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Peripherally-Multiplicative Spectral Preservers Between Function Algebras

Jeffrey Verlyn Johnson
The University of Montana

PERIPHERALLY-MULTIPLICATIVE SPECTRAL PRESERVERS BETWEEN

FUNCTION ALGEBRAS

by

JEFFREY VERLYN JOHNSON

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Approved by:

Sandy Ross, Associate Dean of The Graduate School
Graduate School

Dr. Thomas Tonev, Chair
Mathematical Sciences

Dr. Jennifer Halfpap
Mathematical Sciences

Dr. Karel Stroethoff
Mathematical Sciences

Dr. Eric Chesebro
Mathematical Sciences

Dr. Eijiro Uchimoto
Physics and Astronomy

Johnson, Jeffrey V. Ph.D., May 2013   Mathematics

Peripherally-Multiplicative Spectral Preservers Between Function Algebras

Chairperson: Thomas Tonev, Ph.D.

General sufficient conditions are established for maps between function algebras to be composition or weighted composition operators, which extend previous results regarding spectral conditions for maps between uniform algebras. Let X and Y be a locally compact Hausdorff spaces, where $A \subset C(X)$ and $B \subset C(Y)$ are function algebras, not necessarily with unit. Also let ∂A be the Shilov boundary of A, δA the Choquet boundary of A, and $p(A)$ the set of p-points of A. A map $T\colon A \to B$ is called weakly peripherally-multiplicative if the peripheral spectra of fg and $TfTg$ have non-empty intersection for all f,g in A. (i.e. $\sigma_\pi(fg) \cap \sigma_\pi(TfTg) \neq \varnothing$ for all f, g in A) The map is said to be almost peripherally-multiplicative if the peripheral spectrum of fg is contained in the peripheral spectrum of $TfTg$ (or if the peripheral spectrum of $TfTg$ is contained in the peripheral spectrum of fg) for all f, g in A.

Let X be a locally compact Hausdorff space and $A \subset C(X)$ be a dense subalgebra of a function algebra, not necessarily with unit, such that $\delta A = p(A)$. We show that if $T\colon A \to B$ is a surjective map onto a function algebra $B \subset C(Y)$ that is almost peripherally-multiplicative, then there is a homeomorphism $\psi\colon \delta B \to \delta A$ and a function α on δB so that $(Tf)(y) = \alpha(y) f(\psi(y))$ for all $f \in A$ and $y \in \delta B$, i.e. T is a weighted composition operator where the weight function is a signum function.

We also show that if T is weakly peripherally-multiplicative, and either $\sigma_\pi(f) \subset \sigma_\pi(Tf)$ for all $f \in A$, or, alternatively, $\sigma_\pi(Tf) \subset \sigma_\pi(f)$ for all $f \in A$, then $(Tf)(y) = f(\psi(y))$ for all $f \in A$ and $y \in \delta B$. In particular, if A and B are uniform algebras and $T\colon A \to B$ is a weak peripherally-multiplicative operator, that has a limit, say b, at some $a \in A$ with $a^2 = 1$, then $(Tf)(y) = b(y) a(\psi(y)) f(\psi(y))$ for every $f \in A$ and $y \in \delta B$.

Also, we show that if a weak peripherally-multiplicative map preserving peaking functions in the sense $\mathcal{P}(B) \subset T[\mathbb{T} \cdot \mathcal{P}(A)]$ or $T[\mathcal{P}(A)] \subset \mathbb{T} \cdot \mathcal{P}(B)$ then T is a weighted composition operator with a signum weight function. Finally, for function algebras containing sufficiently many peak functions, including function algebras on metric spaces, it is shown that weak peripherally-multiplicative maps are necessarily composition operators.

Contents

Chapter 1

Introduction

This chapter describes the background and historical context of what are referred to as "spectral preserver problems." In general, one studies maps between Banach algebras and conditions on these maps which imply a structure preserving property of the map, ultimately relating the structure between two Banach algebras.

Structure-preserving maps are ubiquitous throughout mathematics. For example, in algebra, a group is a set with a *structure* determined by the binary operation of the group. If several groups are being considered, it is natural to ask if they have the same, or similar structure.

The topic of this thesis is maps between Banach algebras, and function algebras in particular. A Banach algebra is an object that has a linear, multiplicative, and topological structures. Therefore, maps that preserve all of these structures are most useful in determining whether two Banach algebras are fundamentally different or similar. Structure preserving maps between Banach algebras are called isometric algebra isomorphisms. It is particularly interesting to investigate particular conditions where we may deduce that the map is an isometric algebra isomorphism, since this gives insight into the relationship between the various structures of the algebras.

The goal of this dissertation is to present sufficient conditions for maps between function algebras to be *weighted composition operators*, which operators are a type of structure preserving map that describe the action of a map between algebras in terms of the elements it is acting on (see Definition 4.1.1.) The first two chapters give a brief introduction to the general subject, while the third builds the framework and basic tools for arguments in the later chapters. Much of the theory and many results presented here have been developed previously for uniform algebras. Here we widen our scope to the more general context of function algebras.

The remainder of this chapter provides some historical context and summary of several results that appear in Chapters 4,5 and 6. The terminology and definitions are provided in Chapters 2,3 and 4.

1.1 Linear and Spectral Preserver Problems

The study of real-linear spaces is a historically significant and central topic in mathematics, and it is an important precursor to the general study of Banach spaces. One of the first significant results for real-linear normed vector space is the famed Mazur-Ulam Theorem.

Theorem 1.1.1 (Mazur-Ulam Theorem)**.** *Let $f : X \to Y$ be a distance preserving surjective map between normed vector spaces X, Y over $\mathbb{R}$ such that $f(0) = 0$. That is, $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in X$. Then f is an isometric real-linear transformation.*

More simply, the Mazur-Ulam Theorem states that every isometry between normed vector spaces over the real numbers is an affine transformation. For the Mazur-Ulam theorem, as well as the theorems in Chapters 5 and 6, the assumption of surjectivity is essential since the theorem is not necessarily true for non-surjective mappings. Complex conjugation is a simple counter-example that shows the Mazur-Ulam Theorem does not hold for complex vector spaces.

The linearity of the map f is a consequence of the Mazur-Ulam Theorem stated above. However, many classical results in the theory of Banach algebras assume the linearity of a map and show, under other suitable hypotheses, that the map is multiplicative, thus an algebra isomorphism. The following theorem is an important result in the theory of spectral preserves and provides an example of a linear preserver problem.

Theorem 1.1.2 (Gleason-Kahane-Zelazko Theorem (1973))**.** [26] *Let A be a unital commutative Banach algebra and B a uniform algebra. If $T : A \to B$ is a linear map with $\sigma(Tf) \subset \sigma(f)$, where $\sigma(f)$ is the spectrum of f, for every $f \in A$, then T is multiplicative, i.e., $T(fg) = T(f)T(g)$ for every $f, g \in A$.*

The hypotheses in the Gleason-Kahane-Zelazko Theorem include the *spectral* condition $\sigma(Tf) \subset \sigma(f)$, which is considered an analytic (or topological) condition on the map T (See Definition 2.2.4) and concludes that the map preserves the multiplicative structure of the algebras.

The following theorem is another classical linear preserver problem that describes sufficient conditions for a map between spaces of continuous function to be a weighted composition operator.

Theorem 1.1.3 (Banach-Stone Theorem (1936))**.** *(e.g. in [2]) If X and Y are compact sets and $T : C(X) \to C(Y)$ is a surjective linear isometry, then there is a homeomorphism $\psi : Y \to X$ and a function $\alpha \in C(Y)$ such that $|\alpha(y)| = 1$ for all $y \in Y$ and $(Tf)(y) = \alpha(y)f(\psi(y))$ for all $f \in C(X)$ and $y \in Y$.*

The Banach-Stone Theorem is a linear preserver problem giving sufficient conditions for a map to be a *weighted composition operator*. This is the model used for the theorems presented in Chapters 5 and 6.

1.1.1 Multiplicative Spectral Preserver Problems

Another type of condition on a mapping $T : A \to B$ between algebras of functions relates the spectrum $\sigma(fg)$ of the product fg with the spectrum $\sigma(TfTg)$ of the product of the images $TfTg$. Note that if A and B are algebras with multiplicative identities, 1_A and 1_B respectively, then we say T is a *unital* map if $T(1_A) = 1_B$. The following theorem by Molnár is one of the first to address the sufficient conditions of unital mappings with a multiplicative spectral condition.

Theorem 1.1.4 (Molnár (2001)). [18] *Let X be a first-countable compact space. A surjective, unital mapping $T : C(X) \to C(X)$ for which $\sigma(TfTg) = \sigma(fg)$ for every $f, g \in C(X)$ is an isometric algebra automorphism.*

In 2005, Rao and Roy showed in [23] that the theorem holds for any self map $T : A \to A$ where X is a compact Hausdorff space and $A \subset C(X)$ is a uniform algebra. In [17] from 2007, Luttman and Tonev improved the result to hold for maps between two, possibly different, uniform algebras under a weakened hypothesis requiring only the preservation of a subset of the spectrum, the *peripheral* spectrum (see Definition 2.2.5,) of the products fg and $TfTg$ for each f, g in A. The following theorem appears in Luttman and Tonev's paper *Uniform algebra isomorphisms and peripheral multiplicativity*.

Theorem 1.1.5 (Luttman-Tonev (2007)). [17] *If $T : A \to B$ is a surjective map (not neccesarily linear) between uniform algebras such that*

$$\sigma_\pi(TfTg) = \sigma_\pi(fg)$$

for all $f, g \in A$, then there exists $\kappa \in B$ with $\kappa^2 = 1$ and a homeomorphism $\psi : \delta B \to \delta A$ such that

$$Tf = \kappa(f \circ \psi)$$

on ∂B for all $f \in A$. In particular, $\tilde{T} = \kappa T : A \to B$ is an isometric algebra isomorphism.

Note that the previous theorem of Luttman and Tonev is a generalization of the result by Rao and Roy since if $\sigma(f) = \sigma(g)$, then $\sigma_\pi(f) = \sigma_\pi(g)$ although the converse is not true. That is, the *peripheral multiplicativity* condition $\sigma_\pi(fg) = \sigma_\pi(TfTg)$ is a strictly weaker condition than the spectral condition $\sigma(fg) = \sigma(TfTg)$.

The result was strengthened by Luttman, Lambert and Tonev, again in 2007.

Theorem 1.1.6 (Lambert, Luttman, and Tonev (2007)). [14] *A surjective mapping $T : A \to B$ between uniform algebras with $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and which preserves the peripheral spectra of all elements, i.e. $\sigma_\pi(Tf) = \sigma_\pi(f)$ for all f in A, is an isometric algebra isomorphism.*

If a map satisfies the condition $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \varnothing$ for all f, g in A then T is said to be weakly peripherally-multiplicative (see Definition 5.1.1.) Note that the condition $\sigma_\pi(TfTg) = \sigma_\pi(fg)$ implies the weakly peripherally-multiplicative condition, but not vice-versa. In chapter 5 we establish sufficient conditions for maps between function algebras to be weighted composition operators.

The following results were obtained by the author in the joint paper [12] with T. Tonev in 2011.

Theorem. 5.2.1 (A)[12] *Let $A \subset C(X)$ be a function algebra and $B \subset C(Y)$ a dense subalgebra of a function algebra, not necessarily with units, such that $p(B) = \delta B$, where X and Y are locally compact Hausdorff spaces. If $T : A \to B$ is a surjection such that*

$$\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg) \tag{1.1}$$

for all $f, g \in A$, then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y) f(\psi(y))$$

for every $y \in \delta B$.

The same conclusion follows under alternative symmetric conditions.

Theorem. 5.2.1 (B)[12] *Let $A \subset C(X)$ be a dense subalgebra of a function algebra, and $B \subset C(Y)$ be a function algebra, not necessarily with units, where X and Y are locally compact Hausdorff spaces. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg) \tag{1.2}$$

for all $f, g \in A$, then there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y) f(\psi(y))$$

for every $y \in \delta B$.

We call a map T between function algebras A and B *almost peripherally-multiplicative* if $\sigma_\pi(fg) \subset \sigma_\pi(TfTg)$ or $\sigma_\pi(TfTg) \subset \sigma_\pi(fg)$ for all f, g in A. The previous two theorems can be summarized as follows: An almost peripherally-multiplicative map between function algebras is necessarily a weighted composition operator where the weight function is a signum function.

A clear sufficient condition for T to be an isometric algebra isomorphism is given in the following corollary to the previous theorems.

Corollary 1.1.7. *Let X and Y be locally compact Hausdorff spaces and $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, not necessarily with unit. If $T: A \to B$ is an almost peripheral-multiplicative surjection, and $d(\sigma_\pi(f), \sigma_\pi(Tf)) < 2$ for all $f \in A$, then T is a composition operator.*

1.2 Extensions for Function Algebras

The proofs of the above theorems rely on variations of Bishop's Lemma, a classical result from the theory of uniform algebras. Namely,

Theorem 1.2.1 (Classical Bishop's Lemma). *(e.g. in [4]) Let E be a peak set of a uniform algebra A and $f \in A$ such that $f|_E \not\equiv 0$. Then there is a peaking function $h \in \mathcal{P}_E(A)$ such that fh takes its maximum modulus only within $E = E(h)$.*

In [14], Lambert extended Bishop's Lemma for p-sets in uniform algebras.

Theorem 1.2.2 (Bishop's Lemma for p-sets for Uniform Algebras). *[14] Let $A \subset C(X)$ be a uniform algebra and E a p-set of A. If $f \in A$ is such that $f|_E \not\equiv 0$, then there is a peaking function $h \in \mathcal{P}_E(A)$ such that fh takes its maximum modulus on E.*

In chapter 3 we prove the following theorem for function algebras $A \subset C_0(X)$ where X is a locally compact Hausdorff space.

Theorem 1.2.3 (Bishop's Lemma for p-sets in Function Algebras). *Let X be a locally compact Hausdorff space and $A \subset C_0(X)$ be a function algebra without unit. If $f \in A$ and E is a p-set of A with $f|_E \neq 0$, then there exists a peaking function $h \in \mathcal{P}_E(A)$ such that fh takes its maximum modulus in E.*

As a corollary, we obtain the version of Bishop's lemma for p-points from [24], which is fundamental to the results in chapters 5 and 6.

Theorem 1.2.4 (Strong Multiplicative Bishop's Lemma). *Let X be a locally compact Hausdorff space and $A \subset C(X)$ be a function algebra without unit on $X = \partial A$. If $f \in A$ and $x_0 \in X$ is a p-point of A with $f(x_0) \neq 0$, then there exists a peaking function $h_0 \in \mathcal{P}_{x_0}(A)$ such that*

$$\sigma_\pi(fh_0) = \{f(x_0)\} \tag{1.3}$$

If E is a peak set of A which contains x_0, then h_0 can be chosen so that $E(fh_0) = E(h_0) \subset E$.

In Chapter 4 we state and prove several lemmas needed for the following theorem.

Theorem. 4.2.1 [24] *Let $A \subset C_0(X)$ and $B \subset C_0(Y)$ be dense subalgebras of function algebras without units on X and Y with $p(A) = \delta A$ and $p(B) = \delta B$. If $T: A \to B$ is a surjection such that $\|Tf \cdot Tg\| = \|fg\|$ for all $f, g \in A$, then there is a homeomorphism $\psi: p(B) \to p(A)$ such that*

$$\left| (Tf)(y) \right| = \left| f(\psi(y)) \right| \tag{1.4}$$

for all $f \in A$ and $y \in p(B)$.

It is known that if $A \subset C(X)$ is a function algebra, then $|A|$ separates the points of X (e.g. [23]). We provide an alternative proof of this result. Namely,

Lemma. 4.2.9 *Let X be a locally compact Hausdorff space. If $A \subset C_0(X)$ separates the points of X, then so does $|A|$.*

1.2.1 Weak Peripherally-Multiplicative Maps between Function Algebras

In [14] Lambert, Luttman, and Tonev show that weak peripherally-multiplicative maps between uniform algebras that preserve the peripheral spectrum of each element are isometric algebra isomorphisms. In Chapter 6 we generalize this result to algebras of functions and relax the condition of preserving the peripheral spectra to $\sigma_\pi(f) \subset \sigma_\pi(Tf)$ for all f in A, or $\sigma_\pi(Tf) \subset \sigma_\pi(f)$ for all f in A.

Theorem. *6.0.5 (A)(2011)* [12] *Let X be a locally compact Hausdorff space where $A \subset C(X)$ is a dense subalgebra of a function algebra, not necessarily with unit, such that $X = \partial A$ and $p(A) = \delta A$. If $T: A \to B$ is a surjection onto a function algebra $B \subset C(Y)$ such that*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing \text{ for all } f, g \in A \tag{1.5}$$

and

$$\sigma_\pi(f) \subset \sigma_\pi(Tf) \ \textit{for all } f \in A, \tag{1.6}$$

then T is a bijective ψ-composition operator on δB with respect to a homeomorphism $\psi: \delta B \to \delta A$. That is,

$$(Tf)(y) = f(\psi(y))$$

for all $f \in A$ and $y \in \delta B$. In particular, A is necessarily a function algebra and T is an algebra isomorphism.

Again, the result holds under symmetric conditions.

Theorem. 6.0.5 (B) *(2011)* [12] *Let X be a locally compact Hausdorff space where $A \subset C(X)$ is a function algebra, not necessarily with unit such that $X = \partial A$, and B is a dense subalgebra of a function algebra $B \subset C(Y)$ such that $p(B) = \delta(B)$. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing \ \textit{for all } f, g \in A \tag{1.7}$$

and

$$\sigma_\pi(Tf) \subset \sigma_\pi(f) \ \textit{for all } f \in A, \tag{1.8}$$

then T is a bijective ψ-composition operator on δB with respect to a homeomorphism $\psi: \delta B \to \delta A$. That is,

$$(Tf)(y) = f(\psi(y))$$

for all $f \in A$ and $y \in \delta B$. In particular, B is necessarily a function algebra and T is an algebra isomorphism.

In [9, Theorem 8] it is shown that weak peripherally-multiplicative maps between uniform algebras that are continuous at the unity element are composition operators, and thus algebra isomorphisms. We generalize this result by showing that it is not necessary for T to be continuous at the unity element, but that it must have a limit at a point $a \in A$ with $a^2 = 1$.

Theorem. 6.1.4 *[Johnson and Tonev (2011)]* [12] *Let A and B be uniform algebras on compact Hausdorff spaces X and Y. If $T: A \to B$ is a surjective map such that*

(i) $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$ and

(ii) There exist an $a \in A$ with $a^2 = 1$ such that T has a limit, say b, at a,

then $b^2 = 1$ and $(Tf)(y) = b(y)\, a(\psi(y))\, f(\psi(y))$ for every $f \in A$ and $y \in \delta B$, i.e. the map $f \mapsto bT(af)$ is an isometric algebra isomorphism.

The following two theorems were proved for uniform algebras by Lambert,Luttman, and Tonev in [14]. In chapter 6 we show that the theorems hold for algebras of functions, not necessarily with a unit element.

Theorem. 6.1.6 (A)*(2011) Let A be a dense subalgebra of a function algebra such that $p(A) = \delta A$ and B a function algebra on compact Hausdorff spaces X and Y respectively. Suppose that $T : A \to B$ is surjective, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and*

$$\mathcal{P}(B) \subset T[\mathbb{T} \cdot \mathcal{P}(A)]. \tag{1.9}$$

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $y \in \delta B$.

The same conclusion follows under a symmetric condition.

Theorem. 6.1.6 (B)*(2011) Let A be a function algebra and B a dense subalgebra of a function algebra such that $p(B) = \delta B$ on compact Hausdorff spaces X and Y respectively.*

Suppose that $T : A \to B$ is surjective, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and

$$T[\mathcal{P}(A)] \subset \mathbb{T} \cdot \mathcal{P}(B). \tag{1.10}$$

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $y \in \delta B$.

These theorems also provide us with the following corollaries.

Corollary 1.2.5. *Let A and B be function algebras on compact Hausdorff spaces X and Y. Suppose that T is surjective and*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$$

for all $f, g \in A$ and $\sigma_\pi(Tf)$ is a singleton whenever $\sigma_\pi(f)$ is a singleton (or vice-versa).

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $y \in \delta B$.

Corollary 1.2.6. *Let A and B be function algebras on compact Hausdorff spaces X and Y with the hypotheses of the Theorem 6.1.6 (B). In addition, suppose that $d(\sigma_\pi(f), \sigma_\pi(Tf)) < 2$ for all $f \in A$. Then there exists a homeomorphism $\psi : \delta B \to \delta A$,*

$$(Tf)(y) = f(\psi(y))$$

for every $y \in \delta B$. In other words, the weight function $\alpha : \delta B \to \{1, -1\}$ is identically 1 which

implies that T is a composition operator, thus an isometric algebraic isomorphism.

1.2.2 Function Algebras with Sufficiently many Peak Functions

If the underlying locally compact Hausdorff spaces X, Y are metric spaces, with $A \subset C_0(X)$ and $B \subset C_0(Y)$ function algebras, then we also show that A and B must contain sufficiently many *peak* functions which further describe the behavior of a weakly peripherally-multiplicative map T between A and B.

Theorem. 6.2.2 *Let X, Y be locally compact Hausdorff spaces and let $A \subset C(X), B \subset C(Y)$ be dense subalgebras of function algebras, not necessarily with unit, with $\delta A = p(A)$ and $\delta B = p(B)$. Let for every $f \in A$ and any $x \in \delta A$ there is a peak function $h \in \mathcal{P}_x(A)$ so that $\sigma_\pi(fh) = \{f(x)\}$. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \varnothing \text{ for all } f, g \in A, \qquad (1.11)$$

then T is a weighted composition operator, namely, there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $f \in A$ and $y \in \delta B$.

Then we also have the following corollaries.

Corollary 1.2.7. *Let X and Y be locally compact Hausdorff spaces and let $A \subset C(X)$ and $B \subset C(Y)$ be dense subalgebras of function algebras, not necessarily with unit with $\delta A = p(A)$ and $\delta B = p(B)$. If $T: A \to B$ is a surjection such that $\sigma_\pi(Tf\,Tg)$ is a singleton for every*

f, g ∈ A for which $\sigma_\pi(fg)$ a singleton, and if

$$\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \varnothing$$

for all f, g ∈ A, then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every f ∈ A and y ∈ δB.

The following corollary shows that each weakly peripherally-multiplicative surjective map $T : A \to B$ between function algebras on metric spaces are necessarily weighted composition operators.

Corollary 1.2.8. *Let A be a function algebra and B be a dense subalgebra of function algebra on metric spaces X and Y respectively such that $p(B) = \delta B$. If $T : A \to B$ is a surjection such that*

$$\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \varnothing \text{ for all } f, g \in A,$$

then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every f ∈ A and y ∈ δB.

Chapter 2

Commutative Banach Algebras

Some of the basic definitions and terminology in the theory of Banach Algebras are introduced in this chapter, which provides a short presentation to a reader who is unfamiliar with the subject. The exposition here, in general, follows [7].

2.1 Basic Properties and Definitions

2.1.1 Banach Spaces

Definition 2.1.1. *A Banach Space is a pair $(B, \|\cdot\|)$, where B is a vector space and $\|\cdot\|$ is a norm on B that is complete with respect to the metric defined by the norm.*

Example 2.1.1. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $\mathbb{K}^n$ forms a vector space over $\mathbb{K}$ with norm $\|\mathbf{x}\| = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$. One can show that $\mathbb{K}^n$ is complete with respect to the norm, and is therefore a Banach space.

Example 2.1.2. Suppose $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and fix $p > 0$. Define by ℓ_p all sequences $\mathbf{x} = \{x_n\}_{n=1}^\infty$ in

$\mathbb{K}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

Then ℓ_p is a Banach space with the norm $\|\mathbf{x}\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$.

Definition 2.1.2. *An algebra B (over the complex numbers) is a vector space with a multi-plication operation, making B into a ring. A* Banach algebra *B is a Banach space that is also an algebra, and is equipped with a norm given in B under which*

1. *$\|ab\| \le \|a\|\,\|b\|$ for all $a, b \in B$*

2. *$\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in \mathbb{C}$, $a, b \in B$*

and B is complete with respect to the norm.

The Banach algebra B is unital if there exists a multiplicative identity element for the space B viewed as a ring, i.e. if there exists an element $e \in B$ such that $e \cdot a = a \cdot e = a$ for all $a \in B$. If B has a multiplicative identity e, it is unique and assumed that $\|e\| = 1$. The algebra B is commutative if B is also a commutative ring.

If B is an algebra with unit element e, then the map $\alpha : \mathbb{C} \to B$ defined by $\alpha \mapsto \alpha e$ is clearly an algebra isomorphism and $\|\alpha e\| = |\alpha|$. This identification is commonly used to assume that $\mathbb{C} \subseteq B$.

Example 2.1.3. Suppose X is a compact space. Then $C(X)$, the space of continuous functions on X into $\mathbb{C}$ is a Banach algebra with pointwise operations $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$, and supremum norm

$$\|f\| = \max_{x \in X} |f(x)|.$$

Note that $C(X)$ is unital with $1|_X$ as the unital ring element and the supremum is achieved

since X is compact.

Example 2.1.4. Suppose X is a locally compact space. Define $B(X)$ as the space of all bounded continuous functions $f : X \to \mathbb{C}$ on X, i.e. functions f such that $\|f\| = \sup\{|f(x)| : x \in X\} < \infty$, again with pointwise operations. Then $B(X)$ is complete with respect to the norm and is a Banach algebra.

Example 2.1.5. Let $C_0(X) \subset B(X)$ denote all continuous functions $f : X \to \mathbb{C}$ such that for all $\epsilon > 0$, $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. Then $C_0(X)$ is a closed subalgebra of $B(X)$, thus complete, and therefore is also a Banach algebra.

Note that if X is locally compact (but not compact), then the algebra $C_0(X)$ does not contain the element $1|_X$ and is indeed not unital. Therefore any subalgebra of $C_0(X)$ is also not unital.

For this paper, we are primarily interested in subalgebras of $C(X)$ where X is a compact Hausdorff space, and $C_0(X)$ where X is a locally compact Hausdorff space.

Definition 2.1.3 (Uniform Algebra)**.** [7] *Let X be a compact Hausdorff space and suppose A is a subalgebra of $C(X)$, the Banach algebra of continuous complex-valued functions on X with pointwise operations and equipped with the supremum norm. Then A is a* uniform *algebra if :*

1. *A contains the constant functions*

2. *A separates the points of X, that is, for every $x \neq y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$*

Since X is compact and Hausdorff, it is a *normal* topological space and thus $C(X)$ separates the points of X by Urysohn's lemma, a well-known result of point-set topology. Thus $C(X)$ is a uniform algebra.

Example 2.1.6 (The Disk Algebra)**.** Let $\mathbb{D} = \{z : |z| < 1\}$ be the open disk in the complex plane and let $A(\mathbb{D})$ denote the space of continuous functions in the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \leq 1\}$ that are analytic in $\mathbb{D}$. With pointwise operations and the uniform norm $\|f\| = \max_{x \in \mathbb{D}}\{|f(x)|\}$, $A(\mathbb{D})$ is a commutative Banach algebra called the *disk algebra.*

Many of the tools of Banach algebras require the use of invertible elements. If a Banach algebra B is without unit, the following proposition shows a common method of isometrically embedding B into a unital Banach algebra B' such that the codimension of B in B' is 1.

Proposition 2.1.4. [5] *If B is a Banach algebra without identity, then the space $B' = B \times \mathbb{C}$ with algebraic operations*

1. $(a, \alpha) + (b, \beta) = (a + b, \alpha + \beta)$

2. $\beta(a, \alpha) = (\beta a, \beta \alpha)$

3. $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha \beta)$

and norm $\|(a, \alpha)\| = \|a\| + |\alpha|$ defines a Banach algebra with identity $(0, 1)$. The map $a \mapsto (a, 0)$ is an isometric isomorphism of B into B'. Thus B' is a Banach algebra with identity that naturally contains B such that $\dim B'/B = 1$.

2.1.2 The Dual Space of a Commutative Banach Algebra

Since a Banach algebra B is a vector space, it naturally has a dual space B^*, the set of continuous linear functions from B to $\mathbb{C}$. The elements of this set are commonly referred to as *linear functionals.* The linear structure of B^* is defined as usual.

We say a linear functional φ is *bounded* if $\sup\{|\varphi(x)| : \|x\| \leq 1\} < \infty$. In fact, continuity follows for bounded linear functionals.

Definition 2.1.5. *The dual space, B^*, of the Banach space B is itself a Banach space with norm*

$$\|\varphi\| = \sup\{|\varphi(x)| : \|x\| \leq 1\}, \ \varphi \in B^*$$

Although the dual space of a Banach algebra B is a Banach space, there is no natural operation which turns B^* into a Banach algebra.

There is however, and important topological structure on B^*, called the weak* topology.

Definition 2.1.6. *Let B^{**} denote the double dual of B. For $f \in B$ let $f^{**} : B^* \to \mathbb{C}$ be defined by $f^{**}(\varphi) = \varphi(f)$.*

The weak* topology *on B^* is the coarsest topology such that each $f^{**} : B^* \to \mathbb{C}$ is continuous for every $f^{**} \in B^{**}$.*

It terms of convergence in B^*, the net $\{\varphi_\alpha\}_\alpha \to \varphi$ in the weak* topology if and only if $\varphi_\alpha(f) \to \varphi(f)$ for every $f \in A$. That is, the weak* topology is the weakest topology where convergence is pointwise.

2.2 Invertible Elements and the Spectrum

Definition 2.2.1. *If B is a Banach algebra with unit, then an element $f \in B$ is invertible if there exists an element $g \in B$ such that $fg = e = gf$. That is, e is a unit of B viewed as a ring. Let B^{-1} denote the set of all invertible elements of B.*

The invertible elements play a crucial role in the theory of Banach algebras. Since B is a ring, we observe that B^{-1} forms a group under the multiplication operation.

The following propositions in this section give well-known results in the theory of Banach

algebras, and can be found in chapter 1 of [25].

Proposition 2.2.2. *Let B be a Banach algebra with unit and let f be an element of B such that $\|f\| < 1$. Then $\sum_{n=0}^{\infty} f^n$ is convergent and*

$$\sum_{n=0}^{\infty} f^n = (e - f)^{-1}$$

Proof. Since $\|f\| < 1$, it follows that $\|f^n\| \leq \|f\|^n \to 0$ as $n \to \infty$. Therefore $f^n \to 0$ as $n \to \infty$. Let $g_n = \sum_{k=0}^{n} f^k$ be the nth partial sum of the series, where $f^0 = e$.

If $m < n$, the triangle inequality and the properties in Definition 2.1.2 imply

$$\|g_n - g_m\| = \left\| \sum_{k=m+1}^{n} f^k \right\| \leq \sum_{k=m+1}^{n} \|f^k\| \leq \sum_{k=m+1}^{n} \|f\|^k = \sum_{k=0}^{n} \|f\|^k - \sum_{k=0}^{m} \|f\|^k$$

$$= \frac{1 - \|f\|^{n+1}}{1 - \|f\|} - \frac{1 - \|f\|^{m+1}}{1 - \|f\|} = \frac{\|f\|^{m+1} - \|f\|^{n+1}}{1 - \|f\|} \leq \frac{\|f\|^{m+1}}{1 - \|f\|}.$$

Let $\epsilon > 0$. Since $\|f\| < 1$, the above inequality shows there exists m, n large enough so that $\|g_n - g_m\| < \epsilon$. Therefore $\{g_n\}$ is a Cauchy sequence, and by the completeness of B must converge to some $g \in B$. Let $g = \sum_{n=0}^{\infty} f^n$.

Also,

$$g(e - f) = \left(\sum_{n=0}^{\infty} f^n \right)(e - f) = \left(\lim_{k \to \infty} \sum_{n=0}^{k} f^n \right)(e - f)$$

$$= \lim_{k \to \infty} \sum_{n=0}^{k} (f^n - f^{n+1}) = \lim_{k \to \infty} (e - f^{k+1}) = e - \lim_{k \to \infty} f^{k+1} = e$$

since $f^{k+1} \to 0$ as $k \to \infty$. A similar calculation shows that $(e-f)g = e$, thus $(e-f)$ is invertible with inverse g. $\square$

Proposition 2.2.3. *Let B be a unital Banach algebra, $f \in B$, and let s be a complex number*

such that $|s| > \|f\|$. *Then the element* $se - f$ *is invertible in* B *and*

$$(se - f)^{-1} = \sum_{n=0}^{\infty} \frac{f^n}{s^{n+1}}.$$

Proof. Observe that $s \neq 0$ since $|s| > \|f\| \geq 0$ and define $g = f/s = (1/s)f$. Therefore we have arranged that $\|g\| = (1/|s|)\|f\| < 1$.

From Proposition 2.2.2 we observe that g is invertible with $\sum_{n=0}^{\infty} g^n = (e - g)^{-1}$. Then

$$e = (e - g) \sum_{n=0}^{\infty} g^n = \left(e - \frac{f}{s}\right) \sum_{n=0}^{\infty} \frac{f^n}{s^n} = \frac{se - f}{s} \sum_{n=0}^{\infty} \frac{f^n}{s^n} = (se - f) \sum_{n=0}^{\infty} \frac{f^n}{s^{n+1}}.$$

A similar calculation shows that $\sum_{n=0}^{\infty} \frac{f^n}{s^{n+1}}(se - f) = e$, therefore $se - f$ is invertible with inverse $\sum_{n=0}^{\infty} \frac{f^n}{s^{n+1}}$ as claimed. $\qquad\square$

Definition 2.2.4. *The* spectrum *of an element* f *in a unital Banach algebra is the set*

$$\sigma(f) = \{\lambda \in \mathbb{C} : \lambda e - f \notin B^{-1}\}$$

From Proposition 2.2.3, it follows that $|z| \leq \|f\|$ for all $z \in \sigma(f)$. Consequently, the spectrum of an algebra element is contained in the closure of the disk of radius $\|f\|$, thus $\sigma(f)$ is always a bounded set. Furthermore, $\sigma(f)$ is closed and therefore a compact subset of $\mathbb{C}$.

For some algebras, the spectrum of an element is easily described. Recall example 2.1.6. Suppose $f \in A(\mathbb{D})$ such that $f(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$. Note that the function $1/f$ is defined in all of $\overline{\mathbb{D}}$ is continuous and analytic inside $\mathbb{D}$. Then $1/f \in A(\mathbb{D})$ with the property that $f \cdot 1/f = 1$, so any such f is invertible. Since operations are pointwise, we see that $f - \lambda$ is invertible if and only if $\lambda \notin \mathrm{Ran}(f)$, thus $\sigma(f) = \mathrm{Ran}(f)$.

Definition 2.2.5. *Let* $r_f = \max\{|z| : z \in \sigma(f)\}$ *be called the* spectral radius *of* $f \in B$. *The*

peripheral spectrum *of an element f in a unital Banach algebra B is the set*

$$\sigma_\pi(f) = \{z \in \sigma(f) : |z| = r_f\}$$

The following proof follows [25, Proposition 1.1.11]

Proposition 2.2.6 (Spectral Radius Formula). *Let B be a Banach algebra, not necessarily commutative. The sequence $\{\sqrt[n]{\|f^n\|}\}_1^\infty$ is convergent for every $f \in B$ and*

$$\lim_{n\to\infty} \sqrt[n]{\|f^n\|} = r_f. \tag{2.1}$$

Proof. Let $f \in B$ and $\lambda \in \sigma(f)$.

First we claim $w = \lambda^n \in \sigma(f^n)$. Let $\sqrt[n]{w} = \{\lambda_1, \dots, \lambda_n\}$ be the n roots of w. Note that $\lambda_j^n = w$ for all n. Consider the polynomial expression $f^n - w = (f - \lambda_1)\cdots(f - \lambda_n)$, which shows that $f^n - w$ is invertible if and only if $f - \lambda_j$ is not invertible for some j. That is, $w = \lambda^n \in \sigma(f^n)$ if and only if $\lambda_j \in \sigma(f)$ for some j. Since $\lambda \in \sigma(f)$, it follows that $\lambda^n \in \sigma(f^n)$ as claimed.

Then $|\lambda^n| \le \|f^n\|$, thus

$$|\lambda| = \sqrt[n]{|\lambda|^n} = \sqrt[n]{|\lambda^n|} \le \sqrt[n]{\|f^n\|}$$

for every $n \in \mathbb{N}$. Hence $|\lambda| \le \liminf_{n\in\mathbb{N}} \sqrt[n]{\|f^n\|}$ and thus

$$r_f = \max_{\lambda\in\sigma(f)} |\lambda| \le \liminf \sqrt[n]{\|f^n\|}.$$

From Proposition 2.2.2, we may define $r(\lambda) = (\lambda - f)^{-1} = \sum_{n=0}^\infty \frac{f^n}{\lambda^{n+1}}$ which holds on $\{\lambda \in \mathbb{C} : |\lambda| > \|f\|\}$. Then the function

$$g(z) = \frac{1}{z}r(1/z) = \frac{1}{z}\sum_{n=0}^\infty \frac{f^n}{(1/z)^{n+1}} = \sum_{n=0}^\infty f^n z^n.$$

defines a series which converges absolutely for $|z| < \dfrac{1}{\|f\|}$.

Observe that the function $r(\lambda)$ and $\lambda r(\lambda)$ are analytic in $\mathbb{C} \smallsetminus \sigma(f)$, and are therefore analytic in the subset $\mathbb{C} \smallsetminus \overline{\mathbb{D}}(r_f)$ where $\mathbb{D}(r_f)$ is the open disk of radius r_f. Therefore $g(z) = \sum\limits_{n=0}^{\infty} f^n z^n$ is analytic in $\mathbb{D}(1/r_f)$.

Consider the related series $h(z) = \sum\limits_{n=0}^{\infty} \|f^n\| z^n$, which has radius of convergence

$$R = \frac{1}{\limsup_{n \in \mathbb{N}} \sqrt[n]{\|f^n\|}}.$$

Since $\|f^n z^n\| = \|f^n\| |z^n|$, both series $\sum\limits_{n=0}^{\infty} f^n z^n$ and $\sum\limits_{n=0}^{\infty} \|f^n\| z^n$ are absolutely convergent on the same set. Thus, from complex variables, it follows that the two series have the same radius of absolute convergence, $\mathbb{D}(R) = \{z \in \mathbb{C} : |z| < R\}$. Consequently, $1/r_f \leq R$ and

$$r_f \geq 1/R = \limsup_{n \in \mathbb{N}} \sqrt[n]{\|f^n\|}$$

.

We have obtained

$$r_f \leq \liminf_{n \in \mathbb{N}} \sqrt[n]{\|f^n\|} \leq \limsup_{n \in \mathbb{N}} \sqrt[n]{\|f^n\|} \leq r_f$$

and therefore the sequence $\{ \sqrt[n]{\|f^n\|} \}_{n=1}^{\infty}$ is convergent with limit r_f. $\qquad\square$

2.3 Multiplicative Linear Functionals and the Gelfand Transform

An important technique in the theory of Banach algebras is to represent an algebra as an algebra of continuous functions on a locally compact space. To do this we will need to

investigate an important subset of the dual space, called the maximal ideal space.

2.3.1 Multiplicative Linear Functionals

The following theorem shows that there is, up to isomorphism, only one unital commutative Banach algebra that is a field.

Theorem 2.3.1 (Gelfand-Mazur)**.** *A commutative Banach algebra with identity which is a field must be isometrically isomorphic to the field of complex numbers.*

Definition 2.3.2. *A multiplicative linear functional on B is a non-zero element of the dual space that preserves the multiplication in B. That is, $\varphi \in B^*$ such that*

   1. $\varphi \neq 0$

   2. $\varphi(fg) = \varphi(f)\varphi(g)$

for every $f, g \in B$. Denote by $\mathcal{M}_B$ the set of all non-zero linear multiplicative functionals of B.

Since $\varphi(f) = \varphi(f \cdot e) = \varphi(f)\varphi(e)$, it immediately follows that $\varphi(e) = 1$. Observe that each multiplicative linear functional is a non-trivial complex valued ring homomorphism.

Lemma 2.3.3. *Let $\varphi : B \to \mathbb{C}$ be a multiplicative linear functional from a unital commutative Banach algebra B. Then:*

   1. $\|\varphi\| = 1 = \varphi(e)$

   2. φ *is continuous*

   3. $\ker \varphi$ *is a maximal ideal of B.*

Proof. Let φ be a multiplicative linear functional on B. Then $\varphi(e) = \varphi(e^2) = \varphi(e)\varphi(e)$ which implies that $\varphi(e) \in \{0, 1\}$. However, $\varphi(e) = 0$ implies that $\varphi(f) = \varphi(fe) = \varphi(f)\varphi(e) = 0$ for all $f \in B$. Therefore $\varphi(e) = 1$.

Let $f \in B$. Note that $\lambda - f \in B^{-1}$ for all λ such that $|\lambda| > \|f\|$. Then

$$1 = \varphi(e) = \varphi\left((\lambda - f)(\lambda - f)^{-1}\right) = \varphi(\lambda - f)\varphi((\lambda - f)^{-1})$$

which implies $\varphi((\lambda - f)^{-1}) = \lambda - \varphi(f) \neq 0$. Thus $\varphi(f) \neq \lambda$. Therefore $\varphi(f) \leq \|f\|$ for all $f \in B$ which shows that φ is continuous and that $\|\varphi\| \leq 1$. Since $\varphi(e) = 1$ we have that

$$\|\varphi\| = 1. \tag{2.2}$$

Since φ is a ring homomorphism, basic results from algebra show that $\ker \varphi$ is an ideal of B. Also, φ is clearly onto $\mathbb{C}$ which implies $B/(\ker \varphi)$ is isomorphic to $\mathbb{C}$ by the first isomorphism theorem. Since $B/(\ker \varphi)$ is a field, it follows that $\ker \varphi$ is a maximal ideal. $\qquad \square$

Proposition 2.3.4. *For a unital commutative Banach algebra, $\mathcal{M}_B$ is in bijective correspondence with the maximal ideals of B.*

Proof. Let J be a maximal ideal of B. Then B/J is a field, and by Theorem 2.3.1 is isometrically isomorphic to $\mathbb{C}$. Next we may consider the natural map $\varphi : B \to B/J$ to be a complex homomorphism onto $\mathbb{C}$. Therefore φ is the unique multiplicative linear function defined by this map. Conversely, if φ is a multiplicative linear functional then $\ker \varphi$ is clearly an additive subgroup, and also an ideal since $\varphi(fg) = \varphi(f)\varphi(g) = 0$ for all $f \in A$ and $g \in \ker \varphi$. Since φ is non-zero, it is surjective and therefore $B/\ker \varphi \equiv \mathbb{C}$ by the isomorophism theorem. Since $B/\ker \varphi$ is a field, $\ker \varphi$ is a maximal ideal. Therefore, $\varphi \mapsto \ker \varphi$ defines the bijection required. $\qquad \square$

The set $\mathcal{M}_B$ of multiplicative linear functionals is often called the *maximal ideal space*, which is justified by the previous proposition.

The space $\mathcal{M}_B$ comes equipped with a natural topology, the subspace topology inherited from the weak* topology on B^* from Definition 2.1.6. The topology is described more explicitly in terms of a subbase; form a fundamental system of neighborhoods of $\varphi_0 \in \mathcal{M}_B$ by taking finite intersections of neighborhoods of the form $\{\varphi : |\varphi(f) - \varphi_0(f)| < \epsilon\}$, where $f \in B$ and $\epsilon > 0$. This topology on the maximal ideal space is called the *Gelfand topology*.

Theorem 2.3.5 (Alaoglu)**.** *The closed unit ball of the dual space of a normed vector space is compact in the weak* topology.*

The proof is a classic result in the theory of Banach spaces and is omitted here. It is assumed that the norm on the dual space is described by the norm from Definition 2.1.5.

The theorem by Alaoglu is helpful for the following proposition.

Proposition 2.3.6. *If B is a unital commutative Banach algebra, then $\mathcal{M}_B$ is a compact Hausdorff space.*

The proof follows the arguments from [11].

Proof. By (2.2), $\|\varphi\| = 1$ which implies $\mathcal{M}_B$ is a subset of the unit sphere in B^*.

Recall that for any $\{\varphi_\alpha\}_\alpha$, $\varphi_\alpha \to \varphi$ if and only $\varphi_\alpha(f) \to \varphi(f)$ for every f. Suppose $\varphi_\alpha \to \varphi$ for $\{\varphi_\alpha\} \subset \mathcal{M}_B$ and let $f, g \in B$. Then by the continuity of multiplication in $\mathbb{C}$,

$$\varphi(fg) = \lim_\alpha \varphi_\alpha(fg) = \lim_\alpha (\varphi_\alpha(f)\varphi_\alpha(g)) = \left(\lim_\alpha \varphi_\alpha(f)\right)\left(\lim_\alpha \varphi_\alpha(g)\right) = \varphi(f)\varphi(g).$$

Therefore $\varphi \in \mathcal{M}_B$ which implies that $\mathcal{M}_B$ is a closed subset of the unit ball in B^*, a compact set in the weak* topology by Theorem 2.3.5. Thus $\mathcal{M}_B$ is compact.

Next we show $\mathcal{M}_B$ is homeomorphic to a Hausdorff space. Let $D_f = \{z \in \mathbb{C} : |z| \le \|f\|\}$, the disk of radius $\|f\|$ in $\mathbb{C}$. Define $D = \prod_{f \in B} D_f$ with the product topology, a Hausdorff space. Consider the map

$$\Phi : \mathcal{M}_B \to D, \qquad \text{where } \varphi \mapsto \{\varphi(f)\}_{f \in B} \tag{2.3}$$

Certainly Φ is injective and onto its image, $\Phi(\mathcal{M}_B)$. The convergence in the weak* topology also shows Φ is a homeomorphism onto the subspace $\Phi(\mathcal{M}_B)$ showing that $\mathcal{M}_B$ is indeed Hausdorff. $\square$

2.3.2 The Gelfand Transform

Definition 2.3.7. *Let f be an element of a commutative Banach algebra B. The* Gelfand transform *of f is the function $f : \mathcal{M}_B \to \mathbb{C}$ defined by*

$$\widehat{f}(\varphi) = \varphi(f), \varphi \in \mathcal{M}_B$$

The function $\widehat{f}$ is a continuous function on $\mathcal{M}_B$. If $\varphi_\alpha \to \varphi$ is a net inside $\mathcal{M}_B$, then by the definition of the weak* topology we see that $\widehat{f}(\varphi_\alpha) = \varphi_\alpha(f) \to \varphi(f) = \widehat{f}(\varphi)$. The map $\Lambda : B \to C(\mathcal{M}_B)$ by $f \mapsto \widehat{f}$ is called the Gelfand representation of B. Let $\widehat{B}$ denote the set $\{\widehat{f} : f \in B\}$.

The Gelfand transform gives us an alternative perspective of the spectrum of an element.

Theorem 2.3.8. *Let B be a commutative Banach algebra. Then the Gelfand transformation $\Lambda : B \to C(\mathcal{M}_B)$ by $f \mapsto \widehat{f}$ is an algebraic homomorphism such that $\|\widehat{f}\|_{\mathcal{M}_B} \le \|f\|$ and separates points of $\mathcal{M}_B$. If B is unital, then Λ preserves the constant functions.*

Proof. Let $f, g \in B$. Then by definition and the dual properties of $\varphi \in \mathcal{M}_B$ we have $\widehat{f + g}(\varphi) = \varphi(f + g) = \varphi(f) + \varphi(g) = \widehat{f}(\varphi) + \widehat{g}(\varphi)$. Also $\widehat{fg}(\varphi) = \varphi(fg) = \varphi(f)\varphi(g) = \widehat{f}(\varphi)\widehat{g}(\varphi)$.

We have already seen from the proof of lemma 2.3.3 that $|\varphi(f)| \le \|f\|$ for all $f \in B$. It follows that $|\widehat{f}(\varphi)| \le \|f\|$ for every $\varphi \in \mathcal{M}_B$ thus $\|\widehat{f}\| \le \|f\|$. Therefore $\widehat{f}$ is a bounded continuous function on $\mathcal{M}_B$.

Let $\varphi_1, \varphi_2 \in \mathcal{M}_B$ such that for every $f \in B$ we have $\widehat{f}(\varphi_1) = \widehat{f}(\varphi_2)$. Then $\varphi_1(f) = \varphi_2(f)$ for all $f \in B$ which implies $\varphi_1 = \varphi_2$. Therefore $\widehat{B}$ separates the points of $\mathcal{M}_B$.

Finally, if $e \in B$ is the unit element, then $\widehat{e}(\varphi) = \varphi(e) = 1$. Therefore Λ preserves the unit element and consequently all constant functions in $\mathcal{M}_B$. $\qquad\square$

Proposition 2.3.9. *Let B be a unital commutative Banach algebra. Then the spectrum of any element B coincides with the range of its Gelfand transform $\widehat{f}$, that is*

$$\sigma(f) = \widehat{f}(\mathcal{M}_B) = Ran(\widehat{f}).$$

Proof. [11] Let $\lambda \in \sigma(f)$ and consider the ideal J generated by $f - \lambda e$. Then J is not all of B since $f - \lambda e$ is not invertible. Therefore there exists a maximal ideal I containing J. From the correspondence between maximal ideals and multiplicative linear functionals, we know there exists a multiplicative linear functional φ which vanishes on I. Therefore for all $\lambda \in \sigma(f)$,

$$0 = \varphi(f - \lambda e) = \varphi(f) - \lambda\varphi(e) = \widehat{f}(\varphi) - \lambda = 0$$

which implies $\widehat{f}(\varphi) = \lambda$.

Now suppose that $\lambda \notin \sigma(f)$. Then the element $f - \lambda e$ generates all of B, so in particular there exists some $g \in B$ such that

$$e = g(f - \lambda e).$$

Then for an arbitrary multiplicative linear function φ, we have

$$1 = \varphi(e) = \varphi(g)(\varphi(f) - \varphi(\lambda e)) = \varphi(g)(\varphi(f) - \lambda) = \widehat{g}(\varphi)(\widehat{f}(\varphi) - \lambda)$$

which shows that λ cannot be in the range of $\widehat{f}$. □

Notice that as a consequence of proposition 2.3.9,

$$r_f = \max_{\varphi \in \mathcal{M}_B} |\widehat{f}(\varphi)| \tag{2.4}$$

and we may describe the peripheral spectrum of an element $f \in B$ as

$$\sigma_\pi(f) = \{\lambda \in \mathbb{C} : |\lambda| \geq |z| \text{ for all } z \in \sigma(f)\} = \{\lambda \in \mathbb{C} : |\lambda| \geq |\widehat{f}(\varphi)| \text{ for all } \varphi \in \mathcal{M}_B\}.$$

Proposition 2.3.10. *The Gelfand transformation* $\Lambda : B \to \widehat{B}$ *is an isometry if and only if* $\|f^2\| = \|f\|^2$ *for every* $f \in B$.

Proof. If Λ is an isometry, then by basic properties of the supremum,

$$\|f^2\| = \|\widehat{f^2}\|_{\mathcal{M}_B} = \sup_{\varphi \in \mathcal{M}_B} |\widehat{f^2}(\varphi)| = \left(\sup_{\varphi \in \mathcal{M}_B} |\widehat{f}(\varphi)|\right)^2 = \|f\|^2_{\mathcal{M}_B}.$$

On the other hand, if $\|f^2\| = \|f\|^2$ an inductive argument shows that $\|f^{2^n}\| = \|f\|^{2^n}$ for all $n \in \mathbb{N}$. Consequently,

$$\|f\| = \sqrt[2^n]{\|f\|^{2^n}} = \sqrt[2^n]{\|f^{2^n}\|} \to r_f = \max_{\varphi \in \mathcal{M}_B} |\widehat{f}(\varphi)| = \|f\|_{\mathcal{M}_B}$$

by Proposition 2.2.6 and (2.4).

□

The following standard example (e.g. in [1]) shows that for a commutative Banach algebra, the Gelfand transformation Λ may not be an isometry.

Example 2.3.1. Let $(\mathbb{R}, \Sigma, \mu)$ be a measure space. Consider the set M of all measurable

functions f from $\mathbb{R}$ to $\mathbb{C}$ such that

$$\|f\|_{L^1} = \int_{\infty}^{\infty} |f| \, d\mu < \infty \qquad (2.5)$$

Consider the map $\Phi : M \to \mathbb{R}$ by $f \mapsto \|f\|_{L^1}$. Define $L^1(\mathbb{R})$ to be the quotient space $M/(\ker \Phi)$, where each $f \in L^1(\mathbb{R})$ is considered to be a representative in its equivalence class. The set of such functions forms a vector space with pointwise operations and is a Banach space.

Define the *convolution product* $(f, g) \mapsto f * g$ on $L^1(\mathbb{R})$ by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t) \, dt \qquad (x \in \mathbb{R}). \qquad (2.6)$$

One can verify that $(f * g)(x)$ can be defined almost everywhere on $\mathbb{R}$ with $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$ for every $f, g \in L^1(\mathbb{R})$, and $(L^1(\mathbb{R}), \| \cdot \|_{L^1}, *)$ is a commutative Banach algebra.

Let $f = 2\chi_{[0,1]}$ where $\chi_{[0,1]}$ is the characteristic function on $[0, 1] \subset \mathbb{R}$. That is $\chi_{[0,1]} = 1$ on $[0, 1]$ and $\chi_{[0,1]} = 0$ on $\mathbb{R} \setminus [0, 1]$. Then one can easily see that

$$(f * f)(x) = \begin{cases} 2x & : 0 \leq x \leq 1 \\ 4 - 2x & : 1 < x \leq 2 \\ 0 & : \text{otherwise} \end{cases}$$

where $\|f\|_{L^1} = 2$ and $\|f * f\|_{L^1} = 2$. Therefore $\|f * f\|_{L^1} = 2 < 4 = \|f\|_{L^1}^2$ hence by Proposition 2.3.10, the Gelfand map $L^1(\mathbb{R}) \to \widehat{L^1}(\mathbb{R})$ is not an isometry.

2.4 Function Algebras without Unit

Definition 2.4.1 (Function Algebra)**.** [16] *Let X be a locally compact Hausdorff space and suppose A is a subalgebra of $C_0(X)$, the Banach algebra of continuous complex-valued func-*

tions on X vanishing at infinity, and equipped with the supremum norm. Then A is a function algebra *if it is uniformly closed and separates points in the strong sense. That is, for every $x \neq y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$ and for every $z \in X$, there exists $g \in A$ such that $g(z) \neq 0$.*

Most of the commutative Banach algebras considered in this paper are function algebras, not necessarily with a unit. The need for strong separation condition is apparent in the following proposition.

Let $A \subset C_0(X)$ be a function algebra on the locally compact (but not compact) Hausdorff space X, and let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X. Consider the following family of functions

$$A' = \{f + \lambda : f \in A, \lambda \in \mathbb{C}\} \tag{2.7}$$

where each function $f \in A$ is extended continuously to X_∞ by defining $f(\infty) = 0$.

Proposition 2.4.2. *The family A' is a point-separating subalgebra of $C(X_\infty)$, contains the constant functions, and is closed under the supremum norm.*

Proof. For $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$, we have $(f+\lambda)(g+\mu) = fg+g\lambda+f\mu+\lambda\mu \in A'$ since $fg+g\lambda+f\mu \in A$. Clearly A' preserves sums, thus A' is a subalgebra of $C(X_\infty)$ that contains A. Note that for $x, y \neq \infty$, the points will be separated by a function $f \in A$. However, if $\{\infty, x\} \subset X_\infty$ then by the strong separation, there exists a function $g \in A$ such that $g(x) \neq 0 = g(\infty)$. Therefore A' separates the points of X_∞.

Consider a sequence $\{f_n + \lambda_n\}_n \subset A'$ converging uniformly to $g \in C(X_\infty)$. Then $\|f_n + \lambda_n - g\| \to 0$ uniformly on X_∞. In particular, $f_n(\infty) + \lambda_n(\infty) - g(\infty) = \lambda_n(\infty) - g(\infty) \to 0$. Define $g(\infty) = \lim_{n\to\infty} \lambda_n = \lambda$ and let $f = g - \lambda$. Then taking the uniform limit over X shows

$$0 = \lim_{n\to\infty} \|f_n + \lambda_n - g\| = \|\lim_{n\to\infty}(f_n + \lambda_n - (f + \lambda))\| = \|\lim_{n\to\infty} f_n - f\| = \lim_{n\to\infty} \|f_n - f\|.$$

Since A is complete as a function algebra, this implies that $f \in A$. Therefore $g = f + \lambda \in A'$ as desired. $\qquad\square$

Proposition 2.4.2 shows that any function algebra A without unit is naturally contained in the uniform algebra A'. Consider a function algebra $A \subset C_0(X)$ and define A' as in (2.7). Let $(\varphi_\infty : A' \to \mathbb{C}) \in \mathcal{M}_{A'}$ denote the point evaluation homomorphism at ∞. Then we see that $\ker \varphi_\infty = A$ is closed in A' since φ_∞ is continuous. Certainly A' is unital and $\|f^2\| = \|f\|^2$ for every $f \in A'$, so A' is isometrically isomorphic to the space of Gelfand transforms.

Notice that $\varphi_\infty \notin \mathcal{M}_A$ since $\varphi_\infty|_A = 0$. In fact, the maximal ideal spaces of A and A' are related by $\mathcal{M}_{A'} = \mathcal{M}_A \cup \{\varphi_\infty\} = \mathcal{M}_A \cup \{0\}$. In general, the maximal ideal spaces of non-unital function algebras are not compact but locally compact, i.e. $\mathcal{M}_{A'}$ is the one-point compactification of $\mathcal{M}_A$ and the Gelfand transform $\widehat{f}$ of $f \in A$ is an element of $C_0(\mathcal{M}_A)$.

The spectrum of an element is defined in terms of invertible elements, so an alternative definition from 2.2.4 is needed for commutative Banach algebras without unit. Let A be a function algebra without unit and maximal ideal space $\mathcal{M}_A$. For $f \in A$, define the *spectrum* of f to be $\sigma(f) = \widehat{f}(\mathcal{M}_{A'})$. If $\mathcal{M}_A$ is not compact, then $\widehat{f}(\mathcal{M}_{A'}) = \widehat{f}(\mathcal{M}_A) \cup \{0\}$. This definition, along with several other equivalent algebraic definitions, are given in [16, Theorem 5].

Also note that for a function algebra A, Lemma 2.3.3 holds for the uniform algebra A'. In particular, each $\varphi \in \mathcal{M}_A = \mathcal{M}_{A'} \setminus \{0\}$ is continuous and $\|\varphi\| = 1$.

Chapter 3

Boundaries in Commutative Banach Algebras

This chapter discusses the boundaries of commutative Banach algebras. In particular, the Shilov boundary and Choquet boundary are defined for uniform and function algebras.

3.1   General Boundaries in Commutative Banach Algebras

Let B be a commutative Banach algebra. As we have seen previously, each $f \in B$ can be represented as a function $\widehat{f}$ on the maximal ideal space $\mathcal{M}_B$. It is often useful to focus on subsets of the maximal ideal space which give only the pertinent information for an element in the algebra. This leads to the concept of a boundary.

Definition 3.1.1. *A boundary of a commutative algebra B is a subset E of the maximal ideal space $\mathcal{M}_B$ such that*

$$\max_{\varphi \in E} |\widehat{f}(\varphi)| = \max_{\varphi \in \mathcal{M}_B} |\widehat{f}(\varphi)|$$

Clearly $\mathcal{M}_B$ is always a boundary for B.

Let $A \subset C_0(X)$ be a function algebra without unit. Clearly for each $x \in X$, the point evaluation functional $\varphi_x : A \to \mathbb{C}$ is multiplicative and $x \mapsto \varphi_x$ gives a map from X into $\mathcal{M}_A$. Again consider the uniform algebra A' on X_∞. As discussed in chapter 4 of [16], the correspondence $x \mapsto \varphi_x$ maps X_∞ homeomorphically onto a compact subset of $\mathcal{M}_{A'}$ since A' separates the points of X_∞.

Therefore the point evaluation map identifies X as a closed subspace of $\mathcal{M}_A$. Note that for $x \in X$, the identification implies that $f(x) = \varphi_x(f) = \widehat{f}(\varphi_x)$, therefore it is common to write $\widehat{f}(x)$ with the understanding that x is a member of the maximal ideal space.

Lemma 3.1.2. *Let X be a locally compact Hausdorff space. If $A \subset C_0(X)$ is a function algebra, then $X \hookrightarrow \mathcal{M}_A$ is a boundary for A.*

Proof. Let $f \in B$. Clearly $\max_{\varphi \in \mathcal{M}_B} \{|\widehat{f}(\varphi)|\} \geq \max_{x \in X} \{|f(x)|\}$ where X is regarded as a subset of $\mathcal{M}_B$. On the other hand, from Lemma 2.3.3, we see that

$$|\widehat{f}(\varphi)| = |\varphi(f)| \leq \|f\| = \max_{x \in X}\{|f(x)|\}$$

for all $\varphi \in \mathcal{M}_B$. This implies $\max_{\varphi \in \mathcal{M}_B} \{|\widehat{f}(\varphi)|\} \leq \max_{x \in X}\{|f(x)|\}$, thus $\max_{\varphi \in \mathcal{M}_B}\{|\widehat{f}(\varphi)|\} = \max_{x \in X}\{|f(x)|\}$ as claimed.

$\square$

In light of Lemma 3.1.2, we may identify the boundaries as subsets $E \subset X$.

The following lemma stresses the significance of boundaries for function algebras.

Lemma 3.1.3. *Let A be a function algebra on X and $E \subset \mathcal{M}_A$ be a boundary for A. Then the restriction map $r : A \to A|_E \subset C(E)$ by $r(f) = f|_E$ is an isometric algebra isomorphism*

between function algebras.

Proof. From basic properties of functions, the map is clearly linear, multiplicative and surjective. Since E is a boundary, we know the norm of each function is preserved by E. Thus r is a linear isometric mapping and therefore automatically injective. For A_E to be a function algebra, it only remains to show that A_E separates points, but this is a property that A_E clearly inherits from A. □

3.1.1 The Shilov Boundary

The following theorem due to Shilov asserts the existence of a smallest closed boundary in a commutative Banach algebra.

Theorem 3.1.4 (Shilov)**.** *(e.g. in [8]) The intersection of all closed boundaries of a unital commutative Banach algebra is a closed boundary.*

A somewhat technical lemma is required for the proof of Shilov's theorem. The lemma describes an important relationship between open neighborhoods in the maximal ideal space and closed boundaries.

Lemma 3.1.5. *Let B be a commutative Banach algebra and let V be a fixed Gelfand neighborhood in $\mathcal{M}_B$. Then either V meets every boundary of B or its complement $E \smallsetminus V$ in each closed boundary E of B is also a closed boundary of B.*

Proof. ([25]) Let $V = V(\varphi; f_1, \ldots, f_n; 1) = \{\psi \in \mathcal{M}_B : |\psi(f_j)| < 1, \psi(f_j) = 0, j = 1, \ldots, n\}$. Suppose that E is a closed boundary of B such that $E \smallsetminus V$ is not a boundary. It remains to show that V meets every boundary of B.

First consider the case where $E \smallsetminus V = \varnothing$. Then $V \supset E$. The condition for membership in V guarantees that $|\widehat{f_j}| < 1$ for all j and all $\varphi \in V$, thus $|\widehat{f_j}| < 1$ for all j and all $\varphi \in E$. But E is

a boundary, which implies that $|\widehat{f_j}| < 1$ for all j and $\varphi \in \mathcal{M}_B$. Therefore $V = \mathcal{M}_B$ and clearly meets every boundary of B.

Next suppose that $E \smallsetminus V \neq \varnothing$. By assumption E/V is not a boundary of B, thus

$$\max_{\varphi \in \mathcal{M}_B} |\widehat{f}(\varphi)| = 1 > \max_{\varphi \in E \smallsetminus V} |\widehat{f}(\varphi)|$$

for some $f \in B$. The first equality follows from the fact that $\|\varphi\| = 1$ for all $\varphi \in \mathcal{M}_B$.

Now note that the function $\widehat{f}^n \to 0$ uniformly on the closed set $E \smallsetminus V$. Therefore there exists some integer m such that

$$\max_{\varphi \in E \smallsetminus V} |\widehat{f}^m(\varphi)\|\widehat{f_j}(\varphi)| < 1$$

for all $\varphi \in E \smallsetminus V$ and every j. We also have $|f_j(\varphi)| < 1$ for all j by definition of membership in V and it follows that the inequality above holds on all of E. Since E is a boundary, we have

$$\max_{\varphi \in \mathcal{M}_B} |\widehat{f}^m(\varphi)\widehat{f_j}(\varphi)| = \max_{\varphi \in E} |\widehat{f}^m(\varphi)\widehat{f_j}(\varphi)| < 1$$

for all j.

Since $\widehat{f} : \mathcal{M}_B \to \mathbb{C}$ such that $\|\widehat{f}\| = 1$ and $\mathcal{M}_B$ is compact, there exists some point $\varphi_1 \in \mathcal{M}_B$ such that $|\widehat{f}(\varphi_1) = 1|$. Then $|\widehat{f}^m(\varphi_1)| = 1$ and the inequality

$$1 > |\widehat{f}^m(\varphi_1)\widehat{f_j}(\varphi_1)| = |\widehat{f_j}(\varphi_1)|$$

for all j expresses that $\varphi_1 \in V$. Therefore $\varphi \mapsto |\widehat{f}(\varphi)|$ attains its maximum within V, which shows that V indeed meets every boundary of B. $\qquad \square$

We now have all the ingredients to prove Shilov's theorem.

Proof of Shilov's Theorem. ([25]) Let E_0 be the intersection of all closed boundaries, E, of B.

It is first shown that for any $f \in B$ such that $|\widehat{f}(\varphi)| < 1$ on E_0, necessarily $|\widehat{f}(\varphi)| < 1$ on $\mathcal{M}_B$. It follows immediately that the maximum of $|\widehat{f}|$ taken over E_0 is also the maximum taken over $\mathcal{M}_B$.

By way of contradiction, suppose that the set $K = \{\varphi \in \mathcal{M}_B : |\widehat{f}(\varphi)| \geq 1\}$ is nonempty and let $\varphi_0 \in K$. Then $K \cap E_0 = \varnothing$ which implies $\varphi \notin E_0$. Therefore φ_0 is not in the intersection of all closed boundaries so there exists a closed boundary E such that $\varphi_0 \notin E$. Therefore there exists a Gelfand neighborhood V_0 such that $V_0 \cap E = \varnothing$. By the previous lemma, this implies that $E \smallsetminus V_0$ is a boundary of B.

Since $\mathcal{M}_B$ is compact and $K \subset \mathcal{M}_B$ is closed, K is also a compact set. Therefore there exists a finite integer k, points $\varphi_1, \ldots, \varphi_k$, boundaries $E_1, \ldots, E_k$, and neighborhoods $V_1, \ldots, V_k$ such that $V_i \cap E_i = \varnothing$ and the union of the V_is cover K.

We claim that the set $\tilde{E} = \mathcal{M}_B \smallsetminus \bigcup_{j=1}^n V_j$ is also a nonempty closed boundary of B. Clearly $\tilde{E}$ is a closed set. Also $\tilde{E}$ cannot be the empty set since none of the V_j meet E_0, thus $\bigcup_{j=1}^n V_j$ does not meet E_0 so at least $E_0 \subset \tilde{E}$. If $\mathcal{M}_B \smallsetminus (V_1 \cup V_2)$ is not a boundary, then there exists some $f \in B$ such that $|\widehat{f}| < 1$ on $\mathcal{M}_B \smallsetminus (V_1 \cup V_2)$ but $|\widehat{f}| = 1$ on $V_1 \cup V_2$. This is impossible since both $\mathcal{M}_B \smallsetminus V_i \supset E_i \smallsetminus V_i$ are boundaries for $i = 1, 2$. An inductive argument shows that $\tilde{E}$ is also a boundary.

Since $\bigcup_{j=1}^n V_j$ covers K, we see that $|\widehat{f}| < 1$ on the set $\tilde{E} \subset \mathcal{M}_B \smallsetminus K$. But this implies that $|\widehat{f}| < 1$ holds on all of $\mathcal{M}_B$ since $\tilde{E}$ is a boundary of B. This contradicts the assertion that K was a non-empty set. Therefore E_0 is a boundary as claimed. $\qquad\square$

Definition 3.1.6. *The intersection of all closed boundaries of a commutative Banach algebra is called the* Shilov boundary *of B and is denoted by ∂B.*

By definition 3.1.6, the Shilov boundary is the smallest closed boundary of B, in that it is contained in every closed boundary of B. Since $\mathcal{M}_B$ is compact, it follows that ∂B is also

compact.

Example 3.1.1. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and recall the disk algebra, $A(\mathbb{D})$, in example 2.1.6. For $i = 1, 2$, let $\mathbb{D}_i$ be a copy of the unit disk in $\mathbb{C}$. Define $D = \mathbb{D}_1 \sqcup \mathbb{D}_2$, the disjoint union of $\mathbb{D}_1$ with $\mathbb{D}_2$, and let $A(D)$ be the algebra of functions on D, continuous on $\overline{D}$ and analytic inside D with pointwise operations. Then $A(D)$ is a uniform, hence function algebra such that for each $f \in A(D)$, $f_1 = f|_{\mathbb{D}_1} \in A(\mathbb{D}_1)$ and $f_2 = f|_{\mathbb{D}_2} \in A(\mathbb{D}_2)$. The Shilov boundary of $A(D)$ is given by $\partial A(D) = \mathbb{T}_1 \sqcup \mathbb{T}_2$.

3.2 Peak Sets and Peaking Functions

Suppose $A \subset C(X)$ is a function algebra on a locally compact and Hausdorff set X with $\mathcal{M}_A = X$. There are certain points in the Shilov Boundary that are easily identifiable.

Definition 3.2.1. *A point $x_0 \in X$ is a* peak point *of a function algebra A if there exists a function $f \in A$ such that $f(x_0) = 1$ and $|\widehat{f}(x)| < 1$ for every $x \in X \smallsetminus \{x_0\}$. The function f is called a* peak function.

Clearly each peak point is necessarily contained in every boundary, thus every peak point belongs to ∂A. In general, the set of peak points may be properly contained in ∂A and may not be a boundary for B at all. However, the peak points do form a boundary when the maximal ideal space is metrizable.

Definition 3.2.2. *A* peak set *is a non-empty set $F \subset X$ such that there exists an $f \in A$ with $f(x) = 1$ on F and $|\widehat{f}(x)| < 1$ on $X \smallsetminus F$. Each such function f is called a* peaking function.

Note that $f \in A$ is a peaking function if and only if $\sigma_\pi(f) = \{1\}$. Let $\mathcal{P}(A)$ denote the family of peaking functions in A. For $f \in \mathcal{P}(A)$, we see that $\widehat{f}^{-1}(1)$ is always a peak set. Clearly every singleton peak point is a peak set of A, and every peak function is a peaking function.

Let $K \subset \mathcal{M}_A$ and denote by $\mathcal{P}_K(A) \subset \mathcal{P}(A)$ be the collection of peaking functions $f \in A$ such that $K \subset \widehat{f}^{-1}(1)$. For each $f \in \mathcal{P}_K(A)$ we say that f *peaks on K*.

Let $f \in A \subset C(X)$ be a peaking function with $F = \widehat{f}^{-1}(1) \subset \mathcal{M}_A$. We see immediately that each peak set is closed since it is the inverse image of a closed set under a continuous function. Since $\widehat{f} \in C_0(\mathcal{M}_A)$, there exists a compact set K such that $K \supset \{\varphi \in \mathcal{M}_A : |\widehat{f}(\varphi)| > 1/2\}$. Since $F \subset K$, F is a closed subset of a compact set and is therefore compact.

Another important subset associated to each function $f \in A$ is the *maximizing set* (or *maximum modulus set*),

$$E(f) = \{x \in X : |f(x)| = \|f\|\} \tag{3.1}$$

which is not necessarily a peak set but useful in describing the behavior of functions in A. Note that $E(f)$ is a compact set for each f in the function algebra A.

If X is a locally compact Hausdorff space and $A \subset C_0(X)$ a nonzero function algebra , then the following lemma shows that A contains peaking functions and X contains peak sets. The proof follows the argument in [24], Lemma 2.3(a).

Lemma 3.2.3. *Let A be a dense subalgebra of a function algebra and let $f \in A$ be such that $f \neq 0$. If $\lambda \in \sigma_\pi(f)$, then the function*

$$h = \frac{1}{2\lambda} \cdot \left(\frac{f^2}{\lambda} + f \right) \tag{3.2}$$

is a peaking function in A such that $E(h) = f^{-1}(\lambda)$ and $\sigma_\pi(fh) = \{\lambda\}$.

Proof. Let $f \in A$ such that $\|f\| = |\lambda|$. Consider the function

$$h = \frac{1}{2\lambda} \cdot \left(\frac{f^2}{\lambda} + f \right) \in A. \tag{3.3}$$

Then $h(x) = 1$ for all $x \in f^{-1}\{\lambda\}$ and for all $x \notin f^{-1}\{\lambda\}$ we have

$$|h(x)| = \frac{|f(x)|}{2|\lambda|}\left|\frac{f(x)}{\lambda} + 1\right| \le \frac{1}{2}\left|\frac{f(x)}{\lambda} + 1\right| < 1$$

Hence $h \in \mathcal{P}(A)$ such that $h^{-1}(1) = f^{-1}(\lambda)$, a peak set.

Observe that $f(x)h(x) = \lambda$ whenever $x \in f^{-1}(\lambda)$, and $|f(x)h(x)| = |f(x)||h(x)| < \|fh\|$ when $x \notin f^{-1}(\lambda)$. Therefore $\|fh\| = |\lambda|$ and $\sigma_\pi(fh) = \{\lambda\}$. □

Lemma 3.2.4. *Let A be a dense subalgebra of a function algebra without unit. Then for any $f \in A$, the set $E(f)$ is a disjoint union of peak sets for A.*

Proof. Fix $f \in A$, then without loss of generality we can assume that $\|f\| = 1$. Then

$$E(f) = \bigcup_{\substack{|\lambda|=1 \\ \lambda \in \mathrm{Ran}(f)}} \{f^{-1}\{\lambda\}\} \tag{3.4}$$

Indeed, if $x \in E(f)$ then $f(x) = \lambda$ for some $|\lambda| = 1$. Also if $f(x) = \lambda$ with $|\lambda| = 1$, then $|f(x)| = 1$ which implies $x \in E(f)$. Now fix $\lambda \in \mathrm{Ran}(f)$ with $|\lambda| = 1$ and consider the function

$$g = \frac{1}{2\lambda} \cdot \left(\frac{f^2}{\lambda} + f\right)$$

as in Lemma 3.2.3. Then $g(x) = 1$ for all $x \in f^{-1}\{\lambda\}$, while for all $x \notin f^{-1}\{\lambda\}$ we have $|g(x)| < 1$. Hence $g \in \mathcal{P}(A)$ and $g^{-1}(1) = f^{-1}(\lambda)$, a peak set. The equality (3.4) implies that $E(f)$ is the union of such sets.

□

3.3 The Choquet Boundary

Another important boundary for uniform and function algebras is the *Choquet boundary*. For a function algebra A, the most general characterization defines the Choquet boundary as a certain subset of the unit ball in the dual space, A^*, of A. We give an alternative definition that provides a description based on the topology of the underlying space X. The following construction is due to Phelps in [21].

3.3.1 Classical Definition

Definition 3.3.1. *Let S be a convex set in a vector space V. Then the* extreme points *of S are points $x \in S$ such that x not an interior point of any line segment in S. That is, x is extreme if and only if $x = ty + (1-t)z$ for $t \in (0,1)$ and $z \neq y$ implies that $y \notin S$ or $z \notin S$.*

Let A be a unital Banach algebra. Recall from Proposition 2.3.5 that the unit ball $B_1 = \{\varphi \in A^* : \|\varphi\| = 1\} \subset A^*$ is compact in the weak* topology. It is easy to see that B_1 is a convex subset of the vector space A^* [21]. Denote by $\mathrm{Ext}(B_1)$ the set of extreme points in B_1 and define the point evaluation map $X \to \mathcal{M}_A$ by $x \mapsto \varphi_x$, where $\varphi_x(f) = f(x)$. Then we have the following inclusion

$$\mathrm{Ext}(B_1) \subseteq \{\alpha\varphi_x : |\alpha| = 1, \alpha \in \mathbb{C}, x \in X\}$$

(see e.g. in [6], page 441.)

Definition 3.3.2 (Choquet Boundary)**.** *The* Choquet Boundary *of A, denoted δA, is the set $\delta A = \{x \in X : \varphi_x \in Ext(B_1)\}$.*

3.3.2 Alternative Characterization of the Choquet Boundary

Note that peak functions and peaking functions can be defined for any set of functions. For a locally compact Hausdorff space X, Araujo and Font [3] have shown that the Shilov boundary exists for linear subspaces of $C_0(X)$ that strongly separate points. Here we present a construction of the Choquet boundary for function algebras $A \subset C_0(X)$, which is an adjustment of that in [13] that developes boundaries for arbitrary families of functions in $C_0(X)$ possibly without any algebraic structure.

Definition 3.3.3. *Let $A \subset C_0(X)$ be a function algebra. Then:*

(i) *A non-empty subset $E \subset X$ is called a p-set for A if E is the intersection of a family of peak sets.*

(ii) *A point $x \in \mathcal{M}_A$ is called a p-point of A (or, a generalized peak point, or strong boundary point of A) if it is a singleton p-set. The set of all p-points, or strong boundary points, is denoted $p(A)$.*

Note that every p-set is a compact set in X.

The following theorem demonstrates the significance of the set of strong boundary points for function algebras $A \subset C_0(X)$ where X is a locally compact Hausdorff space.

Theorem 3.3.4. *For a function algebra $A \subset C_0(X)$, the set of p-points $p(A)$ is a boundary for A.*

The proof of Theorem 3.3.4 makes use of some well known topological lemmas.

Lemma 3.3.5. *Let X be a Hausdorff space and $\{E_\alpha\}$ a family of compact sets. Suppose $E := \bigcap_\alpha E_\alpha$. If U an open set that contains E, then there exists $n \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^n$ such that*

$$E \subset \bigcap_{i=1}^n E_{\alpha_i} \subset U.$$

Proof. (e.g. [13]) Fix an $E_{\alpha_0} \in \{E_\alpha\}$. Then by the hypothesis we have

$$E_{\alpha_0} \subset (X \smallsetminus E) \cup U = \left[X \smallsetminus \left(\bigcap_\alpha E_\alpha\right)\right] \cup U = \left[\bigcup_\alpha (X \smallsetminus E_\alpha)\right] \cup U.$$

Therefore $\bigcup_\alpha (X \smallsetminus E_\alpha) \cup U$ is an open cover for the compact set E_{α_0}, thus there exists $n \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^n$ such that

$$E_{\alpha_0} \subset \left[\bigcup_{i=1}^n (X \smallsetminus E_{\alpha_i})\right] \cup U = X \smallsetminus \left[\bigcap_{i=1}^n E_{\alpha_i} \cap (X \smallsetminus U)\right].$$

Thus

$$\bigcap_{i=1}^n E_{\alpha_i} \cap (X \smallsetminus U) \subset (X \smallsetminus E_{\alpha_0}),$$

and taking the intersection with E_{α_0} on both sides of the inclusion shows that $(E_{\alpha_0} \cap \bigcap_{i=1}^n E_{\alpha_i}) \cap (X \smallsetminus U) \subset \varnothing$, thus $E \subset \bigcap_{i=0}^n E_{\alpha_i} \subset U$, which shows the conclusion holds with the set $\{E_{\alpha_i}\}_{i=0}^n$.

$\square$

The previous lemma with $U = \varnothing$ shows the following:

Corollary 3.3.6 (Finite Intersection Property for Hausdorff Spaces). *(e.g. in [19]) A family of compact sets in a Hausdorff space has empty intersection if and only if there is a finite subcollection with empty intersection.*

For any subset $E \subset X$, let $\mathcal{E}_E$ denote the family of all p-sets that are subsets of E. Note that $\mathcal{E}_E$ may be empty, and when $\mathcal{E}_E$ is nonempty it can be partially ordered with respect to inclusion. In particular, $\mathcal{E}_X$ is non-empty for a nonzero function algebra A on X since X contains a peak set by Lemma 3.2.3.

Lemma 3.3.7. *Let X be a locally compact Hausdorff space. If $F \subset X$ and $A \subset C_0(X)$ is a function algebra such that the family $\mathcal{E}_F$ is non-empty, then $\mathcal{E}_F$ has a minimal element.*

Proof. Let $F \subset X$ and $\mathcal{C}$ be a chain in $\mathcal{E}_F$. We aim to show that $\mathcal{C}$ has a lower bound and apply Zorn's lemma. The natural choice for a lower bound is $\bigcap_{E' \in C} E'$, it remains to show that this intersection is non-empty and is a p-set.

Suppose $\{E_j'\}_{j=1}^n$ is a finite collection of sets in $\mathcal{C}$, and without loss of generality, suppose the ordering is consistent with the partial ordering in $\mathcal{C}$ with $j = 1$ smallest. Then $\bigcap_{j=1}^n E_j' = E_1'$ which is a non-empty p-set. Since the finite collection was arbitrary, Lemma 3.3.6 implies that $\bigcap_{E' \in C} E'$ is non-empty. Also by the definition, $\bigcap_{E' \in C} E'$ is an p-set since it is the intersection of p-sets.

Thus every chain has a minimal element, so by Zorn's lemma there exists a minimal element in $\mathcal{E}_F$. $\qquad\square$

Observe that in particular, $\mathcal{E}_X$ has a minimal p-set.

Lemma 3.3.8. *Suppose X is a locally compact Hausdorff space and $A \subset C_0(X)$ is a function algebra. Let E be a p-set. For every open set U containing E there exists $h \in \mathcal{P}_E(A)$ such that $E(h) \subset U$.*

Proof. Let E be a p-set and U a neighborhood of E. Then there exists a family $S \subset \mathcal{P}_E(A)$ such that $E = \bigcap_{f \in S} E(f)$.

Since each $E(f)$ is compact, Lemma 3.3.5 implies that there exists a finite collection $\{h_1, h_2, \ldots, h_n\} \subset S$ such that $E \subset \bigcap_{i=1}^n E(h_i) \subset U$.

Let $h = h_1 \cdot h_2 \cdots h_n$ which is clearly in $\mathcal{P}_E(A)$. Then $E(h) = \bigcap_{i=1}^n E(h_i)$ and it follows that $E(h) = \bigcap_{i=1}^n E(h_i) \subset U$ as desired.. $\qquad\square$

The following theorem is a generalization of a classical result for uniform algebras due to E.

Bishop [4]. The arguments given here are an adjustment of Bishop's lemma for p-sets for uniform algebras found in [15].

Theorem 3.3.9 (Bishop's Lemma for p-sets in Function Algebras)**.** *Let X be a locally compact Hausdorff space and let $A \subset C_0(X)$ be a function algebra without unit. If $f \in A$ and E is a p-set of A with $f|_E \neq 0$, then there exists a peaking function $h \in \mathcal{P}_E(A)$ such that fh takes its maximum modulus in E.*

Proof. Fix $f \in A$ and without loss of generality, we may assume that $\max_{x \in E} |f(x)| = 1$. Therefore we necessarily have $\|f\| \geq 1$.

For every $n \in \mathbb{N}$ define the set

$$U_n = \left\{ x \in X : |f(x)| < 1 + \frac{1}{2^{n+1}} \right\}.$$

Then for every integer $n \geq 1$, U_n is open, $U_n \subset U_{n-1}$ and $U_n \supset E$. Since E is a p-set, Lemma 3.3.8 implies that for each n we can choose a peaking function $k_n \in \mathcal{P}_E(A)$ such that $E(k_n) \subset U_n$.

Then $|k_n(x)| < 1$ on the closed set $X \smallsetminus U_n$, so for each n define h_n to be a power of k_n such that

$$|h_n(x)| < \frac{1}{2^n \|f\|}$$

on $X \smallsetminus U_n$. Define $h = \sum_{n=1}^{\infty} \frac{h_n}{2^n}$.

We claim that h is the desired function. Clearly the series is absolutely convergent, and $\|h\| \leq 1$. Also since each k_n is a peaking function, so is h_n.

Observe that $E(h) = \bigcap_{n=1}^{\infty} E(h_n) \supset E$ since $|h(x)| = 1 = h(x)$ if and only $|h_n(x)| = 1 = h_n(x)$ for every n. Therefore $h \in \mathcal{P}_E(A)$.

If $x \in E$, then $|f(x)h(x)| = |f(x)|$. Consequently

$$\|fh\| = \max_{x \in X} |f(x)h(x)| \geq \max_{x \in E} |f(x)h(x)| = \max_{x \in E} |f(x)| = 1. \tag{3.5}$$

which shows that in fact $\|fh\| \geq 1$. We claim that $\|fh\| \leq 1$. Fix $x \in X$ and let $X = U_0$. There are two cases.

Case 1: $x \in U_{n-1} \setminus U_n$ for some $n \geq 1$.

Then $x \in U_0, U_1, \ldots, U_{n-1}$ but $x \notin U_m$ for $m \geq n$. Therefore $|f(x)| < 1 + \dfrac{1}{2^n}$ and $|h_m(x)| < \dfrac{1}{2^m \|f\|} \leq \dfrac{1}{2}$ for all $m \geq n$. Thus,

$$|h(x)| = \sum_{m=1}^{n-1} \frac{|h_m(x)|}{2^m} + \sum_{m=n}^{\infty} \frac{|h_m(x)|}{2^m} \leq \sum_{m=1}^{n-1} \frac{1}{2^m} + \sum_{m=n}^{\infty} \frac{1/2}{2^m}$$

$$= \left(1 - \frac{1}{2^{n-1}}\right) + \frac{1}{2} \cdot \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}}\left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2^n}$$

Consequently, $|f(x)h(x)| < \left(1 + \dfrac{1}{2^n}\right)\left(1 - \dfrac{1}{2^n}\right) = 1 - \dfrac{1}{4^n} < 1$.

Case 2: $x \in \bigcap\limits_{n=1}^{\infty} U_n$.

Then $x \in U_n$ for all n which implies $|f(x)| < 1 + \dfrac{1}{2^{n+1}}$ for every n. Thus $|f(x)| \leq 1$ which implies that $|f(x)h(x)| \leq 1$ since $h \in \mathcal{P}(A)$. Therefore $\|fh\| \leq 1$ and we have established that $\|fh\| = 1$. Then (3.5) shows that fh indeed takes its maximum modulus on E. $\square$

The following strong version of Bishop's Lemma for p-points and function algebra $A \subset C_0(X)$, not necessarily with unit, follows from Theorem 3.3.9. It plays an important role in the theorems given in chapters 4, 5, and 6.

Theorem 3.3.10 (Strong Multiplicative Bishop's Lemma)**.** [24] *Let X be a locally compact Hausdorff space and $A \subset C(X)$ be a function algebra without unit on $X = \partial A$. If $f \in A$ and $x_0 \in X$ is a p-point of A with $f(x_0) \neq 0$, then there exists a peaking function $h_0 \in \mathcal{P}_{x_0}(A)$ such*

that

$$\sigma_\pi(fh_0) = \{f(x_0)\} \tag{3.6}$$

If E is a peak set of A which contains x_0, then h_0 can be chosen so that $E(fh_0) = E(h_0) \subset E$.

Lemma 3.3.11. *Let X be a locally compact Hausdorff space and $A \subset C_0(X)$ a function algebra. If $x_0 \in X$ is contained in a minimal p-set and $f \in A$ with $f(x_0) \neq 0$, then there exists an $h \in \mathcal{P}_{x_0}(A)$ such that $\sigma_\pi(fh) = \{f(x_0)\}$.*

For uniform algebras Lemma 3.3.11 is proven in [15].

Proof. Let E be the minimal p-set containing x_0. Then by Lemma 3.3.9 there exists a peaking function $k \in \mathcal{P}_E(A)$ such that fk takes its maximal modulus on E, that is $\max_{x \in E} |f(x)k(x)| = \|fk\|$ which implies $E(fk) \cap E \neq \varnothing$.

Now by Lemma 3.2.4, $E(fk)$ can be written as a disjoint union of peak sets

$$E(fk) = \bigcup_{\lambda \in \sigma_\pi(fk)} (fk)^{-1}(\lambda) = \bigcup_{\lambda \in \sigma_\pi(fk)} P_\lambda$$

where $P_\lambda = (fk)^{-1}(\lambda)$.

Therefore $E \cap P_\lambda \neq \varnothing$ for some $\lambda \in \sigma_\pi(fk)$. Since E is a minimal p-set, we must have $E \subset P_\lambda \subset E(fk)$. Therefore $x_0 \in E \subset E(fk)$. Since $k \in \mathcal{P}_E(A)$, we see that $k(x_0) = 1$ and consequently $|f(x_0)| = |f(x_0)k(x_0)| = \|fk\|$. Hence $f(x_0) \in \sigma_\pi(fk)$.

Then by Lemma 3.2.3, $H = (fk)^{-1}(fk(x_0)) = (fk)^{-1}(f(x_0))$ is a peak set of A. Therefore there exists a peaking function $k' \in \mathcal{P}(A)$ such that $E(k') = (fk)^{-1}(f(x_0))$. If $h = kk'$ then $h \in \mathcal{P}(A)$ with $x_0 \in E(h) = E(k) \cap E(k') \subset H$, since $k'(x_0) = 1$ and $k(x_0) = 1$.

Clearly $|f(x_0)| = \|fk\| \geq \|fkk'\| = \|fh\|$, but $fh(x_0) = fkk'(x_0) = f(x_0)$ shows that $|f(x_0)| = \|fh\|$, thus $f(x_0) \in \sigma_\pi(fh)$. Also for any $x \in E(fh) = E(fk) \cap E(k')$, we clearly have $fk(x) =$

$f(x_0)$ since $x \in H$ and $k'(x) = 1$ thus $fh(x) = f(x_0)$. Therefore $\sigma_\pi(fh) = \{f(x_0)\}$ as desired.

$\square$

Lemma 3.3.12. *Let X be a locally compact Hausdorff space and $A \subset C_0(X)$ a function algebra. Then the minimal elements of $\mathcal{E}_X$ are singletons. Moreover, the set of minimal elements of $\mathcal{E}_X$ coincides with $p(A)$.*

Proof. Suppose E is a minimal p-set and $x, y \in E$.

We show that $x = y$ by showing that $f(x) = f(y)$ for every function $f \in A$. Since A separates points, this proves the claim.

If $f(x) = 0 = f(y)$ there is nothing to show. Otherwise let $f(x) \neq 0$. By Lemma 3.3.11 there exists a function $h \in \mathcal{P}_x(B)$ such that $\sigma_\pi(fh) = \{f(x)\}$. Consequently, $x \in E(h) = h^{-1}(1)$, a peak set. Thus $E \cap E(h) \neq \varnothing$. By the minimality of E, we may assume that $E \subset E(h)$. Therefore we also have $y \in E(h) \subset E(fh)$. Then $y \in \sigma_\pi(fh) = \{f(x)\}$ which implies $f(y) = f(y)h(y) = f(x)$. We deduce that $x = y$, which proves the lemma. $\square$

All of the ingredients are now in place to prove the main result of the section, that the set of p-points form a boundary of a function algebra.

Theorem 3.3.13. *Let $A \subset C_0(X)$ be a function algebra. Then $p(A)$ is a boundary for A, and is contained in the Shilov boundary. That is, $p(A) \subset \partial A$.*

The following proof follows [15], Lemma 3.2.16 which establishes the result for uniform algebras.

Proof. Let $f \in A$ and let $\{x\}$ be a minimal element of $\mathcal{E}_{E(f)}$, which is neccesarily nonempty by Lemma 3.2.3. Then $\{x\}$ is an intersection of peak sets by the definition of $\mathcal{E}_{E(f)}$, which

implies $x \in p(A) \cap E(f)$. Since $E(f)$ meets $p(A)$ for every $f \in A$, it follows that

$$\max_{x \in p(A)} |f(x)| = \|f\|$$

for every $f \in A$, which shows that $p(A)$ is a boundary for A.

Let $E \subset X$ be a closed boundary of A. By way of contradiction, suppose that $p(A) \smallsetminus E \neq \varnothing$. Then there exists $x \in (p(A) \smallsetminus E) \subset (X \smallsetminus E)$. Since E is closed, $X \smallsetminus E$ is an open neighborhood of x in X, so by Lemma 3.3.8 there exists a peaking function $h \in \mathcal{P}_x(A)$ such that $x \in h^{-1}(1) \subset X \smallsetminus E$. Therefore $\|h\| = 1$ but $|h(x)| < 1$ on E which contradicts that E is a boundary. Consequently, $p(A) \smallsetminus E = \varnothing$ which implies that $p(A)$ is contained in E. Since E was arbitrary, we see that $p(A)$ is contained in every closed boundary. $\qquad\square$

It is a surprising and convenient fact that the set of strong boundary points, $p(A)$, for a function algebra $A \subset C_0(X)$ coincides with the Choquet boundary in definition 3.3.2.

Theorem 3.3.14. *(e.g. in [23]) If $A \subset C_0(X)$ is a function algebra, then $\delta A = p(A)$.*

The proof will be omitted here, but is provided in [23], pages 2-3. The theorem, together with the results above, show the Choquet boundary is indeed a boundary and is contained in the Shilov boundary. Therefore $\delta B \subset \partial B$ which implies that $\overline{\delta B} \subset \partial B$. The simple observation $\delta B \subset \overline{\delta B}$ shows $\overline{\delta B}$ is a closed boundary, so we also have $\partial B \subset \overline{\delta B}$. It follows that $\overline{\delta B} = \partial B$.

3.4   Dense Subalgebras of Function Algebras

Suppose that $A \subset C_0(X)$ is an algebra such that $\overline{A}$ is a function algebra, not necessarily complete. Then A may inherit several useful properties from $\overline{A}$, but the conclusion of Theorem 3.3.10 may not hold since the construction of the function $h \in \mathcal{P}(A)$ makes use of the completeness of A. In addition, it may not be the case that $p(A) = \delta A$, and in fact it can happen

that $p(A) = \varnothing$. For this reason, the assumption $p(A) = \delta A$ may be added as a hypothesis for some later theorems on dense subalgebras of function algebras.

Also note that $\mathcal{M}_A = \mathcal{M}_{\overline{A}}$. Indeed, every multiplicative linear functional φ is continuous, consider $f \in \overline{A}$ such that $f = \lim f_n$, where $f_n \in A$. Then $\varphi(f)$ is determined as $\lim \varphi(f_n)$. Similarly, $A^* = \overline{A}^*$ and it follows that $\delta A = \delta \overline{A}$ since the Choquet boundary is defined as a subset of the dual space.

Example 3.4.1. Let $C^{(1)}[a, b]$ denote the vector space of continuous complex-valued functions on the interval $[a, b]$ which are differentiable and whose derivatives are continuous on $[a, b]$. Define a norm on $C^{(1)}[a, b]$ by

$$\|f\| = \max_{a \le t \le b} |f(t)|$$

With pointwise addition and multiplication, $C^{(1)}[a, b]$ is a commutative Banach algebra. However, $C^{(1)}[a, b]$ is not uniformly closed in $C[a, b]$.

Example 3.4.2. Let $P[a, b] = \{\sum_{i=0}^{n} a_i x^i : n \in \mathbb{N}, i = 1, \dots, n, a_i \in \mathbb{C}\}$ be the set of polynomials defined on the interval $[a, b]$, again with pointwise operations and the supremum norm. Then $P[a, b]$ is a commutative algebra, but is not complete with respect to the norm. In fact, $\overline{P[a, b]} = C[a, b]$ by the Stone-Weierstrass theorem.

Chapter 4

Norm-Multiplicative Mappings between Function Algebras

Unless otherwise stated, throughout this chapter $A \subset C(X)$ and $B \subset C(Y)$ will be function algebras on locally compact spaces X and Y, respectively.

4.1 Basic Results

Definition 4.1.1. *Let $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, and let $\psi \colon Y \to X$ be a continuous mapping. A map $T \colon A \to B$ is called*

(i) *a ψ-composition operator on Y if $(Tf)(y) = f(\psi(y))$ for all $f \in A$ and $y \in Y$, and*

(ii) *a weighted ψ-composition operator on Y if there is a continuous function α on Y so that $(Tf)(y) = \alpha(y) f(\psi(y))$ for all $f \in A$ and $y \in Y$.*

Proposition 4.1.2. *If $T : A \to B$ is a composition operator between function algebras, i.e.*

Tf = f ∘ ψ for some homeomorphism ψ, then T is linear, multiplicative, injective, and an isometry.

Proof. Let $f, g \in A$ and $\alpha, \beta \in \mathbb{C}$. From the definitions of function operations we immediately see that $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$. If $Tf = Tg$, then $f \circ \psi = g \circ \psi$ and therefore the injectivity of T follows from the injectivity of the homeomorphism ψ. Also $T(fg) = (fg) \circ \psi = (f \circ \psi)(g \circ \psi) = TfTg$ again by basic properties of functions. The fact that T is an isometry follows since

$$\|Tf\| = \max_{y \in Y} |Tf(y)| = \max_{\psi(y) = x \in X} |f(x)| = \|f\| \tag{4.1}$$

for all $f \in A$. □

Note that a *weighted* composition operator T with $|\alpha| = 1$ is also linear, injective and an isometry, and the operator T/α is multiplicative. Observe that if ψ is a homeomorphism and $|\alpha(y)| = 1$ for all $y \in Y$, then $|Tf(y)| = |f(\psi(y))|$ for all $y \in Y$ and therefore T is an isometry as shown in equation (4.1). In [8, 10, 14, 17, 24], techniques were developed for sufficient conditions for composition and weighted composition operators.

In general, an arbitrary isometric algebraic isomorphism between function algebras need not be a composition operator. Recall the disk algebra $A(\mathbb{D})$ identified with its image under the Gelfand transformation $\Lambda : A(\mathbb{D}) \mapsto \widehat{A(\mathbb{D})} \subset C(\mathcal{M}_{A(\mathbb{D})})$. Making the usual identifications, the maximal ideal space was given by $\mathcal{M}_{A(\mathbb{D})} = \overline{\mathbb{D}}$ and the Shilov boundary was found to be $\partial A(\mathbb{D}) = \mathbb{T}$. Lemma 3.1.3 shows that the restriction map $r : A(\mathbb{D}) \to A(\mathbb{D})|_{\mathbb{T}}$ from the disk algebra to the algebra of restrictions is an isometric isomorphism. However a basic result from topology shows that there is no continuous surjection from $\mathbb{T}$ to $\overline{\mathbb{D}}$.

4.2 Norm Multiplicative Operators

Operators $T : A \to B$ between commutative Banach algebras A, B such that

$$\|TfTg\| = \|fg\|$$

for every $f, g \in A$ are called *norm-multiplicative* operators ([14]). Norm-multiplicative oper-
ators were introduced in [14] where a version of the next theorem was proved for surjective
operators between uniform algebras. The version for function algebras plays a crucial role in
the sequel. The complete proof provided here follows the arguments of [24].

Theorem 4.2.1 ([24]). *Let $A \subset C_0(X)$ and $B \subset C_0(Y)$ be dense subalgebras of function
algebras without units on $X = \partial A$ and $Y = \partial B$ with $p(A) = \delta A$ and $p(B) = \delta B$. If $T: A \to B$
is a surjection such that $\|Tf \cdot Tg\| = \|fg\|$ for all $f, g \in A$, then there is a homeomorphism
$\psi: p(B) \to p(A)$ such that*

$$\left|(Tf)(y)\right| = \left|f(\psi(y))\right| \tag{4.2}$$

for all $f \in A$ and $y \in p(B)$.

The proof of Theorem 4.2.1 requires several lemmas. For a function algebra $A \subset C_0(X)$ and
a subset $E \subset X$, define $\mathcal{F}_E(A) = \{f \in A : \|f\| = 1, |f(x)| = 1 \text{ on } E\}$.

Lemma 4.2.2. *Let $A \subset C_0(X)$ be a subalgebra of a function algebra without unit, such that
$X = \partial A$. If $h \in \mathcal{P}(A)$ and $V \subset X$ is an open set containing $E(h)$, then $\sup_{X \smallsetminus V} |h(x)| < 1$.*

Proof. By way of contradiction, suppose that $\max\{|h(x)| : x \in X \smallsetminus V\} = 1$. Consider the
Gelfand transform $\widehat{h}$. As discussed in 2.4, $\widehat{A} \subset C_0(X)$ which implies that $\widehat{h}$ is a continuous
function such that $\lim_{x \to \infty} \widehat{h}(x) = 0$. Therefore $\max_X |\widehat{h}(x)|$ is achieved on the closed set $X \smallsetminus V$,
so there exists $x_0 \in X$ such that $|h(x_0)| = \max\{|h(x)| : x \in X \smallsetminus V\} = 1$. This contradicts the
assumption that $V \supset E(h)$. $\qquad\square$

Lemma 4.2.3. *Let $A \subset C_0(X)$ be a subalgebra of a function algebra without unit, such that $X = \partial A$, and let $E \subset X$ be a non-empty p-set for A. Then $\max_{x \in E} |f(x)| = \inf_{h \in \mathcal{F}_E(A)} \|fh\|$ for any $f \in A$. In particular, if $x_0 \in p(A)$, then $|f(x_0)| = \inf_{h \in \mathcal{F}_E(A)} \|fh\|$.*

Proof. Let $f \in A$ and $\epsilon > 0$. It suffices to find $h \in \mathcal{F}_E(A)$ such that $\max_{x \in E} |f(x)| \leq \|fh\| < \max_{x \in E} |f(x)| + \epsilon$.

Consider the open set $V = \{x \in X : |f(x)| < \max_{\zeta \in E} |f(\zeta)| + \epsilon\}$. Note that $E \subset V$ and let $k \in \mathcal{P}_E$ such that $E(k) \subset V$. By lemma 4.2.2 we have $\sup_{X \setminus V} |k(x)| < 1$. Since f is bounded, there exists $n \in \mathbb{N}$ such that $h := k^n$ where $|f(x)h(x)| \leq \max_{\zeta \in E} |f(\zeta)| + \epsilon$ for all $x \in X \setminus V$. Also, since $\|h\| = 1$, $|f(x)h(x)| \leq |f(x)| < \max_{\zeta \in E} |f(\zeta)| + \epsilon$ for all $x \in V$. Therefore the inequality holds over all of X, thus taking the supremum over all $x \in X$ implies that $\|fh\| \leq \max_{\zeta \in E} |f(\zeta)| + \epsilon$. Therefore,

$$\max_{\zeta \in E} |f(\zeta)| = \max_{\zeta \in E} |f(\zeta)h(\zeta)| \leq \max_{\zeta \in X} |f(\zeta)h(\zeta)| = \|fh\| < \max_{\zeta \in E} |f(\zeta)| + \epsilon$$

which is the desired result. $\qquad\square$

Lemma 4.2.4. *Let $A \subset C_0(X)$ be a dense subalgebra of a function algebra without unit, such that $X = \partial A$ and $p(A) = \delta A$. Then every non-empty set of type $E = \bigcap_\alpha E(f_\alpha)$ where $\|f_\alpha\| = 1$ meets $p(A)$. That is, $E \cap p(A) \neq \varnothing$.*

Proof. Let $E = \bigcap_\alpha E(f_\alpha)$ where $\|f_\alpha\| = 1$ for all α. Choose $x \in E$ and let $x \in \mathcal{P}_\alpha \subset E(f_\alpha)$ for each α. Intersecting both sides over all α gives $x \in \bigcap_\alpha \mathcal{P}_\alpha \subset E$. Therefore x is an element of p-set which is properly contained in E. Since $A \subset \overline{A}$, we see $\bigcap_\alpha \mathcal{P}_\alpha$ is also a peak set for $\overline{A}$. Consequently E meets $\delta \overline{A} = \delta A = p(A)$. $\qquad\square$

Definition 4.2.5. *Suppose that X, Y are locally compact Hausdorff spaces with $A \subset C(X)$ and $B \subset C(Y)$ dense subalgebras of function algebras on $X = \partial A$ and $Y = \partial B$ with $p(A) = \delta A$ and $p(B) = \delta B$.*

- *For a set $S \subset C_0(X)$, let $|S| = \{|f| : f \in S\}$ denote the* modulus *set of S.*

- *An operator $\Phi : |A| \to |B|$ is* increasing *if and only if for all $f, g \in A$, $\Phi(|f|) \leq \Phi(|g|)$ on Y whenever $|f| \leq |g|$ on X.*

- *An operator $\Phi : |A| \to |B|$ is* sup-norm multiplicative *if $\|\Phi(|f|)\Phi(|g|)\| = \|fg\|$ for all $f, g \in A$.*

Lemma 4.2.6. *Let X, Y be locally compact Hausdorff spaces, $A \subset C_0(X), B \subset C_0(Y)$ dense subalgebras of function algebras without units such that $X = \partial A, Y = \partial B$ and $p(A) = \delta A, p(B) = \delta B$. Also suppose $\Phi : |A| \to |B|$ is an increasing bijection. If Φ is sup-norm multiplicative, then there exists a homeomorphism $\psi : p(B) \to p(A)$ with respect to which Φ is a ψ-composition operator on $p(B)$. That is, $\Phi(|f|)(y) = |f(\psi(y))|$ for all $f \in A$ and $y \in p(B)$. In particular, Φ is multiplicative.*

The technique for the proof of the lemma was developed in [22].

Lemma 4.2.7. *For any $y \in p(B)$*

$$E_y = \bigcap_{\substack{f \in A \\ \Phi(|f|) \in |\mathcal{F}_y(B)|}} E(f) \tag{4.3}$$

is nonempty and $E_y \cap p(A) \neq \varnothing$.

Proof. We first show the family $\{E(f) : f \in A, \Phi(|f|) \in |\mathcal{F}_y(B)|\}$ has the finite intersection property. Let $f_1, \ldots, f_n$ such that $\Phi(|f_j|) \in |\mathcal{F}_y(B)|$ for $j = 1, \ldots, n$. Consider the product $\Phi(|f_1|) \cdots \Phi(|f_n|)$. Since Φ is a bijection, there exists $f \in A$ such that $\Phi(|f|) = \Phi(|f_1|) \cdots \Phi(|f_n|)$. Since each $\Phi(|f_j|) \in |\mathcal{F}_y(B)|$, we see that $\Phi(|f|) \in |\mathcal{F}_y(B)|$. Therefore $\Phi(|f|) \leq \Phi(|f_j|)$ for each j, so $|f(\zeta)| \leq |f_j(\zeta)|$ for all $\zeta \in X$ since Φ is increasing.

Note that $\|f\| = |\Phi(|f|)\| = 1$. Certainly $\|\Phi(|f|)\| = 1$ since we have already seen that $\Phi(|f|) \in |\mathcal{F}_y(B)|$. Also the sup-norm multiplicativity implies that $\|f\|^2 = \|f^2\| \|\Phi(|f|)^2\| = \|\Phi(|f|)\|^2$ so

indeed $\|f\| = 1$. Similarly, it follows that $\|f_j\| = \|\Phi(|f_j|)\| = 1$ for all $j = 1, \ldots, n$.

Consequently, we may choose $x \in X$ such that $|f(x)| = 1$. The previous inequality shows that $|f_j(x)| = 1$ for all j. Therefore $E(f) \subset E(f_j)$ for all j, and taking the intersection over all j reveals $E(f) \subset \bigcap_{j=1}^n E(f_j)$. Therefore $\bigcap_{j=1}^n E(f_j) \neq \varnothing$, so the family has the finite intersection property. Since each $E(f)$ is a compact subset of the Hausdorff space X, E_y is also non-empty. Therefore $E_y \cap p(A) \neq \varnothing$ by Lemma 4.2.4. $\qquad \square$

Lemma 4.2.8. *If* $y \in p(B)$ *and* $x \in E_y \cap p(A)$, *then*

$$\Phi(|\mathcal{F}_x(A)|) \subset |\mathcal{F}_y(B)|. \tag{4.4}$$

Proof. Let $h \in \mathcal{F}_x(A)$ and $|k| = \Phi(|h|)$. It suffices to show that $|k(y)| = 1$. Let V be a neighborhood of y and $q \in \mathcal{F}_y(B)$ such that $E(q) \subset V$. If $|p| \in \Phi^{-1}(|q|)$ then $\|p\| = \|q\| = 1$, thus

$$E_y \cap p(A) \subset E_y = \bigcap_{f \in A, \Phi(|f|) \in |\mathcal{F}_y(B)|} E(f) \subset E(p)$$

since $p \in A$ such that $\Phi(|p|) \in |\mathcal{F}_y(B)|$. Therefore $x \in E(p)$ which implies $p \in \mathcal{F}_x(A)$.

Now $1 = |h(x)p(x)| = \|hp\| = \|\Phi(|h|)\Phi(|p|)\| = \|kq\| \leq \|k\|\|q\| = 1$. Therefore there exists some $y_V \in E(q) \subset V$ such that $|k(y_V)q(y_V)| = |k(y_V)||q(y_V)| = 1$, which implies $|k(y_V)| = 1$ and $|q(y_V)| = 1$. Since this holds for every neighborhood V of y, the continuity of k implies that $|k(y)| = 1$ as desired. $\qquad \square$

If A is a function algebra (or uniform algebra), then it separates points of X. Here we provide a proof that A must in fact separate the points in moduli. The arguments given here are an adapation of those found in [20].

Lemma 4.2.9. *Let* X *be a locally compact Hausdorff space. If* $A \subset C_0(X)$ *separates the points of* X, *then so does* $|A|$.

Proof. Suppose that $x, y \in X$ such that $|f(x)| = |f(y)|$ for every $f \in A$. We must show that $x = y$.

Clearly if $f(y) = 0$ then $|f(x)| = |f(y)| = 0$ which implies $f(x) = 0$.

Otherwise choose $f, g \in A$ such that $f(y) \neq 0$ and $g(y) \neq 0$. Define $h = f(y)g - g(y)f$. Note that $h \in A$ and $h(y) = 0$. It follows that $0 = |h(y)| = |h(x)|$ which implies $h(x) = 0$. Therefore $0 = f(y)g(x) - g(y)f(x)$ or $f(x)/f(y) = g(x)/g(y)$. Then for this common ratio $t \in \mathbb{T}$, we have $f(x) = tf(y)$ for every $f \in A$ such that $f(y) \neq 0$.

Consider the identification of x, y with the point evaluation functionals φ_x, φ_y respectively. Then $\varphi_x = t\varphi_y$ on A. In particular, for $f \in A$ such that $f(y) \neq 0$, we can write $\varphi_x(f) = t\varphi_y(f)$ and $\varphi_x(f^2) = t\varphi_y(f^2)$ since $f^2 \in A$. Thus $t = \dfrac{\varphi_x(f)}{\varphi_y(f)}$ and $t = \left(\dfrac{\varphi_x(f)}{\varphi_y(f)}\right)^2$ simultaneously which implies $t = 1$. Therefore $\varphi_x = \varphi_y$ on A which states that $f(x) = f(y)$ for all $f \in A$, and consequently $x = y$ since A separates the points of X. $\qquad\square$

Lemma 4.2.10. *For any $y \in p(B)$, the set $E_y \cap p(A)$ is a singleton.*

Proof. By way of contradiction, suppose $x \in E_y \cap p(A)$ and $z \in (E_y \cap p(A)) \smallsetminus \{x\}$. Since A separates the points of $p(A)$, Lemma 4.2.9 provides the existence of $h \in A$ such that $|h(x)| \neq |h(z)|$. By an appropriate scaling, we can assume without loss of generality that $h \in \mathcal{F}_x(A)$ with $|h(z)| < |h(x)| = 1$. Then $\Phi(|h|) \in \mathcal{F}_y(B)$ so as in the proof of Lemma 4.2.8, we have $E_y \cap p(A) \subset E(h)$. Therefore $z \in E_y \cap p(A)$, a contradiction since then $z \in E(h)$ which implies $|h(z)| = 1$. $\qquad\square$

Therefore for each $y \in p(B)$ we have the well-defined assignment $\psi : p(B) \to p(A)$ by $\{\psi(y)\} = \{E_y \cap p(A)\}$. We can rewrite 4.4 as

$$\Phi(|\mathcal{F}_{\psi(y)}(A)|) \subset |\mathcal{F}_y(B)|. \tag{4.5}$$

Then for any $h \in \mathcal{F}_{\psi(y)}(A)$, we have $\Phi(|h|) \in |\mathcal{F}_y(B)|$ and it follows that

$$|h(\psi(y))| = 1 = |\Phi(h)(y)|. \tag{4.6}$$

It fact, we have that each $k \in |\mathcal{F}_y(B)|$ is in the image of $h \in |\mathcal{F}_{\psi(y)}(A)|$ under Φ.

Corollary 4.2.11. *For any $y \in p(B)$, $\Phi^{-1}(|\mathcal{F}_y(B)|) = |\mathcal{F}_{\psi(y)}(A)|$.*

Proof. First note that (4.5) implies that $|\mathcal{F}_{\psi(y)}(A)| \subset \Phi^{-1}(|\mathcal{F}_y(B)|)$.

Let $y \in p(B)$, $k \in \mathcal{F}_{\psi(y)}(A)$ and suppose $|h| \in \Phi^{-1}(|k|)$. Then it remains to show $h \in \mathcal{F}_{\psi(y)}(A)$. Since $\Phi(|h|) = |k| \in |\mathcal{F}_y(B)|$, (4.3) implies that $\{E_y \cap p(B)\} = \{\psi(y)\} \subset E(h)$. This implies $|h(\psi(y))| = 1 = |k(y)|$ and therefore $\|h\| = 1$, thus $h \in \mathcal{F}_{\psi(y)}(A)$ and the other inclusion is proven. $\qquad\square$

We are now ready to prove Lemma 4.2.6.

Proof of lemma 4.2.6. Let $f \in A$ and $y \in p(B)$. Then lemma 4.2.3, corollary 4.2.11, and the norm-multiplicativity of Φ imply

$$|\Phi(|f|)(y)| = \inf_{k \in \mathcal{F}_y(B)} \|\Phi(|f|) \cdot |k|\| \tag{4.7}$$

$$= \inf_{h \in \mathcal{F}_{\psi(y)}(A)} \|\Phi(|f|)\Phi(|h|)\| \tag{4.8}$$

$$= \inf_{h \in \mathcal{F}_{\psi(y)}(A)} \|fh\| \tag{4.9}$$

$$= |f(\psi(y))|. \tag{4.10}$$

Therefore Φ is a ψ-composition operator.

It remains to show that $\psi : p(B) \to p(A)$ is a homeomorphism. Let $y \in p(B)$ and some constant $0 < c < 1$. Let U be an open neighborhood of $\psi(y)$ in X. Then there exists a peaking function

$h \in \mathcal{P}_{\psi(y)}(A)$ such that $E(h) \subset U$ and $|h(x)| < c$ for all $x \in X \smallsetminus U$. Take $|k| = \Phi(|h|) \in |\mathcal{F}_y(B)|$ and therefore $|k(y)| = |h(\psi(y))| = 1 > c$ by (4.6).

Consider the open set $W = \{\eta \in p(B) : k(\eta) > c\}$ containing y. Then $c < |k(\eta)| = |h(\psi(\eta))|$ for all $\eta \in W$. Since $h(\zeta) < c$ for all $\zeta \in X \smallsetminus U$, it follows that $\psi(\eta) \in U$ for all $\eta \in W$. Thus $\psi(W) \subset U$ which shows that ψ is continuous.

Recall that $\Phi : |A| \to |B|$ is bijective and consider the inverse $\Phi^{-1} : |B| \to |A|$. Then Φ^{-1} is a surjective map that satisfies all the hypothesis of lemma 4.2.6. Following the established symmetric arguments, there exists a continuous map $\phi : p(A) \to p(B)$ such that $\Phi^{-1}(|g|)(x) = |g(\phi(x))|$ for all $g \in B$ and $x \in p(A)$. Since $\Phi^{-1}(|\mathcal{F}_y(B)|) = |\mathcal{F}_{\psi(y)}(A)|$, for each $\psi(y) = x$ we have

$$\{\phi(x)\} = E_{\psi(y)} \cap p(B) \tag{4.11}$$

$$= \Big[\bigcap_{\substack{g \in B \\ \Phi^{-1}(|g|) \in |\mathcal{F}_{\psi(y)}(A)|}} E(g) \Big] \cap p(B) \tag{4.12}$$

$$= \Big[\bigcap_{\substack{g \in B \\ |g| \in \Phi(|\mathcal{F}_{(y)}(A)|)}} E(g) \Big] \cap p(B) \tag{4.13}$$

$$= \{y\} \tag{4.14}$$

Therefore $\phi(\psi(y)) = y$ and similarly $\psi(\psi(x)) = x$, thus $\phi = \psi^{-1}$ and ψ is bijective. Hence ψ is a homeomorphism as claimed.

Also $\Phi(|fg|)(y) = |fg(\psi(y))| = |f(\psi(y))||g(\psi(y))| = (\Phi(|f|)\Phi(|g|))(y)$ for all $y \in p(B)$ so it is multiplicative. $\qquad \square$

We now have all of the tools needed to prove the main result of this section, that a norm multiplicative map T between dense subalgebras of function algebras is a composition operator

in modulus.

Proof of Theorem 4.2.1. Let $f, g \in A$ such that $|Tf| < |Tg|$ on Y. Then by the sup-norm multiplicativity of T, for any $h \in A$ we have $\|fh\| = \|TfTh\| \leq \|TgTh\| = \|gh\|$. Then for any p-point x_0, taking the infimum over all $h \in \mathcal{F}_{x_0}$ and applying Lemma 4.2.3 shows $|f(x_0)| \leq |g(x_0)|$. Therefore $|f| \leq |g|$ on $p(A) = X$.

Now take any $k \in B$ and $h \in T^{-1}(k)$. By the norm multiplicativity of T and since $|f| \leq |g|$, we have $\|Tf \cdot k\| = \|TfTh\| = \|fh\| \leq \|gh\| = \|TgTh\| = \|Tg \cdot k\|$. Again, applying Lemma 4.2.3 implies $|Tf| \leq |Tg|$ on $p(B) = Y$. Therefore $|f| \leq |g|$ on X if and only if $|Tf| \leq |Tg|$ on Y. Note that if $|f| = |g|$ on X, then $|Tf| \leq |Tg|$ on Y but also $\|f\| = \|g\|$ which implies $\|Tf\| = \|Tg\|$. This shows that $|Tf| = |Tg|$ on Y. Conversely we also have $|Tf| = |Tg|$ on Y which implies $|f| = |g|$ on X. Therefore $|f| = |g|$ on X if and only if $|Tf| = |Tg|$ on Y.

Define the map $\Phi : |A| \to |B|$ by $\Phi(|f|) = |Tf|$. The computations above clearly show that Φ is a well-defined, increasing bijection between $|A|$ and $|B|$. Therefore by Lemma 4.2.6, there exists a homeomorphism $\psi : Y \to X$ such that $\Phi(|f|)(y) = |f(\psi(y))|$ for all $y \in Y$. Thus $|Tf(y)| = |f(\psi(y))|$ for all $y \in Y$ and we have shown that T is a ψ-composition operator on $p(B)$ in modulus. $\square$

Chapter 5

Almost Peripherally-Multiplicative Operators between Function Algebras

Since function algebras have algebraic and topological structures, one can impose various conditions on maps T between function algebras A and B to be structure preserving maps. In the spirit of the Mazur-Ulam theorem, this chapter introduces topological constraints on the mappings $T : A \to B$ between function algebras from which it follows that T is a structure preserving map . Specifically, we find spectral conditions which imply that T is a composition, or weighted composition operator.

5.1   Terminology for Maps between Function Algebras

Definition 5.1.1. *Let A, B be function algebras on locally compact spaces X, Y respectively, and suppose $T : A \to B$ is a mapping.*

- *If $\sigma_\pi(TfTg) = \sigma_\pi(fg)$ for all $f, g \in A$, then T is said to be a* peripherally-multiplicative *operator.* [17]

- *If $\sigma_\pi(TfTg) \subset \sigma_\pi(fg)$ for all $f, g \in A$ or $\sigma_\pi(fg) \subset \sigma_\pi(TfTg)$ for all $f, g \in A$, then T is said to be an* almost peripherally-multiplicative *operator.*

- *If $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, then T is said to be a* weakly peripherally-multiplicative *operator.* [14]

If T is a weighted composition operator, i.e. $Tf = \alpha(f \circ \psi)$, such that $\alpha : Y \to \{\pm 1\}$, then T is a peripherally-multiplicative map and thus a weakly peripherally-multiplicative map. Note that a norm-multiplicative operator T only provides information on the *moduli* of the elements of the algebra T is acting on, whereas the peripheral conditions above also provide rotational information regarding the surjective map T.

If T is peripherally-multiplicative map between function algebras on metric spaces X, Y respectively, then T is a weighted composition operators (see Corollary 6.2.5). In general it is not known if weakly peripherally-multiplicative maps are composition operators, but for composition operators, the following converse is always true.

Proposition 5.1.2. *Suppose $T : A \to B$ is a map between function algebras A, B on their locally compact and Hausdorff maximal ideal spaces $\mathcal{M}_A = X$ and $\mathcal{M}_B = Y$ respectively. If there exists a homeomorphism $\psi : Y \to X$ such that $Tf(y) = f(\psi(y))$ for all $y \in Y$, then T is a peripherally-multiplicative map. In particular, T is almost peripherally multiplicative and weak peripherally-multiplicative.*

Proof. For notational convenience, identify element $f \in A$ with $\widehat{f}$, the Gelfand representation of f.

Suppose $z \in \sigma_\pi(Tf \cdot Tg)$. Since $\sigma(Tf \cdot Tg) = \widehat{Tf \cdot Tg}(\mathcal{M}_B)$ there exists $y_0 \in Y$ such that

$Tf(y_0)Tg(y_0) = z$, where

$$|(Tf \cdot Tg)(y)| \leq |z| \tag{5.1}$$

for every $y \in Y$. Since T is a composition operator, it follows that

$z = f(\psi(y_0))g(\psi(y_0)) = (f \cdot g)\psi((y_0)) \in \widehat{f \cdot g}(\mathcal{M}_A) = \sigma(fg)$, where

$$|(f \cdot g)(\psi(y))| \leq |z| \tag{5.2}$$

for every $\psi(y) \in X$, thus every $x \in X$ since ψ surjective. Therefore $z \in \sigma_\pi(f \cdot g)$.

On the other hand, if $z \in \sigma_\pi(f \cdot g)$ then there is a $x = \psi(y) \in X$ such that $(f \cdot g)(\psi(y)) = z$ satisfying (5.2) and also (5.1). The inequalities also hold on X, Y since ψ^{-1} is surjective, which implies $z \in \sigma_\pi(Tf \cdot Tg)$. This shows that both inclusions hold.

Hence $\sigma_\pi(TfTg) = \sigma_\pi(fg)$, i.e. T is peripherally multiplicative, which clearly implies that $\sigma_\pi(TfTg) \subset \sigma_\pi(fg)$ and $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \emptyset$.

$\square$

Example 5.1.1. Let $A(D)$ be as defined in Example 3.1.1. Define a map $T : A(D) \to A(D)$ by

$$Tf = \begin{cases} f_1 & : \text{ on } \mathbb{D}_1 \\ -f_2 & : \text{ on } \mathbb{D}_2 \end{cases}$$

Clearly T is surjective, and for any $f, g \in A(D)$, $TfTg|_{\mathbb{D}_1} = f_1 g_1$ and $TfTg|_{\mathbb{D}_2} = f_2 g_2$. Therefore $\sigma_\pi(TfTg) = \sigma_\pi(fg)$ for all $f, g \in A(D)$. Therefore T is a peripherally multiplicative map, thus almost peripherally-multiplicative and weakly peripherally-multiplicative, which is a weighted composition operator but not a composition operator.

Lemma 5.1.3. *Let $T : A \to B$ be an operator between two function algebras. If T is*

peripherally-multiplicative, almost peripherally-multiplicative, or weakly peripherally-multiplicative,
then T is norm-multiplicative.

The proof of the proposition follows from the observation that for any $\lambda \in \sigma_\pi(fg) \cap \sigma_\pi(TfTg)$, we have $\|fg\| = |\lambda| = \|TfTg\|$ from the definition of the peripheral spectrum.

For uniform algebras $A \subset C(X)$ and $B \subset C(Y)$ on compact Hausdorff spaces X and Y respectively, it has been shown in [17] that a necessary and sufficient condition for a surjective unital operator $T \colon A \to B$ to be a composition operator on δB is for T to be peripherally-multiplicative.

5.2 Almost Peripherally-Multiplicative Conditions

In [14], it is shown if $T \colon A \to B$ is an almost peripherally-multiplicative surjective mapping between two uniform algebras such that

$$\sigma_\pi(Tf) = \sigma_\pi(f) \tag{5.3}$$

for every $f \in A$, then T is a composition operator on δB.

The following two theorems show that almost peripherally-multiplicative maps between arbitrary function algebras are weighted composition operators, without assuming condition (5.3).

Theorem 5.2.1. (A) *(J,T)* [12] *Let $A \subset C(X)$ be a function algebra and $B \subset C(Y)$ a dense subalgebra of a function algebra, not necessarily with units, such that $p(B) = \delta B$, and where X and Y are locally compact Hausdorff spaces. If $T \colon A \to B$ is a surjection such that*

$$\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg) \tag{5.4}$$

for all $f, g \in A$, then there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α

on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)\, f(\psi(y))$$

for every $y \in \delta B$.

Proof. First we show that $(Tf)(y)^2 = f(\psi(y))^2$ for every $f \in A$ and $y \in \delta B$. Let $f \in A$ and $y_0 \in \delta B$. Equality (5.4) and the observation in Lemma 5.1.3 implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi: \delta B \to \delta A$ be the homeomorphism from Theorem 4.2.1, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $f(\psi(y_0)) = 0$.

Suppose $f(\psi(y_0)) \neq 0$. If $V \subset \delta B$ is an arbitrary open neighborhood of y_0 in δA, then, clearly, $U = \psi(V)$ is an open neighborhood of $\psi(y_0)$. By Lemma 3.3.10 there exists a peaking function $h \in \mathcal{P}_{\psi(y_0)}(A)$ with $\sigma_\pi(fh) = \{f(\psi(y_0))\}$ such that $E(fh) = E(h) \subset U$. Denote $k = Th$. Note that $\sigma_\pi(Tf \cdot k) = \{f(\psi(y_0))\}$ since, by (5.4), $\sigma_\pi(Tf \cdot k) \subset \sigma_\pi(fh) = \{f(\psi(y_0))\}$. Therefore, there is a point $y_1 \in \delta B$ so that $(Tf \cdot k)(y_1) = f(\psi(y_0))$, i.e. $(Tf)(y_1)\, k(y_1) = f(\psi(y_0))$. Since T is a composition operator in modulus,

$$|fh(\psi(y_1))| = |f(\psi(y_1))||h(\psi(y_1))| = |(Tf)(y_1)||(Th)(y_1)| = |f(\psi(y_0))| \tag{5.5}$$

and $\sigma_\pi(fh) = \{f(\psi(y_0))\}$, we deduce that the function fh attains the maximum of its modulus at $\psi(y_1)$. Hence $\psi(y_1) \in E(fh) = E(h) \subset U$, which implies $\psi(y_1) \in U$, and therefore, $y_1 \in \psi^{-1}(U) = V$. Thus

$$(Tf)(y_1)^2\, k(y_1)^2 = f(\psi(y_0))^2. \tag{5.6}$$

for some $Th = k \in B$. Then (5.4) implies that $\sigma_\pi(k^2) = \sigma_\pi((Th)^2) \subset \sigma_\pi(h^2) = \{1\}$, and therefore, $\sigma_\pi(k^2) = \{1\}$. Also, $|k^2(y_1)| = |Th^2(y_1)| = |h(\psi(y_1))^2| = 1$ since $y_1 \in E(h)$. It follows that $k^2(y_1) = 1$, and therefore (5.6) becomes

$$(Tf)(y_1)^2 = f(\psi(y_0))^2. \tag{5.7}$$

Since V was an arbitrary neighborhood of y_0, the continuity of Tf, f, ψ imply that (5.7) in fact becomes $(Tf)(y_0)^2 = f(\psi(y_0))^2$ as desired.

Thus there exists a number $\alpha_f(y_0) = \pm 1$, possibly depending on f, such that

$$(Tf)(y_0) = \alpha_f(y_0) f(\psi(y_0)). \tag{5.8}$$

We claim that the number $\alpha_f(y_0)$ does not depend on $f \in A$. First we show that $\alpha_h(y_0)$ has the same value for all peaking functions h in $\mathcal{P}_{\psi(y_0)}(A)$. Indeed, if $h_1, h_2 \in \mathcal{P}_{\psi(y_0)}(A)$, then, by (5.4), $\sigma_\pi(Th_1 \cdot Th_2) \subset \sigma_\pi(h_1 h_2) = \{1\}$, and therefore, $\sigma_\pi(Th_1 \cdot Th_2) = \{1\}$. Since $|(Th_1)(y_0)(Th_2)(y_0)| = |h_1(\psi(y_0))||h_2(\psi(y_0))| = 1$, the function $Th_1 \cdot Th_2$ attains its maximum modulus at y_0. Hence $(Th_1)(y_0)(Th_2)(y_0) \in \sigma_\pi(Th_1 \cdot Th_2) = \{1\}$, and therefore, $(Th_1)(y_0)(Th_2)(y_0) = 1$. Consequently, the numbers $\alpha_{h_i}(y_0) = \alpha_{h_i}(y_0) h_i(\psi(y_0)) = (Th_i)(y_0)$, $i = 1, 2$, have the same sign, thus $\alpha_{h_1}(y_0) = \alpha_{h_2}(y_0)$.

By Lemma 3.3.10 there is an $h \in \mathcal{P}_{\psi(y_0)}(A)$ such that $\sigma_\pi(fh) = \{f(\psi(y_0))\}$. Since, by (5.4), $\sigma_\pi(Tf \cdot Th) \subset \sigma_\pi(fh) = \{f(\psi(y_0))\}$, we have $\sigma_\pi(Tf \cdot Th) = \{f(\psi(y_0))\}$. Hence $|(Tf)(y_0)(Th)(y_0)| = |(Tf)(y_0)||(Th)(y_0)| = |f(\psi(y_0))||h(\psi(y_0))| = |f(\psi(y_0))|$. Consequently, the function $Tf \cdot Th$ attains the maximum of its modulus at y_0, so we must have $(Tf)(y_0)(Th)(y_0) \in \sigma_\pi(Tf \cdot Th) = \{f(\psi(y_0))\}$, thus $(Tf)(y_0)(Th)(y_0) = f(\psi(y_0))$. Therefore,

$$\alpha_f(y_0) \alpha_h(y_0) = \frac{(Tf)(y_0)}{f(\psi(y_0))} \frac{(Th)(y_0)}{h(\psi(y_0))} = \frac{1}{h(\psi(y_0))} = 1.$$

Hence $\alpha_f(y_0) = \alpha_h(y_0)$, thus the number $\alpha_f(y_0)$ has the same value for all $f \in A$ with $f(\psi(y_0)) \neq 0$. Consequently, the function $\alpha(y) = \alpha_f(y)$, $y \in \delta B$, $f \in A$, $f(\psi(y)) \neq 0$, is well defined, and $\alpha^2 = \alpha_h^2 = 1$. Finally we observe (5.8) becomes $(Tf)(y_0) = \alpha(y_0) f(\psi(y_0))$, as desired.

To show that α is continuous at any $y \in \delta B$, let $f \in A$ with $f(\psi(y)) \neq 0$ and let $V \subset \delta B$ be a neighborhood of y such that $f \circ \psi \neq 0$ on V. Since Tf, f and ψ are continuous on V, so is the

function $\alpha = Tf/(f \circ \psi)$. In particular, α is continuous at $y \in V$. $\square$

The theorem holds under the symmetric almost peripherally-multiplicative condition. The proof is analogous to that of Theorem 5.2.1(A).

Theorem. 5.2.1 (B) *(J,T) Let $A \subset C(X)$ be a dense subalgebra of a function algebra with $\delta A = p(A)$, and $B \subset C(Y)$ be a function algebra, not necessarily with units, where X and Y are locally compact Hausdorff spaces. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg) \tag{5.9}$$

for all $f, g \in A$, then there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y) f(\psi(y))$$

for every $y \in \delta B$.

Proof. As before we show first that $(Tf)(y)^2 = f(\psi(y))^2$ for every $f \in A$ and $y \in \delta B$. Let $f \in A$ and $y_0 \in \delta B$. The equality (5.9) implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$, and therefore, Theorem 4.2.1 applies. Let $\psi: \delta B \to \delta A$ be the homeomorphism from Theorem 4.2.1, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Suppose $(Tf)(y_0) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0. By Lemma 3.3.10 there exists a peaking function $k \in \mathcal{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$ and $E(Tf \cdot k) = E(k) \subset V$. Hence for every $h \in T^{-1}(k)$ we have $\sigma_\pi(fh) \subset \sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$, i.e. $\sigma_\pi(fh) = \{(Tf)(y_0)\}$. Therefore, there is a point $x_1 \in \delta A$ so that $(fh)(x_1) = (Tf \cdot k)(y_0) = (Tf)(y_0)$. The surjectivity of ψ implies that there is an $y_1 \in \delta B$ so that $x_1 = \psi(y_1)$. Hence

$f(\psi(y_1)) h(\psi(y_1)) = (Tf)(y_0)$, and squaring both sides yields

$$(f(\psi(y_1)))^2 (h(\psi(y_1)))^2 = ((Tf)(y_0))^2. \tag{5.10}$$

Since $\sigma_\pi(fh) = \{(Tf)(y_0)\}$ and

$$|Tf \cdot k(y_1)| = |(Tf)(y_1)||k(y_1)| = |f(\psi(y_1))||h(\psi(y_1))| = |(Tf)(y_0)|$$

by (4.2), the function $Tf \cdot k$ attains the maximum of its modulus at y_1. Consequently, $y_1 \in E(Tf \cdot k) = E(k) \subset V$. Then $|h(\psi(y_1))^2| = |k(y_1)^2| = 1$ so $\psi(y_1) \in E(h^2)$. Also the condition (5.9) implies that $\sigma_\pi(h^2) \subset \sigma_\pi(k^2) = \{1\}$. Therefore $h(\psi(y_1))^2 = 1$ and (5.10) becomes

$$f(\psi(y_1))^2 = (Tf)(y_0)^2 \tag{5.11}$$

Since V is an arbitrary neighborhood of y_0, the continuity of f, ψ and h imply $f(\psi(y_0))^2 = (Tf)(y_0)^2$ as claimed.

Consequently, there is a number $\alpha_f(y_0) = \pm 1$, possibly dependent on f, such that

$$(Tf)(y_0) = \alpha_f(y_0) f(\psi(y_0)). \tag{5.12}$$

We claim that $\alpha_f(y_0)$ does not depend on $f \in A$. First we show that $\alpha_h(y_0)$ has the same value for any $h \in T^{-1}(k)$ such that $k \in \mathcal{P}_{y_0}(B)$. If $k_1, k_2 \in \mathcal{P}_{y_0}(B)$ and $h_i \in T^{-1}(k_i)$, $i = 1, 2$, then $\sigma_\pi(h_1 h_2) \subset \sigma_\pi(Th_1 \cdot Th_2) = \sigma_\pi(k_1 k_2) = \{1\}$, thus $\sigma_\pi(h_1 h_2) = \{1\}$. Since $|h_1(\psi(y_0))||h_2(\psi(y_0))| = |(Th_1)(y_0)(Th_2)(y_0)| = 1$ it follows that $h_1(\psi(y_0)) h_2(\psi(y_0)) \in \sigma_\pi(h_1 h_2) = \{1\}$, hence $h_1(\psi(y_0)) h_2(\psi(y_0)) = 1$. By (5.12), $\alpha_{h_i}(y_0) h_i(\psi(y_0)) = (Th_i)(y_0) = k_i(y_0) = 1$. Consequently, the numbers $\alpha_{h_i}(y_0) = 1/h_i(\psi(y_0))$, $i = 1, 2$, have the same sign and therefore, $\alpha_{h_1}(y) = \alpha_{h_2}(y_0)$.

Now let $f \in A$ be arbitrary. According to Lemma 3.3.10 there exists a $k \in \mathcal{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{Tf(y_0)\}$. Let $h \in T^{-1}(k)$. Equality (5.9) implies that $\sigma_\pi(fh) \subset \sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$, hence $\sigma_\pi(fh) = \{(Tf)(y_0)\}$. Therefore,

$$|f(\psi(y_0))\, h(\psi(y_0))| = |f(\psi(y_0))||h(\psi(y_0))| = |(Tf)(y_0)\,(Th)(y_0)| = |(Tf)(y_0)|.$$

It follows that the function fh attains the maximum of its modulus at $\psi(y)$, so we must have $f(\psi(y_0))\, h(\psi(y_0)) \in \sigma_\pi(fh)$, thus, $f(\psi(y_0))\, h(\psi(y_0)) = (Tf)(y_0)$. Therefore,

$$\alpha_f(y_0)\, \alpha_h(y_0) = \frac{(Tf)(y_0)}{f(\psi(y_0))} \frac{(Th)(y_0)}{h(\psi(y_0))} = (Th)(y_0) = k(y_0) = 1.$$

Hence $\alpha_f(y_0) = \alpha_h(y_0)$, thus the number $\alpha_f(y_0)$ has the same value for all $f \in A$ with $(Tf)(y_0) \neq 0$.

Consequently, the function $\alpha(y) = \alpha_f(y)$, $y \in \delta B$, $f \in A$, $(Tf)(y) \neq 0$, is well defined. Now (5.12) becomes $(Tf)(y_0) = \alpha(y_0)\, f(\psi(y_0))$. The proof completes as in Theorem 5.2.1(A). $\square$

If both A and B are function algebras, the previous two theorems can be combined as follows.

Theorem 5.2.2. *Let X and Y be a locally compact Hausdorff spaces and $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, not necessarily with unit, such that $X = \partial A$ and $Y = \partial B$. If $T: A \to B$ is an almost peripheral-multiplicative surjection, i.e.*

(a) $\sigma_\pi(Tf \cdot Tg) \subset \sigma_\pi(fg)$ for all $f, g \in A$, or,

(b) $\sigma_\pi(fg) \subset \sigma_\pi(Tf \cdot Tg)$ for all $f, g \in A$,

then T is a weighted composition operator on δB. That is, there is a homeomorphism $\psi: \delta B \to$

δA and a function α on δB with $\alpha^2 = 1$ so that

$$(Tf)(y) = \alpha(y) \, f(\psi(y))$$

for all $f \in A$ and $y \in \delta B$. In particular, T/α is linear and multiplicative operator, i.e. an algebra isomorphism.

If the map $T : A \to B$ in Theorem 5.2.1 is bijective, where both $A \subset C(X)$ and $B \subset C(Y)$ are function algebras, then one can consider the surjective map $T^{-1} : B \to A$ which has the property that $\sigma_\pi(fg) \subset \sigma_\pi(TfTg)$ for all $f, g \in B$. Thus Theorem 5.2.1(B) applies to T^{-1}, and we see that in this case Theorem 5.2.1(B) follows from Theorem 5.2.1(A).

Since $\alpha : \delta B \to \mathbb{C}$ is continuous such that $\alpha^2 = 1$, we see that α naturally separates δB into the open and closed components $S_1 = \alpha^{-1}(1)$ and $S_{-1} = \alpha^{-1}(-1)$ of δB. Therefore the map T takes the form $Tf(y) = f(\psi(y))$ on S_1 and $Tf(y) = -f(\psi(y))$ on S_{-1}.

Recall Example 5.1.1, where $T : A(D) \to A(D)$ defined by

$$Tf = \begin{cases} f_1 & : \text{ on } \mathbb{D}_1 \\ -f_2 & : \text{ on } \mathbb{D}_2 \end{cases}$$

is a peripherally-multiplicative map between the algebra of functions $A(D)$. Here we see that $\delta B = \mathbb{T}_1 \sqcup \mathbb{T}_2$ where δB separates into the open and closed components $S_1 = \alpha^{-1}(1) = \mathbb{T}_1$ and $S_{-1} = \alpha^{-1}(-1) = \mathbb{T}_2$.

The following corollary illustrates a sufficient condition for $T : A \to B$ to be a composition operator.

Corollary 5.2.3. *Let X and Y be a locally compact Hausdorff spaces and $A \subset C(X)$ and $B \subset C(Y)$ be function algebras, not necessarily with unit, such that $X = \partial A$ and $Y = \partial B$. If $T : A \to B$ is an almost peripheral-multiplicative surjection, and $d(\sigma_\pi(f), \sigma_\pi(Tf)) < 2$ for all*

$f \in A$, then T is a composition operator.

Proof. From Theorem 5.2.2, there exists a homeomorphism $\psi : \delta B \to \delta A$ and $\alpha : \delta B \to \mathbb{C}$ with $\alpha^2 = 1$ such that $Tf(y) = \alpha(y)f(\psi(y))$ for every $f \in A$ and $y \in \delta B$. It suffices to show that $\alpha = 1$ on δB.

Case 1: $\sigma_\pi(TfTg) \subset \sigma_\pi(fg)$ for all $f, g \in A$

Let $f \in A$ and fix $y_0 \in \delta B$. Let $h \in \mathcal{P}_{\psi(y_0)}(A)$ and let $k \in T(h)$. Then $\sigma_\pi(h) = \{1\}$ and $d(\sigma_\pi(h), \sigma_\pi(k)) < 2$ imply that

$$|k(y) - 1| < 2 \tag{5.13}$$

for all $y \in E(k)$. However, the almost weak peripheral-multiplicativity of T implies

$$\sigma_\pi(k^2) = \sigma_\pi(Th^2) \subset \sigma_\pi(h^2) = \{1\}.$$

Therefore $\sigma_\pi(k) = \{\pm 1\}$ and together with (5.13) implies $\sigma_\pi(k) = \{1\}$. But $|k(y_0)| = |Th(y_0)| = |h(\psi(y_0))| = 1$ which implies $y_0 \in E(k)$ and thus $k(y_0) = 1$. Therefore

$$1 = k(y_0) = Th(y_0) = \alpha(y_0)h(\psi(y_0)) = \alpha(y_0)$$

and since y_0 was arbitrary in δB we may conclude that $\alpha = 1$ on δB, thus $Tf(y) = f(\psi(y))$ for all $f \in A$ and $y \in \delta B$.

Case 2: $\sigma_\pi(fg) \subset \sigma_\pi(TfTg)$ for all $f, g \in A$

Let $f \in A$ and fix $y_0 \in \delta B$. Let $k \in \mathcal{P}_{y_0}(B)$ and let $h \in A$ such that $Th = k$. Then $\sigma_\pi(k) = \{1\}$ and $d(\sigma_\pi(h), \sigma_\pi(k)) < 2$ imply that

$$|h(x) - 1| < 2 \tag{5.14}$$

for all $x \in E(h)$. However, the almost weak peripheral-multiplicativity of T implies

$$\sigma_\pi(h^2) \subset \sigma_\pi(Th^2) = \sigma_\pi(k^2) = \{1\}.$$

Therefore $\sigma_\pi(h) = \{\pm 1\}$ and together with (5.14) implies $\sigma_\pi(h) = \{1\}$. But $|h(\psi(y_0))| = |Th(y_0)| = |k(y_0)| = 1$ which implies $\psi(y_0) \in E(h)$ and thus $h(\psi(y_0)) = 1$. Therefore $1 = h(\psi(y_0))$, thus

$$1 = k(y_0) = Th(y_0) = \alpha(y_0)h(\psi(y_0)) = \alpha(y_0)$$

and again since y_0 was arbitrary in δB we may conclude that $\alpha = 1$ on δB, thus $Tf(y) = f(\psi(y))$ for all $f \in A$ and $y \in \delta B$. $\qquad \square$

Chapter 6

Weakly Peripherally-Multiplicative Mappings between Function Algebras

In this chapter we investigate the sufficient conditions for weakly peripherally-multiplicative mappings between function algebras to be weighted composition operators. In the case of uniform algebras, it has been shown that if, in addition to the weak peripheral-multiplicativity, T preserves the peripheral spectra of all algebra elements, then T is necessarily a composition operator [14, Proposition 2]. Namely

Proposition 6.0.4 ([14])**.** *If a weakly peripherally-multiplicative surjective map $T : A \to B$ between uniform algebras preserves the peripheral spectra of algebra elements, i.e.*

$$\sigma_\pi(Tf) = \sigma_\pi(f) \tag{6.1}$$

for all $f \in A$, then it is a composition operator on δB, i.e. an isometric algebra isomorphism.

6.1 Basic Results

The following two theorems expand this result to function algebras and weaken condition (6.1).

Theorem 6.1.1. (A)*(J,T) Let X be a locally compact Hausdorff space where $A \subset C(X)$ is a dense subalgebra of a function algebra, not necessarily with unit, such that $X = \partial A$ and $p(A) = \delta A$. If $T : A \to B$ is a surjection onto a function algebra $B \subset C(Y)$ such that*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing \text{ for all } f, g \in A \tag{6.2}$$

and

$$\sigma_\pi(f) \subset \sigma_\pi(Tf) \text{ for all } f \in A, \tag{6.3}$$

then T is a bijective ψ-composition operator on δB with respect to a homeomorphism $\psi : \delta B \to \delta A$. That is,

$$(Tf)(y) = f(\psi(y))$$

for all $f \in A$ and $y \in \delta B$. In particular, A is necessarily a function algebra and T is an algebra isomorphism.

Proof. Let $y_0 \in p(B) = \delta B$. Condition (6.2) implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi : \delta B \to \delta A$ be the homeomorphism from Theorem 4.2.1, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Let $(Tf)(y_0) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0.

According to Lemma 3.3.10, there exists a peaking function $k \in \mathcal{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$ and $E(Tf \cdot k) = E(k) \subset V$. Note that if $h \in T^{-1}(k)$ then $(Tf)(y_0) \in \sigma_\pi(fh)$ since, by (a), $\sigma_\pi(Tf \cdot k) \cap \sigma_\pi(fh) \neq \varnothing$. Therefore, there is a point $x_1 \in \delta A$ so that $(Tf \cdot k)(y_0) =$

$(fh)(x_1)$. Since ψ is surjective, there is an $y_1 \in \delta B$ so that $x_1 = \psi(y_1)$. Hence

$$(Tf)(y_0) = (Tf)(y_0) k(y_0) = f(\psi(y_1)) h(\psi(y_1)). \tag{6.4}$$

Since T is a composition operator in modulus,

$$|(Tf)(y_0)| = |(Tf)(y_0)\|k(y_0)| = |f(\psi(y_1))\|h(\psi(y_1))| = |(Tf)(y_1)\|k(y_1)| = |Tf \cdot k(y_1)|.$$

Hence $y_1 \in E(Tf \cdot k) = E(k) \subset V$. Therefore, $|h(\psi(y_1))| = |k(y_1)| = 1$. Condition (a) implies that $\sigma_\pi(h) \subset \sigma_\pi(k) = \{1\}$, thus $h(\psi(y_1)) \in \sigma_\pi(h)$, hence $h(\psi(y_1)) = 1$. Now the equality (6.4) becomes $(Tf)(y_0) = f(\psi(y_1))$. Since V was an arbitrary neighborhood of y_0, the continuity of f and ψ yield $(Tf)(y_0) = f(\psi(y_0))$ as desired. $\qquad\square$

Theorem. 6.0.5 (B) *(J,T) Let X be a locally compact Hausdorff space where $A \subset C(X)$ is a function algebra, not necessarily with unit such that $X = \partial A$, and B is a dense subalgebra of a function algebra $B \subset C(Y)$ such that $p(B) = \delta(B)$. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing \ \text{for all}\ f, g \in A \tag{6.5}$$

and

$$\sigma_\pi(Tf) \subset \sigma_\pi(f) \ \text{for all}\ f \in A, \tag{6.6}$$

then T is a bijective ψ-composition operator on δB with respect to a homeomorphism $\psi: \delta B \to \delta A$. That is,

$$(Tf)(y) = f(\psi(y))$$

for all $f \in A$ and $y \in \delta B$. In particular, B is necessarily a function algebra and T is an algebra isomorphism.

Proof. Let $y_0 \in p(B) = \delta B$. Condition (6.2) implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi: \delta B \to \delta A$ be the homeomorphism from Theorem 4.2.1, such that $|(Tf)(y)| = |f(\psi(y))|$

for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Let $(Tf)(y_0) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0.

From (6.6), note that $\psi(U)$ is an open neighborhood of $\psi(y_0)$ in δA. Again by Lemma 3.3.10 there exists a peaking function $h \in \mathcal{P}_{\psi(y_0)}(A)$ such that $\sigma_\pi(fh) = \{f(\psi(y_0))\}$ and $E(f \cdot h) = E(h) \subset U$. If $Th = k$, then $f(\psi(y_0)) \in \sigma_\pi(Tf \cdot k)$ since, by (6.5), $\sigma_\pi(Tf \cdot k) \cap \sigma_\pi(fh) \neq \emptyset$. Therefore, there is a point $y_1 \in p(B)$ so that $(Tf \cdot k)(y_1) = f(\psi(y_0))$. Hence

$$f(\psi(y_0)) = f(\psi(y_0)) \, h(\psi(y_0)) = (Tf)(y_1) \, k(y_1). \tag{6.7}$$

Again using that T is a composition operator in modulus we have,

$$|f(\psi(y_0))| = |(Tf)(y_1)||k(y_1)| = |f(\psi(y_1))||(Th)(y_1)| = |f(\psi(y_1))||h(\psi(y_1))|. \tag{6.8}$$

Hence $\psi(y_1) \in E(f \cdot h) = E(h) \subset U$. Thus $|h(\psi(y_1))| = |(Th)(y_1)| = |k(y_1)| = 1$. Since, by condition (6.5), $\sigma_\pi(k) \subset \sigma_\pi(h) = \{1\}$, we deduce that $k(y_1) \in \sigma_\pi(k) = \{1\}$, hence $k(y_1) = 1$. Then the equality (6.7) becomes $f(\psi(y_0)) = (Tf)(y_1)$. Since V was an arbitrary neighborhood of y_0, the continuity of Tf yields $(Tf)(y_0) = f(\psi(y_0))$ as claimed. $\qquad\square$

As mentioned after Theorem 5.2.1(A) and 5.2.1(B), if T is a bijection, Theorem 6.0.5(B) follows directly from Theorem 6.1.1(A).

In [9, Theorem 8] it is shown that the secondary conditions above for weak peripherally-multiplicative maps between uniform algebras can be replaced with a single condition, that T is continuous at the unity element. Namely,

Proposition 6.1.2. [9] *Let $T : A \to B$ be a surjective map between uniform algebras A, B on compact Hausdorff spaces X and Y. If T is a weakly peripherally-multiplicative map that is continuous at 1, then $T(1)$ is a signum function, i.e. $T(1) = \pm 1$, and $f \mapsto T(1)T(f)$ is an*

isometric algebra isomorphism of A onto B.

Theorem 6.1.4 below generalizes this result showing that it is not necessary for T to be continuous at the unity element, but that it must have a limit at a point $a \in A$ with $a^2 = 1$.

Lemma 6.1.3. *Suppose X is a compact Hausdorff space and $A \subset C(X)$ is a uniform algebra. If $\{f_n\}_n$ is a sequence of functions in A such that $\lambda \in \sigma_\pi(f_n)$ for every n and $f_n \to f$ uniformly to $f \in A$, then $\lambda \in \sigma_\pi(f)$.*

Proof. Note that $\|f_n\| = |\lambda|$ for all n and by the reverse triangle inequality, $|\|f\| - \|f_n\|| \leq \|f - f_n\| \to 0$ uniformly as $n \to \infty$. Thus

$$\|f\| = \lim_{n \to \infty} \|f_n\| = \lim_{n \to \infty} |\lambda| = |\lambda|.$$

Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ for every $x \in X$ and $n \geq N$. Also, since $\lambda \in \sigma_\pi(f_n)$ for every n, there exists $x_N \in X$ such that $f_N(x_N) = \lambda$. Therefore $|f(x_N) - \lambda| = |f(x_N) - f_N(x_N)| < \epsilon$. Consequently, there is a sequence $\{x_N\}_N \subset X$ such that $|f(x_N) - \lambda| \to 0$ as $N \to \infty$.

Since X is compact, there exists a convergent subsequence (subnet) $\{x_{N_k}\}_k$ such that $x_{N_k} \to x_0$ as $k \to \infty$ for some $x_0 \in X$. By the continuity of f, we have that $f(x_{N_k}) \to \lambda$ and $f(x_{N_k}) \to f(x_0)$ simultaneously as $k \to \infty$. Since the limits are unique, it follows that $f(x_0) = \lambda$. Therefore we have shown $\lambda \in \sigma_\pi(f)$. $\square$

Theorem 6.1.4. [12]*(J,T) Let A and B be uniform algebras on compact Hausdorff spaces X and Y. If $T: A \to B$ is a surjective map such that*

(i) *$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in A$ and*

(ii) *There exist an $a \in A$ with $a^2 = 1$ such that T has a limit, say b, at a,*

then $b^2 = 1$ and $(Tf)(y) = b(y) a(\psi(y)) f(\psi(y))$ for every $f \in A$ and $y \in \delta B$, i.e. the map $f \mapsto bT(af)$ is an isometric algebra isomorphism.

Proof. First note that for any $\{a_n\} \subset A$ such that $a_n \to a$, we necessarily have $\lim_{n \to \infty} T a_n = b$ in B since B is a uniform algebra and therefore complete.

Condition (i) implies that $\|Tf \cdot Tg\| = \|fg\|$ for all $f, g \in A$. In particular, $\|(Tf)^2\| = \|f^2\|$ and therefore, $\|Tf\| = \|f\|$ for every $f \in A$.

We claim that $\sigma_\pi(f) \subset \sigma_\pi(bT(af))$ for every $f \in A$. Let $f \in A$ and $\lambda \in \sigma_\pi(f)$. If $\lambda = 0$, then $\|f\| = 0$ and so $f = 0$, thus $af = 0$ and hence $\|T(af)\| = 0$. Consequently, $bT(af) = 0$ and, therefore, $\lambda \in \sigma_\pi(bT(af))$.

If $\lambda \neq 0$, then $f^{-1}(\lambda)$ is a peak set in X, so there exists a peaking function $h \in \mathcal{P}(A)$ such that $E(h) = f^{-1}(\lambda)$. Define $h_n = a\dfrac{n+h}{n+1}$.

For $x \in X$ such that $ah_n(x) = 1$, we have $\dfrac{n+h(x)}{n+1} = 1$ since $a^2 = 1$, which clearly implies $h(x) = 1$. Otherwise $|ah_n(x)| < 1$, which implies $\left|\dfrac{n+h(x)}{n+1}\right| < 1$ and consequently $|h(x)| < 1$. Therefore $E(ah_n) = E(h)$ and $ah_n \in \mathcal{P}(A)$ for every n. Therefore $(ah_n)^{-1}(1) = h^{-1}(1) = f^{-1}(\lambda)$ and $\sigma_\pi(ah_n) = \{1\}$ for every n. Consider the function $ah_n f \in A$, and observe that $a(x)h_n(x)f(x) = f(x)$ for $x \in E(h) = f^{-1}(\lambda)$ and $|a(x)h_n(x)f(x)| < |f(x)|$ for $x \notin E(h)$, thus $\sigma_\pi(ah_n f) = \{\lambda\}$ for every n.

Condition (i) implies that $\lambda \in \sigma_\pi(Th_n \cdot T(af))$ for every n. Clearly the function $\dfrac{n+h}{n+1}$ converges uniformly to h, and thus h_n converges uniformly to a. Then $Th_n \to b$ by (ii) and Lemma 6.1.3 implies that $\lambda \in \sigma_\pi(bT(af))$. Consequently,

$$\sigma_\pi(f) \subset \sigma_\pi(bT(af)) \tag{6.9}$$

as claimed.

We claim that $b^2 = 1$. Since $\|Tf \cdot Tg\| = \|fg\|$, Theorem 4.2.1 implies that there exists a homeomorphism $\psi : \delta B \to \delta A$ such that $|Tf(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$. Suppose $\{a_n\}_n \subset A$ be any sequence in A converging to a and fix $y \in \delta B$. Then $Ta_n \to b$ and therefore $|Ta_n(y)| \to |b(y)|$ as $n \to \infty$. On the other hand, $|Ta_n(y)| = |a_n(\psi(y))| \to |a(\psi(y))| = 1$ as $n \to \infty$. Therefore $|b(y)| = 1$ for all $y \in \delta B$.

Fix $y_0 \in \delta B$ and let $K = b^{-1}(b(y_0))$. Since $|b(y_0)| = 1$, $b(y_0) \in \sigma_\pi(b)$ and therefore K is a peak set in Y. Therefore, there exists a peaking function $k \in \mathcal{P}(B)$ with $E(k) = K$. Let $h \in A$ be such that $T(ah) = k$. According to (6.9), $\sigma_\pi(h) \subset \sigma_\pi(bT(ah)) = \sigma_\pi(bk)$. Since $k \in \mathcal{P}(B)$ with $E(k) = K$, we see that $b(y)k(y) = b(y_0)$ for all $y \in E(k)$, and $|b(y)k(y)| < |b(y_0)|$ for all $y \notin E(k)$. Therefore $\sigma_\pi(h) \subset \sigma_\pi(bk) = \{b(y_0)\}$ which implies $\sigma_\pi(h) = \{b(y_0)\}$. Thus $\sigma_\pi(h^2) = \{b(y_0)^2\}$, so by (i), $\{b(y)^2\} \in \sigma_\pi(T(ah)^2) = \sigma_\pi(k^2) = \{1\}$ since $k \in \mathcal{P}(B)$. Since y_0 was arbitrary in δB, we have shown that $b(y)^2 = 1$ for every $y \in \delta B$, thus $b(y) = \pm 1$.

Finally, define the map $\Phi : A \to B$ by $\Phi(f) = bT(af)$. First we show that Φ is surjective. Let $g \in B$. Since T is surjective, there exists $f \in A$ such that $T(f) = bg$. Then if $f' = af \in A$, we have $\Phi(af') = bT(af') = bT(f) = b^2 g = g$. Therefore Φ is surjective.

Also for $f, g \in A$,

$$\Phi(f)\Phi(g) = bT(af) \cdot bT(ag) = T(af)T(ag)$$

and $af \cdot ag = a^2 fg = fg$. Therefore $\sigma_\pi(\Phi(f) \cdot \Phi(g)) = \sigma_\pi(T(af) \cdot T(ag))$ which by (i) has a nonempty intersection with $\sigma_\pi(af \cdot ag) = \sigma_\pi(fg)$. Therefore Φ is weakly peripherally-multiplicative. Furthermore, $\sigma_\pi(f) \subset \sigma_\pi(\Phi(f))$ by (6.9). Therefore Theorem 6.1.1 applies to the map Φ, and the map $f \longmapsto \Phi(f) = bT(af)$ is a ψ-composition operator on δB and thus an isometric algebra isomorphism. Hence $b(y)(T(af)(y) = f(\psi(y))$, and therefore, $T(f)(y) = b(y)\,a(\psi(y))\,f(\psi(y))$ for all $y \in \delta B$ and $f \in A$. $\qquad\square$

Another important property of maps between function algebras is the preservation of the

peaking functions. In [14], the authors discuss various conditions for a weakly peripherally-multiplicative map between uniform algebras to preserve the peaking functions. That is, if $T : A \to B$ for uniform algebras A, B, consider the condition

$$T(\mathcal{P}(A)) = \mathcal{P}(B). \tag{6.10}$$

In the case that the map T is weakly peripherally-multiplicative, one can show that $\mathcal{P}(B) \subset T(\mathcal{P}(A))$ or $T(\mathcal{P}(A)) \subset \mathcal{P}(B) \subset T(A)$ both imply (6.10).

Proposition 6.1.5. [14] *A mapping $T : A \to B$ between uniform algebras is an isometric algebra isomorphism if and only if it is weakly peripherally-multiplicative and preserves the peaking functions.*

In other words, for uniform algebras equation (6.10) and weak peripheral-multiplicativity imply that T is a composition operator, and thus an algebra isomorphism. The following two theorems were proved for uniform algebras in [14], where the arguments relied heavily on the existence of unity element. Here we provide a proof for function algebras possibly without unit.

Theorem 6.1.6. (A) *Let A be a dense subalgebra of a function algebra such that $p(A) = \delta A$ and B a function algebras on locally compact Hausdorff spaces X and Y respectively. Suppose that $T : A \to B$ is a surjective mapping, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and*

$$\mathcal{P}(B) \subset T[\mathbb{T} \cdot \mathcal{P}(A)]. \tag{6.11}$$

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $y \in \delta B$.

Proof. First it will be shown that there exists a homeomorphism $\psi : \delta B \to \delta A$ such that $Tf(y_0)^2 = f(\psi(y_0))^2$ for every $f \in A$ and $y_0 \in \delta B$.

Let $y_0 \in p(B) = \delta B$. The weak peripheral multiplicativity implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi : \delta B \to \delta A$ be the homeomorphism established in Theorem 4.2.1, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Let $(Tf)(y_0) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0. Also let $\psi(y_0) = x_0 \in \delta A$. According to Bishop's Lemma, there exists a peaking function $k \in \mathcal{P}_{y_0}(B)$ such that $\sigma_\pi(Tf \cdot k) = \{(Tf)(y_0)\}$ and $E(Tf \cdot k) = E(k) \subset V$. Since $\mathcal{P}(B) \subset T[\mathbb{T} \cdot \mathcal{P}(A)]$, choose $h \in A$ such that $Th = k$ and $h = e^{i\theta}h'$ where $h' \in \mathcal{P}(A)$, and in fact $h' \in \mathcal{P}_{x_0}(A)$. In particular $\sigma_\pi(h) = \{e^{i\theta}\}$, a singleton.

Then $(Tf)(y_0) \in \sigma_\pi(fh)$ since, $\sigma_\pi(Tf \cdot k) \cap \sigma_\pi(fh) \neq \varnothing$. Therefore, there is a point $x_1 \in \delta A$ so that $(Tf \cdot k)(y_0) = (fh)(x_1)$. Since ψ is surjective, there is an $y_1 \in \delta B$ so that $x_1 = \psi(y_1)$. Hence

$$(Tf)(y_0) = (Tf)(y_0) \, k(y_0) = f(\psi(y_1)) \, h(\psi(y_1)). \tag{6.12}$$

Since T is a weighted composition operator in modulus, $|(Tf)(y_0)| = |(Tf)(y_0)||k(y_0)| = |f(\psi(y_1))||h(\psi(y_1))| = |(Tf)(y_1)||k(y_1)| = |Tf \cdot k(y_1)|$. Hence $y_1 \in E(Tf \cdot k) = E(k) \subset V$. Also, $|h(\psi(y_1))| = |k(y_1)| = 1$ and thus $\psi(y_1) \in E(h)$.

Also clearly $|h(\psi(y_0))| = |Th(y_0)| = |k(y_0)| = 1$ so we also have $\psi(y_0) \in E(h)$. Thus $\psi(y_0), \psi(y_1) \in E(h)$ and therefore $h(\psi(y_0)), h(\psi(y_1)) \in \sigma_\pi(h)$ which is a singleton, thus

$$h(\psi(y_0)) = h(\psi(y_1)) \tag{6.13}$$

Also since $\sigma_\pi(h^2) \cap \sigma_\pi(k^2) \neq \varnothing$ and $\sigma_\pi(k^2) = 1$, we see that $1 \subset \sigma_\pi(h^2)$, and the assumption

that $h = e^{i\theta} h'$ for some $h' \in \mathcal{P}(A)$ implies that $\sigma_\pi(h^2)$ is a singleton. Thus

$$h(x)^2 = 1 \tag{6.14}$$

for all $x \in E(h)$ and it follows that $h(\psi(y_1))^2 = h(\psi(y_0))^2 = 1$.

Then squaring (6.12) gives $Tf(y_0)^2 = f(\psi(y_1))^2$, which by the continuity of Tf, f, ψ implies that $Tf(y_0)^2 = f(\psi(y_0))^2$. Consequently, there exists a number $\alpha_f(y_0) = \pm 1$, possibly depending on f, such that

$$(Tf)(y_0) = \alpha_f(y_0) f(\psi(y_0)). \tag{6.15}$$

We claim that the number $\alpha_f(y_0)$ does not depend on $f \in A$.

First we show that $\alpha_h(y_0)$ has the same value for functions $h \in T^{-1}(k)$ where $k \in \mathcal{P}_{y_0}(B)$. Indeed, if $k_1, k_2 \in \mathcal{P}_{y_0}(B)$ with $Th_i = k_i$ for $i = 1, 2$, then $\sigma_\pi(k_1 k_2) = \{1\}$. Then by the weak peripheral-multiplicativity of T we have

$$\{1\} \cap \sigma_\pi(h_1 h_2) \neq \varnothing, \tag{6.16}$$

and therefore, $1 \in \sigma_\pi(h_1 \cdot h_2)$.

Now by the condition $\mathcal{P}(B) \subset T[\mathbb{T} \cdot \mathcal{P}(A)]$, there exists $h_1 = e^{i\theta_1} h_1'$ and $h_2 = e^{i\theta_2} h_2'$ for some peaking functions $h_1', h_2' \in \mathcal{P}_{x_0}(A)$, so we see that in fact $\sigma_\pi(h_1 h_2) = \{e^{\theta_1 + \theta_2}\}$, a singleton.

Take the modulus yields

$$1 = |k_1(y_0) k_2(y_0)| = |h_1(\psi(y_0)) h_2(\psi(y_0))| = |h_1 h_2(\psi(y_0))| \tag{6.17}$$

which implies $\psi(y_0) \in E(h_1 h_2)$, thus $h_1 h_2(\psi(y_0)) = 1$.

Now, $Th_i(y_0) = k_i(y_0) = \alpha_{h_i}(y_0)h_i(\psi(y_0))$, and multiplying for $i = 1, 2$ gives

$$1 = \alpha_{h_1}(y_0)\alpha_{h_2}(y_0)$$

and consequently, the numbers $\alpha_{h_i}(y_0)$ have the same sign. Define $\alpha_{h_i}(y_0) = \alpha(y_0)$.

Again consider the peaking function $k \in \mathcal{P}_{y_0}$ chosen in the first part of the proof. Then we have seen that

$$Tf(y_0) = \alpha_f(y_0)f(\psi(y_0)). \tag{6.18}$$

Then $h \in \mathcal{P}(A)$ such that $Th = k$ with $\psi(y_1) \in E(h)$ for some $y_1 \in V$. By (6.18) we have $Th(y_1) = \alpha_h(y_1)h(\psi(y_1))$ which becomes $k(y_1) = \alpha_h(y_1)h(\psi(y_0))$ since $h(\psi(y_0)) = h(\psi(y_1))$. But we have already seen that $y_1 \in E(k)$ which implies $k(y_1) = 1$. Thus

$$\alpha_h(y_1) = h(\psi(y_1)) = h(\psi(y_0)) = \alpha_h(y_0) \tag{6.19}$$

So, in fact $\alpha(y_0) = \alpha(y_1)$.

Combining equations (6.18) and (6.12) shows that

$$\alpha_f(y_0)f(\psi(y_0)) = \alpha(y_0)f(\psi(y_1))$$

and since V was an arbitrary neighborhood of y_0 with $y_1 \in V$, the continuity of f, ψ implies that

$$\alpha_f(y_0)f(\psi(y_0)) = \alpha(y_0)f(\psi(y_0))$$

thus $\alpha_f(y_0) = \alpha(y_0)$. Therefore $Tf(y_0) = \alpha(y_0)f(\psi(y_0))$ as desired.

The arguments above also show that α is constant in a neighborhood of y_0, thus is continuous.

□

The next theorem provides the same conclusion as above, but (6.11) is replaced with a slightly different condition.

Theorem. 6.1.6 (B) *Let A be a function algebra and B a dense subalgebra of a function algebra such that $p(B) = \delta B$ on locally compact Hausdorff spaces X and Y respectively. Suppose that $T : A \to B$ is a surjective mapping, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and*

$$T[\mathcal{P}(A)] \subset \mathbb{T} \cdot \mathcal{P}(B). \tag{6.20}$$

Then there exists a homeomorphism $\psi \colon \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $y \in \delta B$.

Proof. Again it will be shown first that there exists a homeomorphism $\psi : \delta B \to \delta A$ such that $Tf(y_0)^2 = f(\psi(y_0))^2$ for every $f \in A$ and $y_0 \in \delta B$.

As in the previous proof, let $y_0 \in p(B) = \delta B$. The weak peripheral multiplicativity implies that $\|Tf \cdot Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi \colon \delta B \to \delta A$ be the homeomorphism established in 4.2.1, such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. Clearly $(Tf)(y_0) = f(\psi(y_0))$ whenever $(Tf)(y_0) = 0$.

Here let $(f(\psi(y_0)) \neq 0$ and let $V \subset \delta B$ be an open neighborhood of y_0. Then $\psi(V) = U$ is an open neighborhood of $\psi(y_0) = x_0$. According to Bishop's Lemma, there exists a peaking function $h \in \mathcal{P}_{x_0}(A)$ such that $\sigma_\pi(f \cdot h) = \{f(\psi(y_0))\}$ and $E(f \cdot h) = E(h) \subset U$. By (6.20), we may choose $Th = k$ such that $k = e^{i\theta}k'$ where $k' \in \mathcal{P}(B)$. Then $f(\psi(y_0)) \in \sigma_\pi(Tf \cdot k)$ by the condition $\sigma_\pi(Tf \cdot k) \cap \sigma_\pi(fh) \neq \varnothing$. Consequently, there is a point $y_1 \in \delta B$ so that

$f(\psi(y_0)) = (Tf \cdot k)(y_1)$. Hence

$$f(\psi(y_0)) = f(\psi(y_0)) \, h(\psi(y_0)) = Tf(y_1) \, k(y_1). \tag{6.21}$$

Taking the modulus of (6.21) and using that T is a composition operator in modulus yields,

$$|f(\psi(y_0))| = |f(\psi(y_0))h(\psi(y_0))| = |Tf(y_1)k(y_1)| = |f(\psi(y_1))||h(\psi(y_1))| = |fh(\psi(y_1))|. \tag{6.22}$$

Hence $\psi(y_1) \in E(fh) = E(h) \subset U$, and it follows that $y_1 \subset V$. Therefore, $1 = |h(\psi(y_1))| = |k(y_1)|$ which shows that $y_1 \in E(k)$ and $\psi(y_1) \in E(h)$.

Note that $1 = |h(\psi(y_0))| = |Th(y_0)| = |k(y_0)|$ so we also have $y_0 \in E(k)$. Thus $y_0, y_1 \in E(k)$ and therefore $k(y_0), k(y_1) \in \sigma_\pi(k)$ which is a singleton by 6.20, thus

$$k(y_0) = k(y_1) \tag{6.23}$$

Since $\sigma_\pi(k)$ is a singleton and $\sigma_\pi(h) = \{1\}$, the weak peripheral multiplicativity of T implies that

$$\sigma_\pi(k^2) \cap \{1\} = \varnothing$$

and it follows that $k(y_1) = \pm 1$. Therefore $Tf(y_0)^2 = f(\psi(y_1))^2$ and since Tf, f, ψ are continuous, we have $Tf(y_0)^2 = f(\psi(y_0))^2$ as desired.

Consequently, there exists a number $\alpha_f(y_0) = \pm 1$, possibly depending on f, such that

$$(Tf)(y_0) = \alpha_f(y_0)f(\psi(y_0)). \tag{6.24}$$

As in the previous theorem, we claim that $\alpha_f(y_0)$ does not depend on $f \in A$.

First we show that $\alpha_h(y_0)$ has the same value for all functions $h \in \mathcal{P}_{\psi(y_0)}(A)$. Indeed, if

$h_1, h_2 \in \mathcal{P}_{\psi(y_0)}(A)$ with $Th_i = k_i$ for $i = 1, 2$, then $\sigma_\pi(k_1 \cdot k_2) \cap \sigma_\pi(h_1 h_2) \neq \emptyset$, and therefore, $1 \in \sigma_\pi(k_1 \cdot k_2)$.

By the assumption (6.20), $k_1 = e^{i\theta_1} k_1'$ and $k_2 = e^{i\theta_2} k_2'$ for some peaking functions $k_1', k_2' \in \mathcal{P}(B)$. But for $i = 1, 2$, the previous shows $k_i(y_0)^2 = h_i(\psi(y_0))^2 = 1$ for each $\psi(y_0) \in E(h)$, so in fact, $k_i = \pm k_i'$. Therefore $\sigma_\pi(k_1 k_2) = \{\pm 1\}$, a singleton. Also for each $\psi(y_0) \in E(h_1 h_2)$, $|h_1(\psi(y_0)) h_2(\psi(y_0))| = |k_1(y_0) k_2(y_0)| = 1$ which implies $y_0 \in E(k_1 k_2)$. Thus $k_1(\psi(y_0)) k_2(\psi(y_0)) = 1$.

Now, $Th_i(y_0) = k_i(y_0) = \alpha_{h_i}(y_0) h_i(\psi(y_0))$, and multiplying for $i = 1, 2$ gives

$$k_1(y_0) k_2(y_0) = \alpha_{h_1}(y_0) \alpha_{h_2}(y_0) h_1(\psi(y_0)) h_2(\psi(y_0))$$

which implies

$$k_1(y_0) k_2(y_0) = 1 = \alpha_{h_1}(y_0) \alpha_{h_2}(y_0)$$

and consequently, the numbers $\alpha_{h_i}(y_0)$ have the same sign. Since α does not depend on the functions $h \in \mathcal{P}(A)$, we may define $\alpha_h(y_0) = \alpha(y_0)$.

Consider the peaking function $h \in \mathcal{P}_{\psi(y_0)}$ chosen in the first part of the proof. Then we have seen that

$$Tf(y_0) = \alpha_f(y_0) f(\psi(y_0)). \tag{6.25}$$

Then $k \in \mathcal{P}(B)$ such that $Th = k$ with $y_1 \in E(k)$ for some $y_1 \in V$. By (6.25) we have $Th(y_1) = \alpha_h(y_1) h(\psi(y_1))$ which becomes $k(y_0) = \alpha_h(y_1) h(\psi(y_1))$ since $k(y_0) = k(y_1)$. But we have already seen that $\psi(y_1) \in E(h)$ which implies $h(\psi(y_1)) = 1$. Thus

$$\alpha_h(y_1) = k(y_1) = k(y_0) = \alpha_h(y_0) \tag{6.26}$$

So, in fact $\alpha(y_0) = \alpha(y_1)$.

Combining equations (6.25) and (6.21) shows that

$$\alpha_f(y_0)f(\psi(y_0)) = \alpha(y_1)f(\psi(y_1)) = \alpha(y_0)f(\psi(y_1))$$

and since V was an arbitrary neighborhood of y_0 with $y_1 \in V$, the continuity of f, ψ implies that

$$\alpha_f(y_0)f(\psi(y_0)) = \alpha(y_0)f(\psi(y_0))$$

thus $\alpha_f(y_0) = \alpha(y_0)$. Therefore $Tf(y_0) = \alpha(y_0)f(\psi(y_0))$ as desired.

The continuity of α follows as in the previous theorem. $\qquad\square$

The following two corollaries are immediate.

Corollary 6.1.7. *Let A and B be function algebras on locally compact Hausdorff spaces X and Y. Suppose that $T : A \to B$ is a surjective mapping, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and*

$$T^{-1}[\mathcal{P}(B)] \subset \mathcal{P}(A). \tag{6.27}$$

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ such that $(Tf)(y) = f(\psi(y))$ for every $y \in \delta B$.

Corollary 6.1.8. *Let A and B be function algebras on locally compact Hausdorff spaces X and Y. Suppose that $T : A \to B$ is a surjective mapping, $\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$ for all $f, g \in A$, and*

$$T[\mathcal{P}(A)] \subset \mathcal{P}(B). \tag{6.28}$$

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ such that $(Tf)(y) = f(\psi(y))$ for every $y \in \delta B$.

Theorem 6.1.6 (B) yields a corollary that shows yet another condition, when accompanied by weak peripheral-multiplicativity, implies that T is a weighted composition operator.

Corollary 6.1.9. *Let A and B be function algebras on locally compact Hausdorff spaces X and Y. Suppose that $T: A \to B$ is a surjective mapping such that*

$$\sigma_\pi(Tf \cdot Tg) \cap \sigma_\pi(fg) \neq \varnothing$$

for all $f, g \in A$ and $\sigma_\pi(Tf)$ is a singleton whenever $\sigma_\pi(f)$ is a singleton.

Then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $y \in \delta B$.

Proof. Let $h \in \mathcal{P}(A)$ and $Th = k$. Then $\sigma_\pi(h) = \{1\}$ is a singleton, and therefore $\sigma_\pi(Th) = \sigma_\pi(k)$ is a singleton by assumption. But the weak perhipheral-multiplicativity implies that T preserves the norm, so $\sigma_\pi(k) = \{e^{i\theta}\}$ for some $\theta \in [0, 2\pi)$. Therefore $k' = e^{-i\theta}k \in \mathcal{P}(B)$ and we see that $Th = e^{i\theta}k' \in \mathbb{T} \cdot \mathcal{P}(B)$. Therefore $T[\mathcal{P}(A)] \subset \mathbb{T} \cdot \mathcal{P}(B)$ and the conclusion follows from Theorem 6.1.6 (B). $\square$

Corollary 6.1.10. *Let A and B be function algebras on locally compact Hausdorff spaces X and Y with the hypotheses of the Theorem 6.1.6 (B). In addition, suppose that $d(\sigma_\pi(f), \sigma_\pi(Tf)) < 2$ for all $f \in A$. Then there exists a homeomorphism $\psi : \delta B \to \delta A$,*

$$(Tf)(y) = f(\psi(y))$$

for every $y \in \delta B$. In other words, the weight function $\alpha : \delta B \to \{1, -1\}$ is identically 1 which implies that T is a composition operator, thus an isometric algebraic isomorphism.

Proof. From Theorem 6.1.6, there exists a homeomorphism $\psi : \delta B \to \delta A$ and $\alpha : \delta B \to \mathbb{C}$ with $\alpha^2 = 1$ such that $Tf(y) = \alpha(y)f(\psi(y))$ for every $f \in A$ and $y \in \delta B$. It suffices to show that $\alpha = 1$ on δB.

Let $f \in A$ and fix $y_0 \in \delta B$. Let $k \in \mathcal{P}_{y_0}(B)$ and choose $h \in T^{-1}(k)$ as in the proof Theorem 6.1.6 (B). Then $\sigma_\pi(k) = \{1\}$ and $d(\sigma_\pi(h), \sigma_\pi(k)) < 2$ imply that

$$|h(x) - 1| < 2 \tag{6.29}$$

for all $x \in E(h)$. However, the weak peripheral-multiplicativity of T implies

$$\sigma_\pi(k^2) \cap \sigma_\pi(h^2) \neq \varnothing$$

so $\sigma_\pi(h^2) = \{1\}$. Therefore $\sigma_\pi(h) = \{\pm 1\}$ and together with (6.29) implies $\sigma_\pi(h) = \{1\}$. Then

$$1 = k(y_0) = Th(y_0) = \alpha(y_0)h(\psi(y_0)) \tag{6.30}$$

and $|h(\psi(y_0))| = |Th(y_0)| = |k(y_0)| = 1$ which implies $\psi(y_0) \in E(h)$ and thus $h(\psi(y_0)) \in \sigma_\pi(h) = \{1\}$. Therefore (6.30) reduces to $\alpha(y_0) = 1$ and we may conclude $\alpha = 1$ on δB as desired. $\square$

6.2   Function Algebras with Sufficiently Many Peak Functions

There is more we can say regarding weakly peripherally-multiplicative maps when there are sufficiently many peak functions h, i.e. peaking functions h such that $E(h)$ is a singleton.

Example 6.2.1. Consider the function algebra $C(X)$, the continuous complex valued functions on a *metric space* X. Fix $x_0 \in X$ and a positive real constant $\ell > 0$. Consider the

following real-valued function

$$f(x) = \begin{cases} 0 & : \ell \le d(x_0, x) \\ \frac{\ell - d(x_0, x)}{\ell} & : \ell > d(x_0, x) \end{cases}$$

Here f is a peaking function such that $E(f) = \{x_0\}$, i.e. f is a peak function, and clearly one can construct such a function for every $x_0 \in X$. Therefore every point in X is a peak point.

Lemma 6.2.1. *Suppose $T : A \to B$ for function algebras A and B on locally compact Hausdorff spaces $X = \partial A$ and $Y = \partial B$ respectively. Also suppose that T is a composition operator in modulus. That is, there exists a homemorphism $\psi : \delta(B) \to \delta(A)$ such that $|Tf(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in Y$. Then for every $f, g \in A$, $E(fg)$ is a singleton implies that $E(TfTg)$ is a singleton.*

Proof. Let $E(fg) = \{x_0\}$, where $x_0 \in \delta A$, and let $(fg)(x_0) = z_0$. Then $|(fg)(x)| < |z_0|$ for all $x \ne x_0$ and hence $\sigma_\pi(fg) = \{z_0\}$. If $x = \psi(y)$ and $x_0 = \psi(y_0)$, where $y, y_0 \in \delta B$ and $\psi : \delta B \to \delta A$ is the assumed homemorphism, then we have $|(TfTg)(y_0)| = |(fg)(x_0)| = |z_0|$ and $|(TfTg)(y)| = |(fg)(x)| < |z_0|$ whenever $y \ne y_0$. Therefore, $E(TfTg) = \{y_0\}$, i.e. $E(TfTg)$ is a singleton. $\qquad\square$

Theorem 6.2.2. *Let X, Y be locally compact Hausdorff spaces and let $A \subset C(X), B \subset C(Y)$ be dense subalgebras of function algebras, not necessarily with unit, with $X = \partial A$, $\delta A = p(A), Y = \partial B$ and $\delta B = p(B)$. Let for every $f \in A$ and any $x \in \delta A$ there is a peak function $h \in \mathcal{P}_x(A)$ so that $\sigma_\pi(fh) = \{f(x)\}$. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \ne \varnothing \text{ for all } f, g \in A, \tag{6.31}$$

then T is a weighted composition operator, namely, there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $f \in A$ and $y \in \delta B$.

Proof. Let $y_0 \in \delta B$. The weak peripheral-multiplicativity implies that $\|Tf\,Tg\| = \|fg\|$ for every $f, g \in A$. Let $\psi: \delta B \to \delta A$ be the homeomorphism from Theorem 4.2.1 for which $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. In particular, if $(Tf)(y_0) = 0$ then $f(\psi(y_0)) = 0$.

Let $(Tf)(y_0) \neq 0$. The equality $|(Tf)(y)| = |f(\psi(y))|$ implies

$$(Tf)(y_0) = \alpha(f, y_0)\, f(\psi(y_0)) \tag{6.32}$$

for some complex number $\alpha(f, y_0)$ with $|\alpha(f, y_0)| = 1$.

For any peak function $h \in \mathcal{P}_{\psi(y_0)}(A)$ we have

$$(Th)^2(y_0) = \alpha^2(h, y_0)\, h^2(\psi(y_0)) = \alpha^2(h, y_0).$$

Since $E(h^2)$ is a singleton by hypothesis, so is $E((Th)^2)$ and also $\sigma_\pi((Th)^2)$, by Lemma 6.2.1. We have $\{1\} \cap \sigma_\pi((Th)^2) = \sigma_\pi(h^2) \cap \sigma_\pi((Th)^2) \neq \varnothing$, hence $\sigma_\pi((Th)^2) = \{1\}$. As a consequence, $(Th)^2(y_0) = 1$ since $|(Th)^2(y_0)| = |h^2(\psi(y_0))| = 1$. Therefore

$$\alpha^2(h, y_0) = \alpha^2(h, y_0)\})\, h^2(\psi(y_0)) = ((Th)^2)(y_0) = 1.$$

Let $h_1, h_2 \in \mathcal{P}_{\psi(y_0)}(A)$. Since $E(h_1 h_2)$ is a singleton, so is $E(Th_1\,Th_2)$ and also $\sigma_\pi(Th_1\,Th_2)$, by Lemma 6.2.1. Consequently, $\sigma_\pi(Th_1\,Th_2) = \{1\}$ since $\{1\} \cap \sigma_\pi(Th_1 Th_2) = \sigma_\pi(h_1 h^2) \cap \sigma_\pi(Th_1 Th_2) \neq \varnothing$. Moreover, $(Th_1 Th_2)(y_0) = 1$ since $|(Th_1 Th_2)(y_0)| = |(h_1 h_2(\psi(y_0))| = 1$. Therefore, $\alpha(h_1, y_0)\, \alpha(h_2, y_0) = \{(Th_1\,Th_2)(y_0)\} = 1$. Consequently, $\alpha(h_1, y_0) = 1/\alpha(h_2, y_0) = \alpha(h_2, y_0)$. Therefore, the number $\alpha(h, y_0)$ is one and the same for any $h \in \mathcal{P}_{\psi(y_0)}(A)$.

Define the function $\alpha: \delta B \to \pm 1$ by $\alpha(y) = \alpha(h, y)$, where $y \in \delta B$ and h is any function in $\mathcal{P}_{\psi(y)}(A)$. Clearly, $\alpha^2 = 1$.

If $f \in A$ then, by the hypotheses, there is an $h \in \mathcal{P}_{\psi(y_0)}(A)$ such that $\sigma_\pi(fh) = \{f(\psi(y_0))\}$. Since $E(fh)$ is a singleton, so is $E(TfTh)$ and also $\sigma_\pi(TfTg)$, by Lemma 6.2.1. We have $\sigma_\pi(TfTh) = \{f(\psi(y_0))\}$ since $\{f(\psi(y_0))\} \cap \sigma_\pi(TfTh) = \sigma_\pi(fh) \cap \sigma_\pi(TfTh) \neq \varnothing$. Hence $\sigma_\pi(TfTh) = \{f(\psi(y_0))\}$ and $(TfTh)(y_0) = f(\psi(y_0))$ since $|(TfTh)(y_0)| = |(fh)(\psi(y_0))| = |(f(\psi(y_0)))|$. According to (6.32),

$$\{\alpha(f, y_0)\, f(\psi(y_0))\, \alpha(y_0)\} = \{\alpha(f, y_0)\, f(\psi(y_0))\, \alpha(y_0)\, h(\psi(y_0))\} = \{(TfTh)(y_0)\} = \{f(\psi(y_0))\}.$$

Consequently, $\alpha(f, y_0) = 1/\alpha(y_0) = \alpha(y_0)$. Now equality (6.32) becomes $(Tf)(y_0) = \alpha(y_0) f(\psi(y_0))$, as desired. $\square$

Lemma 6.2.1 also shows that if $E(fg)$ is a singleton, then so is $E(TfTg)$ which implies that $\sigma_\pi(TfTg)$ is also a singleton.

Corollary 6.2.3. *Let X and Y be locally compact Hausdorff spaces and let $A \subset C(X)$ and $B \subset C(Y)$ be dense subalgebras of function algebras, not necessarily with unit, with $p(A) = \delta A$ and $p(B) = \delta B$. If $T : A \to B$ is a surjection such that $\sigma_\pi(TfTg)$ is a singleton for every $f, g \in A$ for which $\sigma_\pi(fg)$ a singleton, and if*

$$\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \varnothing$$

for all $f, g \in A$, then there exists a homeomorphism $\psi : \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y) f(\psi(y))$$

for every $f \in A$ and $y \in \delta B$.

Proof. Let $y_0 \in \delta B$. Again the weak peripheral-multiplicativity implies that $\|TfTg\| = \|fg\|$ for every $f, g \in A$. Let $\psi : \delta B \to \delta A$ be the homeomorphism such that $|(Tf)(y)| = |f(\psi(y))|$ for all $y \in \delta B$ and $f \in A$. In particular, if $(Tf)(y_0) = 0$ then $f(\psi(y_0)) = 0$.

Let $(Tf)(y_0) \neq 0$. The equality $|(Tf)(y)| = |f(\psi(y))|$ implies

$$(Tf)(y_0) = \alpha(f, y_0) \, f(\psi(y_0)) \tag{6.33}$$

for some complex number $\alpha(f, y_0)$ with $|\alpha(f, y_0)| = 1$.

For any $h \in \mathcal{P}_{\psi(y_0)}(A)$ we have

$$(Th)^2(y_0) = \alpha^2(h, y_0) \, h^2(\psi(y_0)) = \alpha^2(h, y_0).$$

Since $\sigma_\pi(h^2)$ is a singleton, so is $ps((Th)^2)$ by assumption. We have $\{1\} \cap \sigma_\pi((Th)^2) = \sigma_\pi(h^2) \cap \sigma_\pi((Th)^2) \neq \varnothing$, hence $\sigma_\pi((Th)^2) = \{1\}$. As a consequence, $(Th)^2(y_0) = 1$ since $|(Th)^2(y_0)| = |h^2(\psi(y_0))| = 1$. Therefore

$$\alpha^2(h, y_0) = \alpha^2(h, y_0)\}) \, h^2(\psi(y_0)) = ((Th)^2)(y_0) = 1.$$

Let $h_1, h_2 \in \mathcal{P}_{\psi(y_0)}(A)$. Since $\sigma_\pi(h_1 h_2)$ is a singleton, so is $\sigma_\pi(Th_1 Th_2)$ by assumption. Consequently, $\sigma_\pi(Th_1 Th_2) = \{1\}$ since $\{1\} \cap \sigma_\pi(Th_1 Th_2) = \sigma_\pi(h_1 h^2) \cap \sigma_\pi(Th_1 Th_2) \neq \varnothing$. Moreover, $(Th_1 Th_2)(y_0) = 1$ since $|(Th_1 Th_2)(y_0)| = |h_1 h_2(\psi(y_0))| = 1$. Therefore, $\alpha(h_1, y_0) \, \alpha(h_2, y_0) = \{(Th_1 Th_2)(y_0)\} = 1$. Consequently, $\alpha(h_1, y_0) = 1/\alpha(h_2, y_0) = \alpha(h_2, y_0)$. Therefore, the number $\alpha(h, y_0)$ is one and the same for any $h \in \mathcal{P}_{\psi(y_0)}(A)$.

Define the function $\alpha: \delta B \to \pm 1$ by $\alpha(y) = \alpha(h, y)$, where $y \in \delta B$ and h is any function in $\mathcal{P}_{\psi(y)}(A)$. Clearly, $\alpha^2 = 1$.

If $f \in A$ then, by the hypotheses, there is an $h \in \mathcal{P}_{\psi(y_0)}(A)$ such that $\sigma_\pi(fh) = \{f(\psi(y_0))\}$. Again since $\sigma_\pi(fg)$ is a singleton, so is $\sigma_\pi(TfTg)$ by assumption. We have $\sigma_\pi(Tf Th) = \{f(\psi(y_0))\}$ since $\{f(\psi(y_0))\} \cap \sigma_\pi(Tf Th) = \sigma_\pi(fh) \cap \sigma_\pi(Tf Th) \neq \varnothing$. Hence $\sigma_\pi(Tf Th) = \{f(\psi(y_0))\}$ and $(Tf Th)(y_0) = f(\psi(y_0))$ since $|(Tf Th)(y_0)| = |(fh)(\psi(y_0))| = |(f(\psi(y_0))|$. According to (6.33), $\{\alpha(f, y_0) \, f(\psi(y_0)) \, \alpha(y_0)\} = \{\alpha(f, y_0) \, f(\psi(y_0)) \, \alpha(y_0) \, h(\psi(y_0))\} = \{(Tf Th)(y_0)\} =$

$\{f(\psi(y_0))\}$. Consequently, $\alpha(f, y_0) = 1/\alpha(y_0) = \alpha(y_0)$. Now equality (6.33) becomes $(Tf)(y_0) = \alpha(y_0)f(\psi(y_0))$, as desired. □

Proposition 6.2.4. *Let A be a function algebra on a metric space X. Then for every $f \in A$ and $x_0 \in \delta A$ such that $f(x_0) \neq 0$, there exists a* peak *function $h \in \mathcal{P}_{x_0}(A)$ such that $\sigma_\pi(fh) = \{f(x_0)\}$ and $E(h) = \{x_0\}$.*

Proof. Fix $f \in A$ and $x_0 \in \delta(A)$ such that $f(x_0) \neq 0$. Also let $U_n = \{x \in X : d(x, x_0) < 1/n\}$ for each $n \in \mathbb{N}$. By Theorem 3.3.10 there exists $h_n \in \mathcal{P}_{x_0}(A)$ such that $E(h_n) \subset U_n$ and $\sigma_\pi(fh_n) = \{f(x_0)\}$ for every n. Consider the function

$$h = \sum_{n=1}^{\infty} \frac{h_n}{2^n}.$$

in A. Clearly $h(x_0) = 1$ and $|h(x)| < 1$ for any $x \notin U_n$ for some n. Therefore h is a *peak* function such that $E(h) = \{x_0\}$, and $fh(x_0) = f(x_0)$ where $|fh(x)| = |f(x)||h(x)| < |f(x)| = \|f\| = \|fh\|$ for every $x \in E(fh)$ with $x \neq x_0$. Thus $\sigma_\pi(fh) = \{f(x_0)\}$, which shows that h is the desired function. □

Corollary 6.2.5. *Let A be a function algebra and B be a dense subalgebra of function algebra on metric spaces X and Y respectively such that $p(B) = \delta B$. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \varnothing \ \text{ for all } f, g \in A,$$

then there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $f \in A$ and $y \in \delta B$.

Indeed, since A is a function algebra, Theorem 6.2.4 shows that for each $f \in A$ and $x \in \delta A$ such that $f(x) \neq 0$, there is a peak function $h \in \mathcal{P}_x(A)$ so that $E(h) = \{x\}$ and $\sigma_\pi(fh) = \{f(x)\}$.

Since T is also a surjective weakly peripherally-multiplicative map, the result follows from Theorem 6.2.2.

In particular, Corollary 6.2.5 holds for algebras of type $C_0(X)$ on metric and, more general, on first countable spaces X, since in this case $\delta(C_0(X)) = p(C_0(X))$.

Chapter 7

Further Questions

Several theorems from the previous chapters consider secondary conditions for weakly-peripherally multiplicative maps between function algebras. In Chapter 6, the following corollary was proved for metric spaces X and Y.

Corollary 7.0.6. *Let A be a function algebra and B be a dense subalgebra of function algebra on metric spaces X and Y respectively such that $p(B) = \delta B$. If $T: A \to B$ is a surjection such that*

$$\sigma_\pi(Tf\,Tg) \cap \sigma_\pi(fg) \neq \varnothing \text{ for all } f, g \in A,$$

then there exists a homeomorphism $\psi: \delta B \to \delta A$ and a continuous function α on δB with $\alpha^2 = 1$ such that

$$(Tf)(y) = \alpha(y)f(\psi(y))$$

for every $f \in A$ and $y \in \delta B$.

An open question is whether a similar theorem holds for function algebras A and B over general locally compact Hausdorff spaces X and Y. This question also applies to maps between semi-simple commutative Banach algebras.

Again consider a weakly peripheral-multiplicativity on map $T: A \to B$ between function al-gebras A and B. If d is the standard Euclidean metric in $\mathbb{C}$, then the weak peripheral-multiplicativity implies that $d(\sigma_\pi(TfTg), \sigma_\pi(fg)) = 0$ for all $f, g \in A$.

It is interesting to consider what conclusions may be reached if we instead investigate the assumption $d(\sigma_\pi(TfTg), \sigma_\pi(fg)) < \epsilon \|f\| \|g\|$ for all $f, g \in A$ for some fixed $\epsilon > 0$. This condition together with the secondary conditions of chapter 5 and 6 may provide more general sufficient conditions for the map T to be a composition operator, or possibly a *almost composition operator*. A map is said to be an *almost composition operator* on Y if there is an $\epsilon \geq 0$ such that $|Tf(y) - f(\psi(y))| \leq \epsilon |f(\psi(y))|$ for every $f \in A$ for some homeomorphism $\psi : \delta B \to \delta A$.

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