2011

D-colorable digraphs with large girth

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D-COLORABLE DIGRAPHS WITH LARGE GIRTH

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Dissertation

presented in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy
in Mathematics

The University of Montana
Missoula, MT

July 2011

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D-colorable digraphs with large girth

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In 1959 Paul Erdős (*Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38) famously proved, nonconstructively, that there exist graphs that have both arbitrarily large girth and arbitrarily large chromatic number. This result, along with its proof, has had a number of descendants (D. Bokal, G. Fijavž, M. Juvan, P.M. Kayll and B. Mohar, *The circular chromatic number of a digraph*, J. Graph Theory **46** (2004), 227–240; B. Bollobás and N. Sauer, *Uniquely colourable graphs with large girth*, Canad. J. Math. **28** (1976), 1340–1344; J. Nešetřil and X. Zhu, *On sparse graphs with given colorings and homomorphisms*, J. Combin. Theory Ser. B **90** (2004), 161–172; X. Zhu, *Uniquely H-colorable graphs with large girth*, J. Graph Theory **23** (1996), 33–41) that have extended and generalized the result while strengthening the techniques used to achieve it. We follow the lead of Xuding Zhu (op. cit.) who proved that, for a suitable graph H, there exist graphs of arbitrarily large girth that are uniquely H-colorable. We establish an analogue of Zhu's results in a digraph setting.

Let C and D be digraphs. A mapping $f : V(D) \to V(C)$ is a C-coloring if for every arc uv of D, either $f(u)f(v)$ is an arc of C or $f(u) = f(v)$, and the preimage of every vertex of C induces an acyclic subdigraph in D. We say that D is C-colorable if it admits a C-coloring and that D is uniquely C-colorable if it is surjectively C-colorable and any two C-colorings of D differ by an automorphism of C. We prove that if D is a digraph that is not C-colorable, then there exist graphs of arbitrarily large girth that are D-colorable but not C-colorable. Moreover, for every digraph D that is uniquely D-colorable, there exists a uniquely D-colorable digraph of arbitrarily large girth.

Acknowledgments

I would like to give special thanks to my family for their personal support through these years of graduate study and all those years before.

Throughout the preparation of these results I have been supported by teaching assistantships through the University of Montana and one summer by a UM Graduate Student Summer Research Award funded by the George and Dorothy Bryan Endowment.

I would like to thank my collaborators Ararat Harutyunyan, Bojan Mohar, and Mark Kayll with whom the main results of this dissertation will be published. I would particularly like to thank Ararat for his idea to use the Janson inequalities.

I would like to thank my entire committee for their efforts. I would also like to give special thanks for my advisor, Mark Kayll, for his dedication to guiding me through this process and for his patience, support, and friendship.

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Chapter 1

Introduction

In 1959 Paul Erdős [8] famously proved that there exist graphs that have both arbitrarily large girth and arbitrarily large chromatic number. This somewhat paradoxical result is interesting not only because it settled a question in Ramsey theory (cf. [9]), but also because Erdős' proof was an early example of a nonconstructive technique that came to be known as the probabilistic method. In Chapter 2, we discuss this result in detail, as well as two of its descendants, by Bollobás and Sauer [3] and by Zhu [22].

For readers unfamiliar with the basic definitions of graph theory, Chapter 2 also contains the relevant notation and terminology. However, we assume throughout a familiarity with basic probability theory. In particular, we make repeated use of the following result, known as Markov's Inequality.

Theorem 1.1. *If X is a nonnegative random variable with finite mean, then for any real number $t > 0$, we have $\Pr(X \geq t) \leq E[X]/t$.*

The reader unfamiliar with these concepts is encouraged to consult an introductory text on probability theory, e.g. [10, 11].

The purpose of this dissertation is to extend the results of [3, 8, 22] to directed graphs (digraphs). In order to do this we need digraph analogues of the graph notions 'girth' and 'coloring'. The prerequisite digraph definitions are developed in Chapter 3. The reader should note that there are multiple digraph analogues of graph coloring extant in the literature, and the definition we use is the one presented in [2, 18]. The digraph extensions of the results covered in Chapter 2 are stated in the final section of Chapter 3. The proofs of these extensions, which are the main contribution of this work, are presented in Chapter 4. Since these proofs are being published as part of a joint work [12], we shall attempt to be explicit about attributions throughout this dissertation. We close in Chapter 5 with a concrete application of one of our main results, Theorem 3.3, and an indication of where our investigations may lead in the future.

Chapter 2

Notation, terminology, and precursors

In this chapter, we present some of the basic definitions and results concerning graph theory, and specifically graph coloring, that will be useful throughout this dissertation. In our notation and definitions for undirected graphs, we will attempt to stay consistent with [4].

2.1 Basic definitions

In this work, we consider only finite and simple graphs, i.e. those with no multiple edges or loops. A *graph* G is defined to be a finite set $V(G)$ of *vertices* and a finite set $E(G)$, disjoint from $V(G)$, of edges. Each edge is uniquely associated with exactly two distinct vertices and any two such vertices can be associated with at most one edge; we say that these vertices are *incident* to the edge. Two vertices that are both incident to the same edge are said to be *adjacent*.

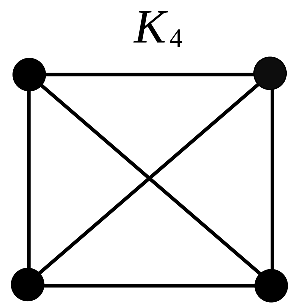

Figure 2.1: K_4, the complete graph on 4 vertices

The graph with r vertices, every two of which are adjacent, is called the *complete* graph on r vertices and denoted by K_r. For example, the complete graph K_4 on four vertices is shown in Figure 2.1.

If X is a subset of the vertex set of G, we define the subgraph $G[X]$ *induced* by X to be the graph with vertex set X and with edge set those edges of G both incident vertices of which are contained in X. If a subset $X \subseteq V(G)$ induces a graph with no edges, we say that X is a *stable* set in G.

A *cycle* on three or more vertices is a graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise. The cycle on r vertices is denoted by C_r.

Usually when we refer to a cycle in a graph, we mean a subgraph that is a cycle. The *girth* of a graph G is the length of a shortest cycle in G. The graph in Figure 2.2 has girth 4. If a graph has no cycles we say that it is *acyclic* and that it has infinite girth.

An *r-vertex-coloring* of a graph G, or simply an *r-coloring*, is a function $\sigma : V(G) \rightarrow \{1, 2, \ldots, r\}$. We say that an r-coloring σ is *proper* if no two adjacent vertices are assigned the same color under σ. A graph is said to be *r-colorable* if it admits a proper r-coloring. In Figure 2.3 we give an example of a (proper) 3-coloring of a graph.

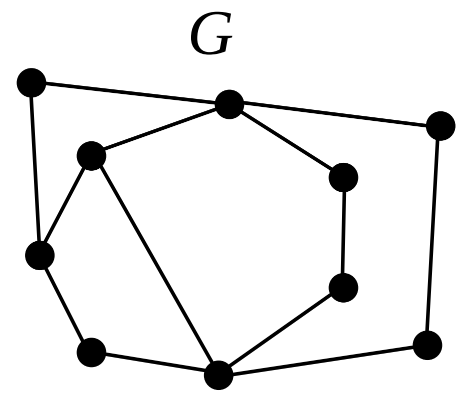

Figure 2.2: A girth 4 graph

Equivalently, an r-coloring can be seen as a partition $\{V_1, V_2, \ldots, V_r\}$ of the vertex set $V(G)$, where, for each $i \in \{1, 2, \ldots, r\}$, the part V_i is the set of vertices assigned color i. In this formulation, the V_i are called *color classes* of the coloring, and the coloring is proper if each V_i is a stable set. In Figure 2.3 the partition $\{\{x_1, x_3, x_4, x_5, x_8\},\ \{x_2, x_6, x_7, x_{10}\},\ \{x_9\}\}$ is an r-coloring (equivalent to the previous one described).

The *chromatic number* $\chi(G)$ of a graph G is the smallest number r of colors such that G admits a proper r-coloring. The graph in Figure 2.3 is 3-colorable (as shown), but it is not 2-colorable since it contains cycles of odd length; therefore, its chromatic number is 3. A graph with chromatic number r is said to be *r-chromatic*.

2.2 Large girth, large chromatic number

Do there exist graphs with arbitrarily large chromatic number? A natural example of an r-chromatic graph is K_r. Since every two vertices are adjacent, in a proper coloring each vertex must be its own color class. Notice that K_r has as many edges as are possible in a simple graph. When one attempts to construct a graph with a large chromatic number, one natural thing to do is to include 'many' edges. In fact, the easiest way to ensure that a graph on n

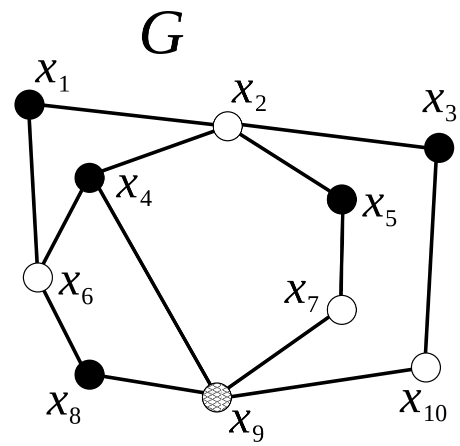

Figure 2.3: A proper 3-coloring of G

vertices is r-chromatic is to make sure it contains K_r as a subgraph.

On the other hand, if one were to attempt to construct a graph with large girth, one natural thing to do would be to include 'few' edges. One example of a girth g graph is C_g, the cycle on g vertices. This graph has girth g and only g edges. In a given graph G, if any g vertices of G induce a subgraph with at least g edges (and there are $\binom{g}{2}$ potential edges in such an induced subgraph), it is impossible for the graph to have girth greater than g.

Intuitively, it may seem like the two properties —large girth and large chromatic number— work against each other. When we want large chromatic number, we want 'lots' of edges; when we want large girth, we want 'few' edges. Is it possible to find a graph with large girth and large chromatic number? Or does having large chromatic number preclude the possibility of having large girth (and vice versa)?

It is certainly difficult to construct a graph with large girth and large chromatic number. In fact, even constructing a graph that has large chromatic number but no triangles (3-cycles) is nontrivial. Tutte, writing under a pseudonym, first found such a construction [5, 6]; this result was rediscovered several times, nominally by Mycielski [20]. He created a construction (now referred to as the Mycielskian) that produces triangle-free graphs of arbitrarily large

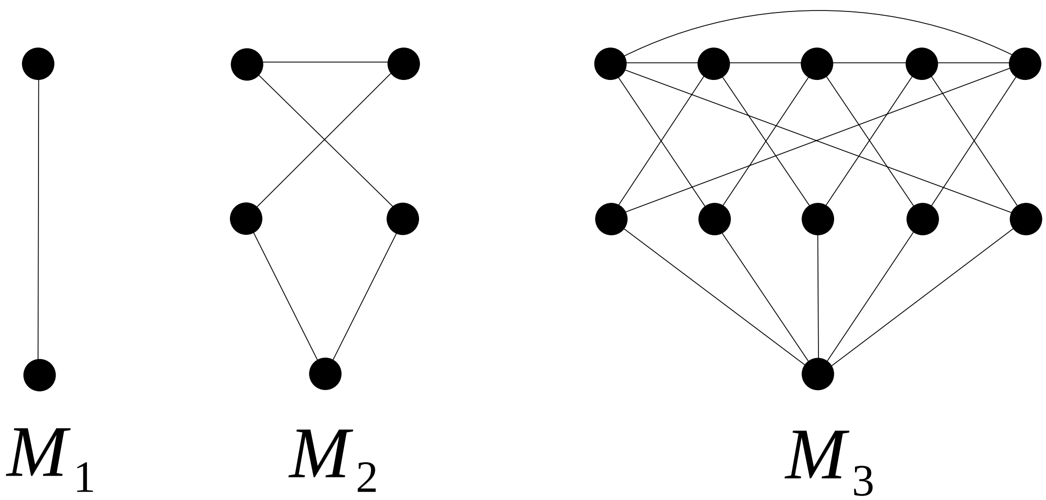

Figure 2.4: The first three Mycielski graphs

chromatic number; see Figure 2.4, where $\chi(M_i) = i + 1$ for $i = 1, 2, 3$. Kelly and Kelly [15] managed to show that graphs of arbitrarily large chromatic number and girth 6 exist (they also rediscovered Tutte's result).

Trying to find a construction that gives graphs of arbitrarily large girth and arbitrarily large chromatic number proved to be quite difficult. Since this dissertation deals with nonconstructive arguments, we will not discuss this further except to note that a constructive approach [17] eventually succeeded, but this constructive argument was not nearly as elegant as the nonconstructive one discovered by Paul Erdős [8].

Theorem 2.1 (Erdős, 1959)**.** *For any integers g,r, there exists a graph of girth at least g and chromatic number at least r.*

Some years before Theorem 2.1 appeared, Erdős [7] had used the probabilistic method to find a lower bound on the diagonal Ramsey numbers. The model used in that proof created a probability space of all graphs on n vertices where each graph was weighted equally. (This was a natural thing to do in that proof and it is a disguised counting argument). In the present setting it turns out to be more convenient to weight the graphs unequally.

The following proof is a modified version of the proof in Alon and Spencer's book [1].

Proof of Theorem 2.1. We will eventually fix a positive integer n large enough to support our estimates. Fix $\epsilon < 1/g$ and $p := n^{\epsilon - 1}$. Let $\mathcal{G}$ be the probability space of all graphs on n vertices, where each edge exists with probability p, independently of all other edges.

We first want to show that in a randomly selected graph $G \in \mathcal{G}$, the probability of there being a 'small' number of short cycles is high. For the prescribed g (which we may assume is at least 3) let X be the number of cycles of length at most g. For each i with $3 \leq i \leq g$, there are $n(n-1)(n-2)\cdots(n-i+1)/(2i)$ possible cycles of length i, each of which is present in G with probability p^i. So the expected number of cycles of length at most g satisfies

$$\begin{aligned} E[X] &= \sum_{i=3}^{g} \frac{n(n-1)(n-2)\cdots(n-i+1)}{2i} p^i \\ &\leq \sum_{i=3}^{g} \frac{n^i}{2i}(n^{\epsilon i - i}) \\ &< n^{\epsilon g}. \end{aligned}$$

By Markov's Inequality (Theorem 1.1 with $t = n/2$), $\Pr(X \geq n/2) \leq E[X]/(n/2) < 2n^{\epsilon g - 1}$. Since $\epsilon g < 1$, $\Pr(X \geq n/2)$ approaches zero as $n \to \infty$. We assume that n is large enough so that $\Pr(X \geq n/2) < 1/2$. Consequently $\Pr(X < n/2) > 1/2$.

We also need to argue that having a large chromatic number has high probability. But instead of obtaining a lower bound on the chromatic number $\chi(G)$, we will obtain an upper bound on the stability number $\alpha(G)$ (the size of a largest stable set). The reason this is helpful is that the stability number is an upper bound on the size of every color class, and so the product of the stability number and the chromatic number is an upper bound on the number of vertices; i.e.,

$$\chi(G)\alpha(G) \geq n. \tag{2.1}$$

Therefore, an upper bound on $\alpha(G)$ will give a lower bound on $\chi(G)$.

Set $x = \lceil (3/p)\log n \rceil + 1$. We will estimate the probability that the stability number of G is at least x. Clearly there are $\binom{n}{x}$ ways to select a subset of x vertices from n. The probability that any particular x-set is stable is $(1-p)^{\binom{x}{2}}$, the probability that none of the possible edges exist. And so we have that

$$\Pr(\alpha(G) \geq x) \leq \binom{n}{x}(1-p)^{\binom{x}{2}} < n^x e^{-px(x-1)/2} = (ne^{-p(x-1)/2})^x.$$

We selected x so that this last upper bound is (super-)exponentially decreasing with n. Therefore we can pick n large enough to make this upper bound less than $1/2$ and so $\alpha(G) \leq x < (4/p)\log n$ with probability greater than $1/2$. (Using the probabilistic method is a balancing act between tightness and clean bounds. In the end, we only care what is happening asymptotically; so while the '+1' was convenient so far, we have absorbed it with a larger coefficient as this will be more convenient starting here.)

In this probability space, the probability that a randomly selected graph has fewer than $n/2$ cycles shorter than g (*short cycles*) is greater than $1/2$, and the probability that a randomly selected graph has stability number at most $(4/p)\log n$ is also greater than $1/2$. Therefore, there must be at least one graph on n vertices with both properties. We select such a graph G and delete one vertex from each short cycle to obtain a graph G^* with at least $n/2$ vertices. The graph G^* has girth at least g and $\alpha(G^*) \leq \alpha(G)$ (clearly the deletion of vertices does not increase the size of a largest stable set).

We now use (2.1) on G^* to achieve a lower bound on $\chi(G^*)$. The chromatic number of G^* satisfies

$$\chi(G^*) \geq \frac{n/2}{\alpha(G^*)} > \frac{n/2}{4n^{1-\epsilon}\log n} = \frac{n^\epsilon}{8\log n}.$$

The lower bound on $\chi(G^*)$ is an unbounded function that increases with n and thus we may choose n large enough so that $n^\epsilon/(8\log n) \geq r$. $\square$

This proof was not just important in that it settled a difficult question. It was also important as an early example of what came to be known as the probabilistic method.

The idea behind the use of probability in this proof is fairly simple; we are trying to prove that there exist graphs that have both large girth and large chromatic number. Once we construct a probability space on graphs with n vertices in which a randomly selected graph has large girth with high probability (greater than 1/2) and has large chromatic number with high probability (greater than 1/2), we may conclude that there exists a graph that has both properties simultaneously. (Of course this is an oversimplification; what was actually found was a graph from which we could break all the short cycles while keeping a large chromatic number.)

Arguments of this type are nonconstructive because we do not actually find such a graph; we simply prove that one exists. Using Markov's Inequality on integer-valued functions is usually called the First Moment Method; see, e.g., [19, Chapter 3]. Generally, a combinatorial argument lurks within a First Moment Method proof. In Erdős' proof, we're in essence estimating how many graphs have both properties and weighting our counts using the language of probability theory.

Although conversion to combinatorial arguments may be possible with First Moment Method proofs, when the second moment is introduced (using variance to establish concentration), such counting parallels are more difficult to find. It is seen then that the tools of probability theory are essential in these proofs and not merely a convenient way to keep track of a counting argument. Most of the arguments in this dissertation use the First Moment Method, but the second moment is needed for part of the proof of Theorem 3.3.

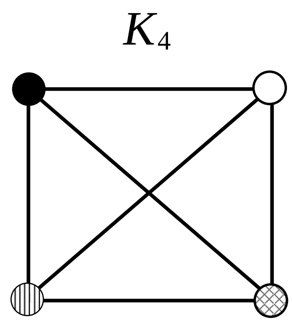

Figure 2.5: This graph is uniquely 4-colorable.

2.3 Unique colorability

Notice that the 3-coloring in Figure 2.3 is not the only possible 3-coloring of that graph. For example, if we assign the colors as in Figure 2.3, then the color classes are $\{x_1, x_3, x_4, x_5, x_8\}$, $\{x_2, x_6, x_7, x_{10}\}$, $\{x_9\}$; another proper 3-coloring would be the partition $\{\{x_4, x_5, x_8\},\ \{x_2, x_6, x_7, x_{10}\},\ \{x_1, x_3, x_9\}\}$, which is clearly distinct from the first coloring. This contrasts with the situation when we properly 4-color K_4; any such coloring, e.g. Figure 2.5, yields essentially the same 4-coloring (the coloring may be nominally different, but the color classes will give the same partition of $V(K_4)$). A graph is *uniquely r-colorable* if it is r-colorable and any proper r-coloring yields the same color classes.

Now it is natural to ask if we can strengthen Theorem 2.1; that is, for any given integer $r \geq 1$, does there exist a graph of arbitrarily large girth that is uniquely r-colorable? This question was answered affirmatively by Bollobás and Sauer in 1976 [3].

Theorem 2.2 (B. Bollobás and N. Sauer, 1976)**.** *For any integers g, r there exists a graph of girth at least g that is uniquely r-colorable.*

Notice that a uniquely r-colorable graph G has $\chi(G) = r$; otherwise, if $\chi(G) < r$, then a $\chi(G)$-coloring σ_1 and a surjective r-coloring σ_2 would give rise to distinct partitions of $V(G)$ into color classes of σ_1 and σ_2, contradicting the unique r-colorability of G. Thus, Theorem 2.2

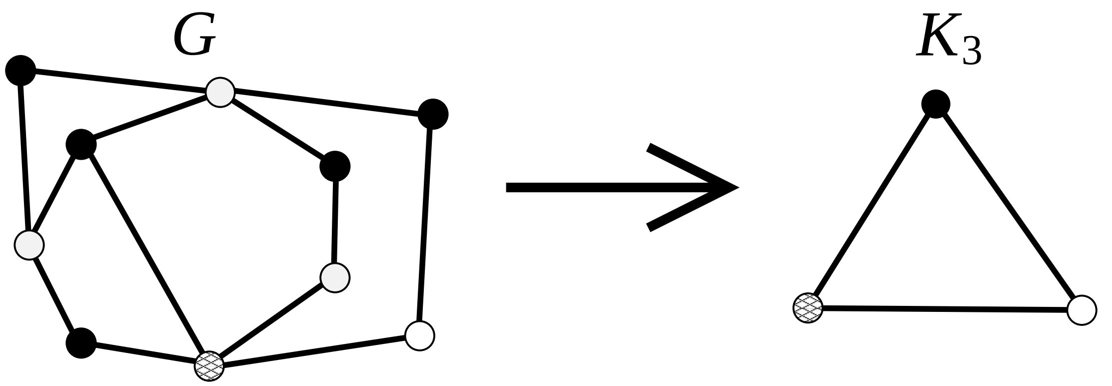

Figure 2.6: A graph homomorphism to K_3

implies Theorem 2.1. Not only is Theorem 2.2 a generalization and strengthening of Erdős' result, but also its proof is a refinement of Erdős' proof. Though the proof uses the probabilistic method, the probability space Bollobás and Sauer employ is slightly different. The model we presented for the proof of Erdős' result (Theorem 2.1) took any possible edge between any two of n vertices with probability p. In order to force probable unique colorability, Bollobás and Sauer started with r stable sets of n vertices, effectively predetermining an r-coloring. Because the n-sets are stable, graphs in the new model are r-colorable. The idea then was to add edges between the stable sets so that no other r-coloring is possible, but not so many edges that a cycle of length less than g appears. Actually, like Erdős, Bollobás and Sauer examined the probability that there are few short cycles and the probability that the graph is uniquely r-colorable after the deletion of a small set of edges (our presented proof of Theorem 2.1 deleted vertices, although Erdős himself also deleted edges).

So the essential change was predetermining the color classes (and there were a couple of new computational tricks). It is perhaps surprising that these refinements produced such a stronger result.

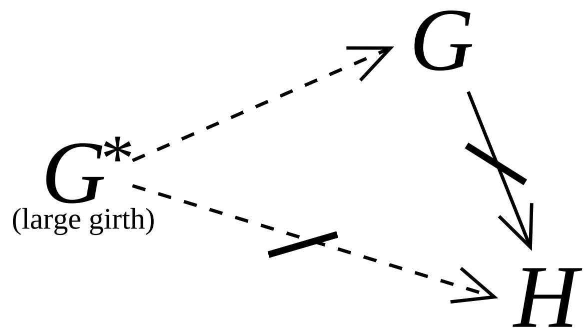

Figure 2.7: Diagram of Theorem 2.3

2.4 Graph homomorphisms

The category theorist views most of mathematics as the study of sets with structure together with structure-preserving morphisms (see, e.g., [16]). While the level of abstraction of category theory is not necessary for this dissertation, it is nonetheless useful to think of a graph homomorphism, qualitatively, as a map between graphs that preserves the 'graph structure'.

A *homomorphism* from a graph G to a graph H is a map $\phi : V(G) \to V(H)$ such that if $uv \in E(G)$ then $\phi(u)\phi(v) \in E(H)$ (cf. [13]). Qualitatively, this means vertices must be mapped to vertices and the map preserves adjacency. If such a map from G to H exists, we say that G is *H-colorable*. The motivation for this terminology follows in the next paragraph.

Notice that in Figure 2.6 the preimages of the vertices define a partition of the vertex set into stable sets. In fact a graph G is r-colorable if and only if there exists a homomorphism from G to K_r. We can see this by explicitly constructing the coloring given such a homomorphism, or likewise explicitly constructing the homomorphism given a coloring. Given a homomorphism to K_r, we simply take the preimage of each vertex as a color class. Conversely, given an r-coloring of G, simply assign each vertex in K_r a color and map that color class there; since no two vertices of a color class are adjacent and the vertices of K_r are pairwise adjacent, this results in a homomorphism. Thus G is r-colorable if and only if G is K_r-colorable. We have

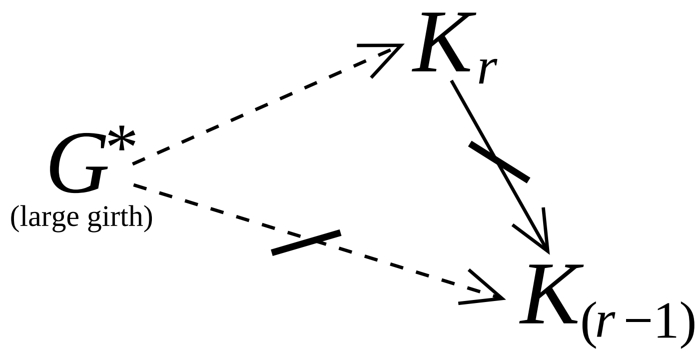

Figure 2.8: Theorem 2.3 implies Theorem 2.1.

overloaded the term 'colorable' to take either a positive integer or a graph as a prefix, but it is always clear from context what we mean.

Now that we have a generalization of coloring, it's natural to ask if Erdős' result, Theorem 2.1, generalizes in this direction. This question was settled by Zhu [22] in 1996.

Theorem 2.3 (X. Zhu, 1996)**.** *If G and H are graphs such that G is not H-colorable, then for any positive integer g, there exists a graph G^* of girth at least g that is G-colorable but not H-colorable.*

Figure 2.7 illustrates Theorem 2.3. To see that Theorem 2.3 implies Theorem 2.1, consider Figure 2.8. Since K_r is not $K_{(r-1)}$-colorable, these graphs satisfy the hypotheses of Theorem 2.3; therefore, for any positive integer g, there exists a graph G^* of girth at least g that is K_r-colorable but not $K_{(r-1)}$-colorable. Since the latter conditions imply that $\chi(G^*) = r$, the graph G^* witnesses the conclusion of Theorem 2.1. Theorem 2.3 is, of course, more general than Theorem 2.1, as G and H are not restricted to being complete graphs under the hypotheses of Theorem 2.3.

A natural question now is whether Bollobás and Sauer's result, Theorem 2.2, generalizes to graph homomorphisms. First we need a notion of unique colorability for homomorphisms. We say that G is *uniquely H-colorable* if it is surjectively H-colorable and any two H-colorings

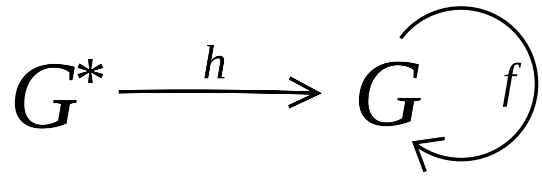

Figure 2.9: G must be a core.

of G differ by an automorphism of H (i.e. for any two H-colorings ϕ_1, ϕ_2 of G, there exists an automorphism σ of H such that $\phi_2 = \sigma \circ \phi_1$). Being uniquely K_r-colorable coincides with being uniquely r-colorable.

That there exist uniquely K_r-colorable graphs (cf. Theorem 2.2) depends on a property enjoyed by complete graphs that is a necessary condition on a general graph G for the existence of uniquely G-colorable graphs. A graph G is a *core* if the only homomorphisms from G to itself are automorphisms. It is easy to check that K_r is a core for every integer $r \geq 1$; any non-automorphism homomorphism from K_r to itself must map at least two vertices to a single vertex, which is impossible since any two vertices are adjacent in K_r. To see that G being a core is necessary for the existence of uniquely G-colorable graphs, consider Figure 2.9. If there is some non-bijective homomorphism $f : V(G) \to V(G)$, then any homomorphism $h : V(G^*) \to V(G)$ from a graph G^* to G gives rise to another $(f \circ h)$ that does not differ from h by an automorphism (because f is not an automorphism).

We conclude that if a graph G is not a core then it is not possible to find uniquely G-colorable graphs. Therefore G being a core is necessary, but is G being a core enough to ensure the existence of uniquely G-colorable graphs? Surprisingly, this necessary condition is also sufficient to ensure the existence of uniquely G-colorable graphs, even ones with large girth.

Theorem 2.4 (X. Zhu, 1996)**.** *If a graph G is a core, then for any positive integer g, there exists a graph G^* of girth at least g that is uniquely G-colorable.*

Since complete graphs are cores, Theorem 2.4 implies Theorem 2.2. Since G is not restricted to complete graphs, Theorem 2.4 implies more. In [22], Zhu uses Theorem 2.4 to show that there exist graphs with arbitrarily large girth and with any prescribed circular chromatic number (see [23] for a survey on the circular chromatic number).

Again, we are not interested in these results simply because they generalize Erdős' Theorem 2.1 and Bollobás and Sauer's Theorem 2.2. Their methods of proof — stepwise refinements of the earlier techniques — attract attention by themselves through their noticeable increase in strength. Each step offered its share of nontrivial insights and is rightfully viewed as a new and significant contribution to mathematics. But it would not be correct to view the successive results in a vacuum, as they are more clearly understood when viewed as an historical progression. The results of this dissertation add another step to this progression. Since we extend these results to directed graphs with certain homomorphisms, the next chapter introduces the necessary concepts, while the succeeding one presents the new contributions.

Chapter 3

Directed graphs and acyclic colorings

The main purpose of this dissertation is to generalize the results covered in the preceding section to directed graphs (digraphs). In order to discuss the new results we will need some basic digraph definitions as well as a notion of digraph coloring. We will then show that our notion of coloring digraphs is a generalization of coloring undirected graphs.

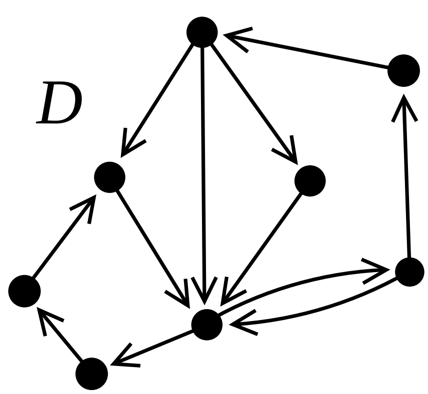

Figure 3.1: A digraph

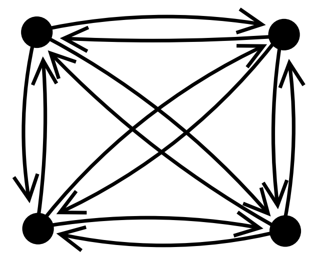

Figure 3.2: K_4, the complete digraph on 4 vertices

3.1 Digraph definitions

Similar to our graph definition, we consider only finite and simple digraphs, i.e. with no multiple arcs or loops. A *digraph* D is defined to be a finite set $V(D)$ of *vertices* and a finite set $E(D)$, disjoint from $V(D)$, of ordered pairs of distinct vertices called *arcs*, each particular ordered pair occurring at most once; these vertices are said to be *incident* to the arc. Note that uv and vu are different arcs, so both may be present. We represent the arc graphically as an arrow; for example, see Figure 3.1. We say that vertices u and v of a digraph D are *adjacent* whenever either uv or vu (or both) is an arc of D.

Many graphical notions have digraph analogues. Two such concepts are 'complete graph' and 'cycle'. A *complete digraph* K_r has r vertices and every ordered pair of two distinct vertices as an arc. (We are overloading this symbol, but it will be clear from context whether we mean the complete graph or the complete digraph.) Figure 3.2 depicts the complete digraph K_4.

A *cycle* on two or more vertices is a simple digraph whose vertices can be arranged in a cyclic sequence in such a way that there is an arc from u to v if and only if u directly precedes v in the sequence. The digraph that is a cycle on r vertices is denoted C_r. (Again, it will be clear from context whether we mean the directed or undirected cycle.)

As in the graph case, usually when we refer to a 'cycle' in a digraph, we mean the subdigraph

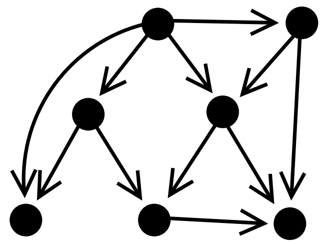

Figure 3.3: An acyclic digraph

that is the specific cycle. An *acyclic* digraph is a digraph containing no cycles; Figure 3.3 depicts such a digraph. The *girth* of a digraph is the length of its shortest cycle. For example, the girth of the digraph in Figure 3.1 is two. The girth of an acyclic digraph is, by convention, infinite.

Now that we have a notion of digraph girth, we will move on to generalize the idea of vertex coloring to digraphs.

3.2 Acyclic colorings

There are a number of ways to generalize vertex colorings to digraphs. One natural one partitions the vertex set into stable sets as we did with graphs, calling each block of the partition a color class. This approach has proved fruitful in studying the subdigraphs of a given digraph; see, e.g., [4, Section 14.1]. The generalization we consider in this dissertation follows [2, 18] and has other interesting consequences for digraphs.

If D is a digraph and $S \subseteq V(D)$, then the subdigraph $D[S]$ of D *induced* by S has vertex set S and arc set the subset of all $e \in E(D)$ such that S contains both incident vertices of e.

We define an *acyclic r-coloring* of D to be a partition of $V(D)$ into r sets (that we will call *color classes*) each of which induces an acyclic subdigraph of D. The *acyclic chromatic number* $\chi(D)$ of D is the smallest number r of colors such that D admits an acyclic r-coloring. The

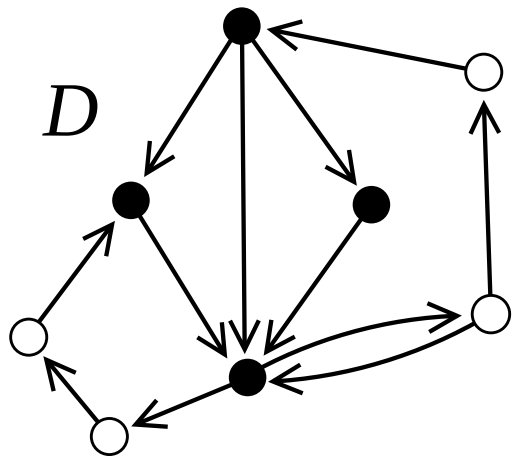

Figure 3.4: An acyclic 2-coloring

digraph in Figure 3.4 is acyclically 2-colorable (as shown), but it is not acyclically 1-colorable since it contains cycles; therefore, its acyclic chromatic number is 2. When only digraphs are being discussed, we will again drop the 'acyclic' modifier and simply say *r-colorable* (when $\chi(D) \leq r$) and *r-chromatic* (when $\chi(D) = r$).

3.3 Acyclic homomorphisms

We will generalize acyclic colorings of digraphs to acyclic homomorphisms of digraphs similarly to the way we generalized colorings of graphs to homomorphisms of graphs. Roughly speaking, an acyclic homomorphism is a map from the vertex set of one digraph to another that preserves the digraph structure and preserves the acyclic structure; that is, cycles are not mapped to vertices. More formally, an *acyclic homomorphism* of a digraph D into a digraph C is a function $\phi\colon V(D) \to V(C)$ such that:

(i) for every vertex $v \in V(C)$, the subdigraph of D induced by $\phi^{-1}(v)$ is acyclic;

(ii) for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$, or $\phi(u)\phi(v)$ is an arc of C.

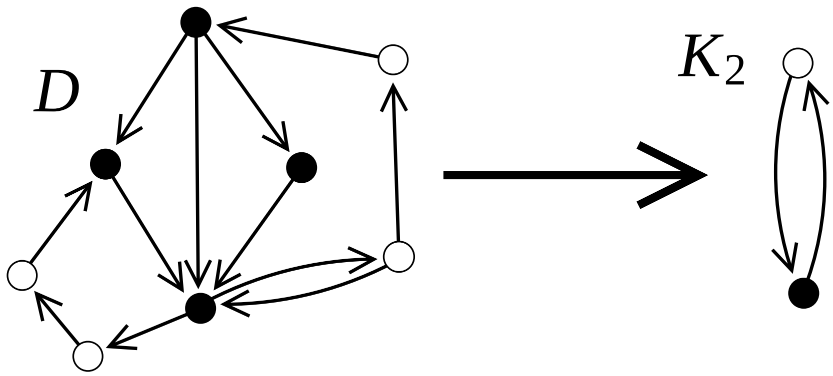

Figure 3.5: An acyclic homomorphism from D to K_2

Notice that in Figure 3.5 the preimages of the vertices of K_r define a partition of the vertex set of D into color classes. In fact, a digraph D is r-colorable if and only if there exists an acyclic homomorphism from D to K_r. We can see this by explicitly constructing the coloring given such an acyclic homomorphism, or likewise explicitly constructing the acyclic homomorphism given a coloring. Given an acyclic homomorphism from D to K_r, we simply take the preimage of each vertex as a color class. The requirement (i) guarantees that this yields an r-coloring of D. Conversely, given an r-coloring of D, simply assign each vertex in K_r a color and map that color class there; since color classes do not induce cycles, this results in an acyclic homomorphism from D to K_r. And so D is r-colorable if and only if it is K_r-colorable.

3.4 A generalization of graph coloring

This section explains how acyclic digraph coloring and acyclic homomorphisms are generalizations of graph coloring and graph homomorphisms.

A digraph D is said to be *symmetric* when, for every pair vertices $u, v \in V(D)$, $uv \in E(D)$ if and only if $vu \in E(D)$. Note that in a symmetric digraph, every arc is part of a two-cycle. Consequently, in a symmetric digraph, the only subsets of vertices that induce an acyclic

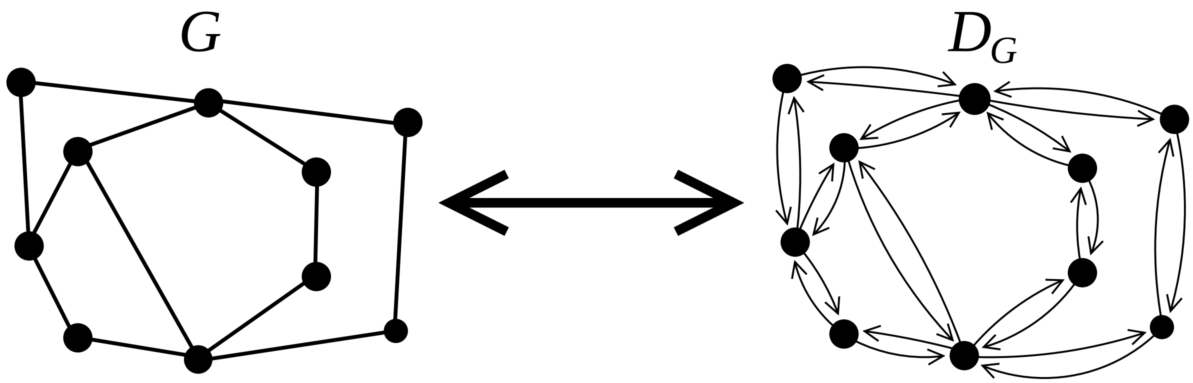

Figure 3.6: An illustration of a cryptomorphism

subdigraph are stable sets.

There is an ontological concept in mathematics called the 'cryptomorphism'. The idea is that when two different mathematical structures are 'essentially' the same, there is a cryptomorphism between them. What this means is that every part of the first structure has a natural analogue in the second, and one can use the cryptomorphism to translate ideas back and forth between the two structures since they are essentially the same in some (rigorous) transcendental sense.

We now describe the cryptomorphism between graphs and symmetric digraphs. Given a graph G, we construct a digraph D_G as follows: set $V(D_G) = V(G)$; then for every edge $uv \in E(G)$ we include the arcs uv and vu in $E(D_G)$. That is, we take every undirected edge in G and replace it with a directed 2-cycle in D_G. Obviously this cryptomorphism is invertible, and for any symmetric digraph D we can construct the cryptomorphically equivalent graph G with $D_G = D$. For example, see Figure 3.6.

Using this construction and cryptomorphism we can see that a coloring of G is an acyclic coloring of D_G and vice versa. Consider a coloring σ of G. Since every color class determines a stable set, this set is acyclic in D_G and so σ is an acyclic coloring of D_G. Consider an acyclic coloring ψ of D_G. Since every color class is a set that induces an acyclic subdigraph of D_G,

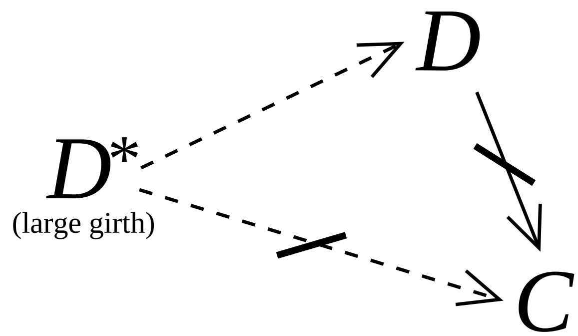

Figure 3.7: Diagram of Theorem 3.2

and any two adjacent vertices in D_G would induce a two-cycle, we have that each color class of ψ is a stable set of D_G and therefore ψ is also a coloring of G. Using this construction, we can likewise see that a graph G is H-colorable if and only if D_G is D_H-colorable (the reasoning is analogous to that just presented for r-coloring).

3.5 New results

Now that we have a generalization of coloring for digraphs, the natural question to ask is whether Erdős' result (Theorem 2.1) generalizes to the new concept; that is to say, does there exist a digraph with arbitrarily large girth and arbitrarily large acyclic chromatic number? This was answered in the affirmative by Bokal et al. in [2, Theorem 4.1] using a probabilistic argument similar to the proof of Theorem 2.1.

Theorem 3.1 (D. Bokal, G. Fijavž, M. Juvan, P.M. Kayll and B. Mohar, 2004)**.** *For any integers g,r, there exists a digraph of girth at least g and acyclic chromatic number at least r.*

One of the main achievements of this dissertation is proving the analogue of Zhu's Theorem 2.3 for digraphs with acyclic homomorphisms.

Theorem 3.2. *If D and C are digraphs such that D is not C-colorable, then for any positive integer g, there exists a digraph D^* of girth at least g that is D-colorable but not C-colorable.*

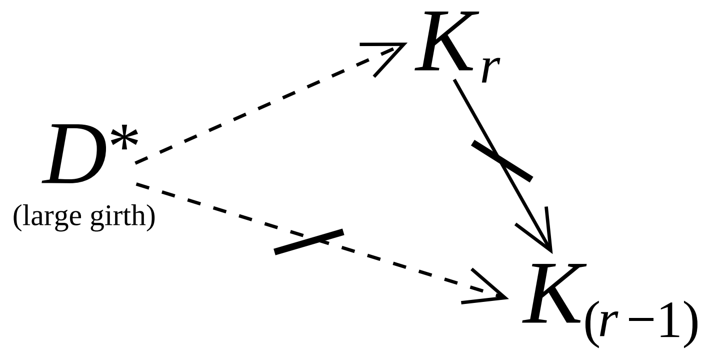

Figure 3.8: Large girth, large acyclic chromatic number

Figure 3.7 illustrates Theorem 3.2. To see that Theorem 3.2 implies [2, Theorem 4.1] consider Figure 3.8. Since K_r is not $K_{(r-1)}$-colorable, these graphs satisfy the hypotheses of Theorem 3.2; therefore, for any positive integer g, there exists a digraph D^* of girth at least g that is K_r-colorable but not $K_{(r-1)}$-colorable. This is equivalent to saying that there exists a digraph with girth at least g and with acyclic chromatic number exactly r. Theorem 3.2 of course is more general than [2, Theorem 4.1], as D and C are not restricted to being complete graphs under its hypotheses.

A natural question now is whether Bollobás and Sauer's result, Theorem 2.2, has a digraph analogue. First, we need a notion of unique colorability for acyclic homomorphisms. We say that D is *uniquely C-colorable* if it is surjectively C-colorable and any two C-colorings of D differ by an automorphism of C (i.e. for any two C-colorings ϕ_1, ϕ_2 of C, there exists an automorphism σ of C such that $\phi_2 = \sigma \circ \phi_1$).

As with graphs, there is an obvious necessary condition on a general digraph D for the existence of uniquely D-colorable digraphs. A digraph D is a *core* if the only acyclic homomorphisms from D to itself are automorphisms. It is easy to check that the digraph K_r is a core for every integer $r \geq 1$; any non-bijective acyclic homomorphism from K_r to itself must map at least two vertices to a single vertex, which is impossible since any two vertices induce a two-cycle in K_r. To see that D being a core is necessary for the existence of uniquely D-

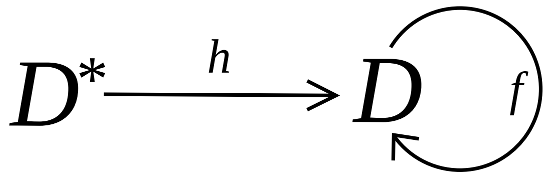

Figure 3.9: D must be a core.

colorable digraphs, consider Figure 3.9. If there is some non-bijective acyclic homomorphism $f : V(D) \to V(D)$, then every homomorphism $h : V(D^*) \to V(D)$ from a digraph D^* to D gives rise to another $(f \circ h)$ that does not differ from h by an automorphism (because f is not an automorphism).

We conclude that if D is not a core, then it is not possible to find uniquely D-colorable digraphs. Therefore G being a core is necessary, but is D being a core enough to ensure the existence of uniquely D-colorable digraphs? As in the graph case, this necessary condition is also sufficient to ensure the existence of uniquely D-colorable digraphs, even ones with large girth.

Theorem 3.3. *For any core D and any positive integer g, there is a digraph D^* of girth at least g that is uniquely D-colorable.*

Since K_r is a core, Theorem 3.3 implies that there exists a digraph of girth at least g that is uniquely r-colorable (an analogue of Bollobás and Sauer's Theorem 2.2 for digraphs with acyclic colorings). Theorem 3.3 is more general than this analogue because D is not restricted to complete graphs.

Now that we have introduced all the prerequisite material, we will prove Theorem 3.2 and Theorem 3.3 in the next chapter.

Chapter 4

D-coloring digraphs

Here we present the main results of this dissertation, namely the proofs of Theorem 3.2 (in Section 4.1) and Theorem 3.3 (in Section 4.3). Both proofs are probabilistic and follow the main ideas of Zhu's Theorems 2.3 and 2.4, which themselves trace back to the proof of Bollobás and Sauer's Theorem 2.2 and ultimately to Erdős' Theorem 2.1. However, just as all of these earlier refinements required new inspiration, new insights and approaches are again needed to move to the digraph and acyclic homomorphism setting.

4.1 Proof of Theorem 3.2

This section and Section 4.3 have heavy overlap with the joint work [12]. Where appropriate, we indicate the passages contributed by collaborators.

We begin by setting up a suitable random digraph model. Suppose that $V(D) = \{1, 2, \ldots, k\}$ and that $q = |E(D)|$. Let n be a (large) positive integer, and let D_n be the digraph obtained from D as follows: replace every vertex i with a (temporarily) stable set V_i of n ordered vertices $v_1, v_2, ..., v_n$, and replace each arc ij of D by the set of all possible n^2 arcs from V_i to V_j; additionally, add each arc $v_r v_s$ such that $v_r, v_s \in V_i$ and $r < s$. Clearly, $|V(D_n)| = kn$ and $|E(D_n)| = qn^2 + k\binom{n}{2}$.

Now fix a positive $\varepsilon < 1/(4g)$. Our random digraph model $\mathcal{D} = \mathcal{D}(D_n, p)$ consists of those spanning subdigraphs of D_n in which the arcs of D_n are chosen randomly and independently with probability $p = n^{\varepsilon - 1}$.

As usual in nonconstructive probabilistic proofs of results of this nature (cf. [3, 21, 22]), the idea is to show that most digraphs in $\mathcal{D}$ have only a few short cycles, and for most digraphs $H \in \mathcal{D}$, the subdigraph of H obtained by removing an arbitrary yet small set of arcs is not C-colorable. Choosing an $H \in \mathcal{D}$ with both these properties, we can force the girth to be large by deleting an arc from each short cycle. Since the set A_0 of deleted arcs is small, the resulting digraph $H - A_0$ satisfies the desired conclusion of Theorem 3.2.

To make this description more precise, let $\mathcal{D}_1$ denote the set of digraphs in $\mathcal{D}$ containing at most $\lceil n^{g\varepsilon} \rceil$ cycles of length less than g, and let $\mathcal{D}_2$ be the set of digraphs $H \in \mathcal{D}$ that have the property that $H - A_0$ is not C-colorable for any set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs. We will show that

$$|\mathcal{D}_1| > \left(1 - n^{-\varepsilon/2}\right) |\mathcal{D}| \tag{4.1}$$

and

$$|\mathcal{D}_2| > \left(1 - e^{-n}\right) |\mathcal{D}| . \tag{4.2}$$

Since (4.1) and (4.2) imply that $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \varnothing$ (for sufficiently large n), there exists a digraph $H \in \mathcal{D}_1 \cap \mathcal{D}_2$. Now $H \in \mathcal{D}_1$ implies that there is a set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs whose removal leaves a digraph $D^* := H - A_0$ of girth at least g, while $H \in \mathcal{D}_2$ means that D^* is not C-colorable. Thus, it remains to establish (4.1) and (4.2).

Proof of (4.1). The expected number N_ℓ of cycles of length ℓ in a digraph $H \in \mathcal{D}$ is at most

$$\binom{kn}{\ell}(\ell-1)!\,p^\ell \tag{4.3}$$

since there are $\binom{kn}{\ell}(\ell-1)!$ ways of choosing a cyclic sequence of ℓ vertices as a candidate for a cycle, and such an ℓ-cycle occurs in $\mathcal{D}$ with probability either 0 or p^ℓ. It is easy to see that the product of the first two factors in (4.3) is smaller than $(kn)^\ell/\ell$. Therefore, if n is large enough, then

$$\sum_{\ell=2}^{g-1} N_\ell \le \sum_{\ell=2}^{g-1} \frac{(kn^\varepsilon)^\ell}{\ell} < k^{g-1}n^{(g-1)\varepsilon} < n^{-\varepsilon/2}n^{g\varepsilon}.$$

Now (4.1) follows easily from Markov's Inequality (Theorem 1.1 with $t = n^{g\varepsilon}$). □

Proof of (4.2). We shall argue that $|\mathcal{D} \setminus \mathcal{D}_2| < e^{-n}|\mathcal{D}|$. If $H \in \mathcal{D} \setminus \mathcal{D}_2$, then there is a set A_0 of at most $\lceil n^{g\varepsilon} \rceil$ arcs of H so that $H - A_0$ admits an acyclic homomorphism h to C (i.e., $H - A_0$ is C-colorable). Let $k' = |V(C)|$. By the pigeonhole principle, for each $i \in V(D)$, there exists a vertex $x_i \in V(C)$ such that $|V_i \cap h^{-1}(x_i)| \ge n/k'$. Define $\phi\colon V(D) \to V(C)$ by setting $\phi(i) = x_i$. Since $n/k' \gg n^{g\varepsilon}$ (this means that $(n/k')/n^{g\varepsilon} \to \infty$ as $n \to \infty$), the set $V_i \cap h^{-1}(x_i)$ contains a subset W_i of cardinality $w := \lceil n/(2k') \rceil$ such that no arc in A_0 has an end vertex in W_i.

Since D is not C-colorable, the function ϕ is not an acyclic homomorphism. Therefore, either there is an arc $ij \in E(D)$ such that $\phi(i) \ne \phi(j)$ and $\phi(i)\phi(j)$ is not an arc of C, or there is a vertex $v \in V(C)$ such that the subdigraph of D induced on $\phi^{-1}(v)$ contains a cycle.

We first consider the case when ij is an arc of D such that $\phi(i) \ne \phi(j)$ and $\phi(i)\phi(j)$ is not an arc of C. Since h is an acyclic homomorphism, there are no arcs from W_i to W_j in $H - A_0$. By the definition of W_i and W_j, neither are there such arcs in H.

Let us now estimate the expected number M of pairs of sets $A \subseteq V_i$, $B \subseteq V_j$, with $|A| = |B| = w$, such that $ij \in E(D)$ and such that there is no arc from A to B in $H \in \mathcal{D}$ (we call such a pair A, B a *bad pair*). By the linearity of expectation, we have

$$M = q\binom{n}{w}^2(1-p)^{w^2} < q\Big(\frac{n^w}{w!}\Big)^2(1-p)^{w^2} = \frac{q(n^2(1-p)^w)^w}{(w!)^2}. \tag{4.4}$$

Since w grows no more (or less) than linearly with n, for sufficiently large n we have

$$n^2(1-p)^w < e^{-2k'} \quad \text{and} \quad \frac{q}{(w!)^2} < \frac{1}{2}.$$

Therefore, Markov's Inequality (Theorem 1.1 with $t = 1$) and (4.4) yield

$$\Pr(\exists \text{ a bad pair}) < \frac{e^{-n}}{2}. \tag{4.5}$$

Suppose now that there is a vertex $v \in V(C)$ such that D contains a cycle Q whose vertices are all in $\phi^{-1}(v)$. Suppose that $Q = i_1 i_2 \cdots i_t$. Observe that $2 \le t \le k$. Since $\phi(Q) = \{v\}$, we conclude that $h(W_{i_1}) = h(W_{i_2}) = \cdots = h(W_{i_t}) = \{v\}$. Since h is an acyclic homomorphism, the subdigraph of H induced on $W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_t}$ is acyclic.

Let us consider all sequences of sets $U_{j_1}, U_{j_2}, \ldots, U_{j_\ell}$ such that, for $r = 1, 2, \ldots, \ell$, we have $U_{j_r} \subseteq V_{j_r}$ and $|U_{j_r}| = w$, and the vertex sequence $j_1 j_2 \cdots j_\ell$ is a cycle in D. Let $U(\ell)$ the subdigraph of H induced on $U_{j_1} \cup U_{j_2} \cup \cdots \cup U_{j_\ell}$, and let $P_\ell := \Pr(U(\ell) \text{ is acyclic})$. We call this sequence of sets *bad* if $U(\ell)$ is acyclic. Since the expected number N of bad sequences is the sum of the corresponding expectations over all possible cycle lengths, we have

$$N \le \sum_{\ell=2}^{k} \binom{k}{\ell}(\ell-1)!\binom{n}{w}^{\ell} P_\ell. \tag{4.6}$$

In order to bound N, we first bound the probabilities P_ℓ.

Lemma 4.1. *For every integer $\ell \in \{2, 3, \ldots, k\}$, $P_\ell \le e^{-n^{1+\varepsilon}/(10(k')^2)}$.*

The following observation will be used in the proof of Lemma 4.1

Lemma 4.2. *A digraph D is acyclic if and only if every induced subdigraph contains a vertex of outdegree* 0*.*

Proof. If D is acyclic, then every induced subdigraph of D must be acyclic and therefore must contain a vertex of outdegree 0. If D is not acyclic, then it must contain a cycle, the vertex set of which induces a subdigraph containing no vertex of outdegree 0. □

Proof of Lemma 4.1. Let E_0 be certain ($\Pr(E_0) = 1$), and let E_j be the event that all induced subdigraphs of $U(\ell)$ with more than $\ell w - j$ vertices have a vertex of outdegree 0 (the outdegree in the induced subdigraph). Lemma 4.2 shows that

$$P_\ell = \Pr(\bigcap_{j=0}^{\ell w} E_j) = \prod_{j=0}^{\ell w-1} \Pr(E_{j+1}|E_j) \leq \prod_{j=0}^{w-1} \Pr(E_{j+1}|E_j). \tag{4.7}$$

We will call a set $S \subseteq V(U(\ell))$ an *acyclic-sink set* if the induced subdigraph $U(\ell)[S]$ is acyclic and there are no arcs in $U(\ell)$ from S to $V(U(\ell)) \smallsetminus S$ (so S acts as a sink in $U(\ell)$).

Claim 1: The union of two acyclic-sink sets in $U(\ell)$ is an acyclic-sink set in $U(\ell)$.

Proof of claim. Let A and B be two acyclic-sink sets in a digraph $U(\ell)$. Since A and B are both sinks in $U(\ell)$, their union $A \cup B$ is a sink since there are no arcs from $A \cup B$ to $V(U(\ell)) \smallsetminus (A \cup B)$. Consider the three sets $A \smallsetminus B$, $B \smallsetminus A$, and $A \cap B$; each is a subset of an acyclic-sink set so each induces an acyclic digraph. Since A is a sink in $U(\ell)$, there can be no arcs from $A \cap B$ to $B \smallsetminus A$. Likewise B is a sink in $U(\ell)$, so there can be no arcs from $A \cap B$ to $A \smallsetminus B$. Therefore, $A \cup B$ induces an acyclic digraph and is consequently an acyclic-sink set in $U(\ell)$. □

Claim 2: There exists an acyclic-sink set $S \subseteq V(U(\ell))$ of cardinality j if and only if E_j occurs.

Proof of claim. If there exists an acyclic-sink set of cardinality j, then a subdigraph of $U(\ell)$ with more than $\ell w - j$ vertices must have a nonempty intersection with it. Any subdigraph that has nonempty intersection with an acyclic-sink set induces a subdigraph containing a vertex of outdegree zero.

If there is no acyclic-sink set of cardinality j, then the largest acyclic-sink set is an $S' \subseteq V(U(\ell))$ such that $|S'| < j$. Then $U(\ell) - S'$ is a subdigraph of $U(\ell)$ with cardinality greater than $\ell w - j$ and with no vertices of outdegree 0 (otherwise we could have added them to S' and had a larger acyclic-sink set). $\square$

Claim 3: If $U(\ell)$ has an acyclic-sink set of cardinality j, then it has an acyclic-sink set of cardinality $j - 1$.

Proof of claim. Suppose that S is an acyclic-sink set in $U(\ell)$ of cardinality j. Then the subdigraph $U(\ell)[S]$ is acyclic, so there must be a vertex v with indegree 0 in $U(\ell)[S]$. Consider the set $S \smallsetminus \{v\}$; this induces an acyclic subdigraph of $U(\ell)$ because it is a subdigraph of an acyclic digraph. There were no arcs from S to $V(U(\ell)) \smallsetminus S$, and there are no arcs from $S \smallsetminus \{v\}$ to v, so $S \smallsetminus \{v\}$ is a sink in $U(\ell)$. Therefore, there exists an acyclic-sink set in $U(\ell)$ of cardinality $j - 1$. $\square$

We now fix j in order to estimate $\Pr(E_{j+1}|E_j)$. Let $I = \left\{1, 2, \ldots, \binom{\ell w}{j}\right\}$ and let $\left\{S_i\right\}_{i \in I}$ be the j-subsets of the ℓw vertices of $U(\ell)$ (in some fixed order). Let B_i be the event that S_i is an acyclic-sink set in $U(\ell)$. By Claim 1, if more than one B_i occurs, there must be an acyclic-sink set of cardinality at least $j + 1$, and so by Claim 3, there exists one of cardinality exactly $j + 1$. Therefore by Claim 2,

$$\Pr\left(E_{j+1} | \bigcap_{i \in Y} B_i\right) = 1 \text{ whenever } Y \subseteq I \text{ and } |Y| \geq 2. \tag{4.8}$$

Now additionally fix a B_i, and we will estimate $\Pr(E_{j+1}|B_i)$. Let F be the event that $U(\ell) - S_i$

contains a vertex of outdegree 0. Then

$$\Pr(E_{j+1}|B_i) = \Pr(E_{j+1}|F \cap B_i)\Pr(F|B_i) + \Pr(E_{j+1}|F^C \cap B_i)\Pr(F^C|B_i). \tag{4.9}$$

The event E_{j+1} occurs when all subsets of $V(U(\ell))$ of cardinality greater than $\ell w - (j+1)$ induce a subdigraph in $U(\ell)$ that has a vertex of outdegree 0. Clearly $U(\ell) - S_i$ has cardinality $\ell w - j$, while F^C is the event that this set induces a subdigraph with no vertex of outdegree zero. Thus $\Pr(E_{j+1}|F^C \cap B_i) = 0$. All sets of cardinality exceeding $\ell w - (j+1)$ that are distinct from $V(U(\ell)) \smallsetminus S_i$ have a nonempty intersection with S_i, which (given B_i) is an acyclic-sink set in $U(\ell)$. Therefore, subdigraphs of $U(\ell)$ induced on these sets have a vertex of outdegree 0, so that $\Pr(E_{j+1}|B_i \cap F) = 1$. Using these observations, (4.9) reduces to $\Pr(E_{j+1}|B_i) = \Pr(F|B_i)$. The event F is independent of the event B_i since the vertices in S_i do not affect the outdegree of vertices in the subdigraph induced by $V(U(\ell)) \smallsetminus S_i$. Therefore, $\Pr(E_{j+1}|B_i) = \Pr(F)$.

Now we estimate the probability of F. The probability that any particular vertex of $U(\ell) - S_i$ has outdegree 0 in the induced subdigraph is bounded from above by $(1-p)^{(w-j)}$. Since these outdegree computations are independent for each vertex, the probability that all vertices have outdegree greater than 0 is bounded from below by $(1-(1-p)^{(w-j)})^{(\ell w - j)}$, so that

$$\begin{aligned} \Pr(E_{j+1}|B_i) = \Pr(F) &\leq 1 - ((1-(1-p)^{(w-j)})^{(\ell w - j)}) \\ &< (\ell w - j)(1-p)^{(w-j)} =: p_j. \end{aligned} \tag{4.10}$$

We also need to estimate $\Pr(E_{j+1}|E_j)$. By Claim 2, E_j occurs if and only if $\bigcup_{i \in I} B_i$ occurs.

Thus we may rewrite $\Pr(E_{j+1}|E_j)$ using inclusion-exclusion:

$$\begin{aligned}
\Pr(E_{j+1}|E_j) &= \Pr\left(E_{j+1}|\bigcup_{i\in I} B_i\right) \\
&= \frac{\Pr\left(E_{j+1}\cap\left(\bigcup_{i\in I} B_i\right)\right)}{\Pr\left(\bigcup_{i\in I} B_i\right)} \\
&= \frac{\Pr\left(\bigcup_{i\in I}(E_{j+1}\cap B_i)\right)}{\Pr\left(\bigcup_{i\in I} B_i\right)} \\
&= \sum_{\varnothing\neq Y\subseteq I}(-1)^{|Y|+1}\frac{\Pr\left(E_{j+1}\cap\left(\bigcap_{y\in Y} B_y\right)\right)}{\Pr\left(\bigcup_{i\in I} B_i\right)} \\
&= \sum_{\varnothing\neq Y\subseteq I}(-1)^{|Y|+1}\frac{\Pr\left(E_{j+1}\cap\left(\bigcap_{y\in Y} B_y\right)\right)}{\Pr\left(\bigcap_{y\in Y} B_y\right)}\frac{\Pr\left(\bigcap_{y\in Y} B_y\right)}{\Pr\left(\bigcup_{i\in I} B_i\right)} \\
&= \sum_{\varnothing\neq Y\subseteq I}(-1)^{|Y|+1}\Pr\left(E_{j+1}|\bigcap_{y\in Y} B_y\right)\Pr\left(\bigcap_{y\in Y} B_y|\bigcup_{i\in I} B_i\right) \\
&= \sum_{y\in I}\Pr(E_{j+1}|B_y)\Pr\left(B_y|\bigcup_{i\in I} B_i\right) \\
&\qquad + \sum_{\substack{Y\subseteq I\\|Y|\geq 2}}(-1)^{|Y|+1}\Pr\left(E_{j+1}|\bigcap_{y\in Y} B_y\right)\Pr\left(\bigcap_{y\in Y} B_y|\bigcup_{i\in I} B_i\right).
\end{aligned}$$

Using (4.8) and (4.10) in the last expression for $\Pr(E_{j+1}|E_j)$ gives

$$\begin{aligned}
\Pr(E_{j+1}|E_j) &\leq p_j\sum_{y\in I}\Pr\left(B_y|\bigcup_{i\in I} B_i\right) + \sum_{\substack{Y\subseteq I\\|Y|\geq 2}}(-1)^{|Y|+1}\Pr\left(\bigcap_{y\in Y} B_y|\bigcup_{i\in I} B_i\right) \\
&= p_j\sum_{y\in I}\Pr\left(B_y|\bigcup_{i\in I} B_i\right) + \left[\Pr\left(\bigcup_{i\in I} B_i|\bigcup_{i\in I} B_i\right) - \sum_{y\in I}\Pr\left(B_y|\bigcup_{i\in I} B_i\right)\right] \\
&= p_j\sum_{y\in I}\Pr\left(B_y|\bigcup_{i\in I} B_i\right) + \left[1 - \sum_{y\in I}\Pr\left(B_y|\bigcup_{i\in I} B_i\right)\right].
\end{aligned}$$

Since $\sum_{y\in I}\Pr(B_y|\bigcup_{i\in I} B_i) \geq 1$ and $p_j - 1 < 0$ we have

$$\Pr(E_{j+1}|E_j) \;\leq\; 1 + \sum_{y\in I}\Pr\left(B_y|\bigcup_{i\in I} B_i\right)(p_j - 1) < p_j.$$

Applying this last estimate to (4.7) yields

$$\begin{aligned}
P_\ell &\leq \prod_{j=0}^{w-1} p_j = \prod_{j=0}^{w-1}(\ell w - j)(1-p)^{(w-j)} \\
&< (\ell w)^w (1-p)^{w(w+1)/2} \\
&\leq (\ell w)^w (1-p)^{w^2/2} \\
&\leq (\ell w)^w e^{-pw^2/2} \\
&\leq \left(\ell w e^{-pw/2}\right)^w \\
&\leq \left(\ell w e^{-n^\varepsilon/(4k')}\right)^w \qquad (4.11)\\
&\leq \left(e^{-n^\varepsilon/(5k')}\right)^w \qquad (4.12)\\
&\leq e^{-n^{1+\varepsilon}/(10(k')^2)}. \qquad (4.13)
\end{aligned}$$

In passing from (4.11) to (4.13), the reader may find it helpful to recall that $n = |V_i|$ (for $1 \leq i \leq k$), $k' = |V(C)|$, ℓ is between 2 and k, $w = \lceil n/(2k')\rceil$, and $p = n^{\varepsilon-1}$, and that these estimates are valid for fixed k' and sufficiently large n. $\square$

We return to our estimation of the expected number N of bad sequences in (4.6), repeated here for convenience:

$$N \;\leq\; \sum_{\ell=2}^{k}\binom{k}{\ell}(\ell-1)!\binom{n}{w}^{\ell} P_\ell.$$

Using Lemma 4.1 to bound the factors P_ℓ in this sum shows that for n large enough,

$$N \;\leq\; \sum_{\ell=2}^{k}\binom{k}{\ell}(\ell-1)!\binom{n}{w}^{\ell} e^{-n^{1+\varepsilon}/(10(k')^2)} \;<\; \sum_{\ell=2}^{k}\frac{e^{-n}}{2k} \;<\; \frac{e^{-n}}{2}. \qquad (4.14)$$

From (4.14) and Markov's Inequality (Theorem 1.1 with $t = 1$), we conclude that

$$\Pr(\exists \text{ a bad sequence}) < \frac{e^{-n}}{2}. \tag{4.15}$$

Since ϕ fails to be an acyclic homomorphism exactly when there exists a bad pair or there exists a bad sequence, (4.5) and (4.15) now show that

$$|\mathcal{D} \smallsetminus \mathcal{D}_2| \leq (\Pr(\exists \text{ bad pair}) + \Pr(\exists \text{ bad sequence}))\, |\mathcal{D}| < e^{-n}\, |\mathcal{D}|\,,$$

which yields (4.2), and hence completes the proof of Theorem 3.2. □

4.2 The Janson Inequalities

In the next section we will need to use the Janson Inequalities, first proved in 1990 [14]. For completeness, in this section we include a brief summary of their presentation in [1, Section 8.1]. For more details we direct the reader to [1], from which we borrow extensively throughout this section.

Frequently we would like to bound the probability that none of a set of 'bad' events B_i, $i \in I$ occur. For example, in the preceding section we needed to bound the probability that a certain type of subset of vertices of our random model induced an acyclic digraph. If the bad events are mutually independent, then

$$\Pr(\bigcap_{i \in I} B_i^C) = \prod_{i \in I} \Pr(B_i^C).$$

The Janson Inequalities are used when the B_i are 'mostly' independent. That is to say that 'rarely' is there a dependence between two particular B_i's, and any such dependence is 'small'. We can then estimate the (small) difference between $\Pr(\cap_{i \in I} B_i^C)$ and $\Pi_{i \in I} \Pr(B_i^C)$.

Let Ω be a finite universal set and let R be a random subset of Ω given by $\Pr(r \in R) = p_r$ (for some probabilities p_r), these events mutually independent over $r \in \Omega$. Let I be a finite index set, and, for $i \in I$, let A_i be a subset of Ω. Let B_i be the event $A_i \subseteq R$. (That is, each point $r \in \Omega$ 'flips a coin' to determine if it is in R; then B_i is the event that the coins for all $r \in A_i$ came up 'heads'.) Let X_i be the indicator random variable for B_i and $X = \sum_{i \in I} X_i$ the number of $A_i \subseteq R$. The events $\cap_{i \in I} B_i^C$ and $\{X = 0\}$ are then identical. For $i, j \in I$ we write $i \sim j$ if $i \neq j$ and $A_i \cap A_j \neq \varnothing$. Note that when $i \neq j$ and $i \nsim j$ then B_i, B_j are independent events since they involve separate coin flips. Furthermore, if $i \notin J \subseteq I$ and $i \nsim j$ for all $j \in J$, then B_i is mutually independent of $\{B_j | j \in J\}$ because the coin flips on A_i and on $\cup_{j \in J} A_j$ are independent. We define

$$\Delta := \sum_{i \sim j} \Pr(B_i \cap B_j).$$

Here the sum is over ordered pairs so that $\Delta/2$ gives the corresponding sum over unordered pairs. We set

$$\mu = E[X] = \sum_{i \in I} \Pr(B_i).$$

We're now ready to state the limited form of the Janson Inequality that we need for the proof of Theorem 3.3.

Theorem 4.3. *If B_i, $i \in I$, Δ and μ are as above, then*

$$\Pr\Big(\bigcap_{i \in I} B_i^C\Big) \leq e^{-\mu + \Delta/2}.$$

Note that when $\Delta \geq 2\mu$, the upper bound of Theorem 4.3 becomes useless because it exceeds 1. Even for Δ slightly less, it is improved by the following result, a simplified version of the so-called Extended Janson Inequality.

Theorem 4.4. *Under the assumptions of Theorem 4.3 and further assumption that $\Delta \geq \mu$,* *we have*

$$\Pr\Big(\bigcap_{i\in I} B_i^C\Big) \le e^{-\mu^2/(2\Delta)}.$$

4.3 Proof of Theorem 3.3

To obtain the conclusion of Theorem 3.3 (unique D-colorability), we shall need to refine the deletion method employed in the proof of Theorem 3.2. We preserve the earlier notation. Let $\mathcal{D}_3$ be the set of digraphs $H \in \mathcal{D}_1$, in which any two cycles of length less than g are disjoint. Let $\mathcal{D}_4$ denote the set of those $H \in \mathcal{D}$ with the property that $H - A_1$ is uniquely D-colorable for any set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ independent arcs. (Here, we call a set $S \subseteq E(H)$ *independent* if no two arcs in S have a vertex in common.) Now we will show that

$$|\mathcal{D}_3| > \left(1 - n^{-\varepsilon/3}\right)|\mathcal{D}| \tag{4.16}$$

and

$$|\mathcal{D}_4| > \left(1 - e^{-n^{\varepsilon}/6}\right)|\mathcal{D}|\,. \tag{4.17}$$

Since (4.16) and (4.17) imply that $\mathcal{D}_3 \cap \mathcal{D}_4 \neq \varnothing$ (for large enough n), we can choose a digraph $H \in \mathcal{D}_3 \cap \mathcal{D}_4$. As $H \in \mathcal{D}_3 \subseteq \mathcal{D}_1$, we can delete a set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ independent arcs from H so that $D^* := H - A_1$ has girth at least g, and $H \in \mathcal{D}_4$ ensures that D^* is uniquely D-colorable. Hence, to complete the proof of Theorem 3.3, it suffices to establish (4.16) and (4.17).

Proof of (4.16). For integers $\ell_1, \ell_2 < g$, we follow [22] and call a digraph an (ℓ_1, ℓ_2)-*double cycle* if it consists of a directed cycle C_{ℓ_1} of length ℓ_1 and a directed path of length ℓ_2 joining two (not necessarily distinct) vertices of C_{ℓ_1}; such a digraph contains $\ell_1 + \ell_2 - 1$ vertices and $\ell_1 + \ell_2$ arcs. Let $\mathcal{D}'$ denote the set of digraphs in $\mathcal{D}$ containing an (ℓ_1, ℓ_2)-double cycle for some $\ell_1, \ell_2 < g$. Notice that $\mathcal{D}_1 \setminus \mathcal{D}_3 \subseteq \mathcal{D}'$, whence

$$|\mathcal{D}_1 \setminus \mathcal{D}_3| \leq |\mathcal{D}'|, \tag{4.18}$$

so we can obtain a lower estimate for $|\mathcal{D}_3|$ by estimating $|\mathcal{D}'|$.

For fixed $\ell_1, \ell_2 < g$, the expected number $N(\ell_1, \ell_2)$ of (ℓ_1, ℓ_2)-double cycles in a digraph $H \in \mathcal{D}$ is less than

$$\ell_1 (kn)^{\ell_1} (kn)^{\ell_2 - 1} p^{\ell_1 + \ell_2},$$

since there are fewer than $\ell_1 (kn)^{\ell_1} (kn)^{\ell_2 - 1}$ ways of choosing such a double cycle Y with $V(Y) \subseteq V$, and each such Y exists with probability 0 or $p^{\ell_1 + \ell_2}$. Since $p = n^{\varepsilon - 1}$ we have

$$N(\ell_1, \ell_2) < \ell_1 k^{\ell_1 + \ell_2} n^{\varepsilon(\ell_1 + \ell_2)} n^{-1}.$$

Since $\varepsilon(\ell_1 + \ell_2) \leq 2g\varepsilon < 1/2$, for large enough n we have

$$\sum_{\substack{2 \leq \ell_1 < g \\ 1 \leq \ell_2 < g}} N(\ell_1, \ell_2) < n^{-1/2}.$$

Markov's Inequality (Theorem 1.1 with $t = 1$) now shows that

$$|\mathcal{D}'| < n^{-1/2} |\mathcal{D}|,$$

so from (4.18) we obtain

$$|\mathcal{D}_3| > |\mathcal{D}_1| - n^{-1/2} |\mathcal{D}|,$$

and (4.1) gives (4.16). □

Proof of (4.17). We will argue that $|\mathcal{D} \setminus \mathcal{D}_4| < e^{-n^{\varepsilon}/6} |\mathcal{D}|$. If $H \in \mathcal{D} \setminus \mathcal{D}_4$, then there is a set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ independent arcs of H so that $H - A_1$ admits an acyclic homomorphism h to D that is not the composition $\sigma \circ c$ of the natural homomorphism $c \colon H - A_1 \to D$ (sending

V_i to i) with an automorphism σ of D. As in the proof of (4.2), we can define a function $\phi\colon V(D) \to V(D)$ such that $\left|V_i \cap h^{-1}(\phi(i))\right| \geq n/k$ for each $i \in V(D)$.

Let us first suppose that ϕ is not an automorphism of D. By hypothesis, D is a core, so any acyclic homomorphism of D to itself must be an automorphism. It follows that ϕ is not an acyclic homomorphism. Therefore, there is an arc $ij \in E(D)$ such that $\phi(i)\phi(j) \notin E(D)$, or there is a vertex $i \in V(D)$ such that $\phi^{-1}(i)$ is not acyclic. Notice that the current arrangement is analogous to the one in the second paragraph in the proof of (4.2). Repeating the earlier argument, with D in the place of C and k in the role of k', we find that most $H \in \mathcal{D}$ do not fall into the present case. More precisely, we reach the following conclusion:

At least $(1 - e^{-n})|\mathcal{D}|$ digraphs $H \in \mathcal{D}$ have the property that for any set A_1 of at most $\lceil n^{g\varepsilon} \rceil$ arcs (independent or otherwise), the digraph $H - A_1$ cannot be D-colored so that ϕ is not an automorphism of D.

Thus, in this case, $|\mathcal{D} \smallsetminus \mathcal{D}_4| < e^{-n}|\mathcal{D}| < e^{-n^{\varepsilon}/6}|\mathcal{D}|$, and (4.17) is proved.

From now on, we treat the case when ϕ is an automorphism of D. Without loss of generality, we may assume that ϕ is the identity, i.e., that

$$\left|V_i \cap h^{-1}(i)\right| \geq n/k \text{ for each } i \in V(D). \tag{4.19}$$

We may assume further that

$$\left|V_j \cap h^{-1}(i)\right| < n/k \text{ for all } j \neq i. \tag{4.20}$$

(Otherwise, we can redefine $\phi(i)$ to be equal to j and fall into the case where ϕ is not an automorphism.)

Since h is not the composition $\sigma \circ c$ of the natural homomorphism $c\colon H - A_1 \to D$ (sending

V_i to i) with an automorphism σ of D, there must be a pair $\{i, j\}$ of distinct vertices of D such that $V_j \cap h^{-1}(i) \neq \varnothing$. Let $\{i_0, j_0\}$ be such a pair that maximizes $|V_{j_0} \cap h^{-1}(i_0)|$. Consider the map $\phi' \colon V(D) \to V(D)$ such that

$$\phi'(x) := \begin{cases} x (= \phi(x)) & \text{if } x \neq j_0 \\ i_0 & \text{if } x = j_0. \end{cases}$$

Clearly ϕ' is not a bijection, and since D is a core, it cannot be an acyclic homomorphism. There are two possibilities.

Case 1: Both $j_0 i_0$ and $i_0 j_0$ are arcs of D (so $\phi'^{-1}(i_0)$ is not acyclic).

Case 2: There exists $v \in V(D)$ such that $v j_0$ is an arc of D but $v i_0$ is not, or $j_0 v$ is an arc of D but $i_0 v$ is not.

We will show that in either case, $|\mathcal{D} \smallsetminus \mathcal{D}_4| < e^{-n^{\varepsilon}/6} |\mathcal{D}|$.

Case 1: Our choice of $\{i_0, j_0\}$ ensures that $h^{-1}(i_0) \cap V_{j_0} \neq \varnothing$. Let $x \in h^{-1}(i_0) \cap V_{j_0}$, and consider the (nonrandom) subdigraph $\widehat{D_n}$ of D_n induced by $\{x\} \cup (h^{-1}(i_0) \cap V_{i_0})$. As V_{i_0} induces no cycles, all cycles of $\widehat{D_n}$ must include x, and since the arcs of A_1 are independent, at most one such arc is incident with x. Furthermore, the constraint on the size of A_1 and our choice of ε (smaller than $1/(4g)$) give

$$|A_1| \leq \lceil n^{g\varepsilon} \rceil < \lceil n^{1/4} \rceil \ll \frac{n}{k}.$$

Because $|h^{-1}(i_0) \cap V_{i_0}| \geq n/k$ (cf. (4.19)), there must be a subset $U \subseteq h^{-1}(i_0) \cap V_{i_0}$ of cardinality $\lfloor n/2k \rfloor$ such that the (random) subdigraph induced by $\{x\} \cup U$ contains no arcs of A_1 and moreover is acyclic (since $h^{-1}(i_0)$ is acyclic). To show that this is unlikely, we first estimate the expected number M of ways to select a vertex $x \in V_{j_0}$ and a subset $U \subseteq V_{i_0}$ of cardinality $\lfloor n/2k \rfloor$ so that the subdigraph $H_{x,U}$ of H that they induce is acyclic and no arc of A_1 is incident with a vertex in U. If $P_{x,U}$ denotes the probability that $H_{x,U}$ is acyclic, then

$$M \leq n \binom{n}{\lfloor n/2k \rfloor} P_{x,U} < n^n P_{x,U}. \tag{4.21}$$

In order to estimate $P_{x,U}$, we employ the Janson Inequalities (cf. Section 4.2). The idea to invoke these inequalities here was originally contributed by Ararat Harutyunyan (cf. [12]). Thus, the first draft of this page and the next two (up to Case 2) was written by him.

Denote by Ω the set of all potential arcs in the subdigraph $D'_{x,U}$ of D_n induced by $\{x\} \cup U$; each arc in Ω appears in $H_{x,U}$ with probability p. Let $\ell > (2+\varepsilon)/\varepsilon$ be a fixed integer. We index those cycles of $D'_{x,U}$ (with the positive integers) that are of length $\ell + 1$ in $D'_{x,U}$. For $j \geq 1$, let S_j be the arc-set of the jth such cycle and B_j be the event that the arcs in S_j all appear (i.e. the cycle determined by S_j is present in $H_{x,U}$). Let X count the B_j that occur; since $\Pr(X = 0)$ is an upper bound for $P_{x,U}$, we can bound $P_{x,U}$ by bounding $\Pr(X = 0)$, and estimating the latter quantity is exactly the purpose of the Janson Inequalities.

In the Janson paradigm, the value of Δ is given by

$$\Delta = \sum_{S_i \sim S_j} \Pr(B_i \cap B_j),$$

where $S_i \sim S_j$ if the two cycles determined by S_i and S_j have at least one arc in common. Since there are at most $\binom{\lfloor n/2k \rfloor}{\ell} < n^\ell$ cycles S_j, if we fix an S_i to maximize $\sum_{j:S_j \sim S_i} \Pr(B_i \cap B_j)$, then

$$\Delta \leq n^\ell \sum_{j:S_j \sim S_i} \Pr(B_i \cap B_j). \tag{4.22}$$

Now we sum over the number r of common arcs an S_j can have with S_i; this fixes at least

$r+1$ vertices of S_j. Thus,

$$\sum_{j:S_j\sim S_i} \Pr(B_i\cap B_j) \;\leq\; \sum_{r=1}^{\ell}\binom{\ell+1}{r}\left\lfloor\frac{n}{2k}\right\rfloor^{\ell-r-1} p^{2(\ell+1)-r}.$$

Using the crude upper estimates $\binom{\ell+1}{r} < 2^{\ell+1}$ and $\lfloor n/2k\rfloor < n$, and replacing p with $n^{\varepsilon-1}$, we obtain

$$\sum_{j:S_j\sim S_i} \Pr(B_i\cap B_j) \;<\; 2^{\ell+1}\sum_{r=1}^{\ell}(np)^{\ell-r-1}p^{\ell+3} \;<\; 2^{\ell+1}\ell(np)^{\ell-2}p^{\ell+3} \;=\; 2^{\ell+1}\ell n^{2\varepsilon\ell+\varepsilon-\ell-3}.$$

This and (4.22) now give

$$\Delta \leq 2^{\ell+1}\ell n^{2\varepsilon\ell+\varepsilon-3}. \tag{4.23}$$

We also need to find a lower bound for $\mu := E[X]$. Since the arcs of $D'_{x,U}$ within U are acyclically oriented, each choice of ℓ vertices within U determines exactly one potential $(\ell+1)$-cycle (viz., through x). It follows that

$$\mu \;=\; \binom{\lfloor n/2k\rfloor}{\ell}p^{\ell+1} \;>\; \left(\frac{\lfloor n/2k\rfloor}{\ell}\right)^{\ell}p^{\ell+1} \;>\; \frac{n^{\varepsilon\ell+\varepsilon-1}}{(4k\ell)^{\ell}}. \tag{4.24}$$

We have two subcases.

Subcase 1(i): $\Delta \geq \mu$.

Here, we have the hypotheses of the Extended Janson Inequality (Theorem 4.4), which, along with (4.23) and (4.24) gives

$$\Pr(X=0) \;\leq\; e^{-\mu^2/(2\Delta)} \;<\; e^{-n^{1+\varepsilon}/(\ell 2^{\ell+2}(4k\ell)^{2\ell})} =: e^{-\beta n^{1+\varepsilon}},$$

where β is the (positive) constant (not depending on n) absorbing the denominator in the preceding exponent.

Subcase 1(ii): $\Delta < \mu$.

Now we have the hypotheses of the Janson Inequality (Theorem 4.3), which, with the help of (4.24) gives

$$\Pr(X = 0) \ \le \ e^{-\mu+\Delta/2} \ < \ e^{-\mu/2} \ < \ e^{-n^{\varepsilon\ell+\varepsilon-1}/(2(4k\ell)^{\ell})}.$$

Recalling our choice of $\ell > (2+\varepsilon)/\varepsilon$, we see that

$$\Pr(X = 0) \ < \ e^{-n^{1+2\varepsilon}/(2(4k\ell)^{\ell})} \ < \ e^{-n^{1+\varepsilon}}.$$

In either subcase, we have that $P_{x,U} \le \Pr(X = 0) < e^{-\beta n^{1+\varepsilon}}$ (since $\beta < 1$), and returning to (4.21), we have

$$M \ < \ n^n P_{x,U} \ < \ n^n e^{-\beta n^{1+\varepsilon}} \ = \ \left(ne^{-\beta n^{\varepsilon}}\right)^n \ < \ e^{-\beta n^{1+\varepsilon}/2}.$$

By Markov's Inequality (Theorem 1.1 with $t = 1$), the probability that there exists such an $\{x\} \cup U$ (that induces an acyclic subdigraph) is less than $e^{-\beta n^{1+\varepsilon}/2} < e^{-n^{\varepsilon}/6}$, and so in Case 1, $|\mathcal{D} \setminus \mathcal{D}_4| < e^{-n^{\varepsilon}/6}|\mathcal{D}|$, as desired.

Case 2: By the hypothesis of this case, there is a vertex v such that either $vj_0 \in E(D)$ and $vi_0 \notin E(D)$, or $j_0v \in E(D)$ and $i_0v \notin E(D)$. We will consider the first of these; the second one yields to similar reasoning. Let us recall that we chose a pair $\{i_0, j_0\}$ of distinct vertices of D so as to maximize $b := |V_{j_0} \cap h^{-1}(i_0)| \neq 0$.

Claim: Every vertex $z \in V(D) \setminus \{i_0\}$ satisfies $\left|V_z \cap h^{-1}(z)\right| \ge n - (k-1)b$.

Proof of claim. Otherwise, some $z \neq i_0$ satisfies $\left|V_z \cap h^{-1}(z)\right| < n-(k-1)b$. By the pigeonhole principle, there is some $u \neq z$ such that $\left|V_z \cap h^{-1}(u)\right| > b$, but this contradicts our choice of $\{i_0, j_0\}$. □

Using the claim, we see that there are sets $U_v \subseteq V_v \cap h^{-1}(v)$ and $U_{j_0} = V_{j_0} \cap h^{-1}(i_0)$ with $|U_v| = n-(k-1)b$ and $|U_{j_0}| = b$. Since $h\colon H - A_1 \to D$ is an acyclic homomorphism and $vi_0 \notin E(D)$, there are at most $\min\{b, \lceil n^{g\varepsilon}\rceil\}$ independent arcs from a vertex in U_v to one in U_{j_0}. We now estimate the expected number $L(b)$ of pairs $U'_v \subseteq V_v$, $U'_{j_0} \subseteq V_{j_0}$ with $|U'_v| = n-(k-1)b = n-(k-1)|U'_{j_0}|$, and at most $\min\{b, \lceil n^{g\varepsilon}\rceil\}$ arcs from U'_v to U'_{j_0}.

For $b < n/k$ (cf. (4.20)) and $s \le \min\{b, \lceil n^{g\varepsilon}\rceil\}$, denote by $L(b,s)$ the expected number of pairs $U'_v \subseteq V_v$, $U'_{j_0} \subseteq V_{j_0}$, $|U'_v| = n-(k-1)b = n-(k-1)|U'_{j_0}|$, and exactly s arcs joining a vertex in U'_v to one in U'_{j_0}. Then

$$\begin{aligned}
L(b,s) &< \binom{n}{n-(k-1)b}\binom{n}{b}\binom{(n-(k-1)b)b}{s}p^s(1-p)^{(n-(k-1)b)b-s}\\
&< n^{(k-1)b}n^b(nb)^s n^{s(\varepsilon-1)}e^{-bn^\varepsilon+n^{\varepsilon-1}((k-1)b^2+s)}\\
&< b^s n^{\varepsilon s}n^{kb}e^{-(bn^\varepsilon)/2}\\
&= b^s n^{\varepsilon s}(n^k e^{-n^\varepsilon/2})^b\\
&< b^s n^{\varepsilon s}e^{-(bn^\varepsilon)/3}\\
&< e^{-n^\varepsilon/4}.
\end{aligned}$$

Letting $L(b) = \sum_{s\le\min\{b,\lceil n^{g\varepsilon}\rceil\}} L(b,s) < \lceil n^{g\varepsilon}\rceil e^{-n^\varepsilon/4} < e^{-n^\varepsilon/5}$, we find that

$$\sum_{1\le b<n/k} L(b) < (n/k)e^{-n^\varepsilon/5} < e^{-n^\varepsilon/6}.$$

This completes the discussion for the case when $vj_0 \in E(D)$ and $vi_0 \notin E(D)$; an identical argument gives the same upper bound in the case when $j_0v \in E(D)$ and $i_0v \notin E(D)$. Thus in Case 2 we also arrive at $|\mathcal{D} \smallsetminus \mathcal{D}_4| < e^{-n^\varepsilon/6}|\mathcal{D}|$.

Combining the estimates obtained above and applying Markov's Inequality (Theorem 1.1 with $t = 1$) finally yields (4.17) and therefore completes the proof of Theorem 3.3. $\square$

Chapter 5

An application and future work

5.1 An application of Theorem 3.3

Different types of digraph coloring such as ordinary coloring, acyclic coloring, and circular coloring can often be described in terms of homomorphisms into certain codomains. For example, we showed in Section 3.3 that the existence of an acyclic r-coloring of a digraph D is equivalent to the existence of an acyclic homomorphism from D into K_r. Since each K_r is a core, Theorem 3.3 implies the existence of uniquely r-colorable digraphs with arbitrarily large girth. In this section, we briefly describe another type of digraph coloring, circular coloring, that also admits a homomorphic description. Here, the codomains are frequently cores, so we can again apply Theorem 3.3.

The circular chromatic number of a graph is a much-studied graph invariant; see [23] for a survey of the research on the circular chromatic number as of 2001. Here we describe the circular chromatic number of a digraph as introduced in [2]. This definition generalizes the circular chromatic number for undirected graphs.

For $q \in \mathbb{Q}^+$, let S_q denote the circle of perimeter q (centered, say, at the origin of $\mathbb{R}^2$). We define a *circular q-coloring* of D to be a map $\phi : V(D) \to S_q$ such that for every $xy \in E(D)$, with $\phi(x) \neq \phi(y)$, the distance $d_S(\phi(x), \phi(y))$ from $\phi(x)$ to $\phi(y)$ in the clockwise direction around S_q is at least 1, and for every $p \in S_q$, the preimage $\phi^{-1}(p)$ induces an acyclic subdigraph of D. It is shown in [2, 18] that there is a rational number $q \in \mathbb{Q}$ such that D has a circular k/d-coloring if and only if $k/d \geq q$. This value q is denoted by $\chi_c(D)$ and called the *circular chromatic number* of D.

In order to implement the results of this dissertation (specifically Theorem 3.3), we will need an equivalent definition of the circular chromatic number in terms of acyclic homomorphisms. Let $d \geq 1$ and $k \geq d$ be integers. Let $C(k, d)$ be the digraph with vertex set $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ and arcs

$$E(C(k, d)) = \{ij \mid j - i \in \{d, d+1, \ldots, k-1\}\},$$

where the subtraction is considered in the cyclic group $\mathbb{Z}_k$ of integers modulo k. An acyclic homomorphism of a digraph D into $C(k, d)$ is called a *(k, d)-coloring* of D. In [2], it is shown that a digraph D has circular chromatic number at most k/d if and only if there exists a (k, d)-coloring.

In [2] it was proved that χ_c assumes all rational values at least one, but the digraphs witnessing this result do not generally have large girth. We observe that if k and d are relatively prime, then $C(k, d)$ is a core (for proof, see [12]), and so we may apply Theorem 3.3 to obtain the following result.

Theorem 5.1. *If k and d are relatively prime integers with $1 \leq d \leq k$, then for every positive integer g, there exists a uniquely $C(k, d)$-colorable digraph of girth at least g (and with circular chromatic number equal to k/d).*

5.2 Future work

Much of our work here was inspired originally by Zhu's results in [22]. In a later paper [21], Nešetřil and Zhu offer a simultaneous generalization of Theorems 2.3 and 2.4. Before we state this result we need an additional definition.

If G is a graph, then a graph H is said to be *G-pointed* if for any two homomorphisms $\phi, \phi' : G \to H$ that satisfy $\phi(x) = \phi'(x)$ for all $x \neq x_0$ (for a fixed vertex $x_0 \in V(G)$) the relation $\phi(x_0) = \phi'(x_0)$ also holds.

Theorem 5.2 (Nešetřil and Zhu, 2004 [21]). *For every graph G and every choice of positive integers k and g there exists a graph G^* together with a surjective homomorphism $c : G^* \to G$ with the following properties:*

(i) The girth of G^* is at least g.

(ii) For every graph H with at most k vertices, there exists a homomorphism $\phi : G^* \to H$ if and only if there exists a homomorphism $f : G \to H$.

(iii) For every G-pointed graph H with at most k vertices and for every homomorphism $\phi : G^* \to H$ there exists a unique homomorphism $f : G \to H$ such that $\phi = f \circ c$.

To see that Theorem 5.2 implies Theorem 2.3, consider a graph G that is not H-colorable. Let k be the maximum of $|V(G)|$ and $|V(H)|$. Then conclusions (i) and (ii) of Theorem 5.2 imply that there is a graph G^* with girth at least g that is G-colorable but not H-colorable. This is the conclusion of Theorem 2.3.

We observe that if G is a core, then any graph homomorphism from G to G must be an automorphism, and so if any two such homomorphisms agree on all but one vertex, they must also agree on that vertex. Therefore G is G-pointed. We take $k = |V(G)|$ and then conclusions

(i) and (iii) of Theorem 5.2 imply that there exists a graph witnessing the conclusion of Theorem 2.4.

There is clearly an analogous concept of 'pointed' for digraphs with acyclic homomorphisms, and so the statement of Theorem 5.2 has a natural analogue in that setting. We pose the question of whether the analogue holds in the digraph setting. An affirmative answer would imply the main results of this dissertation (Theorem 3.2 and 3.3). Investigating this question will be a subject of our future research.

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