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The Categories of Graphs

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THE CATEGORIES OF GRAPHS

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In traditional studies of graph theory, the graphs allow only one edge to be incident to any two vertices, not necessarily distinct, and the graph morphisms must map edges to edges and vertices to vertices while preserving incidence. We refer to these restricted morphisms as *strict morphisms*. We relax the conditions on the graphs by allowing any number of edges to be incident to any two vertices, as well as relaxing the condition on graph morphisms by allowing edges to be mapped to vertices, provided that incidence is still preserved. We call the broader category of these graphs and these morphisms the Category of Conceptual Graphs and Graph Morphisms, denoted **Grphs**. We then define four other concrete categories of graphs created by combinations of restrictions of the graph morphisms as well as restrictions on the allowed graphs.

We determine the categorial structure of these six categories of graphs by characterizing common categorially defined structures and properties and by characterizing six special types of monomorphisms, and dually six special types of epimorphisms. We also establish the Fundamental Morphism Theorem in two of the categories of graphs.

We then provide an Elementary Theory for five categories of graphs, producing a list of first-order axioms that, when taken with the higher-order axiom of the existence of small products and coproducts, characterizes these five categories of graphs. We also provide a result toward Hedetniemi's conjecture that arose from the study of the categories of graphs.

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Notation

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Chapter 1

Introduction - Concrete Categories of Graphs and Their Elementary Theories

Often the study of morphisms of any mathematical object starts with the study of automorphisms. In graph theory, this study produced representation theorems for groups as automorphism groups of graphs. The study of automorphisms also produced characterizations of graphs (e.g. vertex-transitive graphs and distance-transitive graphs). More recently, finding restrictions of the automorphism set for the graph by considering automorphisms that fix vertex colors or by considering automorphisms that fix certain sets of vertices produced useful new graph parameters (the distinguishing number [2] and fixing number [10] for a graph).

The study of morphisms then delves into the study of endomorphisms and finally homomorphisms. In graph theory, a certain class of graph homomorphisms generalizes vertex-coloring, and is now being widely studied. In 2004, a textbook was published about these graph homomorphisms [14].

The most common category considered in (undirected) graph theory is a category where graphs are defined as having at most one edge incident to any two vertices and at most one loop incident to any vertex. We will refer to these graphs as *simple graphs*. The morphisms are usually described as a pair of functions between the vertex sets and edge sets that respect edge incidence. We refer to these restricted morphisms as *strict morphisms*. We call this category the Category of Simple Graphs with Strict Morphisms, denoted **SiStGrphs**.

We will relax the conditions on the graphs by allowing any number of edges to be incident to any two vertices (referred to as *conceptual graphs*), as well as relaxing the condition on graph morphisms by allowing edges to be mapped to vertices, provided that incidence is still preserved. We call the broader category of these graphs and these morphisms the Category of Conceptual Graphs with Graph Morphisms, denoted **Grphs**.

We also consider the restriction on a graph where no edge is allowed to be incident to a single vertex, called *loopless*. Using the restrictions of strict, simple, and loopless, we define four more categories of graphs (for details see Section 2.2).

Category	Full Category Name
Grphs	The Category of Conceptual Graphs with Graph Morphisms
SiGrphs	The Category of Simple Graphs with Graph Morphisms
SiLlGrphs	The Category of Simple Loopless Graphs with Graph Morphisms
StGrphs	The Category of Conceptual Graphs with Strict Morphisms
SiStGrphs	The Category of Simple Graphs with Strict Morphisms
SiLlStGrphs	The Category of Simple Loopless Graphs with Strict Morphisms

We provide an inclusion diagram of these six categories of graphs, with a seventh category of graphs, **Sets** (The Category of Sets and Functions), considered as a graph category consisting of "empty edge graphs" (Figure 1.1). We note that these categories are concrete: the objects are sets with structure and the morphisms are structure preserving functions (see Definition 2.2.1).

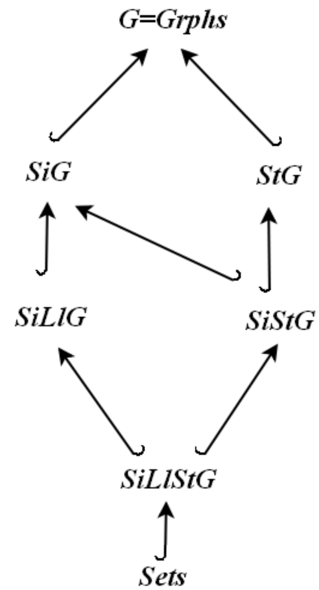


Figure 1.1: An inclusion diagram of the Categories of Graphs

In Section 2.3, we investigate these six concrete categories of graphs and determine the categorial structure by finding the concrete existence of abstractly defined categorial structures and properties or finding counterexamples to their existence (throughout we use the term categorial instead of categorical as categorical is used in model theory to denote a theory with a unique model up to isomorphism, see [11]).

In Subsections 2.3.1 and 2.3.2 we survey the known results about Topos-type constructions (limits, colimits, exponentiation with evaluation, and subobject classifier) and Set-type constructions and properties (natural number object, the axiom of choice, and the number of values in the subobject classifier) and characterize these constructions and properties in the six concrete categories of graphs. Then, in Subsection 2.3.3, we characterize epimorphisms and monomorphisms, noting a difference between surjection and epimorphism in **SiLlGrphs**, **SiStGrphs**, and **SiLlStGrphs** (see Proposition 2.3.14). Last, in Subsection 2.3.4, 2.3.5, and 2.3.6, we characterize free objects, projective objects, generators, and their duals. These characterizations emphasize the variety of categorial structure in the six concrete categories of graphs. We finish Chapter 2 by investigating adjoint functor relationships between the six categories of graphs where we characterize all adjoints to the inclusion functors (see Proposition 2.4.6).

In Chapter 3, we consider the structure of morphisms in the six categories. We find that the (Strong) Fundamental Morphism Theorem, a generalization of the Noether Isomorphism Theorems from Abstract Algebra, holds in **Grphs** and **StGrphs** (see Theorem 3.2.3), but fails to hold in the other four categories of graphs (see the counterexample in Section 3.3, Figure 3.1).

In Chapter 3 we also characterize the relationships between six special types of monomorphisms (split equalizer, coretract, effective monomorphism, regular monomorphism, and extremal monomorphism) and, dually, six special types of epimorphisms (split coequalizer, retract, effective epimorphism, regular epimorphism, and extremal epimorphism) in Theorem 3.6.1 and Theorem 3.6.2 for **Grphs**, **StGrphs**, and **SiGrphs**, Theorem 3.7.4 and Theorem 3.7.5 for **SiLlGrphs** and **SiStGrphs**, and Theorem 3.8.4 and Theorem 3.8.5 for **SiLlStGrphs**.

One asks when given a familiar system of sets with structure and their structure preserving functions if there is an axiomatic system that defines this system. In the main result of thesis, provided in Chapter 4, we answer this question for the categories of graphs giving a characterization of five categories of graphs and their morphisms. We follow the lead and spirit of Lawvere's groundbreaking categorial characterization of the Category of Sets and Functions [16] and Schlomiuk's characterization of the Category of Topological Spaces and Continuous Functions [23].

In both characterizations of the Category of Sets and Functions and the Category of Topological Spaces and Continuous Functions, a list of elementary (or first order) axioms are provided so that when combined with a second order axiom (there exist "small" products and coproducts) a functor equivalence between the axiomatically defined category and the concrete category is formed. We provide such an elementary theory for **Grphs**, **SiGrphs**, **SiLlGrphs**, **StGrphs**, and **SiStGrphs**.

In D. Schlomiuk's Elementary Theory for the Category of Topological Spaces and Continuous Functions, there are three steps in the axiomatization. The first step is to axiomatize a full subcategory of sets (called "discrete spaces") that satisfies Lawvere's Elementary Theory of the Category of Sets and Functions. This step ends with a theorem schema that states that any theorem valid in Lawvere's Elementary Theory of the Category of Sets and Functions holds for discrete spaces. The second step is to provide axioms that find the topological structure of objects in the category. The final step is to provide axioms that ensure there is "enough" structure, in the sense that any topological space is represented unto isomorphism.

We follow this method while giving a "simultaneous" axiomatization of the five categories. We give twelve common axioms to all five categories which compose the first two steps of D. Schlomiuk's method. We then add two to four distinguishing axioms in the third step of D. Schlomiuk's method to distinguish one of the five categories.

The culmination of the elementary theory is provided by a metatheorem for each category, creating a functor equivalence between the concrete category and the axiomatically described category which equates the categorial theory of the concrete category (over von Neumann-Bernays-Gödel set theory) with the theory of the axiomatically described category (which is still over von Neumann-Bernays-Gödel set theory, but could be considered over F.W. Lawvere's Theory of Abstract Categories [17]). This is accomplished for **Grphs** by Metatheorem 2, for **SiGrphs** by Metatheorem 4, for **SiLlGrphs** by Metatheorem 6, for **StGrphs** by Metatheorem 8, and for **SiStGrphs** by Metatheorem 10.

In Chapter 5 we turn to applications of the study of the categories of graphs. We prove a new special case of a graph coloring conjecture due to Hedetniemi in 1966 [13] using a new approach that was found in studying pullbacks. Hedetniemi conjectured that for graphs with finite chromatic number, the chromatic number of the product of the two graphs is equal to the minimum chromatic number of the factors. In Theorem 5.1.5, we establish this to be true if either graph contains a complete subgraph on a number of vertices equal to that minimum chromatic number of the two factors.

We note that if a result is not cited in the statement of the result or in the paragraph directly preceding the result, the result is new.

Chapter 2

The Concrete Categories of Graphs

2.1 Conceptual Graphs and Their Morphisms

In our graphs, we want to start out with as great a generality as possible and add restrictions later. This means we want to allow graphs to have multiple edges between any two vertices and multiple loops at any vertex. We will define our graphs in the style of Bondy and Murty [4], namely, graphs are sets of two kinds of parts: "edges" and "vertices" together with an "incidence" function. We call our graphs conceptual graphs in the sense of F.W. Lawvere's Conceptual Mathematics [18].

Definition 2.1.1. *A* conceptual graph G *consists of*
$G = \langle P(G), V(G); \partial_G : P(G) \to V(G) \veebar V(G), \iota_G : V(G) \hookrightarrow P(G) \rangle$ *where* $P(G)$ *is the set of parts of* G*,* $V(G)$ *is the set of vertices of* G*,* $V(G) \veebar V(G)$ *is the set of unordered pairs of vertices of* G*,* ∂_G *is the incidence map from the set of parts to the unordered pairs of vertices,* ι_G *is the inclusion map of the vertex set into the part set, and for* $\underline{\Delta} : V(G) \to V(G) \veebar V(G)$*, the unordered diagonal map,* $\partial_G \iota_G = \underline{\Delta}$*.*

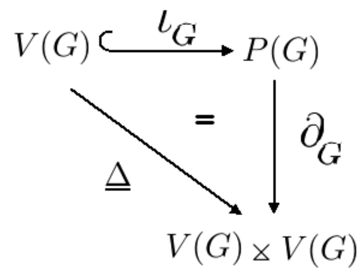


Figure 2.1: Incidence Mappings for Vertices

We define the set of edges of a graph, G, to be $E(G) = P(G)\backslash\iota_G(V(G))$. Henceforth, we will frequently abbreviate conceptual graph to graph. Furthermore, in our study here, we have no need to restrict our edge sets and vertex sets of our graphs to be finite sets.

In [4] a graph does not have the inclusion map, ι, but such a map will be critical when defining a graph homomorphism. In this way, we can think of the vertex "part" of the graph and the edge "part" of the graph in the same "part" set. We do allow $G = \emptyset$, i.e. $P(G) = \emptyset$, the empty graph, to be considered a graph. However, since ∂_G is required to be a function, if $V(G) = \emptyset$ then $P(G) = \emptyset$. We also allow $V(G) \neq \emptyset$ and $E(G) = \emptyset$ ("no edges"), i.e. $P(G) = V(G)$.

We now note the following. First, we naturally use the topologist's "boundary" symbol for incidence. Second, an unordered pair in $V(G)\underline{\times}V(G)$ is denoted u_v or (u_v), for vertices $u, v \in V(G)$. Thus the natural unordered diagonal map $\underline{\Delta} : V(G) \to V(G)\underline{\times}V(G)$ is given by $\underline{\Delta}(v) = v\_v$ or (v_v). Finally, we have chosen to consider our vertex set and edge set to be combined into a "part" set. Thus as an abstract data structure our graphs are a pair of sets: a set of parts with a distinguished subset called "vertices". This is done to make the description of morphisms more natural, i.e. functions between the "over" sets (of parts) that takes the distinguished subset to the other distinguished subset. This is what topologists do in the Category of Topological Pairs of Spaces: for example, an object (X, A) is a topological space X with a subspace A and a morphism $f : (X, A) \to (Y, B)$ is a continuous function from the topological space X to the topological space Y with $f[A] \subseteq f[B]$.

We now define our morphisms for conceptual graphs.

Definition 2.1.2. *$f : G \to H$ is a* graph (homo)morphism *of conceptual graphs from G to H if f is a function $f_P : P(G) \to P(H)$ and $f_V = f_P|_{V(G)} : V(G) \to V(H)$ that preserves incidence, i.e. $\partial_H(f_P(e)) = (f_V(x)\_f_V(y))$ whenever $\partial_G(e) = (x\_y)$, for all $e \in P(G)$ and some $x, y \in V(G)$, or in terms of function composition, $f_P \iota_G = \iota_H f_V$ and $\partial_H f_P = (f_V \veebar f_V)\partial_G$.*

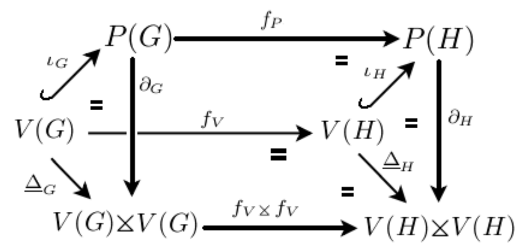


Figure 2.2: The Graph Morphism

This definition allows a graph homomorphism to map an edge to a vertex as long as the incidence of the edges are preserved. As an edge, $e \in E(G)$, can be mapped to the part set of the co-domain graph, H, so that it is the image of a vertex, i.e. $f(e) = \iota_H(v)$ for some $v \in V(H)$.

We will now define some specialized classes of graphs and a specialized graph morphism. Our first restriction is a common restriction in graph theory. The set of graphs is restricted to allow only one edge between any two vertices (see [14]), and at most one edge between a vertex and itself (a loop). We call these graphs *simple graphs* and define them in terms of conceptual graphs.

Definition 2.1.3. *A* simple *graph G is a conceptual graph such that for all $u, v \in V(G)$ with $u \neq v$, there is at most one $e \in P(G)$ such that $\partial_G(e) = (u\_v)$, and for all $w \in V(G)$ there is at most one $f \in E(G) = P(G)\backslash\iota_G(V(G))$ such that $\partial_G(f) = (w\_w)$ (where (u_v) is the unordered pair of vertices u and v).*

Another common restriction is to not allow loops at all. This restriction is often required when discussing vertex coloring. We call these graphs *loopless graphs.*

Definition 2.1.4. *A* loopless *graph G is a conceptual graph such that for all vertices $u \in V(G)$ there is no edge $e \in E(G) = P(G) \backslash \iota_G(V(G))$ such that $\partial_G(e) = (u\_u)$.*

Thus, a graph is simple and loopless if and only if the incidence map is injective.

This is not the usual notion of a "simple" graph often common in graph theory; that notion is the simple and loopless graph by our definition. This is a departure from standard nomenclature, but it fits our categorial discussion better.

We now define the most common notion for a graph morphism in literature, we call it a *strict* morphism because it always takes an edge part to a strict edge part (and not just a part, e.g. a vertex). The following definition is a modified form of the definition presented in [14] to apply to conceptual graphs.

Definition 2.1.5. *Let G and H be conceptual graphs. A* strict graph homomorphism *(or* strict morphism*) $f : G \to H$ is a graph morphism such that the strict edge condition holds: for all edges $e \in E(G)$, $f_P(e) \in E(H)$, i.e. the image under the strict morphism f of an edge is again an edge.*

The condition, $\partial_H(f_P(e)) = (f_V(x)\_f_V(y))$ whenever $\partial_G(e) = (x\_y)$, assures that the incidence of the edges in G is preserved in H under f. Note that the above definition also requires that vertices be mapped to vertices and edges be mapped (strictly) to edges. However, sometimes it may be beneficial to allow edges to be mapped to vertices. Such a morphism would allow a graph to naturally map to the contraction or quotient graph obtained by the contraction of an edge, but this could not be a strict morphism.

We also note that for our figures with graphs, we provide "pictures" (with picture frames) for the graphs. This helps to distinguish the graphs from the morphisms, especially in the case of graphs with multiple components. It also emphasizes that we are often choosing representative graphs from an isomorphism class of graphs. Now that we have defined our graphs and graph homomorphisms, we are ready to discuss the various Categories of Graphs.

2.2 The Categories of Graphs

Definition 2.2.1. [19] ***C*** *is a concrete category if there exists a faithful functor* $|-|: \boldsymbol{C} \rightsquigarrow \boldsymbol{Sets}$.

We will now define six concrete categories of graphs using the various restrictions of the previous section. We do not include all combinations of restrictions, but instead focus on the combinations of restrictions often seen in literature.

Definition 2.2.2. *The* Category of Conceptual Graphs and Graph Morphisms*,* ***Grphs****, is a (concrete) category where the objects are conceptual graphs and the morphisms are graph morphisms.*

Keith Kim Williams [27] proved that the axioms of a category are satisfied by this definition.

Proposition 2.2.3. [27] ***Grphs*** *is a category.*

This category, **Grphs**, we will think of as the big "mother" category of graphs. We now define five other commonly studied concrete subcategories of **Grphs**.

Definition 2.2.4. *The* Category of Simple Graphs with Graph Morphisms*,* ***SiGrphs****, is the (concrete) category where the objects are simple graphs, and the morphisms are conceptual graph morphisms.*

Definition 2.2.5. *The* Category of Simple Loopless Graphs with Graph Morphisms*,* ***SiLl-Grphs****, is the (concrete) category where the objects are simple graphs without loops, and the morphisms are conceptual graph morphisms.*

Definition 2.2.6. *The* Category of Conceptual Graphs with Strict Morphisms*,* ***StGrphs****, is the (concrete) category where the objects are conceptual graphs, and the morphisms are strict graph morphisms.*

Definition 2.2.7. *The* Category of Simple Graphs with Strict Morphisms*,* ***SiStGrphs****, is the (concrete) category where the objects are simple graphs and the morphisms are strict graph morphisms.*

This last category we defined is most often referred to as the "category of graphs" and is the main category of graphs discussed in [8, 14], namely, graphs with at most one edge between vertices, at most one loop at a vertex, and all the morphisms are *strict* (i.e. take and edge or loop *strictly* to an edge or loop).

Definition 2.2.8. *The* Category of Simple Loopless Graphs with Strict Morphisms*,* ***SiLlSt-Grphs****, is the (concrete) category where the objects are simple graphs without loops, and the morphisms are strict graph morphisms.*

As the composition of strict morphisms are strict morphisms, and the identity morphism is a strict morphism, these are in fact categories. We now have a containment picture of our six different (concrete) categories of graphs (Figure 2.3), with our mother category at the top. At the bottom, we have also included as another graph category the category of sets (and functions), where a set is considered as a graph with no edges (i.e. for a set X, $V(X) = P(X) = X$, $\partial_X = \underline{\Delta}_X$, and $\iota_X = 1_X$) and any function is a (strict) morphism of such graphs.

2.3 Constructions in the Concrete Categories of Graphs

In an abstract category there are objects and morphisms but nothing is known about the internal structure of the objects or the morphisms. Often in an abstract category objects are thought of as "dots" and morphisms as "arrows" between the dots. This is done to emphasize that in an arbitrary abstract category nothing is known about the structure or properties of the objects and morphisms other than the information you can get by looking at the dots

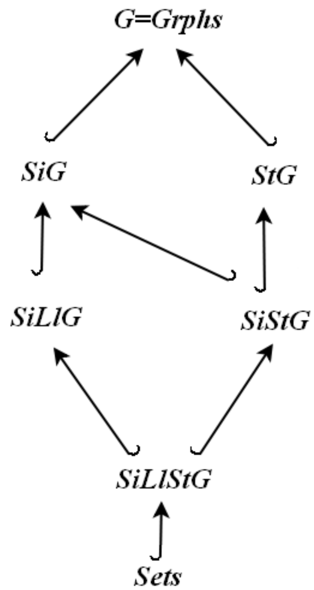


Figure 2.3: The Categories of Graphs

and arrows. Properties in an abstract category can be stated only in terms of objects and morphisms.

In this section, we will investigate common categorial constructions that will help define the properties of the six categories of graphs.

2.3.1 Topos-type Constructions

We first consider the constructions that define a topos.

- **(T1)** Limits (and Colimits).
- **(T2)** Exponentiation with Evaluation.
- **(T3)** A subobject classifier.

Each of these categorial properties are defined abstractly [11]. That is to say, in the definitions only objects and morphisms will be used, not the structure of the objects.

We start with **Grphs**. It has already been shown that **Grphs** contains the constructions for (T1) [27] and (T3) [21], and that **Grphs** fails to have the (T2) construction [5]. We will provide the constructions for (T1) and (T3) for completeness and provide an alternate proof for the failure of the existence of (T2) that has a combinatorial flavor, shows that an exponentiation object can exist, and shows exponentiation with evaluation fails due to an evaluation morphism that fails to satisfy the universal mapping property (as opposed to [5] who show the failure of a necessary adjoint relationship).

Proposition 2.3.1. ***Grphs*** *contains the constructions (T1) and (T3)* [21]*, but fails to have the construction (T2)* [5]*.*

Proof. **Construction (T1) ("limits") exists in Grphs**: It is sufficient to show the existence of all limits by providing the constructions of a terminal object, products, equalizers, and their duals [19]. It is easily shown that the one vertex graph K_1 (the classical "complete graph on one vertex") is the terminal object, which we will denote $\hat{1}$, and the empty vertex set (and edge set) graph $\emptyset$ is the initial object, which we will denote $\hat{0}$.

For products, given two graphs A and B in **Grphs**, the *product*, $A \times B$, is defined by $V(A \times B) = V(A) \times V(B)$ and for $e \in P(A)$ with $\partial_A(e) = (a_1\_a_2)$ and $f \in P(B)$ with $\partial_B(f) = (b_1\_b_2)$ there is an element (e, f) in $P(A \times B)$ with $\partial_{A\times B}((e, f)) = ((a_1, b_1)\_(a_2, b_2))$ and if $a_1 \neq a_2$ and $b_1 \neq b_2$, there is another element $\overline{(e, f)} \in P(A \times B)$ with $\partial_{A\times B}(\overline{(e, f)}) = ((a_1, b_2)\_(a_2, b_1))$ that has the same projections as (e, f).

The coproduct of two graphs in **Grphs** is the disjoint union of the two graphs, and the *equalizer*, $q = eq(f, g)$, of two morphism $f, g : A \to B$ is the inclusion of the subgraph Eq of A defined by $P(Eq) = \{a \in P(A)|f(a) = g(a)$ and if $\partial_A(a) = (a_1\_a_2)$ then $f(a_1) = g(a_1)$ and $f(a_2) = g(a_2)\}$, $V(Eq) = \{a \in V(A)|f_V(a) = g_V(a)\}$, $\iota_{Eq} = \iota_A|_{V(Eq)}$, and $\partial_{Eq} = \partial_A|_{P(Eq)}$.

Given two morphisms $f, g : A \to B$. The *coequalizer*, *coeq*(f, g), of two morphism $f, g : A \to B$ is the natural quotient morphism from B to $Coeq$ defined by $P(Coeq) = P(B)/\sim$ where $\sim$ is the equivalence relation defined by $a \sim b$ if there is a sequence $a_0, a_1, \ldots, a_n \in P(A)$ such that $a = f(a_0), g(a_0) = f(a_1), g(a_1) = f(a_2), \ldots, g(a_{n-1}) = f(a_n)$ and $b = f(a_n)$ or $b = g(a_n)$.

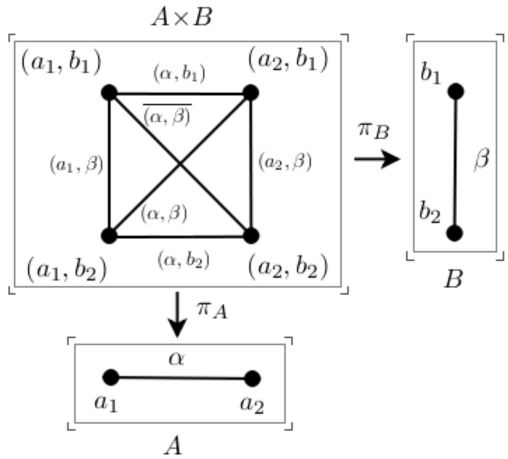


Figure 2.4: An example of the product in **Grphs** with pictures.

Construction (T2) ("exponentiation with evaluation") fails to exist in Grphs: By way of contradiction let us suppose that categorial exponentiation with evaluation (by the universal mapping property of exponentiation with evaluation) exists in **Grphs**. Then **Grphs** has a terminal object, products and exponentiation with evaluation. Hence there is an adjoint functor relationship $hom_{\mathbf{Grphs}}(X \times A, B)$ and $hom_{\mathbf{Grphs}}(X, B^A)$ for all graphs A, B and X. Hence there is a bijection between the set of morphisms $X \times A \to B$ and the set of morphisms $X \to B^A$.

We construct the counterexample to the existence of exponentiation and evaluation in **Grphs** in two steps. First, we use the adjoint functor relationship to completely determine (by a "brute force" count) the vertices, edges, and incidence of B^A as a graph where $B = K_1^\ell$, the graph with a single vertex with a loop on the vertex, and $A = K_2$, the classical "complete graph on two vertices". Second, we show that for all morphisms $B^A \times A \to B$ that satisfy the commuting morphism equations of the evaluation universal mapping property fails to have the uniqueness requirement for the universal mapping property for exponentiation with evaluation.

To begin, we will use the above mentioned adjoint bijection, but for various choices of "test" objects X. First, for $X = \hat{1}$ = terminal object = a single vertex graph, $X \times A \cong A$.

Any morphism from $X \times A$ to B must send both vertices to the single vertex in B, the edge may be sent to either the loop or to the vertex. So there are two maps here. Therefore, there must be two morphisms from $\hat{1}$ to B^A. Since $\hat{1}$ is just a single vertex, B^A must have exactly two vertices.

Second, suppose $X = K_1^\ell$ is a vertex with a single loop. Then $X \times A$ is a graph on two vertices, with a loop at each vertex, and two edges incident to the two distinct vertices.

Again, both the vertices of $X \times A$ must be sent to the single vertex of B. Now there are four edges in $X \times A$, each edge maybe sent to either the loop or to the vertex (independent of where the other edges are sent). So by the multiplication principle there are $2^4 = 16$ morphisms here. Therefore there must be exactly 16 morphisms from X to B^A. There are exactly two morphisms which send both the edge and the vertex of X to a single vertex (since we have already determined that there are only two vertices in B^A). Which leaves 14 more morphisms to account for. Since the vertex of X must be sent to a vertex, and the loop must be either sent to a loop or a vertex, we conclude that there are 14 loops distributed between the two vertices (we do not know how they are divided between the two, but we know that there are 14 of them).

Third, suppose $X = K_2$ is two vertices with a non-loop edge between them. Then $X \times A \cong K_4$

Again, all four vertices of $X \times A$ must be sent to the single vertex of B. The six edges of $X \times A$ can be sent to either the loop or to the vertex (independent of where the other ones are sent). So by the multiplication principle there are $2^6 = 64$ morphisms here.

Therefore there must be 64 morphisms from X to B^A. X has only one edge, it can either be sent to a vertex, a loop, or a non-loop edge. There are two ways to send the edge to a vertex (and this will force both its vertices to be sent to this vertex to preserve incidence). Since there are 14 loops, there are 14 ways to send the edge to a loop (and since incidence must be preserved both the vertices of X must be sent to the vertex incident on this loop, it is worth noting that we still don't know where these 14 loops are, but it doesn't matter counting these morphisms). Which leaves us with 48 morphisms to account for, which must send the edge of X to a non-loop edge of B^A. There are only two vertices in B^A so there is only place to send a non-loop edge. Also, each non-loop edge in B^A will give us two morphisms from X to that edge (once you decide which of the vertices to send one vertex of X to, the edge must be sent to the edge and the other vertex of X to the other vertex of B^A). Therefore there must be precisely 24 non-loop edges connecting the two vertices of B^A. We still do not know where the 14 loops are in B^A but we have a pretty good idea of what it must look like.

Now we will test what B^A must be by testing with one more X to determine the placement of the loops. So for the fourth test choice of X, suppose X is two vertices with one loop and one non-loop edge.

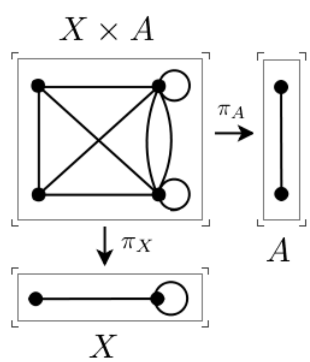


Figure 2.5: A picture of $X \times A$, for the fourth test choice of X.

Again, all four vertices of $X \times A$ must be sent to the single vertex in A, and each of the 9 edges can either be sent to the loop or to the vertex (independent of where the other edges go). So there are $2^9 = 512$ morphisms here.

Now we will count the number of morphisms $X \to B^A$ by considering the following six mutually exclusive types of morphisms whose union are all the morphisms: everything in X can be sent to a single vertex, everything in X but the loop can be sent to a vertex with the loop to a loop, the non-loop edge can be sent to a loop with everything else sent to a vertex, the non-loop edge can be sent to a non-loop edge with the loop sent to a vertex, the loop can be sent to a loop and the non-loop edge sent to a non-loop edge, or both the loop and the non-loop edge can be sent to loops.

There are two ways to send everything in X to a vertex since B^A only has two vertices (this was the first X we tested against). There are 14 ways to send the loop of X to a loop and everything else to a vertex since B^A has 14 loops (this was the second X we tested against). Likewise there are 14 ways to send the non-loop edge to a loop with everything else going to the incident vertex. As discussed before, there are 48 ways to send the non-loop edge to a non-loop edge and the loop to a vertex (this was the third X we tested against).

Now we will count the number of ways to send the loop of X to a loop in B^A and the non-loop edge of X to a non-loop edge of B^A. There are 14 choices of where to send the loop, and this choice determines where vertex incident on the loop is sent. After this choice is made, there will be 24 non-loop edges in B^A to send the non-loop edge of X to (note again that we don't know which vertex the loops are on, but it does not effect our count of this type of morphism). So by the multiplication principle there are $14 \times 24 = 336$ of this type of morphism.

We have now accounted for $2+14+14+48+336 = 414$ morphisms, which leaves $512-414 = 98$ morphisms to account for. The only other type of morphism is one which sends both the loop and the non-loop edge of X to loops in B^A. Suppose there are m loops on one vertex of B^A and n loops on the other. Then there is $m^2 + n^2 = 98$ morphisms which send both edges of X to a loop, and $m + n = 14$. Solving this system of equations yields the unique solution of $m = 7$ and $n = 7$. Hence the 14 loops are distributed evenly between the two vertices of B^A.

So we now have a complete description of what we will call the "exponential object" B^A, for the given A and given B. (This assumed that categorial exponentiation with evaluation exists).

We've determined that B^A is a graph with two vertices (which we will label u and v) 24 non-loop edges (which we will label e_i for $i = 1, \ldots, 24$), and 7 loops on each vertex (which we will label u_{ℓ_j} and v_{ℓ_j} for $j = 1, \ldots, 7$). It will also help us to label the graphs A and B. Label the vertices of $A \cong K_2$ as a_1 and a_2, and the edge as e_a. Label the vertex of B by b and the loop by ℓ_b.

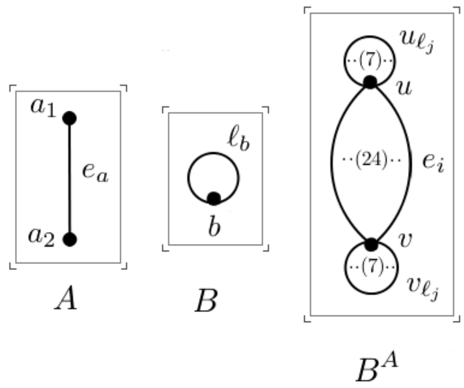


Figure 2.6: Pictures of the graphs for the counterexample to categorial "exponentiation with evaluation" in **Grphs**.

But even though this exponential object exists, we have yet to show that its evaluation satisfies the uniqueness feature of the universal mapping property for exponentiation with evaluation. So we investigate evaluation by again using test objects in the universal mapping property for exponentiation with evaluation, which states that there exists $ev : B^A \times A \to B$ such that for all X and $g : X \to B$, there is a unique $\overline{g} : X \to B^A$ such that $g = ev(\overline{g} \times 1_A)$.

For the first choice of test objects, let X be the single vertex graph K_1 (which we have denoted $\hat{1}$) with vertex x. Then as $X(= \hat{1})$ is the terminal object, $X \times A \cong A$. Thus there are two morphisms from $X \times A$ to B. Let $g_1 : X \times A \to B$ be the morphism which maps all of $P(X \times A)$ to the vertex b of B, and let $g_2 : X \times A \to B$ by the morphism that maps the edge of $X \times A$ to the loop ℓ_b of B.

Consider g_1, by the universal mapping property there is a unique $\overline{g} : X \to B^A$ such that $ev(\overline{g} \times 1_A) = g$. There are two morphisms from X to B^A, $\overline{g}(x) = u$ or $\overline{g}(x) = v$. If $\overline{g}(x) = u$, then $ev(\overline{g} \times 1_A((x, e_a))) = ev((u, e_a)) = g(x) = b$. If $\overline{g}(x) = v$, then $ev(\overline{g} \times 1_A((x, e_a))) = ev((v, e_a)) = g(x) = b$.

As $\overline{g}$ is unique, only one of the two above possibilities holds. Since there is an automorphism of $B^A \times A$ that exchanges (u, e_a) and (v, e_a) (exchange all labels of u and v and swap (e_i, e_a) with $\overline{(e_i, e_a)}$), without loss of generality we can choose $ev((v, e_a)) = b$. Then as $\overline{g}$ is unique, $ev((v, e_a)) \neq ev((u, e_a))$. Hence $ev((u, e_a)) = \ell_b$.

For the second choice of test objects, test with $X = A(= K_2)$ to achieve a contradiction.

We claim that $g((a_1, e_a)) = b$ if and only if $\overline{g}(a_1) = v$ and $g((a_1, e_a)) = \ell_b$ if and only if $\overline{g}(a_1) = u$. For, since $ev((v, e_a)) = b$ and $ev((u, e_a)) = \ell_b$, $g((a_1, e_a)) = ev(\overline{g} \times 1_A((a_1, e_a))) = ev((\overline{g}(a_1), e_a))$. Hence $g((a_1, e_a)) = b$ if and only if $\overline{g}(a_1) = v$ and $g((a_1, e_a)) = \ell_b$ if and only if $\overline{g}(a_1) = u$.

A similar argument shows $g((a_2, e_a)) = b$ if and only if $\overline{g}(a_2) = v$ and $g((a_2, e_a)) = \ell_b$ if and only if $\overline{g}(a_2) = u$.

Then for $g : A \times A \to B$ with $g((a_1, e_a)) = b$ and $g((a_2, e_a)) = \ell_b$, we have $\overline{g}(a_1) = v$ and $\overline{g}(a_2) = u$. Hence for such a g, as $\overline{g}$ must preserve incidence, $\overline{g}(e_a) = e_i$ for some $i = 1, \ldots, 24$. We now notice the following useful observation.

(1) If $\overline{g}(e_a) = e_i$ for some $i = 1, \ldots, 24$ then $\overline{g} \times 1_A((e_a, a_1)) = (e_i, a_1)$, $\overline{g} \times 1_A((e_a, a_2)) = (e_i, a_2)$, $\overline{g} \times 1_A((e_a, e_a)) = (e_i, e_a)$, and $\overline{g} \times 1_A(\overline{(e_a, e_a)}) = \overline{(e_i, e_a)}$.

For each $i = 1, \ldots, 24$ there are two choices of where to map each of (e_i, a_1), (e_i, a_2), (e_i, e_a), and $\overline{(e_i, e_a)}$ in a morphism from $B^A \times A \to B$ (either to b or ℓ_b). Thus for a fixed i, there are 16 possible ways to map the edges (e_i, a_1), (e_i, a_2), (e_i, e_a), and $\overline{(e_i, e_a)}$ to B. However, there are 24 such indicies. Thus by the pigeonhole principle,

(2) there exists $i, j \in \{1, \ldots, 24\}$ with $i \neq j$, $ev((e_i, a_1)) = ev((e_j, a_1))$, $ev((e_i, a_2)) = ev((e_j, a_2))$, $ev((e_i, e_a)) = ev((e_j, e_a))$, and $ev(\overline{(e_i, e_a)}) = ev(\overline{(e_j, e_a)})$.

So define a morphism $g : A \times A \to B$ by $g(x) = b$ for all vertices $b \in V(A \times A)$, $g((a_1, e_a)) = b$, $g((a_2, e_a)) = \ell_b$, $g((e_a, a_1)) = ev((e_j, a_1))$, $g((e_a, a_2)) = ev((e_j, a_2))$, $g((e_a, e_a)) = ev((e_j, e_a))$, and $g(\overline{(e_a, e_a)}) = ev(\overline{(e_j, e_a)})$ (incidence is trivially preserved). Then there is a unique $\overline{g} : A \to B^A$ such that $ev(\overline{g} \times 1_A) = g$.

However, by (1) $\overline{g}(a_1) = v$ and $\overline{g}(a_2) = u$, $\overline{g}(e_a) = e_j$ is such a morphism and by (2) $\overline{\overline{g}}(a_1) = v$, $\overline{\overline{g}}(a_2) = u$, and $\overline{\overline{g}}(e_a) = e_i$ is another. Hence no such unique morphism exists and (T2) does not exist in **Grphs**.

Construction (T3) ("subobject classifier") exists in Grphs: In **Grphs** the subobject classifier is the following graph:

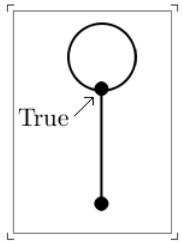


Figure 2.7: A picture of the subobject classifier Ω in **Grphs**

together with the canonical morphism $\top : \hat{1} \to \Omega$ from the terminal object $\hat{1}$ to the subobject classifier Ω, which maps the vertex of $\hat{1}$ to the vertex labeled "True" in the above picture. $\square$

We now to turn to **SiGrphs** and **SiLlGrphs**. These categories have rarely been studied.

Proposition 2.3.2. ***SiGrphs*** *has the constructions (T1) and (T3), but fails to have the construction (T2).*

Proof. **Construction (T1) ("limits") exists in SiGrphs**: The proof of existence products and coproducts in **SiGrphs** follows similarly to the proof of existence of products and coproducts in **Grphs** using same constructions, by identifying any multiple edges that occur as a single edge and any multiple loops that occur as a single loop.

Construction (T2) ("exponentiation with evaluation") fails to exist in SiGrphs: Suppose that exponentiation exists in **SiGrphs**. Then **SiGrphs** has a terminal object, products and exponentiation. Thus there is a standard adjoint functor relationship creating a bijection between the set of morphisms $X \times B \to A$ and the set of morphisms $X \to A^B$.

To construct our counterexample to the existence of exponentiation in **SiGrphs**, let both A be K_1^ℓ the graph with a single vertex with a loop, and B be K_2. To begin, we will let X be a single vertex (this is the multiplicative identity, $\hat{1}$, in **SiGrphs** and hence $X \times B = B$). As K_2 admits 2 morphisms to K_1^ℓ in **SiGrphs**, then A^B has 2 vertices, identified by X.

Now let X be K_1^ℓ. Then $X \times B$ is a graph with two vertices with a loop at each vertex and an edge between the two vertices. $X \times B$ admits 8 morphism to K_1^ℓ, as the edge and each loop can be mapped to either the vertex or the loop. Hence K_1^ℓ admits 8 morphisms to A^B. However, in **SiGrphs**, K_1^ℓ admits at most 4 morphisms to any graph on two vertices, 2 morphisms that map the loop to a vertex, and 2 that map the loop to another loop. Hence we have a contradiction and exponentiation and evaluation does not exist in **SiGrphs**.

Construction (T3) ("subobject classifier") exists in in SiGrphs: The subobject classifier for **SiGrphs** is the same as it is for **Grphs**. □

Proposition 2.3.3. ***SiLlGrphs*** *has the constructions (T1) and (T2) but fails to have the construction (T3).*

Proof. **Construction (T1) ("limits") exists in SiLlGrphs**: To show the existence of limits, we note that the terminal object, products, and equalizers are defined as in **SiGrphs**. For colimits, the initial object, and the coproduct are the same as in **SiGrphs** with the coequalizer being the construction given in **Grphs** with multiple edges identified as a single edge, and loops identified with the incident vertex.

Construction (T2) ("exponentiation with evaluation") exists in SiLlGrphs: Given graphs G and H, define H^G by $V(H^G) = \hom_{\mathbf{SiLlGrphs}}(G, H)$, and $e \in P(H^G)$ with $\partial_{H^G}(e) = (f_1 \_ f_2)$ if for all $d \in P(G)$ with $\partial_G(d) = (d_1 \_ d_2)$, there exists $d' \in P(H)$ with $\partial_H(d') = (f_1(d_1) \_ f_2(d_2))$.

Then define $ev : H^G \times G \to H$ by $ev((f, v)) = f(v)$ for all vertices $(f, v) \in V(H^G \times G)$ and for $e \in P(H^G \times G)$ such that $\partial_{H^G \times G}(e) = ((f, v) \_ (g, u))$ define $ev(e) = d$ for $d \in P(H)$ with $\partial_H(d) = (f(v) \_ g(u))$. Such a d exists by construction of H^G, and by construction of H^G, ev is a graph morphism.

Now let X be a graph with morphism $g : X \times G \to H$. We must show there is a unique morphism $\overline{g} : X \to H^G$ such that $g = ev(\overline{g} \times 1_G)$.

Let $x \in V(X)$ and consider $\{x\} \times G := \{(x, v) | (x, v) \in V(X \times G) \text{ for some } v \in V(G)\} \subseteq V(X \times G)$. Then $g|_{\{x\} \times G}$ induces a function $f_x : V(G) \to V(H)$ defined by $f_x(v) = g((x, v))$.

We uniquely extend f_x to a morphism for if there exists $e \in P(G)$ with $\partial_G(e) = (u\_v)$, then there exists $(x, e) \in P(X \times G)$ with $\partial_{X \times G}((x, e)) = ((x, u)\_(x, v))$ and as g is a morphism $g(e) = d$ for some $d \in P(H)$ with $\partial_H(d) = (g(x, u)\_g(x, v))$ which uniquely defines $f_x(e) = d$ to preserve incidence. Then for $g = ev(\overline{g} \times 1_G)$ to hold, define $\overline{g}(x) = f_x$ and $\overline{g}$ is a vertex set function uniquely determined by g.

Now let $e \in P(X)$ with $\partial_X(e) = (x_1\_x_2)$. Consider $\{e\} \times G := \{d \in P(X \times G) | \partial_{X \times G}(d) = ((x_1, u)\_(x_2, v))$ for some $u, v \in V(G)\} \subseteq P(X \times G)$. Note that for a part $d \in \{e\} \times G$, $\partial_{X \times G}(d) = ((x_1, u)\_(x_2, v))$ for some $u, v \in V(G)$ implies there is a part $d' \in P(G)$ such that $\partial_G(d') = (u\_v)$.

For such a d, since g preserves incidence, $\partial_H(g(d)) = (g(x_1, u)\_g(x_2, v)) = (f_{x_1}(u)\_f_{x_2}(v))$. Then for $g = ev(\overline{g} \times 1_G)$ to hold, define $\overline{g}(e) = a$ where $\partial_{H^G}(a) = (f_{x_1}\_f_{x_2})$ which exists by definition of H^G, and is uniquely determined by g. Clearly $\overline{g}$ is a morphism in **SiLlGrphs** and is uniquely determined by g.

Construction (T3) ("subobject classifier") fails to exist in SiLlGrphs: Assume a subobject classifier, Ω, exists with morphism $\top : \hat{1} \to \Omega$. Consider K_2^c having vertices a and b with $!_{K_2^c} : K_2^c \to \hat{1}$ the unique morphism to the terminal object. Let $i : K_2^c \hookrightarrow K_2$ be inclusion where K_2 is K_2^c with edge e. Then there exists a unique $\chi_{K_2^c} : K_2 \to \Omega$ such that K_2^c is the pullback of $\top$ and $\chi_{K_2^c}$. Then $\top !_{K_2^c} = \chi_{K_2^c} i$.

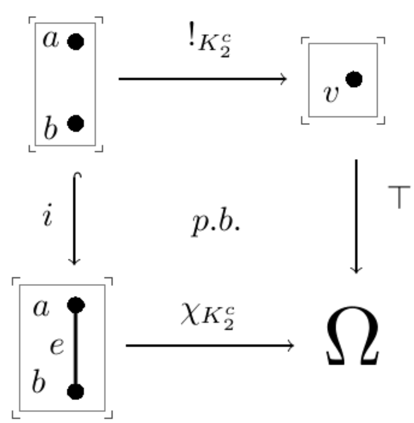


Figure 2.8: A picture for the counterexample to the existence of a subobject classifier in **SiLlGrphs**.

Since $!_{K_2^c}(a) = !_{K_2^c}(b) = v$ for v the vertex of $\hat{1}$ and, and since morphisms much map vertices to vertices, $\top(!_{K_2^c}(a)) = \top(!_{K_2^c}(b)) = \top(v)$. Since $\top !_{K_2^c} = \chi_{K_2^c} i$, $\chi_{K_2^c}(i(a)) = \chi_{K_2^c}(i(b)) = \top(v)$. Since graphs in **SiLlGrphs** are loopless and incidence must be preserved, $\chi_{K_2^c}(e) = \top(v)$.

Now consider the pullback of $\chi_{K_2^c}$ and $\top$. It is the vertex induced subgraph of $K_2 \times \hat{1}$ on $V(Pb) = \{(c,v) \in V(K_2 \times \hat{1}) | \chi_{K_2^c}(\pi_{K_2}((c,v))) = \top(\pi_{\hat{1}}((c,v)))\}$. However, since $K_2 \times \hat{1} \cong K_2$ and $\chi_{K_2^c}(\pi_{K_2^\ell}((a,v))) = \chi_{K_2^c}(a) = \top(v) = \chi_{K_2^c}(b) = \chi_{K_2^c}(\pi_{K_2}((b,v)))$, $V(Pb) = \{(a,v),(b,v)\}$ and $Pb \cong K_2$. This contradicts that K_2^c is the pullback of $\chi_{K_2^c}$ and $\top$. Hence no subobject classifier exists. □

We now focus on the three categories with strict morphisms. We note that **StGrphs** has already been shown to have the constructions (T1) and (T3) while failing to have the construction (T2) [21]. We provide the constructions for (T1) and (T3).

Proposition 2.3.4. [21] ***StGrphs*** *contains the constructions (T1) and (T3), but fails to have the construction (T2).*

Proof. The graph with a single vertex and a loop, K_1^ℓ, is the terminal object and the empty vertex set (and part set) graph $\emptyset$ is the initial object.

For products, we follow the construction of **Grphs** but delete all pairs (e,f) if exactly one of e or f is a vertex.

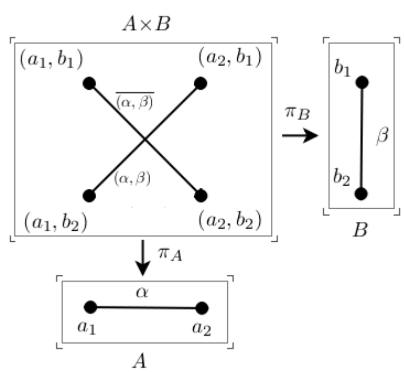


Figure 2.9: An example of the product in **StGrphs** with pictures.

The coproduct, equalizer and coequalizer in **StGrphs** is precisely the same as in **Grphs**. The subobject classifier in **StGrphs** is the following graph (Figure 2.10):

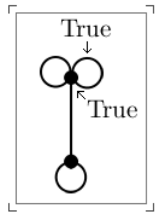


Figure 2.10: A picture of the subobject classifier Ω in **StGrphs**

together with the canonical strict morphism $\top : \hat{1} \to \Omega$ from the terminal object $\hat{1}$, which sends the vertex and the loop of $\hat{1}$ to the vertex and loop labeled "True" above. $\square$

We next turn to **SiStGrphs**. As **SiStGrphs** is the most common category of graphs studied in literature, constructions (T1) and (T2) are already known [8, 14], and the failure of **SiStGrphs** to have construction (T3) is also known [20].

Proposition 2.3.5. ***SiStGrphs*** *has constructions (T1) and (T2)* [8, 14] *but fails to have the construction (T3)* [20].

Last we consider **SiLlStGrphs**. It has been shown that **SiLlStGrphs** fails to have any of the Topos-type constructions [20].

Proposition 2.3.6. [20] ***SiLlStGrphs*** *fails to have the constructions of (T1), (T2) and (T3).*

We note that products, coproducts, equalizers, and an initial object exist in **SiLlStGrphs**, by the constructions in **SiStGrphs**.

2.3.2 Set-type Constructions and Properties

We now consider one construction and two properties whose existence combined with that with the topos constructions, defines **Sets** [7, 16, 25].

- **(S1)** A natural number object.
- **(S2)** The the Axiom of Choice.
- **(S3)** The subobject classifier is two-valued.

Like the topos constructions, each of these categorial properties are also defined abstractly [11]. We again start with **Grphs**.

Proposition 2.3.7. [21] ***Grphs*** *has the construction (S1) but fails to have the properties (S2), and (S3).*

In the next two propositions we show **SiGrphs** and **SiLlGrphs** have the same fate.

Proposition 2.3.8. ***SiGrphs*** *has the construction (S1) but fails to have the properties (S2) and (S3).*

Proof. **The construction (S1) ("natural number object") exists in SiGrphs**: The natural number object for **SiGrphs** is the same as it is for **Grphs**. Namely, the natural number object, N is the graph with no edges, and a countably infinite number of vertices labeled by the natural numbers, coupled with the initial morphism $\ulcorner 0 \urcorner : \hat{1} \to N$ from the terminal object $\hat{1}$ defined by mapping the single vertex of the terminal object to the vertex labeled 0, and successor function $\sigma : N \to N$ where given a vertex labeled n, $\sigma(n) = n + 1$.

Properties (S2) and (S3) ("choice" and "two-valued") fail for SiGrphs: Consider the graph morphism in Figure 2.11 where $f(a_1) = b_1$ and $f(a_2) = b_2$.

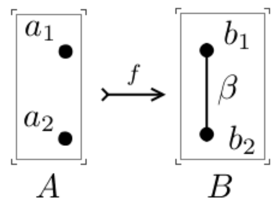


Figure 2.11: A picture for the counterexample to (S2) - "choice" in **SiGrphs**

For any $g : B \to A$, g must send the edge β to one of the vertices. Without loss of generality, assume $g(\beta) = a_1$, then we must have $g(b_1) = a_1$ and $g(b_2) = a_1$ to preserve incidence. But then $fgf \neq f$. So we have an example where there does not exist a $g : B \to A$ such that $fgf = f$.

As for "two-valued", by applying the definition of terminal object and coproduct we have that $\hat{1} + \hat{1}$ is a two vertex graph with no edges. But Ω has edges so it can not be isomorphic to $\hat{1} + \hat{1}$ and is therefore not two-valued. □

Proposition 2.3.9. ***SiLlGrphs*** *has the construction (S4) but fails to have the properties (S2) and (S3).*

Proof. **Construction (S1) ("natural number object") exists in SiLlGrphs**, **properties (S2) and (S3) ("choice" and "two-valued") fails for SiLlGrphs**: The natural number object of **SiLlGrphs** is the same as in **SiGrphs**, and the counterexample to choice in **SiGrphs** applies as well. Since **SiLlGrphs** does not have a subobject classifier, (S3) does not apply. □

We now turn to the categories with strict morphisms.

Proposition 2.3.10. [21] ***StGrphs*** *has the construction (S1) but fails to have properties (S2) and (S3).*

Proposition 2.3.11. ***SiStGrphs*** *has the construction (S1) but fails to have properties (S2) and (S3).*

Proof. **Construction (S1) ("natural number object") exists in SiStGrphs**: The natural number object in **SiStGrphs** is the same as it is in **StGrphs**. Namely the natural number object, N, is countably many vertices with loops labeled with the natural numbers, coupled with the initial morphism $\ulcorner 0 \urcorner : \hat{1} \to N$ defined by mapping the single vertex and loop of the terminal object to the vertex and loop labeled 0, and successor function $\sigma : N \to N$ where given a vertex with a loop labeled n, $\sigma(n) = n+1$. The natural number object works similarly as it did in **SiGrphs**.

Properties (S2) and (S3) ("choice" and "two-valued") fails for SiStGrphs: The counterexample in **SiGrphs** of choice applies here as well. Since **SiStGrphs** does not have a subobject classifier, (S3) does not apply. □

Proposition 2.3.12. ***SiLlStGrphs*** *fails to have construction (S1) and properties (S2) and (S3).*

Proof. As no terminal object exists in **SiLlStGrphs**, nor does a natural number object. Since no subobject classifier exists, (S3) does not apply. The same counterexample for choice in **SiGrphs** applies here. □

We provide a reference table for the Topos-type and Set-type constructions and properties (Table 2.1 - Y=Yes, N=No, **G**=**Grphs**).

Table 2.1: Topos-type and Set-type constructions/properties for categories of graphs.

	Sets	**SiLlStG**	**SiLlG**	**SiStG**	**SiG**	**StG**	**G**
(T1) Limits	Y	N	Y	Y	Y	Y	Y
(Colimits)	Y	N	Y	Y	Y	Y	Y
$\hat{1}$	Y	N	Y	Y	Y	Y	Y
$(\hat{0})$	Y	Y	Y	Y	Y	Y	Y
$\times$	Y	Y	Y	Y	Y	Y	Y
$(+)$	Y	Y	Y	Y	Y	Y	Y
Equalizer	Y	Y	Y	Y	Y	Y	Y
(Coequalizer)	Y	N	Y	Y	Y	Y	Y
(T2) Exp. with Eval.	Y	N	Y	Y	N	N	N
(T3) Subobj. Classifier	Y	N	N	N	Y	Y	Y
(S1) Nat. Num. Obj.	Y	N	Y	Y	Y	Y	Y
(S2) Choice	Y	N	N	N	N	N	N
(S3) 2-valued	Y	N	N	N	N	N	N

2.3.3 Characterizations for Epimorphisms and Monomorphisms

We begin our investigation of other properties of the concrete categories of graphs by first giving characterizations of epimorphisms and monomorphisms in the categories of graphs. These characterizations of epimorphisms and monomorphisms are known in **Grphs** [27].

Proposition 2.3.13. *A morphism in **Grphs**, **StGrphs**, and **SiGrphs** is an epimorphism if and only if it is a surjective function of the part sets, and a morphism in the above categories is a monomorphism if and only if it is an injective function of the part sets.*

Proof. As graphs are just part sets (with an incidence relation), and in concrete categories surjections are always epimorphisms and injections are always monomorphisms, we must only prove the converses.

So let $f : A \to B$ be an epimorphism in **Grphs**, and suppose f is not surjective. Then there exists $e \in P(B) \backslash Im(f)$.

First suppose $e \in V(B)$. Construct the graph C by appending a vertex e' to B such that e' is adjacent to every vertex e is adjacent to. By construction B is a subgraph of C.

Since $e \in P(B) \backslash Im(f)$, no edge incident to e is in the image of f. Now consider $i : B \to C$ the inclusion morphism and $g : B \to C$ defined by $g(u) = i(u)$ for all $u \in V(B) \backslash \{e\}$, $g(e) = e'$, $g(m) = i(m)$ for all edges m not incident to e, and for edge n incident to e, set $g(n)$ to be the corresponding edge incident to e'. This is clearly a morphism (actually it is strict). Then $if = gf$ but $i \neq g$, a contradiction to f being an epimorphism.

Now suppose e is an edge of B. Construct the graph C by appending an edge e' to B such that e' has the same incidence as e. Then by construction B is a subgraph of C.

Now consider $i : B \to C$ the inclusion morphism and $g : B \to C$ defined by $g(u) = i(u)$ for all $u \in P(B) \backslash \{e\}$ and $g(e) = e'$. As the incidence of e' is the same as e this is a morphism (it is actually strict). Then $if = gf$ but $i \neq g$, a contradiction to f being an epimorphism. Hence epimorphisms in **Grphs** are surjective functions of the corresponding edge sets. A similar proof applies or **StGrphs**.

For **SiGrphs**, a similar proof applies. However, in the case that $e \in P(B) \backslash Im(f)$ is an edge, a different construction is required. Let C be K_1^{ℓ}, the graph with one vertex and one loop. Let $h : B \to C$ be the morphism that maps everything to the vertex, and $g : B \to C$ be the morphism that maps everything except e to the vertex, and maps e to the edge. Then $hf = gh$ but $h \neq g$, a contradiction.

Now let $f : A \to B$ be an monomorphism in **Grphs**, and suppose f is not injective. Then there exists $d, e \in P(A)$ such that $f(d) = f(e)$. Consider $g, h : K_2 \to A$ where g maps the edge to d, and the vertices to the vertices incident to d and h maps the edge to e and the vertices to the vertices incident to e. Then as f must preserve incidence, $fg = fh$ but $g \neq h$, a contradiction to f being a monomorphism. A similar proof applies to **StGrphs** and **SiGrphs** □

This changes if we add enough restrictions, as seen in the following proposition. The following result for epimorphisms in **SiStGrphs** and **SiLlStGrphs** is known [20].

Proposition 2.3.14. *A morphism in* ***SiStGrphs****,* ***SiLlGrphs****, and* ***SiLlStGrphs*** *is an epimorphism if and only if it is a surjective function of vertex sets, and a morphism in the above categories is a monomorphism if and only if it is an injective function of the vertex sets.*

Proof. Let $f : A \to B$ be an epimorphism in **SiLlGrphs**. Suppose f_V is not surjective. Then there exists $v \in V(B)\backslash Im(f_V)$. Construct the graph C by appending a vertex v' to B such that v' is adjacent to every vertex v is adjacent to. By construction B is a subgraph of C.

Since $v \in V(B)\backslash Im(f_V)$, no edge incident to v is in the image of f_E. Now consider $i : B \to C$ the inclusion morphism and $g : B \to C$ defined by $g(u) = i(u)$ for all $u \in V(B)\backslash\{v\}$, $g(v) = v'$, $g(e) = i(e)$ for all edges e not incident to v, and for edge f incident to v, set $g(f)$ to be the corresponding edge incident to v'. Then $if = gf$ but $i \neq g$, a contradiction to f being an epimorphism. Hence epimorphisms in **SiLlGrphs** have surjective vertex set functions. A similar proof applies to **SiStGrphs** and **SiLlStGrphs**.

Suppose $f : A \to B$ is a morphism in **SiLlGrphs** and f_V is surjective. Consider morphisms $h, k : B \to C$ such that $hf = kf$. Since f_V is surjective and $h_V f_V = k_V f_V$, $h_V = k_V$. So if $h \neq k$ there exists an edge $e \in E(B)$ such that $h(e) \neq k(e)$, even though $h_V = k_V$. There are two possibilities for $h(e)$ and $k(e)$, either as different vertices or edges.

If $h(e)$ and $k(e)$ are different vertices, as $h_V = k_V$, the incident vertices to e in B are both mapped to the same vertex, so for incidence to hold $h(e)$ and $k(e)$ would also be mapped to that vertex and $h(e) = k(e)$. If $h(e)$ and $k(e)$ are mapped to different edges, since $h_V = k_V$ they must have the same incidence. Since graphs in **SiLlGrphs** are simple and loopless, $h(e) = k(e)$. Hence both possibilities lead to contradictions. A similar proof holds for **SiLlStGrphs**, and for **SiStGrphs** a third possiblity arises for $h(e)$ and $k(e)$ to be different loops. However, in this case, as simple graphs have only one loop and $h_V = k_V$, they must be mapped to the same loop.

Now let $f : A \to B$ be a monomorphism in **SiLlGrphs**. Suppose f_V is not injective.

Then there exists $u, v \in V(A)$ such that $f(u) = f(v)$. Then consider the two morphisms $j, k : K_1 \to A$ defined by j mapping the single vertex of K_1 to u, and k mapping the single vertex of K_1 to v. Clearly $fj = fk$ but $j \neq k$ a contradiction to f being a monomorphism. Hence monomorphisms in **SiLlGrphs** have injective vertex set functions. A similar proof applies to **SiStGrphs** and **SiLlStGrphs**.

Suppose $f : A \to B$ is a morphism in **SiLlGrphs** and f_V is injective. Consider morphisms $j, k : C \to A$ such that $fj = fk$. Since f_V is injective, $j_V = k_V$. Thus if there exists $e \in P(C)$ such that $j(e) \neq k(e)$, then e must be an edge of C. Since f_V is injective and $fj = fk$, $j(e)$ and $k(e)$ cannot both be vertices in A. Without loss of generality assume $k(e)$ is an edge.

Note that $j(e)$ cannot be a vertex of A, for both incident vertices of e in C are mapped to $j(e)$ as well. Then since $j_V = k_V$, morphisms preserve incidence, and the graphs are loopless, $k(e)$ is mapped to a vertex. Hence $j(e)$ must be an edge of A. Since $j_V = k_V$, $j(e)$ and $k(e)$ have the same incident vertices, and since the graphs are simple, $k(e) = j(e)$, a contradiction. Hence f is a monomorphism.

□

2.3.4 Free Objects and Cofree Objects

We now consider the underlying vertex set functor $|-|_V$: **-Grphs** $\rightsquigarrow$ **Sets** defined for each of the categories of graphs, where $|G|_V = V(G)$ for a graph G and $|f|_V = f_V$ for a morphism f. We define the free graph functor $F(-)$: **Sets** $\rightsquigarrow$ **-Grphs** to be such that $F(-)$ is left adjoint to $|-|_V$. We similarly define the cofree graph functor $C(-)$: **Sets** $\rightsquigarrow$ **-Grphs** to be such that $|-|_V$ is left adjoint to $C(-)$. The following two proposition characterizes the free and cofree objects in the categories of graphs. These characterizations are known in **SiStGrphs** and **SiLlStGrphs** [20].

Proposition 2.3.15. *Given a set X, in all six categories of graphs, the free graph on X is just the empty edge graph on the vertex set X, and the free objects are the empty edge set* *graphs.*

Proof. Let X be a set in **Sets**, and let $F(X) = K^c$ the empty edge set graph with vertex set $V(K^c) = X$. Now let G be a graph such that there is a function $g : X \to |G|_V$. We must show there is a unique graph morphism $\overline{g} : F(X) \to G$ such that $g = |\overline{g}|_V u$ for some $u : X \to |F(X)|_V$. Note that $|F(X)|_V = V(F(X)) = X$. Hence define the function $u : X \to |F(X)|_V$ as $u = 1_X$.

Let $\overline{g}$ be the function map $\overline{g} = g$ and $\overline{g}_V = g$. Since there are no edges in $F(X)$, incidence is clearly preserved (and the morphism is strict). Then since $g = |\overline{g}|_V u$ must hold, $u = 1_X$, and $|\overline{g}|_V = \overline{g}_V = g$, $\overline{g}$ is uniquely determined by g. □

Proposition 2.3.16. *Given a set X,*

1. *in **Grphs**, **SiGrphs**, and **SiLlGrphs** the cofree graph on X is the complete graph on the vertex set X and the cofree objects are the complete graphs,*
2. *in **StGrphs**, **SiStGrphs** the cofree graph on X is a complete graph with a single loop on each vertex having the vertex set X, and the cofree objects are the complete graphs with a loop at each vertex,*
3. *in **SiLlStGrphs** no cofree graph exists.*

Proof. **Part 1**: Let X be a set in **Sets** and define $C(X)$ as the complete graph with the vertex set $V(C(X)) = X$. Let G be a graph in **Graphs** with set function $g : |G|_V \to X$. We must show that there is a unique graph morphism $\overline{g} : G \to C(X)$ such that $g = c|\overline{g}|_V$ for some set function $c : |C(X)|_V \to X$ Note that $|C(X)|_V = V(C(X)) = X$. Hence we define c as 1_X.

For $g = 1_X|\overline{g}|_V$ to hold, $\overline{g}_V = g$ is uniquely determined. Then let e be an edge of G incident to vertices $x, y \in V(G)$ where x and y are not necessarily distinct. Then since graph morphisms must preserve incidence, for $\overline{g}$ to be a morphism, $\overline{g}(e)$ must map to the part e' of $C(X)$ incident to vertices $g(x)$ and $g(y)$. By the definition of $C(X)$ such a part e' exists, even if it is a vertex. Hence $\overline{g}$ exists and is uniquely determined by g. A similar proof applies for **SiGrphs** and **SiLlGrphs**.

Part 2: Let X be a set in **Sets** and define $C(X)$ as the complete graph with a loop at every vertex with the vertex set $V(C(X)) = X$. Let G be a graph in **StGraphs** with set function $g : |G|_V \to X$. We must show that there is a unique strict graph homomorphism $\overline{g} : G \to C(X)$ such that $g = c|\overline{g}|_V$ for some set function $c : |C(X)|_V \to X$ Note that $|C(X)|_V = V(C(X)) = X$. Hence we define c as 1_X.

For $g = 1_X|\overline{g}|_V$ to hold, $\overline{g}_V = g$ is uniquely determined. Then let e be an edge of G incident to vertices $x, y \in V(G)$ where x and y are not necessarily distinct. Then since strict graph homomorphisms must send edges to edges and preserve incidence, for $\overline{g}$ to be a strict graph homomorphism, $\overline{g}(e)$ must map to the edge e' of $C(X)$ incident to vertices $g(x)$ and $g(y)$. By the definition of $C(X)$ such an edge e' exists. Hence $\overline{g}$ exists and is uniquely determined by g. A similar proof applies for **SiStGrphs**.

Part 3: Assume cofree graphs exist. Let $X = \{x\}$ in **Sets** and $C(X)$ be the cofree graph associated with X and function $c : |C(X)|_V \to X$. Consider K_2 with vertices a and b and edge e and set function $g : |K_2|_V \to X$ defined by $g(a) = g(b) = x$. Then since $C(X)$ is a cofree object, there is a unique morphism in **SiLlStGraphs**, $\overline{g} : K_2 \to C(X)$, such that $g = c|\overline{g}|_V$. Since $\overline{g}$ is a strict graph homomorphism, it must send e to an edge in $C(X)$. Thus $\overline{g}(e) = f$ for some $f \in E(C(X))$. Since graph homomorphisms preserve incidence, f is incident to $\overline{g}(a) = |\overline{g}|_V(a) = a'$ for some vertex $a' \in V(C(X))$ and $\overline{g}(b) = |\overline{g}|_V(b) = b'$ for some vertex $b' \in V(C(X))$.

Since $C(X)$ is loopless, $a' \neq b'$. Then since $g = c|\overline{g}|_V$, $g(a) = c(|\overline{g}|_V(a)) = x$ and $g(b) = c(|\overline{g}|_V(b)) = x$, $c(a') = c(b') = x$. Now consider the morphism $\overline{h} : K_2 \to C(X)$ defined by $\overline{h}(e) = f$, $\overline{h}(a) = b'$ and $\overline{h}(b) = a'$. Clearly $\overline{h} \neq \overline{g}$. Then $c(|\overline{h}|_V(a)) = c(b') = x = g(a)$ and $c(|\overline{h}|_V(b)) = c(a') = x = g(b)$. Thus $c|\overline{h}|_V = g$, and $\overline{g}$ is not unique, which is a contradiction to the universal mapping property of the cofree object. □

We also consider the underlying part set functor $|-|_P :$ **-Grphs** $\rightsquigarrow$ **Sets** defined for each of the categories of graphs, where $|G|_P = P(G)$ for a graph G and $|f|_P = f_P$ for a morphism f. We show there is no corresponding free functor in the six categories of graphs, and there is not a corresponding cofree functor for **SiGrphs**, **SiLlGrphs**, **SiStGrphs** and **SiLlStGrphs**. We will show there is a corresponding cofree functor for **Grphs** and **StGrphs**.

Proposition 2.3.17. *In all six categories of graphs, $|-|_P$ does not have a left adjoint.*

Proof. If $|-|_P$ had a left adjoint F_P, then $|-|_P$ would commute with limits. However as $P(K_2 \times K_2) \neq P(K_2) \times P(K_2)$ in all six categories of graphs, this is not the case. □

Proposition 2.3.18. *In **SiGrphs**, **SiLlGrphs**, **SiStGrphs** and **SiLlStGrphs**, $|-|_P$ does not have a right adjoint.*

Proof. If $|-|_P$ had a right adjoint C_P, then $|-|_P$ would commute with colimits. So consider P_3 the path on three vertices with vertices a, b and c and edges j, k with incidence $\partial_{P_3}(j) = (a\_b)$ and $\partial_{P_3}(k) = (b\_c)$, and P_3^c the empty edge graph on the vertices of P_3. Define two morphisms $f : P_3^c \to P_3$ and $g : P_3^c \to P_3$ where f is inclusion and $g(a) = c, g(b) = b$, and $g(c) = a$. We note this particular coequalizer exists in **SiLlStGrphs** and it is isomorphic to K_2. Then $P(Coeq(f, g)) \neq Coeq(P(f), P(g))$ in any of the four categories and C_P does not exist. □

Proposition 2.3.19. *Given a set X,*

1. *in **Grphs** the part cofree graph on X, $C_P(X)$, is the graph with $V(C_P(X)) = X$ and for each $u \in V(C_P(X))$ and for all $v \in X$ if $u \neq v$ then there exists $\ell_{u,v} \in P(C_P(X))$ with $\partial_{C_P(X)}(\ell_{u,v}) = (u\_u)$, and for each $u, v \in V(C_P(X))$ with $u \neq v$ then for each $w \in X$ there exists $e_{\{u,v\},w} \in P(C_P(X))$ with $\partial_{C_P(X)}(e_{\{u,v\},w}) = (u\_v)$.*

2. *in **StGrphs** the part cofree graph on X is the same construction as in **Grphs** with the additional loop for each vertex $u \in V(C_P(X))$ $\ell_{u,u} \in P(C_P(X))$ with $\partial_{C_P(X)}(\ell_{u,u}) = (u\_u)$.*

Proof. We prove the result in **Grphs** and the proof for **StGrphs** follows similarly.

First, we define $c : P(C_P(X)) \to X$ by $c(x) = x$ for all $x \in \iota_{C_P(X)}(X)$, $c(\ell_{u,v}) = v$, and $c(e_{\{u,v\},w}) = w$.

Let A be a graph in **Grphs** and let $g : P(A) \to X$ be any set function. This induces a function $g_V = g|_{\iota_A(V(A))}$. Hence we define a morphism in **Grphs** $\overline{g} : A \to C_P(X)$ as follows. For a vertex $v \in \iota_A(V(A))$ define $\overline{g}(a) = g(a)$. For an edge $a \in E(A)$ with $\partial_A(a) = (a_1\_a_2)$, if $g(a_1) = g(a_2) = g(a)$, define $\overline{g}(a) = g(a_1)$ (in **StGrphs** we would define $\overline{g}(a) = \ell_{g(a_1),g(a_1)}$), or if $g(a_1) = g(a_2) \neq g(a)$ define $\overline{g}(a) = \ell_{g(a_1),g(a)}$, otherwise $\overline{g}(a) = e_{\{g(a_1),g(a_2)\},g(a)}$.

As incidence is preserved and vertices are mapped to vertices, $\overline{g}$ is a graph morphism. We now show $\overline{g}$ is the unique morphism such that $g = c|\overline{g}|_P$.

By definition of $\overline{g}$, $g = c|\overline{g}|_P$. So let $h : A \to C_P(X)$ be a morphism in **Grphs** such that $c|h|_P = g$. As h is a morphism $h(a) \in V(C_P(X))$ for all $a \in V(A)$. Thus, for $a \in V(A)$, $c(|h(a)|_P) = h(a)$ for $h(a) \in X = V(C_P(X))$. As $c|h|_P = c|\overline{g}|_P$, $h(a) = c(|h(a)|_P) = c|\overline{g}|_P = g(a) = \overline{g}(a)$. Hence h and $\overline{g}$ agree on vertices.

Now let $a \in E(A)$. As h is a graph morphism, the incidence of a under h is preserved. Thus for $\partial_A(a) = (a_1\_a_2)$, $\partial_{C_P(X)}(h(a)) = (h(a_1)\_h(a_2)) = (\overline{g}(a_1)\_\overline{g}(a_2))$. First, if $h(a)$ is a loop, then $h(a) = \ell_{g(a_1),b}$ for some $b \in X$ such that $c(|h(a)|_P) = g(a) = b$. Hence as $\overline{g}(a) = \ell_{g(a_1),b}$ by construction $\overline{g}$ and h agree on loops. Second, if $h(a)$ is a non-loop edge, then $h(a_1) \neq h(a_2)$ and therefore $h(a) = e_{\{h(a_1),h(a_2)\},b}$ for $c(e_{\{h(a_1),h(a_2)\},b}) = g(a) = b$. Then as $(h(a_1)\_h(a_2)) = (g(a_1)\_g(a_2))$, $\overline{g}(a) = e_{\{h(a_1),h(a_2)\},b}$ and $\overline{g} = h$.

Thus $C_P(X)$ is the part cofree graph for set X. □

2.3.5 Projective and Injective Objects

The definitions for free objects and cofree objects are dependent on the category being a concrete category. We move on to other categorial constructions that are defined for any abstract category. We start with the injective objects and projective objects [1].

These results for projective and injective objects are known in **Grphs** [27]. We will provide an alternate proof. The results for projective and injective objects are also known in **SiStGrphs** and **SiLlStGrphs** [20]. We provide their construction for completeness.

Proposition 2.3.20. *1. In **Grphs**, **SiGrphs**, and **StGrphs**, all graphs with at most 1 non-loop edge per component are precisely the projective objects, and there are enough projective objects.*

*2. In **SiLlGrphs**, **SiStGrphs**, and **SiLlStGrphs**, the projective objects are precisely the free objects, and there are a enough projective objects.*

Proof. **Part 1**: First note that if $f : A \to B$ is an epimorphism in **Grphs** then f is a surjective map of the associated part sets. So let A be a graph with at most one non-loop edge in each component with morphism $g : A \to G$ for some graph G. Let H be a graph with an epimorphism $h : H \to G$.

Consider a component of A. If the component is composed of a single vertex, v, then since g is a surjection, there exists $v' \in V(H)$ such that $h(v') = g(v)$. If the component is composed of a single vertex v with a loop ℓ, and under g the loop is identified with $g(v)$, then as before there exists $v' \in H$ such that $h(v') = g(v) = g(\ell)$.

If the component has an edge e, and two vertices u and v, and under g the two vertices are identified with the edge, then there exists $v' \in V(H)$ such that $h(v') = g(v) = g(u) = g(e)$. If under g the two vertices are identified, and the edge is sent to a loop, then there exists $v' \in V(H)$ and $\ell' \in E(H)$ with $\partial_H(\ell') = (v' \_ v')$ such that $h(v') = g(v) = g(u)$ and $h(\ell') = g(e)$. If under g the two vertices are not identified then there exists a non-loop edge $d \in E(H)$ with incident vertices $a, b \in V(H)$ such that $h(d) = g(e)$, $h(a) = g(u)$ and $h(b) = g(v)$. Then the definition for $\overline{g}$ such that $h\overline{g} = g$ is obvious, and since each component can be mapped independently from other components, this is a graph morphism.

Now suppose G is a graph with at least two edges in some component, called e and f. Consider the graph H created by "splitting" G at each vertex incident to more than two edges. That is, for every vertex v incident to at least two edges a and b, create v_1 and v_2 in

H such that a is incident to v_1 and b is incident to v_2 with no edge between v_1 and v_2. Then H admits an epimorphism h to G by re-identifying these split vertices.

However, with morphism $1_G : G \to G$, G does not admit a morphism $\overline{g}$ to H such that $h\overline{g} = 1_G$ as edges e and f must be sent to the same component to preserve incidence. Hence G is not projective.

If G has a component with a loop $\ell \in E(G)$ incident to $x \in V(G)$, then consider $H = G' + K_2$ where G' is formed from G by deleting the loop ℓ. Then define $f : H \to G$ by $f(a) = a$ if $a \in P(G')$ and for $u, v \in V(K_2)$ $f(u) = f(v) = x$, and for $e \in E(K_2)$, $f(e) = \ell$. As f preserves incidence and is a surjection on part sets, f is an epimorphism. However, for $1_G : G \to G$, there is no $h : G \to H$ such that $1_G = fh$. Hence G is not projective.

Let G be a graph in **Grphs**. To show there are enough projectives, we must show there is a projective object H and an epimorphism $e : H \to G$. As above, construct H by "splitting" G and then turning all components containing a single vertex with a loop into a copy of K_2. Then H admits an epimorphism to G, and since H does not have more than one non-loop edge per component, H is projective. A similar proof applies to **SiStGrphs** and **SiLlStGrphs**

Part 2: First note that if $f : A \to B$ is an epimorphism in **SiLlGrphs** then the vertex set function f_V is surjective. We show that the free objects are projective objects. Clearly the empty graph $\emptyset$ is projective since it is the initial object. Now let X be a non-empty set in **Sets**, G be a graph with a morphism $h : F(X) \to G$, and H be a graph with an epimorphism $g : H \to G$. We must show that there is a morphism $\overline{h} : F(X) \to H$ such that $g\overline{h} = h$.

Since g is an epimorphism, g_V is a surjective function. Hence for all $v_i \in V(F(X))$, there is a $u_i \in V(H)$ such that $g(u_i) = h(v_i)$. Then define $\overline{h}(v_i) = u_i$ for every $v_i \in V(F(X))$. Then $g(\overline{h}(v_i)) = g(u_i) = h(v_i)$ for every vertex v_i of $F(X)$. Since $F(X)$ contains no edges, $\overline{h}$ is a graph morphism (and strict). Thus $F(X)$ is projective.

Now let A be a graph with at least 1 edge, and consider K, the complete graph on $V(A)$, with an inclusion morphism $h : A \to K$. By Proposition 2.3.14, there is an epimorphism $e : K^c \to K$ for K^c the empty edge graph on $V(A)$. Since A has an edge any morphism from A to K^c must identify at least two vertices, and hence no such morphism $f : A \to K^c$ exists such that $h = ef$. Thus A is not projective.

Let G be a graph in **SiLlGrphs**. To show there are enough projectives, we must show there is a projective object H and an epimorphism $e : H \to G$. By Proposition 2.3.14, the projective object K^c admits an epimorphism to G. A similar proof applies to **SiStGrphs** and **SiLlStGrphs**

□

Proposition 2.3.21. *1. In **Grphs**, **SiGrphs**, and **SiLlGrphs** the injective objects are precisely the graphs containing the cofree objects as spanning subgraphs and there are enough injective objects.*

*2. In **StGrphs** and **SiStGrphs**, the injective objects are precisely the graphs containing the cofree objects as spanning subgraphs and there are enough injective objects.*

*3. In **SiLlStGrphs**, there are no injective objects.*

The injective objects in **Grphs** are known [27], we provide an alternate proof.

Proof. **Part 1**: Let A be a graph that contains a cofree spanning subgraph in **Grphs**, and let G, H be graphs in **Grphs** with a morphism $f : G \to A$ and a monomorphism $g : G \to H$. We must show there is a morphism $\overline{f} : H \to A$ such that $f = \overline{f}g$.

Since g is a monomorphism, it is an injection of the part sets. Then for all $v \in P(Im(g))$ there is a unique $v' \in P(G)$ such that $g(v') = v$. Since A is non-empty, it has a vertex x. Define $\overline{f} : H \to A$ by $\overline{f}(v) = f(v')$ if $v \in P(Im(g))$ and $\overline{f}(v) = x$ if v is not in the image and a vertex. If v is an edge that is not in the image with $\partial_G(v) = (u_1 \_ u_2)$, then define $\overline{f}(v) = e$ where e is some part with $\partial_A(e) = (\overline{f}(u_1) \_ \overline{f}(u_2))$. One exists since A contains a spanning cofree subgraph, and in the case that $u_1 = u_2$ the vertex suffices. By this construction, $\overline{f}$ is a morphism and $\overline{f}g = f$.

Now let G be a graph in **Grphs** that does not contain a cofree spanning subgraph. Assume it is an injective object of **Grphs**. Then there are distinct vertices $u, v \in V(G)$ such that there is no edge e with $\partial_G(e) = (u\_v)$.

Then consider K_2^c with morphism $f : K_2^c \to G$ defined by $f(a) = u$ and $f(b) = v$, for a and b the two vertices of K_2^c, and $i : K_2^c \to K_2$ the inclusion morphism. Since the inclusion morphism is a monomorphism, there is a morphism $\overline{f} : K_2 \to G$ such that $\overline{f}i = f$. Then $\overline{f}(i(a)) = \overline{f}(a) = u$ and $\overline{f}(i(b)) = \overline{f}(b) = v$. Since morphisms preserve incidence, $\partial_G(\overline{f}(e)) = (\overline{f}(a)\_\overline{f}(b)) = (u\_v)$, and there is an edge e' such that $\partial_G(e') = (u\_v)$, a contradiction. Hence G is not an injective object.

To show there are enough injective objects we must show that for any graph G in **Grphs**, there is an injective object H with a monomorphism $f : G \to H$. If G is not the initial object, $C(V(G))$ is an injective object and $i : G \to C(V(G))$, the inclusion morphism, is a monomorphism. If $G = \emptyset$ then $\emptyset \hookrightarrow K_1$ suffices. Hence there are enough injective objects in **Grphs**. A similar proof applies to **SiGrphs** as well as **SiLlGrphs** that relies on monomorphisms as injections of the vertex sets and the fact that there is at most one edge between any two distinct vertices.

Part 2: Let A be a graph that contains a cofree spanning subgraph in **StGrphs**, and let G, H be graphs in **StGrphs** with a morphism $f : G \to A$ and a monomorphism $g : G \to H$. We must show there is a morphism $\overline{f} : H \to A$ such that $f = \overline{f}g$.

Since g is a monomorphism, it is an injection of the edge sets. Then for all $v \in P(Im(g))$ there is a unique $v' \in P(G)$ such that $g(v') = v$. Since A is non-empty, it has a vertex x. Define $\overline{f} : H \to A$ by $\overline{f}(v) = f(v')$ if $v \in P(Im(g))$ and $\overline{f}(v) = x$ if v is not in the image and a vertex. If v is an edge that is not in the image with $\partial_G(v) = (u_1\_u_2)$, then define $\overline{f}(v) = e$ where e is some edge with $\partial_A(e) = (\overline{f}(u_1)\_\overline{f}(u_2))$. One exists since A contains a spanning cofree subgraph (it may be a loop). By this construction, $\overline{f}$ is a strict graph morphism and $\overline{f}g = f$.

Now let G be a graph in **StGrphs** that does not contain a cofree spanning subgraph. Assume it is an injective object of **StGrphs**. Then there are vertices $u, v \in V(G)$ (not necessarily distinct) such that there is no edge $e \in E(G)$ with $\partial_G(e) = (u\_v)$.

Then consider K_2^c with morphism $f : K_2^c \to G$ defined by $f(a) = u$ and $f(b) = v$, for a

and b the two vertices of K_2^c, and $i : K_2^c \to K_2$ the inclusion morphism. Since the inclusion morphism is a monomorphism, there is a morphism $\overline{f} : K_2 \to G$ such that $\overline{f}i = f$. Then $\overline{f}(i(a)) = \overline{f}(a) = u$ and $\overline{f}(i(b)) = \overline{f}(b) = v$. Since morphisms preserve incidence, $\partial_G(\overline{f}(e)) = (\overline{f}(a)\_\overline{f}(b)) = (u\_v)$, there is an edge e' such that $\partial_G(e') = (u\_v)$, a contradiction. Hence G is not an injective object.

To show there are enough injective objects we must show that for any graph G in **StGrphs**, there is an injective object H with a monomorphism $f : G \to H$. If G is not the initial object, $C(V(G))$ is an injective object and $i : G \to C(V(G))$, the inclusion morphism, is a monomorphism. If $G = \emptyset$ then $\emptyset \hookrightarrow K_1^\ell$ suffices. Hence there are enough injective objects in **StGrphs**. A similar proof applies to **SiStGrphs** that relies on monomorphisms as injections of the vertex set and the fact that there is at most one edge between any two (not necessarily distinct) vertices. □

2.3.6 Generators and Cogenerators

The last construction of this chapter that we characterize in the six categories of graphs is a classification of generators and cogenerators (as separators and coseparators in [1]). The results for **SiStGrphs** and **SiLlStGrphs** are known [20] and we provide their constructions for completeness.

Proposition 2.3.22. *1. In **Grphs** and **SiGrphs**, all graphs containing a non-loop edge are precisely the generators,*

*2. in **SiLlGrphs** all nonempty graphs are generators,*

*3. in **StGrphs** no generators exist,*

*4. in **SiStGrphs** and **SiLlStGrphs**, the empty edge graphs, K^c, are precisely the generators (for $V(K^c) \neq \emptyset$).*

Proof. **Part 1**: Let A be a graph in **Grphs** with a non-loop edge, e, with vertices u_1 and u_2 incident to e. Consider K_2 with vertices v_1 and v_2 with incident edge e'. Then A has an epimorphism $f : A \to K_2$ defined by $f(u_2) = v_2$ and $f(y) = v_1$ for all vertices $y \in V(A)\backslash\{u_2\}$ and where every loop incident to u_2 is mapped to v_2 and every non-loop edge incident to v_2 (including e) is mapped to e', and all other edges mapped to v_1.

Hence, we only need to show K_2 is a generator. Let $f, g : G \to H$ be such that $f \neq g$. Hence $f(a) \neq g(a)$ for some $a \in P(G)$. First suppose a is a vertex. Then the morphism $\ulcorner a \urcorner$ from K_2 to G mapping the two vertices and edge of K_2 to a suffices. If a is an edge of G, then the morphism that maps the edge of K_2 to a and the incident vertices of the edge to the incident vertices of a suffices.

Now suppose A is a graph containing no non-loop edges. Then no morphism from A to K_2 can distinguish between $f, g : K_2 \to K_1^\ell$, where f maps the two vertices and edge of K_2 to the vertex of K_1^ℓ and g maps the two vertices of K_2 to the single vertex of K_1^ℓ and the edge to the loop. Hence A is not a generator. The same proof applies to **SiGrphs**.

Part 2: The empty graph is in the initial object of **SiLlGrphs** and thus cannot be a generator.

Since in all graphs of **SiLlGrphs** there is at most one edge between any two distinct vertices, if $f, g : G \to H$ agree on the vertex sets, the agree on the edge sets and $f = g$. Hence if $f, g : G \to H$ are distinct, then for some vertex v of G, $f(v) \neq g(v)$. So let A be a non-empty graph. Then A has a vertex and the morphism $h : A \to G$, where h maps all of $P(A)$ to the vertex v suffices.

Part 3: Suppose a generator G in **StGrphs** exists. Then consider $f, g : K_1 \to K_2^c$, where f maps the vertex of K_1 to one vertex of K_2^c and g maps the vertex of K_1 to the other vertex of K_2^c. Since G is a generator, it admits a morphism $h : G \to K_1$ such that $fh \neq gh$. Since morphisms are strict, edges must be mapped to edges. However, K_1 has no edge, and thus G is edgeless. Hence $G \cong K^c$ for some empty edge graph K^c.

Now consider K_2 and A, where A is a graph consisting of two vertices with two parallel edges between the two vertices. Define $j, k : K_2 \to A$ by j mapping the edge of K_2 to one edge of A, and k mapping the edge of K_2 to the other edge of A, but mapping the vertices of K_2 in tandum. Then no morphism from $G \cong K^c$ can distinguish between j and k. Hence G is not a generator, a contradiction, and no generators exist in **StGrphs**. □

Proposition 2.3.23. *1. In **Grphs** and **SiGrphs**, the graphs containing a loop and a non-loop edge are precisely the cogenerators,*

*2. In **SiLlGrphs**, the graphs containing an edge are the cogenerators,*

*3. in **StGrphs**, the graphs containing both a vertex with two distinct loops and containing a subgraph isomorphic to K_2^ℓ are precisely the cogenerators,*

*4. in **SiStGrphs**, the cogenerators are precisely the graphs containing a subgraph isomorphic to K_2^ℓ,*

*5. in **SiLlStGrphs** no cogenerators exist.*

Proof. **Part 1**: Let A be a graph composed of two disconnected components. One component contains a vertex v with a loop ℓ, and the other component contains two vertices u_1 and u_2 with an edge e between them. If A is a cogenerator, then any graph containing a loop and a non-loop edge is also a cogenerator.

Let $f, g : G \to H$ be two distinct morphisms in **Grphs**. Then $f(e') \neq g(e')$ for some $e' \in P(G)$. If both $f(e')$ and $g(e')$ are vertices, define $h : H \to A$ by $h(f(e')) = u_2$, $h(y) = u_1$ for all $y \in V(H) \backslash \{f(e')\}$, $h(a) = e$ for all non-loop edges a incident to $f(e')$ in H, $h(a) = u_2$ for all loops a incident to $f(e')$, and $h(a) = u_1$ for all other edges a of H. Hence $hf \neq hg$.

If $f(e')$ is a vertex of H and $g(e')$ is not, define h by $h(f(e')) = v$, $h(g(e')) = \ell$ and $h(y) = v$ for all $y \in P(H) \backslash \{f(e'), g(e')\}$. Hence $hf \neq hg$. If $f(e')$ is an edge of H, then define h by $h(f(e')) = \ell$ and $h(y) = v$ for all $y \in P(H) \backslash \{f(e')\}$. Hence $hf \neq hg$, and A is a cogenerator.

If C is a graph not containing any loops, then no morphism exists from K_1^ℓ to C that can distinguish between $id, f : K_1^\ell \to K_1^\ell$, where id is the identity morphism, and f is the morphism that maps the loop and vertex of K_1^ℓ to the vertex of K_1^ℓ. If C is a graph not containing any non-loop edges, then no morphism exists from K_2 to C that can distinguish between $j, k : K_1 \to K_2$ where j maps the single vertex of K_1 to one vertex of K_2, and k maps the single vertex of K_1 to the other vertex of K_2. Hence all cogenerators require a loop and a non-loop edge. A similar proof applies to **SiGrphs**.

Part 2: Let A be a graph with an edge e incident to vertices u_1 and u_2. As in the proof for generators in **SiLlGrphs**, if $f, g : G \to H$ are distinct, then there is a vertex $v \in V(G)$ such that $f(v) \neq g(v)$. Define $h : H \to A$ by $h(f(v)) = u_2$, $h(y) = u_1$ for all $y \in V(H)\backslash\{f(v)\}$, $h(a) = e$ for all edges of H incident to $f(v)$, and $h(a) = u_1$ for all other edges in H. Hence $hf \neq hg$, and A is a cogenerator.

If C is a graph not containing any edges, then no morphism exists from K_2 to C that can distinguish between $j, k : K_1 \to K_2$ where j maps the single vertex of K_1 to one vertex of K_2, and k maps the single vertex of K_1 to the other vertex of K_2. Hence all cogenerators in **SiLlGrphs** require an edge.

Part 3: Let A be a graph composed of two disconnected components. One component contains a vertex v with two loops ℓ_1 and ℓ_2, and the other component contains two vertices u_1 and u_2 with an edge e between them, and a loop ℓ_{u1} and ℓ_{u2} on u_1 and u_2 respectively. If A is a cogenerator, then any graph containing a vertex with 2 loops incident to it and a non-loop edge with a loop incident to each incident vertex of the edge is also a cogenerator.

Let $f, g : G \to H$ be two distinct morphisms in **StGrphs**. Then $f(e') \neq g(e')$ for some $e' \in P(G)$. If $f(e')$ is a vertex, define $h : H \to A$ by $h(f(e')) = u_2$, $h(y) = u_1$ for all $y \in V(H)\backslash\{f(e')\}$, $h(a) = e$ for all non-loop edges a incident to $f(e')$ in H, $h(a) = \ell_{u2}$ for all loops a incident to $f(e')$, and $h(a) = \ell_{u1}$ for all other edges a of H. Hence $hf \neq hg$.

If $f(e')$ is an edge of H, then define h by $h(f(e')) = \ell_1$ and $h(y) = v$ for all $y \in V(H)$ and $h(a) = \ell_2$ for all other edges a in H. Hence $hf \neq hg$, and A is a cogenerator.

Now suppose C is a graph that has no vertex that is incident to two loops. Then consider $f, g : K_1^\ell \to B$, where B is a graph composed of one vertex with two loops ℓ_a and ℓ_b, f maps the loop of K_1^ℓ to ℓ_a and g maps the loop to ℓ_b. No morphism exists from B to C that can distinguish between f and g.

Now suppose C is a cogenerator. Then let K_1 have vertex x and K_2^ℓ, the complete graph on 2 vertices with a loop incident to each vertex, have vertices a and b. Define $j : K_1 \to K_2^\ell$ by $j(x) = a$ and define $k : K_1 \to K_2^\ell$ by $k(x) = b$. Then there exists $h : K_2^\ell \to C$ such that $hj \neq hk$. Hence $hj(x) \neq hk(x)$ and C has at least two vertices. As h must preserve incidence, $\partial_C(h(e)) = (hj(x)\_hk(x))$ and C has a non-loop edge. Furthermore, as morphisms are strict, the loops of K_2^ℓ must be mapped to loops adjacent to $hj(x)$ and $hk(x)$. Thus C has a component with a non-loop edge with a loop incident to each vertex incident to that edge. □

2.4 Adjoint Functor Relationships

We now will explore the adjoint functor relationships between the categories of graphs. We have already considered adjoint functor relationships between the six categories of graphs and **Sets** in section 2.3.4. We have seen that all six categories of graphs have a left adjoint to the underlying vertex set functor and every category except **SiLlStGrphs** has a right adjoint to the underlying vertex set functor. However, only in **Grphs** and **StGrphs** is there an adjoint (specifically a right adjoint) to the underlying part set functor.

We now consider the existence of adjoints to the inclusion functors of the categories of graphs (see Figure 2.3). We provide the following known results for completeness.

Proposition 2.4.1. [20] *The inclusion functor **SiLlStGrphs**$\hookrightarrow$**Grphs** does not have a left or right adjoint.*

This proposition is easy to see if you consider that inclusion having a left adjoint means that $\hookrightarrow (K_2 + K_2) \cong \hookrightarrow (K_2 \times K_2) \cong \hookrightarrow (K_2) \times \hookrightarrow (K_2) \cong K_2 \times K_2$ in **Grphs** (a clear contradiction), and the right adjoint to the inclusion functor would have to commute with $\hat{1}$.

Proposition 2.4.2. [20] *The inclusion functor **SiStGrphs**$\hookrightarrow$**Grphs** does not have a left adjoint.*

This proposition is also easy to see if you consider the same counterexample to inclusion having a left adjoint as above. Using the above counterexample, this proposition can be easily extended as follows.

Proposition 2.4.3. *The inclusion functor **SiStGrphs**$\hookrightarrow$**SiGrphs** does not have a left adjoint, nor does the inclusion functor **StGrphs**$\hookrightarrow$**Grphs**.*

Proposition 2.4.4. [20] *For the functor S_1 :**Grphs**$\rightsquigarrow$**SiLlGrphs** defined by $S_1(G)$ the underlying simple loopless graph of G (identify loops with their incident vertices, identify all edges between the same distinct vertices) and for $f : G \to H$ in **Grphs** $S_1(f) = f'$ the morphism induced by f on the underlying simple loopless graphs of G and H, S_1 is left adjoint to inclusion, i.e. $S_1 \dashv \hookrightarrow$.*

Proposition 2.4.5. [20] *For the functor S_2 :**StGrphs**$\rightsquigarrow$**SiStGrphs** defined by $S_1(G)$ the underlying simple graph of G (identify all edges between the same, not necessarily distinct, vertices) and for $f : G \to H$ in **StGrphs** $S_1(f) = f'$ the morphism induced by f on the underlying simple graphs of G and H, S_2 is left adjoint to inclusion, i.e. $S_2 \dashv \hookrightarrow$*

We proceed to characterize the adjoints to the inclusion functors.

Proposition 2.4.6. *1.* [20] *There is a left adjoint S_1 to the inclusion **SiLlGrphs**$\hookrightarrow$**Grphs**.*

2. [20] *There is a left adjoint S_2 to the inclusion functor **SiStGrphs**$\hookrightarrow$**StGrphs**.*

*3. There is a left adjoint S_3 to the inclusion functor **SiLlGrphs**$\hookrightarrow$**SiGrphs***

*4. There is a right adjoint C to the inclusion functor **StGrphs**$\hookrightarrow$**Grphs**.*

5. No other left or right adjoint exists for all other inclusion functors.

Proof. **(Part 3):** Define S_3 :**SiGrphs**$\rightsquigarrow$**SiLlGrphs** as follows. For a graph G, define $S_3(G)$ as the subgraph of G formed by identifying any loop $\ell \in E(G)$ with its incident vertex $v_\ell \in V(G)$. For $f : G \to H$, define $S_3(f) : S_3(G) \to S_3(H)$ as $S_3(f) = f|_{P(S_3(G))}$. Clearly

$S_3(G)$ is a simple loopless graph, $S_3(f)$ is a graph morphism, and $S_3(fg) = S_3(f)S_3(g)$.

Define $\phi_{A,B} : \hom_{\mathbf{SiGrphs}}(A, \hookrightarrow(B)) \to \hom_{\mathbf{SiLlGrphs}}(S_3(A), B)$ for $f : A \to \hookrightarrow (B)$ by $\phi_{A,B}(f) = S_3(f)$. We note that as f is a morphism with a simple and loopless codomain, for any loops $\ell \in P(A)$ with incident vertex $v_\ell \in V(A)$, $f(\ell) = f(v_\ell) \in V(\hookrightarrow (B))$. Hence $S_3(f)$ is a morphism from $S_3(A)$ to B. We will show this is a natural bijection.

For $g : S_3(A) \to B$ define $\phi^{-1}(g) : A \to\hookrightarrow (B)$ as $\phi^{-1}_{A,B}(g)(a) = g(a)$ for $a \in P(S_3(A))$ and for all loops $\ell \in P(A)$ with incident vertex v_ℓ define $\phi^{-1}_{A,B}(g)(\ell) = g(v_\ell)$. Clearly incidence is preserved and $\phi^{-1}_{A,B}(g)$ is a graph morphism. This induces the function $\phi^{-1}_{A,B}$: $\hom_{\mathbf{SiLlGrphs}}(S_3(A), B) \to \hom_{\mathbf{SiGrphs}}(A, \hookrightarrow (B))$. Then by definition $\phi_{A,B}\phi^{-1}_{A,B}(g) = g$ and $\phi^{-1}_{A,B}\phi_{A,B}(f) = f$ and $\phi_{A,B}$ is a bijection.

Now consider $h : A \to A'$ in **SiGrphs**, and B a graph in **SiLlGrphs**. Consider the following diagram.

$$\begin{array}{ccc}
\hom_{\mathbf{SiGrphs}}(A', \hookrightarrow (B)) & \xrightarrow{\phi_{A',B}} & \hom_{\mathbf{SiLlGrphs}}(S_3(A'), B) \\
\downarrow{\scriptstyle \hom_{\mathbf{SiGrphs}}(h,\hookrightarrow(B))} & & \downarrow{\scriptstyle \hom_{\mathbf{SiLlGrphs}}(S_3(h),B)} \\
\hom_{\mathbf{SiGrphs}}(A, \hookrightarrow (B)) & \xrightarrow{\phi_{A,B}} & \hom_{\mathbf{SiLlGrphs}}(S_3(A), B)
\end{array}$$

Let $f : A \to\hookrightarrow (B)$. As $\hookrightarrow (B)$ is a loopless graph, $f(\ell) = f(v_\ell)$ for any loops $\ell \in P(A)$ with incident vertex $v_\ell \in V(A)$. Then $\phi_{A,B} \hom_{\mathbf{SiGrphs}}(h, \hookrightarrow (B))(f) = \phi_{A,B}(fh) = S_3(fh) = S_3(f)S_3(h) = \hom_{\mathbf{SiLlGrphs}}(S_3(h), B)S_3(f) = \hom_{\mathbf{SiLlGrphs}}(S_3(h), B)\phi_{A',B}(f)$, and hence the diagram commutes and $\phi_{A,B}$ is natural in A. A similar proof shows $\phi_{A,B}$ is natural in B as $S_3(\hookrightarrow (h)) = h$ for all morphisms h of **SiLlGrphs**. Hence $S_3 \dashv \hookrightarrow$.

(Part 4): Define C :**Grphs**$\rightsquigarrow$**StGrphs** as follows. For a graph G in **Grphs**, define $C(G) = G^\ell$ the graph formed from G by appending a new loop ℓ_v to every vertex $v \in V(G)$. For $f : G \to H$, define $C(f) : G^\ell \to H^\ell$ by $C(f)(v) = f(v)$ for $v \in V(G)$, $C(f)(\ell_v) = \ell_{f(v)}$ for ℓ_v the appended loops in G^ℓ, and for $e \in E(G)$, define $C(f)(e) = f(e)$ if $f(e) \in E(H)$ and define $C(f)(e) = \ell_{f(e)}$ if $f(e) \in V(H)$. Clearly $C(G)$ is a graph in **StGrphs**, and as $C(f)$ preserves incidence and maps edges to edges, $C(f)$ is a strict morphism.

Now let $g : G \to H$ and $f : H \to K$ be morphisms in **Grphs**. We consider $C(fg)$.

If $v \in V(G)$, then $C(fg)(v) = (fg)(v) = f(g(v)) = C(f)C(g)$. If $e \in E(G)$ then three cases arise, $fg(e) \in E(K)$, $fg(e) \in V(K)$ and $g(e) \in E(H)$, and $fg(e) \in V(K)$ and $g(e) \in V(H)$. In the first case $C(fg)(e) = fg(e) = C(f)C(g)(e)$. In the second case, as $fg(e) \in V(K)$, $C(fg)(e) = \ell_{fg(e)}$, and as $g(e) \in E(H)$, $C(f)C(g)(e) = C(f)g(e) = \ell_{f(g(e))} = \ell_{fg(e)}$ as desired. In the last case $C(fg)(e) = \ell_{f(g(e))}$, and as $g(e) \in V(H)$, $C(f)C(g)(e) = C(f)(\ell_{g(e)}) = \ell_{f(g(e))} = \ell_{fg(e)}$ as desired. Hence C is a functor.

Let A be a graph in **StGrphs** and B be a graph in **Grphs**. For $f :\hookrightarrow (A) \to B$ define $\phi_{A,B}(f) : A \to C(B)$ as $\phi_{A,B}(f) = C(f)|_A$ for A the subgraph of $C(A) = A^{\ell}$. Then $\phi_{A,B}$ induces a function $\phi_{A,B} : \hom_{\mathbf{Grphs}}(\hookrightarrow (A), B) \to \hom_{\mathbf{StGrphs}}(A, C(B))$. Now for $g : A \to C(B)$ in **StGrphs** define $\phi^{-1}_{A,B}(g) :\hookrightarrow (A) \to B$ by $\phi^{-1}_{A,B}(g)(v) = g(v)$ for all vertices $v \in V(A)$, $\phi^{-1}_{A,B}(g)(e) = v$ if $g(e) = \ell_v$, and $\phi^{-1}_{A,B}(g)(e) = g(e)$ otherwise. $\phi^{-1}_{A,B}(g)$ preserves incidence and maps vertices to vertices, so $\phi^{-1}_{A,B}(g)$ is a morphism of **Grphs**.

Now consider $\phi^{-1}_{A,B}\phi_{A,B}(f)$ for $f :\hookrightarrow (A) \to B$. We first note that $\phi^{-1}_{A,B}\phi_{A,B}(f) = \phi^{-1}_{A,B}(C(f)|_A)$, and for $a \in P(A)$ three cases arise, $a \in V(A)$, $a \in E(A)$ and $C(f)(a) = \ell_{f(a)}$, and $a \in E(A)$ and $C(f)(a) = f(a)$. In the first and third case, as $C(f)(a) = f(a)$ and $f(a) \neq \ell_v$ for any vertex $v \in V(C(H))$, $\phi^{-1}_{A,B}(C(f)|_A(a)) = \phi^{-1}_{A,B}(f(a)) = f(a)$. In the second case, as $C(f)(a) = \ell_{f(a)}$, $\phi^{-1}_{A,B}(C(f)|_A(a)) = \phi^{-1}_{A,B}(\ell_{f(a)}) = f(a)$. Hence $\phi^{-1}_{A,B}\phi_{A,B}(f) = f$.

Now consider $\phi_{A,B}\phi^{-1}_{A,B}(g)$ for $g : A \to C(B)$. If $a \in V(A)$, $\phi_{A,B}\phi^{-1}_{A,B}(g(a)) = \phi_{A,B}(g(a)) = g(a)$. If $a \in E(A)$ and $g(a) = \ell_v$ for some vertex $v \in V(C(B))$, then $\phi^{-1}_{A,B}(g)(a) = v$ and $\phi_{A,B}\phi^{-1}_{A,B}(g)(a) = \ell_v = g(a)$. If $a \in E(A)$ and $g(a) \neq \ell_v$, then $\phi_{A,B}\phi^{-1}_{A,B}(g)(a) = \phi_{A,B}(g)(a)$ and as g is strict, $g(a) \in E(B)$. Hence $\phi_{A,B}(g)(a) = C|_A(g(a)) = g(a)$. Thus, $\phi_{A,B}\phi^{-1}_{A,B}(g) = g$ and $\phi_{A,B}$ is a bijection.

We now show this bijection is natural in A and B. Let $h : A \to A'$ in **StGrphs** and B be a graph in **Grphs**. Consider the following diagram.

$$\begin{array}{ccc}
\hom_{\mathbf{Grphs}}(\hookrightarrow (A'), B) & \xrightarrow{\phi_{A',B}} & \hom_{\mathbf{StGrphs}}(A', C(B)) \\
\Big\downarrow{\scriptstyle \hom_{\mathbf{Grphs}}(\hookrightarrow(h),B)} & & \Big\downarrow{\scriptstyle \hom_{\mathbf{StGrphs}}(h,C(B))} \\
\hom_{\mathbf{Grphs}}(\hookrightarrow (A), B) & \xrightarrow{\phi_{A,B}} & \hom_{\mathbf{StGrphs}}(A, C(B))
\end{array}$$

Let $f :\hookrightarrow (A') \to B$. Then $\phi_{A,B}\,\hom_{\mathbf{Grphs}}(\hookrightarrow (h), B)(f) = \phi_{A,B}(fh) = C(fh)|_A = C(f)|_{A'}C(h)|_A$, as h is strict implying $C(h)|_A = h : A \to A'$. Thus $C(f)|_{A'}C(h)|_A = \hom_{\mathbf{StGrphs}}(h, C(B))C(f)|_{A'} = \hom_{\mathbf{StGrphs}}(h, C(B))\phi_{A',B}(f)$, and the diagram commutes. Hence $\phi_{A,B}$ is natural in A. A similar proof shows $\phi_{A,B}$ is natural in B.

(Part 5): Proposition 2.4.1 can easily be extended to show that the inclusion of **SiLlSt-Grphs** into the other four graph categories does not have a right adjoint. Proposition 2.4.1 can also be extended to show that the inclusion of **SiLlStGrphs** into **Grphs**, **SiGrphs** and **SiLlGrphs** does not have a left adjoint.

Suppose the inclusion functor **SiLlStGrphs**$\hookrightarrow$**StGrphs** has a left adjoint R :**StGrphs**$\rightsquigarrow$**SiLlStGrphs**. Then there is a natural bijection $\hom_{\mathbf{SiLlStGrphs}}(R(A), B) \cong \hom_{\mathbf{StGrphs}}(A, \hookrightarrow (B))$. Consider $A = K_1^\ell$ the graph of a vertex and a loop incident to that vertex. Suppose $B = K$ where K is the complete graph on $V(R(A))$. Then as $\hom_{\mathbf{StGrphs}}(K_1^\ell, K) = \emptyset$, there is no morphism from $R(A)$ to K. As every simple loopless graph, G, admits the inclusion morphism into the complete graph on $V(G)$, we reach a contradiction and no such left adjoint exists. A similar proof will show the inclusion functor from **SiLlStGrphs** to **SiStGrphs** does not have a left adjoint.

By Propositions 2.4.2 and 2.4.3, the inclusion functor from **SiStGrphs** to **Grphs** and **SiGrphs** does not have a left adjoint. We will show no right adjoint exists.

Suppose **SiStGrphs**$\hookrightarrow$**Grphs** has a right adjoint C :**Grphs**$\rightsquigarrow$**SiStGrphs**. Then there is a natural bijection $\hom_{\mathbf{Grphs}}(\hookrightarrow (A), B) \cong \hom_{\mathbf{SiStGrphs}}(A, C(B))$. Let $B = K_1^\ell$ the graph with a vertex and a loop incident to the vertex. Using a "test" object $A = K_1$ the graph of a single vertex, as $\hom_{\mathbf{Grphs}}(\hookrightarrow (K_1), K_1^\ell)$ has a single element, $\hom_{\mathbf{SiStGrphs}}(K_1, C(K_1^\ell))$ has a single element. Hence $C(K_1^\ell)$ has a single vertex. Thus $C(K_1^\ell)$ is K_1 or K_1^ℓ. Now using the "test" object $A = K_2$, as $\hom_{\mathbf{Grphs}}(\hookrightarrow (K_1), K_1^\ell)$ has two elements, K_2 admits two morphisms to $C(K_1^\ell)$ in **SiStGrphs**. However, K_2 does not admit two morphisms to either K_1 or K_1^ℓ, a contradiction. Thus no right adjoint exists. The same proof also holds to show no right adjoint exists to the inclusion functor **SiStGrphs**$\hookrightarrow$**SiGrphs**.

Now suppose there is a right adjoint to the inclusion functor **SiStGrphs**$\hookrightarrow$**StGrphs**,

C :**StGrphs**$\rightsquigarrow$**SiStGrphs**. Then there is a natural bijection $\hom_{\mathbf{StGrphs}}(\hookrightarrow(A), B) \cong \hom_{\mathbf{SiStGrphs}}(A, C(B))$. Letting B be the graph of a single vertex with two loops at that vertex, we can use a similar "test" object argument as we did above using $A = K_1$ to determine $C(B)$ has one vertex and $A = K_2$ to derive a contradiction. Using a similar argument with the same graph B and the same test objects $A = K_1$ and $A = K_2$ we can derive a similar contradiction to the right adjoint to the inclusion functor from **SiGrphs** to **Grphs**. Using a similar argument with the graph $B = K_1^\ell$ and test objects $A = K_1$ and $A = K_2$ we can derive a similar contradiction to a right adjoint to the inclusion functor from **SiLlGrphs** to **Grphs** and **SiGrphs**.

By Proposition 2.4.3 the inclusion functor from **StGrphs** to **Grphs** does not have a left adjoint. So we must only consider a left adjoint to the inclusion functor from **SiGrphs** to **Grphs**. Suppose there is a left adjoint R :**Grphs**$\rightsquigarrow$**SiGrphs**. Then $\hookrightarrow$ must commute with limits. So consider the product $K_1^\ell \times K_2$ in **SiGrphs** and **Grphs**. Their constructions are shown below.

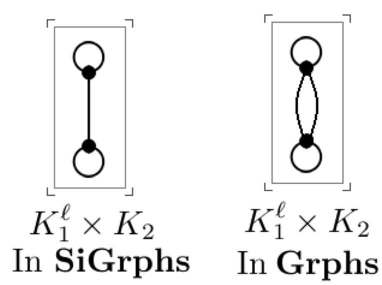


Figure 2.12: A counterexample to the inclusion functor **SiGrphs**$\hookrightarrow$**Grphs** preserving products.

Hence $\hookrightarrow(K_1^\ell \times K_2) \ncong \hookrightarrow(K_1^\ell) \times \hookrightarrow(K_2)$, a contradiction. Thus no left adjoint exists. $\square$

This provides us with the following "big picture" with the adjoints to inclusion included.

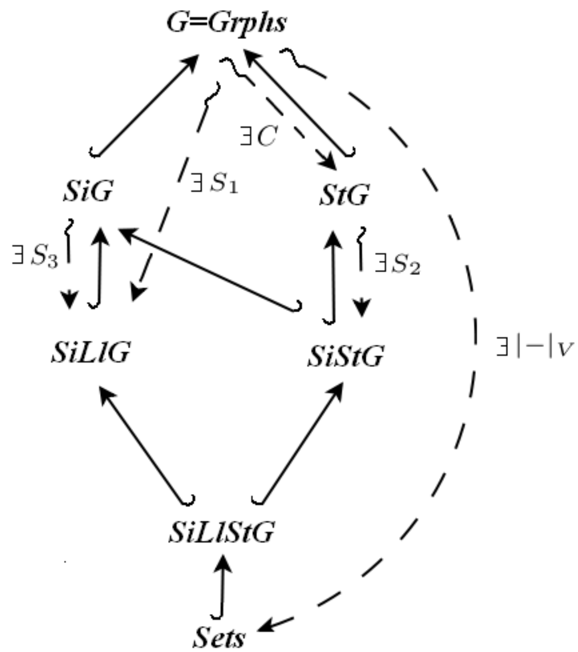


Figure 2.13: The Categories of Graphs with all adjoints to the inclusion functors

Chapter 3

The Fundamental Morphism Theorem and the Classification of Special Morphisms

3.1 Preliminaries for the Fundamental Morphism Theorem

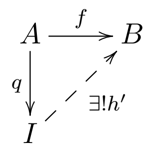
In the next three sections, as **SiLlStGrphs** does not have the coequalizer construction (e.g. for $p_0, p_1 : K_2 \times K_2 \to K_2$ the canonical projection morphisms, $coeq(p_0, p_1)$ does not exist), we will only consider the five categories **Grphs**, **SiGrphs**, **SiLlGrphs**, **StGrphs** and **SiStGrphs**.

In a category with products, coproducts, equalizers, and coequalizers, for a morphism $f : A \to B$ we can form the following construction [16],

$$\begin{array}{ccccccccccc} R_f & \xrightarrow{k} & A \times A & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & A & \xrightarrow{f} & B & \overset{i_0}{\underset{i_1}{\rightrightarrows}} & B + B & \xrightarrow{k^*} & R_f^* \\ & & & & {\scriptstyle q}\downarrow & = & \uparrow{\scriptstyle q^*} & & & & \\ & & & & I & \underset{\exists! h}{\dashrightarrow} & I^* & & & & \end{array} \tag{3.1}$$

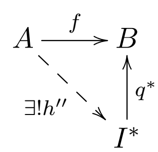
where $k = eq(fp_0, fp_1)$, $q = coeq(p_0k, p_1k)$, $k^* = coeq(i_0f, i_1f)$, and $q^* = eq(k^*i_0, k^*i_1)$. This construction yields a unique morphism $h : I \to I^*$ which makes the diagram commute as follows.

As $q = coeq(p_0k, p_1k)$ and $k = eq(fp_0, fp_1)$, $qp_0k = qp_1k$ and $fp_0k = fp_1k$. Then by the UMP of q, there is a unique morphism $h' : I \to B$ such that $h'q = f$ and the following diagram commutes.



(3.2)

Now since $q^* = eq(k^*i_0, k^*i_1)$ and $k^* = coeq(i_0f, i_1f)$, $k^*i_0q^* = k^*i_1q^*$ and $k^*i_0f = k^*i_1f$. Then by the UMP of q^* there is a unique morphism $h'' : A \to I^*$ such that $q^*h'' = f$ and the following diagram commutes.



(3.3)

Now by the UMP of q^* there is a unique morphism $h : I \to I^*$ such that $h' = hq^*$ and the diagram (3.1) commutes, since $h'q = f = q^*h''$, $k^*i_0h'q = k^*i_1h'q$, and q is an epimorphism.

This construction and the resulting unique arrow h is known as the Weak Fundamental Morphism Theorem. The (Strong) Fundamental Morphism Theorem [16] asserts that $h : I \to I^*$ is an isomorphism. The three Noether Isomorphism Theorems follow as corollary of the Fundamental Morphism Theorem. K.K. Williams developed the three Noether Isomorphism Theorems for **Grphs** directly [27].

Recall from Chapter 2 the product, coproduct, equalizer, and coequalizer constructions in the five categories of graphs.

Given two graphs A and B in **Grphs**, the *product*, $A \times B$, is defined by $V(A \times B) = V(A) \times V(B)$ and for $e \in P(A)$ with $\partial_A(e) = (a_1\_a_2)$ and $f \in P(B)$ with $\partial_B(f) = (b_1\_b_2)$ there is an element (e, f) in $P(A \times B)$ with $\partial_{A\times B}((e, f)) = ((a_1, b_1)\_(a_2, b_2))$ and if $a_1 \neq a_2$ and $b_1 \neq b_2$, there is another element $\overline{(e, f)} \in P(A \times B)$ with $\partial_{A\times B}(\overline{(e, f)}) = ((a_1, b_2)\_(a_2, b_1))$

that has the same projections as (e, f).

In **SiGrphs** we follow the same construction and identify any parallel edges to a single edge and any multiple loops to a single loop, and in **SiLlGrphs** we also identify any loops with their incident vertex.

In **StGrphs** and **SiStGrphs** we follow the construction of **Grphs** but delete all pairs (e, f) if exactly one of e or f is a vertex.

In all five categories of graphs the *coproduct*, $A + B$, of two graphs A and B is the disjoint union of the two graphs, and the *equalizer*, $q = eq(f, g)$, of two morphism $f, g : A \to B$ is the inclusion of the subgraph Eq of A defined by $P(Eq) = \{a \in P(A) | f(a) = g(a)$ and if $\partial_A(a) = (a_1 \_ a_2)$ then $f(a_1) = g(a_1)$ and $f(a_2) = g(a_2)\}$.

In **Grphs** and **StGrphs** the *coequalizer*, $coeq(f, g)$, of two morphism $f, g : A \to B$ is the natural quotient morphism from B to $Coeq$ defined by $P(Coeq) = P(B)/\sim$ where $\sim$ is the equivalence relation defined by $a \sim b$ if there is a sequence $a_0, a_1, \ldots, a_n \in P(A)$ such that $a = f(a_0), g(a_0) = f(a_1), g(a_1) = f(a_2), \ldots, g(a_{n-1}) = f(a_n)$ and $b = f(a_n)$ or $b = g(a_n)$.

In **SiGrphs** and **SiStGrphs** we follow the same construction for the coequalizer but we also identify any parallel edges to a single edge and any multiple loops to a single loop, and in **SiLlGrphs** we also identify any loops to their incident vertex.

Since products, coproducts, equalizers, and coequalizers exist in the five categories of graphs, for any morphism in the category we can follow the construction in diagram (3.1).

Theorem 3.1.1 (The Weak Fundamental Morphism Theorem)**.** *In* ***Grphs, SiGrphs, SiLl-Grphs, StGrphs*** *and* ***SiStGrphs*** *the construction in diagram (3.1) yields the unique arrow h that makes the diagram commute.*

3.2 The Fundamental Morphism Theorem in Grphs and StGrphs

We will establish the (Strong) Fundamental Morphism Theorem in **Grphs** and **StGrphs** but we first need a lemma concerning the properties of morphisms in these two categories. We will also need to make use of this property for **SiGrphs** later, so we will include it with the lemma.

Lemma 3.2.1. ***Grphs**, **StGrphs**, and **SiGrphs** are balanced categories.*

Proof. Let $f : A \to B$ be both a monomorphism and an epimorphism in **Grphs**. Then by Proposition 2.3.13, $f : P(A) \to P(B)$ is a bijection and there is a set function $f^{-1} : P(B) \to P(A)$ such that $ff^{-1} = 1_{P(B)}$ and $f^{-1}f = 1_{P(A)}$ as set functions. It suffices to show f^{-1} is a graph morphism.

As f is a morphism, f maps vertices to vertices, and as f is a bijection, f^{-1} maps vertices to vertices. Further as monomorphisms are trivially strict morphisms, both f and f^{-1} map edges to edges. Now let $e \in E(B)$ with $\partial_B(e) = (b_1\_b_2)$ for some $b_1, b_2 \in V(B)$, then there is an edge $e' \in E(A)$ with $\partial_A(e') = (a_1\_a_2)$ such that $f(e') = e$. Since f is a morphism, incidence is preserved and $(b_1\_b_2) = (f(a_1)\_f(a_2))$. Hence $f^{-1}(e) = e'$ and $\partial_A(f^{-1}(e)) = (f^{-1}(b_1)\_f^{-1}(b_2)) = (f^{-1}(f(a_1))\_f^{-1}(f(a_2))) = (a_1\_a_2) = \partial_A(e')$, and incidence is preserved. Thus f^{-1} is a graph morphism, and f is an isomorphism.

This proof also holds in **SiGrphs** and **StGrphs**. □

We note that since **Grphs** and **StGrphs** are balanced, all epimorphisms are extremal epimorphisms (defined below) in these two categories (see Theorem 3.5.1).

Definition 3.2.2. *A morphism $f : A \to B$ is an* extremal epimorphism *if f does not factor through any proper monomorphism, i.e. if $f = me$ with m a monomorphism and e an epimorphism, then m is an isomorphism.*

We now proceed with the Fundamental Morphism Theorem.

Theorem 3.2.3 (The Fundamental Morphism Theorem)**.** *In **Grphs** and **StGrphs** the unique morphism $h : I \to I^*$ in the construction given by (3.1) is an isomorphism.*

Proof. Consider the construction in **Grphs**. We proceed by establishing five claims:

Claim 1: $k^* = coeq(i_0 f, i_1 f)$ identifies parts $i_0(e)$ and $i_1(e)$ for $e \in P(B)$ if and only if $e \in P(Im(f))$ where $Im(f)$ is the image of f, a subgraph of B.

Claim 2: $I^* = Im(f)$.

Claim 3: $q = coeq(p_0 k, p_1 k)$ identifies $a, b \in P(A)$ if and only if $f(a) = f(b)$.

Claim 4: $h : I \to I^*$ is a monomorphism.

Claim 5: h'' defined as in (3.3) is an epimorphism (and by Lemma 3.2.1 an extremal epimorphism).

Once these claims are established then as $h'' = hq$ is a proper epimorphism factorization of an extremal epimorphism, h is an isomorphism.

Proof of Claim 1. ($\Leftarrow$) Let $v \in V(Im(f))$, then there is a vertex $u \in V(A)$ such that $v = f(u)$ for if v is the image of an edge, then v is also the image of the edge's incident vertices. Hence, as $i_0 f(u) = i_0(v)$ and $i_1 f(u) = i_1(v)$, $k^* i_0(v) = k^* i_1(v)$.

Let $e \in E(Im(f))$, then there is an edge $e' \in E(A)$ with $f(e') = e$, and hence as $i_0 f(e') = i_0(e)$ and $i_1 f(e') = i_1(e)$, $k^* i_0(e) = k^* i_1(e)$.

($\Rightarrow$) We prove the converse by contrapositive. Let $b \in P(B) \backslash P(Im(f))$. Then for all parts $a \in P(A)$, $f(a) \neq b$, and hence $i_0 f(a) \neq i_0(b)$ and $i_1 f(a) \neq i_1(b)$. Thus $i_0(b) \nsim i_1(b)$ and $k^* i_0(b) \neq k^* i_1(b)$ as there is no sequence formed in the construction of the coequalizer between $i_0(b)$ and $i_1(b)$.

Proof of Claim 2. We first show $P(I^*) = P(Im(f))$ as sets. Let $e \in P(I^*)$, then as $q^* = eq(k^* i_0, k^* i_1)$, $k^* i_0 q^*(e) = k^* i_1 q^*(e)$. As q^* is inclusion, $k^* i_0(e) = k^* i_1(e)$ and by Claim 1 $e \in P(Im(f))$.

Now let $e \in P(Im(f))$. Then by Claim 1, $k^* i_0(e) = k^* i_1(e)$. If $e \in E(Im(f))$ then so are the vertices incident to e, and as q^* is an equalizer $e \in P(I)$. Hence as sets $P(I) = P(Im(f))$. As q^* is a morphism, incidence is preserved and they are equal as graphs.

Proof of Claim 3. We first note that as $p_0((a,b)) = a$ and $p_1((a,b)) = b$, $P(R_f) = \{(a,b) \in P(A \times A) | f(a) = f(b)$ and if $\partial_{A \times A}((a,b)) = ((u_a, u_b)_-(v_a, v_b))$ then $f(u_a) = f(u_b)$ and $f(v_a) = f(v_b)\}$.

($\Rightarrow$) Let $a, b \in P(A)$ be such that $q(a) = q(b)$. Then there is a sequence $a_1, a_2, \ldots, a_n \in P(R_f)$ with $a = p_0k(a_1), p_1k(a_1) = p_0k(a_2), p_1k(a_2) = p_0k(a_3), \ldots, p_1k(a_{n-1}) = p_0k(a_n)$ and $b = p_0k(a_n)$ or $b = p_1k(a_n)$. As $k : R_f \to A \times A$ is inclusion, $a = p_0(a_1), p_1(a_1) = p_0(a_2), p_1(a_2) = p_0(a_3), \ldots, p_1(a_{n-1}) = p_0(a_n)$ and $b = p_0(a_n)$ or $b = p_1(a_n)$. Then since $a = p_0(a_1)$, $a_1 = (a, c_1)$ for some $c_1 \in P(A)$. As $p_0(a_2) = p_1(a_1)$, $a_2 = (c_1, c_2)$ for some $c_2 \in P(A)$, and inductively $p_0(a_i) = p_1(a_{i-1})$ implies $a_i = (c_{i-1}, c_i)$ for $3 \leq i \leq n$. Then as $b = p_0(a_n)$ or $b = p_1(a_n)$, $b = c_{n-1}$ or $b = c_n$ respectively. Since $a_1, a_2, \ldots, a_n \in P(R_f)$, the object of an equalizer, $f(a) = f(c_1), f(c_1) = f(c_2), \ldots, f(c_{n-1}) = f(c_n)$ and transitively $f(a) = f(b)$.

($\Leftarrow$) Let $a, b \in P(A)$ with $f(a) = f(b)$. We consider two cases.

Case 1: One of a or b is a vertex or a loop.

Without loss of generality, let a be a vertex or a loop. Then $\partial_A(a) = (u\_u)$ for some $u \in V(A)$. Let $\partial_A(b) = (u_b\_v_b)$ for some $u_b, v_b \in V(A)$. Since $f(a) = f(b)$ and morphisms preserve incidence, $f(u) = f(u_b) = f(v_b)$. Thus $(a, b) \in P(R_f)$ and as $q = coeq(p_0k, p_1k)$, $p_1k((a, b)) = b$ and $p_0k((a, b)) = a$, $q(a) = q(b)$.

Case 2: a and b are non-loop edges.

Let $\partial_A(a) = (u_a\_u_b)$ and $\partial_A(b) = (v_b\_v_b)$ for some $u_a, u_b, v_a, v_b \in V(A)$. Since $f(a) = f(b)$, $(f(u_a)\_f(v_a)) = (f(u_b)\_f(v_b))$ and hence either $f(u_a) = f(u_b)$ and $f(v_a) = f(v_b)$ or $f(u_a) = f(v_b)$ and $f(u_b) = f(v_a)$. In the first case $(a, b) \in P(R_f)$ and in the second case $\overline{(a, b)} \in P(R_f)$. As k is inclusion, $p_0((a, b)) = p_0(\overline{(a, b)}) = a$, and $p_1((a, b)) = p_1(\overline{(a, b)}) = b$, $q(a) = q(b)$.

Proof of Claim 4: Let $a, b \in V(I)$ with $a \neq b$. As q is a coequalizer, q is an epimorphism and by Proposition 2.3.13 surjective on part sets. Hence there is $u, v \in P(A)$ such that $q(u) = a$ and $q(v) = b$. By Claim 3, as $a \neq b$, $f(u) \neq f(v)$. Then since $f = q^*hq$ and q^* is inclusion, $h(a) = q^*h(a) = q^*hq(u) = f(u) \neq f(v) = q^*hq(v) = q^*h(b) = h(b)$, and h is an injection on part sets. Then by Proposition 2.3.13, h is a monomorphism.

Proof of Claim 5: By Claim 2, $I^* = Im(f)$. So define $\overline{h} : A \to I^*$ by $\overline{h}(e) = f(e)$ for all $e \in P(A)$. As $Im(f) = I^*$ and f is a morphism, $\overline{h}$ is well defined and a morphism. Since q^* is inclusion, $q^*\overline{h}(a) = q^*f(u) = f(a)$ for all $a \in P(A)$. Thus $q^*\overline{h} = f$. However, h'' is the unique morphism such that $q^*h'' = f$. Therefore $\overline{h} = h''$. As $\overline{h}$ is a surjection on part sets, so is h''. Thus by Proposition 2.3.13 h'' is an epimorphism.

The proof for **StGrphs** follows similarly. □

3.3 Restricted Categories of Graphs

As we add restrictions on the graphs in our categories, the coequalizer morphism identifies parallel edges and loops. The following figure of the construction (3.1) applied to $f : K_2^c \to K_2$ the injection of the two vertices into K_2 provides a counterexample to the (Strong) Fundamental Morphism Theoerem for all three of **SiGrphs**, **SiStGrphs**, and **SiLlGrphs**. We note, by the discussion in section 2, a unique morphism $\overline{f} : I \to I^*$ exists, but it is not necessarily an isomorphism.

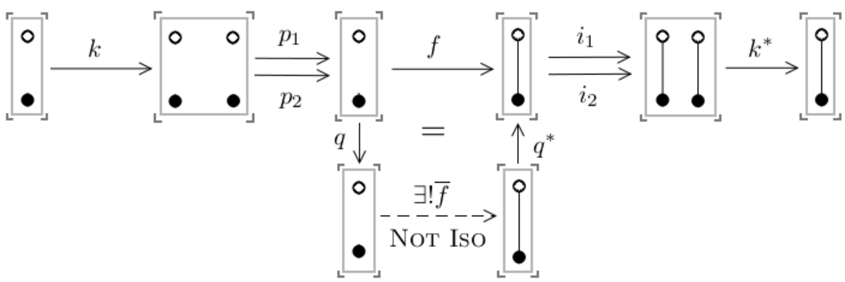


Figure 3.1: A counterexample to the (Strong) Fundamental Morphism Theorem in **SiGrphs**, **SiStGrphs**, and **SiLlGrphs**

3.4 Classifications of Special Morphisms in an Abstract Category

In this section, we will define five special classifications of monomorphisms, and dually five special classifications of epimorphisms. In the next section, we will review their relationships in general categories, as well as categories with extra properties. We will characterize these classifications in **Grphs**, **StGrphs** and **SiGrphs**, **SiLlGrphs**, **SiStGrphs** and **SiLlStGrphs**.

We now define the five special classifications for monomorphisms in a category.

Definition 3.4.1. *A morphism $f : A \to B$ is a* split equalizer *if $f = eq(q_1, q_2)$ for two morphisms $q_1, q_2 : B \to C$ for some object C, and there exists morphisms $s' : C \to B$ and $s : B \to A$ such that $sf = 1_A$, $s'q_1 = 1_B$, and $s'q_2 = fs$.*

An example of a split equalizer in **Top** is an embedding into a space that can be continuously deformed into the embedded space.

Definition 3.4.2. *A morphism $f : A \to B$ is a* coretract *if f has a left inverse g, i.e. there is a morphism $g : B \to A$ such that $gf = 1_A$.*

An example of a coretract in **Top** is an embedding into a space that can be continuously mapped into the image of the embedding. In **Grphs** the inclusion of a vertex into a graph is a coretract.

Definition 3.4.3. *A morphism $f : A \to B$ is an* effective monomorphism *if $f = eq(cokp(f))$ where $cokp(f)$ is the cokernel pair of f, the canonical pair of morphisms $c_1, c_2 : A \to A +_B A$ where $A +_B A$ is the pushout of f with itself.*

Definition 3.4.4. *A morphism $f : A \to B$ is a* regular monomorphism *if f is an equalizer, i.e. there exists morphisms $q_1, q_2 : B \to C$ for some object C such that $f = eq(q_1, q_2)$.*

We remark that a regular monomorphism is not the same thing as the equalizer, as a functor can preserve regular monomorphisms, but not preserve equalizers. For example, consider $1_{K_2}, t_w : K_2 \to K_2$ in **Grphs**. $|Eq(1_{K_2}, t_w)|_P = \emptyset$, but $Eq(|1_{K_2}|_P, |t_w|_P) = e$ for e the edge of K_2.

Definition 3.4.5. *A morphism $f : A \to B$ is an* extremal monomorphism *if f does not factor through any proper epimorphism, i.e. if $f = me$ with m a monomorphism and e an epimorphism, then e is an isomorphism.*

In **Top**, extremal monomorphisms are embeddings. In **SiStGrphs** and **SiLlGrphs** an extremal monomorphism is the inclusion of a vertex induced subgraph.

We now dually define the five special classifications for epimorphisms in a category.

Definition 3.4.6. *A morphism $f : A \to B$ is a* split coequalizer *if $f = coeq(p_1, p_2)$ for two morphisms $p_1, p_2 : C \to A$ for some object C, and there exists morphisms $s' : A \to C$ and $s : B \to A$ such that $fs = 1_B$, $p_1 s' = 1_A$, and $p_2 s' = sf$.*

Definition 3.4.7. *A morphism $f : A \to B$ is a* retract *if f has a right inverse g, i.e. there is a morphism $g : B \to A$ such that $fg = 1_B$.*

An example of a retract in **Grphs** is the constant vertex morphism. In fact, for **SiStGrphs** there is large theory involving retracts from a graph to its subgraphs [14].

Definition 3.4.8. *A morphism $f : A \to B$ is an* effective epimorphism *if $f = coeq(kp(f))$ where $kp(f)$ is the kernel pair of f, the canonical pair of morphisms $c_1, c_2 : A \times_B A \to A$, where $A \times_B A$ is the pullback of f with itself.*

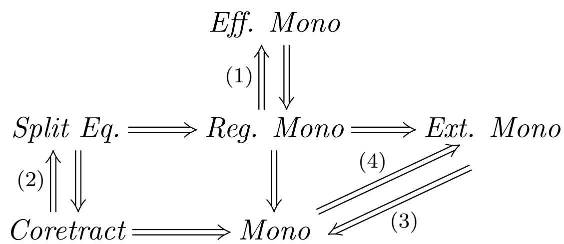
Definition 3.4.9. *A morphism $f : A \to B$ is a* regular epimorphism *if f is an coequalizer, i.e. there exists morphisms $p_1, p_2 : C \to A$ for some object C such that $f = coeq(p_1, p_2)$.*

We will also consider extremal epimorphisms which we defined earlier (Definition 3.2.2). In **Top**, extremal epimorphisms are identifications.

3.5 Categorial characterizations of special morphism classifications

We begin with a general characterization theorem for the classification of special types of monomorphisms.

Theorem 3.5.1. [Classification of Monomorphisms] [1] *In any category,*



where (1) and (2) hold if there are pushouts, (3) holds if there are coequalizers, and (4) holds if the category is balanced.

In **Sets** all of these classifications are equivalent, as **Sets** is balanced, has pushouts, has coequalizers, and has "cochoice" (note cochoice implies monos are coretracts).

However, in other categories differences emerge. For example, in **Top** we have the following

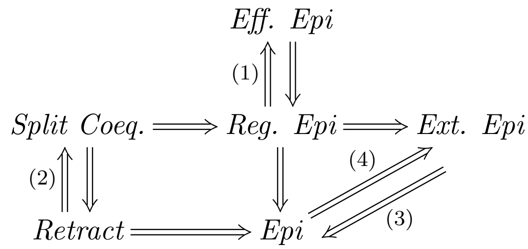
$$\text{Split Eq.} = \text{Coretract} \Longrightarrow \text{Eff. Mono} = \text{Reg. Mono} = \text{Ext. Mono} \Longrightarrow \text{Mono}$$

where the "$\Rightarrow$" is strict in each instance. In abelian categories the morphism classifications characterize as

$$\text{Split Eq.} = \text{Coretract} \Longrightarrow \text{Eff. Mono} = \text{Reg. Mono} = \text{Ext. Mono} = \text{Mono}$$

with "$\Rightarrow$" strict. We now consider the dual, the classification of epimorphisms.

Theorem 3.5.2. [Classification of Epimorphisms] [1] *In any category,*



where (1) and (2) hold if there are pullbacks, (3) holds if there are equalizers, and (4) holds if the category is balanced.

In a similar vein, in **Sets** all of these classifications are equivalent, and the dual results for **Top** and abelian categories also hold.

3.6 The classification of special morphisms in Grphs, StGrphs and SiGrphs

In the three least restricted categories of graphs, the characterization of classifications of special types of monomorphisms is similar to abelian categories.

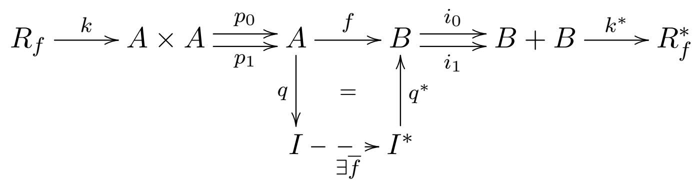
Theorem 3.6.1. *In **Grphs**, **StGrphs** and **SiGrphs***

$$\textit{Split Eq.} = \textit{Coretract} \Longrightarrow \textit{Eff. Mono} = \textit{Reg. Mono} = \textit{Ext. Mono} = \textit{Mono}.$$

Proof. We will handle the case of **SiGrphs** separately from **Grphs** and **StGrphs**.

In **Grphs** and **StGrphs** pushouts and coequalizers exist. Furthermore these two categories are balanced (Lemma 3.2.1). So it suffices to show that extremal monomorphisms are effective monomorphisms, and monomorphisms are not coretracts.

Let $f : A \to B$ be an extremal monomorphism, then consider the following construction,



where $k = eq(fp_0, fp_1)$, $q = coeq(p_0k, p_1k)$, $k^* = coeq(i_0f, i_1f)$, and $q^* = eq(k^*i_0, k^*i_1)$. By the Fundamental Morphism Theorem (Theorem 3.2.3), $\overline{f}$ is an isomorphism. Since $\overline{f}$ is an isomorphism, it is a monomorphism. Hence as $q : A \to I$ is an epimorphism and $q^* : I^* \to B$ is a monomorphism, $f = (q^*f)q$ is an epi-mono factorization of f. Thus as f is an extremal monomorphism, q is an isomorphism and $I^* \cong A$. Thus $f = coeq(k^*i_0, k^*i_1)\overline{f}q = eq(cokp(f))$ and f is an effective monomorphism.

For a counterexample to "monomorphisms are coretracts" consider the inclusion of $K_2^c \hookrightarrow K_2$.

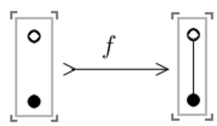
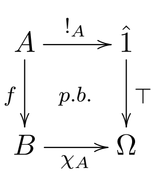


Figure 3.2: A counterexample to the "monomorphisms are coretracts" in **Grphs**, **StGrphs**, and **SiGrphs**.

As f is injective on part sets it is clearly a monomorphism, but it has no left inverse.

For **SiGrphs** we note that the above counterexample still applies to show monomorphisms are not coretracts. As pushouts and coequalizers exist in **SiGrphs** and **SiGrphs** is balanced (Lemma 3.2.1), we must only show that monomorphisms are regular monomorphisms.

Let $f : A \to B$ be a monomorphism. Then as **SiGrphs** has a subobject classifier (Proposition 2.3.2), Ω, there exists a morphism $\chi_A : B \to \Omega$ such the following diagram is a pullback.



However, as products and equalizers exist in **SiGrphs**, we can construct the pullback of χ_A and $\top$ by first constructing $\hat{1} \times B$ with projections $p_B : \hat{1} \times B \to B$ and $!_{\hat{1}\times B} : \hat{1} \times B \to \hat{1}$ and then constructing the equalizer $q = eq(\chi_A p_B, \top !_{\hat{1}\times B}) : Eq(\chi_A p_B, \top !_{\hat{1}\times B}) \to \hat{1} \times B$. As pullbacks are unique upto isomorphism and $\hat{1} \times B \cong B$, we have isomorphisms $\psi : A \to Eq(\chi_A p_B, \top !_{\hat{1}\times B})$ and $\phi : B \to \hat{1} \times B$, with $\phi f = q\psi$. Hence $f = eq(\chi_A p_B \phi, \top !_{\hat{1}\times B}\phi)$ and f is a regular monomorphism. □

We note that the proof that monomorphisms are regular monomorphisms given in **SiGrphs** also holds in **Grphs** and **StGrphs**. We chose to use the Fundamental Morphism Theorem to exhibit that in a category with the Fundamental Morphism Theorem, extremal monomorphisms are effective monomorphisms.

Dually, the characterization of the classification of special epimorphisms in **Grphs**, **StGrphs** and **SiGrphs** is also similar to abelian categories.

Theorem 3.6.2. *In **Grphs**, **StGrphs** and **SiGrphs***

$$\textit{Split Coeq.} = \textit{Retract} \Longrightarrow \textit{Eff. Epi} = \textit{Reg. Epi} = \textit{Ext. Epi} = \textit{Epi.}$$

Proof. We will handle the case of **SiGrphs** separately from **Grphs** and **StGrphs**.

In **Grphs** and **StGrphs** pullbacks and equalizers exist. Furthermore these two categories are balanced (Lemma 3.2.1). So it suffices to show that extremal epimorphisms are effective epimorphisms, and epimorphisms are not retracts.

Let $f : A \to B$ be an extremal epimorphism, then consider the following construction,

$$\begin{array}{ccccccccccc} R_f & \xrightarrow{k} & A \times A & \underset{p_1}{\overset{p_0}{\rightrightarrows}} & A & \xrightarrow{f} & B & \underset{i_1}{\overset{i_0}{\rightrightarrows}} & B + B & \xrightarrow{k^*} & R_f^* \\ & & & & {\scriptstyle q}\downarrow & = & \uparrow{\scriptstyle q^*} & & & & \\ & & & & I & \underset{\exists \bar{f}}{\dashrightarrow} & I^* & & & & \end{array}$$

where $k = eq(fp_0, fp_1)$, $q = coeq(p_0 k, p_1 k)$, $k^* = coeq(i_0 f, i_1 f)$, and $q^* = eq(k^* i_0, k^* i_1)$. By the Fundamental Morphism Theorem (Theorem 3.2.3), $\overline{f}$ is an isomorphism. Since $\overline{f}$ is an isomorphism, it is an epimorphism. Hence as $q : A \to I$ is an epimorphism and $q^* : I^* \to B$ is a monomorphism, $f = q^*(fq)$ is an epi-mono factorization of f. Thus as f is an extremal epimorphism, q^* is an isomorphism and $I \cong B$. Thus $f = q^*\overline{f}coeq(p_0k, p_1k) = coeq(kp(f))$ and f is an effective epimorphism. For a counterexample to "epimorphisms are retracts" consider the following morphism of $K_2 + K_2 \to P_3$ where P_3 is the path on three vertices,

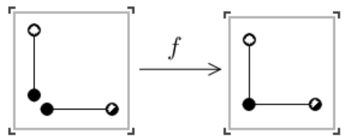


Figure 3.3: A counterexample to the "epimorphisms are retracts" in **Grphs**, **StGrphs**, and **SiGrphs**.

where each vertex is mapped to the vertex with the same coloring. As f is surjective on part sets it is an epimorphism, but no right inverse of f exists.

For **SiGrphs** we note that the above counterexample still applies to show epimorphisms are not retracts. As pullbacks and equalizers exist in **SiGrphs** and **SiGrphs** is balanced (Lemma 3.2.1), we must only show that epimorphisms are regular epimorphisms.

Let $f : A \to B$ be an epimorphism in **SiGrphs**. Then f is a surjection on part sets, and for $b \in P(B)$, $f^{-1}(b) \subseteq P(A)$. We will construct a graph C with two morphisms $c_1, c_2 : C \to A$ such that $f = coeq(c_1, c_2)$.

We define the graph C by creating a K_2 component of C, with the edge labeled x and vertices x_1, x_2 with $\partial_C(x) = (x_1 \_ x_2)$ for every $x \in P(A)$. For each $x \in P(A)$ fix an ordering of the incident vertices so that $\partial_A(x) = (x_{a1} \_ x_{a2})$ with $x_{a1} \leq x_{a2}$.

Now define $c_1 : C \to A$ by $c_1(x) = x$ and for $\partial_A(x) = (x_{a1} \_ x_{a2})$, $c_1(x_1) = x_{a1}$ and $c_1(x_2) = x_{a2}$. Incidence is trivially preserved, and c_1 is a graph morphism.

For all $b \in P(B)$ fix an ordering of $f^{-1}(b)$ by $f^{-1}(b) = \{x_{b_i}\}_{i \in I_b}$ for some well-ordered index set I_b (which exists by the axiom of choice). Now define $c_2 : C \to A$ by $c_2(x) = c_1(x')$, $c_2(x_1) = c_1(x'_1)$ and $c_2(x_2) = c_1(x'_2)$ where $x = x_{b_i}$ for some $b \in P(A)$ and $x' = x_{b_j}$ for j the least element of I_b with $j > i$ if it exists and $c_2(x) = c_1(x), c_2(x_1) = c_1(x_1)$ and $c_2(x_2) = c_1(x_2)$

otherwise. As each K_2 component of C is mapped as a component by c_1 in c_2, and as c_1 is a morphism, c_2 is a graph morphism.

By construction, $fc_1 = fc_2$. We now show that $f = coeq(c_1, c_2)$.

Let Y be a graph with $y : A \to Y$ such that $yc_1 = yc_2$. Define $\overline{y} : B \to Y$ by $\overline{y}(b) = y(a_b)$ where a_b is a fixed element of $f^{-1}(b)$. Let $a'_b \in f^{-1}(b)$ and suppose $a_b \leq a'_b$ under the ordering of I_b. Let $a_b < a_1 < a_2 < \cdots < a_n < a'_b$ be all elements in the ordering of I_b from a_b to a'_b. Then as $c_2(a_b) = c_1(a_1)$, $c_2(a_1) = c_1(a_2)$, $\ldots$ $c_2(a_n) = c_1(a'_b)$, and $yc_1 = yc_2$, $\overline{y}(b) = y(a_b) = yc_1(a_b) = yc_2(a_b) = yc_1(a_1) = yc_2(a_1) = yc_1(a_2) = yc_2(a_2) = \cdots = yc_2(a_n) = yc_1(a'_b) = y(a'_b)$. If $a'_b \leq a_b$ we just reverse the sequence and $\overline{y}$ is well defined.

By construction $\overline{y}f = y$, and $\overline{y}$ is uniquely determined by y. We must only show incidence is preserved. Let $b \in P(B)$ with $\partial_B(b) = (b_1 \_ b_2)$. By a similar argument to showing $\overline{y}$ is well defined, for any element $w \in f^{-1}(b_1)$, $y(w) = y(a_{b_1})$ and for any element $z \in f^{-1}(b_2)$, $y(z) = y(a_{b_2})$. Hence $\partial_Y(\overline{y}(b)) = (\overline{y}(b_1) \_ \overline{y}(b_2))$ and incidence is preserved. Hence $f = coeq(c_1, c_2)$ and f is a regular epimorphism. □

3.7 The classification of special morphisms in SiLlGrphs and SiStGrphs

Before we give the characertization of the classification of special morphisms in **SiLlGrphs** and **StStGrphs** we require the following three lemmas.

Lemma 3.7.1. *In **SiLlGrphs** and **SiStGrphs**, a morphism $f : A \to B$ is an extremal monomorphism if and only if f is a monomorphism and $Im(f)$ is a vertex induced subgraph of B.*

Proof. We will prove this lemma for **SiStGrphs**, and as monomorphisms are trivially strict, our proof will hold for **SiLlGrphs**.

($\Rightarrow$) We proceed by contradiction. We first note f is a monomorphism as coequalizers exist in **SiStGrphs** (and **SiLlGrphs**). Hence we assume that $Im(f)$ is not a vertex induced subgraph of B. Hence there is an edge $b \in E(B)$ with incidence $\partial_B(b) = (f(b_1)\_f(b_2))$ for some $b_1, b_2 \in V(A)$ and for all edges $a \in E(A)$, $\partial_A(a) \neq (b_1\_b_2)$.

We then construct a graph A' by appending an edge b' to A such that $\partial_{A'}(b') = (b_1\_b_2)$. By construction A' is a simple graph and there is a natural inclusion monomorphism $i : A \to A'$.

Now define $f' : A' \to B$ by $f'(x) = f(x)$ for all $x \in P(A')\backslash\{b'\}$, and $f'(b') = b$. As f is a monomorphism, it is injective on vertex sets. Hence f' is injective on vertex sets and is a monomorphism. Furthermore as $V(A) = V(A')$, $i : A \to A'$ is surjective on vertex sets and hence an epimorphism. Thus $f'i = f$ is a proper epi-mono factorization of f and i is not an isomorphism, a contradiction to f being an extremal epimorphism.

($\Leftarrow$) Conversely, let f be a monomorphism such that $Im(f)$ is a vertex induced subgraph of B. Suppose $e : A \to B$ and $m : C \to B$ are such that e is an epimorphism, m is a monomorphism and $f = me$. Then $f = me$ is a epi-mono factorization of f.

As e is an epimorphism it is surjective on vertices. Hence for all $c \in V(C)$ there is a vertex $a \in V(A)$ such that $e(a) = c$. As f and m are monomorphisms, they are injective on vertex sets, and hence for all $c \in V(C)$ and $a_1, a_2 \in V(A)$ with $f(a_1) = f(a_2) = m(c)$, $f(a_1) = me(a_1) = m(c) = me(a_2) = f(a_2)$ implies $a_1 = a_2$ and e is injective on vertices.

As e is strict (in **SiLlGrphs** it is a monomorphism and hence strict) for all $a \in E(A)$, $e(a) \in E(C)$. Furthermore, as e is bijective on vertices, if $a, a' \in E(A)$ are such that $e(a) = e(a')$ and $\partial_A(a) = (a_1\_a_2)$ and $\partial_A(a') = (a'_1\_a'_2)$, then $(e(a_1)\_e(a_2)) = \partial_C(e(a)) = \partial_C(e(a')) = (e(a'_1)\_e(a'_2))$ implies that $a_1 = a'_1$ and $a_2 = a'_2$ or that $a_1 = a'_2$ and $a_2 = a'_1$. In either case, as A is simple, $a = a'$ and e is injective on part sets.

Now let $c \in E(C)$ with $\partial_C(c) = (c_1\_c_2)$ for some $c_1, c_2 \in V(C)$. Then as e is surjective on vertices, there are vertices $a_1, a_2 \in V(A)$ such that $e(a_1) = c_1$ and $e(a_2) = c_2$. Then as m is strict (as m is a monomorphism it is strict in **SiLlGrphs**) there is an edge $m(c) \in E(B)$ with $\partial_B(m(c)) = (m(c_1)\_m(c_2)) = (me(a_1)\_me(a_2)) = (f(a_1)\_f(a_2))$. As $im(f)$ is a vertex induced subgraph of B, there is an edge $a \in E(A)$ such that $f(a) = m(c)$. Hence, as e is strict, incidence is preserved, and C is simple, $e(a) = c$. Hence e is bijective on part sets.

So we define $e^{-1} : P(C) \to P(A)$ by $e^{-1}(c) = a$ for $a \in P(A)$ the unique part such that $e(a) = c$. It clearly preserves incidence, maps vertices to vertices, and maps edges to edges. Hence e^{-1} is a graph morphism and e is an isomorphism. Hence f is an extremal monomorphism. □

Lemma 3.7.2. *In **SiLlGrphs** and **SiStGrphs**, a morphism $f : A \to B$ is an extremal epimorphism if and only if $Im(f) = B$.*

Proof. (⇒) We proceed by the contrapositive. Suppose $Im(f) \neq B$. If f is not surjective on vertices it is not an epimorphism and hence as equalizers exist f is not an extremal epimorphism and we are done. So assume f is surjective on vertices. Then as $Im(f) \neq B$, there in an edge $b \in E(B) \backslash E(Im(f))$. Then for $i : Im(f) \to B$ the inclusion of $Im(f)$ into B and for $\overline{f} : A \to Im(f)$ defined by $\overline{f}(a) = f(a)$ for all $a \in P(A)$, $f = i\overline{f}$ is a proper epi-mono factorization of f but i is not an isomorphism. Hence f is not an extremal epimorphism.

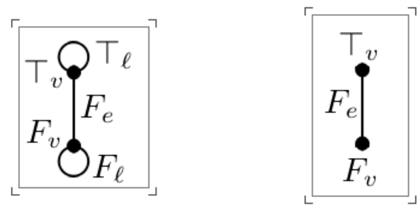
(⇐) Let $Im(f) = B$, and let $e : A \to C$ and $m : C \to B$ be such that e is an epimorphism, m is a monomorphism and $f = me$. Hence $f = me$ is an epi-mono factorization of f.

As $Im(f) = B$, f is surjective on vertices and hence f is an epimorphism. Then as $f = me$, and f is an epimorphism, so is me. Hence m is an epimorphism. As m is both a monomorphism and an epimorphism, m is bijective on vertices.

Let $b \in E(B)$. As $Im(f) = B$, there is an edge $a \in E(A)$ such that $f(a) = b$. Thus $f(a) = me(a) = b$. As b is an edge and m is a monomorphism (and hence strict) $e(a) \in E(C)$. Hence m is surjective on edges.

Now let $c, c' \in E(C)$ with $\partial_C(c) = (c_1\_c_2)$ and $\partial_C(c') = (c'_1\_c'_2)$ for some $c_1, c_2, c'_1, c'_2 \in V(C)$ be such that $m(c) = m(c')$. Then as incidence is preserved $(m(c_1)\_m(c_2)) = \partial_B(m(c)) = \partial_B(m(c')) = (m(c'_1)\_m(c'_2))$, and thus either $m(c_1) = m(c'_1)$ and $m(c_2) = m(c'_2)$ or $m(c_1) = m(c'_2)$ and $m(c_2) = m(c'_1)$. In either case, as m is injective on vertices, $(c_1\_c_2) = (c'_1\_c'_2)$. As C is simple, $c = c'$ and m is injective on part sets. Hence m is a bijection of part sets, and similar to the proof of Lemma 3.7.1, m is an isomorphism. Thus f is an extremal epimorphism. □

Lemma 3.7.3. *In* ***SiLlGrphs*** *and* ***SiStGrphs****, an extremal mono subobject classifier exists.*



In **SiStGrphs** In **SiLlGrphs**

Figure 3.4: The extremal monomorphism subobject classifier in **SiStGrphs** and in **SiLlGrphs**.

Proof. Let Ω be the proposed extremal monomorphism subobject classifier. In **SiStGrphs** we define $\top : \hat{1} \to \Omega$ by mapping the vertex of $\hat{1}$ to $\top_v$ and the loop of $\hat{1}$ to $\top_\ell$. In **SiLlGrphs** we define $\top : \hat{1} \to \Omega$ by sending the single vertex of $\hat{1}$ to $\top_v$. We proceed in **SiStGrphs**.

Let $f : A \to B$ be an extremal monomorphism. We define $\chi_A : B \to \Omega$ as follows. For $b \in f(V(A))$ define $\chi_A(b) = \top_V$, for $b \in f(E(A))$ define $\chi_A(b) = \top_\ell$, for $b \in V(B) \backslash f(V(A))$ define $\chi_A(b) = F_v$, and for $b \in E(B) \backslash f(E(A))$ define $\chi_A(b) = F_e$ if $\partial_B(b) = (b_1 \_ b_2)$ such that $b_1 \in f(V(A))$ or $b_2 \in f(V(A))$ (only one of b_1 and b_2 are in $f(V(A))$ as $Im(f)$ is a vertex induced subgraph of B by Lemma 3.7.1), and define $\chi_A(b) = F_\ell$ otherwise. As $Im(f)$ is a vertex induced subgraph of B, χ_A is a well defined strict graph morphism.

Let $!_A : A \to \hat{1}$ be the unique morphism from A to the terminal object. Then as f, $!_A$ and $\top$ are strict, for all $a \in V(A)$ $\top !_A(a) = \top_v = \chi_A f(a)$ and for all $a' \in E(A)$, $\top !_A(a') = \top_\ell = \chi_A f(a')$. Hence $\top !_A = \chi_A f$. We now show the following commuting diagram is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{!_A} & \hat{1} \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle \top} \\ B & \xrightarrow[\chi_A]{} & \Omega \end{array}$$

Let $h : X \to \hat{1}$ and $k : X \to B$ be such that $\top h = \chi_A k$. We note that as $\hat{1}$ is the terminal object, h is unique. Hence for $x \in V(X)$, $\chi_A k(x) = \top h(x) = \top_V$, and for $x' \in E(X)$, $\chi_A k(x') = \top h(x') = \top_\ell$. Then as $\chi_A k(x) = \top_v$ for all $x \in V(X)$, $k(x) \in f(V(A))$ and there is a vertex $a \in V(A)$ such that $k(x) = f(a)$. As f is a monomorphism, it is injective on vertices and a is uniquely determined. Furthermore, as $\chi_a k(x') = \top_\ell$ for all $x' \in E(X)$, $Im(f)$ is a vertex induced subgraph, and $x \in f(V(A))$ for all $x \in V(X)$, $k(x') \in f(E(A))$. Hence there is an edge $a' \in E(A)$ such that $f(a') = k(x')$.

We show a' is uniquely determined. Let $a, a' \in E(A)$ such that $f(a) = f(a')$, and let $\partial_A(a) = (a_1\_a_2)$, $\partial_A(a') = (a'_1\_a'_2)$ for some $a_1, a_2, a'_1, a'_2 \in V(A)$. Then $(f(a_1)\_f(a_2)) = \partial_B(f(a)) = \partial_B(f(a')) = (f(a'_1)\_f(a'_2))$. As f is injective on vertices, this implies that $a_1 = a'_1$ and $a_2 = a'_2$ or that $a_1 = a'_2$ and $a_2 = a'_1$. In either case as A is simple, $a = a'$. Hence for any $x' \in E(X)$, $a' \in E(A)$ with $f(a') = k(x)$ is uniquely determined.

So we define $\overline{k} : X \to A$ by $\overline{k}(x) = a$ for a the unique part such that $f(a) = k(x)$ for all $x \in P(X)$. By construction $\overline{k}$ maps vertices to vertices and edges to edges.

Let $x \in P(X)$ with $\partial_X(x) = (x_1\_x_2)$ for some $x_1, x_2 \in V(X)$. Then there exists $a_1, a_2 \in V(A)$ and $a \in P(A)$ such that $\overline{k}(x_1) = a_1$, $\overline{k}(x_2) = a_2$, and $\overline{k}(x) = a$ for $k(x_1) = f(a_1)$, $k(x_2) = f(a_2)$, and $k(x) = f(a)$. As f preserves incidence, A is simple, and $\partial_B(f(a)) = \partial_B(k(x)) = (h(x_1)\_h(x_2)) = (f(a_1)\_f(a_2))$, $\partial_A(a) = (a_1\_a_2)$ and $\overline{k}$ preserves incidence. Hence $\overline{k}$ is a strict graph morphism uniquely determined by h and k. Therefore the following diagram is a pullback and Ω is the extremal monomorphism subobject classifier.

$$\begin{array}{ccc} A & \xrightarrow{!_A} & \hat{1} \\ {\scriptstyle f}\downarrow & p.b. & \downarrow{\scriptstyle \top} \\ B & \xrightarrow[\chi_A]{} & \Omega \end{array}$$

The proof in **SiLlGrphs** follows similarly, noting that in the definition of $\chi_A : B \to \Omega$, edges $b \in f(E(A))$ will now be mapped to $\top_v$ and edges $b \in E(B) \backslash f(E(A))$ with no incident vertex in $f(V(A))$ will now be mapped to F_v. □

Theorem 3.7.4. *In **SiLlGrphs** and **SiStGrphs***

$$\textit{Split Eq.} = \textit{Coretract} \Longrightarrow \textit{Eff. Mono} = \textit{Reg. Mono} = \textit{Ext. Mono} \Longrightarrow \textit{Mono.}$$

Proof. As pushouts and coequalizers exist in **SiLlGrphs** and **SiStGrphs**, we must only show that monomorphisms are not extremal monomorphisms, extremal monomorphisms are not coretracts, and extremal monomorphisms are regular monomorphisms.

To show that there are monomorphisms that are not extremal monomorphisms, consider the following inclusion of K_2^c into K_2. By Lemma 2.3.14 this inclusion is a monomorphisms and by Lemma 3.7.1 it is not an extremal monomorphism.

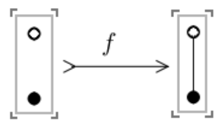


Figure 3.5: A counterexample to the "monomorphisms are extremal monomorphisms" in **SiLlGrphs** and **SiStGrphs**.

To show that there are extremal monomorphism that are no coretracts, consider the following inclusion of P_4, the path on 4 vertices, into C_5 the cycle on 5 vertices.

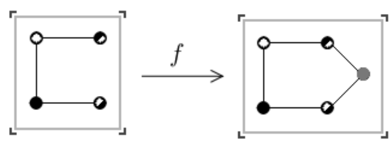


Figure 3.6: A counterexample to the "extremal monomorphisms are coretracts" in **SiLlGrphs** and **SiStGrphs**.

As f is injective on vertices, it is a monomorphism, and as $Im(f)$ is a vertex induced subgraph of C_5, by Lemma 3.7.1 f is an extremal monomorphism. However, by inspection, there is no coretract for f.

To show that extremal monomorphisms are regular monomorphism, we note that by Lemma 3.7.3, an extremal monomorphism subobject classifier exists in **SiLlGrphs** and **SiStGrphs**. Then the proof provided for "monomorphisms are regular monomorphisms" in **SiGrphs** (as part of Theorem 3.6.1) provides the proof that "extremal monomorphisms are regular monomorphisms" in **SiLlGrphs** and **SiStGrphs**. □

Theorem 3.7.5. *In **SiLlGrphs** and **SiStGrphs***

$$\textit{Split Coeq.} = \textit{Retract} \Longrightarrow \textit{Eff. Epi} = \textit{Reg. Epi} = \textit{Ext. Epi} \Longrightarrow \textit{Epi.}$$

Proof. As pullbacks and equalizers exist in **SiLlGrphs** and **SiStGrphs**, we must only show that epimorphisms are not extremal epimorphisms, extremal epimorphisms are not retracts, and extremal epimorphisms are regular epimorphisms.

To show that there are epimorphisms that are not extremal epimorphisms, consider the inclusion of K_2^c into K_2. By Lemma 2.3.14, this is an epimorphism, and by Lemma 3.7.2 as $Im(f) \neq K_2$, it is not an extremal epimorphism.

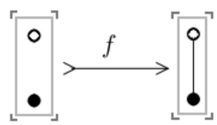


Figure 3.7: A counterexample to the "epimorphisms are extremal epimorphisms" in **SiLl-Grphs** and **SiStGrphs**.

For a counterexample to "extremal epimorphisms are retracts" consider the following morphism of $K_2 + K_2 \to P_3$ where P_3 is the path on three vertices,

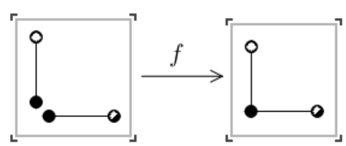


Figure 3.8: A counterexample to the "extremal epimorphisms are retracts" in **SiLlGrphs**, and **SiStGrphs**.

where each vertex is mapped to the vertex with the same coloring. As f is surjective on vertex sets it is an epimorphism, and as $Im(f) = P_3$ by Lemma 3.7.2 f is an extremal epimorphism. However no right inverse of f exists.

To show that extremal epimorphisms are regular epimorphisms, we note that by Lemma 3.7.2, extremal epimorphisms are surjective on part sets. Hence the proof provided for "epimorphisms are regular epimorphisms" in **SiGrphs** (part of Theorem 3.6.2) extends to show "extremal epimorphisms are regular epimorphisms" in **SiLlGrphs**. As the morphisms of **SiStGrphs** are strict, we must only modify the argument by creating K_1 components for vertices of A and K_2 components for edges in A of the construction of the graph C provided in the proof of Theorem 3.6.2. □

3.8 The classification of special morphisms in SiLlStGrphs

We now characterize the classification of special morphisms for **SiLlStGrphs**. **SiLlStGrphs** does not have all limits and colimits. In fact, it does not have a terminal object nor coequalizers, but it has been shown to have products, equalizers, pullbacks, coproducts, and an initial object. Hence all of the special conditions of Theorem 3.5.1 do not apply. We proceed with 3 lemmas to show that some of the lemmas from Section 3.7 still apply.

Lemma 3.8.1. *In **SiLlStGrphs** an extremal monomorphism is a monomorphism (and hence Lemma 3.7.1 applies to **SiLlStGrphs**).*

Proof. Let $f : A \to B$ be an extremal monomorphism, and suppose there exists $e : A \to C$, and $m : C \to B$ be such that $f = me$ with m a monomorphism and e an epimorphism. Then $f = me$ is an epi-mono factorization and hence e is an isomorphism (and hence a monomorphism). Hence f is a the composition of two monomorphisms and is a monomorphism. Therefore, it suffices to show that every morphism $f : A \to B$ has an epi-mono factorization.

We proceed by showing $Im(f)$ is a graph, and then for $i : Im(f) \to B$ inclusion, and $\overline{f} : A \to Im(f)$ defined by $\overline{f}(a) = f(a)$ for all $a \in P(A)$, $f = i\overline{f}$ is an epi-mono factorization of f.

So define $Im(f) = \langle P(Im(f)), V(Im(f)); \partial_{Im(f)}, \iota_{Im(f)} \rangle$, by $P(Im(f)) = \{b \in P(B) | \exists a \in P(A) \text{ with } f(a) = b\}$, $V(Im(f)) = \{b \in V(B) | \exists a \in V(A) \text{ with } f(a) = b\}$, $\partial_{Im(f)}(b) = \partial_B(b)$, and $\iota_{Im(f)}(b) = \iota_B(b)$.

We note that if $b \in P(Im(f))$ with $b_1, b_2 \in V(B)$ such that $\partial_B b = (b_1\_b_2)$, then as f preserves incidence, for $a \in P(A)$ with $f(a) = b$, there are $a_1, a_2 \in V(A)$ such that $(a_1\_a_2) = \partial_B(f(a)) = \partial_B(b) = (b_1\_b_2)$, and $b_1, b_2 \in V(Im(f))$. Hence $\partial_{Im(f)}$ and $\iota_{Im(f)}$ are well defined. Furthermore, as B is simple and loopless, so is $Im(f)$. Hence $Im(f)$ is a graph in **SiLlStGrphs**.

Then an extremal monomorphisms are monomorphisms and by Proposition 2.3.14, the proof for Lemma 3.7.1 holds for **SiLlStGrphs**, and $f : A \to B$ is an extremal monomorphism if and only if f is a monomorphism and $Im(f)$ is a vertex induced subgraph of B. □

Lemma 3.8.2. *If $f : A \to B$ is an extremal monomorphism in **SiLlStGrphs** then $B +_A B$, the pushout of f with itself, exists.*

Proof. Let $f : A \to B$ be an extremal monomorphism. We construct $B +_A B$ by appending vertex v' to B for all $v \in V(B) \backslash V(Im(f))$, appending an edge e' to B with incidence $\partial_{B+_AB}(e') = (v_1'\_v_2')$ for all $e \in E(B)$ with $\partial(e) = (v_1\_v_2)$ with $v_1, v_2 \in V(B) \backslash V(Im(f))$, and by appending an edge $\overline{e}$ to B with incidence $\partial_{B+_AB}(\overline{e}) = (v_1'\_v_2)$ for all $e \in E(B)$ with $\partial_B(e) = (v_1\_v_2)$ with $v_1 \in V(B) \backslash V(Im(f))$ and $v_2 \in V(Im(f))$. As B is a simple and loopless graph, so is $B +_A B$.

By construction there is a natural inclusion morphism $i : B \to B +_A B$. We define a morphism $g : B \to B +_A B$ by

$$g(x) = \begin{cases} x & \text{if} \quad x \in P(Im(f)) \\ x' & \text{if} \quad x \in V(B) \backslash V(Im(f)) \\ x' & \text{if} \quad x \in E(B) \text{ with } \partial_B(x) = (x_1\_x_2) \text{ such that } x_1, x_2 \in V(B) \backslash V(Im(f)) \\ \overline{x} & \text{if} \quad x \in E(B) \text{ with } \partial_B(x) = (x_1\_x_2) \\ & \qquad \text{such that } x_1 \in V(B) \backslash V(Im(f)) \text{ and } x_2 \in V(Im(f)) \end{cases}$$

As f is an extremal monomorphism, by Lemma 3.8.1 $Im(f)$ is a vertex induced subgraph of B. Hence g is a well defined strict morphism. As both i and g preserve $Im(f)$, $if = gf$, and the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & = & \downarrow{\scriptstyle i} \\ B & \xrightarrow[g]{} & B +_A B \end{array}$$

We now show this diagram is indeed a pushout. Let $h_1 : B \to Y$ and $h_2 : B \to Y$ be such that $h_1 f = h_2 f$. Then define $\overline{h} : B +_A B \to Y$ by $\overline{h}(x) = h_1(x)$ if $x \in P(Im(i))$ and $\overline{h}(x') = h_2(x)$ if $x' \in P(B +_A B) \backslash P(Im(i))$ with x' appended to B in the construction of $B +_A B$ derived from $x \in P(B)$. As $h_1 f = h_2 f$, h_1 and h_2 agree on all $x \in P(Im(f))$. Hence for all $x \in i(P(Im(f)))$, $\overline{h}(x) = h_1(x) = h_2(x)$, and by construction of $B +_A B$, $\overline{h}$ is a well-defined graph morphism.

By definition, $\overline{h}i = h_1$ and as h_1 and h_2 agree on $Im(f)$, $\overline{h}g = h_2$. As $\overline{h}$ is defined by h_1 and h_2, it is uniquely defined. Thus the following diagram is a pushout.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f}\downarrow & p.o. & \downarrow{\scriptstyle i} \\ B & \xrightarrow[g]{} & B +_A B \end{array}$$

□

Lemma 3.8.3. *In **SiLlStGrphs**, a morphism $f : A \to B$ is an extremal epimorphism if and only if $Im(f) = B$.*

Proof. The proof given for Lemma 3.7.2 also applies to **SiLlStGrphs**. □

Theorem 3.8.4. *In **SiLlStGrphs***

$$\textit{Split Eq.} = \textit{Coretract} \Longrightarrow \textit{Eff. Mono} = \textit{Reg. Mono} = \textit{Ext. Mono} \Longrightarrow \textit{Mono}.$$

Proof. We must show the following four implications: coretracts are split equalizers, regular monomorphisms are effective monomorphisms, extremal monomorphisms are monomorphisms (given by Lemma 3.8.1), and extremal monomorphisms are regular monomorphisms. We must also show that extremal monomorphisms are not coretracts, and that monomorphisms are not extremal monomorphisms.

We begin by showing coretracts are split equalizers. Let $f : A \to B$ be a coretract. Then there is a morphism $g : B \to A$ such that $gf = 1_A$. We claim $f = eq(1_B, fg)$.

As $1_B f = f = fgf = (fg)f$, f equalizes 1_B and fg. Now let $d : D \to B$ be such that $1_B d = fgd$. Define $\overline{d} = gd : D \to A$. Then $f\overline{d} = fgd = 1_B d = d$. We show $\overline{d}$ is unique.

Suppose $d' : D \to A$ is such that $fd' = d$. Let $v \in V(D)$, then $f(d'(v)) = d(v) = f(\overline{d}(v))$ and as f is a monomorphism $\overline{d}(v) = d'(v)$. Hence d and d' agree on vertices.

Let $e \in E(D)$ with $\partial_d(e) = (e_1 \_ e_2)$ for some $e_1, e_2 \in V(D)$. As morphisms are strict $\overline{d}(e) \in E(A)$ and $d'(e) \in E(A)$. As incidence is preserved and $\overline{d}$, d' agree on vertices $\partial_A(d'(e)) = (d'(e_1) \_ d'(e_2)) = (\overline{d}(e_1) \_ \overline{d}(e_2)) = \partial_A(\overline{d}(e))$. As the graphs are simple, $\overline{d}(e) = d'(e)$ and $\overline{d} = d'$. Hence $\overline{d}$ is unique and $f = eq(1_B, fg)$.

Let $s = g$ and $s' = 1_B$, then $sf = gf = 1_A$, $s'1_B = 1_B 1_B = 1_B$ and $s'fg = 1_B fg = fg = fs$. Thus f is a split equalizer.

Let $f : A \to B$ be an regular monomorphism. Then f is an extremal monomorphism by Theorem 3.5.1. Hence by Lemma 3.8.2 the pushout of f with itself exists, and the general proof for Theorem 3.5.1 part (1) applies, and f is an effective monomorphism.

We note by Lemma 3.8.1 extremal monomorphisms are monomorphisms. By the proof of Lemma 3.8.2, given $f : A \to B$ an extremal monomorphism and for $i, g : B \to B +_A B$ defined in the proof, i, g only agree on $Im(f)$ by construction. Hence f satisfies the classical definition of equalizer for i, g in **SiLlStGrphs**.

The counterexamples to monomorphisms are extremal monomorphisms and to extremal monomorphisms are coretracts are the same for **SiLlStGrphs** as the counterexamples in **SiStGrphs** (Theorem 3.7.4). □

Theorem 3.8.5. *In* ***SiLlStGrphs***

$$\textit{Split Coeq.} = \textit{Retract} \Longrightarrow \textit{Eff. Epi} = \textit{Reg. Epi} = \textit{Ext. Epi} \Longrightarrow \textit{Epi.}$$

Proof. As equalizers and pullbacks exist in **SiLlStGrphs**, we must only show that extremal epimorphisms are regular epimorphisms, epimorphisms are not extremal epimorphisms, and extremal epimorphisms are not retracts. These proofs follow directly from the proofs given in Theorem 3.7.5 for **SiStGrphs** where we note that an extremal epimorphism is surjective on part sets by Lemma 3.8.3. □

Chapter 4

The Elementary Theory of the Categories of Graphs

4.1 The Elementary Theory

In this chapter we will follow the style and notation found in [23] with the exception that we compose on the left, i.e $g : A \to B$ and $f : B \to C$ composes as $fg = f \circ g : A \to C$ and that we denote the identity morphism from an object A to itself as $id_A = A : A \to A$.

We will simultaneously axiomatize five categories of graphs: **Grphs**, **SiGrphs**, **SiLlGrphs**, **StGrphs**, and **SiStGrphs**. We provide twelve axioms that will hold in all five categories and then using two to four distinguishing axioms we specify which category of graphs is axiomatized. The methods to axiomatize these five categories of graphs fail to axiomatize **SiLlStGrphs** as **SiLlStGrphs** fails to have many of the constructions required in this axiomatization (see Axiom 1 and Axiom 10).

We will always assume that we have the axioms which define an abstract category. We will refer to these axioms as Axiom 0.

Axiom 1. *There exists an initial object (denoted $\hat{0}$) and terminal object (denoted $\hat{1}$); every pair of objects has a product (denoted $\times$) and a coproduct (denoted $+$); every pair of morphisms has an equalizer and a coequalizer.*

We note that this axiom implies that all finite limit and colimits exist.

Definition 1. *Given a class $\mathscr{H}$ of objects in a category, a* mimimal *$\mathscr{H}$-object, A, is such that $A \in \mathscr{H}$ and for every other object C in $\mathscr{H}$, there is a monomorphism $A \to C$.*

Definition 2. *A* base point object*, $\hat{B}$, is the minimal non-initial projective object.*

We note that in the category of topological spaces with continuous functions, the base point object is a one point space, in the category of abelian groups and group homomorphisms, the base point object is $(\mathbb{Z}, +)$, and in the category of modules over an invariant basis ring, the base point object is a free module on one generator.

Axiom 2. *There is a base point object, $\hat{V}$, such that if there is a morphism $f : A \to \hat{V}$ from any object A it is unique. We call $\hat{V}$ the* vertex object*.*

We note that as there is a morphism $\hat{V} : \hat{V} \to \hat{V}$ and it is unique, it is the only endomorphism of $\hat{V}$. We require the only endomorphism of $\hat{V}$ to be the identity in order to later show that $\hat{0}$ has no vertices, which we now define.

Definition 3. *For a category with a base point object $\hat{V}$, a* vertex*, v, of an object A is a morphism $v : \hat{V} \to A$. We denote $\hom(\hat{V}, A)$ by $V(A)$, and use the notation $v \in V(A)$ if v is a vertex of A.*

We remark that in the category of topological spaces, $V(A)$ corresponds to the underlying set of the topological space A, in the category of abelian groups with group homomorphisms $V(A)$ corresponds the underlying set of the abelian group A, and in the category of modules over an invariant basis ring $V(A)$ corresponds to the underlying set of the module A.

Corollary 1. *$\hat{0}$ has no vertex, and $\hat{1}$ has only a single vertex.*

Proof. $\hat{1}$ having a single vertex is trivial by the definition of terminal object.

For $\hat{0}$, suppose $\hat{0}$ has a vertex $x : \hat{V} \to \hat{0}$. Then as $\hat{0}$ is the initial object, there is a unique morphism $!_{\hat{V}} : \hat{0} \to \hat{V}$ and a unique morphism $\hat{0} : \hat{0} \to \hat{0}$. Furthermore, as $x!_{\hat{V}} : \hat{0} \to \hat{0}$ is an endomorphism of $\hat{0}$, $x!_{\hat{V}} = \hat{0}$. As there is a unique endomorphism of $\hat{V}$ and $!_{\hat{V}}x : \hat{V} \to \hat{V}$, $!_{\hat{V}}x = \hat{V}$ and $\hat{V} \cong \hat{0}$. This contradicts the definition of a base point object. □

Proposition 1. *In a category with Axiom 2, the vertex object $\hat{V}$ is unique up to isomorphism.*

Proof. Suppose $\hat{V}_1$ and $\hat{V}_2$ are vertex objects. As they are both minimal projective objects, there exists monomorphisms $m_1 : \hat{V}_1 \to \hat{V}_2$ and $m_2 : \hat{V}_2 \to \hat{V}_1$. Then $m_2m_1 : \hat{V}_1 \to \hat{V}_1$ is an endomorphism of $\hat{V}_1$ and as $\hat{V}_1$ has a unique endomorphism, $m_2m_1 = \hat{V}_1$. Similarly $m_1m_2 = \hat{V}_2$. Hence $\hat{V}_1 \cong \hat{V}_2$. □

Proposition 2. *In a category with Axiom 2, any morphism $f : \hat{V} \to A$ is a monomorphism.*

Proof. As every morphism to $\hat{V}$ is unique, this proposition holds trivially. □

Proposition 3. *Let $f : A \to B$ be a morphism from object A to object B in a category with Axiom 2,*

(i) *if f is an epimorphism, then f is surjective on vertices, i.e. for every $x \in V(B)$, there is a $y \in V(A)$ such that $fy = x$.*

(ii) *if f is a monomorphism, then f is injective on vertices, i.e. for every pair of vertices $x, y \in V(A)$ such that $x \neq y$, we have $fx \neq fy$.*

Proof. **Part (i)**: Let $f : A \to B$ be an epimorphism. If $x : \hat{V} \to B$ is a vertex of B, then as $\hat{V}$ is projective, there is a morphism $y : \hat{V} \to A$ such that $fy = x$.

Part (ii): Let $x, y \in V(A)$ with $fx = fy$, then as f is a monomorphism, it is left cancelable and $x = y$. □

We note that our Proposition 1 is not bi-conditional. It actually is only bi-conditional in **SiLlGrphs**, **SiStGrphs**, and **SiStLlGrphs** (see Proposition 2.3.14).

Axiom 3. *Every vertex of a coproduct $A+B$ can be factored through one of the injections $i_A : A \to A+B$, $i_B : B \to A+B$, i.e. if $x \in V(A+B)$, then there is a vertex $t \in V(A)$ or $t \in V(B)$ such that $x = i_A t$ or $x = i_B t$.*

Definition 4. *In a category with terminal objects and coproducts, an object A is* connected *if any morphism $f : A \to \hat{1} + \hat{1}$ factors through $\iota_0, \iota_1 : \hat{1} \to \hat{1} + \hat{1}$, i.e. $\iota_0 !_A = f$ or $\iota_1 !_A = f$.*

This definition is different than the categorial definition usually given for connected. The standard categorial definition states that an object is connected if it does not admit an epimorphism to $\hat{1} + \hat{1}$. However, the graph consisting of just two vertices, K_2^c, satisfies that definition in **StGrphs**, but it is not path-connected - the standard "connected" in Graph Theory.

However, in most categories our definition of connected does imply the standard categorial definition of connected.

Definition 5. *An* element *of an object A is a morphism $x : \hat{1} \to A$.*

Proposition 4. *In a category that has an object with more than one element, if A is connected then A does not admit an epimorphism to $\hat{1} + \hat{1}$.*

Proof. Let X be the object in the category with more than one element. Let $a, b : \hat{1} \to X$ be two distinct elements of X, i.e. $a \neq b$. Then by the universal mapping property of the coproduct, there is a unique morphism $(a+b) : \hat{1} + \hat{1} \to X$ such that $(a+b)\iota_0 = a$ and $(a+b)\iota_1 = b$. If $\iota_0 = \iota_1$ then $a = (a+b)\iota_0 = (a+b)\iota_1 = b$, a contradiction. Hence $\iota_0 \neq \iota_1$.

Now consider a morphism $f : A \to \hat{1} + \hat{1}$. As A is connected, f factors through ι_0 or ι_1. Without loss of generality, let $f = \iota_0 !_A$. Now define $\bar{\iota}_0 : \hat{1} + \hat{1} \to \hat{1} + \hat{1}$ as the unique morphism prescribed by the universal mapping property of coproduct such that $\bar{\iota}_0 \iota_0 = \iota_0$ and $\bar{\iota}_0 \iota_1 = \iota_0$.

Consider $\bar{\iota}_0 f$: $\bar{\iota}_0 f = \bar{\iota}_0 \iota_0 !_A = \iota_0 !_A = f = (\hat{1}+\hat{1})f$. Hence $\bar{\iota}_0 f = (\hat{1}+\hat{1})f$. However, as $(\hat{1}+\hat{1})\iota_1 = \iota_1$ and $\bar{\iota}_0 \iota_1 = \iota_0$, $\hat{1}+\hat{1} \neq \bar{\iota}_0$ and f is not an epimorphism. $\square$

We note that in the category of topological spaces with continuous morphisms, connected is equivalent to Lennes connected.

Definition 6. *In a category with terminal objects, coproducts and a vertex object, an* arc-edge object *is a minimal, 2-vertex connected object, where a 2-vertex object is an object, A, with exactly two morphisms from vertex object $\hat{V}$ to A, $x, y : \hat{V} \to A$ with $x \neq y$.*

We note that the arc-edge object is isomorphic to K_2 in any of the six graph categories, and isomorphic to a directed K_2 in directed graph categories. We add an extra condition to distinguish this object as an edge object.

Axiom 4. *There is a unique up to isomorphism arc-edge object, $\hat{E}$, along with an automorphism $\tau : \hat{E} \to \hat{E}$ such that for the two vertices $a, b \in V(\hat{E})$, $\tau a = b$ and $\tau b = a$. We call $\hat{E}$ the* edge object.

We also include the definition of an edge, but the reason for including this axiom this early is to have objects with more than one vertex in the category.

Definition 7. *An* edge*, e, of an object, A, is an unordered pair of distinct morphisms $e = (e_1 : \hat{E} \to A \ \_e_2 : \hat{E} \to A)$ with $e_1 = e_2\tau$.*

We now continue with the elementary theory.

Definition 8. *Define the object $\hat{2}_{\hat{V}} = \hat{V} + \hat{V}$*

Proposition 5. *The two injections $i_0 : \hat{V} \to \hat{2}_{\hat{V}}$ and $i_1 : \hat{V} \to \hat{2}_{\hat{V}}$ are different and they are the only vertices of $\hat{2}_{\hat{V}}$.*

Proof. As $a, b : \hat{V} \to \hat{E}$ with $a \neq b$, there is a morphism $(a+b) : \hat{2}_{\hat{V}} \to \hat{E}$ such that $(a+b)i_0 = a$ and $(a+b)i_1 = b$. If $i_0 = i_1$, then $a = (a+b)i_0 = (a+b)i_1 = b$. Hence $i_0 \neq i_1$.

Now let $c : \hat{V} \to \hat{2}_{\hat{V}}$. By Axiom 3, there is a morphism $t : \hat{V} \to \hat{V}$ such that $c = i_0 t$ or $c = i_1 t$. However, as t is a endomorphism of $\hat{V}$ it is the identity, and $c = i_0$ or $c = i_1$. □

We now turn to axiomatizing **Sets** as a subcategory of the six categories of graphs, using discrete objects. Our definition of discrete objects is an augmentation of Schlomiuk's definition [23].

Definition 9. *In a category with a vertex object and pullbacks, an object A is a* discrete object *if for all $x \in V(A)$, there exists a morphism $f_x : A \to \hat{2}_{\hat{V}}$ such that $f_x y \neq f_x x$ for all $y \in V(A)$ with $y \neq x$, and for $i : \hat{V} \to \hat{2}_{\hat{V}}$ with $f_x x = i$, the following diagram is a pullback.*

$$\begin{array}{ccc} \hat{V} & \xrightarrow{x} & A \\ {\scriptstyle \hat{V}}\downarrow & p.b. & \downarrow{\scriptstyle f_x} \\ \hat{V} & \xrightarrow[i]{} & \hat{2}_{\hat{V}} \end{array}$$

In the category of topological spaces with continuous morphisms (as well as in **Sets**), the above definition of a discrete object concretely coincides with the discrete objects defined by Schlomiuk. We require the added pullback property in **Grphs** to discount graphs with loops on each vertex (but no non-loop edges) as being discrete.

Proposition 6. *$\hat{0}$, $\hat{V}$, and $\hat{2}_{\hat{V}}$ are discrete objects.*

Proof. We first note that $\hat{0}$ satisfies the conditions vacuously.

As $\hat{V} : \hat{V} \to \hat{V}$ is unique, it has a single vertex and $f_{\hat{V}} = i_0 : \hat{V} \to \hat{2}_{\hat{V}}$ satisfies the condition of $f_{\hat{V}}\hat{V} \neq f_{\hat{V}}y$ for all $y \in V(\hat{V})$ with $y \neq \hat{V}$ vacuously. Now consider the following commuting diagram.

$$\begin{array}{ccc} \hat{V} & \xrightarrow{\hat{V}} & \hat{V} \\ {\scriptstyle \hat{V}}\downarrow & = & \downarrow{\scriptstyle i_0} \\ \hat{V} & \xrightarrow[i_0]{} & \hat{2}_{\hat{V}} \end{array}$$

For any object X with $f, g : X \to \hat{V}$ with $i_0 f = i_0 g$, by definition of $\hat{V}$, $f = g$ and $f : X \to \hat{V}$ is the unique morphism such that $\hat{V}f = f = \hat{V}g$ (it is the unique morphism period). Thus the diagram is a pullback.

Now we consider $\hat{2}_{\hat{V}}$. We note that by Proposition 4 $V(\hat{2}_{\hat{V}}) = \{i_0, i_1\}$.

Define $f_{i_0} = f_{i_1} = \hat{2}_{\hat{V}} : \hat{2}_{\hat{V}} \to \hat{2}_{\hat{V}}$. Clearly $\hat{2}_{\hat{V}} i_0 \neq \hat{2}_{\hat{V}} i_1$. Consider the following commuting diagram.

$$\begin{array}{ccc} \hat{V} & \xrightarrow{i_0} & \hat{2}_{\hat{V}} \\ {\scriptstyle \hat{V}}\downarrow & = & \downarrow{\scriptstyle \hat{2}_{\hat{V}}} \\ \hat{V} & \xrightarrow[i_0]{} & \hat{2}_{\hat{V}} \end{array}$$

Let $g : X \to \hat{V}$ and $h : X \to \hat{2}_{\hat{V}}$ such that $i_0 g = \hat{2}_{\hat{V}} h = h$. Hence $g : X \to \hat{V}$ is the unique morphism such that $i_0 g = h$ (and $g = g\hat{V}$). Thus the diagram is a pullback. $\square$

Axiom 5. *For every morphism $f : A \to B$ such that $A \neq \hat{0}$ and B is discrete, there exists $g : B \to A$ such that $fgf = f$.*

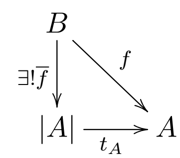
This axiom is provides the Axiom of Choice for discrete codomains, the weaker form of the Axiom of Choice that holds in the categories of graphs.

Axiom 6. *All objects A with $A \ncong \hat{0}$ have $V(A) \neq \emptyset$.*

Proposition 7. *If $A \ncong \hat{0}$ is a discrete object, then there is a morphism $A \to \hat{V}$ (and it is unique).*

Proof. As $A \ncong \hat{0}$, by Axiom 6 there is a morphism $x : \hat{V} \to A$. By Axiom 5, there exists a $g : A \to \hat{V}$ such that $xgx = x$. Then by Axiom 2, g is unique. □

Axiom 7. *For every object A, there exists a discrete object $|A|$ together with a morphism $t_A : |A| \to A$ such that for every discrete object B and morphism $f : B \to A$, there exists a unique $\overline{f} : B \to |A|$ such that $f = t_A\overline{f}$ and the following diagram commutes.*



We note that $|A|$ is unique up to isomorphism and $| - |$ is functorial. Furthermore as a functor, $| - |$ is right adjoint to the inclusion functor of discrete objects (it is a coreflector).

Proposition 8. *In a category with Axiom 7, if A is a discrete object then $A \cong |A|$.*

Proof. Let A be a discrete object, then $t_A = A : A \to A$ trivially satisfies the universal mapping property. Let $|A|$ and $t_A : |A| \to A$ be the object prescribed by Axiom 7. Then as $A : A \to A$ is a morphism from a discrete object to A, there exists a unique $\bar{t}_A : A \to |A|$ such that $t_A\bar{t}_A = A$. Then as A satisfies the universal mapping property, $t_A : |A| \to A$ is unique, and $\bar{t}_A t_A : |A| \to |A|$ is the unique morphism such that $t_A = t_A\bar{t}_A t_A$. However, as $t_A = t_A|A|$, $\bar{t}_A t_A = |A|$. Hence $A \cong |A|$. □

Proposition 9. *In a category with Axiom 1 and Axiom 7, the finite limit of discrete objects is discrete.*

Proof. As $|-|$ has a left adjoint (inclusion) it is left continuous. Hence $|-|$ commutes with left limits. Furthermore, if A is discrete then $|A| \cong A$. Hence $\lim_{\leftarrow}(A_i) \cong \lim_{\leftarrow}(|A_i|) \cong |\lim_{\leftarrow}(A_i)|$. □

Proposition 10. *In a category with Axiom 2 and Axiom 6, if A is discrete and $f : A \to \hat{V}$ is a monomorphism, then $A \cong \hat{0}$ or $A \cong \hat{V}$.*

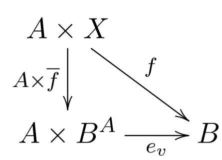
Proof. If $A \cong \hat{0}$ then the proposition holds, so assume $A \ncong \hat{0}$ and $f : A \to \hat{V}$ is a monomorphism. Then as $A \ncong \hat{0}$, $V(A) \neq \emptyset$ by Axiom 6. Hence there exists $x : \hat{V} \to A$. As f is a monomorphism, x is unique. For if there exists a $y : \hat{V} \to A$, $fx = fy : A \to \hat{V}$ and $x = y$.

By Axiom 2 the only endomorphism of $\hat{V}$ is $\hat{V}$, then as $fx : \hat{V} \to \hat{V}$ is an endomorphism of $\hat{V}$, $fx = \hat{V}$. To show that $xf = A$, we must first show A has only one endomorphism. Let $g, h : A \to A$. Then as $xf : A \to A$ is unique, $xfg = xfh$. By hypothesis and proposition 2, both f and x are monomorphisms and $g = h$. Then as A only has one endomorphism, $xf = A$. Thus $A \cong \hat{V}$. □

Proposition 11. *In a category with terminal objects and Axioms 2,5,6, and 7,* $|\hat{1}| \cong \hat{V}$.

Proof. By proposition 7, every discrete object A has a unique morphism $\hat{V}_A : A \to \hat{V}$. Then by proposition 9, $|\hat{1}|$ is the terminal object for discrete objects. Hence, as the terminal object is unique up to isomorphism, $|\hat{1}| \cong \hat{V}$. □

Axiom 8. *For every discrete object A and object B, there exists an object B^A and a morphism $e_v : A \times B^A \to B$ (called evaluation) such that for every object X and morphism $f : A \times X \to B$ there exists a unique morphism $\overline{f} : X \to B^A$ such that $e_v(A \times \overline{f}) = f$ and the below diagram commutes.*



Note, in a category with Axiom 8 and Axiom 1, we have the law of *discrete* exponents and if A is discrete, then $A \times (B + C) \cong (A \times B) + (A \times C)$.

Proposition 12. *If A and B are discrete objects, then $|B^A|$ and $|e_v| : A \times |B^A| \to A$ satisfies the definition of exponentiation with evaluation for discrete objects X.*

Proof. Let A, B and X be discrete. We apply $|-|$ to the diagram in Axiom 8.

$$\begin{array}{ccc} |A \times X| & & \\ {\scriptstyle |A\times\overline{f}|}\downarrow & \searrow^{|f|} & \\ |A \times B^A| & \xrightarrow[|e_v|]{} & |B| \end{array}$$

By propositions 8 and 9, $|A \times X| \cong A \times X$, $|B| \cong B$ and $|A \times B^A| \cong A \times |B^A|$. Using these isomorphisms we obtain the following commuting diagram.

$$\begin{array}{ccc} A \times X & & \\ {\scriptstyle A\times|\overline{f}|}\downarrow & \searrow^{f} & \\ A \times |B^A| & \xrightarrow[|e_v|]{} & B \end{array}$$

We now note that $|\overline{f}| : X \to |B^A|$ is the unique morphism such that $t_{B^A}|\overline{f}| = \overline{f}$ by Axiom 7, and is unique as $\overline{f}$ is unique. □

Axiom 9. *There exists a discrete object N together with morphisms $\ulcorner 0 \urcorner : \hat{V} \to N$ and $s : N \to N$ such that for every morphism $\ulcorner x \urcorner : \hat{V} \to X$ and $r : X \to X$ with X a discrete object, there exists a unique $\overline{x} : N \to X$ such that $\overline{x}\ulcorner 0 \urcorner = \ulcorner x \urcorner$ and $\overline{x}s = r\overline{x}$ so that the following diagram commutes.*

$$\begin{array}{ccccc} \hat{V} & \xrightarrow{\ulcorner 0 \urcorner} & N & \xrightarrow{s} & N \\ & {\scriptstyle \ulcorner x \urcorner}\searrow & \downarrow{\scriptstyle \overline{x}} & & \downarrow{\scriptstyle \overline{x}} \\ & & X & \xrightarrow{r} & X \end{array}$$

We note that Axiom 9 is the axiom of the "Natural Number Object" for discrete objects as $|\hat{1}| \cong \hat{V}$ which is given by Proposition 11.

We include the following definition for completeness (see Definition 3.4.5).

Definition 10. *A morphism $f : A \to B$ is an* extremal monomorphism *if f does not factor through any proper epimorphism, i.e. if $f = me$ with m a monomorphism and e an epimorphism, then e is an isomorphism.*

Proposition 13. *If $f : A \to B$ be a monomorphism and B be discrete then f is an extremal monomorphism.*

Proof. Let $e : A \to C$ be an epimorphism and $m : C \to A$ be a monomorphism such that $f = me$, i.e. me is a epi-mono factorization of f. Suppose $A \cong \hat{0}$. If $V(C) \neq \emptyset$, then for $x : \hat{V} \to C$, as $\hat{V}$ is projective, there exists $\overline{x} : \hat{V} \to \hat{0}$ such that $e\overline{x} = x$. However $\hat{0}$ has no vertex by Corollary 1, a contradiction. Thus $V(C) = \emptyset$, and by the contrapositive of Axiom 6, $C \cong \hat{0}$ and e is an isomorphism.

Suppose $A \ncong \hat{0}$. Then by Axiom 5, there exists a morphism $g : B \to A$ such that $fgf = f$. As f is a monomorphism, we left-cancel to obtain $gf = A$. We consider $gm : C \to A$. First we note $(gm)e = gf = A$. Then as $e(gm)e = egf = e$ and e is an epimorphism, we right-cancel to obtain $e(gm) = C$. Hence e is an isomorphism and f is an extremal monomorphism. □

Axiom 10. *There exists an extremal monomorphism subobject classifier Ω. That is, there exists and object Ω with morphism $\top : \hat{1} \to \Omega$ such that for any $f : A \to B$ an extremal monomorphism, there exists a unique $\chi_A : B \to \Omega$ such that the following diagram is a pullback.*

$$\begin{array}{ccc} A & \xrightarrow{!_A} & \hat{1} \\ {\scriptstyle f}\downarrow & p.b. & \downarrow{\scriptstyle \top} \\ B & \xrightarrow[\chi_A]{} & \Omega \end{array}$$

Proposition 14. $|\Omega|$ *is a subobject classifier for discrete objects.*

Proof. Let A and B be discrete objects with a monomorphism $f : A \to B$. Then by proposition 13, f is an extremal monomorphism. Then by Axiom 10, there exists a unique $\chi_A : B \to \Omega$ such that the following diagram is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{!_A} & \hat{1} \\ {\scriptstyle f}\downarrow & p.b. & \downarrow{\scriptstyle \top} \\ B & \xrightarrow[\chi_A]{} & \Omega \end{array}$$

Applying $|-|$ to the diagram preserves the diagram being a pullback by proposition 9. We obtain the following.

$$\begin{array}{ccc} |A| & \xrightarrow{|!_A|} & |\hat{1}| \\ {\scriptstyle |f|}\downarrow & p.b. & \downarrow{\scriptstyle |\top|} \\ |B| & \xrightarrow[|\chi_A|]{} & |\Omega| \end{array}$$

By propositions 8 and 11, $|A| \cong A$, $|B| \cong B$, and $|\hat{1}| \cong \hat{V}$. Letting $\hat{V}_A : A \to \hat{V}$, $\top_{\hat{V}} : \hat{V} \to |\Omega|$ and $|\chi|_A : B \to |\Omega|$ be given by the isomorphisms for $|!_A|$, $|\top|$ and $|\chi_A|$ respectively, we obtain the following pullback diagram.

$$\begin{array}{ccc} A & \xrightarrow{\hat{V}_A} & \hat{V} \\ {\scriptstyle f}\downarrow & p.b. & \downarrow{\scriptstyle \top_{\hat{V}}} \\ B & \xrightarrow[|\chi|_A]{} & |\Omega| \end{array}$$

By proposition 7, $\hat{V}$ is the terminal object for discrete objects. So we must only show $|\chi|_A$ is unique. So we note that $|\chi|_A : B \to |\Omega|$ is the unique morphism such that $t_\Omega|\chi|_A = \chi_A$ by Axiom 7, and is unique as χ_A is unique. □

Theorem Schema 1. *If Φ is a theorem of the elementary theory of the category of sets ([7, 16, 25]) and $\overline{\Phi}$ is obtained from Φ by replacing "object" with "discrete object", then $\overline{\Phi}$ is a theorem in any category satisfying Axioms 1-10.*

Proof. It suffices to prove the theorem in the case that Φ is an axiom of the elementary theory of the category of sets. We choose to use the six Lawvere-Tierney axioms for the category of sets [7, 16, 25]. They are as follows.

(LT 1) There exists finite limits.

(LT 2) There exists exponentiation with evaluation.

(LT 3) There exists a subobject classifier.

(LT 4) There exists the axiom of choice.

(LT 5) There exists a natural number object.

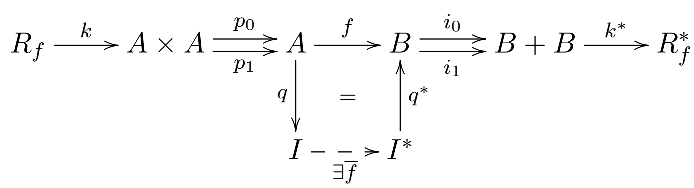
(LT 6) There exists 2-valued internal logic (i.e. the only subobjects of $\hat{1}$ are $\hat{0}$ and $\hat{1}$).

We note that (LT 1) holds by proposition 9, (LT 2) holds by proposition 12, (LT 3) holds by proposition 14, (LT 4) holds by Axiom 5, (LT 5) holds by Axiom 9, and (LT 6) holds by propositions 10 and 11. □

We now can obtain the following propositions from the above schema.

Proposition 15 ([16] $|\Omega|$ is Boolean)**.** $|\Omega| \cong \hat{V} + \hat{V} = \hat{2}_{\hat{V}}$

Proposition 16 ([16] The Fundamental Morphism Theorem for Discrete Objects)**.** *If $f : A \to B$ is a morphism with A and B discrete, then there exists an isomorphism $\overline{f}$ such that the following diagram commutes,*



where $k = eq(fp_0, fp_1)$, $q = coeq(p_0k, p_1k)$, $k^* = coeq(i_0f, i_1f)$, and $q^* = eq(k^*i_0, k^*i_1)$.

Proposition 17 ([15] Cantor-Schroeder-Bernstein Theorem)**.** *If X and Y are discrete objects with monomorphisms $m_1 : X \rightarrowtail Y$ and $m_2 : Y \rightarrowtail X$, then $X \cong Y$.*

Proposition 18 ([16] $\hat{1}$ is a generator)**.** *If A and B are discrete with $f, g : A \to B$ and $f \neq g$, then there is a vertex $v : \hat{V} \to A$ such that $fv \neq gv$.*

Proposition 19 ([16] - Theorem 5)**.** *Let A and X be discrete objects with monomorphism $\alpha : A \rightarrowtail X$. Then there exists a discrete object A' with monomorphism $\alpha' : A' \to X$ such that $X \cong A + A'$ with α, α' the injections.*

We now concern ourselves with graph properties, we have defined a vertex in Definition 3 and an edge in Definition 7. We now work to define a loop.

Definition 11. *Given a vertex $b \in V(B)$, a* constant vertex-b morphism *from an object A to B, denoted $\ulcorner b \urcorner : A \to B$, is such that there exists $\hat{V}_A : A \to \hat{V}$ with $\ulcorner b \urcorner = b\hat{V}_A$. If a morphism factors through $\hat{V}$ in this way, we call the morphism a* constant vertex morphism.

With this definition we can use the edge object to define a loop.

Definition 12. *A* loop *of an object A is a morphism $\ell : \hat{E} \to A$ such that $\ell\tau = \ell$ and ℓ is not a constant vertex morphism (i.e. ℓ does not factor through $\hat{V}$).*

We will now concern ourselves with defining incidence. To do this, we will first require a categorial definition of "unordered product".

Definition 13. *The* twist morphism *of a self product $V \times V$, is the unique morphism $t_w : V \times V \to V \times V$ such that for $p_0, p_1 : V \times V \to V$ the canonical projection morphisms, $p_0 t_w = p_1$ and $p_1 t_w = p_0$.*

Definition 14. *Given an object V, the* unordered product $V \veebar V$ *is defined as the coequalizer object, $V \veebar V \cong Coeq(V \times V, t_w)$ of $V \times V, t_w : V \times V \to V \times V$.*

We now show the unordered product is functorial.

Proposition 20. *Given $f : X \to Y$, there exists a unique $f \veebar f : X \veebar X \to Y \veebar Y$ such that $(f \veebar f) coeq(X \times X, t_{w_X}) = coeq(Y \times Y, t_{w_Y})(f \times f)$.*

Proof. Let $c_0 = coeq(X \times X, t_{w_X}) : X \times X \to X \veebar X$, and $c_1 = coeq(Y \times Y, t_{w_Y}) : Y \times Y \to Y \veebar Y$. Hence $c_0 t_{w_X} = c_0(X \times X)$ and $c_1 t_{w_Y} = c_1(Y \times Y)$. Let $p_0, p_1 : X \times X \to X$, $\pi_0, \pi_1 : Y \times Y \to Y$ be the canonical projection morphisms. Then $p_0 t_{w_X} = p_1$, $p_1 t_{w_X} = p_0$, $\pi_0 t_{w_Y} = \pi_1$, and $\pi_1 t_{w_Y} = \pi_0$. Furthermore, as product is functorial, there exists $f \times f : X \times X \to Y \times Y$ such that $\pi_0(f \times f) = f p_0$ and $\pi_1(f \times f) = f p_1$.

Consider $\pi_0 t_{w_Y}(f \times f) t_{w_X}$: $\pi_0 t_{w_Y}(f \times f) t_{w_X} = \pi_1(f \times f) t_{w_X} = f p_1 t_{w_X} = f p_0$. Similarly $\pi_1 t_{w_Y}(f \times f) t_{w_X} = f p_1$. However, $f \times f$ is the unique morphism such that $\pi_0(f \times f) = f p_0$ and $\pi_1(f \times f) = f p_1$. Hence $f \times f = t_{w_Y}(f \times f) t_{w_X}$.

Consider $c_1(f \times f) t_{w_X}$: $c_1(f \times f) t_{w_X} = c_1(Y \times Y)(f \times f) t_{w_X} = c_1 t_{w_Y}(f \times f) t_{w_X} = c_1(f \times f) = c_1(f \times f)(X \times X)$. Thus by the universal mapping property of $X \veebar X$, there exists a unique morphism $f \veebar f : X \veebar X \to Y \veebar Y$ such that $(f \veebar f) c_0 = c_1(f \times f)$. $\square$

Definition 15. *The* unordered diagonal morphism $\underline{\Delta} : X \to X \veebar X$ *is defined as* $\underline{\Delta} = coeq(X \times X, t_{w_X})\Delta$ *for* $\Delta : X \to X \times X$ *the diagonal morphism.*

We will first determine the incidence of each part of an object and then work on an incidence morphism.

Definition 16. *Let* $a \in V(\hat{E})$.

1. *Let* $(e_1\_e_2)$ *be an edge of an object* A. *We say* $(e_1\_e_2)$ *is* incident *to vertex* $|e_1a| : \hat{V} \to |A|$ *and vertex* $|e_2a| : \hat{V} \to |A|$ *for* $|e_1a|$, $|e_2a|$ *the unique vertices prescribed by Axiom 7 such that* $t_A|e_1a| = e_1a$ *and* $t_A|e_2a| = e_2a$.

2. *Let* ℓ *be a loop of an object* A. *We say loop* ℓ *is* incident *to vertex* $|\ell a|$ *for* $|\ell a|$ *the unique vertex prescribed by Axiom 7 such that* $t_A|\ell a| = \ell a$.

3. *Let* v *be a vertex of* A. *We say vertex* v *is* incident *to* $|v|$ *for* $|v|$ *the unique vertex prescribed by Axiom 7 such that* $t_A|v| = v$.

We note that the choice of $a \in V(\hat{E})$ wont effect the vertices the part is incident to (provided the pair of vertices an edge is incident to is unordered), as for $b \in V(\hat{E})$ the other vertex of $\hat{E}$, $e_1b = e_1\tau a = e_2a$, $e_2b = e_2\tau a = e_1a$, and $\ell b = \ell\tau a = \ell a$.

We now create a "part set" object and define incidence.

Axiom 11. *Let $a \in V(\hat{E})$. Given an object X, there exists a minimal discrete object P_X along with a monomorphism $\iota_X : |X| \rightarrowtail P_X$ and morphism $\partial_X : P_X \to |X| \veebar |X|$ such that,*

1. $\partial_X \iota_X = \underline{\Delta}_X$,
2. *for any edge $(e_1\_e_2)$ of X with incident vertices $(|e_1 a|\_|e_2 a|)$, there exists $e : \hat{V} \to P_X$ with $\partial_X e = coeq(t_w, |X| \times |X|)(|e_1 a| \times |e_2 a|)$ and for any other distinct edge $(f_1\_f_2)$, with $f \in V(P_X)$ the corresponding vertex, $e \neq f$,*
3. *and for any loop ℓ of X incident to $|\ell a| : \hat{V} \to |X|$, there exists vertex $\ell^* : \hat{V} \to P_X$ with $\partial_X \ell^* = \underline{\Delta}_X |\ell a|$, $\ell^* \neq \iota_A |\ell a|$, and for any other distinct loop j, with $j^* \in V(P_X)$ the corresponding vertex, $\ell^* \neq j^*$.*

Again we note that the choice of $a \in V(\hat{E})$ does not effect incidence, as $coeq(t_w, |X| \times |X|)(|e_1 a| \times |e_2 a|) = coeq(t_w, |X| \times |X|) t_w(|e_1 a| \times |e_2 a|) = coeq(t_w, |X| \times |X|)(|e_2 a| \times |e_1 a|)$. We now proceed to prove a useful property about P_X.

Proposition 21. *Given an object X, the discrete object P_X prescribed by Axiom 11 is unique up to isomorphism.*

Proof. Suppose P_X and P'_X both satisfy the conditions in Axiom 11 for X. Then as they are both minimal such objects, there are monomorphisms $m_1 : P_X \rightarrowtail P'_X$ and $m_2 : P'_X \rightarrowtail P_X$. Hence, by Proposition 17, $P_X \cong P'_X$. □

We now work to provide an axiom to extend a morphism between objects A and B to a morphism between P_A and P_B.

Proposition 22. *If* $(e_1\_e_2)$ *is an edge of* A *and* $f : A \to B$*, then*

1. *if* $fe_1 \neq fe_2$ *then* $(fe_1\_fe_2)$ *is an edge of* B*,*

2. *or if* $fe_1 = fe_2$ *then* fe_1 *is a loop of* B *or a constant vertex morphism.*

Proof. First suppose $fe_1 \neq fe_2$, then as $fe_1\tau = fe_2$, $(fe_1\_fe_2)$ is an edge of B. If $fe_1 = fe_2$ then $fe_1\tau = fe_2 = fe_1$ and fe_1 is either a loop or a constant vertex morphism by definition of a loop. □

Proposition 23. *If* ℓ *is a loop of* A *and* $f : A \to B$ *then* $f\ell$ *is either a loop of* B *or constant vertex morphism.*

Proof. As $\ell\tau = \ell$, $f\ell\tau = f\ell$ and by definition of a loop, $f\ell$ is a loop or constant vertex morphism. □

Axiom 12. *Let* $f : A \to B$*, then there exists a morphism* $f_P : P_A \to P_B$ *such that* $f_P\iota_A = \iota_B|f|$ *and* $\partial_B f_P = (|f|\veebar|f|)\partial_A$*, for* $|f| : |A| \to |B|$ *the unique morphism prescribed by Axiom 7 such that* $ft_A = t_B|f|$*, such that for* $p \in V(P_A)$

1. *if* p *corresponds to an edge* $(e_1\_e_2)$ *of* A*, then* $(fe_1\_fe_2)$ *is an edge of* B *and* $f_Pp = fe$ *for* fe *the vertex of* P_B *corresponding to* $(fe_1\_fe_2)$*,* fe_1 *is a loop of* B *and for* fe_1^* *the corresponding vertex in* P_B*,* $f_Pp = fe_1^*$*, or* fe_1 *is a constant vertex morphism such that* $fe_1 = v\hat{V}_{\hat{E}}$ *for* $v \in V(B)$ *and* $f_Pp = \iota_B|v|$ *for* $|v|$ *the unique vertex of* $|B|$ *such that* $v = t_B|v|$*,*

2. *or if* p *corresponds to a loop* ℓ *of* A*, then* $f\ell$ *is a loop of* B *and* $f_Pp = f\ell^*$ *the corresponding vertex of* P_B*, or* $f\ell$ *is a constant vertex morphism such that* $f\ell = v\hat{V}_{\hat{E}}$ *for some* $v \in V(B)$ *and* $f_Pp = \iota_B|v|$ *for* $|v|$ *the unique vertex of* $|B|$ *such that* $v = t_B|v|$*.*

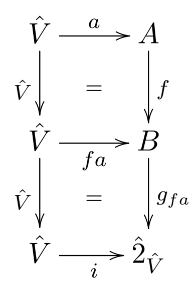
We note that for $f : A \to B$ and $p \in V(P_A)$ such that there is a vertex $v \in V(|A|)$ with $p = \iota_A v$, then $f_P p = \iota_B |f| v$, for as $f_P \iota_A = \iota_B |f|$, $f_P p = f_P \iota_A v = \iota_B |f| v$.

Before providing the last axioms for each category, we need another two propositions that will hold in all five categories.

Proposition 24. *If $f : A \rightarrowtail B$ is a monomorphism and B is discrete, then A is discrete.*

Proof. As f is a monomorphism, by Proposition 3 f is injective on vertices. For $x \in V(B)$ let $g_x : B \to \hat{2}_{\hat{V}}$ be the morphism given in Definition 9 such that $g_x y \neq g_x x$ for all $y \in V(B)$ with $y \neq x$. Hence as f is injective on vertices, for all vertices $a \in V(A)$, $g_{fa} fa \neq g_{fa} fz$ for all vertices $z \in V(A)$ with $z \neq a$. Hence the first condition of discrete objects is satisfied by A.

Now consider the following diagram where i is the injection morphism given by Definition 9 such that $i = g_{fa} fa$. We will show it is a pullback.



We note that the bottom square is a pullback by Definition 9. We show the top square is a pullback.

So let $h : C \to A$ and $k : C \to \hat{V}$ be such that $fh = fak$. As f is a monomorphism, $h = ak$. Hence $k : C \to \hat{V}$ is such that $h = ak$ and $\hat{V}k = k$. As morphisms to $\hat{V}$ are unique by Axiom 2, k is unique, and the top square is a pullback. Hence

$$\begin{array}{ccc} \hat{V} & \xrightarrow{a} & A \\ {\scriptstyle \hat{V}}\downarrow & p.b. & \downarrow{\scriptstyle f} \\ \hat{V} & \xrightarrow[fa]{} & B \\ {\scriptstyle \hat{V}}\downarrow & p.b. & \downarrow{\scriptstyle g_{fa}} \\ \hat{V} & \xrightarrow[i]{} & \hat{2}_{\hat{V}} \end{array}$$

and by the pullback lemma [1], the diagram is a pullback, and by definition A is discrete. $\square$

Proposition 25. *If $f : A \to B$ is a morphism with A and B discrete objects and f injective on vertices (i.e. for all $x, y \in V(A)$ such that $x \neq y$, $fx \neq fy$) then f is a monomorphism.*

Proof. If $A \cong \hat{0}$ then the proposition holds vacuously. So suppose $A \not\cong \hat{0}$, then by Axiom 5 there exists $g : B \to A$ such that $fgf = f$. Consider gf. If $gf \neq A$, then by Proposition 18 there is a vertex $x \in V(A)$ such that $gfx \neq Ax$ or $gfx \neq x$. As f is injective on vertices, $fgfx \neq fx$ and $fgf \neq f$, a contradiction. Hence $gf = A$ and f is a monomorphism. $\square$

The following axioms will be used to specialize the category we are in. The first set of axioms describe what objects correspond to a pair of discrete objects P_A, V_A with a monomorphism $\iota_A : V_A \rightarrowtail P_A$ and an incidence morphism $\partial_A : P_A \to V_A \veebar V_A$. The second set of axioms provide morphisms between objects if there are certain morphisms between their associated part discrete objects.

Axiom 13 (Grphs). *For every pair of discrete objects $\overline{P_A}, \overline{V_A}$ with monomorphism $\overline{\iota_A} : \overline{V_A} \rightarrowtail \overline{P_A}$ and morphism $\overline{\partial_A} : \overline{P_A} \to \overline{V_A} \veebar \overline{V_A}$ such that $\overline{\partial_A}\,\overline{\iota_A} = \underline{\Delta}_A$ for $\underline{\Delta}_A : \overline{V_A} \to \overline{V_A} \veebar \overline{V_A}$ the unordered diagonal morphism, there exists an object A such that there are isomorphisms $\phi_A : |A| \to \overline{V_A}$ and $\psi_A : P_A \to \overline{P_A}$ such that $\overline{\iota_A}\phi_A = \iota_A \psi_A$ and $(\phi_A \veebar \phi_A)\partial_A = \overline{\partial_A}\psi_A$.*

This axiom for **Grphs** establishes that there is always an object that corresponds to any incidence morphism ∂ and vertex inclusion morphism ι with $\partial\iota = \underline{\Delta}$.

Axiom 13 (SiGrphs). *For every pair of discrete objects $\overline{P_A}, \overline{V_A}$ with monomorphism $\overline{\iota_A} : \overline{V_A} \rightarrowtail \overline{P_A}$ and morphism $\overline{\partial_A} : \overline{P_A} \to \overline{V_A} \veebar \overline{V_A}$ such that $\overline{\partial_A}\overline{\iota_A} = \overline{\underline{\Delta}_A}$ for $\overline{\underline{\Delta}_A} : \overline{V_A} \to \overline{V_A} \veebar \overline{V_A}$ the unordered diagonal morphism, and for all $a, b \in V(\overline{P_A})$ such that for all $v \in V(\overline{V_A})$ $a \neq \overline{\iota_A} v$ and $b \neq \overline{\iota_A} v$, we have $a \neq b$ implies $\overline{\partial_A} a \neq \overline{\partial_A} b$, there exists an object A such that there are isomorphisms $\phi_A : |A| \to \overline{V_A}$ and $\psi_A : P_A \to \overline{P_A}$ such that $\overline{\iota_A}\phi_A = \iota_A \psi_A$ and $(\phi_A \veebar \phi_A)\partial_A = \overline{\partial_A}\psi_A$.*

Axiom 13 (SiLlGrphs). *For every pair of discrete objects $\overline{P_A}, \overline{V_A}$ with monomorphism $\overline{\iota_A} : \overline{V_A} \rightarrowtail \overline{P_A}$ and morphism $\overline{\partial_A} : \overline{P_A} \to \overline{V_A} \veebar \overline{V_A}$ such that $\overline{\partial_A}\overline{\iota_A} = \overline{\underline{\Delta}_A}$ for $\overline{\underline{\Delta}_A} : \overline{V_A} \to \overline{V_A} \veebar \overline{V_A}$ the unordered diagonal morphism, for all $a, b \in V(\overline{P_A})$ such that for all $v \in V(\overline{V_A})$ $a \neq \overline{\iota_A} v$ and $b \neq \overline{\iota_A} v$, we have $a \neq b$ implies $\overline{\partial_A} a \neq \overline{\partial_A} b$, and for all $a \in V(\overline{P_A})$ such that for all $v \in V(\overline{V_A})$, $\iota_A v \neq a$, we have $\overline{\partial_A} a \neq \underline{\Delta}_A y$ for all $y \in V(\overline{V_A})$, there exists an object A such that there are isomorphisms $\phi_A : |A| \to \overline{V_A}$ and $\psi_A : P_A \to \overline{P_A}$ such that $\overline{\iota_A}\phi_A = \iota_A \psi_A$ and $(\phi_A \veebar \phi_A)\partial_A = \overline{\partial_A}\psi_A$.*

Axiom 14 (Grphs). *If A and B are objects such that there are morphisms $f_P : P_A \to P_B$ and $f_V : |A| \to |B|$ for P_A, P_B the discrete objects prescribed by Axiom 11 such that $f_P \iota_A = \iota_B f_V$ and $\partial_B f_P = (f_V \veebar f_V)\partial_A$, then there exists a unique morphism $g : A \to B$ such that there is a morphism g_P prescribed by Axiom 12 with $g_P = f_P$ and $|g| = f_V$ for $|g| : |A| \to |B|$ the unique morphism prescribed by Axiom 7 such that $g t_A = t_B |g|$.*

This axiom establishes a morphism between objects if there are certain morphisms between part discrete objects.

Proposition 26 (**Grphs**)**.** *If A and B are objects with isomorphisms $\psi_A : P_A \to P_B$ and $\phi_A : |A| \to |B|$ with $\iota_B \phi_A = \psi_A \iota_A$ and $\partial_B \psi_A = (\phi_A \veebar \phi_A)\partial_A$, then $A \cong B$.*

Proof. By Axiom 14 (**Grphs**) there exists a unique $g : A \to B$ such that there is a morphism

$g_P = \psi_A$ and $|g| = \phi_A$. Now consider $\psi_A^{-1} : P_B \to P_A$ and $\phi_A^{-1} : |B| \to |A|$. As $\iota_B \phi_A = \psi_A \iota_A$, $\psi_A^{-1} \iota_B \phi_A = \iota_A$ and $\psi_A^{-1} \iota_B = \iota_a \phi_A^{-1}$. As $\partial_B \psi_A = (\phi_A \Join \phi_A) \partial_A$, $(\phi_A^{-1} \Join \phi_A^{-1}) \partial_B \psi_A = \partial_A$ and $(\phi_A^{-1} \Join \phi_A^{-1}) \partial_B = \partial_A \psi_A^{-1}$. Hence Axiom 14 (**Grphs**) applies and there exists a unique $f : B \to A$ such that there is a morphism $f_P = \psi_A^{-1}$ and $|f| = \phi_A^{-1}$.

Then as $\phi_A \phi_A^{-1} = |B|$, $\psi_A \psi_A^{-1} = P_B$, $P_B \iota_B = \iota_B |B|$, $\partial_B P_B = (|B| \Join |B|) \partial_B$, and $B : B \to B$ is the unique morphism that satisfies Axiom 14 (**Grphs**), $gf = B$. Similarly $fg = A$, and $A \cong B$. □

Axiom 14 (StGrphs). *If A and B are objects such that there are morphisms $f_P : P_A \to P_B$ and $f_V : |A| \to |B|$ for P_A, P_B the discrete objects prescribed by Axiom 11 such that $f_P \iota_A = \iota_B f_V$ $\partial_B f_P = (f_V \Join f_V) \partial_A$, and for all $a \in V(P_A)$ such that for all $v \in V(|A|)$, $a \neq \iota_A v$ we have $f_P a \neq \iota_B y$ for all $y \in V(|B|)$, then there exists a unique morphism $g : A \to B$ such that there is a morphism g_P prescribed by Axiom 12 with $g_P = f_P$ and $|g| = f_V$ for $|g| : |A| \to |B|$ the unique morphism prescribed by Axiom 7 such that $g t_A = t_B |g|$.*

Proposition 26 (**StGrphs**). *If A and B are objects with isomorphisms $\psi_A : P_A \to P_B$ and $\phi_A : |A| \to |B|$ with $\iota_B \phi_A = \psi_A \iota_A$ and $\partial_B \psi_A = (\phi_A \Join \phi_A) \partial_A$, then $A \cong B$.*

Proof. It suffices show the isomorphism $\psi_A : P_A \to P_B$ respects the property that for all $a \in P_A$ such that for all $v \in |A|$ $a \neq \iota_A v$ we have $\psi_A a \neq \iota_B y$ for all $y \in |B|$. The proof will then follow from the proof of Proposition 26 (**Grphs**).

We proceed by contrapositive, suppose there exists a $y \in V(|B|)$ such that $\psi_A a = \iota_B y$ for $a \in V(P_A)$. Then as $\iota_B \phi_A = \psi_A \iota_A$ and ϕ_A is an isomorphism, $\iota_B = \psi_A \iota_A \phi_A^{-1}$. Hence $\psi_A a = \psi_A \iota_A \phi_A^{-1} y$ and as ψ_A is an isomorphism $a = \iota_A \phi_A^{-1} y$ and for $v = \phi_A^{-1} y$, $a = \iota_A v$ for $v \in V(|A|)$. □

Axiom 15 (SiGrphs). *For any object A and for P_A, ι_A, and ∂_A given by Axiom 11, for any $a, b \in V(P_A)$ such that $a \neq b$, a corresponds to an edge or loop, and b corresponds to an edge or loop, then $\partial_A a \neq \partial_A b$.*

Axiom 16 (SiLlGrphs). *No object has a loop and for any object A and for P_A, ι_A, and ∂_A given by Axiom 11, for any $a \in V(P_A)$ such that a corresponds to an edge, $\partial_A a \neq \underline{\Delta}_A y$ for all $y \in V(|A|)$.*

Axiom 16 (StGrphs). *Given a morphism $f : A \to B$, there exists an f_P provided by Axiom 12 such that if $x \in V(P_A)$ such that for all $v \in V(|A|)$, $\iota_A v \neq x$, then for all $y \in V(|B|)$, $f_P x \neq \iota_B y$.*

4.2 Metatheorems

In this section we show that the elementary system of axioms which we have constructed, along with one non-elementary axiom form a characterization of the five categories of graphs. The proofs are informal but could be formalized with sufficiently strong set theory.

We will restrict our discussion to locally small categories, i.e. categories with the property that the class of mappings from an object A to an object B is a set. We will characterize each of the five categories using two metatheorems. The first metatheorem will show a functor equivalence between the standard set theory category of graphs, and a category of graphs constructed over Lawvere's system of set theory [16]. The second metatheorem in the pair will show a functor equivalence between our elementary system with the axiom of completeness to the category of graphs constructed over Lawvere's system of set theory. We first define these categories of graphs over Lawvere's system of set theory.

Definition 17. *Suppose* $\mathbf{D}$ *is a category satisfying Lawvere's elementary axioms for the category of* ***Sets****. Denote by* $\boldsymbol{Grphs}_{\mathbf{D}}$ *the category whose objects X are ordered 4-tuples, of two objects followed by two morphisms in* $\mathbf{D}$, $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \veebar V_X)$ *such that for $\underline{\Delta}_X : V_X \to V_X \veebar V_X$, the unordered diagonal morphism, $\partial_X \iota_X = \underline{\Delta}_X$, and whose morphisms $f : A \to B$ are ordered pairs of morphisms in* $\mathbf{D}$ $(f_P : P_A \to P_B, f_V : V_A \to V_B)$ *such that $f_P \iota_A = \iota_B f_V$ and $\partial_B f_P = (f_V \veebar f_V)\partial_A$.*

Proposition 27. $\boldsymbol{Grphs}_{\mathbf{D}}$ *is indeed a category.*

Proof. We first note that composition of morphisms is a morphism. Let $(P_A, V_A; \iota_A, \partial_A)$, $(P_B, V_B; \iota_B, \partial_B)$, and $(P_C, V_C; \iota_C, \partial_C)$, be given objects with morphisms $(f_P : P_A \to P_B, f_V : V_A \to V_B)$ and $(g_P : P_B \to P_C, g_V : V_B \to V_C)$. Then $(g_P, g_V)(f_P, f_V) = (g_P f_P, g_V f_V)$ is a morphism as $g_P f_P \iota_A = g_P \iota_B f_V = \iota_C g_V f_V$ and $\partial_C g_P f_P = (g_V \veebar g_V)\partial_B f_P = (g_V \veebar g_V)(f_V \veebar f_V)\partial_A$, and by the universal mapping property of unordered product of morphisms $(g_V \veebar g_V)(f_V \veebar f_V) = g_V f_V \veebar g_V f_V$.

Composition is associative as composition of morphisms in $\mathbf{D}$ is associative. Given an object $(P_A, V_A; \iota_A, \partial_A)$, (P_A, V_A) forms the local identity. As $P_A \iota_A = \iota_A = \iota_A V_A$ and $\partial_A P_A = \partial_A = (V_A \veebar V_A)\partial_A$, (P_A, V_A) is a morphism. □

Definition 18. *Given a 4-tuple $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \veebar V_X)$ of objects and morphisms in $\mathbf{D}$, a category satisfying the Lawvere's elementary axioms for the category of **Sets**, the **simple restriction** on these 4-tuples is for $a, b : \hat{1} \to P_X$ with $a \neq b$ such that for all $v : \hat{1} \to V_X$, $\iota_X v \neq a$ and $\iota_X v \neq b$, we have $\partial_X a \neq \partial_X b$.*

Definition 19. *Given a 4-tuple $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \veebar V_X)$ of objects and morphisms in $\mathbf{D}$, a category satisfying the Lawvere's elementary axioms for the category of **Sets**, the **loopless restriction** on these 4-tuples is for $a : \hat{1} \to P_X$ such that for all $v : \hat{1} \to V_X$, $\iota_X v \neq a$ we have $\partial_X a \neq \underline{\Delta}_X y$ for all $y : \hat{1} \to V$.*

Definition 20. *Given two 4-tuples $(P_A, V_A; \iota_A : V_A \rightarrowtail P_A, \partial_A : P_A \to V_A \veebar V_A)$ and $(P_B, V_B; \iota_B : V_B \rightarrowtail P_B, \partial_B : P_B \to V_B \veebar V_B)$ of objects and morphisms in $\mathbf{D}$, a category satisfying the Lawvere's elementary axioms for the category of **Sets**, along with two morphisms $f_P : P_A \to P_B$ and $f_V : V_A \to V_B$, the **strict restriction** on the morphism pair (f_P, f_V) is for $a : \hat{1} \to P_X$ such that for all $v : \hat{1} \to V_X$, $\iota_X v \neq a$ we have that for all $x : \hat{1} \to V_B$, $f_P a \neq \iota_B x$.*

Definition 21. *Suppose **D** is a category satisfying Lawvere's elementary axioms for the category of **Sets**. Denote by **SiGrphs**$_{\mathbf{D}}$ the category whose objects X are ordered 4-tuples, of two objects followed by two morphisms in D, $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \veebar V_X)$ such that $\partial_X \iota_X = \underline{\Delta}_X$ with the **simple restriction**, and whose morphisms $f : A \to B$ are* *ordered pairs of morphisms in* $\mathbf{D}$ $(f_P : P_A \to P_B, f_V : V_A \to V_B)$ *such that* $f_P \iota_A = \iota_B f_V$ *and* $\partial_B f_P = (f_V \veebar f_V)\partial_A$.

We note that $\mathbf{SiGrphs_D}$ is a category as the proof given in Proposition 27 holds here.

Definition 22. *Suppose* $\boldsymbol{D}$ *is a category satisfying Lawvere's elementary axioms for the category of* ***Sets****. Denote by* $\boldsymbol{SiLlGrphs_D}$ *the category whose objects* X *are ordered 4-tuples, of two objects followed by two morphisms in* D*,* $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \veebar V_X)$ *such that* $\partial_X \iota_X = \underline{\Delta}_X$ *with the* ***simple restriction*** *and the* ***loopless restriction****, and whose morphisms* $f : A \to B$ *are ordered pairs of morphisms in* $\mathbf{D}$ $(f_P : P_A \to P_B, f_V : V_A \to V_B)$ *such that* $f_P \iota_A = \iota_B f_V$ *and* $\partial_B f_P = (f_V \veebar f_V)\partial_A$.

We note that $\mathbf{SiLlGrphs_D}$ is a category as the proof given in Proposition 27 holds here.

Definition 23. *Suppose* $\boldsymbol{D}$ *is a category satisfying Lawvere's elementary axioms for the category of* ***Sets****. Denote by* $\boldsymbol{StGrphs_D}$ *the category whose objects* X *are ordered 4-tuples, of two objects followed by two morphisms in* D*,* $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \veebar V_X)$ *such that for* $\underline{\Delta}_X : V_X \to V_X \veebar V_X$*, and whose morphisms* $f : A \to B$ *are ordered pairs of morphisms in* $\mathbf{D}$ $(f_P : P_A \to P_B, f_V : V_A \to V_B)$ *such that* $f_P \iota_A = \iota_B f_V$ *and* $\partial_B f_P = (f_V \veebar f_V)\partial_A$ *and* (f_P, f_V) *have the* ***strict restriction****.*

Proposition 28. $\boldsymbol{StGrphs_D}$ *is indeed a category.*

Proof. Using the proof given in Proposition 27, we must only show that the composition of two morphisms with the **strict restriction** still has the **strict restriction**. So let $(f_P, f_V) : (P_A, V_A; \iota_A, \partial_A) \to (P_B, V_B; \iota_B, \partial_B)$ and $(g_P, g_V) : (P_B, V_B; \iota_B, \partial_B) \to (P_C, V_C; \iota_C, \partial_C)$, and consider $(g_P, g_V)(f_P, f_V) = (g_P f_P, g_V f_V)$. Let $a : \hat{1} \to P_A$ be such that for all $v : \hat{1} \to V_A$ has $a \neq \iota_A v$. Then $f_P a$ is such that for all $x : \hat{1} \to V_B$, $f_P a \neq \iota_B x$. Hence for all $y : \hat{1} \to V_C$, $g_p f_p a \neq \iota_c y$. Hence $(g_P, g_V)(f_P, f_V)$ has the strict restriction and $\mathbf{StGrphs_D}$ is a category. □

Definition 24. *Suppose **D** is a category satisfying Lawvere's elementary axioms for the category of **Sets**. Denote by **SiStGrphs**$_{\mathbf{D}}$ the category whose objects X are ordered 4-tuples, of two objects followed by two morphisms in D, $(P_X, V_X; \iota_X : V_X \rightarrowtail P_X, \partial_X : P_X \to V_X \underline{\times} V_X)$ such that for $\underline{\Delta}_X : V_X \to V_X \underline{\times} V_X$ with the **simple restriction**, $\partial_X \iota_X = \underline{\Delta}_X$, and whose morphisms $f : A \to B$ are ordered pairs of morphisms in $\mathbf{D}$ $(f_P : P_A \to P_B, f_V : V_A \to V_B)$ such that $f_P \iota_A = \iota_B f_V$ and $\partial_B f_P = (f_V \underline{\times} f_V)\partial_A$ and (f_P, f_V) have the **strict restriction**.*

We note that $\mathbf{SiStGrphs_D}$ is a category as the proof from Proposition 28 holds here.

We now proceed to the metatheorems.

Metatheorem 1. *Let $\mathbf{D}$ be any locally small category such that $\mathbf{D}$ is a model of Lawvere's system of axioms for the category of sets. If $\mathbf{D}$ is complete then **Grphs**$_{\mathbf{D}}$ is equivalent to **Grphs**.*

Proof. As $\mathbf{D}$ satisfies Lawvere's axioms and $\mathbf{D}$ is complete, there exists a functor equivalence $H^1 : \mathbf{D} \rightsquigarrow \mathbf{Sets}$, by $H^1(A) = \hom_{\mathbf{D}}(\hat{1}, A)$ and for $f : A \to B$,
$H^1(f) : \hom_{\mathbf{D}}(\hat{1}, A) \to \hom_{\mathbf{D}}(\hat{1}, B)$ by $a \mapsto fa$.

We define $F : \mathbf{Grphs_D} \rightsquigarrow \mathbf{Grphs}$ as follows.
$F((P_A, V_A; \iota_A, \partial_A)) = (H^1(P_A), H^1(V_A), H^1(\iota_A), \psi_A H^1(\partial_A))$ where
$\psi_A : H^1(V_A \underline{\times} V_A) \to H^1(V_A) \underline{\times} H^1(V_A)$ is the canonical isomorphism given as H^1 preserves limits and colimits. For morphisms $F((f_P, f_V)) = (H^1(f_P), H^1(f_V))$.

We now proceed to show F is functor. As H^1 is an equivalence of categories, $H^1(\iota_A)$ is a monomorphism as ι_A is a monomorphism and the preservation of limits and colimits yields, for $\underline{\Delta}_{H^1(A)} : H^1(V_A) \to H^1(V_A) \underline{\times} H^1(V_A)$, $\underline{\Delta}_{H^1(A)} = \psi_A H^1(\underline{\Delta}_A)$. As $\underline{\Delta}_A = \partial_A \iota_A$, $\underline{\Delta}_{H^1(A)} = \psi_A H^1(\underline{\Delta}_A) = \psi_A H^1(\partial_A \iota_A) = \psi_A H^1(\partial_A) H^1(\iota_A)$. Hence $F((P_A, V_A; \iota_A, \partial_A))$ is an object in $\mathbf{Grphs}$.

Now consider $F((f_P, f_V))$ for (f_P, f_V) a morphism from $(P_A, V_A; \iota_A, \partial_A)$ to $(P_B, V_B; \iota_B, \partial_B)$. Then $H^1(f_P)H^1(\iota_A) = H^1(f_P \iota_A) = H^1(\iota_B f_V) = H^1(\iota_B)H^1(f_V)$ and $\psi_B H^1(\partial_B)H^1(f_P) = \psi_B H^1(\partial_B f_P) = \psi_B H^1((f_V \underline{\times} f_V)\partial_A) = \psi_B H^1(f_V \underline{\times} f_V)H^1(\partial_A) = (H^1(f_V) \underline{\times} H^1(f_V))\psi_A H^1(\partial_A)$, where $\psi_B H^1(f_V \times f_V) = (H^1(f_V) \times H^1(f_V))\psi_A$ as ψ_A, ψ_B are the canonical isomorphisms given by H^1 preserving limits an colimits. Hence $(H^1(f_P), H^1(f_V))$ is a morphism in **Grphs**.

We note that as H^1 is a functor, $F((P_A, V_A)) = (H^1(P_A), H^1(V_A))$ and local identities are preserved. We are left with checking that composition is preserved.

Let $(f_P, f_V) : (P_A, V_A; \iota_A, \partial_A) \to (P_B, V_B; \iota_B, \partial_B)$ and

$(g_P, g_V) : (P_B, V_B; \iota_B, \partial_B) \to (P_B, V_B; \iota_B, \partial_B)$ and consider $F((g_P, g_V)(f_P, f_V))$:

$F((g_P, g_V)(f_P, f_V)) = F((g_P f_P, g_V f_V)) = (H^1(g_P f_P), H^1(g_V f_V))$

$= (H^1(g_P)H^1(f_P), H^1(g_V)H^1(f_V)) = (H^1(g_P), H^1(g_V))(H^1(f_P), H^1(f_V))$

$= F((g_P, g_V))F((f_P, f_V))$. Hence F is a functor.

We now establish a functor equivalence by proving F is faithful, full, and dense (every object C in **Grphs** is isomorphic to $F(A)$ for some object A in $\mathbf{Grphs_D}$).

We begin by showing F is faithful. Let $(f_P, f_V), (g_P, g_V) : (P_A, V_A; \iota_A, \partial_A) \to (P_B, V_B; \iota_B, \partial_B)$ be such that $F((f_P, f_V)) = F((g_P, g_V))$. Then as H^1 is an equivalence of categories, it is faithful and $H^1(f_P) = H^1(g_P)$ implies $f_P = g_P$ and $H^1(f_V) = H^1(g_V)$ implies $f_V = g_V$. Thus $(f_P, f_V) = (g_P, g_V)$ and F is faithful.

We now establish F is full. Let $(f_P, f_V) : F((P_A, V_A; \iota_A, \partial_A)) \to F((P_B, V_B, \iota_B, \partial_B))$. Hence $f_P : H^1(P_A) \to H^1(P_B)$ and $f_V : H^1(V_A) \to H^1(V_B)$ such that $f_P H^1(\iota_A) = H^1(\iota_B) f_V$ and $(f_V \times f_V)\psi_A H^1(\partial_A) = \psi_B H^1(\partial_B) f_P$. As H^1 is an equivalence of categories, H^1 is a full functor, and hence there exists $g_P : P_A \to P_B$ and $g_V : V_A \to V_B$ such that $H^1(g_P) = f_P$ and $H^1(g_V) = f_V$. We show (g_P, g_V) is a morphism of $\mathbf{Grphs_D}$.

As H^1 is faithful and $H^1(g_P \iota_A) = f_P H^1(\iota_A) = H^1(\iota_B) f_V = H^1(\iota_B g_V)$, $g_P \iota_A = \iota_B g_V$. Now consider $\psi_B H^1(\partial_B g_P)$: $\psi_B H^1(\partial_B g_P) = \psi_B H^1(\partial_B) f_P = (f_V \times f_V)\psi_A H^1(\partial_A)$

$= (H^1(g_V) \times H^1(g_V))\psi_A H^1(\partial_A) = \psi_B H^1(g_V \times g_V) H^1(\partial_A) = \psi_B H^1((g_V \times g_V)\partial_A)$. As ψ_B is an isomorphism and H^1 is a faithful functor, $\partial_B g_P = (g_V \times g_V)\partial_A$ and g is a morphism. Hence F is full.

We establish that F is dense. Let $(P_X, V_X; \iota_X, \partial_X)$ be a graph in **Grphs**. As H^1 is an equivalence of categories, there exists P_D, V_D such that there are isomorphisms $\phi_P : H^1(P_D) \to P_X$ and $\phi_V : H^1(V_D) \to V_X$. Then as H^1 is full, for $\alpha = \phi_P^{-1} \iota_X \phi_V : H^1(V_D) \to H^1(P_D)$, there exists $\iota_D : V_D \to P_D$ such that $H^1(\iota_D) = \alpha$. Furthermore, as α is a monomorphism and H^1 is an equivalence of categories ι_D is a monomorphism. Again as H^1 is a full functor, for $\gamma = \psi_D^{-1}(\phi_V^{-1} \boxtimes \phi_V^{-1})\partial_X \phi_P$, there exists $\partial_D : P_D \to V_D \boxtimes V_D$ such that $H^1(\partial_B) = \gamma$.

We now show $(P_D, V_D; \iota_D, \partial_D)$ is an object of $\mathbf{Grphs_D}$. First, consider $\underline{\Delta}_X \phi_V : H^1(V_D) \to V_X \boxtimes V_X$. Let $a \in H^1(V_D)$, then $\underline{\Delta}_X \phi_V(a) = \underline{\Delta}_X(\phi_V(a)) = (\phi_V(a)\_\phi_V(a))$. As $(\phi_V \boxtimes \phi_V)\psi_D H^1(\underline{\Delta}_D)(a) = (\phi_V \boxtimes \phi_V)\psi_D(\underline{\Delta}_D a) = (\phi_V \boxtimes \phi_V)(a\_a) = (\phi_V(a)\_\phi_V(a))$, $\underline{\Delta}_X \phi_V = (\phi_V \boxtimes \phi_V)\psi_D H^1(\underline{\Delta}_D)$ and $\psi_D^{-1}(\phi_V^{-1} \boxtimes \phi_V^{-1})\underline{\Delta}_X \phi_V = H^1(\underline{\Delta}_D)$.

Now consider $\psi_D H^1(\partial_D \iota_D)$: $\psi_D H^1(\partial_D \iota_D) = \psi_D H^1(\partial_D) H^1(\iota_D) = \psi_D \psi_D^{-1}(\phi_V^{-1} \boxtimes \phi_V^{-1})\partial_X \phi_P \phi_P^{-1} \iota_X \phi_V = \psi_D \psi_D^{-1}(\phi_V^{-1} \boxtimes \phi_V^{-1})\partial_X \iota_X \phi_V = \psi_D \psi_D^{-1}(\phi_V^{-1} \boxtimes \phi_V^{-1})\underline{\Delta}_X \phi_V = \psi_D H^1(\underline{\Delta}_D)$. As ψ_D is an isomorphism and H^1 a faithful functor, $\partial_D \iota_D = \underline{\Delta}_D$, and $(P_D, V_D; \iota_D, \partial_D)$ is an object of $\mathbf{Grphs_D}$.

Finally we show that $F((P_D, V_D; \iota_D, \partial_D)) \cong (P_X, V_X; \iota_X, \partial_X)$ by showing (ϕ_P, ϕ_V) is a morphism in **Grphs** and is hence an isomorphism. As $\phi_P H^1(\iota_D) = \phi_P \phi_P^{-1} \iota_X \phi_V = \iota_X \phi_V$, and $(\phi_V \boxtimes \phi_V)\psi_D H^1(\partial_D) = (\phi_V \boxtimes \phi_V)\psi_D \psi_D^{-1}(\phi_V^{-1} \boxtimes \phi_V^{-1})\partial_X \phi_P = \partial_X \phi_P$, (ϕ_P, ϕ_V) is a morphism. Hence there exists a functor equivalence between $\mathbf{Grphs_D}$ and **Grphs**. □

Metatheorem 2. *Let* $\boldsymbol{C}$ *be a locally small category such that:*

(i) $\boldsymbol{C}$ *is a model of Axioms 1-12, Axiom 13 (**Grphs**), and Axiom 14 (**Grphs**).*

(ii) for every family $\{A_j\}_{j \in J}$ *of objects in* $\boldsymbol{C}$ *there exists a product and a coproduct in* $\boldsymbol{C}$*.*

Then for $\mathbf{D}$ *the full subcategory of discrete objects of* $\boldsymbol{C}$*,* $\mathbf{D}$ *is a model of Lawvere's axioms for the category of sets,* $\mathbf{D}$ *is complete and* $\boldsymbol{C}$ *is equivalent to* $\boldsymbol{Grphs}_{\mathbf{D}}$ *(and thus equivalent to **Grphs**).*

We will first require three lemmas.

Lemma 1. *Let **C** be a locally small category such that:*

*(i) **C** is a model of Axioms 1-12.*

*(ii) for every family $\{A_j\}_{j \in J}$ of objects in **C** there exists a product and a coproduct in **C**.*

Then if each member of a family of objects, A_i, is discrete for $i \in I$ then $\sum_{i \in I} A_i$ is discrete.

Proof of Lemma 1. Let $X = \sum_{i \in I} A_i$. Consider $|X|$ with $t_X : |X| \to X$. As A_i is discrete for all $i \in I$, and $\iota_{A_i} : A_i \to X$ are the canonical injection morphisms, by Axiom 7 there exists $|\iota_{A_i}| : A_i \to |X|$ such that $t_A|\iota_{A_i}| = \iota_{A_i}$.

As X is a coproduct and for all $i \in I$ there are morphisms $|\iota_{A_i}| : A_i \to |X|$, by the universal mapping property of coproduct there exists a unique $\phi : X \to |X|$ such that for all $i \in I$ $|\iota_{A_i}| = \phi\iota_{A_i}$.

Hence $t_X\phi\iota_{A_i} = t_X|\iota_{A_i}| = \iota_{A_i}$ for all $i \in I$. However, as $X : X \to X$ is the unique morphism such that $X\iota_{A_i} = \iota_{A_i}$, $t_X\phi = X$. Thus ϕ is a monomorphism and by Proposition 24, X is discrete. □

Lemma 2. *Let **C** be a locally small category such that:*

*(i) **C** is a model of Axioms 1-12.*

*(ii) for every family $\{A_j\}_{j \in J}$ of objects in **C** there exists a product and a coproduct in **C**.*

*Then given an object A in **C**, there exists a choice for P_A prescribed by Axiom 11 such that if $v \in V(P_A)$, then v corresponds to a vertex (i.e. $v = \iota_A|v|$ for some $|v| \in V(|A|)$), loop, or edge of A.*

Proof of Lemma 2. If $A \cong \hat{0}$, then the result holds trivially. So suppose $A \ncong \hat{0}$, and hence $V(P_A) \neq \emptyset$. Let $v_j : \hat{V}_j \to P_A$, $\hat{V}_j = \hat{V}$ for $j \in J$, be the collection of vertices of P_A such that v_j corresponds to a vertex, edge, or loop of A. Consider $\sum_{j \in J} \hat{V}_j = X$ with injections $i_j : \hat{V}_j \to X$.

By Lemma 1, X is discrete as $\hat{V}_j$ is discrete for all $j \in J$. By Proposition 2, $v_j : \hat{V}_j \to P_A$ is a monomorphism for all $j \in J$. Hence by the universal mapping property of coproduct, there exists a unique $u : X \to P_A$ such that for all $j \in J$, $ui_j = v_j$. As $v_j \neq v_k$ for $j, k \in J$ with $j \neq k$, $ui_j \neq ui_k$ for $j, k \in J$ with $j \neq k$. Let $x, y \in V(X)$ with $x \neq y$. By Axiom 3, there exist $\hat{V}_j$ and $\hat{V}_k$ such that $x = \hat{V}_k i_k = \hat{V} i_k = i_k$ and $y = \hat{V}_j i_j = \hat{V} i_j = i_j$. We note that $i_k \neq i_j$, otherwise using the universal mapping property of coproduct on $a : \hat{V}_j \to \hat{E}$ and the morphisms $b : \hat{V}_l \to \hat{E}$ for all $l \in J$ with $l \neq j$, a and b the two vertices of $\hat{E}$, yields $(a + b) : X \to \hat{E}$ and $a = (a + b)i_j = (a + b)i_k = b$ a contradiction to Axiom 4. Thus $ux = ui_j \neq ui_k = uy$ and u is injective on vertices. Then by Proposition 25, u is a monomorphism.

Then by Proposition 19, there exists discrete object X' with monomorphism $u' : X' \to P_A$ such that $P_A \cong X + X'$ with injections u and u'. We will now show X satisfies the conditions of Axiom 11 for A.

Now consider $V_X = \sum_{|v| \in V(|A|)} \hat{V}_{|v|}$ with injections $\iota_{|v|} : \hat{V}_{|v|} \to V_X$ for $\hat{V} = \hat{V}_{|v|}$. By Lemma 1, V_X is discrete, and as $|v| : \hat{V}_{|v|} \to |A|$ for all $|v| \in V(|A|)$, there exists a unique $l : V_X \to |A|$ such that $l\iota_{|v|} = |v|$ for all $|v| \in V(|A|)$. Using a similar argument as the argument for u being a monomorphism, l is a monomorphism. Furthermore, as for every $|v| \in V(|A|)$, there is a $v_j \in V(P_A)$ such that $\iota_A|v| = v_j$, there is an $i_j \in V(X)$ such that $\iota_A|v| = ui_j$. Hence by the universal mapping property of coproduct, there exists $r : V_X \to X$ such that $r\iota_{|v|} = i_j$ for $i_j \in V(X)$ such that $\iota_A|v| = ui_j$. By a similar argument to u being a monomorphism r is a monomorphism.

Consider $l : V_X \to |A|$. By Axiom 5 there exists $s : |A| \to V_X$ such that $lsl = l$. As l is a monomorphism, $sl = V_X$. Let $|v| \in V(|A|)$. Consider $\iota_A ls|v|$. $\iota_A ls|v| = \iota_A lsl\iota_{|v|} = \iota_A l\iota_{|v|} = \iota_A|v|$. As ι_A is a monomorphism $ls|v| = |v|$. Then by the contrapositive of Proposition 18, as $ls|v| = |v|$ for all $|v| \in V(|A|)$, $ls = |A|$.

Define $\partial_X = \partial_A u$ and $\iota_X : |A| \rightarrowtail X$ by $\iota_X = rs$ (both r and s are monomorphisms hence rs is a monomorphism). We now check the required properties of Axiom 11.

Let $|v| \in V(|A|)$ and consider $\partial_X \iota_X|v|$ with $i_j \in V(X)$ such that $r\iota_{|v|} = i_j$: $\partial_X \iota_X|v| = \partial_A urs|v| = \partial_A ursl\iota_{|v|} = \partial_A ur\iota_{|v|} = \partial_A ui_j = \partial_A \iota_A|v| = \underline{\Delta}_A|v|$. Hence $\partial_X \iota_X|v| = \underline{\Delta}_A|v|$ for all $|v| \in V(|A|)$. Then by the contrapositive to Proposition 18, $\partial_X \iota_X = \underline{\Delta}_A$.

Let $(e_1\_e_2)$ be an edge of A with incident vertices $(|e_1a|\_|e_2a|)$. Then by Axiom 11, there exists $e \in V(P_A)$ such that $\partial_A e = coeq(t_w, |A|\times|A|)(|e_1a|\times|e_2a|)$. Then there exists $i_j \in V(X)$ such that $ui_j = e$, and $\partial_X i_j = \partial_A ui_j = \partial_A e = coeq(t_w, |A| \times |A|)(|e_1a| \times |e_2a|)$. For another other distinct edge $(f_1\_f_2)$ of A with corresponding $f \in V(P_A)$, as $e \neq f$, for $i_k \in V(X)$ such that $ui_k = f$, $ui_j = e \neq f = ui_k$ implies $i_j \neq i_k$.

Let ℓ be a loop of A incident to $|\ell a|$, then there exists a vertex $\ell^* \in V(P_A)$ such that $\partial_A \ell^* = \underline{\Delta}_A|\ell a|$ and $\ell^* \neq \iota_A|\ell a|$. Thus there exists a $i_j \in V(X)$ such that $ui_j = \ell^*$. As $\iota_A|\ell a| \neq \ell^*$, $\iota_X|\ell a| = rs|\ell a| = rsl\iota_{|\ell a|} = r\iota_{|\ell a|} = i_k$ for $i_k \in V(X)$ such that $ui_k = \iota_A|\ell a|$. Hence as $ui_j = \ell^* \neq \iota_A|\ell a| = ui_k$, $i_j \neq \iota_X|\ell a|$. Furthermore $\partial_X i_j = \partial_A ui_j = \partial_A \ell^* = \underline{\Delta}_A|\ell a|$.

Hence X satisfies all the properties of Axiom 11. Then as P_A is a minimum such object, there exists a monomorphism $m_1 : P_A \to X$. However as $u : X \to P_A$ is a monomorphism, by Proposition 17, $|X| \cong P_A$. □

Lemma 3. *Let* $\boldsymbol{C}$ *be a locally small category such that:*

(i) $\boldsymbol{C}$ *is a model of Axioms 1-12.*

(ii) for every family $\{A_j\}_{j\in J}$ *of objects in* $\boldsymbol{C}$ *there exists a product and a coproduct in* $\boldsymbol{C}$*.*

Then given a morphism $f : A \to B$ *in* $\boldsymbol{C}$*, using the choices of* P_A *and* P_B *given in Lemma 2,* $f_P : P_A \to P_B$ *prescribed by Axiom 12 is unique.*

Proof of Lemma 3. Let P_A and P_B be chosen by Lemma 2. Then let $g_P, f_P : P_A \to P_B$ be morphisms that satisfy the conditions of Axiom 12. If $g_P \neq f_P$ then by Proposition 18, there exists a vertex $v \in V(P_A)$ such that $g_P v \neq f_P v$. However, by Lemma 2, v corresponds to a vertex, loop, or edge of A.

Suppose first that v corresponds to a vertex, that is $v = \iota_A|v|$ for some $|v| \in V(|A|)$. Then as $|f|$ is unique, $f_P \iota_A|v| = \iota_B|f||v| = g_P \iota_A|v|$ and $f_P v = g_P v$ a contradiction.

Next suppose v corresponds to a loop ℓ of A. If fv is a loop, j, of B, then by Axiom 12 $f_P v = fj^* = g_P v$ for j^* the corresponding vertex of P_B, a contradiction. If fv is a constant vertex morphism with vertex x in B such that $f\ell = x\hat{V}_{\hat{E}}$, then $f_P v = \iota_B|x| = g_P v$ for $|x|$ the unique vertex of $|B|$ such that $x = t_B|x|$, a contradiction.

Finally suppose v corresponds to an edge $(e_1{}_-e_2)$ of A. If $(fe_1{}_-fe_2)$ is an edge of B, then for $fe \in V(P_B)$ the corresponding vertex, $f_P v = fe = g_P e$, a contradiction. If fe_1 is a loop of B, then for $fe_1^* \in V(P_B)$ the corresponding vertex, $f_P v = fe_1^* = g_P v$ a contradiction. Finally if fe_1 is a constant vertex morphism with $x \in V(B)$ such that $fe_1 = x\hat{V}_{\hat{E}}$, $f_P v = \iota_B|x| = g_P v$ for $|x|$ the unique vertex of $V(|B|)$ such that $x = |x|t_B$, a contradiction. Thus $f_P = g_P$ and the morphism is unique. □

We now proceed with the proof of the metatheorem.

Proof of Metatheorem 2. Let **D** be the full subcategory of discrete objects of **C**. By Theorem Schema 1, **D** is a model for the elementary axioms of Lawvere's system of sets. Since **C** has arbitrary products and $|-| :$**C**$\rightsquigarrow$ **D** is product preserving, **D** has arbitrary products. By Lemma 1, **D** has arbitrary coproducts. Hence **D** is complete.

Define $F :\mathbf{C}\rightsquigarrow\mathbf{Grphs_D}$ on objects A by $F(A) = (P_A, |A|; \iota_A, \partial_A)$ for $|A|$ prescribed by Axiom 7 and P_A, ι_A, and ∂_A chosen by Lemma 2, and on morphisms $f : A \to B$ by
$F(f) = (f_P, |f|)$ for f_P chosen by Lemma 3, and $|f|$ the unique morphism such that $ft_A = t_B|f|$. We show F is a functor.

By Axiom 11, $F(A) = (P_A, |A|; \iota_A, \partial_A)$ is such that $\partial_A \iota_A = \underline{\Delta}_A$, and $\iota_A : |A| \rightarrowtail P_A$ is a monomorphism. Hence $F(A)$ is an object of $\mathbf{Grphs_D}$, and by Axiom 12 and Lemma 3, $F(f) = (f_P, |f|)$ is well defined, $f_P \iota_A = \iota_B |f|$, and $\partial_B f_P = (|f| \asymp |f|)\partial_A$. Hence $(f_P, |f|)$ is a morphism of $\mathbf{Grphs_D}$.

Consider $A : A \to A$. Let $F(A) = (f, g)$, for $f : P_A \to P_A$ and $g : |A| \to |A|$. Then as the morphism f that satisfies Axiom 12 is unique by Lemma 3, the morphism g is unique by Axiom 7, and $P_A : P_A \to P_A$ and $|A| : |A| \to |A|$ satisfy the axioms, $f = P_A$, $g = |A|$ and identities are preserved. Now let $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$. Then $F(g)F(f) = (g_P, |g|)(f_P, |f|) = (g_P f_P, |g||f|)$ and as the choices of $g_P, f_P, |g|$ and $|f|$ are unique, $F(g)F(f) = (g_P f_P, |g||f|) = ((gf)_P, |gf|) = F(gf)$. Hence F is a functor.

Now define $G : \mathbf{Grphs_D} \rightsquigarrow \mathbf{C}$ by $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A})) = A$ for A prescribed by Axiom 13 (**Grphs**), and for $(f_P, f_V) : (\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}) \to (\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B})$, $G((f_P, f_V)) = g : A \to B$ the unique morphism prescribed by Axiom 14 (**Grphs**) for $A = G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}))$, $B = G((\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B}))$, $\psi_B^{-1} f_P \psi_A : P_A \to P_B$, and $\phi_B^{-1} f_V \phi_A : |A| \to |B|$. We show G is a functor.

Clearly $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}))$ is an object of $\mathbf{C}$ and $G((f_P, f_V))$ is a morphism of $\mathbf{C}$. As the morphism prescribed by Axiom 14 (**Grphs**) is unique, identities are trivially preserved. Let $(f_P, f_V) : (\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}) \to (\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B})$ and
$(h_P, h_V) : (\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B}) \to (\overline{P_C}, \overline{V_C}; \overline{\iota_C}, \overline{\partial_C})$. Then $G((h_P, h_V))G((f_P, f_V)) = kg$ for $k : B \to C$ formed from $\psi_C^{-1} h_P \psi_B : P_B \to P_C$ with $\phi_C^{-1} h_V \phi_B : |B| \to |C|$ and $g : A \to B$ formed from $\psi_B^{-1} f_P \psi_A : P_A \to P_B$ with $\phi_B^{-1} f_V \phi_A : |A| \to |B|$. Then $g_P = \psi_B^{-1} f_P \psi_A$, $|g| = \phi_B^{-1} f_V \phi_A$, $k_P = \psi_C^{-1} h_P \psi_B$ and $|k| = \phi_C^{-1} h_V \phi_B$. By Lemma 3, g_P and k_P are unique. Then as $\psi_C^{-1} h_P \psi_B \psi_B^{-1} f_P \psi_A = \psi_C^{-1} h_P f_P \psi_A : P_A \to P_C$ and $\phi_C^{-1} h_V \phi_B \phi_B^{-1} f_V \phi_A = \phi_C^{-1} h_V f_V \phi_A : |A| \to |C|$, by Axiom 14 (**Grphs**) there exists a unique morphism $A \to C$ formed from $\psi_C^{-1} h_P f_P \psi_A$ and $\phi_C^{-1} h_V f_V \phi_A$, however kg is such a morphism. Hence $G((h_P, h_V))G((f_P, f_V)) = kg = G((h_P, h_V)(f_P, f_V)) = G((h_P f_P, h_V f_V))$ and G is a functor.

We now will show that F and G form a functor equivalence. Consider $FG((\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A})) = F(A) = (P_A,|A|;\iota_A,\partial_A)$ with isomorphisms (in $\mathbf{C}$) $\psi_A : P_A \to \overline{P_A}$ and $\phi_A : |A| \to \overline{V_A}$ such that $\overline{\iota_A}\phi_A = \iota_A\psi_A$ and $(\phi_A \veebar \phi_A)\partial_A = \overline{\partial_A}\psi_A$. By definition (ψ_A,ϕ_A) is a morphism in $\mathbf{Grphs_D}$. We show this is a natural isomorphism.

Given $(f_P,f_V) : (\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A}) \to (\overline{P_B},\overline{V_B};\overline{\iota_B},\overline{\partial_B})$, $FG((f_P,f_V))$
$= (g_P,|g|) : (P_A,|A|;\iota_A,\partial_A) \to (P_B,|B|;\iota_B,\partial_B)$ with isomorphisms
$(\psi_A,\phi_A) : (P_A,|A|;\iota_A,\partial_A) \to (\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A})$ and
$(\psi_B,\phi_B) : (P_B,|B|;\iota_B,\partial_B) \to (\overline{P_B},\overline{V_B};\overline{\iota_B},\overline{\partial_B})$. By Axiom 14 (**Grphs**), $g_P = \psi_B^{-1} f_P \psi_A$ and $|g| = \phi_B^{-1} f_V \phi_A$. Hence $\psi_B g_P = f_P\psi_A$ and $\phi_B|g| = f_V\phi_A$. Therefore $(\psi_B,\phi_B)(g_P,|g|) = (f_P,f_V)(\psi_A,\phi_A)$ and there is a natural isomorphism $FG \cong \mathbf{Grphs_D}$.

Now consider $GF(A) = G((P_A,|A|;\iota_A,\partial_A)) = A'$ such that there are isomorphisms (in $\mathbf{C}$) $\psi_{A'} : P_{A'} \to P_A$ and $\phi_{A'} : |A'| \to |A|$ such that $\iota_A\phi_{A'} = \iota_{A'}\psi_{A'}$ and $(\phi_{A'} \veebar \phi_{A'})\partial_{A'} = \partial_A\psi_{A'}$. By Proposition 26 (**Grphs**) there is an isomorphism $\gamma_{A'} : A' \to A$ such that $\gamma_{A'_P} = \psi_{A'}$ and $|\gamma_{A'}| = \phi_{A'}$. We show this is a natural isomorphism.

Let $f : A \to B$, then $GF(f) = g : A' \to B'$ such that $g_P = \psi_{B'}^{-1} f_P \psi_{A'}$ and $|g| = \phi_{B'}^{-1} f_V \phi_{A'}$. Hence $\psi_{B'} g_P = f_P\psi_{A'}$ and $\phi_{B'}|g| = f_V\phi_{A'}$. Therefore, as the morphism prescribed by Axiom 14 (**Grphs**) is unique, $\gamma_{B'} g = f\gamma_{A'}$ and there is a natural isomorphism $GF \cong \mathbf{C}$ □

Metatheorem 3. *Let* $\mathbf{D}$ *be any locally small category such that* $\mathbf{D}$ *is a model of Lawvere's system of axioms for the category of sets. If* $\mathbf{D}$ *is complete then* $\mathbf{SiGrphs_D}$ *is equivalent to* ***SiGrphs****.*

Proof. We begin by defining $F : \mathbf{SiGrphs_D} \rightsquigarrow \mathbf{SiGrphs}$ as in the proof of Metatheorem 1. We must only show that $F((P_A,V_A;\iota_A,\partial_A))$ is a simple graph and the rest of the proof follows exactly as in the proof of Metatheorem 1.

Let $x,y \in H^1(P_A)\backslash Im(H^1(\iota_A))$ such that $x \neq y$. Then as $x,y \in H^1(P_A)\backslash Im(H^1(\iota_A))$, for $v \in H^1(V_A)$ (note $H^1(V_A) = \hom_{\mathbf{D}}(\hat{1},V_A)$), $\iota_A v \neq x$ and $\iota_A v \neq y$. Hence $\partial_A x \neq \partial_A y$ by the **simple restriction**. As H^1 is faithful and ψ_A is an isomorphism (and hence a monomorphism) $\psi_A H^1(\partial_A)x \neq \psi_A H^1(\partial_a)y$. Hence no two edges share the same incidence and $F((P_A, V_A; \iota_A, \partial_A))$ is a simple graph. □

Metatheorem 4. *Let **C** be a locally small category such that:*

*(i) **C** is a model of Axioms 1-12, Axiom 13 (**SiGrphs**), Axiom 14 (**Grphs**) and Axiom 15 (**SiGrphs**).*

*(ii) for every family $\{A_j\}_{j \in J}$ of objects in **C** there exists a product and a coproduct in **C**.*

*Then for $\mathbf{D}$ the full subcategory of discrete objects of **C**, $\mathbf{D}$ is a model of Lawvere's axioms for the category of sets, $\mathbf{D}$ is complete and **C** is equivalent to $\boldsymbol{SiGrphs}_{\mathbf{D}}$ (and thus equivalent to **SiGrphs**).*

Proof. As Axioms 1-12 apply, so do Lemmas 1-3. Let $\mathbf{D}$ be the full subcategory of discrete objects of $\mathbf{C}$. By Theorem Schema 1, $\mathbf{D}$ is a model for the elementary axioms of Lawvere's system of sets. Since $\mathbf{C}$ has arbitrary products and $|-| : \mathbf{C} \rightsquigarrow \mathbf{D}$ is product preserving, $\mathbf{D}$ has arbitrary products. By Lemma 1, $\mathbf{D}$ has arbitrary coproducts. Hence $\mathbf{D}$ is complete.

Define $F : \mathbf{C} \rightsquigarrow \mathbf{SiGrphs}_{\mathbf{D}}$ on objects A by $F(A) = (P_A, |A|; \iota_A, \partial_A)$ for $|A|$ prescribed by Axiom 7, P_A, ι_A, and ∂_A chosen by Lemma 2, and for $f : A \to B$ $F(f) = (f_P, |f|)$ for f_P the unique morphism guaranteed by Lemma 3 and $|f|$ the unique morphism such that $ft_A = t_B|f|$.

By Axiom 11, $F(A) = (P_A, |A|; \iota_A, \partial_A)$ has ι_A a monomorphism and $\partial_A \iota_A = \underline{\Delta}_A$. We now show $(P_A, |A|; \iota_A, \partial_A)$ satisfies the **simple restriction**.

Suppose there exists $a, b : \hat{1} \to P_A$ (in $\mathbf{D}$) such that for all $v : \hat{1} \to |A|$ $a \neq \iota_A v$, $b \neq \iota_A v$, and $\partial_A a = \partial_A b$. Then by Lemma 2, a corresponds to a vertex, loop, or edge of A and b corresponds to a vertex, loop, or edge A. As $a \neq \iota_A v$ and $b \neq \iota_A v$, a and b are not vertices. Hence a corresponds to an edge or a loop of A and b corresponds to an edge or loop of A. Hence by Axiom 15 (**SiGrphs**) $a = b$. Thus $F(A)$ is an object of $\mathbf{SiGrphs}_{\mathbf{D}}$.

The fact F is a functor now follows from the proof given for F in Metatheorem 2. Now define $G : \mathbf{SiGrphs}_{\mathbf{D}} \rightsquigarrow \mathbf{C}$ by $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A})) = A$ for A prescribed by Axiom 13 (**SiGrphs**) and for $(f_P, f_V) : (\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}) \to (\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B})$ define $G((f_P, f_V)) = g : A \to B$ the unique morphism prescribed by Axiom 14 (**Grphs**) for $A = G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}))$, $B = G((\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B}))$, $\psi_B^{-1} f_P \psi_A : P_A \to P_B$, and $\phi_B^{-1} f_V \phi_A : V_A \to P_B$. We note that $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A})) = A$ is an object in **C**. Then the rest of the proof for the Metatheorem follows exactly as the proof in Metatheorem 2. $\square$

Metatheorem 5. *Let* **D** *be any locally small category such that* **D** *is a model of Lawvere's system of axioms for the category of sets. If* **D** *is complete then* ***SiLlGrphs***$_{\mathbf{D}}$ *is equivalent to* ***SiLlGrphs****.*

Proof. We define F :**SiLlGrphs**$_{\mathbf{D}}$ $\rightsquigarrow$**SiLlGrphs** as in the proof of Metatheorem 1. We must only show that $F((P_A, V_A; \iota_A, \partial_A))$ is a loopless graph, as the proof of Metatheorem 3 establishes that it is a simple graph. Then the proof from Metatheorem 1 applies.

So let $a \in H^1(P_A) \backslash Im(\iota_A)$. Then for all $v : \hat{1} \to V_A$, $\iota_A v \neq a$. Thus by the **loopless restriction** for all $x : \hat{1} \to V_A$, $\underline{\Delta}_A x \neq \partial_A a$. We note that $H^1(a) : \hom_{\mathbf{D}}(\hat{1}, \hat{1}) \to \hom_{\mathbf{D}}(\hat{1}, P_A)$ by $\hat{1} \mapsto a\hat{1} = a$. Hence $H^1(a)\hat{1} = a$. Similarly, $H^1(x)\hat{1} = x$. Then as H^1 is faithful, ψ_A an isomorphism (and hence a monomorphism) and $\hat{1} : \hat{1} \to \hat{1}$ is an epimorphism, $\psi_A H^1(\partial_A)a = \psi_A H^1(\partial_A) H^1(a)\hat{1} = \psi_A H^1(\partial_A a)\hat{1} \neq \psi_A H^1(\underline{\Delta}_A x)\hat{1} = \psi_A H^1(\underline{\Delta}_A) H^1(x)\hat{1} = \psi_A H^1(\underline{\Delta}_A)x$. Hence for all $y \in H^1(V_A)$, $y : \hat{1} \to V_A$ and $\psi_A H^1(\partial_A)a \neq \psi_A H^1(\underline{\Delta}_A)y = \underline{\Delta}_{H^1(A)} y = (y\_y)$ Thus each edge is incident to two distinct vertices and $F((P_A, V_A; \iota_A, \partial_A))$ is loopless. $\square$

Metatheorem 6. *Let* $\boldsymbol{C}$ *be a locally small category such that:*

(i) $\boldsymbol{C}$ *is a model of Axioms 1-12, Axiom 13 (**SiLlGrphs**), Axiom 14 (**Grphs**), Axiom 15 (**SiGrphs**), and Axiom 16 (**SiLlGrphs**).*

(ii) *for every family* $\{A_j\}_{j\in J}$ *of objects in* $\boldsymbol{C}$ *there exists a product and a coproduct in* $\boldsymbol{C}$*.*

Then for $\mathbf{D}$ *the full subcategory of discrete objects of* $\boldsymbol{C}$*,* $\mathbf{D}$ *is a model of Lawvere's axioms for the category of sets,* $\mathbf{D}$ *is complete and* $\boldsymbol{C}$ *is equivalent to* $\boldsymbol{SiLlGrphs}_{\mathbf{D}}$ *(and thus equivalent to **SiLlGrphs**).*

Proof. As Axioms 1-12 apply, so do Lemmas 1-3. Let $\mathbf{D}$ be the full subcategory of discrete objects of $\mathbf{C}$. By Theorem Schema 1, $\mathbf{D}$ is a model for the elementary axioms of Lawvere's system of sets. Since $\mathbf{C}$ has arbitrary products and $|-|:\mathbf{C}\rightsquigarrow\mathbf{D}$ is product preserving, $\mathbf{D}$ has arbitrary products. By Lemma 1, $\mathbf{D}$ has arbitrary coproducts. Hence $\mathbf{D}$ is complete.

Define $F:\mathbf{C}\rightsquigarrow\mathbf{SiLlGrphs_D}$ on objects A by $F(A)=(P_A,|A|;\iota_A,\partial_A)$ for $|A|$ prescribed by Axiom 7, P_A,ι_A, and ∂_A chosen by Lemma 2, and for $f:A\to B$ $F(f)=(f_P,|f|)$ for f_P the unique morphism guaranteed by Lemma 3 and $|f|$ the unique morphism such that $ft_A=t_B|f|$.

By Axiom 11, $F(A)=(P_A,|A|;\iota_A,\partial_A)$ has ι_A a monomorphism and $\partial_A\iota_A=\underline{\Delta}_A$. As in the proof of Metatheorem 4, Axiom 15 (**SiGrphs**) guarantees $F(A)$ satisfies the **simple restriction**. We show $F(A)$ satisfies the **loopless restriction**.

Suppose there exists $a:\hat{1}\to P_A$ (in $\mathbf{D}$) such that for all $v:\hat{1}\to|A|$ $a\neq\iota_A v$. Then by Lemma 2 a corresponds to a vertex, edge, or loop of A. As $a\neq\iota_A v$ for all $v\in|A|$, a does not correspond to a vertex. Furthermore by Axiom 16 (**SiLlGrphs**), a cannot correspond to a loop. Hence a corresponds to an edge, and by Axiom 16 (**SiLlGrphs**) $a\neq\underline{\Delta}_A y$ for all $y\in V(|A|)$. Thus $F(A)$ is an object of $\mathbf{SiLlGrphs_D}$.

The fact F is a functor now follows from the proof given for F in Metatheorem 2. Now define $G:\mathbf{SiLlGrphs_D}\rightsquigarrow\mathbf{C}$ by $G((\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A}))=A$ for A prescribed by Axiom 13

(**SiLlGrphs**) and for $(f_P, f_V) : (\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}) \to (\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B})$ define $G((f_P, f_V)) = g : A \to B$ the unique morphism prescribed by Axiom 14 (**Grphs**) for $A = G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}))$, $B = G((\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B}))$, $\psi_B^{-1} f_P \psi_A : P_A \to P_B$, and $\phi_B^{-1} f_V \phi_A : V_A \to P_B$. We note that $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A})) = A$ is an object in **C**. Then the rest of the proof for the Metatheorem follows exactly as the proof in Metatheorem 2. □

Metatheorem 7. *Let* **D** *be any locally small category such that* **D** *is a model of Lawvere's system of axioms for the category of sets. If* **D** *is complete then* $\textbf{\textit{StGrphs}}_{\mathbf{D}}$ *is equivalent to* ***StGrphs****.*

Proof. We define F :$\mathbf{StGrphs}_{\mathbf{D}} \rightsquigarrow$**StGrphs** as in the proof of Metatheorem 1. We must only show that $F((f_P, f_V))$ is a strict graph morphism for $(f_P, f_V) : (P_A, V_A; \iota_A, \partial_A) \to (P_B, V_B; \iota_B, \partial_B)$. Then the proof of Metatheorem 1 applies (as isomorphisms are trivially strict).

So consider $F((f_P, f_V)) = (H^1(f_P), H^1(f_V))$. Let $a \in H^1(P_A) \backslash Im(\iota_A)$. Then for all $v : \hat{1} \to V_A$, $a \neq \iota_A v$. Hence by the **strict restriction** $f_P a \neq \iota_B x$ for all $x : \hat{1} \to V_B$. As H^1 is faithful and $\hat{1} : \hat{1} \to \hat{1}$ is an epimorphism, $H^1(f_P)a = H^1(f_P)H^1(a)\hat{1} = H^1(f_P a)\hat{1} \neq H^1(\iota_B x)\hat{1} = H^1(\iota_B)H^1(x)\hat{1} = H^1(\iota_B)x$. Hence for all $y \in V_B$, $y : \hat{1} \to V_B$ and $H^1(\iota_B)y \neq H^1(f_P)a$. Hence $H^1(f_P)(a) \in P_B \backslash Im(\iota_B)$ and (f_P, f_V) is a strict morphism. □

Metatheorem 8. *Let* ***C*** *be a locally small category such that:*

(i) ***C*** *is a model of Axioms 1-12, Axiom 13 (****Grphs****), Axiom 14 (****StGrphs****), and Axiom 16 (****StGrphs****).*

(ii) for every family $\{A_j\}_{j \in J}$ *of objects in* ***C*** *there exists a product and a coproduct in* ***C****.*

Then for **D** *the full subcategory of discrete objects of* ***C****,* **D** *is a model of Lawvere's axioms for the category of sets,* **D** *is complete and* ***C*** *is equivalent to* $\textbf{\textit{StGrphs}}_{\mathbf{D}}$ *(and thus equivalent to* ***StGrphs****).*

Proof. As Axioms 1-12 apply, so do Lemmas 1-3. Let **D** be the full subcategory of discrete objects of **C**. By Theorem Schema 1, **D** is a model for the elementary axioms of Lawvere's system of sets. Since **C** has arbitrary products and $|-|:\mathbf{C}\rightsquigarrow \mathbf{D}$ is product preserving, **D** has arbitrary products. By Lemma 1, **D** has arbitrary coproducts. Hence **D** is complete.

Define $F:\mathbf{C}\rightsquigarrow\mathbf{StGrphs_D}$ on objects A by $F(A) = (P_A, |A|; \iota_A, \partial_A)$ for $|A|$ prescribed by Axiom 7, P_A, ι_A, and ∂_A chosen by Lemma 2, and for $f : A \to B$ $F(f) = (f_P, |f|)$ for f_P the unique morphism guaranteed by Lemma 3 and $|f|$ the unique morphism such that $ft_A = t_B|f|$. By the proof of Metatheorem 2, $F(A)$ is an object of $\mathbf{StGrphs_D}$. We show that for $f : A \to B$, $F(f) = (f_P, |f|)$ is a morphism of $\mathbf{StGrphs_D}$.

By Axiom 12 and Lemma 3, $F(f) = (f_P, |f|)$ is well defined, $f_P\iota_A = \iota_B|f|$, and $\partial_B f_P = (|f|\times|f|)\partial_A$. We show it satisfies the **strict restriction**. Let $x \in V(P_A)$ be such that for all $v \in V(|A|)$, $\iota_A v \neq x$, then by Axiom 16 (**StGrphs**) $\iota_B y \neq f_P x$ for all $y \in V(|B|)$. It follows from the proof of Metatheorem 2 that F is a functor.

Now define $G:\mathbf{StGrphs_D}\rightsquigarrow\mathbf{C}$ by $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A})) = A$ for A prescribed by Axiom 13 (**Grphs**) and for $(f_P, f_V) : (\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}) \to (\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B})$ define $G((f_P, f_V)) = g : A \to B$ the unique morphism prescribed by Axiom 14 (**StGrphs**) for $A = G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A}))$, $B = G((\overline{P_B}, \overline{V_B}; \overline{\iota_B}, \overline{\partial_B}))$, $\psi_B^{-1} f_P \psi_A : P_A \to P_B$, and $\phi_B^{-1} f_V \phi_A : V_A \to P_B$. We note that $G((\overline{P_A}, \overline{V_A}; \overline{\iota_A}, \overline{\partial_A})) = A$ is an object in **C**. Then the rest of the proof for the Metatheorem follows exactly as the proof in Metatheorem 2. □

Metatheorem 9. *Let* **D** *be any locally small category such that* **D** *is a model of Lawvere's system of axioms for the category of sets. If* **D** *is complete then* ***SiStGrphs*$_\mathbf{D}$** *is equivalent to* ***SiStGrphs***.

Proof. Define $F:\mathbf{SiStGrphs_D}\rightsquigarrow\mathbf{SiStGrphs}$ as in the proof of Metatheorem 1. By the proof of Metatheorem 3, $F((P_A, V_A; \iota_A, \partial_A))$ is a simple graph, and by the proof of metatheorem 7, for (f_P, f_V) a morphism of $\mathbf{SiStGrphs_D}$, $F((f_P, f_V))$ is a strict morphism. Hence the rest of the proof follows similarly to the proof of Metatheorem 1. □

Metatheorem 10. *Let* $\boldsymbol{C}$ *be a locally small category such that:*

(i) $\boldsymbol{C}$ *is a model of Axioms 1-12, Axiom 13 (**SiGrphs**), Axiom 14 (**StGrphs**), Axiom 15 (**SiGrphs**), and Axiom 16 (**StGrphs**).*

(ii) *for every family* $\{A_j\}_{j\in J}$ *of objects in* $\boldsymbol{C}$ *there exists a product and a coproduct in* $\boldsymbol{C}$*.*

Then for $\mathbf{D}$ *the full subcategory of discrete objects of* $\boldsymbol{C}$*,* $\mathbf{D}$ *is a model of Lawvere's axioms for the category of sets,* $\mathbf{D}$ *is complete and* $\boldsymbol{C}$ *is equivalent to* $\boldsymbol{SiStGrphs}_{\mathbf{D}}$ *(and thus equivalent to **SiStGrphs**).*

Proof. As Axioms 1-12 apply, so do Lemmas 1-3. Let $\mathbf{D}$ be the full subcategory of discrete objects of $\mathbf{C}$. By Theorem Schema 1, $\mathbf{D}$ is a model for the elementary axioms of Lawvere's system of sets. Since $\mathbf{C}$ has arbitrary products and $|-|:\mathbf{C}\rightsquigarrow\mathbf{D}$ is product preserving, $\mathbf{D}$ has arbitrary products. By Lemma 1, $\mathbf{D}$ has arbitrary coproducts. Hence $\mathbf{D}$ is complete.

Define $F:\mathbf{C}\rightsquigarrow\mathbf{SiStGrphs_D}$ on objects A by $F(A)=(P_A,|A|;\iota_A,\partial_A)$ for $|A|$ prescribed by Axiom 7, P_A,ι_A, and ∂_A chosen by Lemma 2, and for $f:A\to B$ $F(f)=(f_P,|f|)$ for f_P the unique morphism guaranteed by Lemma 3 and $|f|$ the unique morphism such that $ft_A=t_B|f|$. $F(A)$ is an object of $\mathbf{SiStGrphs_D}$ by the proof of Metatheorem 4, and $F(f)$ is a morphism of $\mathbf{SiStGrphs_D}$ by the proof of Metatheorem 8. F is a functor by the proof of Metatheorem 2.

Now define $G:\mathbf{StGrphs_D}\rightsquigarrow\mathbf{C}$ by $G((\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A}))=A$ for A prescribed by Axiom 13 (**SiGrphs**) and for $(f_P,f_V):(\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A})\to(\overline{P_B},\overline{V_B};\overline{\iota_B},\overline{\partial_B})$ define $G((f_P,f_V))=g:A\to B$ the unique morphism prescribed by Axiom 14 (**StGrphs**) for $A=G((\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A}))$, $B=G((\overline{P_B},\overline{V_B};\overline{\iota_B},\overline{\partial_B}))$, $\psi_B^{-1}f_P\psi_A:P_A\to P_B$, and $\phi_B^{-1}f_V\phi_A:V_A\to P_B$. We note that $G((\overline{P_A},\overline{V_A};\overline{\iota_A},\overline{\partial_A}))=A$ is an object in $\mathbf{C}$. Then the rest of the proof for the Metatheorem follows exactly as the proof in Metatheorem 2. $\square$

Chapter 5

An Application to Graph Theory

5.1 A Result Toward Hedetniemi's Conjecture

We give an application of the study of the categories of graphs to graph theory. In 1966, Hedetniemi conjectured that the chromatic number of the categorial product of two graphs with finite chromatic number is the minimum chromatic number of the two graphs. In 1985, A. Hajnal found graphs requiring an uncountable color set to provide a proper vertex coloring whose product only required a countable color set [12]. Hedetniemi's original conjecture remains open today.

Conjecture 5.1.1. [13] *Given graphs G and H with $\chi(G) < \infty$ and $\chi(H) < \infty$, $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.*

This conjecture has since produced much research [14, 22, 24, 28] focused primarily on the following restatement in **SiStGrphs**.

Conjecture 5.1.2. [24] *For all finite cardinals κ, $(G \not\to K_\kappa \wedge H \not\to K_\kappa \Rightarrow G \times H \not\to K_\kappa)$.*

The equivalence of these two statements comes from the following elementary result about strict morphisms.

Proposition 5.1.3. [14] *If $G \to K_n$ then $\chi(G) \leq n$.*

Another immediate result of this proposition is the following.

Proposition 5.1.4. [14] *If $G \to H$ then $\chi(G) \leq \chi(H)$.*

With this proposition, using projections and composition, it is trivial to see that $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$, the conjecture proposes the equality.

In **SiStGrphs**, we establish the following special case of Hedetiemi's conjecture directly without need for the restatement. Recall that a clique is a complete subgraph [4].

Theorem 5.1.5. *If A or B contains a $\min\{\chi(A), \chi(B)\}$-clique, then $\chi(A \times B) = \min\{\chi(A), \chi(B)\}$.*

This result is in a similar flavor of the following two results.

Theorem 5.1.6. [6] *Let G be a graph such that every vertex of G is in an n-clique. For every graph H, if $\chi(G \times H) = n$ then $\min\{\chi(G), \chi(H)\} = n$.*

Theorem 5.1.7. [9, 26] *Let G and H be connected graphs containing n-cliques. If $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$.*

We will first need an observation and lemma before the proof of Theorem 5.1.5.

Observation 5.1.8. *If $\chi(A) = k$, then the subgraph A' created by deleting all vertices of one color class has $\chi(A') = k - 1$.*

Proof. As the k-coloring of A is a $(k-1)$-coloring of A', $\chi(A') \leq k - 1$. Suppose that there is a l-coloring of A' with $l < k - 1$. Then as the vertices deleted from A are from the same color class, using the l colors of A' and the single color class that was deleted, we achieve a $l + 1 < k$ coloring of A, a contradiction. □

Lemma 5.1.9. *If* $\chi(A) = k$*, then* $\chi(A \times K_k) = k$*.*

Proof. We proceed by induction on $\chi(A)$ on: "If $\chi(A) = k$, then there is a monomorphism $j : A \to A \times K_k$." If this result is established, so is the lemma.

The base case, $k = 1$, is trivial. So suppose that $k > 1$ and the result holds for all graphs, B, with $\chi(B) = k - 1$.

Let A' be the subgraph of A formed by deleting the vertices of one color class of A. Then by the previous observation $\chi(A') = k - 1$. Hence by I.H. there is a monomorphism $\overline{j} : A' \to A' \times K_{k-1}$. Then there is a monomorphism $A' \xrightarrow{\overline{j}} A' \times K_{k-1} \xrightarrow{g \times m} A \times K_k$, where $g : A' \hookrightarrow A$ and $m : K_{k-1} \hookrightarrow K_k$ are inclusion morphisms. So we "lift" $\overline{j}$. Define $j : A \to A \times K_k$, by $j(a) = \overline{j}(a)$ if $a \in P(A') \subseteq P(A)$, $j(a) = (a, u)$ for $a \in V(A) \backslash V(A')$ and u the single vertex of $V(K_k) \backslash V(K_{k-1})$.

We show that for any edge $e \in E(A)$ with $\partial_A(e) = (a\_b)$ for $a \in V(A) \backslash V(A')$, then there is an edge of $A \times K_k$ between (a, u) and $\overline{j}(b)$ ($b \in V(A')$ as no edges exist between members of the color class of a). As u is adjacent to every other vertex of K_k, and $\pi_{K_{k-1}} \overline{j}(b) \in V(K_{k-1})$, there is an edge $e' \in E(K_k)$ with $\partial_{K_k}(e') = (u\_\pi_{K_{k-1}} \overline{j}(b))$. Hence, for each edge $e \in E(A)$ incident to a, there is the required edge (e, e') in $A \times K_k$. So we define $j(e) = (e, e')$. Hence j preserves incidence and is a monomorphism. □

Now we proceed with the proof of Theorem 5.1.5.

Proof of Theorem 5.1.5. As $\chi(A \times B) \leq \min\{\chi(A), \chi(B)\}$, it suffices to establish $\chi(A \times B) \geq \min\{\chi(A), \chi(B)\}$. Without loss of generality let $\chi(A) \leq \chi(B)$. Set $k = \min\{\chi(A), \chi(B)\}$. If B contains the k-clique, then by the previous lemma, $\chi(A \times K_k) = k$ and $A \times K_k$ is a subgraph of $A \times B$. Hence $\chi(A \times B) \geq k$. If A contains the k-clique, then create B' a subgraph of B by deleting vertices of $\chi(B) - k$ color classes. Then $\chi(B') = k$ and by the previous lemma

$\chi(K_k \times B') = k$. As $K_k \times B'$ is a subgraph of $A \times B$, $\chi(A \times B) \geq k$. $\square$

Chapter 6

Further Directions

In Chapter 4, we provided an Elementary Theory for the Categories of Graphs, and in doing so supplied a sufficient list of axioms to characterize five categories of graphs. However, this list may not be necessary.

One way of showing an axiom is independent is to remove the axiom and find multiple models that satisfy the remaining axioms. This was done to show Euclid's parallel postulate was independent. For example, as both **Grphs** and **SiGrphs** satisfy Axioms 1-12 and Axiom 14 (**Grphs**), Axiom 13 (**Grphs**) is independent from Axioms 1-12 and Axiom 14 (**Grphs**). Hence a direction of future research is to either show the axioms are independent, or find and remove dependences to produce a necessary and sufficient list.

Another future direction of research is to develop an Elementary Theory of the Category of Simple and Loopless Graphs with Strict Morphisms. As finite limits and colimits fail to exist, as well as most "quotient" objects, much of the theory presented in Chapter 4 does not apply to **SiLlStGrphs**. However, **SiLlStGrphs** does contain a full subcategory of **Sets**, a vertex object, and an edge object.

We have been focused on undirected graphs, but using similar restrictions on objects and morphisms we can define six categories of directed graphs, **DiGrphs**, **SiDiGrphs**,

SiLlDiGrphs, **StDiGrphs**, **SiStDiGrphs**, and **SiLlStDiGrphs**, where strict directed graph morphisms must map arcs to arcs, and simple directed graphs must have at most one arc incident to any pair of, not necessarily distinct, nodes.

DiGrphs has been the focus of much study in Category Theory, as transitively closed directed graphs form diagram categories, and most categorial constructions can be viewed in terms of directed graphs. Most textbooks in Category Theory include sections or chapters on directed graphs [1, 3, 19]. Furthermore **DiGrphs** is a topos by the Fundamental Theorem of Topoi [11] as **DiGrphs** can be viewed as a functor category from the diagram category $\cdot \overset{s}{\underset{t}{\rightrightarrows}} \cdot$ to **Sets** [3].

Many of the constructions created in our Elementary Theory of the Categories of Graphs also apply to directed graphs. For example, the vertex object (Chapter 4, Axiom 2) also serves as a node object in the categories of directed graphs. The arc-edge object (Chapter 4, Definition 6) can be used to determine the arcs of an object in the categories of directed graphs. Another direction of further research would be to extend an Elementary Theory of the Categories of Graphs to an Elementary Theory of the Categories of Directed Graphs.

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