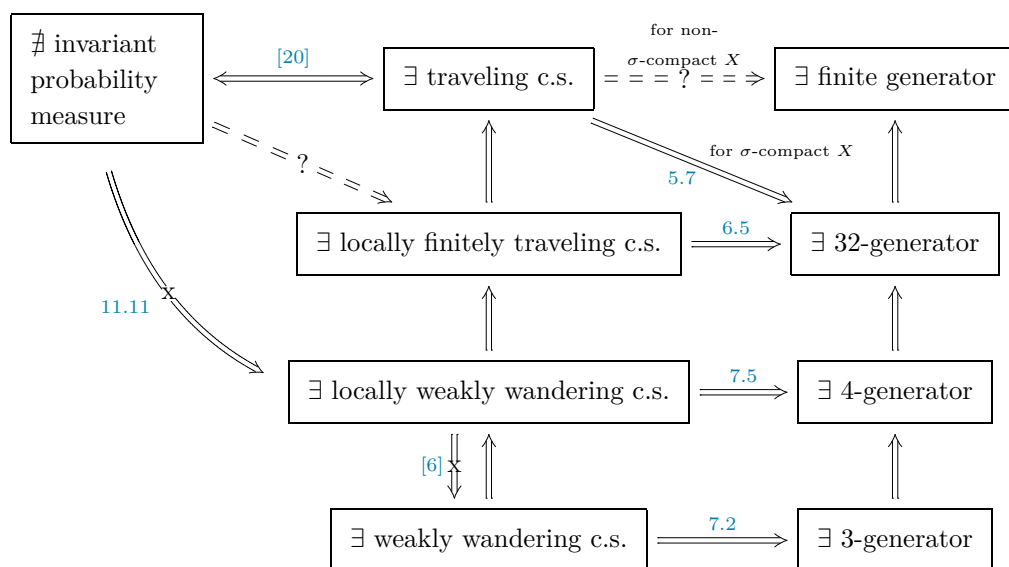
Question 1.22 (Eigen–Hajian–Nadkarni, 1993). *Let X be a Borel \mathbb{Z} -space. If X does not admit an invariant probability measure, is there a countably generated (by Borel sets) partition of X into invariant sets, each of which admits a weakly wandering Borel complete section?*

We define a related notion of *locally weakly wandering* sets (Definition 7.1) — a generalization of weakly wandering sets, and show that the presence of a locally weakly

wandering Borel complete section still gives a 4-generator ([Theorem 7.5](#)). However, we give an example ([Corollary 11.11](#)) of a Polish G -space that does not support an invariant probability measure, but neither does it admit a locally weakly wandering Borel complete section nor a partition as in [Question 1.22](#), thus providing a negative answer to the latter question. This result is a consequence of a criterion for the nonexistence of non-meager weakly wandering sets ([Theorem 11.7](#)), which also implies that the analogues of the Hajian–Kakutani–Itô theorem fail not only in the Borel, but also in the Baire category setting. Ben Miller pointed out that he had also obtained a negative answer to [Question 1.22](#) in [[18, Example 3.13](#)].

We further generalize locally weakly wandering sets to *traveling sets* (Definition 3.1) and observe as an immediate consequence of Nadkarni’s theorem (see Subsection 1.10 and Theorem 1.26) that a Borel G -space does not admit an invariant probability measure if and only if it admits a Borel traveling complete section (Proposition 3.7). Thus, Question 1.13 amounts to whether the existence of a Borel traveling complete section implies the existence of a finite generator. Regarding this formulation of the question, the strongest result obtained in the current paper is as follows: we define an intermediate (between traveling and locally weakly wandering) notion of *locally finitely traveling sets* (Definition 6.2) and prove that the presence of a Borel locally finitely traveling complete section implies the existence of a 32-generator (Corollary 6.5). Unlike the analogous results with weakly wandering and locally weakly wandering sets (Proposition 7.2 and Theorem 7.5), this result is nonconstructive — its proof uses the same machinery as that of Theorem 5.7.

The following diagram summarizes the results mentioned in this subsection; here “c.s.” stands for “complete section”.



1.9. Separating smooth-many sets and maps to the aperiodic part of the shift action

In the proofs of the dichotomy theorems mentioned above ([Theorems 9.3 and 9.5](#)), we needed to apply the following result to the equivalence relation E of being in the same component of the ergodic decomposition; we mention it here as it may be of interest on its own.

Theorem 8.12. *Let X be an aperiodic Borel G -space and let E be a smooth (see [Definition 8.11](#)) G -invariant equivalence relation on X . There exists a partition \mathcal{P} of X into 4 Borel sets such that $G\mathcal{P}$ separates any two E -nonequivalent points in X , i.e. for all $x, y \in X$, $[x]_E \neq [y]_E$ implies $f_{\mathcal{P}}(x) \neq f_{\mathcal{P}}(y)$.*

The proof of this theorem in its turn uses the following result, which may also be of independent interest.

Theorem 8.8. *Any aperiodic Borel G -space admits a G -equivariant Borel map to the aperiodic part of the shift action $G \curvearrowright 2^G$.*

1.10. Nadkarni's theorem

In order to explain the idea behind the proof of [Theorem 5.7](#), we now present an equivalent condition to the hypothesis of [Question 1.13](#), i.e. to the nonexistence of invariant measures. It was proved by Nadkarni in [\[20\]](#) and it is the analogue of Tarski's theorem about paradoxical decompositions (see [\[23\]](#)) for countably additive measures.

Let X be a Borel G -space and denote the set of invariant Borel probability measures on X by $\mathcal{M}_G(X)$. Also, for $S \subseteq X$, let $[S]_G$ denote the saturation of S , i.e. $[S]_G = \bigcup_{g \in G} gS$.

The following definition makes no reference to any invariant measure on X , yet provides a sufficient condition for the measure of two sets to be equal (resp. \leq or $<$).

Definition 1.23. Two Borel sets $A, B \subseteq X$ are said to be equidecomposable (denoted by $A \sim B$) if there are Borel partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B , respectively, and $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ such that $g_n A_n = B_n$. We write $A \preceq B$ if $A \sim B' \subseteq B$, and we write $A \prec B$ if moreover $[B \setminus B']_G = [B]_G$.

The following explains the above definition.

Proposition 1.24. *Let $A, B \subseteq X$ be Borel sets and $\mu \in \mathcal{M}_G(X)$.*

- (a) *If $A \sim B$, then $\mu(A) = \mu(B)$.*
- (b) *If $A \preceq B$, then $\mu(A) \leq \mu(B)$.*
- (c) *If $A \prec B$, then either $\mu(A) = \mu(B) = 0$ or $\mu(A) < \mu(B)$.*

Proof. The only part worth proving is (c) and to this end, let $B' \subseteq B$ be such that $A \sim B'$ and $[B \setminus B']_G = [B]_G$. If $C := B \setminus B'$ is μ -null, then so is B , and hence A , because $B \subseteq [C]_G = \bigcup_{g \in G} gC$. On the other hand, if $\mu(C) > 0$, then, because μ is a finite measure, we have $\mu(A) = \mu(B') = \mu(B) - \mu(C) < \mu(B)$. \square

Definition 1.25. A Borel set $A \subseteq X$ is called compressible if $A \prec A$.

In other words, $A \prec A$ means that there is a Borel witness to the fact that each orbit of A is Dedekind infinite, i.e. is equinumerous to its proper subset.

It is clear from (c) of Proposition 1.24 that if a Borel set $A \subseteq X$ is compressible, then $\mu(A) = 0$ for all $\mu \in \mathcal{M}_G(X)$. In particular, if X itself is compressible then $\mathcal{M}_G(X) = \emptyset$. Thus compressibility is an apparent obstruction to having an invariant probability measure. It turns out that it is the only one:

Theorem 1.26 (Nadkarni, 1991). *A Borel G -space X admits an invariant Borel probability measure if and only if it is not compressible.*

The proof of this first appeared in [20] for $G = \mathbb{Z}$ and is also presented in Chapter 4 of [1] for an arbitrary countable group G . Although we do not explicitly use this theorem in our arguments, we largely use ideas from its proof.

1.11. Outline of the proof of Theorem 5.7

In our attempt to positively answer Question 1.13, we take the non-constructive approach and try to prove the contrapositive:

No finite generator $\Rightarrow \exists$ an invariant probability measure.

When constructing an invariant measure (e.g. Haar measure), one usually needs some notion of “largeness” so that X is “large” (e.g. having nonempty interior, being incompressible). So we aim at something like this:

No 32-generator	\exists an invariant probability measure
\searrow	\nearrow
X is not “small” = X is “large”	

In the definition of equidecomposability of sets A and B , the partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ belong to the Borel σ -algebra. For $i \geq 1$, we define a finer notion of equidecomposability by restricting the Borel σ -algebra to *some* σ -algebra that is generated by the G -translates of i -many Borel sets. In this case we say that A and B are *i -equidecomposable* and denote by $A \sim_i B$. In other words, $A \sim_i B$ if i -many Borel sets are enough to generate a G -invariant σ -algebra that is sufficiently fine to carve out partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ witnessing $A \sim B$.

As before, we say that a set A is i -compressible if $A \prec_i A$. Taking i -compressibility as our notion of “smallness”, we prove the following:

$$\begin{array}{ll} \text{No 32-generator} & \exists \text{ an invariant probability measure} \\ (1) \not\asymp & \not\asymp (2) \\ X \text{ is not 4-compressible} & \end{array}$$

We prove the contrapositive of Step (1). More precisely, assuming i -compressibility, we construct a 2^{i+1} -generator by hand (see [Proposition 3.13](#)), thus obtaining:

$$\text{No } 2^5\text{-generator} \Rightarrow X \text{ is not 4-compressible.}$$

Step (2) is proving an analogue of Nadkarni’s theorem for i -compressibility:

$$X \text{ is not 4-compressible} \Rightarrow \exists \text{ an invariant probability measure.}$$

To accomplish this step, firstly, we show that i -compressibility is indeed a notion of “smallness”, i.e. that the set of i -compressible sets (roughly speaking) forms a σ -ideal (see [Proposition 2.19](#)). The difficulty here is to prevent i from growing when taking unions.

Secondly, we assume that X is not 4-compressible and give a construction of a measure (see [Section 4](#)) reminiscent of the one in the proof of Nadkarni’s theorem or the existence of Haar measure. But unfortunately, our proof only yields a family of *finitely additive* invariant probability measures because here we cannot prevent i from growing when taking countable unions. However, with the additional assumption that X is σ -compact, we are able to concoct a countably additive invariant probability measure out of this family of finitely additive measures, thus obtaining [Theorem 5.7](#).

1.12. Open questions

Here are some open questions that arose in this research. Let X denote a Borel G -space.

- (A) Is X being compressible equivalent to X being i -compressible for some $i \geq 1$?
- (B) Does the existence of a Borel traveling complete section imply the existence of a Borel locally finitely traveling complete section?
- (C) Can we get a 2-generator instead of a 32-generator in [Theorem 5.7](#)?

A positive answer to any of Questions (A) and (B) would imply a positive answer to [Question 1.13](#) since (A) is just a rephrasing of [Question 1.13](#) because of [Corollary 3.18](#), and for (B), it follows from [Proposition 3.5](#) and [Corollary 6.5](#).

2. The theory of i -compressibility

Throughout this section, let X be a Borel G -space and let E_G denote the orbit equivalence relation on X induced by the action of G . For a set $A \subseteq X$ and G -invariant set $P \subseteq X$, put $A^P := A \cap P$.

For an equivalence relation E on X and $A \subseteq X$, let $[A]_E$ denote the saturation of A with respect to E , i.e. $[A]_E = \{x \in X : \exists y \in A (xEy)\}$. In case $E = E_G$, we use $[A]_G$ instead of $[A]_{E_G}$.

Let \mathfrak{B} denote the (proper) class of all Borel subsets of standard Borel spaces, i.e.

$$\mathfrak{B} = \{B : B \text{ is a Borel subset of some standard Borel space } X\}.$$

Also, let Γ be a class σ -algebra of subsets of standard Borel spaces containing \mathfrak{B} and closed under Borel preimages, i.e. if X, Y are standard Borel spaces and $f : X \rightarrow Y$ is a Borel map, then for a subset $A \subseteq Y$, if $A \in \Gamma$ then $f^{-1}(A)$ is also in Γ . For example, $\Gamma = \mathfrak{B}, \sigma(\Sigma_1^1)$, universally measurable sets.

For a standard Borel space X , let $\Gamma(X)$ denote the set of all subsets of X that belong to Γ . In particular, $\mathfrak{B}(X)$ denotes the set of all Borel subsets of X .

2.1. The notion of \mathcal{I} -equidecomposability

A countable partition of X is called Borel if all the sets in it are Borel. For a finite Borel partition $\mathcal{I} = \{A_i : i < k\}$ of X , let $F_{\mathcal{I}}$ denote the equivalence relation of not being separated by $G\mathcal{I} := \{gA_i : g \in G, i < k\}$, more precisely, $\forall x, y \in X$,

$$xF_{\mathcal{I}}y \Leftrightarrow f_{\mathcal{I}}(x) = f_{\mathcal{I}}(y),$$

where $f_{\mathcal{I}}$ is the symbolic representation map for (X, G, \mathcal{I}) defined above. Note that if \mathcal{I} is a generator, then $F_{\mathcal{I}}$ is just the equality relation.

For $A \subseteq X$, put

$$\Gamma(X)|_A = \{A' \subseteq A : \exists B \in \Gamma(X) (A' = B \cap A)\}.$$

Also, for an equivalence relation E on X and $A, B \subseteq X$, say that A is E -invariant relative to B or just $E|_B$ -invariant if $[A]_E \cap B = A \cap B$.

Definition 2.1 (\mathcal{I} -equidecomposability). Let $A, B \subseteq X$, and \mathcal{I} be a finite Borel partition of X . A and B are said to be equidecomposable with Γ pieces (denote by $A \sim^{\Gamma} B$) if there are $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ and partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B , respectively, such that for all $n \in \mathbb{N}$

- $g_n A_n = B_n$,
- $A_n \in \Gamma(X)|_A$ and $B_n \in \Gamma(X)|_B$.

If moreover,

- A_n and B_n are $F_{\mathcal{I}}$ -invariant relative to A and B , respectively,

then we will say that A and B are \mathcal{I} -equidecomposable with Γ pieces and denote it by $A \sim_{\mathcal{I}}^{\Gamma} B$. If $\Gamma = \mathfrak{B}$, we will not mention Γ and will just write \sim and $\sim_{\mathcal{I}}$.

Note that for any finite Borel partition \mathcal{I} of X and Borel sets $A, B \subseteq X$, A and B are \mathcal{I} -equidecomposable if and only if $f_{\mathcal{I}}(A)$ and $f_{\mathcal{I}}(B)$ are equidecomposable (although the images of Borel sets under $f_{\mathcal{I}}$ are analytic, they are Borel relative to $f_{\mathcal{I}}(X)$ due to the Luzin Separation Theorem for analytic sets, see [13, 14.7]). Also note that if \mathcal{I} is a generator, then $\sim_{\mathcal{I}}$ coincides with \sim .

Observation 2.2. Below let $\mathcal{I}, \mathcal{I}_0, \mathcal{I}_1$ denote finite Borel partitions of X , and $A, B, C \in \Gamma(X)$.

- (Quasi-transitivity) If $A \sim_{\mathcal{I}_0}^{\Gamma} B \sim_{\mathcal{I}_1}^{\Gamma} C$, then $A \sim_{\mathcal{I}}^{\Gamma} C$ with $\mathcal{I} = \mathcal{I}_0 \vee \mathcal{I}_1$ (the least common refinement of \mathcal{I}_0 and \mathcal{I}_1).
- ($F_{\mathcal{I}}$ -disjoint countable additivity) Let $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ be partitions of A and B , respectively, into Γ sets such that $\forall n \neq m, [A_n]_{F_{\mathcal{I}}} \cap [A_m]_{F_{\mathcal{I}}} = [B_n]_{F_{\mathcal{I}}} \cap [B_m]_{F_{\mathcal{I}}} = \emptyset$. If $\forall n \in \mathbb{N}, A_n \sim_{\mathcal{I}}^{\Gamma} B_n$, then $A \sim_{\mathcal{I}}^{\Gamma} B$.

If $A \sim B$, then there is a Borel isomorphism ϕ of A onto B with $\phi(x)E_G x$ for all $x \in A$; namely $\phi(x) = g_n x$ for all $x \in A_n$, where A_n, g_n are as in Definition 1.23. It is easy to see that the converse is also true, i.e. if such ϕ exists, then $A \sim B$. In Proposition 2.5 we prove the analogue of this for $\sim_{\mathcal{I}}^{\Gamma}$, but first we need the following lemma and definition that take care of definability and $F_{\mathcal{I}}$ -invariance, respectively.

For a Polish space Y , $f : X \rightarrow Y$ is said to be Γ -measurable if the preimages of open sets under f are in Γ . For $A \in \Gamma(X)$ and $h : A \rightarrow G$, define $\hat{h} : A \rightarrow X$ by $x \mapsto h(x)x$.

Lemma 2.3. If $h : A \rightarrow G$ is Γ -measurable, then the images and preimages of sets in Γ under \hat{h} are in Γ .

Proof. Let $B \subseteq A, C \subseteq X$ be in Γ . For $g \in G$, set $A_g = h^{-1}(g)$ and note that $\hat{h}(B) = \bigcup_{g \in G} g(A_g \cap B)$ and $\hat{h}^{-1}(C) = \bigcup_{g \in G} g^{-1}(gA_g \cap C)$. Thus $\hat{h}(B)$ and $\hat{h}^{-1}(C)$ are in Γ by the assumptions on Γ . \square

The following technical definition is needed in the proofs of Propositions 2.5 and 2.9.

Definition 2.4. For $A \subseteq X$ and a finite Borel partition \mathcal{I} of X , we say that \mathcal{I} is A -sensitive or that A respects \mathcal{I} if A is $F_{\mathcal{I}}$ -invariant relative to $[A]_G$, i.e. $[A]_{F_{\mathcal{I}}}^{[A]_G} = A$.

For example, if \mathcal{I} is finer than $\{A, A^c\}$, then \mathcal{I} is A -sensitive. Note that if $A \sim_{\mathcal{I}} B$ and A respects \mathcal{I} , then so does B .

Proposition 2.5. *Let $A, B \in \Gamma(X)$ and let \mathcal{I} be a Borel partition of X that is A -sensitive. Then, $A \sim_{\mathcal{I}}^{\Gamma} B$ if and only if there is an $F_{\mathcal{I}}$ -invariant Γ -measurable map $\gamma : A \rightarrow G$ such that $\hat{\gamma}$ is a bijection between A and B . We refer to such γ as a witnessing map for $A \sim_{\mathcal{I}}^{\Gamma} B$. The same holds if we delete “ $F_{\mathcal{I}}$ -invariant” and “ \mathcal{I} ” from the statement.*

Proof. \Rightarrow : If $\{g_n\}_{n \in \mathbb{N}}$, $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ are as in Definition 2.1, then define $\gamma : A \rightarrow G$ by setting $\gamma|_{A_n} \equiv g_n$.

\Leftarrow : Let γ be as in the lemma. Fixing an enumeration $\{g_n\}_{n \in \mathbb{N}}$ of G with no repetitions, put $A_n = \gamma^{-1}(g_n)$ and $B_n = g_n A_n$. It is clear that $\{A_n\}_{n \in \mathbb{N}}$, $\{B_n\}_{n \in \mathbb{N}}$ are partitions of A and B , respectively, into Γ sets. Since γ is $F_{\mathcal{I}}$ -invariant, each A_n is $F_{\mathcal{I}}$ -invariant relative to A and hence relative to $P := [A]_G = [B]_G$ because A respects \mathcal{I} . It remains to show that each B_n is $F_{\mathcal{I}}$ -invariant relative to B . To this end, let $y \in [B_n]_{F_{\mathcal{I}}} \cap B$ and thus there is $x \in A_n$ such that $y F_{\mathcal{I}} g_n x$. Hence $z := g_n^{-1} y F_{\mathcal{I}} g_n^{-1} g_n x = x$ and therefore $z \in A_n$ because A_n is $F_{\mathcal{I}}$ -invariant relative to P . Thus $y = g_n z \in B_n$. \square

In the rest of the subsection we work with $\Gamma = \mathfrak{B}$.

Next we prove that \mathcal{I} -equidecomposability can be extended to $F_{\mathcal{I}}$ -invariant Borel sets. First we need the following separation lemma for analytic sets⁸:

Lemma 2.6 (*Invariant analytic separation*). *Let E be an analytic equivalence relation on X . For any disjoint family $\{A_n\}_{n \in \mathbb{N}}$ of E -invariant analytic sets, there exists a disjoint family $\{B_n\}_{n \in \mathbb{N}}$ of E -invariant Borel sets such that $A_n \subseteq B_n$.*

Proof (Vaught). We give the proof for two disjoint E -invariant analytic sets A_0, A_1 since this easily implies the statement for countably many. Recursively define analytic sets $C_n \subseteq X$ and Borel sets $D_n \subseteq X$ such that for every $n \in \mathbb{N}$ we have

- (i) $A_0 \subseteq C_n \subseteq D_n \subseteq C_{n+1} \subseteq A_1^c$,
- (ii) C_n is E -invariant.

To do this, let $C_0 = A_0$, and, assuming that C_n is defined, define D_n, C_{n+1} as follows: since C_n and A_1 are disjoint analytic sets, there is a Borel set D_n separating them (by the Luzin separation theorem), i.e. $D_n \supseteq C_n$ and $D_n \cap A_1 = \emptyset$. Let $C_{n+1} = [D_n]_E$, and note that C_{n+1} is analytic and disjoint from A_1 since A_1 is E -invariant and disjoint from D_n . This finishes the construction.

⁸ My original argument used Π_1^1 reflection principles, but it was pointed out to me by Shashi Srivastava that one could use analytic separation instead. I chose to present this latter argument here since analytic separation may be more transparent for non-logicians than Π_1^1 reflection principles.

Now let $B = \bigcup_{n \in \mathbb{N}} D_n$; hence B is Borel, contains A_0 and is disjoint from A_1 . On the other hand, $B = \bigcup_{n \in \mathbb{N}} C_n$ and thus is E -invariant. \square

Proposition 2.7 (*$F_{\mathcal{I}}$ -invariant extensions*). Let \mathcal{I} be a Borel partition of X and let $A, B \subseteq X$ be Borel sets with $[A]_{F_{\mathcal{I}}} \cap [B]_{F_{\mathcal{I}}} = \emptyset$. If $A \sim_{\mathcal{I}} B$, then there exists Borel sets $A' \supseteq A$ and $B' \supseteq B$ such that A', B' are $F_{\mathcal{I}}$ -invariant and $A' \sim_{\mathcal{I}} B'$. In fact, if $\{g_n\}_{n \in \mathbb{N}}, \{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ witness $A \sim_{\mathcal{I}} B$, then there are $F_{\mathcal{I}}$ -invariant Borel partitions $\{A'_n\}_{n \in \mathbb{N}}, \{B'_n\}_{n \in \mathbb{N}}$ of A' and B' respectively, such that $g_n A'_n = B'_n$ and $A'_n \supseteq A_n$ (and hence $B'_n \supseteq B_n$).

Proof. Let $\{g_n\}_{n \in \mathbb{N}}, \{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ be as in Definition 2.1 and put $\bar{A}_n = [A_n]_{F_{\mathcal{I}}}$. It is easy to see that for $n \neq m \in \mathbb{N}$,

- (i) $\bar{A}_n \cap \bar{A}_m = \emptyset$;
- (ii) $g_n \bar{A}_n \cap g_m \bar{A}_m = \emptyset$.

Put $\bar{A} = [A]_{F_{\mathcal{I}}}$ and note that $\{\bar{A}_n\}_{n \in \mathbb{N}}$ is a partition of \bar{A} . Although \bar{A}_n and \bar{A} are $F_{\mathcal{I}}$ -invariant, they are analytic and in general not Borel. We obtain Borel analogues of these sets using invariant analytic separation as follows: Lemma 2.6 applied to $\{A_n\}_{n \in \mathbb{N}}$ and $\{g_n A_n\}_{n \in \mathbb{N}}$ (separately), gives us sequences $\{C_n\}_{n \in \mathbb{N}}$ and $\{D_n\}_{n \in \mathbb{N}}$ of $F_{\mathcal{I}}$ -invariant Borel sets such that $C_n \supseteq A_n$, $D_n \supseteq g_n A_n$ and $C_n \cap C_m = D_n \cap D_m = \emptyset$ for $n \neq m$. Taking $A'_n = C_n \cap g_n^{-1} D_n$, we see that $\{A'_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint family of $F_{\mathcal{I}}$ -invariant Borel sets such that $A'_n \supseteq A_n$. Moreover, $\{g_n A'_n\}_{n \in \mathbb{N}}$ is also a pairwise disjoint family. Thus, taking $B'_n = g_n A'_n$, we are done. \square

Lemma 2.8 (*Orbit-disjoint unions*). Let $A_k, B_k \in \mathfrak{B}(X)$, $k = 0, 1$, be such that $[A_0]_G$ and $[A_1]_G$ are disjoint and put $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$. If \mathcal{I} is an A, B -sensitive finite Borel partition of X such that $A_k \sim_{\mathcal{I}} B_k$ for $k = 0, 1$, then $A \sim_{\mathcal{I}} B$. Moreover, if $\gamma_0 : A_0 \rightarrow G$ is a Borel map witnessing $A_0 \sim_{\mathcal{I}} B_0$, then there exists a Borel map $\gamma : A \rightarrow G$ extending γ_0 that witnesses $A \sim_{\mathcal{I}} B$.

Proof. First assume without loss of generality that $X = [A]_G (= [B]_G)$ since the statement of the lemma is relative to $[A]_G$. Thus A, B are $F_{\mathcal{I}}$ -invariant.

Applying Proposition 2.7 to $A_0 \sim_{\mathcal{I}} B_0$, we get $F_{\mathcal{I}}$ -invariant $A'_0 \supseteq A_0$, $B'_0 \supseteq B_0$ such that $A' \sim_{\mathcal{I}} B'$. Moreover, by the second part of the same lemma, if $\gamma_0 : A_0 \rightarrow G$ is a witnessing map for $A_0 \sim_{\mathcal{I}} B_0$, then there is a witnessing map $\delta : A'_0 \rightarrow G$ for $A' \sim_{\mathcal{I}} B'$ extending γ_0 . Put $C = A'_0 \cap A$ and note that C is $F_{\mathcal{I}}$ -invariant since so are A'_0 and A . Finally, put $\bar{A}_0 = \{x \in C : C^{[x]_G} = A^{[x]_G} \wedge \hat{\delta}(C^{[x]_G}) = B^{[x]_G}\}$ and note that $\bar{A}_0 \supseteq A_0$ since $\delta \supseteq \gamma_0$ and $[A_0]_G \cap [A_1]_G = \emptyset$.

Claim. \bar{A}_0 is $F_{\mathcal{I}}$ -invariant.

Proof of Claim. First note that for any $F_{\mathcal{I}}$ -invariant $D \subseteq X$ and $z \in X$, $[D^{[z]G}]_{F_{\mathcal{I}}} = D^{[[z]_{F_{\mathcal{I}}}G}]$. Furthermore, if $D \subseteq C$, then $[\hat{\delta}(D)]_{F_{\mathcal{I}}} = \hat{\delta}([D]_{F_{\mathcal{I}}})$ since $\hat{\delta}$ and its inverse map $F_{\mathcal{I}}$ -invariant sets to $F_{\mathcal{I}}$ -invariant sets.

Now take $x \in \bar{A}_0$ and let $Q = [[x]_{F_{\mathcal{I}}}]_G$. Since A, B, C are $F_{\mathcal{I}}$ -invariant, $C^Q = [C^{[x]G}]_{F_{\mathcal{I}}} = [A^{[x]G}]_{F_{\mathcal{I}}} = A^Q$. Furthermore, $\hat{\delta}(C^Q) = \hat{\delta}([C^{[x]G}]_{F_{\mathcal{I}}}) = [\hat{\delta}(C^{[x]G})]_{F_{\mathcal{I}}} = [B^{[x]G}]_{F_{\mathcal{I}}} = B^Q$. Thus, $\forall y \in [x]_{F_{\mathcal{I}}}$, $C^{[y]G} = A^{[y]G}$ and $\hat{\delta}(C^{[y]G}) = B^{[y]G}$; hence $[x]_{F_{\mathcal{I}}} \subseteq \bar{A}_0$. \square

Put $\bar{A}_1 = A \setminus \bar{A}_0$, $\alpha_0 = \delta|_{\bar{A}_0}$, $\alpha_1 = \gamma_1|_{\bar{A}_1}$, where γ_1 is a witnessing map for $A_1 \sim_{\mathcal{I}} B_1$. It is clear from the definition of \bar{A}_0 that \bar{A}_0 is E_G -invariant relative to A and hence $[\bar{A}_0]_G \cap [\bar{A}_1]_G = \emptyset$. Thus, for $k = 0, 1$, it follows that α_k witnesses $\bar{A}_k \sim_{\mathcal{I}} \bar{B}_k$, where $\bar{B}_k = \hat{\alpha}_k(\bar{A}_k)$. Furthermore, it is clear that $B^{[\bar{A}_k]G} = \bar{B}_k$ and, since $[\bar{A}_0]_G \cup [\bar{A}_1]_G = X$, $\bar{B}_0 \cup \bar{B}_1 = B$. Now since \bar{A}_k are $F_{\mathcal{I}}$ -invariant, $\gamma = \alpha_0 \cup \alpha_1$ is $F_{\mathcal{I}}$ -invariant and hence witnesses $A \sim_{\mathcal{I}} B$. Finally, $\alpha_0|_{A_0} = \delta|_{A_0} = \gamma_0$ and hence $\alpha_0 \supseteq \gamma_0$. \square

Proposition 2.9 (*Orbit-disjoint countable unions*). For $k \in \mathbb{N}$, let $A_k, B_k \in \mathfrak{B}(X)$ be such that $[A_k]_G$ are disjoint and put $A = \bigcup_{k \in \mathbb{N}} A_k$, $B = \bigcup_{k \in \mathbb{N}} B_k$. Suppose that \mathcal{I} is an A, B -sensitive finite Borel partition of X such that $A_k \sim_{\mathcal{I}} B_k$ for all k . Then $A \sim_{\mathcal{I}} B$.

Proof. We recursively apply Lemma 2.8 as follows. Put $\bar{A}_n = \bigcup_{k \leq n} A_k$ and $\bar{B}_n = \bigcup_{k \leq n} B_k$. Inductively define Borel maps $\gamma_n : \bigcup_{k \leq n} A_k \rightarrow G$ such that γ_n is a witnessing map for $\bar{A}_n \sim_{\mathcal{I}} \bar{B}_n$ and $\gamma_n \sqsubseteq \gamma_{n+1}$. Let γ_0 be a witnessing map for $A_0 \sim_{\mathcal{I}} B_0$. Assume γ_n is defined. Then γ_{n+1} is provided by Lemma 2.8 applied to \bar{A}_n and A_{n+1} with γ_n as a witness for $\bar{A}_n \sim_{\mathcal{I}} \bar{B}_n$. Thus $\gamma_n \sqsubseteq \gamma_{n+1}$ and γ_{n+1} witnesses $\bar{A}_{n+1} \sim_{\mathcal{I}} \bar{B}_{n+1}$.

Now it just remains to show that $\gamma := \bigcup_{n \in \mathbb{N}} \gamma_n$ is $F_{\mathcal{I}}$ -invariant since then it follows that γ witnesses $A \sim_{\mathcal{I}} B$. Let $x, y \in A$ be $F_{\mathcal{I}}$ -equivalent. Then there is n such that $x, y \in \bar{A}_n$. By induction on n , γ_n is $F_{\mathcal{I}}$ -invariant and, since $\gamma|_{\bar{A}_n} = \gamma_n$, $\gamma(x) = \gamma(y)$. \square

Corollary 2.10 (*Finite quasi-additivity*). For $k = 0, 1$, let $A_k, B_k \in \mathfrak{B}(X)$ be such that $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$ and put $A = A_0 \cup A_1$, $B = B_0 \cup B_1$. Let \mathcal{I}_k be an A_k, B_k -sensitive finite Borel partition of X . If $A_0 \sim_{\mathcal{I}_0} B_0$ and $A_1 \sim_{\mathcal{I}_1} B_1$, then $A \sim_{\mathcal{I}_0 \vee \mathcal{I}_1} B$.

Proof. Put $\mathcal{I} = \mathcal{I}_0 \vee \mathcal{I}_1$, $P = [A_0]_G \cap [A_1]_G$, $Q = [A_0]_G \setminus [A_1]_G$ and $R = [A_1]_G \setminus [A_0]_G$. Then A_k^P, B_k^P respect \mathcal{I} , and thus $[A_0]_{F_{\mathcal{I}}}^P \cap [A_1]_{F_{\mathcal{I}}}^P = \emptyset$, $[B_0]_{F_{\mathcal{I}}}^P \cap [B_1]_{F_{\mathcal{I}}}^P = \emptyset$. Hence $A^P \sim_{\mathcal{I}} B^P$ since the sets that are $F_{\mathcal{I}}$ -invariant relative to A_k^P are also $F_{\mathcal{I}}$ -invariant relative to A^P , and the same is true for B_k^P and B^P . Also, $A^Q \sim_{\mathcal{I}} B^Q$ and $A^R \sim_{\mathcal{I}} B^R$ because $A^Q = A_0$, $B^Q = B_0$, $A^R = A_1$, $B^R = B_1$. Now since P, Q, R are pairwise disjoint, it follows from Proposition 2.9 that $A \sim_{\mathcal{I}} B$. \square

2.2. The notion of i -compressibility

For a finite collection \mathcal{F} of subsets of X , let $\langle \mathcal{F} \rangle$ denote the partition of X generated by \mathcal{F} .

Definition 2.11 (*i-equidecomposability*). For $i \geq 1$, $A, B \subseteq X$, we say that A and B are i -equidecomposable with Γ pieces (write $A \sim_i^\Gamma B$) if there is an A -sensitive partition \mathcal{I} of X generated by i Borel sets such that $A \sim_{\mathcal{I}}^\Gamma B$ (in particular, \mathcal{I} must also be B -sensitive). For a collection \mathcal{F} of Borel sets, we say that \mathcal{F} witnesses $A \sim_i^\Gamma B$ if $|\mathcal{F}| = i$, $\mathcal{I} := \langle \mathcal{F} \rangle$ is A -sensitive and $A \sim_{\mathcal{I}}^\Gamma B$.

Remark 2.12. In the above definition, it might seem more natural to have i be the cardinality of the partition \mathcal{I} instead of the cardinality of the collection \mathcal{F} generating \mathcal{I} . However, our definition above of i -equidecomposability is needed in order to show that the collection \mathfrak{C}_i defined below forms a σ -ideal. More precisely, the presence of \mathcal{F} is needed in the definition of i^* -compressibility, which ensures that the partition \mathcal{I} in the proof of Proposition 2.19 is B -sensitive.

For a family \mathcal{F} of subsets of X , let $\sigma_G(\mathcal{F})$ denote the σ -algebra generated by $G\mathcal{F}$.

Remark 2.13. Sławomir Solecki pointed out that for $i \geq 1$ and Borel sets $A, B \subseteq X$, $A \sim_i B$ if and only if $A \sim B$ and the partitions $\{A_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ witnessing the equidecomposability of A and B can be taken from a σ -algebra generated by the G -translates of i -many Borel sets. More precisely, $A \sim_i B$ if and only if there are a family \mathcal{F} of i -many Borel sets, a sequence $\{g_n\}_{n \in \mathbb{N}} \subseteq G$, and partitions $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ of A and B , respectively, such that $A_n, B_n \in \sigma_G(\mathcal{F})$ and $g_n A_n = B_n$. Thus, i -equidecomposability is obtained from equidecomposability by restricting the Borel σ -algebra to *some* σ -algebra generated by the G -translates of i -many Borel sets. Finally, note that every instance of \sim_i uses a (potentially) different σ -algebra.

For $i \geq 1$, $A, B \subseteq X$, we write $A \preceq_i^\Gamma B$ if there is a Γ set $B' \subseteq B$ such that $A \sim_i^\Gamma B'$. If moreover $[A \setminus B]_G = [A]_G$, then we write $A \prec_i^\Gamma B$. If $\Gamma = \mathfrak{B}$, we simply write $\sim_i, \preceq_i, \prec_i$.

Definition 2.14 (*i-compressibility*). For $i \in \mathbb{N}$, $A \subseteq X$, we say that A is i -compressible with Γ pieces if $A \prec_i^\Gamma A$.

Unless specified otherwise, we will be working with $\Gamma = \mathfrak{B}$, in which case we simply say i -compressible.

For a collection of sets \mathcal{F} and a G -invariant set P , set $\mathcal{F}^P = \{A^P : A \in \mathcal{F}\}$. We will use the following observations without mentioning.

Observation 2.15. Let $i, j \geq 2$, $A, A', B, B', C \in \mathfrak{B}$. Let $P \subseteq [A]_G$ denote a G -invariant Borel set and $\mathcal{F}, \mathcal{F}_0, \mathcal{F}_1$ denote finite collections of Borel sets.

- (a) If $A \sim_i B$ then $A^P \sim_i B^P$.
- (b) If \mathcal{F} witnesses $A \sim_i B$, then so does $\mathcal{F}^{[A]_G}$.

- (c) If $A \sim_i B \sim_j C$, then $A \sim_{(i+j)} C$. In fact, \mathcal{F}_0 and \mathcal{F}_1 witness $A \sim_i B$ and $B \sim_j C$, respectively, then $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ witnesses $A \sim_{(i+j)} C$.
- (d) If $A \preceq_i B \preceq_j C$, then $A \preceq_{(i+j)} C$. If one of the first two \preceq is \prec then $A \prec_{(i+j)} C$.
- (e) If $A \sim_i B$ and $A' \sim_j B'$ with $A \cap A' = B \cap B' = \emptyset$, then $A \cup A' \sim_{(i+j)} B \cup B'$.

Proof. Part (e) follows from [Corollary 2.10](#), and the rest follows directly from the definition of i -equidecomposability and [Observation 2.2](#). \square

Lemma 2.16. *If a Borel set $A \subseteq X$ is i -compressible, then so is $[A]_G$. In fact, if \mathcal{F} is a finite collection of Borel sets witnessing the i -compressibility of A , then it also witnesses that of $[A]_G$.*

Proof. Let $B \subseteq A$ be a Borel set such that $[A \setminus B]_G = [A]_G$ and $A \sim_i B$. Furthermore, let \mathcal{I} be an A, B -sensitive partition generated by a collection \mathcal{F} of i Borel sets such that $A \sim_{\mathcal{I}} B$. Let $\gamma : A \rightarrow G$ be a witnessing map for $A \sim_I B$. Put $A' = [A]_G$, $B' = B \cup (A' \setminus A)$ and note that A', B' respect \mathcal{I} . Define $\gamma' : A' \rightarrow G$ by setting $\gamma'|_{A' \setminus A} = id|_{A' \setminus A}$ and $\gamma'|_A = \gamma$. Since A' respects \mathcal{I} and $id|_{A' \setminus A}, \gamma$ are $F_{\mathcal{I}}$ -invariant, γ' is $F_{\mathcal{I}}$ -invariant and thus clearly witnesses $A' \sim_{\mathcal{I}} B'$. \square

The following is a technical refinement of the definition of i -compressibility that is (again) necessary for \mathfrak{C}_i , defined below, to be a σ -ideal.

Definition 2.17 (i^* -compressibility). For $i \geq 1$, we say that a Borel set A is i^* -compressible if there is a Borel set $B \subseteq A$ such that $[A \setminus B]_G = [A]_G =: P$, $A \sim_i B$, and the latter is witnessed by a collection \mathcal{F} of Borel sets such that $B \in \mathcal{F}^P$.

Finally, for $i \geq 1$, put

$$\mathfrak{C}_i = \{A \subseteq X : \text{there is a } G\text{-invariant Borel set } P \supseteq A \text{ such that } P \text{ is } i^*\text{-compressible}\}.$$

Lemma 2.18. *Let $i \geq 1$ and $A \subseteq X$ be Borel. If $A \prec_i A$, then $A \in \mathfrak{C}_{i+1}$.*

Proof. Setting $P = [A]_G$ and applying [Lemma 2.16](#), we get that $P \prec_i P$, i.e. there is $B \subseteq P$ such that $[P \setminus B]_G = P$ and $P \sim_i B$. Let \mathcal{F} be a collection of Borel sets witnessing the latter fact. Then $\mathcal{F}' = \mathcal{F} \cup \{B\}$ witnesses $P \sim_{(i+1)} B$ and contains B . \square

Proposition 2.19. *For all $i \geq 1$, \mathfrak{C}_i is a σ -ideal.*

Proof. We only need to show that \mathfrak{C}_i is closed under countable unions. For this it is enough to show that if $A_n \in \mathfrak{B}(X)$ are i^* -compressible G -invariant Borel sets, then so is $A := \bigcup_{n \in \mathbb{N}} A_n$.

We may assume that A_n are pairwise disjoint since we could replace each A_n by $A_n \setminus (\bigcup_{k < n} A_k)$. Let $B_n \subseteq A_n$ be a Borel set and $\mathcal{F}_n = \{F_k^n\}_{k < i}$ be a collection of Borel

sets with $(F_0^n)^{A_n} = B_n$ such that \mathcal{F}_n witnesses $A_n \sim_i B_n$ and $[A_n \setminus B_n]_G = A_n$. Using part (b) of [Observation 2.15](#), we may assume that $\mathcal{F}_n^{A_n} = \mathcal{F}_n$; in particular, $F_0^n = B_n$.

Put $B = \bigcup_{n \in \mathbb{N}} B_n$ and $F_k = \bigcup_{n \in \mathbb{N}} F_k^n$, $\forall k < i$; note that $F_0 = B$. Set $\mathcal{F} = \{F_k\}_{k < i}$ and $\mathcal{I} = \langle \mathcal{F} \rangle$. Since $B \in \mathcal{F}$ and A is G -invariant, \mathcal{I} is A , B -sensitive. Furthermore, since $\mathcal{F}^{A_n} = \mathcal{F}_n$, $A_n \sim_{\mathcal{I}} B_n$ for all $n \in \mathbb{N}$. Thus, by [Proposition 2.9](#), $A \sim_{\mathcal{I}} B$ and hence A is i^* -compressible. \square

3. Traveling sets and finite generators

Throughout this section, we again let X be a Borel G -space and Γ be a σ -algebra of subsets of standard Borel spaces as in the previous section.

3.1. Traveling and i -traveling sets

Definition 3.1. Let $A \in \Gamma(X)$.

- We call A a traveling set with Γ pieces if there exists pairwise disjoint sets $\{A_n\}_{n \in \mathbb{N}}$ in $\Gamma(X)$ such that $A_0 = A$ and $A \sim^\Gamma A_n$, $\forall n \in \mathbb{N}$.
- For a finite Borel partition \mathcal{I} , we say that A is \mathcal{I} -traveling with Γ pieces if A respects \mathcal{I} and the above condition holds with \sim^Γ replaced by $\sim_{\mathcal{I}}^\Gamma$.
- For $i \geq 1$, we say that A is i -traveling if it is \mathcal{I} -traveling for some A -sensitive partition \mathcal{I} generated by a collection of i Borel sets.

Definition 3.2. For a set $A \subseteq X$, a function $\gamma : A \rightarrow G^{\mathbb{N}}$ is called a travel guide for A if $\forall x \in A$, $\gamma(x)(0) = 1_G$ and $\forall (x, n) \neq (y, m) \in A \times \mathbb{N}$, $\gamma(x)(n)x \neq \gamma(y)(m)y$.

For $A \in \Gamma(X)$, a Γ -measurable map $\gamma : A \rightarrow G^{\mathbb{N}}$ and $n \in \mathbb{N}$, set $\gamma_n := \gamma(\cdot)(n) : A \rightarrow G$ and note that γ_n is also Γ -measurable.

Observation 3.3. Suppose $A \in \Gamma(X)$ and \mathcal{I} is an A -sensitive finite Borel partition of X . Then A is \mathcal{I} -traveling with Γ pieces if and only if it has a Γ -measurable $F_{\mathcal{I}}$ -invariant travel guide.

Proof. Follows from definitions and [Proposition 2.5](#). \square

Now we establish the connection between compressibility and traveling sets.

Lemma 3.4. Let \mathcal{I} be a finite Borel partition of X , $P \in \Gamma(X)$ be a Borel G -invariant set and let A, B be Γ subsets of P . If $P \sim_{\mathcal{I}}^\Gamma B$, then $P \setminus B$ is \mathcal{I} -traveling with Γ pieces. Conversely, if A is \mathcal{I} -traveling with Γ pieces, then $P \sim_{\mathcal{I}}^\Gamma (P \setminus A)$. The same is true if we replace $\sim_{\mathcal{I}}^\Gamma$ and “ \mathcal{I} -traveling” with \sim^Γ and “traveling”, respectively.

Proof. For the first statement, let $\gamma : P \rightarrow G$ be a witnessing map for $P \sim_{\mathcal{I}}^{\Gamma} B$. Put $A' = P \setminus B$ and note that A' respects \mathcal{I} since so does P and hence B . We show that A' is \mathcal{I} -traveling. Put $A_n = (\hat{\gamma})^n(A')$, for each $n \geq 0$. It follows from injectivity of $\hat{\gamma}$ that the A_n are pairwise disjoint. For $n \in \mathbb{N}$, recursively define $\delta_n : A' \rightarrow G$ as follows

$$\begin{cases} \delta_0 = \gamma|_{A'} \\ \delta_{n+1} = \gamma \circ \hat{\delta}_n \end{cases}.$$

It follows from $F_{\mathcal{I}}$ -invariance of γ that each δ_n is $F_{\mathcal{I}}$ -invariant. It is also clear that $\hat{\delta}_n = (\hat{\gamma})^n$ and hence δ_n is a witnessing map for $A' \sim_{\mathcal{I}}^{\Gamma} A_n$. Thus A' is i -traveling with Γ pieces.

For the converse, assume that A is \mathcal{I} -traveling and let $\{A_n\}_{n \in \mathbb{N}}$ be as in [Definition 3.1](#). In particular, each A_n respects \mathcal{I} and $A_n \sim_{\mathcal{I}}^{\Gamma} A_m$, for all $n, m \in \mathbb{N}$. Let $P' = \bigcup_{n \in \mathbb{N}} A_n$ and $B' = \bigcup_{n \geq 1} A_n$. Since $A_n \sim_{\mathcal{I}}^{\Gamma} A_{n+1}$, part (b) of [Observation 2.2](#) implies that $P' \sim_{\mathcal{I}}^{\Gamma} B'$. Moreover, since $P \setminus P' \sim_{\mathcal{I}}^{\Gamma} P \setminus P'$, we get $P \sim_{\mathcal{I}}^{\Gamma} (B' \cup (P \setminus P')) = P \setminus A$. \square

For a G -invariant set P and $A \subseteq P$, we say that A is a complete section for P if $[A]_G = P$. The above lemma immediately implies the following.

Proposition 3.5. *Let $P \in \Gamma(X)$ be G -invariant and $i \geq 1$. P is i -compressible with Γ pieces if and only if there exists a complete section for P that is i -traveling with Γ pieces. The same is true with “ i -compressible” and “ i -traveling” replaced by “compressible” and “traveling”.*

We need the following lemma in the proofs of [Propositions 3.7 and 3.8](#).

Lemma 3.6. *Suppose $A \subseteq X$ is an invariant analytic set that does not admit an invariant Borel probability measure. Then there is an invariant Borel set $A' \supseteq A$ that still does not admit an invariant Borel probability measure.*

Proof. Let \mathcal{M} denote the standard Borel space of G -invariant Borel probability measures on X (see [\[13, Section 17\]](#)). Let $\Phi \subseteq \text{Pow}(X)$ be the following predicate:

$$\Phi(W) \Leftrightarrow \forall \mu \in \mathcal{M} (\mu(W) = 0).$$

Claim. *There is a Borel set $B \supseteq A$ with $\Phi(B)$.*

Proof of Claim. By the dual form of the First Reflection Theorem for Π_1^1 (see [\[13, the discussion following 35.10\]](#)), it is enough to show that Φ is Π_1^1 on Σ_1^1 . To this end, let Y be a Polish space and $D \subseteq Y \times X$ be analytic. Then, for any $n \in \mathbb{N}$, the set

$$H_n = \left\{ (\mu, y) \in \mathcal{M} \times Y : \mu(D_y) > \frac{1}{n} \right\},$$

is analytic by a theorem of Kondô–Tugué (see [13, 29.26]), and hence so are the sets $H'_n := \text{proj } Y(H_n)$ and $H := \bigcup_{n \in \mathbb{N}} H'_n$. Finally, note that

$$\{y \in Y : \Phi(A_y)\} = \left\{y \in Y : \exists \mu \in \mathcal{M} \exists n \in \mathbb{N} \left(\mu(A_y) > \frac{1}{n} \right)\right\}^c = H^c,$$

and so $\{y \in Y : \Phi(A_y)\}$ is Π_1^1 . \square

Now put $A' = (B)_G$, where $(B)_G = \{x \in B : [x]_G \subseteq B\}$. Clearly, A' is an invariant Borel set, $A' \supseteq A$, and $\Phi(A')$ since $A' \subseteq B$ and $\Phi(B)$. \square

Proposition 3.7. *Let X be a Borel G -space. The following are equivalent:*

- (1) X is compressible with universally measurable pieces;
- (2) There is a universally measurable complete section that is a traveling set with universally measurable pieces;
- (3) There is no G -invariant Borel probability measure on X ;
- (4) X is compressible with Borel pieces;
- (5) There is a Borel complete section that is a traveling set with Borel pieces.

Proof. Equivalence of (1) and (2) as well as (4) and (5) is asserted in Proposition 3.5, (4) \Rightarrow (1) is trivial, and (3) \Rightarrow (4) follows from Nadkarni's theorem (see Theorem 1.26). It remains to show (1) \Rightarrow (3). To this end, suppose $X \sim^\Gamma B$, where $B^c = X \setminus B$ is a complete section and Γ is the class of universally measurable sets. If there was a G -invariant Borel probability measure μ on X , then $\mu(X) = \mu(B)$ and hence $\mu(B^c) = 0$. But since B^c is a complete section, $X = \bigcup_{g \in G} gB^c$, and thus $\mu(X) = 0$, a contradiction. \square

Now we prove an analogue of this for i -compressibility.

Proposition 3.8. *Let X be a Borel G -space. For $i \geq 1$, the following are equivalent:*

- (1) X is i -compressible with universally measurable pieces;
- (2) There is a universally measurable complete section that is an i -traveling set with universally measurable pieces;
- (3) There is a partition \mathcal{I} of X generated by i Borel sets such that $Y = f_{\mathcal{I}}(X) \subseteq |\mathcal{I}|^G$ does not admit a G -invariant Borel probability measure;
- (4) X is i -compressible with Borel pieces;
- (5) There is a Borel complete section that is an i -traveling set with Borel pieces.

Proof. Equivalence of (1) and (2) as well as (4) and (5) is asserted in Proposition 3.5 and (4) \Rightarrow (1) is trivial. It remains to show (1) \Rightarrow (3) \Rightarrow (5).

(1) \Rightarrow (3): Suppose $X \sim_{\mathcal{I}}^\Gamma B$, where $B^c = X \setminus B$ is a complete section, \mathcal{I} is a partition of X generated by i Borel sets, and Γ denotes the class of universally measurable

sets. Let $\gamma : X \rightarrow G$ be a witnessing map for $X \sim_i^F B$. By the Jankov-von Neumann uniformization theorem (see [13, 18.1]), $f_{\mathcal{I}}$ has a $\sigma(\Sigma_1^1)$ -measurable (hence universally measurable) right inverse $h : Y \rightarrow X$. Define $\delta : Y \rightarrow G$ by $\delta(y) = \gamma(h(y))$ and note that δ is universally measurable being a composition of such functions. Letting $B' = \hat{\delta}(Y)$, it is straightforward to check that $\hat{\delta} \circ f_{\mathcal{I}} = f_{\mathcal{I}} \circ \hat{\gamma}$ and thus $B' = f_{\mathcal{I}}(\hat{\gamma}(X)) = f_{\mathcal{I}}(B)$. Now it follows that δ is a witnessing map for $Y \sim^F B'$ and hence Y is compressible with universally measurable pieces. Finally, (1) \Rightarrow (3) of Proposition 3.7 implies that Y does not admit an invariant Borel probability measure.

(3) \Rightarrow (5): Assume Y is as in (3). Then by Lemma 3.6, there is a Borel G -invariant $Y' \supseteq Y$ that does not admit a G -invariant Borel probability measure. Viewing Y' as a Borel G -space, we apply (3) \Rightarrow (4) of Proposition 3.7 and get that Y' is compressible with Borel pieces; thus there is a Borel $B' \subseteq Y'$ with $[Y' \setminus B']_G = Y'$ such that $Y' \sim B'$. Let $\delta : Y' \rightarrow G$ be a witnessing map for $Y' \sim B'$. Put $B = f_{\mathcal{I}}^{-1}(B')$ and $\gamma = \delta \circ f_{\mathcal{I}}$. By definition, γ is $F_{\mathcal{I}}$ -invariant. In fact, it is straightforward to check that γ is a witnessing map for $X \sim_{\mathcal{I}} B$ and $[X \setminus B]_G = [f_{\mathcal{I}}^{-1}(Y \setminus B')]_G = f_{\mathcal{I}}^{-1}([Y \setminus B']_G) = f_{\mathcal{I}}^{-1}(Y) = X$. Hence X is \mathcal{I} -compressible. \square

We now give an example of a 1-traveling set. First we need some definitions.

Definition 3.9. Let X be a Borel G -space. A set $A \subseteq X$ is called

- aperiodic if it intersects every orbit in either 0 or infinitely many points;
- a partial transversal if it intersects every orbit in at most one point;
- a transversal if it intersects every orbit in exactly one point;
- smooth if there is a Borel partial transversal $T \subseteq A$ such that $[T]_G = [A]_G$.

Finally, the action $G \curvearrowright X$ is called smooth if X is a smooth set, i.e. admits a Borel transversal.

Proposition 3.10. Let X be an aperiodic Borel G -space and $T \subseteq X$ be Borel. If T is a partial transversal, then T is $\langle T \rangle$ -traveling.

Proof. Let $G = \{g_n\}_{n \in \mathbb{N}}$ with $g_0 = 1_G$. For each $n \in \mathbb{N}$, define $\bar{n} : X \rightarrow \mathbb{N}$ and $\gamma_n : T \rightarrow G$ recursively in n as follows:

$$\begin{cases} \bar{n}(x) = \text{the least } k \text{ such that } g_k x \notin \{\hat{\gamma}_i(x) : i < n\} \\ \gamma_n(x) = g_{\bar{n}(x)} \end{cases}.$$

Clearly, \bar{n} and γ_n are well-defined and Borel. Define $\gamma : T \rightarrow G^{\mathbb{N}}$ by setting $\gamma(\cdot)(n) = \gamma_n$. It follows from the definitions that γ is a Borel travel guide for T and hence, T is a traveling set. It remains to show that γ is $F_{\mathcal{I}}$ -invariant, where $\mathcal{I} = \langle T \rangle$. For this it is enough to show that \bar{n} is $F_{\mathcal{I}}$ -invariant, which we do by induction on n . Since it trivially

holds for $n = 0$, we assume it is true for all $0 \leq k < n$ and show it for n . To this end, suppose $x, y \in T$ with $x F_{\mathcal{I}} y$, and assume for contradiction that $m := \bar{n}(x) < \bar{n}(y)$. Thus it follows that $g_m y = \hat{\gamma}_k(y) \in \hat{\gamma}_k(T)$, for some $k < n$. By the induction hypothesis, $\hat{\gamma}_k(T)$ is $F_{\mathcal{I}}$ -invariant and hence, $g_m x \in \hat{\gamma}_k(T)$, contradicting the definition of $\bar{n}(x)$. \square

Corollary 3.11. *Let X be an aperiodic Borel G -space. If a Borel set $A \subseteq X$ is smooth, then $A \in \mathfrak{C}_1$.*

Proof. Let $P = [A]_G$ and let T be a Borel partial transversal with $[T]_G = P$. By Proposition 3.10, T is \mathcal{I} -traveling, where $\mathcal{I} = \langle T \rangle$. Hence, $P \sim_{\mathcal{I}} P \setminus T$, by Lemma 3.4. This implies that P is 1^* -compressible since $\mathcal{I} = \langle T^c \rangle$ and $P \setminus T \in \{T^c\}^P$. \square

3.2. Constructing finite generators using i -traveling sets

Lemma 3.12. *Let $A \in \mathfrak{B}(X)$ be a complete section and \mathcal{I} be an A -sensitive finite Borel partition of X . If A is \mathcal{I} -traveling (with Borel pieces), then there is a Borel $2|\mathcal{I}|$ -generator. If moreover $A \in \mathcal{I}$, then there is a Borel $(2|\mathcal{I}| - 1)$ -generator.*

Proof. Let γ be an $F_{\mathcal{I}}$ -invariant Borel travel guide for A . Fix a countable family $\{U_n\}_{n \in \mathbb{N}}$ generating the Borel structure of X and let $B = \bigcup_{n \geq 1} \hat{\gamma}_n(A \cap U_n)$. By Lemma 2.3, each $\hat{\gamma}_n$ maps Borel sets to Borel sets and hence B is Borel. Set $\mathcal{J} = \langle B \rangle$, $\mathcal{P} = \mathcal{I} \vee \mathcal{J}$ and note that $|\mathcal{P}| \leq 2|\mathcal{I}|$. A and B are disjoint since $\{\hat{\gamma}_n(A)\}_{n \in \mathbb{N}}$ is a collection of pairwise disjoint sets and $\hat{\gamma}_0(A) = A$; thus if $A \in \mathcal{I}$, $|\mathcal{P}| \leq 1 + 2(|\mathcal{I}| - 1) = 2|\mathcal{I}| - 1$. We show that \mathcal{P} is a generator, that is $G\mathcal{P}$ separates points in X .

Let $x \neq y \in X$ and assume they are not separated by $G\mathcal{I}$, thus $x F_{\mathcal{I}} y$. We show that $G\mathcal{J}$ separates x and y . Because A is a complete section, multiplying x by an appropriate group element, we may assume that $x \in A$. Since A respects \mathcal{I} , A is $F_{\mathcal{I}}$ -invariant and thus $y \in A$. Also, because γ is $F_{\mathcal{I}}$ -invariant, $\gamma_n(x) = \gamma_n(y)$, $\forall n \in \mathbb{N}$. Let $n \geq 1$ be such that $x \in U_n$ but $y \notin U_n$. Put $g = \gamma_n(x)$ ($= \gamma_n(y)$). Then $gx = \hat{\gamma}_n(x) \in \hat{\gamma}_n(A \cap U_n)$ while $gy = \hat{\gamma}_n(y) \notin \hat{\gamma}_n(A \cap U_n)$. Hence, $gx \in B$ and $gy \notin B$ because $\gamma_m(A) \cap \gamma_n(A) = \emptyset$ for all $m \neq n$ and $gy = \hat{\gamma}_n(y) \in \hat{\gamma}_n(A)$. Thus $G\mathcal{J}$ separates x and y . \square

Now Proposition 3.8 and Lemma 3.12 together imply the following.

Proposition 3.13. *Let X be a Borel G -space and $i \geq 1$. If X is i -compressible then there is a Borel 2^{i+1} -generator.*

Proof. By Proposition 3.8, there exists a Borel i -traveling complete section A . Let \mathcal{I} witness A being i -traveling and thus, by Lemma 3.12, there is a $2|\mathcal{I}| \leq 2 \cdot 2^i = 2^{i+1}$ -generator. \square

Example 3.14. For $2 \leq n \leq \infty$, let \mathbb{F}_n denote the free group on n generators and let X be the boundary of \mathbb{F}_n , i.e. the set of infinite reduced words. Clearly, the product

topology makes X a Polish space and \mathbb{F}_n acts continuously on X by left concatenation and cancellation. We show that X is 1-compressible and thus admits a Borel $2^2 = 4$ -generator by [Proposition 3.13](#). To this end, let a, b be two of the n generators of \mathbb{F}_n and let X_a be the set of all words in X that start with a . Then $X = (X_{a^{-1}} \cup X_{a^{-1}}^c) \sim_{\mathcal{I}} Y$, where $Y = bX_{a^{-1}} \cup aX_{a^{-1}}^c$ and $\mathcal{I}\langle X_{a^{-1}} \rangle$. Hence $X \sim_1 Y$. Since $X \setminus Y \supseteq X_{a^{-1}}$, $[X \setminus Y]_{\mathbb{F}_n} = X$ and thus X is 1-compressible.

Now we obtain a sufficient condition for the existence of an embedding into a finite Bernoulli shift.

Corollary 3.15. *Let X be a Borel G -space and $k \in \mathbb{N}$. If there exists a G -equivariant Borel map $f : X \rightarrow k^G$ such that $Y = f(X)$ does not admit a G -invariant Borel probability measure, then there is a G -equivariant Borel embedding of X into $(2k)^G$.*

Proof. Let $\mathcal{I} = \mathcal{I}_f$ and hence $f = f_{\mathcal{I}}$. By (3) \Rightarrow (5) of [Proposition 3.8](#) (or rather the proof of it), X admits a Borel \mathcal{I} -traveling complete section. Thus by [Lemma 3.12](#), X admits a $2|\mathcal{I}| = 2k$ -generator and hence, there is a G -equivariant Borel embedding of X into $(2k)^G$. \square

Lemma 3.16. *Let \mathcal{I} be a partition of X into n Borel sets. Then \mathcal{I} is generated by $k = \lceil \log_2(n) \rceil$ Borel sets.*

Proof. Since $2^k \geq n$, we can index \mathcal{I} by the set $2^{\mathbf{k}}$ of all k -tuples of $\{0, 1\}$, i.e. $\mathcal{I} = \{A_{\sigma}\}_{\sigma \in 2^{\mathbf{k}}}$. For all $i < k$, put

$$B_i = \bigcup_{\sigma \in 2^{\mathbf{k}} \wedge \sigma(i)=1} A_{\sigma}.$$

Now it is clear that for all $\sigma \in 2^{\mathbf{k}}$, $A_{\sigma} = \bigcap_{i < k} B_i^{\sigma(i)}$, where $B_i^{\sigma(i)}$ is equal to B_i if $\sigma(i) = 1$, and equal to B_i^c , otherwise. Thus $\mathcal{I} = \langle B_i : i < k \rangle$. \square

Proposition 3.17. *If X is compressible and there is a Borel n -generator, then X is $\lceil \log_2(n) \rceil$ -compressible.*

Proof. Let \mathcal{I} be an n -generator and hence, by [Lemma 3.16](#), \mathcal{I} is generated by $\lceil \log_2(n) \rceil$ Borel sets. Since $G\mathcal{I}$ separates points in X , each $F_{\mathcal{I}}$ -class is a singleton and hence $X \prec_{\mathcal{I}} X$ implies $X \prec_{\mathcal{I}} X$. \square

From [Propositions 3.13 and 3.17](#) we immediately get the following corollary, which justifies the use of i -compressibility in studying [Question 1.13](#).

Corollary 3.18. *Let X be a Borel G -space that is compressible (equivalently, does not admit an invariant Borel probability measure). X admits a finite generator if and only if X is i -compressible for some $i \geq 1$.*

4. Finitely additive invariant measures and i -compressibility

This section is mainly devoted to the following theorem, together its corollaries and proof.

Theorem 4.1. *Let X be a Borel G -space. If X is aperiodic, then there exists a function $m : \mathfrak{B}(X) \times X \rightarrow [0, 1]$ satisfying the following properties for all $A, B \in \mathfrak{B}(X)$:*

- (a) $m(A, \cdot)$ is Borel;
- (b) $m(X, x) = 1$, $\forall x \in X$;
- (c) If $A \subseteq B$, then $m(A, x) \leq m(B, x)$, $\forall x \in X$;
- (d) $m(A, x) = 0$ off $[A]_G$;
- (e) $m(A, x) > 0$ on $[A]_G$ modulo \mathfrak{C}_4 ;
- (f) $m(A, x) = m(gA, x)$, for all $g \in G$, $x \in X$ modulo \mathfrak{C}_3 ;
- (g) If $A \cap B = \emptyset$, then $m(A \cup B, x) = m(A, x) + m(B, x)$, $\forall x \in X$ modulo \mathfrak{C}_4 .

Remark 4.2. A version of this theorem is what lies at the heart of the proof of Nadkarni's theorem. The conclusions of our theorem are modulo \mathfrak{C}_4 , which is potentially a smaller σ -ideal than the σ -ideal of sets contained in compressible Borel sets used in Nadkarni's version. However, the price we pay for this is that part (g) asserts only finite additivity instead of countable additivity asserted by Nadkarni's version.

Before proceeding with the proof of this theorem, we draw a couple of corollaries. [Theorem 4.1](#) will only be used via [Corollary 4.4](#).

Definition 4.3. Let X be a Borel G -space. $\mathcal{B} \subseteq \mathfrak{B}(X)$ is called a Boolean G -algebra, if it is a Boolean algebra, i.e. is closed under finite unions and complements, and is closed under the G -action, i.e. $G\mathcal{B} = \mathcal{B}$.

Corollary 4.4. *Let X be a Borel G -space and let $\mathcal{B} \subseteq \mathfrak{B}(X)$ be a countable Boolean G -algebra. For any $A \in \mathcal{B}$ with $A \notin \mathfrak{C}_4$, there exists a G -invariant finitely additive probability measure μ on \mathcal{B} with $\mu(A) > 0$. Moreover, μ can be taken such that there is $x \in A$ such that $\forall B \in \mathcal{B}$ with $B \cap [x]_G = \emptyset$, $\mu(B) = 0$.*

Proof. Let $A \in \mathcal{B}$ be such that $A \notin \mathfrak{C}_4$. We may assume that $X = [A]_G$ by setting the (to be constructed) measure to be 0 outside $[A]_G$.

If X is not aperiodic, then by assigning equal point masses to the points of a finite orbit, we will have a probability measure on all of $\mathfrak{B}(X)$, so assume X is aperiodic.

Since \mathfrak{C}_4 is a σ -ideal and \mathcal{B} is countable, [Theorem 4.1](#) implies that there is a $P \in \mathfrak{C}_4$ such that (a)–(g) of the same theorem hold on $X \setminus P$ for all $A, B \in \mathcal{B}$. Since $A \notin \mathfrak{C}_4$, there exists $x_A \in A \setminus P$. Hence, letting $\mu(B) = m(B, x_A)$ for all $B \in \mathcal{B}$, conditions (b), (f) and (g) imply that μ is a G -invariant finitely additive probability measure on \mathcal{B} .

Moreover, since $x_A \in [A]_G \setminus P$, $\mu(A) = m(A, x_A) > 0$. Finally, the last assertion follows from condition (d). \square

Corollary 4.5. *Let X be a Borel G -space. For every Borel set $A \subseteq X$ with $A \notin \mathfrak{C}_4$, there exists a G -invariant finitely additive Borel probability measure μ (defined on all Borel sets) with $\mu(A) > 0$.*

Proof. The statement follows from [Corollary 4.4](#) and a standard application of the Compactness Theorem of propositional logic. Here are the details.

We fix the following set of propositional variables

$$\mathcal{P} = \{P_{A,r} : A \in \mathfrak{B}(X), r \in [0, 1]\},$$

with the following interpretation in mind:

$$P_{A,r} \Leftrightarrow \text{“the measure of } A \text{ is } \geq r\text{”}.$$

Define the theory T as the following set of sentences: for each $A, B \in \mathfrak{B}(X)$, $r, s \in [0, 1]$ and $g \in G$,

- (i) “ $P_{A,0}$ ” $\in T$;
- (ii) if $r > 0$, then “ $\neg P_{\emptyset,r}$ ” $\in T$;
- (iii) if $s \geq r$, then “ $P_{A,s} \rightarrow P_{A,r}$ ” $\in T$;
- (iv) if $A \cap B = \emptyset$, then “ $(P_{A,r} \wedge P_{B,s}) \rightarrow P_{A \cup B, r+s}$ ”, “ $(\neg P_{A,r} \wedge \neg P_{B,s}) \rightarrow \neg P_{A \cup B, r+s}$ ” $\in T$;
- (v) “ $P_{X,1}$ ” $\in T$;
- (vi) “ $P_{A,r} \rightarrow P_{gA,r}$ ” $\in T$.

If there is an assignment of the variables in \mathcal{P} satisfying T , then for each $A \in \mathfrak{B}(X)$, we can define

$$\mu(A) = \sup\{r \in [0, 1] : P_{A,r}\}.$$

Note that due to (i), μ is well defined for all $A \in \mathfrak{B}(X)$. In fact, it is straightforward to check that μ is a finitely additive G -invariant probability measure. Thus, we only need to show that T is satisfiable, for which it is enough to check that T is finitely satisfiable, by the Compactness Theorem of propositional logic (or by Tychonoff’s theorem).

Let $T_0 \subseteq T$ be finite and let \mathcal{P}_0 be the set of propositional variables that appear in the sentences in T_0 . Let \mathcal{B} denote the Boolean G -algebra generated by the sets that appear in the indices of the variables in \mathcal{P}_0 . By [Corollary 4.4](#), there is a finitely additive G -invariant probability measure μ defined on \mathcal{B} . Consider the following assignment of the variables in \mathcal{P}_0 : for all $P_{A,r} \in \mathcal{P}_0$,

$$P_{A,r} \text{ :} \Leftrightarrow \mu(A) \geq r.$$

It is straightforward to check that this assignment satisfies T_0 , and hence, T is finitely satisfiable. \square

We now start working towards the proof of [Theorem 4.1](#), following the general outline of Nadkarni's proof of [Theorem 1.26](#). The construction of $m(A, x)$ is somewhat similar to that of Haar measure. First, for sets A, B , we define a Borel function $[A/B] : X \rightarrow \mathbb{N} \cup \{-1, \infty\}$ that basically gives the number of copies of $B^{[x]_G}$ that fit in $A^{[x]_G}$ when moved by group elements (piecewise). Then we define a decreasing sequence of complete sections (called a fundamental sequence below), which serves as a gauge to measure the size of a given set.

Assume throughout that X is an aperiodic Borel G -space (although we only use the aperiodicity assumption in [Proposition 4.16](#) to assert that smooth sets are in \mathfrak{C}_1).

4.1. Measuring the size of a set relative to another

Lemma 4.6 (*Comparability*). $\forall A, B \in \mathfrak{B}(X)$, there is a partition $X = P \cup Q$ into G -invariant Borel sets such that for any A, B -sensitive finite Borel partition \mathcal{I} of X , $A^P \prec_{\mathcal{I}} B^P$ and $B^Q \preceq_{\mathcal{I}} A^Q$.

Proof. It is enough to prove the lemma assuming $X = [A]_G \cap [B]_G$ since we can always include $[B]_G \setminus [A]_G$ in P and $X \setminus [B]_G$ in Q .

Fix an enumeration $\{g_n\}_{n \in \mathbb{N}}$ for G . We recursively construct Borel sets A_n, B_n, A'_n, B'_n as follows. Set $A'_0 = A$ and $B'_0 = B$. Assuming A'_n, B'_n are defined, set $B_n = B'_n \cap g_n A'_n$, $A_n = g_n^{-1} B_n$, $A'_{n+1} = A'_n \setminus A_n$ and $B'_{n+1} = B'_n \setminus B_n$.

It is easy to see by induction on n that for any A, B -sensitive \mathcal{I} , A_n, B_n are $F_{\mathcal{I}}$ -invariant since so are A, B . Thus, setting $A^* = \bigcup_{n \in \mathbb{N}} A_n$ and $B^* = \bigcup_{n \in \mathbb{N}} B_n$, we get that $A^* \sim_{\mathcal{I}} B^*$ since $B_n = g_n A_n$.

Let $A' = A \setminus A^*$, $B' = B \setminus B^*$ and set $P = [B']_G$, $Q = X \setminus P$.

Claim. $[A']_G \cap [B']_G = \emptyset$.

Proof of Claim. Assume for contradiction that $\exists x \in A'$ and $n \in \mathbb{N}$ such that $g_n x \in B'$. It is clear that $A' = \bigcap_{k \in \mathbb{N}} A'_k$, $B' = \bigcap_{k \in \mathbb{N}} B'_k$; in particular, $x \in A'_n$ and $g_n x \in B'_n$. But then $g_n x \in B_n$ and $x \in A_n$, contradicting $x \in A'$. \square

Let \mathcal{I} be an A, B -sensitive partition. Then $A^P = (A^*)^P$ and hence $A^P \prec_{\mathcal{I}} B^P$ since $(A^*)^P \sim_{\mathcal{I}} (B^*)^P \subseteq B^P$ and $[B^P \setminus (B^*)^P]_G = [B']_G = P = [B^P]_G$. Similarly, $B^Q = (B^*)^Q$ and hence $B^Q \preceq_{\mathcal{I}} A^Q$ since $(B^*)^Q \sim_{\mathcal{I}} (A^*)^Q \subseteq A^Q$. \square

Definition 4.7 (*Divisibility*). Let $n \leq \infty$, $A, B, C \in \mathfrak{B}(X)$ and \mathcal{I} be a finite Borel partition of X .

- Write $A \sim_{\mathcal{I}} nB \oplus C$ if there are Borel sets $A_k \subseteq A$, $k < n$, such that $\{A_k\}_{k < n} \cup \{C\}$ is a partition of A , each A_k is $F_{\mathcal{I}}$ -invariant relative to A and $A_k \sim_{\mathcal{I}} B$.
- Write $nB \preceq_{\mathcal{I}} A$ if there is $C \subseteq A$ with $A \sim_{\mathcal{I}} nB \oplus C$, and write $nB \prec_{\mathcal{I}} A$ if moreover $[C]_G = [A]_G$.
- Write $A \preceq_{\mathcal{I}} nB$ if there is a Borel partition $\{A_k\}_{k < n}$ of A such that each A_k is $F_{\mathcal{I}}$ -invariant relative to A and $A_k \preceq_{\mathcal{I}} B$. If moreover, $A_k \prec_{\mathcal{I}} B$ for at least one $k < n$, we write $A \prec_{\mathcal{I}} nB$.

For $i \geq 1$, we use the above notation with \mathcal{I} replaced by i if there is an A , B -sensitive partition \mathcal{I} generated by i sets for which the above conditions hold.

Proposition 4.8 (Euclidean decomposition). *Let $A, B \in \mathfrak{B}(X)$ and put $R = [A]_G \cap [B]_G$. There exists a partition $\{P_n\}_{n \leq \infty}$ of R into G -invariant Borel sets such that for any A , B -sensitive finite Borel partition \mathcal{I} of X and $n \leq \infty$, $A^{P_n} \sim_{\mathcal{I}} nB^{P_n} \oplus C_n$ for some C_n such that $C_n \prec_{\mathcal{I}} B^{P_n}$, if $n < \infty$.*

Proof. We repeatedly apply Lemma 4.6. For $n < \infty$, recursively define R_n, P_n, A_n, C_n satisfying the following:

- R_n are invariant decreasing Borel sets such that $nB^{R_n} \preceq_{\mathcal{I}} A^{R_n}$ for any A , B -sensitive \mathcal{I} ;
- $P_n = R_n \setminus R_{n+1}$;
- $A_n \subseteq R_{n+1}$ are pairwise disjoint Borel sets such that for any A , B -sensitive \mathcal{I} , every A_n respects \mathcal{I} and $A_n \sim_{\mathcal{I}} B^{R_{n+1}}$;
- $C_n \subseteq P_n$ are Borel sets such that for any A , B -sensitive \mathcal{I} , every C_n respects \mathcal{I} and $C_n \prec_{\mathcal{I}} B^{P_n}$.

Set $R_0 = R$. Given $R_n, \{A_k\}_{k < n}$ satisfying the above properties, let $A' = A^{R_n} \setminus \bigcup_{k < n} A_k$. We apply Lemma 4.6 to A' and B^{R_n} , and get a partition $R_n = P_n \cup R_{n+1}$ such that $(A')^{P_n} \prec_{\mathcal{I}} B^{P_n}$ and $B^{R_{n+1}} \preceq_{\mathcal{I}} (A')^{R_{n+1}}$. Set $C_n = (A')^{P_n}$. Let $A_n \subseteq (A')^{R_{n+1}}$ be such that $B^{R_{n+1}} \sim_{\mathcal{I}} A_n$. It is straightforward to check (i)–(iv) are satisfied.

Now let $P_{\infty} = \bigcap_{n \in \mathbb{N}} R_n$ and $C_{\infty} = (A \setminus \bigcup_{n \in \mathbb{N}} A_n)^{P_{\infty}}$. It follows from (i)–(iv) that for all $n \leq \infty$, $\{A_k^{P_n}\}_{k < n} \cup \{C_n\}$ is a partition of A^{P_n} witnessing $A^{P_n} \sim_{\mathcal{I}} nB \oplus C_n$, and for all $n < \infty$, $C_n \prec_{\mathcal{I}} B^{P_n}$. \square

For $A, B \in \mathfrak{B}(X)$, let $\{P_n\}_{n \leq \infty}$ be as in the above proposition. Define

$$[A/B](x) = \begin{cases} n & \text{if } x \in P_n, n < \infty \\ \infty & \text{if } x \in P_{\infty} \text{ or } x \in [A]_G \setminus [B]_G \\ 0 & \text{if } x \in [B]_G \setminus [A]_G \\ -1 & \text{otherwise} \end{cases}.$$

Note that $[A/B] : X \rightarrow \mathbb{N} \cup \{-1, \infty\}$ is a Borel function by definition.

4.2. Properties of $[A/B]$

Lemma 4.9 (*Infinite divisibility \Rightarrow compressibility*). Let $A, B \in \mathfrak{B}(X)$ with $[A]_G = [B]_G$, and let \mathcal{I} be a finite Borel partition of X . If $\infty B \preceq_{\mathcal{I}} A$, then $A \prec_{\mathcal{I}} A$.

Proof. Let $C \subseteq A$ be such that $A \sim_{\mathcal{I}} \infty B \oplus C$ and let $\{A_k\}_{k < \infty}$ be as in Definition 4.7. $A_k \sim_{\mathcal{I}} B \sim_{\mathcal{I}} A_{k+1}$ and hence $A_k \sim_{\mathcal{I}} A_{k+1}$. Also trivially $C \sim_{\mathcal{I}} C$. Thus, letting $A' = \bigcup_{k < \infty} A_{k+1} \cup C$, we apply (b) of Observation 2.2 to A and A' , and get that $A \sim_{\mathcal{I}} A'$. Because $[A \setminus A']_G = [A_0]_G = [B]_G = [A]_G$, we have $A \prec_{\mathcal{I}} A$. \square

Lemma 4.10 (*Ambiguity \Rightarrow compressibility*). Let $A, B \in \mathfrak{B}(X)$ and \mathcal{I} be a finite Borel partition of X . If $nB \preceq_{\mathcal{I}} A \prec_{\mathcal{I}} nB$ for some $n \geq 1$, then $A \prec_{\mathcal{I}} A$.

Proof. Let $C \subseteq A$ be such that $A \sim_{\mathcal{I}} nB \oplus C$ and let $\{A_k\}_{k < n}$ be a partitions of $A \setminus C$ witnessing $A \sim_{\mathcal{I}} nB \oplus C$. Also let $\{A'_k\}_{k < n}$ be witnessing $A \prec_{\mathcal{I}} nB$ with $A'_0 \prec_{\mathcal{I}} B$. Since $A'_k \preceq_{\mathcal{I}} B \sim_{\mathcal{I}} A_k$, $A'_k \preceq_{\mathcal{I}} A_k$, for all $k < n$ and $A'_0 \prec_{\mathcal{I}} A_0$. Note that it follows from the hypothesis that $[A]_G = [B]_G$ and hence $[A_0]_G = [A]_G$ since $[A_0]_G = [B]_G$. Thus it follows from (b) of Observation 2.2 that $A = \bigcup_{k < n} A'_k \prec_{\mathcal{I}} \bigcup_{k < n} A_k \subseteq A$. \square

Proposition 4.11. Let $n \in \mathbb{N}$ and $A, A', B, P \in \mathfrak{B}(X)$, where P is invariant.

- (a) $[A/B] \in \mathbb{N}$ on $[B]_G$ modulo \mathfrak{C}_3 .
- (b) If $A \subseteq A'$, then $[A/B] \leq [A'/B]$.
- (c) If $[A/B] = n$ on P then $nB^P \preceq_{\mathcal{I}} A^P \prec_{\mathcal{I}} (n+1)B^P$, for any finite Borel partition \mathcal{I} that is A, B -sensitive. In particular, $nB^P \preceq_2 A^P \prec_2 (n+1)B^P$ by taking $\mathcal{I} = \langle A, B \rangle$.
- (d) For $n \geq 1$, if $A^P \prec_i nB^P$, then $[A/B] < n$ on P modulo \mathfrak{C}_{i+1} ;
- (e) If $A^P \subseteq [B]_G$ and $nB^P \preceq_i A^P$, then $[A/B] \geq n$ on P modulo \mathfrak{C}_{i+1} .

Proof. For (a), notice that Lemmas 2.18 and 4.9 imply that $P_{\infty} \in \mathfrak{C}_3$.

For part (b), it is enough to note the following: if $X = P \cup Q$ and $X = P' \cup Q'$ are the partitions provided by Lemma 4.6 when applied to A, B and A', B , respectively, then it follows from the construction in the proof of that lemma that $Q' \supseteq Q$.

Part (c) follows from the definition of $[A/B]$.

For (d), let \mathcal{I} be an A, B -sensitive partition of X generated by i Borel sets such that $A^P \prec_{\mathcal{I}} nB^P$, and put $Q = \{x \in P : [A/B](x) \geq n\}$. By (c), $nB^Q \preceq_{\mathcal{I}} A^Q$. Thus, by Lemma 4.10, $A^Q \prec_{\mathcal{I}} A^Q$ and hence, by Lemma 2.18, $[A^Q]_G = Q \in C_{i+1}$.

For (e), let \mathcal{I} be an A, B -sensitive partition of X generated by i Borel sets such that $nB^P \preceq_{\mathcal{I}} A^P$, and put $Q = \{x \in P : [A/B](x) < n\}$. By (c), $A^Q \prec_{\mathcal{I}} nB^Q$. Thus, by Lemma 4.10, $A^Q \prec_{\mathcal{I}} A^Q$ and hence, by Lemma 2.18, $[A^Q]_G = Q \in C_{i+1}$. \square

Lemma 4.12 (Almost cancellation). For any $A, B, C \in X$,

$$[A/B][B/C] \leq [A/C] < ([A/B] + 1)([B/C] + 1)$$

on $R := [B]_G \cap [C]_G$ modulo \mathfrak{C}_4 .

Proof. Let $\mathcal{I} = \langle A, B, C \rangle$.

$[A/B][B/C] \leq [A/C]$: Fix integers $i, j > 0$ and let $P = \{x \in X : [A/B](x) = i \wedge [B/C](x) = j\}$. Since $i, j > 0$, $P \subseteq [A]_G \cap [B]_G \cap [C]_G$ and we work in P . By (c) of Proposition 4.11, $iB \preceq_{\mathcal{I}} A$ and $jC \preceq_{\mathcal{I}} B$. Thus it follows that $ijC \preceq_{\mathcal{I}} A$ and hence $[A/C] \geq ij$ modulo \mathfrak{C}_4 by (e) of Proposition 4.11.

$[A/C] < ([A/B] + 1)([B/C] + 1)$: By (a) of Proposition 4.11, $[A/C], [A/B], [B/C] \in \mathbb{N}$ on R modulo \mathfrak{C}_3 . Fix $i, j \in \mathbb{N}$ and let $Q = \{x \in R : [A/B](x) = i \wedge [B/C](x) = j\}$. We work in Q . By (c) of 4.11, $A \prec_{\mathcal{I}} (i+1)B$ and $B \prec_{\mathcal{I}} (j+1)C$. Thus $A \prec_{\mathcal{I}} (i+1)(j+1)C$ and hence $[A/C] < (i+1)(j+1)$ modulo \mathfrak{C}_4 by (d) of Proposition 4.11. \square

Lemma 4.13 (Invariance). For $A, F \in \mathfrak{B}(X)$, $\forall g \in G$, $[A/F] = [gA/F]$, modulo \mathfrak{C}_3 .

Proof. We may assume that $X = [A]_G \cap [F]_G$. Fix $g \in G$, $n \in \mathbb{N}$, and put $Q = \{x \in X : [gA/F](x) = n\}$. We work in Q . Let $\mathcal{I} = \langle A, F \rangle$ and hence A, gA, F respect \mathcal{I} . By (c) of Proposition 4.11, $nF \preceq_{\mathcal{I}} gA$. But clearly $gA \sim_{\mathcal{I}} A$ and hence $nF \preceq_{\mathcal{I}} A$. Thus, by (e) of Proposition 4.11, $[A/F] \geq n = [gA/F]$, modulo \mathfrak{C}_3 . By symmetry, $[gA/F] \geq [A/F]$ (modulo \mathfrak{C}_3) and the lemma follows. \square

Lemma 4.14 (Almost additivity). For any $A, B, F \in X$ with $A \cap B = \emptyset$, $[A/F] + [B/F] \leq [A \cup B/F] \leq [A/F] + [B/F] + 1$ modulo \mathfrak{C}_4 .

Proof. Let $\mathcal{I} = \langle A, B, F \rangle$.

$[A/F] + [B/F] \leq [A \cup B/F]$: Fix $i, j \in \mathbb{N}$ not both 0, say $i > 0$, and let $S = \{x \in X : [A/F](x) = i \wedge [B/F](x) = j\}$. Since $i > 0$, $S \subseteq [A]_G \cap [F]_G$ and we work in S . By (c) of Proposition 4.11, $iF^S \preceq_{\mathcal{I}} A^S$ and $jF^S \preceq_{\mathcal{I}} B^S$. Hence $(i+j)F^S \preceq_{\mathcal{I}} (A \cup B)^S$ and thus, by (e) of Proposition 4.11, $[A \cup B/F] \geq i+j$, modulo \mathfrak{C}_4 .

$[A \cup B/F] \leq [A/F] + [B/F] + 1$: Outside $[F]_G$, the inequality clearly holds. Fix $i, j \in \mathbb{N}$ and let $M = \{x \in [F]_G : [A/F](x) = i \wedge [B/F](x) = j\}$. We work in M . By (c) of Proposition 4.11, $A \prec_{\mathcal{I}} (i+1)F$ and $B \prec_{\mathcal{I}} (j+1)F$. Thus it is clear that $A \cup B \prec_{\mathcal{I}} (i+j+2)F$ and hence $[A \cup B/F] < i+j+2$, modulo \mathfrak{C}_4 , by (d) of Proposition 4.11. \square

4.3. Fundamental sequence

Definition 4.15. A sequence $\{F_n\}_{n \in \mathbb{N}}$ of decreasing Borel complete sections with $F_0 = X$ and $[F_n/F_{n+1}] \geq 2$ modulo \mathfrak{C}_3 is called fundamental.

Proposition 4.16. There exists a fundamental sequence.

Proof. Take $F_0 = X$. Given any complete Borel section F , its intersection with every orbit is infinite modulo a smooth set (if the intersection of an orbit with a set is finite, then we can choose an element from each such nonempty intersection in a Borel way and get a Borel transversal). Thus, by [Corollary 3.11](#), F is aperiodic modulo \mathfrak{C}_1 . Now use [Lemma 8.1](#) (the proof of this does not use any results from the current paper, so there is no loop) to write $F = A \cup B$, $A \cap B = \emptyset$, where A, B are also complete sections. Let now P, Q be as in [Lemma 4.6](#) for A, B , and hence $A^P \prec_2 B^P$, $B^Q \preceq_2 A^Q$ because we can take $\mathcal{I} = \langle A, B \rangle$. Let $A' = A^P \cup B^Q$, $B' = B^P \cup A^Q$. Then $F = A' \cup B'$, $A' \cap B' = \emptyset$, $A' \preceq B'$ and A' is also a complete Borel section. By (e) of [Proposition 4.11](#), $[F/A'] \geq 2$ modulo \mathfrak{C}_3 . Iterate this process to inductively define F_n . \square

4.4. Definition and properties of $m(A, x)$

Fix a fundamental sequence $\{F_n\}_{n \in \mathbb{N}}$ and for any $A \in \mathfrak{B}(X)$, $x \in X$, define

$$m(A, x) = \lim_{n \rightarrow \infty} \frac{[A/F_n](x)}{[X/F_n](x)}, \quad (4.17)$$

if the limit exists, and 0 otherwise. In the above fraction we define $\frac{\infty}{\infty} = 1$. We will prove in [Proposition 4.19](#) that this limit exists modulo \mathfrak{C}_4 . But first we need a lemma.

Lemma 4.18. *For any $A \in \mathfrak{B}(A)$,*

$$\lim_{n \rightarrow \infty} [A/F_n] = \begin{cases} \infty & \text{on } [A]_G \\ 0 & \text{on } X \setminus [A]_G \end{cases}, \text{ modulo } \mathfrak{C}_4.$$

Proof. The part about $X \setminus [A]_E$ is clear, so work in $[A]_E$, i.e. assume $X = [A]_G$. By (a) of [Proposition 4.11](#) and [Lemma 4.12](#), we have

$$\infty > [F_1/A] \geq [F_1/F_n][F_n/A] \geq 2^{n-1}[F_n/A], \text{ modulo } \mathfrak{C}_4,$$

which holds for all n at once since \mathfrak{C}_4 is a σ -ideal. Thus $[F_n/A] \rightarrow 0$ modulo \mathfrak{C}_4 and hence, as $[F_n/A] \in \mathbb{N}$, $[F_n/A]$ is eventually 0, modulo \mathfrak{C}_4 . So if

$$B_k := \{x \in [A]_G : [F/A](x) = 0\},$$

then $B_k \nearrow X$, modulo \mathfrak{C}_4 . Now it follows from [Lemma 4.6](#) that $[A/F_k] > 0$ on B_k modulo \mathfrak{C}_4 . But

$$[A/F_{k+n}] \geq [A/F_k][F_k/F_{k+n}] \geq 2^n[A/F_k], \text{ modulo } \mathfrak{C}_4,$$

so for every k , $[A/F_n] \rightarrow \infty$ on B_k modulo \mathfrak{C}_4 . Since $B_k \nearrow X$ modulo \mathfrak{C}_4 , we have $[A/F_n] \rightarrow \infty$ on X , modulo \mathfrak{C}_4 . \square

Proposition 4.19. *For any Borel set $A \subseteq X$, the limit in (4.17) exists and is positive on $[A]_G$, modulo \mathfrak{C}_4 .*

Proof.

Claim. *Suppose $B, C \in \mathfrak{B}(X)$, $i \in \mathbb{N}$ and $D_i = \{x \in X : [C/F_i](x) > 0\}$. Then*

$$\overline{\lim} \frac{[B/F_n]}{[C/F_n]} \leq \frac{[B/F_i] + 1}{[C/F_i]}$$

on D_i , modulo \mathfrak{C}_4 .

Proof of Claim. Working in D_i and using Lemma 4.12, $\forall j$ we have (modulo \mathfrak{C}_4)

$$\begin{aligned} [B/F_{i+j}] &\leq ([B/F_i] + 1)([F_i/F_{i+j}] + 1) \\ [C/F_{i+j}] &\geq [C/F_i][F_i/F_{i+j}] > 0, \end{aligned}$$

so

$$\begin{aligned} \frac{[B/F_{i+j}]}{[C/F_{i+j}]} &\leq \frac{[B/F_i] + 1}{[C/F_i]} \cdot \frac{[F_i/F_{i+j}] + 1}{[F_i/F_{i+j}]} \\ &\leq \frac{[B/F_i] + 1}{[C/F_i]} \cdot \left(1 + \frac{1}{2^j}\right), \end{aligned}$$

from which the claim follows. \square

Applying the claim to $B = A$ and $C = X$ (hence $D_i = X$), we get that for all $i \in \mathbb{N}$

$$\overline{\lim}_{n \rightarrow \infty} \frac{[A/F_n](x)}{[X/F_n](x)} \leq \frac{[A/F_i](x) + 1}{[X/F_i](x)} \quad (\text{modulo } \mathfrak{C}_4).$$

Thus

$$\overline{\lim}_{n \rightarrow \infty} \frac{[A/F_n]}{[X/F_n]} \leq \lim_{i \rightarrow \infty} \frac{[A/F_i] + 1}{[X/F_i]} = \lim_{i \rightarrow \infty} \frac{[A/F_i]}{[X/F_i]}$$

since $\lim_{i \rightarrow \infty} \frac{1}{[X/F_i]} = 0$.

To see that $m(A, x)$ is positive on $[A]_E$ modulo \mathfrak{C}_4 we argue as follows. We work in $[A]_G$. Applying the above claim to $B = X$ and $C = A$, we get

$$\frac{1}{m(A, x)} = \lim_{n \rightarrow \infty} \frac{[X/F_n]}{[A/F_n]} \leq \frac{[X/F_i] + 1}{[A/F_i]} < \infty \text{ on } D_i \text{ (modulo } \mathfrak{C}_4).$$

Thus $m(A, x) > 0$ on $\bigcup_{i \in \mathbb{N}} D_i$, modulo \mathfrak{C}_4 . But $D_i \nearrow [A]_G$ because $[A/F_i] \rightarrow \infty$ as $i \rightarrow \infty$, and hence $m(A, x) > 0$ on $[A]_G$ modulo \mathfrak{C}_4 . \square

4.5. Proof of Theorem 4.1

Fix $A, B \in \mathfrak{B}(X)$. The fact that $m(A, x) \in [0, 1]$ and parts (b) and (d) follow directly from the definition of $m(A, x)$. Part (a) follows from the fact that $[A/F_n]$ is Borel for all $n \in \mathbb{N}$. (c) follows from (b) of Proposition 4.11, and (e) and (f) are asserted by Proposition 4.19 and Lemma 4.13, respectively.

To show (g), we argue as follows. By Lemma 4.14, $[A/F_n] + [B/F_n] \leq [A \cup B/F_n] \leq [A/F_n] + [B/F_n] + 1$, modulo \mathfrak{C}_4 , and thus

$$\frac{[A/F_n]}{[X/F_n]} + \frac{[B/F_n]}{[X/F_n]} \leq \frac{[A \cup B/F_n]}{[X/F_n]} \leq \frac{[A/F_n]}{[X/F_n]} + \frac{[B/F_n]}{[X/F_n]} + \frac{1}{[X/F_n]},$$

for all n at once, modulo \mathfrak{C}_4 (using the fact that \mathfrak{C}_4 is a σ -ideal). Since $[X/F_n] \geq 2^n$, passing to the limit in the inequalities above, we get $m(A, x) + m(B, x) \leq m(A \cup B, x) \leq m(A, x) + m(B, x)$. QED (Theorem 4.1)

5. Finite generators in the case of σ -compact spaces

In this section we prove that the answer to Question 1.13 is positive in case X has a σ -compact realization. To do this, we first prove Proposition 5.3, which shows how to construct a countably additive invariant probability measure on X using a finitely additive one. We then use Corollary 4.4 to conclude the result.

For the next two statements, let X be a second countable Hausdorff topological space equipped with a continuous action of G .

Lemma 5.1. *Let $\mathcal{U} \subseteq \text{Pow}(X)$ be a countable basis for X closed under the G -action and finite unions/intersections. Let ρ be a G -invariant finitely additive probability measure on the G -algebra generated by \mathcal{U} . For every $A \subseteq X$, define*

$$\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \rho(U_n) : U_n \in \mathcal{U} \wedge A \subseteq \bigcup_{n \in \mathbb{N}} U_n \right\}.$$

Then:

- (a) μ^* is a G -invariant outer measure.
- (b) If $K \subseteq X$ is compact, then K is metrizable and μ^* is a metric outer measure on K (with respect to any compatible metric).

Proof. It is a standard fact from measure theory that μ^* is an outer measure. That μ^* is G -invariant follows immediately from G -invariance of ρ and the fact that \mathcal{U} is closed under the action of G .

For (b), first note that by Urysohn metrization theorem, K is metrizable, and fix a metric on K . If $E, F \subseteq K$ are a positive distance apart, then so are \bar{E} and \bar{F} . Hence

there exist disjoint open sets U, V such that $\bar{E} \subseteq U, \bar{F} \subseteq V$. Because \bar{E} and \bar{F} are compact, U, V can be taken to be finite unions of sets in \mathcal{U} and therefore $U, V \in \mathcal{U}$.

Now fix $\epsilon > 0$ and let $W_n \in \mathcal{U}$, be such that $E \cup F \subseteq \bigcup_n W_n$ and

$$\sum_n \rho(W_n) \leq \mu^*(E \cup F) + \epsilon \leq \mu^*(E) + \mu^*(F) + \epsilon. \quad (5.2)$$

Note that $\{W_n \cap U\}_{n \in \mathbb{N}}$ covers E , $\{W_n \cap V\}_{n \in \mathbb{N}}$ covers F and $W_n \cap U, W_n \cap V \in \mathcal{U}$. Also, by finite additivity of ρ ,

$$\rho(W_n \cap U) + \rho(W_n \cap V) = \rho(W_n \cap (U \cup V)) \leq \rho(W_n).$$

Thus

$$\mu^*(E) + \mu^*(F) \leq \sum_n \rho(W_n \cap U) + \sum_n \rho(W_n \cap V) \leq \sum_n \rho(W_n),$$

which, together with (5.2), implies that $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ since ϵ is arbitrary. \square

Proposition 5.3. *Suppose there exist a countable basis $\mathcal{U} \subseteq \text{Pow}(X)$ for X and a compact set $K \subseteq X$ such that the G -algebra generated by $\mathcal{U} \cup \{K\}$ admits a finitely additive G -invariant probability measure ρ with $\rho(K) > 0$. Then there exists a countably additive G -invariant Borel probability measure on X .*

Proof. Let K, \mathcal{U} and ρ be as in the hypothesis. We may assume that \mathcal{U} is closed under the G -action and finite unions/intersections. Let μ^* be the outer measure provided by Lemma 5.1 applied to \mathcal{U}, ρ . Thus μ^* is a metric outer measure on K and hence all Borel subsets of K are μ^* -measurable (see [19, 13.2]). This implies that all Borel subsets of $Y = [K]_G = \bigcup_{g \in G} gK$ are μ^* -measurable because μ^* is G -invariant. By Carathéodory's theorem, the restriction of μ^* to the Borel subsets of Y is a countably additive Borel measure on Y , and we extend it to a Borel measure μ on X by setting $\mu(Y^c) = 0$. Note that μ is G -invariant and $\mu(Y) \leq 1$.

It remains to show that μ is nontrivial, which we do by showing that $\mu(K) \geq \rho(K)$ and hence $\mu(K) > 0$. To this end, let $\{U_n\}_{n \in \mathbb{N}} \subseteq \mathcal{U}$ cover K . Since K is compact, there is a finite subcover $\{U_n\}_{n < N}$. Thus $U := \bigcup_{n < N} U_n \in \mathcal{U}$ and $K \subseteq U$. By finite additivity of ρ , we have

$$\sum_{n \in \mathbb{N}} \rho(U_n) \geq \sum_{n < N} \rho(U_n) \geq \rho(U) \geq \rho(K),$$

and hence, it follows from the definition of μ^* that $\mu^*(K) \geq \rho(K)$. Thus $\mu(K) = \mu^*(K) > 0$. \square

Corollary 5.4. *Let X be a second countable Hausdorff topological G -space whose Borel structure is standard. For every compact set $K \subseteq X$ not in \mathfrak{C}_4 , there is a G -invariant countably additive Borel probability measure μ on X with $\mu(K) > 0$.*

Proof. Fix any countable basis \mathcal{U} for X and let \mathcal{B} be the Boolean G -algebra generated by $\mathcal{U} \cup \{K\}$. By Corollary 4.4, there exists a G -invariant finitely additive probability measure ρ on \mathcal{B} such that $\rho(K) > 0$. Now apply Proposition 5.3. \square

As a corollary, we derive the analogue of Nadkarni's theorem for \mathfrak{C}_4 in case of σ -compact spaces.

Corollary 5.5. *Let X be a Borel G -space that admits a σ -compact realization. $X \notin \mathfrak{C}_4$ if and only if there exists a G -invariant countably additive Borel probability measure on X .*

Proof. \Leftarrow : If $X \in \mathfrak{C}_4$, then it is compressible in the usual sense and hence does not admit a G -invariant Borel probability measure.

\Rightarrow : Suppose that X is a σ -compact topological G -space and $X \notin \mathfrak{C}_4$. Then, since X is σ -compact and \mathfrak{C}_4 is a σ -ideal, there is a compact set K not in \mathfrak{C}_4 . Now apply Corollary 5.4. \square

Remark 5.6. For a Borel G -space X , let \mathcal{K} denote the collection of all subsets of invariant Borel sets that admit a σ -compact realization (when viewed as Borel G -spaces). Also, let \mathfrak{C} denote the collection of all subsets of invariant compressible Borel sets. It is clear that \mathcal{K} and \mathfrak{C} are σ -ideals, and what Corollary 5.5 implies is that $\mathfrak{C} \cap \mathcal{K} \subseteq \mathfrak{C}_4$.

Theorem 5.7. *Let X be a Borel G -space that admits a σ -compact realization. If there is no G -invariant Borel probability measure on X , then X admits a Borel 32-generator.*

Proof. By Corollary 5.5, $X \in \mathfrak{C}_4$ and hence, X is 4-compressible. Thus, by Proposition 3.13, X admits a Borel 2^5 -generator. \square

Example 5.8. Let LO denote the set of all linear orderings of \mathbb{N} with the ordering relation symbol $<$; this can be modeled as a closed subset of $2^{\mathbb{N}^2}$, so it is a compact Polish space. We think of each $x \in LO$ as a structure $(\mathbb{N}, <_x)$, where $<_x$ is the linear ordering of \mathbb{N} according to x and we write, for example, $7 <_x 5$ to mean that 7 is less than 5 according to x .

Letting G be the group of finite permutations of elements of \mathbb{N} , we see that G is countable and acts continuously on LO in the natural way:

$$n <_{gx} m \Leftrightarrow g^{-1}(n) <_x g^{-1}(m),$$

for $n, m \in \mathbb{N}$, $g \in G$, $x \in LO$; this is referred to as *the logic action*.

Put $Y = LO \setminus DLO$, where DLO denotes the set of all dense linear orderings without endpoints (copies of \mathbb{Q}). It is straightforward to see that DLO is a G_δ subset of LO , hence Y is F_σ and therefore σ -compact since LO is compact. Also clearly Y is G -invariant.

Let μ be the unique measure on LO defined by $\mu(V_{(F, <_F)}) = \frac{1}{n!}$, where $(F, <_F)$ is a finite linearly ordered subset of \mathbb{N} of cardinality n and $V_{(F, <_F)}$ is the set of all linear orderings of \mathbb{N} extending the order $<_F$ on F . It is not hard to check (shown in [9]) that μ is the unique invariant measure for the action of G on LO and $\mu(Y) = 0$. Thus, there is no G -invariant Borel probability measure on Y and hence, by Theorem 5.7, Y admits a 32-generator. However, as pointed out by Todor Tsankov and the referee, LO (and hence also Y) already has an obvious 2-generator: namely, the partition generated by the set $\{x \in LO : 0 <_x 1\}$.

Nevertheless, we can modify this example to make the application of Theorem 5.7 more fruitful by considering multiple relations instead of just one. For example, for $n \geq 1$, let LO_n be the set of all n -tuples of linear orderings of \mathbb{N} with the ordering relation symbols $<^0, <^1, \dots, <^{n-1}$, and, as before, consider the natural (logic) action of G on LO_n ; this can be modeled as the coordinatewise (diagonal) action of G on $LO_n := LO^n$, i.e. $g(x_0, x_1, \dots, x_{n-1}) = (gx_0, gx_1, \dots, gx_{n-1})$. The Polish G -space LO_n has an obvious 2^n -generator: namely, the partition generated by the sets $\{x \in LO_n : 0 <^i_x 1\}$, $i < n$. However, letting $X = LO^n \setminus DLO^n$, we again see that it is G -invariant and σ -compact. Moreover, X does not admit an invariant probability measure because otherwise, one of the sets of the form $LO^i \times Y \times LO^{n-i-1}$ would have positive measure, so the pushforward measure under the projection onto the i th coordinate would be a nontrivial finite invariant measure on Y , but we argued above that such a measure does not exist. Thus, by Theorem 5.7, X has a 32-generator, which gives us something new when $n > 5$.

6. Finitely traveling sets

Throughout this section, let X be a Borel G -space.

Definition 6.1. Let $A, B \in \mathfrak{B}(X)$ be equidecomposable, i.e. there are $N \leq \infty$, $\{g_n\}_{n < N} \subseteq G$ and Borel partitions $\{A_n\}_{n < N}$ and $\{B_n\}_{n < N}$ of A and B , respectively, such that $g_n A_n = B_n$ for all $n < N$. A, B are said to be

- locally finitely equidecomposable (denote by $A \sim_{\text{lfm}} B$), if $\{A_n\}_{n < N}$, $\{B_n\}_{n < N}$, $\{g_n\}_{n < N}$ can be taken so that for every $x \in A$, $A_n \cap [x]_G = \emptyset$ for all but finitely many $n < N$;
- finitely equidecomposable (denote by $A \sim_{\text{fm}} B$), if N can be taken to be finite.

The notation \prec_{fm} , \prec_{lfm} and the notions of finite and locally finite compressibility are defined analogous to Definitions 1.23 and 1.25.

Definition 6.2. A Borel set $A \subseteq X$ is called (locally) finitely traveling if there exists pairwise disjoint Borel sets $\{A_n\}_{n \in \mathbb{N}}$ such that $A_0 = A$ and $A \sim_{\text{fin}} A_n$ ($A \sim_{\text{lfin}} A_n$), $\forall n \in \mathbb{N}$.

Proposition 6.3. *If X is (locally) finitely compressible then X admits a (locally) finitely traveling Borel complete section.*

Proof. We prove for finitely compressible X , but note that everything below is also locally valid (i.e. restricted to every orbit) for a locally compressible X .

Run the proof of the first part of [Lemma 3.4](#) noting that a witnessing map $\gamma : X \rightarrow G$ of finite compressibility of X has finite image and hence the image of each δ_n (in the notation of the proof) is finite, which implies that the obtained traveling set A is actually finitely traveling. \square

Proposition 6.4. *If X admits a locally finitely traveling Borel complete section, then $X \in \mathfrak{C}_4$.*

Proof. Let A be a locally finitely traveling Borel complete section and let $\{A_n\}_{n \in \mathbb{N}}$ be as in [Definition 6.2](#). Let $\mathcal{I}_n = \{C_k^n\}_{k \in \mathbb{N}}$, $\mathcal{J}_n = \{D_k^n\}_{k \in \mathbb{N}}$ be Borel partitions of A and A_n , respectively, that together with $\{g_k^n\}_{k \in \mathbb{N}} \subseteq G$ witness $A \sim_{\text{lfin}} A_n$ (as in [Definition 6.1](#)). Let \mathcal{B} denote the Boolean G -algebra generated by $\{X\} \cup \bigcup_{n \in \mathbb{N}} (\mathcal{I}_n \cup \mathcal{J}_n \cup \{A_n\})$.

Now assume for contradiction that $X \notin \mathfrak{C}_4$ and hence, $A \notin \mathfrak{C}_4$. Thus, applying [Corollary 4.4](#) to A and \mathcal{B} , we get a G -invariant finitely additive probability measure μ on \mathcal{B} with $\mu(A) > 0$. Moreover, there is $x \in A$ such that $\forall B \in \mathcal{B}$ with $B \cap [x]_G = \emptyset$, $\mu(B) = 0$.

Claim. $\mu(A_n) = \mu(A)$, for all $n \in \mathbb{N}$.

Proof of Claim. For each n , let $\{C_{k_i}^n\}_{i < K_n}$ be the list of those C_k^n such that $C_k^n \cap [x]_G \neq \emptyset$ ($K_n < \infty$ by the definition of locally finitely traveling). Set $B = A \setminus (\bigcup_{i < K_n} C_{k_i}^n)$ and note that by finite additivity of μ ,

$$\mu(A) = \mu(B) + \sum_{i < K_n} \mu(C_{k_i}^n).$$

Similarly, set $B' = A_n \setminus (\bigcup_{i < K_n} D_{k_i}^n)$ and hence

$$\mu(A_n) = \mu(B') + \sum_{i < K_n} \mu(D_{k_i}^n).$$

But $B \cap [x]_G = \emptyset$ and $B' \cap [x]_G = \emptyset$, and thus $\mu(B) = \mu(B') = 0$. Also, since $g_{k_i}^n C_{k_i}^n = D_{k_i}^n$ and μ is G -invariant, $\mu(C_{k_i}^n) = \mu(D_{k_i}^n)$. Therefore

$$\mu(A) = \sum_{i < K_n} \mu(C_{k_i}^n) = \sum_{i < K_n} \mu(D_{k_i}^n) = \mu(A_n). \quad \square$$

This claim contradicts μ being a probability measure since for large enough N , $\mu(\bigcup_{n < N} A_n) = N\mu(A) > 1$, contradicting $\mu(X) = 1$. \square

This, together with [Proposition 3.13](#), implies the following.

Corollary 6.5. *Let X be a Borel G -space. If X admits a locally finitely traveling Borel complete section, then there is a Borel 32-generator.*

7. Locally weakly wandering sets and other special cases

Assume throughout the section that X is a Borel G -space.

Definition 7.1. We say that $A \subseteq X$ is

- weakly wandering with respect to $H \subseteq G$ if $(hA) \cap (h'A) = \emptyset$, for all distinct $h, h' \in H$;
- weakly wandering, if it is weakly wandering with respect to an infinite subset $H \subseteq G$ (by shifting H , we can always assume $1_G \in H$);
- locally weakly wandering if for every $x \in X$, $A^{[x]_G}$ is weakly wandering.

7.1. Weakly wandering sets and finite generators

The following is a prototypical/toy example of a construction of a finite generator, and it has served as a driving idea for a number of constructions in the current paper.

Proposition 7.2. *If a G -space X admits a Borel weakly wandering complete section, then it admits a 3-generator.*

Proof. Let W be a Borel complete section that is weakly wandering with respect to an infinite sequence $(g_n)_{n \in \mathbb{N}} \subseteq G$, where $g_0 = 1_G$. Let $(U_n)_{n \geq 1}$ be a sequence of Borel sets that generate the Borel σ -algebra of X (e.g. a countable basis of open sets in a compatible Polish topology) and put

$$V = \bigcup_{n \geq 1} (g_n W \cap g_n U_n).$$

Thus, $W \cap V = \emptyset$ and we claim that the partition $\{W, V, (W \cup V)^c\}$ is a generator. Indeed, fix distinct points $x, y \in X$. Because W intersects the orbit of x , there is $g \in G$ with $gx \in W$, so by replacing x, y with gx, gy , we may assume that x was in W to begin with. Now if y is not in W , then W separates x and y , and we are done; so suppose y is also in W . Then, since $\{U_n\}_{n \geq 1}$ separates points, there must be $n \geq 1$ such that $x \in U_n$ but $y \notin U_n$. But then $g_n x \in g_n W \cap g_n U_n$, and hence $g_n x \in V$, whereas $g_n y \notin g_n U_n$, so $g_n y \notin V$ because $g_n y \in g_n W$ and $g_n W \cap V \subseteq g_n U_n$. Thus $g_n^{-1} V$ separates x and y . \square

Example 7.3.

- (a) Let $X = \mathbb{R}$ and let \mathbb{Z} act on \mathbb{R} by translation. Then any interval is weakly wandering and any interval of length greater than 1 is a complete section. Thus, by the above proposition, this Polish \mathbb{Z} -space admits a 3-generator.
- (b) Let $X = \mathcal{N}$ (the Baire space) and \tilde{E}_0 be the equivalence relation of eventual agreement of sequences of natural numbers. We find a countable group G of homeomorphisms of X such that $E_G = \tilde{E}_0$. For each $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $s \perp t$ (i.e. $s \not\sqsubseteq t$ and $t \not\sqsubseteq s$) or $s = t$, let $\phi_{s,t} : X \rightarrow X$ be defined as follows:

$$\phi_{s,t}(x) = \begin{cases} t \smallfrown y & \text{if } x = s \smallfrown y \\ s \smallfrown y & \text{if } x = t \smallfrown y \\ x & \text{otherwise} \end{cases},$$

and let G be the group generated by $\{\phi_{s,t} : s, t \in \mathbb{N}^{<\mathbb{N}}, |s| = |t|\}$. It is clear that each $\phi_{s,t}$ is a homeomorphism of X and $E_G = \tilde{E}_0$. Now for $n \in \mathbb{N}$, let $X_n = \{x \in X : x(0) = n\}$ and let $g_n = \phi_{0,n}$. Then the X_n are pairwise disjoint and $g_n X_0 = X_n$. Hence X_0 is a weakly wandering complete section and thus X admits a Borel 3-generator by [Proposition 7.2](#).

- (c) Let $X = 2^{\mathbb{N}}$ (the Cantor space) and E_t be the tail equivalence relation on X , that is:

$$xE_t y \Leftrightarrow (\exists n, m \in \mathbb{N})(\forall k \in \mathbb{N})x(n+k) = y(m+k).$$

Let G be the group generated by $\{\phi_{s,t} : s, t \in 2^{<\mathbb{N}}, s \perp t\}$, where the $\phi_{s,t}$ are defined as above. To see that $E_G = E_t$ fix $x, y \in X$ with $xE_t y$. Thus, there are nonempty $s, t \in 2^{<\mathbb{N}}$ and $z \in X$ such that $x = s \smallfrown z$ and $y = t \smallfrown z$. If $s \perp t$, then $y = \phi_{s,t}(x)$. Otherwise, assume, say, $s \sqsubseteq t$ and let $s' \in 2^{<\mathbb{N}}$ be such that $s \perp s'$ (exists since $s \neq \emptyset$). Then $s' \perp t$ and $y = \phi_{s',t} \circ \phi_{s,s'}(x)$.

Now for $n \in \mathbb{N}$, let $s_n = \underbrace{11\dots 1}_n 0$ and $X_n = \{x \in X : x = s_n \smallfrown y, \text{ for some } y \in X\}$.

Note that the s_n are pairwise incompatible and hence the X_n are pairwise disjoint. Letting $g_n = \phi_{s_0, s_n}$, we see that $g_n X_0 = X_n$. Thus X_0 is a weakly wandering complete section and hence X admits a Borel 3-generator by [Proposition 7.2](#).

7.2. Localization

Let $F(G^{\mathbb{N}})$ denote the Effros space of $G^{\mathbb{N}}$, i.e. the standard Borel space of closed subsets of $G^{\mathbb{N}}$, see [\[13, 12.C\]](#). Below we use a Borel selector for $F(G^{\mathbb{N}})$, i.e. a Borel function $s : F(G^{\mathbb{N}}) \rightarrow G^{\mathbb{N}}$ with $s(F) \in F$ for each nonempty $F \in F(G^{\mathbb{N}})$. It is a basic fact of descriptive set theory that such functions exist; see, for example, [\[13, Theorem 12.13\]](#).

For $A \subseteq X$ and $x \in A$, put

$$\Delta_A(x) = \{(g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}} : g_0 = 1_G \wedge \forall n \neq m (g_n A^{[x]_G} \cap g_m A^{[x]_G} = \emptyset)\}.$$

Proposition 7.4. *Let $A \in \mathfrak{B}(X)$.*

- (a) $\forall x \in X$, $\Delta_A(x)$ is a closed set in $G^{\mathbb{N}}$.
- (b) $\Delta_A : A \rightarrow F(G^{\mathbb{N}})$ is $\sigma(\Sigma_1^1)$ -measurable and hence universally measurable.
- (c) Δ_A is F_A -invariant, i.e. $\forall x, y \in A$, if xF_Ay then $\Delta_A(x) = \Delta_A(y)$.
- (d) If A is locally weakly wandering, then it is 1-traveling with $\sigma(\Sigma_1^1)$ -pieces. In fact, for any Borel selector $s : F(G^{\mathbb{N}}) \rightarrow G^{\mathbb{N}}$, the function $\gamma := s \circ \Delta_A$ is a $\sigma(\Sigma_1^1)$ -measurable F_A - and G -invariant travel guide for A .

Proof.

- (a) $\Delta_A(x)^c$ is open since being in it is witnessed by two coordinates.
- (b) For $s \in G^{<\mathbb{N}}$, let $B_s = \{F \in F(G^{\mathbb{N}}) : F \cap V_s \neq \emptyset\}$, where $V_s = \{\alpha \in G^{\mathbb{N}} : \alpha \supseteq s\}$. Since $\{B_s\}_{s \in G^{<\mathbb{N}}}$ generates the Borel structure of $F(G^{\mathbb{N}})$, it is enough to show that $\Delta_A^{-1}(B_s)$ is analytic, for every $s \in G^{<\mathbb{N}}$. But $\Delta_A^{-1}(B_s) = \{x \in X : \exists (g_n)_{n \in \mathbb{N}} \in V_s[g_0 = 1_G \wedge \forall n \neq m (g_n A^{[x]_G} \cap g_m A^{[x]_G} = \emptyset)]\}$ is clearly analytic.
- (c) Assume for contradiction that xF_Ay , but $\Delta_A(x) \neq \Delta_A(y)$ for some $x, y \in A$. We may assume that there is $(g_n)_{n \in \mathbb{N}} \in \Delta_A(x) \setminus \Delta_A(y)$ and thus $\exists n \neq m$ such that $g_n A^{[y]_G} \cap g_m A^{[y]_G} \neq \emptyset$. Hence $A^{[y]_G} \cap g_n^{-1} g_m A^{[y]_G} \neq \emptyset$ and let $y', y'' \in A^{[y]_G}$ be such that $y'' = g_n^{-1} g_m y'$. Let $g \in G$ be such that $y' = gy$. Since $y' = gy$, $y'' = g_n^{-1} g_m gy$ are in A , xF_Ay , and A is F_A -invariant, gx , $g_n^{-1} g_m gx$ are in A as well. Thus $A^{[x]_G} \cap g_n^{-1} g_m A^{[x]_G} \neq \emptyset$, contradicting $g_n A^{[y]_G} \cap g_m A^{[y]_G} = \emptyset$ (this holds since $(g_n)_{n \in \mathbb{N}} \in \Delta_A(x)$).
- (d) Follows from parts (b) and (c), and the definition of Δ_A . \square

Theorem 7.5. *Let X be a Borel G -space. If there is a locally weakly wandering Borel complete section for X , then X admits a Borel 4-generator.*

Proof. By part (d) of Proposition 7.4 and Proposition 3.8, X is 1-compressible. Thus, by Proposition 3.13, X admits a Borel 2^2 -finite generator. \square

7.3. Countable unions of weakly wandering sets

We can do even better in the case when a locally weakly wandering complete section is actually a countable union of weakly wandering sets.

Corollary 7.6. *Let X be a Borel G -space. If X admits a complete section that is a countable union of weakly wandering Borel sets, then X admits a Borel 3-generator.*

Proof. Let $(W_n)_{n \in \mathbb{N}}$ be a sequence of Borel weakly wandering sets such that $\bigcup_{n \in \mathbb{N}} W_n$ is a complete section. By replacing each W_n with $W_n \setminus \bigcup_{i < n} [W_i]_G$, we may assume that the $[W_n]_G$ are pairwise disjoint and hence $A := \bigcup_{n \in \mathbb{N}} W_n$ is a locally weakly wandering complete section. Using countable choice, take a function $p : \mathbb{N} \rightarrow G^{\mathbb{N}}$ such that

$\forall n \in \mathbb{N}$, $p(n) \in \bigcap_{x \in W_n} \Delta_{W_n}(x)$ (we know that $\bigcap_{x \in W_n} \Delta_{W_n}(x) \neq \emptyset$ since W_n is weakly wandering).

Define $\gamma : A \rightarrow G^{\mathbb{N}}$ by

$$x \mapsto \text{the smallest } k \text{ such that } p(k) \in \Delta_A(x).$$

The condition $p(k) \in \Delta_A(x)$ is Borel because it is equivalent to $\forall n, m \in \mathbb{N}$, $y, z \in A \cap [x]_G$, $p(k)(n)y = p(k)(m)z \Rightarrow n = m \wedge x = y$; thus γ is a Borel function. Note that γ is a travel guide for A by definition. Moreover, it is F_A -invariant because if $\Delta_A(x) = \Delta_A(y)$ for some $x, y \in A$, then conditions $p(k) \in \Delta_A(x)$ and $p(k) \in \Delta_A(y)$ hold or fail together. Since Δ_A is F_A -invariant, so is γ . Hence, [Lemma 3.12](#) applied to $\mathcal{I} = \langle A \rangle$ gives a Borel $(2 \cdot 2 - 1)$ -generator. \square

In the light of this last corollary, we now record a version of the Hajian–Kakutani–Itô theorem (see [Theorem 1.21](#)) as a corollary of the same theorem.

Corollary 7.7. *Let (X, μ) be a standard probability space equipped with a nonsingular Borel action of G . There is no invariant Borel probability measure absolutely continuous with respect to μ if and only if, modulo μ -NULL, there is a complete section that is a countable union of weakly wandering Borel sets.*

Proof. The right-to-left direction immediately follows from the fact that if W is a weakly wandering Borel set and ν is an invariant finite measure, then $\nu([W]_G) = 0$. For the left-to-right direction, we use a standard measure exhaustion argument based on iterative applications of the Hajian–Kakutani–Itô theorem. By recursion on $n \in \mathbb{N}$, we will define a decreasing sequence $(X_n)_{n \in \mathbb{N}}$ of invariant Borel sets as well as a disjoint sequence $(W_n)_{n \in \mathbb{N}}$ of μ -positive weakly wandering Borel sets such that

- (i) the X_n are μ -vanishing, i.e. $\bigcap_{n \in \mathbb{N}} X_n$ is μ -null,
- (ii) W_n is a complete section for $X_n \setminus X_{n+1}$, i.e. $[W_n]_G = X_n \setminus X_{n+1}$.

To this end, put $X_0 := X$, and assuming that X_n is defined, apply the Hajian–Kakutani–Itô theorem to X_n and get that the set

$$\mathcal{W}_n := \{W \subseteq X_n : W \text{ is Borel and } \mu(W) > 0\}$$

is nonempty. Thus there is $W_n \in \mathcal{W}_n$ with $\mu(W_n) > \frac{1}{2}w_n$, where $w_n := \sup_{W \in \mathcal{W}_n} \mu(W)$. Putting $X_{n+1} := X_n \setminus [W_n]_G$, we are through with the construction. However, we still have to check that $X_\infty := \bigcap_{n \in \mathbb{N}} X_n$ is μ -null. If $\mu(X_\infty) > 0$, an application of the Hajian–Kakutani–Itô theorem to X_∞ would provide a μ -positive weakly wandering Borel set $W_\infty \subseteq X_\infty$. But the sequence $(w_n)_{n \in \mathbb{N}}$ is summable since the W_n are pairwise disjoint and $\mu(W_n) > \frac{1}{2}w_n$, so for large enough $n \in \mathbb{N}$, $w_n < \mu(W_\infty)$, contradicting the definition of w_n .

Finally, putting $W = \bigcup_{n \in \mathbb{N}} W_n$, we get a complete section for $X \setminus X_\infty$ that is a countable union of weakly wandering Borel sets. \square

Corollaries 7.6 and 7.7 immediately imply the following version of the Krengel–Kuntz theorem (see **Theorem 1.18**) with a 3-generator instead of 2.

Corollary 7.8. *Let (X, μ) be a standard probability space equipped with a nonsingular Borel action of G . If there is no invariant Borel probability measure absolutely continuous with respect to μ , then X admits a 3-generator modulo μ -NULL.*

7.4. Further special cases

Using the function Δ defined above, we give another proof of **Proposition 3.10**.

Proposition 3.10. *Let X be an aperiodic Borel G -space and $T \subseteq X$ be Borel. If T is a partial transversal, then T is $\langle T \rangle$ -traveling.*

Proof. By definition, T is locally weakly wandering.

Claim. Δ_T is Borel.

Proof of Claim. Using the notation of the proof of part (b) of **Proposition 7.4**, it is enough to show that $\Delta_T^{-1}(B_s)$ is Borel for every $s \in G^{<\mathbb{N}}$. But since $\forall x \in T$, $T \cap [x]_G$ is a singleton, $\Delta_T(x) \in B_s$ is equivalent to $s(0) = 1_G \wedge (\forall n < m < |s|) s(m)x \neq s(n)x$. The latter condition is Borel, hence so is $\Delta_T^{-1}(B_s)$. \square

By part (d) of **Proposition 7.4**, $\gamma = s \circ \Delta_T$ is a Borel F_T -invariant travel guide for T . \square

Corollary 7.9. *Every aperiodic and smooth Borel G -space X admits a Borel 3-generator.*

Proof. Let $T \subseteq X$ be a Borel transversal. By **Proposition 3.10**, T is $\langle T \rangle$ -traveling. Thus, by **Lemma 3.12**, there is a Borel $(2 \cdot 2 - 1)$ -generator. \square

Lastly, in case of smooth free actions, a direct construction gives the optimal result as the following proposition shows.

Proposition 7.10. *Let X be a Borel G -space. If the G -action is free and smooth, then X admits a Borel 2-generator.*

Proof. Let $T \subseteq X$ be a Borel transversal. Also let $G \setminus \{1_G\} = \{g_n\}_{n \in \mathbb{N}}$ be such that $g_n \neq g_m$ for $n \neq m$. Because the action is free, $g_n T \cap g_m T = \emptyset$ for $n \neq m$.

Define $\pi : \mathbb{N} \rightarrow \mathbb{N}$ recursively as follows:

$$\pi(n) = \begin{cases} \min\{m : g_m \notin \{g_{\pi(i)} : i < n\}\} & \text{if } n = 3k \\ \min\{m : g_m, g_m g_k \notin \{g_{\pi(i)} : i < n\}\} & \text{if } n = 3k + 1 \\ \text{the unique } l \text{ s.t. } g_l = g_{\pi(3k+1)} g_k & \text{if } n = 3k + 2 \end{cases}.$$

Note that π is a bijection. Fix a countable family $\{U_n\}_{n \in \mathbb{N}}$ generating the Borel sets and put $A = \bigcup_{k \in \mathbb{N}} g_{\pi(3k)}(T \cap U_k) \cup \bigcup_{k \in \mathbb{N}} g_{\pi(3k+1)}T$. Clearly, A is Borel, and we show that $\mathcal{I} = \langle A \rangle$ is a generator. Fix distinct $x, y \in X$. Note that since T is a complete section, we can assume that $x \in T$.

First assume $y \in T$. Take k with $x \in U_k$ and $y \notin U_k$. Then $g_{\pi(3k)}x \in g_{\pi(3k)}(T \cap U_k) \subseteq A$ and $g_{\pi(3k)}y \in g_{\pi(3k)}(T \setminus U_k)$. However $g_{\pi(3k)}(T \setminus U_k) \cap A = \emptyset$ and hence $g_{\pi(3k)}y \notin A$.

Now suppose $y \notin T$. Then there exists $y' \in T^{[y]_G}$ and k such that $g_k y' = y$. Now $g_{\pi(3k+1)}x \in g_{\pi(3k+1)}T \subseteq A$ and $g_{\pi(3k+1)}y = g_{\pi(3k+1)}g_k y' = g_{\pi(3k+2)}y' \in g_{\pi(3k+2)}T$. But $g_{\pi(3k+2)}T \cap A = \emptyset$, hence $g_{\pi(3k+1)}y \notin A$. \square

Corollary 7.11. *Let H be a Polish group and G be a countable subgroup of H . If G admits an infinite discrete subgroup, then the translation action of G on H admits a 2-generator.*

Proof. Let G' be an infinite discrete subgroup of G . Clearly, it is enough to show that the translation action of G' on H admits a 2-generator. Since G' is discrete, it is closed. Indeed, if d is a left-invariant compatible metric on H , then $B_d(1_H, \epsilon) \cap G' = \{1_H\}$, for some $\epsilon > 0$. Thus every d -Cauchy sequence in G' is eventually constant and hence G' is closed. This implies that the translation action of G' on H is smooth and free (see [13, 12.17]), and hence Proposition 7.10 applies. \square

8. Separating partitions

Assume throughout this section that X is a Borel G -space.

8.1. Aperiodic separation and G -equivariant maps to 2^G

Lemma 8.1. *If X is aperiodic then it admits a countably infinite partition into Borel complete sections.*

Proof. The following argument is also given in [14, proof of Theorem 13.1]. By the marker lemma (see [14, 6.7]), there exists a vanishing sequence $\{B_n\}_{n \in \mathbb{N}}$ of decreasing Borel complete sections, i.e. $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$. For each $n \in \mathbb{N}$, define $k_n : X \rightarrow \mathbb{N}$ recursively as follows:

$$\begin{cases} k_0(x) = 0 \\ k_{n+1}(x) = \min\{k \in \mathbb{N} : B_{k_n(x)} \cap [x]_G \not\subseteq B_k\} \end{cases},$$

and define $A_n \subseteq X$ by

$$x \in A_n \Leftrightarrow x \in B_{k_n(x)} \setminus B_{k_{n+1}(x)}.$$

It is straightforward to check that A_n are pairwise disjoint Borel complete sections. \square

For $A \in \mathfrak{B}(X)$, if $\mathcal{I} = \langle A \rangle$ then we use the notation F_A and f_A instead of $F_{\mathcal{I}}$ and $f_{\mathcal{I}}$, respectively.

We now work towards strengthening the above lemma to yield a countably infinite partition into F_A -invariant Borel complete sections.

Definition 8.2 (*Aperiodic separation*). For Borel sets $A, Y \subseteq X$, we say that A aperiodically separates Y if $f_A([Y]_G)$ is aperiodic (as an invariant subset of the shift 2^G). If such A exists, we say that Y is aperiodically separable.

Proposition 8.3. *For $A \in \mathfrak{B}(X)$, if A aperiodically separates X , then X admits a countably infinite partition into Borel F_A -invariant complete sections.*

Proof. Let $Y = \{y \in 2^G : |[y]_G| = \infty\}$ and hence $f_A(X)$ is a G -invariant subset of Y . By Lemma 8.1 applied to Y , there is a partition $\{B_n\}_{n \in \mathbb{N}}$ of Y into Borel complete sections. Thus $A_n = f_{\mathcal{I}}^{-1}(B_n)$ is a Borel F_A -invariant complete section for X and $\{A_n\}_{n \in \mathbb{N}}$ is a partition of X . \square

Let \mathfrak{A} denote the collection of all subsets of aperiodically separable Borel sets.

Lemma 8.4. \mathfrak{A} is a σ -ideal.

Proof. We only have to show that if Y_n are aperiodically separable Borel sets, then $Y = \bigcup_{n \in \mathbb{N}} Y_n \in \mathfrak{A}$. Let A_n be a Borel set aperiodically separating Y_n . Since A_n also aperiodically separates $[Y_n]_G$ (by definition), we can assume that Y_n is G -invariant. Furthermore, by taking $Y'_n = Y_n \setminus \bigcup_{k < n} Y_k$, we can assume that Y_n are pairwise disjoint. Now letting $A = \bigcup_{n \in \mathbb{N}} (A_n \cap Y_n)$, it is easy to check that A aperiodically separates Y . \square

Let \mathfrak{S} denote the collection of all subsets of smooth sets. By a similar argument as the one above, \mathfrak{S} is a σ -ideal.

Lemma 8.5. *If X is aperiodic, then $\mathfrak{S} \subseteq \mathfrak{A}$.*

Proof. Let $S \in \mathfrak{S}$ and hence there is a Borel transversal T for $[S]_G$. Fix $x \in S$ and let $y \neq z \in [x]_G$. Since T is a transversal, there is $g \in G$ such that $gy \in T$, and hence $gz \notin T$. Thus $f_T(y) \neq f_T(z)$, and so $f_T([x]_G)$ is infinite. Therefore T aperiodically separates $[S]_G$. \square

For the rest of this subsection, fix an enumeration $G = \{g_n\}_{n \in \mathbb{N}}$ and let F_A^n be following equivalence relation:

$$yF_A^n z \Leftrightarrow \forall k < n (g_k y \in A \leftrightarrow g_k z \in A).$$

Note that F_A^n has no more than 2^n equivalence classes and that $yF_A z$ if and only if $\forall n (yF_A^n z)$.

Lemma 8.6. *For $A, Y \in \mathfrak{B}(X)$, A aperiodically separates Y if and only if $(\forall x \in Y)(\forall n)(\exists y, z \in Y^{[x]_G})[yF_A^n z \wedge \neg(yF_A z)]$.*

Proof. \Rightarrow : Assume that for all $x \in Y$, $f_A([x]_G)$ is infinite and thus $F_A|_{[x]_G}$ has infinitely many equivalence classes. Fix $n \in \mathbb{N}$ and recall that F_A^n has only finitely many equivalence classes. Thus, by the Pigeon Hole Principle, there are $y, z \in Y^{[x]_G}$ such that $yF_A^n z$ yet $\neg(yF_A z)$.

\Leftarrow : Assume for contradiction that $f_A(Y^{[x]_G})$ is finite for some $x \in Y$. Then it follows that $F_A = F_A^n$, for some n , and hence for any $y, z \in Y^{[x]_G}$, $yF_A^n z$ implies $yF_A z$, contradicting the hypothesis. \square

Theorem 8.7. *If X is an aperiodic Borel G -space, then $X \in \mathfrak{A}$.*

Proof. By Lemma 8.1, there is a partition $\{A_n\}_{n \in \mathbb{N}}$ of X into Borel complete sections. We will inductively construct Borel sets $B_n \subseteq C_n$, where C_n should be thought of as the set of points colored (black or white) at the n^{th} step, and B_n as the set of points colored black (thus $C_n \setminus B_n$ is colored white).

Define a function $\# : X \rightarrow \mathbb{N}$ by $x \mapsto m$, where m is such that $x \in A_m$. Fix a countable family $\{U_n\}_{n \in \mathbb{N}}$ of sets generating the Borel σ -algebra of X .

Assuming that for all $k < n$, C_k, B_k are defined, let $\bar{C}_n = \bigcup_{k < n} C_k$ and $\bar{B}_n = \bigcup_{k < n} B_k$. Put $P_n = \{x \in A_0 : \forall k < n (g_k x \in \bar{C}_n) \wedge g_n x \notin \bar{C}_n\}$ and set $F_n = F_{\bar{B}_n}^n|_{P_n}$, that is for all $x, y \in P_n$,

$$yF_n z \Leftrightarrow \forall k < n (g_k y \in \bar{B}_n \leftrightarrow g_k z \in \bar{B}_n).$$

Now put $C'_n = \{x \in P_n : \#(g_n x) = \min \#((g_n P_n)^{[x]_G})\}$, $C''_n = \{x \in C'_n : \exists y, z \in (C'_n)^{[x]_G} (y \neq z \wedge yF_n z)\}$ and $C_n = g_n C''_n$. Note that it follows from the definition of P_n that C_n is disjoint from \bar{C}_n .

Now in order to define B_n , first define a function $\bar{n} : X \rightarrow \mathbb{N}$ by

$$x \mapsto \text{the smallest } m \text{ such that there are } y, z \in C''_n \cap [x]_G$$

$$\text{with } yF_n z, y \in U_m \text{ and } z \notin U_m.$$

Note that \bar{n} is Borel and G -invariant. Lastly, let $B'_n = \{x \in C''_n : x \in U_{\bar{n}(x)}\}$ and $B_n = g_n B'_n$. Clearly $B_n \subseteq C_n$. Now let $B = \bigcup_{n \in \mathbb{N}} B_n$ and $D = [\bigcup_{n \in \mathbb{N}} (C'_n \setminus C''_n)]_G$. We

show that B aperiodically separates $Y := X \setminus D$ and $D \in \mathfrak{S}$. Since $\mathfrak{S} \subseteq \mathfrak{A}$ and \mathfrak{A} is an ideal, this will imply that $X \in \mathfrak{A}$.

Claim 1. $D \in \mathfrak{S}$.

Proof of Claim. Since \mathfrak{S} is a σ -ideal, it is enough to show that for each n , $[C'_n \setminus C''_n]_G \in \mathfrak{S}$, so fix $n \in \mathbb{N}$. Clearly $(C'_n \setminus C''_n)^{[x]_G}$ is finite, for all $x \in X$, since there can be at most 2^n pairwise F_n -nonequivalent points. Thus, fixing some Borel linear ordering of X and taking the smallest element from $(C'_n \setminus C''_n)^{[x]_G}$ for each $x \in C'_n \setminus C''_n$, we can define a Borel transversal for $[C'_n \setminus C''_n]_G$. \square

By Lemma 8.6, to show that B aperiodically separates Y , it is enough to show that $(\forall x \in Y)(\forall n)(\exists y, z \in [x]_G)[yF_B^n z \wedge \neg(yF_B z)]$. Fix $x \in Y$.

Claim 2. $(\exists^\infty n)(C''_n)^{[x]_G} \neq \emptyset$.

Proof of Claim. Assume for contradiction that $(\forall n)(C''_n)^{[x]_G} = \emptyset$. Since $x \notin D$, it follows that $(\forall n)P_n^{[x]_G} = \emptyset$. Since A_0 is a complete section and $\bar{C}_0 = \emptyset$, $P_0^{[x]_G} \neq \emptyset$. Let N be the largest number such that $P_N^{[x]_G} \neq \emptyset$. Thus for all $n > N$, $C_n^{[x]_G} = \emptyset$ and hence for all $n > N$, $\bar{C}_n^{[x]_G} = \bar{C}_{N+1}^{[x]_G}$. Because $C_N^{[x]_G} \neq \emptyset$, there is $y \in A_0^{[x]_G}$ such that $\forall k \leq N(g_k y \in \bar{C}_{N+1})$; but because $P_{N+1}^{[x]_G} = \emptyset$, $g_{N+1}y$ must also fall into \bar{C}_{N+1} . By induction on $n > N$, we get that for all $n > N$, $g_n y \in \bar{C}_n$ and thus $g_n y \in \bar{C}_{N+1}$.

On the other hand, it follows from the definition of C'_n that for each n , $(C'_n)^{[x]_G}$ intersects exactly one of A_k . Thus $\bar{C}_{N+1}^{[x]_G}$ intersects at most $N+1$ of A_k and hence there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $\bar{C}_{N+1}^{[x]_G} \cap A_k = \emptyset$. Since $\exists^\infty n(g_n y \in \bigcup_{k \geq K} A_k)$, $\exists^\infty n(g_n y \notin \bar{C}_{N+1})$, a contradiction. \square

Now it remains to show that for all $n \in \mathbb{N}$, $(C''_n)^{[x]_G} \neq \emptyset$ implies that $\exists y, z \in [x]_G$ such that $yF_B^n z$ but $\neg(yF_B z)$. To this end, fix $n \in \mathbb{N}$ and assume $(C''_n)^{[x]_G} \neq \emptyset$. Thus there are $y, z \in (C''_n)^{[x]_G}$ such that $yF_n z$, $y \in U_{\bar{n}(x)}$ and $z \notin U_{\bar{n}(x)}$; hence, $g_n y \in B_n$ and $g_n z \notin B_n$, by the definition of B_n . Since the C_k are pairwise disjoint, $B_n \subseteq C_n$ and $g_n y, g_n z \in C_n$, it follows that $g_n y \in B$ and $g_n z \notin B$, and therefore $\neg(yF_B z)$. Finally, note that $F_n = F_B^n|_{P_n}$ and hence $yF_B^n z$. \square

It is worth explicitly stating the previous theorem in terms of the coding map f_A :

Theorem 8.8. Any aperiodic Borel G -space admits a G -equivariant Borel map to the aperiodic part of the shift action $G \curvearrowright 2^G$.

Corollary 8.9. For an aperiodic Borel \mathbb{Z} -space X , there is a Borel set $A \subseteq X$ such that $G\langle A \rangle$ separates points in each orbit, i.e. $f_A|_{[x]_{\mathbb{Z}}}$ is one-to-one, for all $x \in X$.

Proof. Let A be a Borel set aperiodically separating X (exists by [Theorem 8.7](#)) and put $Y = f_A(X)$. Then $Y \subseteq 2^{\mathbb{Z}}$ is aperiodic and hence the action of \mathbb{Z} on Y is free. But this implies that for all $y \in Y$, $f_A^{-1}(y)$ intersects every orbit in X at no more than one point, and hence f_A is one-to-one on every orbit. \square

From [Proposition 8.3](#) and [Theorem 8.7](#) we immediately get the following strengthening of [Lemma 8.1](#).

Corollary 8.10. *Any aperiodic Borel G -space X admits a countably infinite partition into Borel F_A -invariant complete sections, for some Borel set $A \subseteq X$.*

8.2. Separating smooth-many invariant sets

The following is a useful tool in constructing generators and we will apply it in the next section to the equivalence relation E of being in the same component of the ergodic decomposition.

Definition 8.11 (*Smooth equivalence relations*). An equivalence relation E on a standard Borel space X is called smooth if there is a Borel map $h : X \rightarrow \mathbb{R}$ reducing E to the equality relation on \mathbb{R} , i.e. for each $x, y \in X$,

$$xEy \iff h(x) = h(y).$$

Note that since any two uncountable Polish spaces are Borel isomorphic (see [\[13, Theorem 15.6\]](#)), in the above definition \mathbb{R} can be replaced with any other uncountable Polish space. Also note that a smooth equivalence relation E is automatically Borel (as a subset of X^2) because $E = \bar{h}^{-1}(\Delta(\mathbb{R}))$, where $\Delta(\mathbb{R}) = \{(r, r) : r \in \mathbb{R}\}$ and $\bar{h} : X^2 \rightarrow \mathbb{R}^2$ is defined by $(x, y) \mapsto (h(x), h(y))$.

It is a theorem of Burgess (see [\[2\]](#)) that for a Borel G -space X , the orbit equivalence relation E_G is smooth if and only if the action $G \curvearrowright X$ is smooth in the sense of [Definition 3.9](#), that is: X admits a Borel transversal.

For a Borel G -space X , an equivalence relation E on X is called G -invariant if $E_G \subseteq E$; in other words, each E -class is a union of G -orbits.

Theorem 8.12. *Let X be an aperiodic Borel G -space and let E be a smooth G -invariant equivalence relation on X . There exists a partition \mathcal{P} of X into 4 Borel sets such that $G\mathcal{P}$ separates any two E -nonequivalent points in X , i.e. for all $x, y \in X$, $[x]_E \neq [y]_E$ implies $f_{\mathcal{P}}(x) \neq f_{\mathcal{P}}(y)$.*

Proof. By [Corollary 8.10](#), there is $A \in \mathfrak{B}(X)$ and a Borel partition $\{A_n\}_{n \in \mathbb{N}}$ of X into F_A -invariant complete sections. Fix an enumeration $G = \{g_n\}_{n \in \mathbb{N}}$, and for each $n \in \mathbb{N}$, define a function $\bar{n} : X \rightarrow \mathbb{N}$ by

$$x \mapsto \text{the smallest } m \text{ such that } \exists x' \in A_0^{[x]_G} \text{ with } g_m x' \in A_n.$$

Clearly \bar{n} is Borel, and because all of A_k are F_A -invariant, \bar{n} is also F_A -invariant, i.e. for all $x, y \in X$, $x F_A y \rightarrow \bar{n}(x) = \bar{n}(y)$. Also, \bar{n} is G -invariant by definition.

Put $A'_n = \{x \in A_0 : g_{\bar{n}(x)}x \in A_n\}$ and note that A'_n is F_A -invariant Borel since so are \bar{n} , A_0 and A_n . Moreover, A'_n is clearly a complete section. Define $\gamma_n : A'_n \rightarrow A_n$ by $x \mapsto g_{\bar{n}(x)}x$. Clearly, γ_n is Borel and one-to-one.

Since E is smooth, there is a Borel $h : X \rightarrow \mathbb{R}$ such that for all $x, y \in X$, $x E y \leftrightarrow h(x) = h(y)$. Let $\{V_n\}_{n \in \mathbb{N}}$ be a countable family of subsets of \mathbb{R} generating the Borel σ -algebra of \mathbb{R} and put $U_n = h^{-1}(V_n)$. Because each equivalence class of E is G -invariant, so is h and hence so is U_n .

Now let $B_n = \gamma_n(A'_n \cap U_n)$ and note that B_n is Borel being a one-to-one Borel image of a Borel set. It follows from the definition of γ_n that $B_n \subseteq A_n$. Put $B = \bigcup_{n \in \mathbb{N}} B_n$ and $\mathcal{P} = \langle A, B \rangle$; in particular, $|\mathcal{P}| \leq 4$. We show that \mathcal{P} is what we want. To this end, fix $x, y \in X$ with $\neg(x E y)$. If $\neg(x F_A y)$, then $G\langle A \rangle$ (and hence $G\mathcal{P}$) separates x and y .

Thus assume that $x F_A y$. Since $h(x) \neq h(y)$, there is n such that $h(x) \in V_n$ and $h(y) \notin V_n$. Hence, by invariance of U_n , $gx \in U_n \wedge gy \notin U_n$, for all $g \in G$. Because A'_n is a complete section, there is $g \in G$ such that $gx \in A'_n$ and hence $gy \in A'_n$ since A'_n is F_A -invariant. Let $m = \bar{n}(gx) (= \bar{n}(gy))$. Then $g_m gx \in B_n$ while $g_m gy \notin B_n$ although $g_m gy \in \gamma_n(A'_n) \subseteq A_n$. Thus $g_m gx \in B$ but $g_m gy \notin B$ and therefore $G\mathcal{P}$ separates x and y . \square

9. Potential dichotomy theorems

In this section we prove dichotomy theorems assuming Weiss's question has a positive answer for $G = \mathbb{Z}$. In the proofs we use the Ergodic Decomposition Theorem (see [7,22]) and a Borel/uniform version of Krieger's finite generator theorem, so we first state both of the theorems and sketch the proof of the latter.

For a Borel G -space X , let $\mathcal{M}_G(X)$ denote the set of G -invariant Borel probability measures on X and let $\mathcal{E}_G(X)$ denote the set of ergodic ones among those. Clearly both are Borel subsets of $P(X)$ (the standard Borel space of Borel probability measures on X) and thus are themselves standard Borel spaces.

Ergodic Decomposition Theorem 9.1 (Farrell, Varadarajan). *Let X be a Borel G -space. If $\mathcal{M}_G(X) \neq \emptyset$ (and hence $\mathcal{E}_G(X) \neq \emptyset$), then there is a Borel surjection $x \mapsto e_x$ from X onto $\mathcal{E}_G(X)$ such that:*

- (i) $x E_G y \Rightarrow e_x = e_y$;
- (ii) For each $e \in \mathcal{E}_G(X)$, if $X_e = \{x \in X : e_x = e\}$ (hence X_e is invariant Borel), then $e(X_e) = 1$ and $e|_{X_e}$ is the unique ergodic invariant Borel probability measure on X_e ;
- (iii) For each $\mu \in \mathcal{M}_G(X)$ and $A \in \mathfrak{B}(X)$, we have $\mu(A) = \int e_x(A) d\mu(x)$.

For the rest of the section, let X be a Borel \mathbb{Z} -space.

For $e \in \mathcal{E}_{\mathbb{Z}}(X)$, if we let h_e denote the entropy of (X, \mathbb{Z}, e) , then the map $e \mapsto h_e$ is Borel. Indeed, if $\{\mathcal{P}_k\}_{k \in \mathbb{N}}$ is a refining sequence of partitions of X that generates the Borel σ -algebra of X , then, by [5, 4.1.2], $h_e = \lim_{k \rightarrow \infty} h_e(\mathcal{P}_k, \mathbb{Z})$, where $h_e(\mathcal{P}_k, \mathbb{Z})$ denotes the entropy of \mathcal{P}_k . By [13, 17.21], the function $e \mapsto h_e(\mathcal{P}_k)$ is Borel and thus so is the map $e \mapsto h_e$.

For all $e \in \mathcal{E}_{\mathbb{Z}}(X)$ with $h_e < \infty$, let N_e be the smallest integer such that $\log_2 N_e > h_e$. The map $e \mapsto N_e$ is Borel because so is $e \mapsto h_e$.

Krieger's Finite Generator Theorem 9.2 (Uniform version). *Let X be a Borel \mathbb{Z} -space. Suppose $\mathcal{M}_{\mathbb{Z}}(X) \neq \emptyset$ and let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Assume also that all measures in $\mathcal{E}_{\mathbb{Z}}(X)$ have finite entropy and let $e \mapsto N_e$ be the map defined above. Then there is a partition $\{A_n\}_{n \leq \infty}$ of X into Borel sets such that*

- (i) A_{∞} is invariant and does not admit an invariant Borel probability measure;
- (ii) For each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $\{A_n \cap X_e\}_{n < N_e}$ is a generator for $X_e \setminus A_{\infty}$, where $X_e = \rho^{-1}(e)$.

Sketch of Proof. Note that it is enough to find a Borel invariant set $X' \subseteq X$ and a Borel \mathbb{Z} -map $\phi : X' \rightarrow \mathbb{N}^{\mathbb{Z}}$, such that for each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, we have

- (I) $e(X \setminus X') = 0$;
- (II) $\phi|_{X_e \cap X'}$ is one-to-one and $\phi(X_e \cap X') \subseteq (N_e)^{\mathbb{Z}}$, where $(N_e)^{\mathbb{Z}}$ is naturally viewed as a subset of $\mathbb{N}^{\mathbb{Z}}$.

Indeed, assume we had such X' and ϕ , and let $A_{\infty} = X \setminus X'$ and $A_n = \phi^{-1}(V_n)$ for all $n \in \mathbb{N}$, where $V_n = \{y \in \mathbb{N}^{\mathbb{Z}} : y(0) = n\}$. Then it is clear that $\{A_n\}_{n \in \mathbb{N}}$ satisfies (ii). Also, (I) and part (iii) of the Ergodic Decomposition Theorem imply that (i) holds for A_{∞} .

To construct such a ϕ , we use the proof of Krieger's theorem presented in [5, Theorem 4.2.3], and we refer to it as Downarowicz's proof. For each $e \in \mathcal{E}_{\mathbb{Z}}(X)$, the proof constructs a Borel \mathbb{Z} -embedding $\phi_e : X' \rightarrow N_e^{\mathbb{Z}}$ on an e -measure 1 set X' . We claim that this construction is uniform in e in a Borel way and hence would yield X' and ϕ as above.

Our claim can be verified by inspection of Downarowicz's proof. The proof uses the existence of sets with certain properties and one has to check that such sets exist with the properties satisfied for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$ at once. For example, the set C used in [5, proof of Lemma 4.2.5] can be chosen so that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $C \cap X_e$ has the required properties for e (using the Shannon–McMillan–Brieman theorem). Another example is the set B used in the proof of the same lemma, which is provided by Rohlin's lemma. By inspection of the proof of Rohlin's lemma (see [8, 2.1]), one can verify that we can get a Borel B such that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $B \cap X_e$ has the required properties for e . The sets in these two examples are the only kind of sets whose exis-

tence is used in the whole proof; the rest of the proof constructs the required ϕ “by hand”. \square

Theorem 9.3 (Dichotomy I). *Suppose the answer to Question 1.13 is positive and let X be an aperiodic Borel \mathbb{Z} -space. Then exactly one of the following holds:*

- (1) *there exists an invariant ergodic Borel probability measure with infinite entropy;*
- (2) *there exists a partition $\{Y_n\}_{n \in \mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator.*

Proof. We first show that the conditions above are mutually exclusive. Indeed, assume there exist an invariant ergodic Borel probability measure e with infinite entropy and a partition $\{Y_n\}_{n \in \mathbb{N}}$ of X into invariant Borel sets such that each Y_n has a finite generator. By ergodicity, e would have to be supported on one of the Y_n . But Y_n has a finite generator and hence the dynamical system (Y_n, \mathbb{Z}, e) has finite entropy by the Kolmogorov–Sinai theorem (see Theorem 1.9). Thus so does (X, \mathbb{Z}, e) since these two systems are isomorphic (modulo e -NULL), contradicting the assumption on e .

Now we prove that at least one of the conditions holds. Assume that there is no invariant ergodic measure with infinite entropy. Now, if there was no invariant Borel probability measure at all, then, since the answer to Question 1.13 is assumed to be positive, X would admit a finite generator, and we would be done. So assume that $\mathcal{M}_{\mathbb{Z}}(X) \neq \emptyset$ and let $\{A_n\}_{n \leq \infty}$ be as in Krieger’s Finite Generator Theorem 9.2. Furthermore, let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Set $X' = X \setminus A_{\infty}$, $Y_{\infty} = A_{\infty}$, and for all $n \in \mathbb{N}$,

$$Y_n = \{x \in X' : N_{e_x} = n\},$$

where the map $e \mapsto N_e$ is as above. Note that the sets Y_n are invariant since ρ is invariant, so $\{Y_n\}_{n \leq \infty}$ is a countable partition of X into invariant Borel sets. Since Y_{∞} does not admit an invariant Borel probability measure, by our assumption, it has a finite generator.

Let E be the equivalence relation on X' defined by ρ , i.e. $\forall x, y \in X'$,

$$xEy \Leftrightarrow \rho(x) = \rho(y).$$

By definition, E is a smooth Borel equivalence relation with $E \supseteq E_{\mathbb{Z}}$ since ρ respects the \mathbb{Z} -action. Thus, by Theorem 8.12, there exists a partition \mathcal{P} of X' into 4 Borel sets such that $\mathbb{Z}\mathcal{P}$ separates any two points in different E -classes.

Now fix $n \in \mathbb{N}$ and we will show that $\mathcal{I} = \mathcal{P} \vee \{A_i\}_{i < n}$ is a generator for Y_n . Indeed, take distinct $x, y \in Y_n$. If x and y are in different E -classes, then $\mathbb{Z}\mathcal{P}$ separates them and hence so does $\mathbb{Z}\mathcal{I}$. Thus we can assume that xEy . Then $e := \rho(x) = \rho(y)$, i.e. $x, y \in X_e = \rho^{-1}(e)$. By the choice of $\{A_i\}_{i \in \mathbb{N}}$, $\{A_n \cap X_e\}_{n < N_e}$ is a generator for X_e and

hence $\mathbb{Z}\{A_i\}_{i < N_e}$ separates x and y . But $n = N_e$ by the definition of Y_n , so $\mathbb{Z}\mathcal{I}$ separates x and y . \square

Proposition 9.4. *Let X be a Borel \mathbb{Z} -space. If X admits invariant ergodic probability measures of arbitrarily large entropy, then it admits an invariant probability measure of infinite entropy.*

Proof. For each $n \geq 1$, let μ_n be an invariant ergodic probability measure of entropy $h_{\mu_n} > n2^n$ such that $\mu_n \neq \mu_m$ for $n \neq m$, and put

$$\mu = \sum_{n \geq 1} \frac{1}{2^n} \mu_n.$$

It is clear that μ is an invariant probability measure, and we show that its entropy h_μ is infinite using an argument pointed out by the referee of the current paper (the author's original argument was less general).

Claim. *For any invariant probability measures ν, μ on X and $c \leq 1$, if $c\nu \leq \mu$ then $ch_\nu \leq h_\mu$.*

Proof of Claim. First note that the entropy of any atomic invariant probability measure ρ is 0: indeed, such a measure is supported on countably many orbits, each of which is finite, and hence for any partition \mathcal{I} , the sequence $h_\rho(\bigvee_{i=-n}^n T^i \mathcal{I})$ converges as $n \rightarrow \infty$; therefore, the time-average entropy (see (1.7)) is 0. It follows now that the nonatomic parts of ν, μ do not contribute in the calculation of their entropies, so we may assume without loss of generality that ν, μ are nonatomic.

The function $g(x) = -x \log_2 x$ used in the calculation of the entropy is convex on $(0, 1)$ and satisfies $cg(x) \leq g(cx) \leq g(y)$ for $0 < x, y < 1/2$ with $cx \leq y$. In (1.6), x is equal to the measure of a piece of a partition; since, by the convexity of $g(x)$, refining a partition only increases its static entropy and our measures are nonatomic, in (1.8) we can take the supremum merely over the partitions with pieces of measure less than $1/2$. This yields $ch_\nu \leq h_\mu$. \square

For any $n \geq 1$, we have $\frac{1}{2^n} \mu_n \leq \mu$, so the above claim gives $h_\mu \geq \frac{1}{2^n} h_{\mu_n} \geq n$ and hence $h_\mu = \infty$. \square

Theorem 9.5 (Dichotomy II). *Suppose the answer to Question 1.13 is positive and let X be an aperiodic Borel \mathbb{Z} -space. Then exactly one of the following holds:*

- (1) *there exists an invariant Borel probability measure with infinite entropy;*
- (2) *X admits a finite generator.*

Proof. The Kolmogorov–Sinai theorem implies that the conditions are mutually exclusive, and we prove that at least one of them holds. Assume that there is no invariant measure with infinite entropy. If there was no invariant Borel probability measure at all, then, by our assumption, X would admit a finite generator. So assume that $\mathcal{M}_{\mathbb{Z}}(X) \neq \emptyset$ and let $\{A_n\}_{n \leq \infty}$ be as in Krieger’s Finite Generator [Theorem 9.2](#). Furthermore, let ρ be the map $x \mapsto e_x$ as in the Ergodic Decomposition Theorem. Set $X' = X \setminus A_\infty$ and $X_e = \rho^{-1}(e)$, for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$.

By our assumption, A_∞ admits a finite generator \mathcal{P} . Also, by [Proposition 9.4](#), there is $N \geq 1$ such that for all $e \in \mathcal{E}_{\mathbb{Z}}(X)$, $N_e \leq N$ and hence $\mathcal{Q} := \{A_n\}_{n < N}$ is a finite generator for X_e ; in particular, \mathcal{Q} is a partition of X' . Let E be the following equivalence relation on X :

$$xEy \Leftrightarrow (x, y \in A_\infty) \vee (x, y \in X' \wedge \rho(x) = \rho(y)).$$

By definition, E is a smooth equivalence relation with $E \supseteq E_{\mathbb{Z}}$ since ρ respects the \mathbb{Z} -action and A_∞ is \mathbb{Z} -invariant. Thus, by [Theorem 8.12](#), there exists a partition \mathcal{J} of X into 4 Borel sets such that $\mathbb{Z}\mathcal{J}$ separates any two points in different E -classes.

We now show that $\mathcal{I} := \langle \mathcal{J} \cup \mathcal{P} \cup \mathcal{Q} \rangle$ is a generator. Indeed, fix distinct $x, y \in X$. If x and y are in different E -classes, then $\mathbb{Z}\mathcal{J}$ separates them. So we can assume that xEy . If $x, y \in A_\infty$, then $\mathbb{Z}\mathcal{P}$ separates x and y . Finally, if $x, y \in X'$, then $x, y \in X_e$, where $e = \rho(x)$ ($= \rho(y)$), and hence $\mathbb{Z}\mathcal{Q}$ separates x and y . \square

Remark 9.6. It is likely that the above dichotomies are also true for any amenable group using a uniform version of Krieger’s theorem for amenable groups (see [\[4\]](#)), but the author has not checked the details.

10. Finite generators on comeager sets

This section is devoted to the proof of the following:

Theorem 10.1. *Any aperiodic Polish G -space admits a 4-generator on an invariant comeager set.*

Throughout this section, let X be an aperiodic Polish G -space. We use the notation $\forall^* x$ to mean “for comeager many x ”.

Having advertised the Kuratowski–Ulam method in the introduction, let us point out that a “blind” application of it would not give us the statement of the above theorem. Indeed, assume for a moment that we have found a parametrized construction of finite partitions \mathcal{P}_α , for $\alpha \in \mathbb{N}^{\mathbb{N}}$, and let

$$\Phi(\mathcal{P}_\alpha, x, y) :\Leftrightarrow \text{“if } x \neq y, \text{ then } G\mathcal{P}_\alpha \text{ separates } x \text{ and } y\text{”}.$$

If we apply the Kuratowski–Ulam method to this Φ , we will get that for comeager many $\alpha \in \mathbb{N}^{\mathbb{N}}$, we have:

$$\forall^*(x, y) \in X^2 \Phi(\mathcal{P}_\alpha, x, y),$$

while we want a comeager set $D \subseteq X$ such that

$$\forall(x, y) \in D^2 \Phi(\mathcal{P}_\alpha, x, y).$$

The problem is that a 2-dimensional comeager set may not contain a square of a 1-dimensional comeager set. To get around this, we transform our 2-dimensional problem into two 1-dimensional problems, and here is the first of them:

Lemma 10.2. *There exists $A \in \mathfrak{B}(X)$ such that $G \langle A \rangle$ separates points in each orbit of a comeager G -invariant set D , i.e. for each $x \in D$, the restriction of the coding map $f_{\langle A \rangle}$ to $[x]_G$ is one-to-one.*

Proof. Fix a countable basis $\{U_n\}_{n \in \mathbb{N}}$ for X with $U_0 = \emptyset$ and let $\{A_n\}_{n \in \mathbb{N}}$ be a partition of X provided by Lemma 8.1. For each $\alpha \in \mathcal{N}$ (the Baire space), define

$$B_\alpha = \bigcup_{n \in \mathbb{N}} (A_n \cap U_{\alpha(n)}).$$

Claim. $\forall^* \alpha \in \mathcal{N} \forall^* z \in X \forall x, y \in [z]_G (x \neq y \Rightarrow \exists g \in G (gx \in B_\alpha \nleftrightarrow gy \in B_\alpha))$.

Proof of Claim. By Kuratowski–Ulam, it is enough to show the statement with places of the quantifiers $\forall^* \alpha \in \mathcal{N}$ and $\forall^* z \in X$ switched. Also, since orbits are countable and a countable intersection of comeager sets is comeager, we can also switch the places of the quantifiers $\forall^* \alpha \in \mathcal{N}$ and $\forall x, y \in [z]_G$. Thus we fix $z \in X$ and $x, y \in [z]_G$ with $x \neq y$ and show that $C = \{\alpha \in \mathcal{N} : \exists g \in G (gx \in B_\alpha \nleftrightarrow gy \in B_\alpha)\}$ is dense open.

To see that C is open, take $\alpha \in C$ and let $g \in G$ be such that $gx \in B_\alpha \nleftrightarrow gy \in B_\alpha$. Let $n, m \in \mathbb{N}$ be such that $gx \in A_n$ and $gy \in A_m$. Then for all $\beta \in \mathcal{N}$ with $\beta(n) = \alpha(n)$ and $\beta(m) = \alpha(m)$, we have $gx \in B_\beta \nleftrightarrow gy \in B_\beta$. But the set of such β is open in \mathcal{N} and contained in C .

For the density of C , let $s \in \mathbb{N}^{<\mathbb{N}}$ and set $n = |s|$. Since A_n is a complete section, $\exists g \in G$ with $gx \in A_n$. Let $m \in \mathbb{N}$ be such that $gy \in A_m$. Take any $t \in \mathbb{N}^{\max\{n, m\}+1}$ with $t \supseteq s$ satisfying the following condition:

Case 1: $n > m$. If $gy \in U_{s(m)}$ then set $t(n) = 0$. If $gy \notin U_{s(m)}$, then let k be such that $gx \in U_k$ and set $t(n) = k$.

Case 2: $n \leq m$. Let k be such that $gx \in U_k$ but $gy \notin U_k$ and set $t(n) = t(m) = k$.

Now it is easy to check that in any case $gx \in B_\alpha \nleftrightarrow gy \in B_\alpha$, for any $\alpha \in \mathcal{N}$ with $\alpha \supseteq t$, and so $\alpha \in C$ and $\alpha \supseteq s$. Hence C is dense. \square

By the claim, $\exists \alpha \in \mathcal{N}$ such that

$$D := \{z \in X : \forall x, y \in [z]_G \text{ with } x \neq y, G\langle B_\alpha \rangle \text{ separates } x \text{ and } y\}$$

is comeager and clearly invariant, which completes the proof. \square

The reader is invited to compare this last lemma with [Theorem 8.7](#) and [Corollary 8.9](#). In fact, in the proof of [Theorem 10.1](#) below, we only use [Lemma 10.2](#) to deduce the conclusion of [Theorem 8.7](#) modulo MEAGER, which we could also deduce from [Theorem 8.7](#) itself. However, we still included [Lemma 10.2](#) here to keep this section self-contained, and also because its proof is easier than that of [Theorem 8.7](#).

Proof of Theorem 10.1. Let A and D be provided by [Lemma 10.2](#). Throwing away an invariant meager set from D , we may assume that D is dense G_δ and hence Polish in the relative topology. Therefore, we may assume without loss of generality that $X = D$.

Thus A aperiodically separates X and hence, by [Proposition 8.3](#), there is a partition $\{A_n\}_{n \in \mathbb{N}}$ of X into F_A -invariant Borel complete sections (the latter could be inferred directly from [Corollary 8.10](#) without using [Lemma 10.2](#)). Fix an enumeration $G = \{g_n\}_{n \in \mathbb{N}}$ and a countable basis $\{U_n\}_{n \in \mathbb{N}}$ for X . Denote $\mathcal{N}_2 = (\mathbb{N}^2)^\mathbb{N}$ and for each $\alpha \in \mathcal{N}_2$, define

$$B_\alpha = \bigcup_{n \geq 1} (A_n \cap g_{(\alpha(n))_0} U_{(\alpha(n))_1}).$$

Claim. $\forall^* \alpha \in \mathcal{N}_2 \forall^* x \in X \forall l \in \mathbb{N} \exists n, k \in \mathbb{N} (\alpha(n) = (k, l) \wedge g_k x \in A_n)$.

Proof of Claim. By Kuratowski–Ulam, it is enough to show that $\forall x \in X$ and $\forall l \in \mathbb{N}$, $C = \{\alpha \in \mathcal{N}_2 : \exists k, n \in \mathbb{N} (\alpha(n) = (k, l) \wedge g_k x \in A_n)\}$ is dense open.

To see that C is open, note that for fixed $n, k, l \in \mathbb{N}$, $\alpha(n) = (k, l)$ is an open condition in \mathcal{N}_2 .

For the density of C , let $s \in (\mathbb{N}^2)^{<\mathbb{N}}$ and set $n = |s|$. Since A_n is a complete section, $\exists k \in \mathbb{N}$ with $g_k x \in A_n$. Any $\alpha \in \mathcal{N}_2$ with $\alpha \supseteq s$ and $\alpha(n) = (k, l)$ belongs to C . Hence C is dense. \square

By the claim, there exists $\alpha \in \mathcal{N}_2$ such that $Y = \{x \in X : \forall l \in \mathbb{N} \exists k, n \in \mathbb{N} (\alpha(n) = (k, l) \wedge g_k x \in A_n)\}$ is comeager.

Let $\mathcal{I} = \langle A, B_\alpha \rangle$, and so $|\mathcal{I}| \leq 4$. We show that \mathcal{I} is a generator on Y . Fix distinct $x, y \in Y$. If x and y are separated by $G\langle A \rangle$ then we are done, so assume otherwise, that is $x F_A y$. Let $l \in \mathbb{N}$ be such that $x \in U_l$ but $y \notin U_l$. Then there exists $k, n \in \mathbb{N}$ such that $\alpha(n) = (k, l)$ and $g_k x \in A_n$. Since $g_k x F_A g_k y$ and A_n is F_A -invariant, $g_k y \in A_n$. Furthermore, since $g_k x \in A_n \cap g_k U_l$ and $g_k y \notin A_n \cap g_k U_l$, $g_k x \in B_\alpha$ while $g_k y \notin B_\alpha$. Hence $G\langle B_\alpha \rangle$ separates x and y , and thus so does $G\mathcal{I}$. Therefore \mathcal{I} is a generator. \square

Corollary 10.3. *Let X be a Polish G -space. If X is aperiodic, then it is 2-compressible modulo MEAGER.*

Proof. By [14, Theorem 13.1], X is compressible modulo MEAGER. Also, by the above theorem, X admits a 4-generator modulo MEAGER. Thus Proposition 3.17 implies that X is 2-compressible modulo MEAGER. \square

11. The nonexistence of non-meager weakly wandering sets

Throughout this section, let X be a Polish \mathbb{Z} -space and T be the homeomorphism corresponding to the action of $1 \in \mathbb{Z}$.

11.1. An arithmetic criterion

Observation 11.1. Let $A \subseteq X$ be weakly wandering with respect to $H \subseteq \mathbb{Z}$. Then A is weakly wandering with respect to

- (a) any subset of H ;
- (b) $r + H$, $\forall r \in \mathbb{Z}$;
- (c) $-H$.

Definition 11.2. Let $d \geq 1$ and $F = \{n_i\}_{i < k} \subseteq \mathbb{Z}$, where $n_0 < n_1 < \dots < n_{k-1}$ are increasing. F is called d -syndetic if $n_{i+1} - n_i \leq d$ for all $i < k - 1$. In this case we say that the length of F is $n_{k-1} - n_0$ and denote it by $\|F\|$.

Lemma 11.3. Let $d \geq 1$ and $F \subseteq \mathbb{Z}$ be a d -syndetic set. For any $H \subseteq \mathbb{Z}$, if $|H| = d + 1$ and $\max(H) - \min(H) < \|F\| + d$, then F is not weakly wandering with respect to H (viewing \mathbb{Z} as a \mathbb{Z} -space).

Proof. Using (b) and (c) of Observation 11.1, we may assume that H is a set of non-negative numbers containing 0. Let $F = \{n_i\}_{i < k}$ with n_i increasing.

Claim. $\forall h \in H$, $(h + F) \cap [n_{k-1}, n_{k-1} + d) \neq \emptyset$.

Proof of Claim. Fix $h \in H$. Since $0 \leq h < \|F\| + d$,

$$n_0 + h < n_0 + (\|F\| + d) = n_{k-1} + d.$$

We prove that there is $0 \leq i \leq k - 1$ such that $n_i + h \in [n_{k-1}, n_{k-1} + d)$. Otherwise, because $n_{i+1} - n_i \leq d$, one can show by induction on i that $n_i + h < n_{k-1}$, $\forall i < k$, contradicting $n_{k-1} + h \geq n_{k-1}$. \square

Now $|H| = d + 1 > d = |\mathbb{Z} \cap [n_{k-1}, n_{k-1} + d)|$, so by the Pigeon Hole Principle there exists $h \neq h' \in H$ such that $(h + F) \cap (h' + F) \neq \emptyset$ and hence F is not weakly wandering with respect to H . \square

Definition 11.4. Let $d, l \geq 1$ and $A \subseteq X$. We say that A contains a d -syndetic set of length l if there exists $x \in X$ such that $\{n \in \mathbb{Z} : T^n(x) \in A\}$ contains a d -syndetic set of length $\geq l$. This is equivalent to $\bigcap_{n \in F} T^n(A) \neq \emptyset$, for some d -syndetic set $F \subseteq \mathbb{Z}$ of length $\geq l$.

For $A \subseteq X$, define $s_A : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$d \mapsto \sup\{l \in \mathbb{N} : A \text{ contains a } d\text{-syndetic set of length } l\}.$$

Also, for infinite $H \subseteq \mathbb{Z}$, define a width function $w_H : \mathbb{N} \rightarrow \mathbb{N}$ by

$$d \mapsto \min\{\max(H') - \min(H') : H' \subseteq H \wedge |H'| = d + 1\}.$$

Proposition 11.5. If $A \subseteq X$ is weakly wandering with respect to an infinite $H \subseteq \mathbb{Z}$ then $\forall d \in \mathbb{N}$, $s_A(d) + d \leq w_H(d)$.

Proof. Let H be an infinite subset of \mathbb{Z} and $A \subseteq X$, and assume that $s_A(d) + d > w_H(d)$ for some $d \in \mathbb{N}$. Thus $\exists x \in X$ such that $\{n \in \mathbb{Z} : T^n(x) \in A\}$ contains a d -syndetic set F of length l with $l + d > w_H(d)$ and $\exists H' \subseteq H$ such that $|H'| = d + 1$ and $\max(H') - \min(H') = w_H(d)$. By Lemma 11.3 applied to F and H' , F is not weakly wandering with respect to H' and hence neither is A . Thus A is not weakly wandering with respect to H . \square

Corollary 11.6. If $A \subseteq X$ contains arbitrarily long d -syndetic sets for some $d \geq 1$, then it is not weakly wandering.

Proof. If A and d are as in the hypothesis, then $s_A(d) = \infty$ and hence, by Proposition 11.5, A is not weakly wandering with respect to any infinite $H \subseteq \mathbb{Z}$. \square

Theorem 11.7. Let X be a Polish G -space. Suppose for every nonempty open $V \subseteq X$ there exists $d \geq 1$ such that V contains arbitrarily long d -syndetic sets, i.e. $\bigcap_{n \in F} T^n(V) \neq \emptyset$ for arbitrarily long d -syndetic sets $F \subseteq \mathbb{Z}$. Then X does not admit a non-meager Baire measurable weakly wandering subset.

Proof. Let A be a non-meager Baire measurable subset of X . By the Baire property, there exists a nonempty open $V \subseteq X$ such that A is comeager in V . By the hypothesis, there exists arbitrarily long d -syndetic sets $F \subseteq \mathbb{Z}$ such that $\bigcap_{n \in F} T^n(V) \neq \emptyset$. Since A is comeager in V and T is a homeomorphism, $\bigcap_{n \in F} T^n(A)$ is comeager in $\bigcap_{n \in F} T^n(V)$, and hence $\bigcap_{n \in F} T^n(A) \neq \emptyset$ for any F for which $\bigcap_{n \in F} T^n(V) \neq \emptyset$. Thus A also contains arbitrarily long d -syndetic sets and hence, by Corollary 11.6, A is not weakly wandering. \square

Corollary 11.8. *Let X be a Polish G -space. Suppose for every nonempty open $V \subseteq X$ there exists $d \geq 1$ such that $\{T^{nd}(V)\}_{n \in \mathbb{N}}$ has the finite intersection property. Then X does not admit a non-meager Baire measurable weakly wandering subset.*

Proof. Fix nonempty open $V \subseteq X$ and let $d \geq 1$ such that $\{T^{nd}(V)\}_{n \in \mathbb{N}}$ has the finite intersection property. Then for every N , $F = \{kd : k \leq N\}$ is a d -syndetic set of length Nd and $\bigcap_{n \in F} T^n(V) \neq \emptyset$. Thus Theorem 11.7 applies. \square

11.2. A negative answer to the Eigen–Hajian–Nadkarni question

Lemma 11.9. *Let X be a generically ergodic Polish G -space. If there is a non-meager Baire measurable locally weakly wandering subset then there is a non-meager Baire measurable weakly wandering subset.*

Proof. Let A be a non-meager Baire measurable locally weakly wandering subset. By generic ergodicity, we may assume that $X = [A]_G$. Throwing away a meager set from A we can assume that A is G_δ . Then, by (d) of Proposition 7.4, there exists a $\sigma(\Sigma_1^1)$ -measurable (and hence Baire measurable) G -invariant travel guide $\gamma : A \rightarrow G^\mathbb{N}$. By generic ergodicity, γ must be constant on a comeager set, i.e. there is $(g_n)_{n \in \mathbb{N}} \in G^\mathbb{N}$ such that $Y := \gamma^{-1}((g_n)_{n \in \mathbb{N}})$ is comeager. But then $W := A \cap Y$ is non-meager and is weakly wandering with respect to $\{g_n\}_{n \in \mathbb{N}}$. \square

Let $X = \{\alpha \in 2^\mathbb{N} : \alpha \text{ has infinitely many 0-s and 1-s}\}$ and T be the odometer transformation on X . We will refer to this \mathbb{Z} -space as the odometer space.

Corollary 11.10. *The odometer space does not admit a non-meager Baire measurable locally weakly wandering subset.*

Proof. Let $\{U_s\}_{s \in 2^{<\mathbb{N}}}$ be the standard basis. Then for any $s \in 2^{<\mathbb{N}}$, $T^d(U_s) = U_s$, where $d = 2^{|s|}$. Thus $\{T^{nd}(U_s)\}_{n \in \mathbb{N}}$ has the finite intersection property, in fact $\bigcap_{n \in \mathbb{N}} T^{nd}(U_s) = U_s$. Hence, we are done by Corollary 11.8 and Lemma 11.9. \square

The following corollary shows the failure of the analogue of the Hajian–Kakutani–Itô theorem in the context of Baire category as well as gives a negative answer to Question 1.22.

Corollary 11.11. *There exists a generically ergodic Polish \mathbb{Z} -space Y (namely an invariant dense G_δ subset of the odometer space) with the following properties:*

- (i) *there does not exist an invariant Borel probability measure on Y ;*
- (ii) *there does not exist a non-meager Baire measurable locally weakly wandering set;*

(iii) *there does not exist a Baire measurable countably generated partition of Y into invariant sets, each of which admits a Baire measurable weakly wandering complete section.*

Proof. By the Kechris–Miller theorem (see [Theorem 1.20](#)), there exists an invariant dense G_δ subset Y of the odometer space that does not admit an invariant Borel probability measure. Now (ii) is asserted by [Corollary 11.10](#). By generic ergodicity of Y , for any Baire measurable countably generated partition of Y into invariant sets, one of the pieces of the partition has to be comeager. But then that piece does not admit a Baire measurable weakly wandering complete section since otherwise it would be non-meager, contradicting (ii). \square

Acknowledgments

First and foremost, I thank my advisor Alexander Kechris for his help, support and encouragement, and for suggesting the problems and guiding me throughout the research. Many thanks to Slawek Solecki for his positive feedback and valuable remarks, and especially for pointing out a nice way of thinking about the notion of i -equidecomposability. I also thank Benjamin Weiss, Mahendra Nadkarni and Shashi Srivastava for their positive feedback, useful remarks and corrections, as well as for pointing out that an instance of Π_1^1 -reflection in one of the proofs could be replaced with analytic separation. Many thanks to the UCLA and Caltech logic groups for running a series of seminars in which I presented my work. I also thank Patrick Allen, Ben Miller, Yiannis Moschovakis, Itay Neeman, Justin Palumbo, Todor Tsankov, and Robin Tucker-Drob for useful conversations and comments. Finally, I thank the referee for very helpful remarks, suggestions and corrections, which led to improvements in many places.

References

- [1] H. Becker, A.S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Note Series, vol. 232, Cambridge Univ. Press, 1996.
- [2] J.P. Burgess, A selection theorem for group actions, *Pacific J. Math.* 80 (1979) 333–336.
- [3] C.T. Conley, A.S. Kechris, B.D. Miller, Stationary probability measures and topological realizations, *Israel J. Math.* 198 (1) (2013) 333–345.
- [4] A.I. Danilenko, K.K. Park, Generators and Bernoullian factors for amenable actions and cocycles on their orbits, *Ergodic Theory Dynam. Systems* 22 (2002) 1715–1745.
- [5] T. Downarowicz, *Entropy in Dynamical Systems*, New Mathematical Monographs Series, vol. 18, Cambridge Univ. Press, 2011.
- [6] S. Eigen, A. Hajian, M. Nadkarni, Weakly wandering sets and compressibility in a descriptive setting, *Proc. Indian Acad. Sci.* 103 (3) (1993) 321–327.
- [7] R.H. Farrell, Representation of invariant measures, *Illinois J. Math.* 6 (1962) 447–467.
- [8] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, 2003.
- [9] E. Glasner, B. Weiss, Minimal actions of the group $S(\mathbb{Z})$ of permutations of the integers, *Geom. Funct. Anal.* 12 (2002) 964–988.
- [10] A.B. Hajian, Y. Itô, Weakly wandering sets and invariant measures for a group of transformations, *J. Math. Mech.* 18 (1969) 1203–1216.

- [11] A.B. Hajian, S. Kakutani, Weakly wandering sets and invariant measures, *Trans. Amer. Math. Soc.* 110 (1964) 136–151.
- [12] S. Jackson, A.S. Kechris, A. Louveau, Countable Borel equivalence relations, *J. Math. Log.* 2 (1) (2002) 1–80.
- [13] A.S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, 1995.
- [14] A.S. Kechris, B. Miller, *Topics in Orbit Equivalence*, Lecture Notes in Math., vol. 1852, 2004.
- [15] W. Krieger, On entropy and generators of measure-preserving transformations, *Trans. Amer. Math. Soc.* 149 (1970) 453–464.
- [16] U. Krengel, Transformations without finite invariant measure have finite strong generators, in: *First Midwest Conference, Ergodic Theory and Probability*, in: Springer Lecture Notes, vol. 160, 1970, pp. 133–157.
- [17] A.J. Kuntz, Groups of transformations without finite invariant measures have strong generators of size 2, *Ann. Probab.* 2 (1) (1974) 143–146.
- [18] B.D. Miller, Full groups, classification, and equivalence relations, PhD thesis, University of California at Los Angeles, 2004.
- [19] M.E. Munroe, *Introduction to Measure and Integration*, Addison–Wesley, 1953.
- [20] M.G. Nadkarni, On the existence of a finite invariant measure, *Proc. Indian Acad. Sci. Math. Sci.* 100 (1991) 203–220.
- [21] D. Rudolph, *Fundamentals of Measurable Dynamics*, Oxford Univ. Press, 1990.
- [22] V.S. Varadarajan, Groups of automorphisms of Borel spaces, *Trans. Amer. Math. Soc.* 109 (1963) 191–220.
- [23] S. Wagon, *The Banach–Tarski Paradox*, Cambridge Univ. Press, 1993.
- [24] B. Weiss, Countable generators in dynamics-universal minimal models, in: *Measure and Measurable Dynamics*, in: *Contemp. Math.*, vol. 94, 1987, pp. 321–326.