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Norm-Preserving Criteria for Uniform Algebra Isomorphisms

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NORM–PRESERVING CRITERIA FOR UNIFORM ALGEBRA ISOMORPHISMS

by

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Norm–Preserving Criteria for Uniform Algebra Isomorphisms

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Let $A \subset C(X)$ and $B \subset C(Y)$ be uniform algebras with Choquet boundaries δA and δB. We establish sufficient conditions for a surjective map $T\colon A \to B$ to be an algebra isomorphism. In particular, we show that if $T\colon A \to B$ is a surjection that preserves the norm of the sums of the moduli of algebra elements, then T induces a homoemorphism ψ between the Choquet boundaries of A and B such that $|Tf| = |f \circ \psi|$ on the Choquet boundary of B. If, in addition, T preserves the norms of all linear combinations of algebra elements and either preserves both *1* and i or the peripheral spectra of $\mathbb{C}$–peaking functions, then T is a composition operator and thus an algebra isomorphism. We also show that if a surjection T that preserves the norm of the sums of the moduli of algebra elements also preserves the norms of sums of algebra elements as well as either preserving both *1* and i or preserving the peripheral spectra of $\mathbb{C}$–peaking functions, then T is a composition operator and thus an algebra isomorphism. In the process, we generalize the additive analog of Bishop's Lemma.

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Contents

Chapter 1

Introduction and History of the Subject

The study of transformations between spaces that preserve a particular property of the spaces, often called "preserver problems," has a long and distinguished history in many branches of mathematics. Algebraists study homomorphisms and isomorphisms, topologists study homeomorphisms, category theorists study morphisms, functional analysists study linear functionals, and the list continues. This thesis addresses the topic of isomorphisms between uniform algebras, mappings that preserve the structure of the algebras, from the perspective of functional analysis and Banach algebra theory. Isomorphisms are very useful to mathematicians (and others) since two objects that are isomorphic can be identified and the isomorphism can be used to translate results between the objects. As an example, the classical Gelfand–Mazur Theorem states that any commutative Banach algebra in which every nonzero element is invertible is isomorphic to the field of complex numbers, so every element in such a Banach algebra behaves as a complex number.

The goal of this thesis is to provide sufficient conditions under which mappings between algebras will be algebra isomorphisms. Initially, the focus was on mappings that preserve properties related to the spectrum of algebra elements, often merely a subset of the spectrum and a subset of the algebra elements. In the course of our research, that focus has expanded to mappings that satisfy certain norm conditions instead of spectral conditions. These questions are along the lines of "spectral preserver problems" (see [10]); however, because our main results deal mostly with the preservation of the norms of certain quantities as opposed to the preservation of spectra, we term them "norm–preserver problems."

This chapter will review several of the key results in the history of these problems and summarize our results. The necessary terms and notations are given in Chapters 2 and 3.

1.1 Preserver Problems

One of the most basic questions in the area of preserver problems is whether an operator between two spaces with the same structure (e.g., two groups, two rings, two algebras) is a homomorphism. Specifically, we ask whether the operator $f\colon X \to Y$ preserves the operation in those spaces, i.e., we ask if $f(x * y) = f(x) * f(y)$, where $*$ is the operation in X in the first case and the operation in Y in the second case. If f does preserve the operation, then we call f a homomorphism and can thus apply any of the results that we know concerning the spaces X and Y and homomorphisms between them.

The following classical theorem by Mazur and Ulam provides a more interesting example of a preserver problem.

Theorem 1.1.1 (Mazur–Ulam Theorem (1932))**.** [15] *Let $f\colon X \to Y$ be a surjective mapping between normed vector spaces over $\mathbb{R}$ such that $f(0_X) = 0_Y$ and $\|f(x_1) - f(x_2)\|_Y = \|x_1 - x_2\|_X$ for every $x_1, x_2 \in X$. Then f is an $\mathbb{R}$–linear isometry.*

In this example, f is required to be surjective, which can be considered a preservation in the sense that all of the elements of the space Y must be the image of some element from X (so f "preserves" all of Y in its image), and to preserve the zero element and distances between elements. The last property is merely the definition of an isometry, so we are not surprised by the conclusion that f is an isometric transformation; however, the fact that linearity follows from the fact that f is an isometry is much more powerful than we may have expected because f being an isometry only makes a statement about norms.

1.2 Linear Preserver Problems

Although the Mazur–Ulam Theorem mentioned in the previous section does not assume that a map is linear from the beginning, many of the classical results in the area of preserver problems fall into the category of "linear preserver problems," problems in which maps are assumed to be linear and to preserve some other property that then leads to a conclusion categorizing such maps. The following theorem is one such classical result.

Theorem 1.2.1 (Gleason–Kahane–Żelazko Theorem (1973))**.** [23] *Let A be a unital commutative Banach algebra and B be a uniform algebra. If $T\colon A \to B$ is a linear map with $\sigma(Tf) \subset \sigma(f)$ for every $f \in A$, then T is multiplicative, i.e., $T(fg) = T(f)T(g)$ for every $f, g \in A$.*

Although this theorem only allows us to conclude that the mapping is multiplicative, if we strengthen the conditions to require that T is surjective, maps from a uniform algebra to a uniform algebra, and preserves the spectra of algebra elements, as in the next corollary, then we may conclude that T preserves the distances between algebra elements (T is an isometry) and the structure of the algebras (T is an algebra isomorphism).

Corollary 1.2.2. (*e.g.* [21]) *A surjective, linear mapping $T\colon A \to B$ between uniform algebras that preserves the spectra of algebra elements (i.e. $\sigma(Tf) = \sigma(f)$ for every $f \in A$) is an isometric algebra isomorphism.*

The following classical theorem constructs an isometric algebra isomorphism between the spaces of continuous functions on compact spaces.

Theorem 1.2.3 (Banach–Stone Theorem (1936))**.** *(e.g. [2]) If X and Y are compact spaces and $T\colon C(X) \to C(Y)$ is a surjective linear isometry, then there exists a homeomorphism $\tau\colon Y \to X$ and a function $\alpha \in C(Y)$ such that $|\alpha(y)| = 1$ for every $y \in Y$ and*

$$(Tf)(y) = \alpha(y)f(\tau(y))$$

for every $f \in C(X)$ and $y \in Y$, and thus $\overline{\alpha}T$ is an isometric algebra isomorphism.

1.3 Spectral Preserver Problems

The Gleason–Kahane–Żelazko Theorem (Theorem 1.2.1) has a spectral condition, but it also requires the mapping $T\colon A \to B$ to be a linear operator. In this section, we present several results that require preservation of all or part of the spectra of the elements of the algebra or a subset of the elements of the algebra but do not require that the mapping T be linear. The first such result requires that the spectrum of the difference between algebra elements be preserved in order to have the mapping preserve the algebraic structure as well as the distances between algebra elements.

Theorem 1.3.1 (Kowalski and Słodkowski (1980))**.** [9] *A surjective mapping $T\colon A \to B$ between semisimple commutative Banach algebras such that $T(0) = 0$ and $\sigma(Tf - Tg) = \sigma(f - g)$ for every $f, g \in A$ is an isometric algebra isomorphism.*

The isometry conclusion here may not be too surprising since the spectral condition implies that $\|Tf - Tg\| = \|f - g\|$ for every $f, g \in A$, i.e., T preserves distances between the algebra elements. Thus we also see that the Mazur–Ulam Theorem (Theorem 1.1.1) implies that T is an $\mathbb{R}$–linear mapping, so the additivity requirement for an isomorphism is met.

1.3.1 Multiplicative Spectral Preserver Problems

Several of the following results require that the mapping $T\colon A \to B$ between unital algebras preserve the unit element, i.e., $T(1_A) = 1_B$. Such mappings are referred to as *unital* operators. The next result by Molnár is the first in the multiplicative direction among the spectral preserver problems.

Theorem 1.3.2 (Molnár (2001)). [16] *Let X be a first-countable compact topological space. A surjective, unital mapping $T\colon C(X) \to C(X)$ for which $\sigma(TfTg) = \sigma(fg)$ for every $f, g \in C(X)$ is an isometric algebra automorphism.*

Rao and Roy were able to extend this result to surjective self–maps from any uniform algebra to itself and for an arbitrary compact Hausdorff set X.

Theorem 1.3.3 (Rao and Roy (2005)). [17] *A surjective, unital mapping $T\colon A \to A$ from a uniform algebra to itself such that $\sigma(TfTg) = \sigma(fg)$ for every $f, g \in A$ is an algebra automorphism.*

A year later, Luttman and Tonev significantly improved Rao and Roy's result by allowing T to be an operator between any two uniform algebras instead of requiring it to be a self–map and by only requiring the preservation of a subset of the spectra (the peripheral spectra) of products of algebra elements.

Theorem 1.3.4 (Luttman and Tonev (2006)). [6,13] *A surjective, unital mapping $T\colon A \to B$ between uniform algebras for which $\sigma_\pi(TfTg) = \sigma_\pi(fg)$ for every $f, g \in A$ is an isometric algebra isomorphism.*

We note that if $\sigma(f) = \sigma(g)$ for algebra elements f and g, then $\sigma_\pi(f) = \sigma_\pi(g)$, but not vice versa, so Rao and Roy's result follows directly from this more general result. Luttman and Tonev later extended this theorem to standard operator algebras [14].

Lambert joined with Luttman and Tonev to show that instead of the preservation of the peripheral spectra of products of algebra elements, T need only preserve at least one element of the peripheral spectra of products. They also removed the requirement that T be unital and added the requirement that T preserve the peripheral spectra of all algebra elements. However, this is not requiring more than the previous result because Theorem 1.3.4 requires that T be unital, in which case $\sigma_\pi(Tf) = \sigma_\pi(TfT1) = \sigma_\pi(f \cdot 1) = \sigma_\pi(f)$, so a map that satisfies the hypotheses of Theorem 1.3.4 does in fact preserve the peripheral spectra of algebra elements.

Theorem 1.3.5 (Lambert, Luttman, and Tonev (2007))**.** [11] *A surjective map* $T\colon A \to B$ *between uniform algebras for which* $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \varnothing$ *for every* $f, g \in A$ *and which preserves the peripheral spectra of all algebra elements* ($\sigma_\pi(Tf) = \sigma_\pi(f)$ *for every* $f \in A$) *is an isometric algebra isomorphism.*

The proofs of these theorems are based on variations of the following classical result by E. Bishop:

Lemma (**3.4.1**, Classical Bishop's Lemma)**.** (*e.g.* [1, p. 102]) *Let* E *be a peak set of a uniform algebra* A *and* $f \in A$ *such that* $f|_E \not\equiv 0$. *Then there is a peaking function* $h \in \mathcal{P}_E(A)$ *such that* fh *takes its maximum modulus only within* $E = E(h)$.

Lambert refined this result in his paper with Luttman and Tonev [11] as follows:

Lemma (**3.4.2**, Bishop's Lemma for p-sets)**.** *Let* $A \subset C(X)$ *be a uniform algebra and* E *be a* p*-set of* A. *If* $f \in A$ *is such that* $f|_E \not\equiv 0$, *then there is a peaking function* $h \in \mathcal{P}_E(A)$ *such that* fh *takes its maximum modulus on* E.

In Chapter 3, we prove the following even stronger version of this lemma.

Lemma (**3.4.3**, Strong Version of the Multiplicative Bishop's Lemma)**.** *Let* A *be a uniform algebra on a compact Hausdorff space* X. *Let* $f \in A$ *and* $x_0 \in \delta A$. *If* $f(x_0) \neq 0$, *then there* *exists an* $h \in \mathcal{P}_{x_0}(A)$ *such that*

$$|(fh)(x)| < |f(x_0)| \tag{1.1}$$

for every $x \notin E(h)$ *and* $|(fh)(x)| = f(x_0)$ *for every* $x \in E(h)$. *If* U *is any neighborhood of* x_0, *then* h *can be chosen such that* $E(h) \subset U$.

1.3.2 Additive Spectral Preserver Problems

In the additive direction, in 2006 Rao, Tonev, and Toneva showed that a surjection that preserves the peripheral spectra of sums of algebra elements as well as the sup-norms of the sums of the moduli of algebra elements will preserve the distances between algebra elements as well as the structure of the algebra.

Theorem (**Corollary 4.2.9**, Rao, Tonev, and Toneva (2006))**.** [6, 18] *A surjective mapping* $T\colon A \to B$ *between uniform algebras that satisfies* $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$ *and* $\||Tf| + |Tg|\| = \||f| + |g|\|$ *for every* $f, g \in A$ *is an isometric algebra isomorphism.*

The proof of this result is based on the following additive version of Bishop's Lemma (Lemma 3.4.1), which is proven in [18].

Lemma (**3.4.4**, Additive Version of Bishop's Lemma)**.** [18] *If* $E \subset X$ *is a peak set for* A, *and* $f \not\equiv 0$ *on* E *for some* $f \in A$, *then there exists a function* $h \in \mathcal{P}(A)$ *that peaks on* E *and satisfies the inequality*

$$|f(x)| + N\,|h(x)| < \max_{\xi \in E} |f(\xi)| + N$$

for any $x \in X \setminus E$ *and any real number* $N \geq \|f\|$.

We strengthen this result in Chapter 3 as follows:

Lemma (**3.4.5**, Strong Version of the Additive Bishop's Lemma)**.** *For any* $f \in A$, $x_0 \in \delta A$

and real number $r > 1$, there exists an $\mathbb{R}$-peaking function $h \in r\,\|f\| \cdot \mathcal{P}_{x_0}(A)$ such that

$$|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| \tag{1.2}$$

for every $x \notin E(h)$ and $|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$ for every $x \in E(h)$. In particular, $\||f(x)| + |h(x)|\| = |f(x_0)| + |h(x_0)|$.

If U is a neighborhood of x_0, then h can be chosen such that $E(h) \subset U$.

We also prove a useful corollary to this result in Chapter 3.

Corollary (**3.4.8**, Tonev and Yates (2009))**.** [22] *Let $f \in A$, $x_0 \in \delta A$, and $r > 1$, and let the function $h_0 \in r\,\|f\| \cdot \mathcal{P}_{x_0}(A)$ be as in Lemma 3.4.5. Then*

$$|f(x_0)| + r\,\|f\| = |f(x_0)| + |h_0(x_0)| = \||f| + |h_0|\| = \inf_{\substack{h \in \mathcal{E}_{x_0}(A) \\ \|h\| = r\|f\|}} \||f| + |h|\|\,. \tag{1.3}$$

In Chapter 4, we consider additive spectral preserver problems and improve the result of Rao, Tonev, and Toneva (Corollary 4.2.9) in the same spirit in which Theorem 1.3.5 improves Theorem 1.3.4 by requiring that T preserve at least one element of the peripheral spectra of sums of algebra elements. The proof of this result uses Lemma 3.2 and Corollary 3.4.8.

Theorem (**Corollary 4.2.16**, Tonev and Yates (2009))**.** [22] *A surjective mapping $T\colon A \to B$ between uniform algebras that satisfies $\sigma_\pi(Tf+Tg) \cap \sigma_\pi(f+g) \neq \varnothing$, $\||Tf| + |Tg|\| = \||f| + |g|\|$ for every $f, g \in A$ and either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) $\sigma_\pi(Th) = \sigma_\pi(h)$ for every $\mathbb{C}$–peaking function h in A

is an isometric algebra isomorphism.

We note that Corollary 4.2.9 implicitly requires that the peripheral spectra of all elements of the algebra A be preserved because any T that satisfies the hypotheses of Corollary 4.2.9 must also preserve zero, so $\sigma_\pi(Tf) = \sigma_\pi(Tf + T0) = \sigma_\pi(f + 0) = \sigma_\pi(f)$. Thus, Corollary 4.2.16 extends and encompasses the result given in Corollary 4.2.9. We also show the following more compact result:

Theorem (**Corollary 4.2.17**, Tonev and Yates (2009))**.** [22] *A surjective mapping $T\colon A \to B$ between uniform algebras that satisfies $\sigma_\pi(\lambda Tf + \mu Tg) \cap \sigma_\pi(\lambda f + \mu g) \neq \varnothing$ for every $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$ and for which either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) $\sigma_\pi(Th) = \sigma_\pi(h)$ for every $\mathbb{C}$–peaking function h in A

is an isometric algebra isomorphism.

1.4 Norm–Preserver Problems

We note that if $\sigma_\pi(f) = \sigma_\pi(g)$ for two algebra elements f and g, then $\|f\| = \|g\|$, but the converse does not necessarily hold. Thus, a natural question arising from the results in the previous section is whether the peripheral spectra conditions can be replaced by norm conditions. Lambert, Luttman, and Tonev proved the following result for operators that preserve the norms of products of algebra elements.

Theorem (**5.0.18**, Lambert, Luttman, and Tonev (2007))**.** [11] *A mapping $T\colon A \to B$ between uniform algebras that preserves the peaking functions of the algebra (i.e., $T(\mathcal{P}(A)) = \mathcal{P}(B)$) satisfies the equation $\|TfTg\| = \|fg\|$ for every $f, g \in A$ if and only if there exists a homeomorphism $\psi\colon \delta B \to \delta A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$.*

In Chapter 4, we prove an additive analogue of this result:

Theorem (**4.1.10**, Tonev and Yates (2009))**.** [22] *If an $\mathbb{R}^+$–homogeneous bijection $T\colon A \to B$ satisfies $\||Tf| + |Tg|\| = \||f| + |g|\|$ for every $f, g \in A$, then there exists a homeomorphism $\psi\colon \delta B \to \delta A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$.*

Corollary 4.2.16 is encompassed by the next result, which is proven in Chapter 4. Since Corollary 4.2.16 requires that $\sigma_\pi(Tf + Tg) \cap \sigma_\pi(f + g) \neq \varnothing$ for every $f, g \in A$, it follows that $\|Tf + Tg\| = \|f + g\|$ for every $f, g \in A$, so any operator $T\colon A \to B$ that satisfies the hypotheses of Corollary 4.2.16 will also satisfy the hypotheses of the following theorem.

Theorem (**4.2.7**, Tonev and Yates (2009))**.** [22] *If a surjective mapping $T\colon A \to B$ between uniform algebras satisfies $\|Tf + Tg\| = \|f + g\|$ and $\||Tf| + |Tg|\| = \||f| + |g|\|$ for every $f, g \in A$, then there exists a homeomorphism $\psi\colon \delta B \to \delta A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$. If, in addition, either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) $\sigma_\pi(Th) = \sigma_\pi(h)$ for every $\mathbb{C}$–peaking function h in A

is an isometric unital algebra isomorphism.

Since, as we show in Chapter 4, any surjective mapping $T\colon A \to B$ such that $\|\lambda Tf + \mu Tg\| = \|\lambda f + \mu g\|$ for every $\lambda, \mu \in \mathbb{C}$ and $f, g \in A$ and that preserves either the peripheral spectra of $\mathbb{C}$–peaking functions of A or the constant functions *1* and i satisfies the hypotheses of Theorem 4.2.7, the next more compact result actually follows directly from Theorem 4.2.7.

Theorem 1.4.1 (**4.2.11**, Tonev and Yates (2009))**.** [22] *If a surjection $T\colon A \to B$ between uniform algebras satisfies $\|\lambda Tf + \mu Tg\| = \|\lambda f + \mu g\|$ for every $f, g \in A$ and $\lambda, \mu \in C$, then there exists a homeomorphism $\psi\colon \delta B \to \delta A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$. If, in addition, either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) $\sigma_\pi(Th) = \sigma_\pi(h)$ *for every* $\mathbb{C}$*–peaking function* h *in* A

is an isometric unital algebra isomorphism.

We obtain an extension of the Gleason–Kahane–Żelazko Theorem (Theorem 1.2.1) to uniform algebras as a corollary to these results.

Corollary (**4.2.14**, Tonev and Yates (2009))**.** [22] *Any norm–preserving, linear surjection* $T\colon A \to B$ *between two uniform algebras such that*

(a) $T(1) = 1$ *and* $T(i) = i$ *or*

(b) T *preserves the peripheral spectra of all* $\mathbb{C}$*–peaking functions of* A

is automatically multiplicative and, in fact, an algebra isomorphism.

Chapter 2

Commutative Banach Algebras

2.1 Introduction to Commutative Banach Algebras

In this chapter, we present some of the basic definitions and results concerning Banach algebras that will be useful throughout this thesis.

Definition 2.1.1. *An* algebra *over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ with a multiplication that makes A into a ring such that if $\alpha \in \mathbb{F}$ and $a, b \in A$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$.*

Example 1. Cleary, $\mathbb{R}$ with itself as the scalar field and the usual multiplication is an algebra, as is $\mathbb{C}$ with $\mathbb{R}$ or $\mathbb{C}$ as the scalar field.

Example 2. The collection of $n \times n$ matrices with real entries, $M_n(\mathbb{R})$, with $\mathbb{R}$ as the scalar field, forms a vector space that is a ring under normal matrix multiplication. For any scalar α and matrices A, B in this vector space, we have $\alpha(AB) = (\alpha A)B = A(\alpha B)$, so this, too, is an algebra (in this case noncommutative). $M_n(\mathbb{C})$ is also a noncommutative algebra.

Example 3. The algebras with which we are primarily concerned here are function algebras; the simplest example of such is the set $C(X)$ of continuous functions on a space (generally compact and Hausdorff in our case) under pointwise operations with either $\mathbb{R}$ or $\mathbb{C}$ as the scalar field. The product of two continuous functions is itself a continuous function, so we have a multiplication that makes $C(X)$ into a ring in which $\alpha(fg) = (\alpha f)g = f(\alpha g)$. Thus, $C(X)$ is an algebra. Because function multiplication is commutative, $C(X)$ is in fact a commutative algebra.

Example 4. Similarly, the space $C^n[a, b]$ of all continuous $\mathbb{C}$-valued functions on the closed interval $[a, b]$ that have continuous derivatives up to and including order n is a commutative function algebra when considered with pointwise operations.

Definition 2.1.2. *An algebra B over $\mathbb{C}$ with a norm $\|\cdot\|$ with respect to which B is a Banach space and for which*

$$\|ab\| \leq \|a\| \, \|b\|$$

holds for every $a, b \in B$ is called a Banach algebra*. If its multiplication is commutative, the Banach algebra is called* commutative*, and if there exists a unit element with respect to the multiplication (usually written e or 1) such that $\|e\| = 1$, the algebra is said to be* with unit *or* unital*.*

Example 5. Both $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces under the norm $|\cdot|$; because $|ab| = |a||b|$ for any $a, b \in \mathbb{R}$ or $a, b \in \mathbb{C}$, $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$ are Banach algebras that are clearly commutative and unital.

Example 6. We have seen that $M_n(\mathbb{F})$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) is an algebra. If we endow it with the operator norm $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$, then we have $\|AB\| = \sup\{\|ABx\| : \|x\| = 1\} \leq \sup\{\|A\| \, \|Bx\| : \|x\| = 1\} = \|A\| \, \|B\|$, so $M_n(\mathbb{F})$ under the operator norm is a Banach algebra. Because matrix multiplication is not commutative, this is not a commutative algebra. The identity matrix serves as the unit element, so $M_n(\mathbb{F})$ is unital.

Example 7. The function algebra $C(X)$ with X a compact Hausdorff space is a Banach space under the supremum (or uniform) norm $\|f\| = \sup_{x\in X} |f(x)|$ and

$$\|fg\| = \sup_{x\in X} |(fg)(x)| \le \sup_{x\in X} |f(x)| \sup_{x\in X} |g(x)| = \|f\| \, \|g\| \, ;$$

thus $C(X)$ is a commutative Banach algebra. The constant function $f(x) = 1$ is a unit element with respect to multiplication, so $C(X)$ is also unital.

Example 8. (e.g. [21]) The function algebra $C^n[a,b]$ mentioned previously is a commutative Banach algebra with respect to the norm $\|f\|_{C^n[a,b]} = \sum_{k=0}^{n} \frac{1}{k!} \max_{a\le t\le b} \left|f^{(k)}(t)\right|$ since under this norm $C^n[a,b]$ is a Banach space and satisfies the Banach algebra product inequality as follows: for $f, g \in C^n[a,b]$, we have

$$\begin{aligned}
\|fg\|_{C^n[a,b]} &= \sum_{k=0}^{n} \frac{1}{k!} \max_{a\le t\le b} \left|(fg)^{(k)}(t)\right| \\
&\le \sum_{k=0}^{n} \frac{1}{k!} \max_{a\le t\le b} \left(\sum_{i=0}^{k} \binom{k}{i} \left|(f)^{(i)}(t)\right| \left|(g)^{(k-i)}(t)\right| \right) \\
&\le \sum_{k=0}^{n} \left(\sum_{i=0}^{k} \frac{1}{i!} \max_{a\le t\le b} \left|(f)^{(i)}(t)\right| \frac{1}{(k-i)!} \max_{a\le t\le b} \left|(g)^{(k-i)}(t)\right| \right) \\
&= \sum_{i=0}^{n} \left(\sum_{k=i}^{n} \frac{1}{i!} \max_{a\le t\le b} \left|(f)^{(i)}(t)\right| \frac{1}{(k-i)!} \max_{a\le t\le b} \left|(g)^{(k-i)}(t)\right| \right) \\
&= \sum_{i=0}^{n} \left(\frac{1}{i!} \max_{a\le t\le b} \left|(f)^{(i)}(t)\right| \sum_{k=i}^{n} \frac{1}{(k-i)!} \max_{a\le t\le b} \left|(g)^{(k-i)}(t)\right| \right) \\
&= \sum_{i=0}^{n} \left(\frac{1}{i!} \max_{a\le t\le b} \left|(f)^{(i)}(t)\right| \sum_{k=0}^{n} \frac{1}{k!} \max_{a\le t\le b} \left|(g)^{(k)}(t)\right| \right) \\
&= \left(\sum_{i=0}^{n} \frac{1}{i!} \max_{a\le t\le b} \left|(f)^{(i)}(t)\right| \right) \left(\sum_{k=0}^{n} \frac{1}{k!} \max_{a\le t\le b} \left|(g)^{(k)}(t)\right| \right) = \|f\|_{C^n[a,b]} \, \|g\|_{C^n[a,b]} \, .
\end{aligned}$$

For the rest of this chapter, we will be mainly concerned with general commutative Banach algebras. However, most of our results occur in a specific type of Banach algebra called a uniform algebra, so we give a definition and examples here.

Definition 2.1.3. (*e.g.* [21]) *Let X be a compact Hausdorff space. A commutative Banach* *algebra A over $\mathbb{C}$ is a* uniform algebra *on X if*

1. *the algebra A consists of continuous complex-valued functions defined on X; i.e., $A \subset C(X)$,*
2. *all the constant functions on X are in A (so in particular, the function $1|_X \in A$),*
3. *the operations in A are pointwise addition and multiplication,*
4. *the algebra A is closed in $C(X)$ with respect to the uniform norm*

$$\|f\| = \max_{x \in X} |f(x)| \quad \textit{for} \quad f \in A, \quad \textit{and}$$

5. *the algebra A separates the points of X; i.e., for every $x_1 \neq x_2$ in X, there is a function $f \in A$ such that $f(x_1) \neq f(x_2)$.*

Example 9. When X is a compact Hausdorff space, $C(X)$ itself under the supremum (uniform) norm is clearly a uniform algebra. In fact, because X is compact and Hausdorff, each function attains its maximum value at some point in X, so the norm can be written as $\|f\| = \max_{x \in X} |f(x)|$.

Example 10. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in $\mathbb{C}$ and let $A(\mathbb{D})$ be the space of continuous functions in $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ that are analytic in $\mathbb{D}$. If we consider $A(\mathbb{D})$ with pointwise operations and the uniform norm, then $A(\mathbb{D})$ is a uniform algebra (commonly called the *disc algebra*). Clearly, $A(\mathbb{D}) \subset C(\overline{\mathbb{D}})$ and $A(\mathbb{D})$ is closed under the uniform norm because uniform limits of continuous functions on $\overline{\mathbb{D}}$ that are analytic in $\mathbb{D}$ are themselves continuous on $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$.

Definition 2.1.4. *An element f in a commutative Banach algebra B is* invertible *if there is some $g \in B$ such that $fg = 1$. Then g is uniquely defined and called the* inverse *of f in B, denoted by f^{-1}. The set of all invertible elements in B is denoted by B^{-1}.*

The inverse of an element in a commutative Banach algebra is unique since if $fg_1 = 1 = fg_2$, then $g_1 = 1 \cdot g_1 = (fg_2)g_1 = (g_2f)g_1 = g_2(fg_1) = g_2(1) = g_2$. Also, the set of invertible elements B^{-1} forms a subgroup of B under multiplication: let $f, g \in B^{-1}$. Then there exist $f^{-1}, g^{-1} \in B^{-1}$ such that $ff^{-1} = 1 = gg^{-1}$, so we need only show that B^{-1} is closed under multiplication. We see that $(fg)(g^{-1}f^{-1}) = f(gg^{-1})f^{-1} = ff^{-1} = 1$, so fg is invertible, i.e., $fg \in B^{-1}$ and B^{-1} is closed under multiplication.

Example 11. (e.g. [21]) The *exponents*, defined by the convergent power series

$$e^f = 1 + \frac{f}{1!} + \frac{f^2}{2!} + \cdots + \frac{f^n}{n!} + \cdots ,$$

of a commutative Banach algebra (these elements obey the normal exponent laws $e^{f+g} = e^f e^g$) are invertible with $(e^f)^{-1} = e^{-f}$. The set of all exponents in B is denoted by e^B.

Proposition 2.1.5. *Let B be a commutative Banach algebra with unit element e, and let $f \in B$.*

(a) If $\|f\| < 1$, then $e - f \in B^{-1}$ and $(e-f)^{-1} = \sum_{n=0}^{\infty} f^n$ (where $f^0 = 1$).

(b) If $\lambda \in \mathbb{C}$ with $|\lambda| > \|f\|$, then $(\lambda \cdot e - f) \in B^{-1}$ and $(\lambda \cdot e - f)^{-1} = \sum_{n=0}^{\infty} \frac{f^n}{\lambda^{n+1}}$.

Proof. [21] (a) Let $\|f\| < 1$. If $m < n$, then applying the triangle inequality and the Banach algebra inequality gives

$$\left\| \sum_{k=0}^{n} f^k - \sum_{k=0}^{m} f^k \right\| = \left\| \sum_{k=m+1}^{n} f^k \right\| \leq \sum_{k=m+1}^{n} \left\| f^k \right\| \leq \sum_{k=m+1}^{n} \|f\|^k .$$

Because $\|f\| < 1$, this is a convergent geometric series with

$$\sum_{k=m+1}^{n} \|f\|^k = \frac{\|f\|^{m+1} - \|f\|^{n+1}}{1 - \|f\|} \leq \frac{\|f\|^{m+1}}{1 - \|f\|}.$$

Because $\|f\| < 1$, for every $\epsilon > 0$, we can choose n and m sufficiently large such that $\left\|\sum_{k=0}^{n} f^k - \sum_{k=0}^{m} f^k\right\| < \epsilon$, so $\left\{\sum_{k=0}^{n} f^k\right\}$ is a Cauchy sequence in B, a complete space. Thus, $\lim_{n\to\infty}\sum_{k=0}^{n} f^k = \sum_{n=0}^{\infty} f^n \in B$. Hence,

$$\left(\sum_{n=0}^{\infty} f^n\right)(e-f) = \left(\lim_{k\to\infty}\sum_{n=0}^{k} f^n\right)(e-f)$$
$$= \lim_{k\to\infty}\sum_{n=0}^{k}\left(f^n - f^{n+1}\right) = \lim_{k\to\infty}\left(e - f^{k+1}\right) = e - \lim_{k\to\infty} f^{k+1} = 1$$

because $\|f\| < 1$ implies that $\lim_{k\to\infty}\left\|f^{k+1}\right\| \le \lim_{k\to\infty}\|f\|^{k+1} = 0$, so $\lim_{k\to\infty} f^{k+1} = 0$. Therefore, $(e-f)^{-1} = \sum_{n=0}^{\infty} f^n$.

(b) If $|\lambda| > \|f\|$, then $\left\|\frac{f}{\lambda}\right\| = \left\|\left(\frac{1}{\lambda}\right)f\right\| = \frac{\|f\|}{|\lambda|} < 1$. By part (a), we have that $e - \frac{f}{\lambda}$ is invertible with $\left(e-\frac{f}{\lambda}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{f}{\lambda}\right)^n$, which implies that

$$1 = \left(e-\frac{f}{\lambda}\right)\sum_{n=0}^{\infty}\left(\frac{f}{\lambda}\right)^n = \left(e-\frac{f}{\lambda}\right)\sum_{n=0}^{\infty}\frac{f^n}{\lambda^n} = \frac{\lambda\cdot e - f}{\lambda}\sum_{n=0}^{\infty}\frac{f^n}{\lambda^n} = (\lambda\cdot e - f)\sum_{n=0}^{\infty}\frac{f^n}{\lambda^{n+1}}.$$

Hence, $\lambda\cdot e - f \in B^{-1}$ and $(\lambda\cdot e - f)^{-1} = \sum_{n=0}^{\infty}\frac{f^n}{\lambda^{n+1}}$. $\square$

2.2 The Spectrum of an Element

Definition 2.2.1. *The* spectrum *of an element f in a Banach algebra B is the set*

$$\sigma(f) = \{\lambda\in\mathbb{C}\colon \lambda\cdot e - f \notin B^{-1}\} = \{\lambda\in\mathbb{C}\colon\ ((\lambda\cdot e - f)\cdot B)\cap B^{-1} = \varnothing\}.$$

From this point onward, we will employ the notation-simplifying convention of suppressing the unit element when writing the Banach algebra element $\lambda \cdot e$, where $\lambda \in \mathbb{C}$. In particular, $\sigma(f) = \{\lambda \in \mathbb{C} \colon \lambda - f \notin B^{-1}\}$.

Example 12. Let B be a commutative Banach algebra. Then for $0 \in B$, $\sigma(0) = \{\lambda \in \mathbb{C} \colon \lambda - 0 = \lambda \notin B^{-1}\}$ implies that $\lambda = 0$, i.e., $\sigma(0) = \{0\}$. In fact, for any constant $c \in \mathbb{C}$, $\sigma(c) = \{c\}$.

Proposition 2.1.5 implies that the set $\{\lambda - f : |\lambda| > \|f\|\}$ is a subset of B^{-1}, i.e., contains only invertible elements. Hence, the spectrum is a set bounded by a circle centered at the origin of radius $\|f\|$, i.e., $\sigma(f) \subset \{z \in \mathbb{C} : |z| \leq \|f\|\}$. In a Banach algebra over $\mathbb{C}$, the spectrum is also nonempty, as we will prove following Singh's arguments in [19]. We first prove a preliminary lemma.

Lemma 2.2.2. [19] *Let $\phi(z)$ be a continuous function in $\mathbb{C}$ that is analytic in $\mathbb{C}\backslash\{0\}$, and define $g(r,\theta) : [0,\infty) \times [0,2\pi] \to \mathbb{C}$ such that $g(r,\theta) = \phi(re^{i\theta})$. Then the function*

$$F(r) = \int_0^{2\pi} g(r,\theta)d\theta$$

is constant on $[0,\infty)$ and its value is $2\pi\phi(0)$.

Proof. [19] Because ϕ is analytic on $\mathbb{C}\backslash\{0\}$, the function $g(r,\theta)$ has continuous partial derivatives with respect to r and θ on $(0,\infty) \times (0,2\pi)$, which we compute as follows:

$$\frac{\partial g}{\partial \theta}(r,\theta) = \frac{\partial \phi(re^{i\theta})}{\partial \theta} = \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial \theta} = \frac{\partial \phi}{\partial z} ire^{i\theta}$$

and

$$\frac{\partial g}{\partial r}(r,\theta) = \frac{\partial \phi(re^{i\theta})}{\partial r} = \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial \phi}{\partial z} e^{i\theta} = \frac{1}{ir}\frac{\partial g}{\partial \theta}(r,\theta).$$

Because ϕ is continuous and analytic, the derivative $\dfrac{\partial g}{\partial r}$ must be continuous and bounded on

$(0, 2\pi)$ for every $r > 0$, so we can differentiate under the integral sign:

$$\begin{aligned}\frac{dF}{dr} &= \int_0^{2\pi} \frac{\partial}{\partial r} g(r,\theta)d\theta = \int_0^{2\pi} \frac{1}{ir}\frac{\partial g}{\partial \theta}(r,\theta)d\theta \\ &= \frac{1}{ir}\left(g(r,2\pi) - g(r,0)\right) = \frac{1}{ir}\left(\phi(re^{2\pi i}) - \phi(re^{0})\right) = \frac{1}{ir}\left(\phi(r) - \phi(r)\right) = 0\end{aligned}$$

Thus, $F(r)$ is constant on $(0, \infty)$. We note that for $r < 1$, as $r \to 0$, the function $g(r,\theta) \to g(0,\theta)$ and $|g(r,\theta)| = |\phi(re^{i\theta})| \leq \max_{|z|\leq 1} |\phi(z)|$, which is bounded on $[0, 2\pi] \times [0, 1]$. Hence, $F(r)$ is continuous at 0, so $F(r)$ is in fact constant on $[0, \infty)$. Specifically,

$$F(r) = F(0) = \int_0^{2\pi} g(0,\theta)d\theta = \int_0^{2\pi} \phi(0)d\theta = 2\pi\phi(0).$$

$\square$

Theorem 2.2.3. *If a is an element of a complex Banach algebra B, then $\sigma(a)$ is nonempty.*

Proof. [19] Suppose $a = 0$. Then, as shown in Example 12, $0 \in \sigma(0)$, so $\sigma(0)$ is nonempty. Thus, we may suppose that $a \neq 0$. We proceed by contradiction: assume $\sigma(a) = \varnothing$. This implies that $\widetilde{a}(z) := (a - z)^{-1}$ is in B (i.e., $a - z$ is invertible) for every $z \in \mathbb{C}$. Let f be any bounded linear functional on B. We define the function $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi(z) = f(\widetilde{a}(z))$. We will show that ϕ is differentiable with respect to z. Let $h \in \mathbb{C}$. Then, because f is linear and continuous, for all $z \in \mathbb{C}$ we have

$$\begin{aligned}
\frac{d\phi}{dz} = \lim_{h\to 0}\frac{\phi(z+h)-\phi(z)}{h} &= \lim_{h\to 0}\frac{f(\widetilde{a}(z+h))-f(\widetilde{a}(z))}{h} \\
&= \lim_{h\to 0}\frac{f((a-(z+h))^{-1})-f((a-z)^{-1})}{h} \\
&= \lim_{h\to 0}\frac{f((a-(z+h))^{-1}-(a-z)^{-1})}{h} \\
&= \lim_{h\to 0}\frac{f\left((a-z)^{-1}\left((a-z)-(a-z-h)\right)(a-z-h)^{-1}\right)}{h} \\
&= \lim_{h\to 0}\frac{f\left((a-z)^{-1}h(a-z-h)^{-1}\right)}{h} \\
&= \lim_{h\to 0}\frac{hf\left((a-z)^{-1}(a-z-h)^{-1}\right)}{h} \\
&= \lim_{h\to 0} f\left((a-z)^{-1}(a-z-h)^{-1}\right) \\
&= f\left(\lim_{h\to 0}\left((a-z)^{-1}(a-z-h)^{-1}\right)\right) \\
&= f\left(\left((a-z)^{-1}\right)^2\right) = f(\widetilde{a}(z)^2).
\end{aligned}$$

Thus, ϕ is an entire function and $\dfrac{d\phi}{dz} = f(\widetilde{a}(z)^2)$. If $g(r,\theta) := \phi(re^{i\theta}) = f(\widetilde{a}(re^{i\theta}))$, then Lemma 2.2.2 implies that the function $\int_0^{2\pi} g(r,\theta)d\theta = \int_0^{2\pi} f(\widetilde{a}(re^{i\theta}))d\theta$ is constant on $[0,\infty)$, and its value is $2\pi\phi(0) = 2\pi f(\widetilde{a}(0)) = 2\pi f(a^{-1})$. By the Hahn-Banach Theorem we can choose a bounded linear functional f on B such that $f(a^{-1}) \neq 0$. Because f is linear, we have

$$f(\widetilde{a}(re^{i\theta})) = f((a-re^{i\theta})^{-1}) = f\left(\frac{1}{re^{i\theta}}\left(\frac{a}{re^{i\theta}}-1\right)^{-1}\right) = \frac{1}{re^{i\theta}}f\left(\left(\frac{a}{re^{i\theta}}-1\right)^{-1}\right).$$

When $r\to\infty$, we have $\dfrac{a}{re^{i\theta}} \to 0$, which implies that $\left(\dfrac{a}{re^{i\theta}}-1\right)^{-1} \to -1$ because the map $x \mapsto x^{-1}$ is continuous on the group of invertible elements. Thus, we can make $|f(\widetilde{a}(re^{i\theta}))|$ arbitrarily small uniformly in θ by choosing r as large as necessary.

We now fix r such that $|f(\widetilde{a}(re^{i\theta}))| < \dfrac{|f(a^{-1})|}{2}$. Then we have

$$2\pi|f(a^{-1})| = \left|\int_0^{2\pi} f(\widetilde{a}(re^{i\theta}))d\theta\right| \le \int_0^{2\pi} |f(\widetilde{a}(re^{i\theta}))|d\theta \le \int_0^{2\pi} \frac{|f(a^{-1})|}{2} d\theta \le \pi|f(a^{-1})|.$$

Hence, $f(a^{-1}) = 0$, which contradicts our choice of f. Thus, $\sigma(a) \neq \varnothing$. □

We next show that any commutative Banach algebra whose set of invertible elements includes every element other than 0 is effectively the field of complex numbers.

Theorem 2.2.4 (Gelfand–Mazur Theorem)**.** *A commutative Banach field B (a Banach algebra with $B^{-1} = B\backslash\{0\}$) is isometrically isomorphic to $\mathbb{C}$.*

Proof. (e.g. [21]) Let B be a commutative Banach algebra such that $B^{-1} = B\backslash\{0\}$ and consider the subalgebra $\mathbb{C}e = \{f \in B : f = \lambda e\}$. Because $\|\lambda e\| = |\lambda|$, $\mathbb{C}e$ is isometrically isomorphic to $\mathbb{C}$, implying that $\mathbb{C}e$ is one-dimensional. Because $\sigma(f)$ is nonempty for every $f \in B$, there is at least one $z_f \in \sigma(f) \subset \mathbb{C}$. Then $z_f e - f$ is not invertible, so $z_f e \notin B\backslash\{0\}$; i.e., $z_f e - f = 0$, so $f = z_f e$. Thus, every element of B is of the form ze for some $z \in \mathbb{C}$, so B is isometrically isomorphic to the subalgebra $\mathbb{C}e$ and, therefore, isometrically isomorphic to $\mathbb{C}$. □

2.3 Linear Multiplicative Functionals

Throughout this section, B will be a unital commutative Banach algebra.

Definition 2.3.1. *A* linear multiplicative functional *of B is a linear function $\phi : B \to \mathbb{C}$ for which $\phi(ab) = \phi(a)\phi(b)$ for every $a, b \in B$.*

Any linear multiplicative functional other than $\phi \equiv 0$ preserves the identity and takes nonzero values for invertible elements of the algebra, as the next lemma demonstrates.

Lemma 2.3.2. *If $\phi \not\equiv 0$ is a linear multiplicative functional on B, then $\phi(e) = 1$ and $\phi(a) \neq 0$ for all $a \in B^{-1}$.*

Proof. Let ϕ be a linear multiplicative functional on B. Then $\phi(e) = \phi(e \cdot e) = \phi(e)\phi(e)$, which implies that $\phi(e)(1 - \phi(e)) = 0$ so either $\phi(e) = 0$ or $\phi(e) = 1$. If $\phi(e) = 0$, then $\phi(a) = \phi(a \cdot e) = \phi(a)\phi(e) = 0$ for every $a \in B$, so $\phi \equiv 0$. Otherwise, $\phi(e) = 1$.

If $\phi \not\equiv 0$, then $1 = \phi(e) = \phi(a \cdot a^{-1}) = \phi(a)\phi(a^{-1})$ for every $a \in B^{-1}$. Thus, $\phi(a) \neq 0$ whenever $a \in B^{-1}$ and $\phi \not\equiv 0$. $\square$

Additionally, every linear multiplicative functional on B is continuous and has norm 1.

Lemma 2.3.3. *Let ϕ be a nonzero linear multiplicative functional on B. Then ϕ is continuous on B and $\|\phi\| = 1$.*

Proof. Let ϕ be a nonzero linear multiplicative functional on B and $f \in B$. We must show that ϕ is bounded (and thus continuous). It suffices to show that $|\phi(f)| \leq \|f\|$. We consider $\phi(f) - f$. Because ϕ is linear, we have $\phi(\phi(f) - f) = \phi(f) - \phi(f) = 0$, so by Lemma 2.3.2, $\phi(f) - f \notin B^{-1}$. This implies that $\phi(f) - f \in \sigma(f)$, so $|\phi(f)| \leq \|f\|$. This proves the claim and that $\|\phi\| \leq 1$. Lemma 2.3.2 implies that $\phi(e) = 1$, so $\|\phi\| \geq |\phi(e)| = 1$, which means that $\|\phi\| = 1$. $\square$

Example 13. Let X be a compact Hausdorff space and fix an x in X. Let $\phi_x \in C(X)$ be the point evaluation at x defined by $\phi_x(f) = f(x)$ for every $f \in C(X)$. Clearly, $\phi_x : C(X) \to \mathbb{C}$, so ϕ_x is a functional. Let $f, g \in C(X)$ and $\lambda, \mu \in \mathbb{C}$. Then

$$\phi_x(\lambda f + \mu g) = (\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x) = \lambda \phi_x f + \mu \phi_x g$$

and

$$\phi_x(fg) = (fg)(x) = f(x)g(x) = \phi_x(f)\phi_x(g),$$

so ϕ_x is a linear multiplicative functional on $C(X)$.

In fact, every linear multiplicative functional on $C(X)$ can be considered as a point evaluation for some $x \in X$, as we see in the next theorem.

Theorem 2.3.4. *If ϕ is a linear multiplicative functional on $C(X)$, then there is an $x_0 \in X$ such that $\phi = \phi_{x_0}$.*

Proof. (e.g. [21]) Suppose that ϕ is a linear multiplicative functional on $C(X)$ that is not of type ϕ_x for some $x \in X$.

We first claim that for every $x \in X$ there is some $f_x \in X$ such that $\phi(f_x) = 0$ but $f_x(x) \neq 0$ and prove this by contradiction. Assume that there is an $x_0 \in X$ such that $f(x_0) = 0$ for every $f \in \{f \in C(X) \colon \phi(f) = 0\}$. We fix an $f \in C(X)$ and consider the function $f^* = f - \phi(f)$. Evaluating ϕ at f^*, we have $\phi(f^*) = \phi(f - \phi(f)) = \phi(f) - \phi(\phi(f)) = \phi(f) - \phi(f)\phi(1) = 0$, which implies that $0 = f^*(x_0) = f(x_0) - \phi(f)$. Thus, $\phi(f) = f(x_0)$ for every $f \in C(X)$. This contradicts our choice of ϕ not being a point evaluation, so we have proven that if ϕ is a linear multiplicative functional that is not a point evaluation, then for every $x \in X$ there is some $f_x \in X$ such that $\phi(f_x) = 0$ but $f_x(x) \neq 0$.

Because f_x is a continuous function, $|f_x|^2 > 0$ on some neighborhood U_x of x and $\phi(|f_x|^2) = \phi(f_x \overline{f_x}) = \phi(f_x)\phi(\overline{f_x}) = 0$. We choose a finite covering for X of neighborhoods $\{U_{x_j}\}_{j=1}^n$ with $x_j \in X$ such that $\phi(f_{x_j}) = 0$ but $f_{x_j}(x_j) \neq 0$ and consider the function $g(x) = |f_{x_1}(x)|^2 + \cdots + |f_{x_n}(x)|^2 = f_{x_1}(x)\overline{f_{x_1}}(x) + \cdots + f_{x_n}(x)\overline{f_{x_n}}(x)$. Clearly, $g(x) > 0$ on X and so $\dfrac{1}{g(x)} \in C(X)$, i.e., $g \in C(X)^{-1}$. This contradicts the fact that $\phi(g) = \phi(f_{x_1}\overline{f_{x_1}} + \cdots + f_{x_n}\overline{f_{x_n}}) = \phi(f_{x_1})\phi(\overline{f_{x_1}}) + \cdots + \phi(f_{x_n})\phi(\overline{f_{x_n}}) = 0$. Thus, ϕ must be of type ϕ_{x_0} for some $x_0 \in X$. $\square$

Example 14. Functionals that act as point evaluations are also linear multiplicative functionals in the uniform algebra $A(\mathbb{D})$ as well as in the commutative Banach algebra $C^n[a, b]$. In fact, in both cases all linear multiplicative functionals can be expressed as point evaluations.

2.4 Maximal Ideals

The significance of linear multiplicative functionals in a commutative Banach algebra lies in their connection to the maximal ideals of the algebra. We recall the definition of an ideal of an algebra. Again in this section, B is a unital commutative Banach algebra.

Definition 2.4.1. (*e.g.* [4]) *A linear subset $J \subset B$ is an* ideal *of B if it is closed with respect to multiplication by arbitrary elements of B; i.e., $ab \in J$ for any $a \in B$ and $b \in J$. An ideal J is* proper *if $J \neq B$ and* maximal *if it is proper and if $J \subseteq I$ implies that $J = I$ or $I = B$ for any ideal I of B.*

The kernel of a linear multiplicative functional is an important example of an ideal of B.

Lemma 2.4.2. *If ϕ is a linear multiplicative functional on B, then $\ker \phi = \{b \in B : \phi(b) = 0\}$ is a proper ideal of B.*

Proof. Let $a, b \in \ker \phi$ and $c \in B$. Then $\phi(a - b) = \phi(a) - \phi(b) = 0 - 0 = 0$ and $\phi(ac) = \phi(a)\phi(c) = 0 \cdot \phi(c) = 0$. Hence, $a - b \in \ker \phi$ and $ac \in \ker \phi$, so $\ker \phi$ an ideal. Because $\phi(e) = 1$, $e \notin \ker \phi$, implying that $\ker \phi$ is a proper ideal. $\square$

We note that e is not in any proper ideal of B (else $a \cdot e = a$ would be in the ideal for any $a \in B$, so the ideal would no longer be proper). Also, if J is an ideal of B and $a \in B^{-1} \cap J$, then $aa^{-1} = e \in J$, so J must equal B. Therefore, a proper ideal contains no invertible elements and an ideal of type aB is proper if and only if $a \notin B^{-1}$.

The following important result from abstract algebra will be useful in our proof of the connection between linear multiplicative functionals and maximal ideals.

Theorem 2.4.3. *Every proper ideal J is contained in a maximal ideal of B.*

Proof. (e.g. [4]) We will prove this theorem using Zorn's Lemma. Let $\mathcal{J}$ be the set of all proper ideals in B that contain J. Then $\mathcal{J}$ is nonempty because $J \in \mathcal{J}$ and $\mathcal{J}$ is partially ordered by inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{J}$, and define $N := \bigcup_{A \in \mathcal{C}} A$ to be the union of all the ideals in $\mathcal{C}$.

We claim that N is an ideal in B. Because $J \in \mathcal{C}$, we have that N is nonempty; specifically, $0 \in N$ since 0 is in every ideal. Let $a, b \in N$. Then there are some ideals $J_1, J_2 \in \mathcal{C}$ such that $a \in J_1$ and $b \in J_2$. Because $\mathcal{C}$ is a chain, either $J_1 \subset J_2$ or $J_2 \subset J_1$. Without loss of generality, we may assume that $J_1 \subset J_2$, so $a \in J_2$, which implies that $a - b \in J_2$ because J_2 is closed under subtraction. Also, for any $\lambda \in \mathbb{C}$ and any $f \in B$, we have $\lambda b \in J_2$ and $fb \in J_2$, so $\lambda b, fb \in N$. Hence, N is an ideal.

Additionally, N is a proper ideal since $1 \notin J_i$ for every $J_i \in N$, so $1 \notin N$. Therefore, $N \in \mathcal{J}$ and N is an upper bound for $\mathcal{C}$. Thus, every chain in $\mathcal{J}$ has an upper bound, so by Zorn's Lemma, $\mathcal{J}$ has a maximal element that is a maximal ideal containing J. $\square$

We next prove that the kernel of any nonzero linear multiplicative functional is a maximal ideal.

Theorem 2.4.4. *Let $\phi \not\equiv 0$ be a linear multiplicative functional on B. Then $\ker\phi$ is a maximal ideal of B.*

Proof. Let ϕ be a nonzero linear multiplicative functional on B. Then ϕ is an algebra homomorphism, so by the First Isomorphism Theorem, the quotient algebra $B/\ker\phi$ is isomorphic to $\phi(B)$. Let $\lambda \in \mathbb{C}$. Because B is unital, we have $\phi(\lambda) = \lambda\phi(1) = \lambda$, so ϕ is surjective from B onto $\mathbb{C}$. Thus, $B/\ker\phi \cong \mathbb{C}$, a field, implying that $\ker\phi$ is a maximal ideal in B. $\square$

In fact, every maximal ideal of a commutative Banach algebra is the kernel of a linear multiplicative functional, as we prove in the next theorem.

Theorem 2.4.5. *Every maximal ideal J of a commutative Banach algebra B coincides with the kernel of some linear multiplicative functional.*

Proof. Let J be a maximal ideal of B. Because the closure of an ideal is itself an ideal, $\overline{J}$ is either B or J. By Proposition 2.1.5, since $a = e - (e - a)$, if $\|e - a\| < 1$, then a is invertible. As we noted earlier, an ideal J contains no invertible elements, so we must have $\|e - a\| \geq 1$ for every $a \in J$. This implies that $J \subset \{a \in B : \|e - a\| \geq 1\}$ and so $\overline{J} \subset \{a \in B : \|e - a\| \geq 1\}$. Because $\|e - e\| = 0$ implies that $e \notin \overline{J}$, we have $\overline{J} \neq B$, so $\overline{J} = J$. Thus, every maximal ideal J is a closed subset of B, so the quotient space B/J is a Banach space. Since J is a maximal ideal, B/J is a field, so the Gelfand–Mazur Theorem (Theorem 2.2.4) implies that B/J is isometrically isomorphic to $\mathbb{C}$. Then the mapping

$$\phi_j = \gamma \circ \pi : B \xrightarrow{\pi} B/J \xrightarrow{\gamma} \mathbb{C},$$

where π is the natural projection from B onto B/J and γ is the isomorphism from the Gelfand-Mazur Theorem, is a homeomorphism of B into $\mathbb{C}$ since it is the composition of two homeomorphisms. As such, ϕ_j is a linear multiplicative functional on B with kernel J. $\square$

We summarize the relationship between linear multiplicative functionals and maximal ideals in a commutative Banach algebra with the following theorem.

Theorem 2.4.6. *The correspondence $\phi \mapsto \ker \phi$ is a bijective mapping between $\mathcal{M}_B$, the set of nonzero linear multiplicative functionals on B, and the set of all maximal ideals of B.*

Proof. (e.g. [21]) By Theorem 2.4.5, we know that every $\phi \in \mathcal{M}_B$ determines a maximal ideal M_ϕ of B, namely the ideal $M_\phi = \ker \phi$ and, conversely, that every maximal ideal $M \subset B$ determines a linear multiplicative functional, namely $\phi_M \in \mathcal{M}_B$. Because $\ker \phi_{M_\phi} = M_\phi = \ker \phi$ and $\phi_{M_\phi}(1) = 1 = \phi(1)$, we have $\phi_{M_\phi} = \phi$. Also, by construction of ϕ_M, we have $M_{\phi_M} = \ker \phi_M = M$, so every maximal ideal M of B is of type $\ker(\phi_M)$ for some $\phi_M \in \mathcal{M}_B$. Hence, the set of maximal ideals of B and the family of kernels of linear multiplicative functionals on B are in bijective correspondence. $\square$

Chapter 3

Boundaries and Peaking Functions

3.1 The Maximal Ideal Space and the Gelfand Transform

As we have seen, the set of linear multiplicative functionals on a commutative Banach algebra and the set of maximal ideals for that algebra are in bijective correspondence, so we can make the following definition.

Definition 3.1.1. *Let B be a commutative Banach algebra with unit. The set $\mathcal{M}_B$ of all nonzero linear multiplicative functionals of B is called the* maximal ideal space *of B.*

Though the space $\mathcal{M}_B$ does not possess a natural algebraic structure, we can equip it with the weak–* topology it inherits as a subset of B^*, the collection of all bounded linear functionals on B. When applied to the maximal ideal space, we call this topology the *Gelfand topology.* We recall that under this topology, a net of elements $\{\phi_\alpha\}$ in $\mathcal{M}_B$ tends to $\phi \in \mathcal{M}_B$ if and only if $\phi_\alpha(f) \to \phi(f)$ for every $f \in B$. Thus, under the Gelfand topology, convergence of functionals in $\mathcal{M}_B$ is pointwise convergence. A weak–* limit of linear multiplicative functionals is itself a non-zero linear multiplicative functional because $\left(\lim_\alpha \phi_\alpha\right)(1) = \lim_\alpha \phi_\alpha(1) = 1$ (e.g. [21]).

We also note that the space $\mathcal{M}_B$ is compact in the weak–* topology by the Banach-Alaoglu theorem.

Definition 3.1.2. (*e.g.* [21]) *Let f be an element in a commutative Banach algebra B. The* Gelfand transform of f *is the function $\widehat{f}$ on $\mathcal{M}_B$ defined by*

$$\widehat{f}(\phi) = \phi(f) \quad \textit{for every} \quad \phi \in \mathcal{M}_B.$$

The Gelfand transform of f is clearly continuous on $\mathcal{M}_B$ with respect to the Gelfand topology since if $\phi_\alpha \to \phi$, then $\phi_\alpha(f) \to \phi(f)$, which implies that $\widehat{f}(\phi_\alpha) \to \widehat{f}(\phi)$.

3.2 The Shilov Boundary

We recall that the maximum modulus principle for analytic functions implies that the non-constant functions in the disc algebra $A(\mathbb{D})$ take their maximum moduli only at points on the unit circle $\mathbb{T}$, which is the topological boundary of the unit disc. This notion of a boundary is extended to any commutative Banach algebra in the following definition.

Definition 3.2.1. (*e.g.* [6]) *A subset E in the maximal ideal space of a commutative Banach algebra B is called a* boundary *of B if, for every $f \in B$, there is a $\phi_0 \in E$ such that* $\left|\widehat{f}(\phi_0)\right| = \max_{\phi \in \mathcal{M}_B} \left|\widehat{f}(\phi)\right|$.

According to this definition, any boundary of B is nonempty.

It is clear that the maximal ideal space $\mathcal{M}_B$ is a boundary for B. The following theorem gives us a more interesting boundary, which Shilov introduced in the 1940s.

Theorem 3.2.2 (Shilov's Theorem)**.** *The intersection of all closed boundaries of a commutative Banach algebra B is a boundary of B.*

Before we present a proof for this theorem, we will prove the following lemma.

Lemma 3.2.3. (*e.g.* [20]) *Let B be a commutative Banach algebra on X and let*

$$V = V(\phi_0; f_1, \cdots, f_n; 1) = \{\phi \in \mathcal{M}_B \colon\ |\phi(f_j)| < 1,\ \phi_0(f_j) = 0,\ j = 1, \ldots, n\}$$

be a fixed Gelfand neighborhood in $\mathcal{M}_B$. Then either V meets every boundary of B, or the complement $E \setminus V$ of V in every closed boundary E of B is also a closed boundary of B.

Proof. Suppose that E is a closed boundary of B and $E \setminus V$ is not a boundary. If $E \setminus V = \varnothing$, then $V \supset E$ and thus $\left|\widehat{f}_j(\phi)\right| < 1$ on $\mathcal{M}_B$ for every $j = 1, \ldots, n$ because $\left|\widehat{f}_j(\phi)\right| < 1$ on the boundary E of B. Therefore $V = \mathcal{M}_B$, so V must meet each boundary of B.

If, on the other hand, $E \setminus V \neq \varnothing$, then there is some $f \in B$ such that

$$\max_{\phi \in \mathcal{M}_B} \left|\widehat{f}(\phi)\right| = 1 > \max_{\phi \in E \setminus V} \left|\widehat{f}(\phi)\right|$$

since $E \setminus V$ is not a boundary of B by assumption. Because $\widehat{f^n} \to 0$ uniformly as $n \to \infty$ on $E \setminus V$, there is an $m \in \mathbb{Z}$ such that

$$\max_{\phi \in E \setminus V} \left|\widehat{f^m}(\phi)\right| \left|\widehat{f}_j(\phi)\right| < 1$$

holds for every $j = 1, \ldots, n$. The inequality $\left|\widehat{f^m}(\phi)\right| \left|\widehat{f}_j(\phi)\right| < 1$ also holds on V for every $j = 1, \ldots, n$ because $\left|\widehat{f}_j(\phi)\right| < 1$ for every $\phi \in V$. Thus, this inequality holds on E for every $j = 1, \ldots, n$. Because E is a boundary of B by hypothesis, the inequalities $\left|\widehat{f^m}(\phi)\right| \left|\widehat{f}_j(\phi)\right| < 1$ hold for all $j = 1, \ldots, n$ everywhere on $\mathcal{M}_B$. Now choose $\phi_1 \in \mathcal{M}_B$ such that $\left|\widehat{f}(\phi_1)\right| = 1$. Then

$$1 > \left|\widehat{f^m}(\phi_1)\widehat{f}_j(\phi_1)\right| = \left|\widehat{f}_j(\phi_1)\right| \quad \text{for} \quad j = 1, \ldots, n$$

implies that $\phi_1 \in V$. Therefore, the positive function $\phi \mapsto \left|\widehat{f}(\phi)\right|$ attains its maximum only within V and every boundary of B meets V. $\square$

We can now prove Shilov's theorem (Theorem 3.2.2).

Proof. (e.g. [20]) Let E_0 be the intersection of all closed boundaries of B and let f be a fixed element in B such that $\left|\widehat{f}(\phi)\right| < 1$ for every $\phi \in E_0$. We will show that $\left|\widehat{f}(\phi)\right| < 1$ on $\mathcal{M}_B$. Suppose instead that the set $K = \left\{\phi \in \mathcal{M}_B \colon \left|\widehat{f}(\phi)\right| \geq 1\right\} \neq \varnothing$. Let $\phi_0 \in K$. Then $\phi_0 \notin E_0$ because $K \cap E_0 = \varnothing$. Thus, there is a closed boundary E of B that does not contain ϕ_0, which implies that there is a Gelfand neighborhood of ϕ_0, call it V_E, which does not meet E. By Lemma 3.2.3, $\mathcal{M}_B \setminus V_E$ is also a boundary of B. Because K is a compact set, there are finitely many closed boundaries E_j of B and corresponding open subsets $V_{E_1}, \ldots, V_{E_k}$ in $\mathcal{M}_B$ such that $V_{E_j} \cap E_j = \varnothing$ and whose union covers K. This implies that the sets $\mathcal{M}_B \setminus E_j$ are also boundaries of B. By induction, we see that $\mathcal{M}_B \setminus \bigcup_{j=1}^{n} V_{E_j}$ is also a nonempty boundary of B. By the definition of K, we have that the inequality $\left|\widehat{f}(\phi)\right| < 1$ holds on the boundary $\mathcal{M}_B \setminus \bigcup_{j=1}^{n} V_{E_j} \subset \mathcal{M}_B \setminus K$ of B and so it must hold everywhere on $\mathcal{M}_B$. This is a contradiction to the assumption that K is nonempty. Hence, K must be empty, so $\left|\widehat{f}(\phi)\right| < 1$ on $\mathcal{M}_B$ and E_0 is a boundary of B. $\square$

Definition 3.2.4. *The intersection of all closed boundaries of a commutative Banach algebra B is called the* Shilov boundary of B *and is denoted by ∂B.*

Clearly, ∂B is the smallest closed boundary of B and is contained in every closed boundary of B. It is also compact because it is a closed subset of the compact set $\mathcal{M}_B$. The next lemma gives us a useful characterization of the points in the Shilov boundary.

Lemma 3.2.5. *A point $x \in X$ is in the Shilov boundary of a commutative Banach algebra B if and only if for every open neigborhood U of x, there is an $f \in B$ such that*

$$\left\|f|_{X \setminus U}\right\|_\infty < \|f|_U\|_\infty .$$

Proof. [8] Let $x \in X \setminus \partial B$. Then $X \setminus \partial B$ is an open neighborhood of x and ∂B is a boundary for B, so for every $f \in B$, we have $\left\|f|_{X\setminus\partial B}\right\|_\infty \leq \|f\|_\infty = \max_{x\in X}|f(x)| = \left\|f|_{X\setminus\partial B}\right\|_\infty = \left\|f|_{X\setminus(X\setminus\partial B)}\right\|_\infty$.

On the other hand, let $x \in \partial B$ and suppose there is some open neighborhood U of x such that $\left\|f|_{X\setminus U}\right\|_\infty \geq \|f|_U\|_\infty$ for every $f \in B$. Then $X \setminus U$ is a boundary for B, so ∂B must be contained in $X \setminus U$, which contradicts $x \in \partial B$. □

Example 15. The Shilov boundary of $C(X)$ is X itself. This is clear since by Urhysohn's lemma, given any neighborhood U of a point $x \in X$, we can find a function $f \in C(X)$ such that f has norm 1 on U and less than 1 on $X \setminus U$. Thus by Lemma 3.2.5, X is the Shilov boundary of $C(X)$.

Example 16. Let $\lambda \in \mathbb{T}$, the unit circle. Then the function $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ given by $f(z) = 1 + \overline{\lambda}z$ takes its maximum modulus at $z = \lambda$, so $\mathbb{T}$ is contained in any closed boundary of $A(\mathbb{D})$. Hence, $\mathbb{T}$ is the Shilov boundary of $A(\mathbb{D})$.

3.3 Peripheral Spectra and Peaking Functions

In this section, $A \subset C(X)$ will be a uniform algebra on a compact Hausdorff space X.

We now define a subset of the spectrum of an algebra element that will appear in many of the conditions for isomorphisms in Chapter 4.

Definition 3.3.1. *The* peripheral spectrum *of $f \in A$ is the set*

$$\sigma_\pi(f) = \sigma(f) \cap \{z \in \mathbb{C}\colon\ |z| = \|f\|\}$$

of elements in the spectrum of f with maximal modulus.

Example 17. The peripheral spectrum of the constant function c is c itself: $\sigma_\pi(c) = \{c\}$. If z is the identity function in $A(\mathbb{D})$, then $\sigma_\pi(z) = \mathbb{T}$, the unit circle.

We note that if $\sigma(f) = \sigma(g)$ for some $f, g \in A$, then clearly $\sigma_\pi(f) = \sigma_\pi(g)$. However, the converse is not true: equality of peripheral spectra does not necessarily imply equality of spectra.

We also give a name to the set of elements in X that a particular algebra element maps to its peripheral spectrum.

Definition 3.3.2. *For any $f \in A$, we call the set $E(f)$ of all $x \in X$ at which f attains its maximum (extreme) modulus the* maximum modulus set *of f, i.e.,*

$$E(f) = \{x \in X \colon |f(x)| = \|f\|\} = \{x \in X \colon f(x) \in \sigma_\pi(f)\} = f^{-1}(\sigma_\pi(f)).$$

The set of algebra elements that take their maximum modulus at a given $x \in X$ will be denoted by $\mathcal{E}_x(A) = \{f \in A \colon |f(x)| = \|f\|\}$. We note that $\mathcal{E}_x(A)$ is the set of all algebra elements f for which $x \in E(f)$. In his proof of the Banach–Stone Theorem (Theorem 1.2.3) in [7], Holsztyński made use of families of sets similar to $\mathcal{E}_x(A)$ in the case where $A = C(X)$.

Algebra elements whose peripheral spectra are singletons will also be useful to us in Chapter 4.

Definition 3.3.3. *An element $h \in A$ is called a* peaking function *of A if $\sigma_\pi(h) = \{1\}$, i.e., if $\|h\| = 1$ and $|h(x)| < 1$ whenever $h(x) \neq 1$. We denote the set of all peaking functions in A by $\mathcal{P}(A)$. Given an $x \in X$, we let $\mathcal{P}_x(A)$ denote the set of all peaking functions of A that peak on x, i.e., $h(x) = 1$.*

Example 18. The constant function *1* is clearly a peaking function.

Example 19. Given any $c > 0$, the function $f(z) = \dfrac{z+c}{1+c}$ in $A(\mathbb{D})$ is a peaking function since $\max\limits_{z \in \overline{\mathbb{D}}} \left| \dfrac{z+c}{1+c} \right| = 1$ at $z = 1$ and is less than 1 for every other z.

Example 20. If (X, d) is a compact metric space such that $d(x, y) \leq 1$ for every $x, y \in X$ and if we fix a point $x_0 \in X$, then the function $h(x) = 1 - d(x, x_0)$ takes the value 1 at the point x_0 and has value less than 1 at every other point in X, so h is a peaking function of (X, d).

Definition 3.3.4. *The maximum modulus set* $E(h) = \{x \in X \colon h(x) = 1\} = h^{-1}\{1\}$ *of a peaking function* h *is called the* peak set *of* h*. Nonempty intersections of peak sets of* A (*in general, neither finite nor countable*) *are called* generalized peak sets *or* p-sets *of* A*. If* E *is a subset of* X *such that* $E \subset E(h)$ *for some peaking function* h*, we say that* h peaks on E.

Example 21. The peak sets of the peaking functions given in Examples 18, 19, and 20 are as follows: $E(1) = X$, $E\left(\dfrac{z+c}{1+c}\right) = \{1\}$ and $E(h) = \{x_0\}$.

If we multiply a peaking function by an element of $\mathbb{C}$ (or $\mathbb{R}$), the result is clearly still a function with a singleton peripheral spectrum.

Definition 3.3.5. *We call the elements of* $\mathbb{C} \cdot \mathcal{P}(A)$ (*or* $\mathbb{R} \cdot \mathcal{P}(A)$) $\mathbb{C}$-peaking functions (*or* $\mathbb{R}$-peaking functions) *of* A.

Example 22. Clearly, any constant function c is a $\mathbb{C}$–peaking function. The function $z + c$, $c \in \mathbb{C}$ is a $\mathbb{C}$–peaking function for $A(\mathbb{D})$.

If $h \in \mathcal{P}_x(A)$, then $x \in E(h)$, so $h \in \mathcal{E}_x(A)$. Thus, $\mathcal{P}_x(A) \subset \mathcal{E}_x(A)$. It is clear that $\mathbb{C} \cdot \mathcal{P}_x(A) \subset \mathcal{E}_x(A)$ as well.

Definition 3.3.6. *A point* $x \in X$ *is called a* generalized peak point, *or* p-point, *of* A *if for every neighborhood* V *of* x *there is a peaking function* h *with* $x \in E(h) \subset V$*. The set* δA *of all generalized peak points of* A *is called the* Choquet boundary (*or the* strong boundary) *of* A.

We note that although this definition is not one of the standard definitions of the Choquet boundary, Dales proves that this definition is equivalent to the standard definitions when A is a uniform algebra, as in our case, in [3, p. 448].

Clearly, every p-point belongs to the Shilov boundary of A, so $\delta A \subset \partial A$. The Choquet boundary is a boundary for A, but it is not necessarily closed. However, the closure of the Choquet boundary is closed, so it must coincide with the Shilov boundary: $\overline{\delta A} = \partial A$ (e.g. [5]).

3.4 Bishop's Lemma

One of the tools often used in Chapter 4 and in other work on this subject is Bishop's Lemma, which allows us to take an element of an algebra and multiply it by a peaking function in order to make the peak set of the product occur strictly within a desired peak set.

Lemma 3.4.1 (Classical Bishop's Lemma)**.** (*e.g.* [1, p. 102]) *Let E be a peak set of a uniform algebra A and $f \in A$ such that $f|_E \not\equiv 0$. Then there is a peaking function $h \in \mathcal{P}_E(A)$ such that fh takes its maximum modulus only within $E = E(h)$.*

Luttman and Tonev made use of this lemma in [13] and then refined it in their paper [11] with Lambert to a result that allowed them to multiply any element of a uniform algebra by a peaking function and have the product take its maximum modulus inside a desired p-set instead of peak set.

Lemma 3.4.2 (Bishop's Lemma for p-sets)**.** [11] *Let $A \subset C(X)$ be a uniform algebra and E be a p-set of A. If $f \in A$ is such that $f|_E \not\equiv 0$, then there is a peaking function $h \in \mathcal{P}_E(A)$ such that fh takes its maximum modulus on E.*

Each of these versions of Bishop's Lemma follows as a corollary to the following stronger result:

Lemma 3.4.3 (Strong Version of the Multiplicative Bishop's Lemma)**.** *Let A be a uniform algebra on a compact Hausdorff space X. Let $f \in A$ and $x_0 \in \delta A$. If $f(x_0) \neq 0$, then there* *exists an $h \in \mathcal{P}_{x_0}(A)$ such that*

$$|(fh)(x)| < |f(x_0)| \tag{3.1}$$

for every $x \notin E(h)$ and $|(fh)(x)| = |f(x_0)|$ for every $x \in E(h)$. If U is any neighborhood of x_0, then h can be chosen such that $E(h) \subset U$.

Proof. For every $n \in \mathbb{N}$, we define the set

$$U_n = \left\{ x \in X : |f(x) - f(x_0)| < \frac{|f(x_0)|}{2^{n+1}} \right\}.$$

Then $x_0 \in U_n \subset U_{n-1}$ and U_n is open in X for every integer $n > 1$. For each $n \in \mathbb{N}$, we choose a peaking function $k_n \in \mathcal{P}_{x_0}(A)$ such that $E(k_n) \subset U_n$ and let h_n be a large enough power of k_n such that $|h_n(x)| < \dfrac{|f(x_0)|}{2^n \|f\|}$ on $X \setminus U_n$. The function $h = \displaystyle\sum_{n=1}^{\infty} \frac{h_n}{2^n}$ belongs to $\mathcal{P}_{x_0}(A)$. We claim that h is the desired function that satisfies (3.1).

We note that $E(h) \subset \displaystyle\bigcap_{n=1}^{\infty} E(h_n) \subset \bigcap_{n=1}^{\infty} U_n$. In fact, $\displaystyle\bigcap_{n=1}^{\infty} U_n = f^{-1}(f(x_0))$: clearly, if $x \in f^{-1}(f(x_0))$, then $x \in \displaystyle\bigcap_{n=1}^{\infty} U_n$. On the other hand, if $x \in \displaystyle\bigcap_{n=1}^{\infty} U_n$, then $|f(x) - f(x_0)| < \dfrac{|f(x_0)|}{2^{n+1}}$ for every $n \in \mathbb{N}$, so $f(x) = f(x_0)$, i.e., $x \in f^{-1}(f(x_0))$. Hence, $E(h) \subset \displaystyle\bigcap_{n=1}^{\infty} U_n \subset f^{-1}(f(x_0))$, and if $x \in E(h)$, then $|f(x)h(x)| = |f(x_0)|$. If $x \in f^{-1}(f(x_0)) \setminus E(h)$, then $|f(x)h(x)| = |f(x_0)h(x)| < |f(x_0)|$.

When $x \notin f^{-1}(f(x_0))$, there are two possibilities. In the first case, suppose $x \notin U_1$. Then $x \notin U_n$ for every $n \in \mathbb{N}$, so $|h_n(x)| < \dfrac{|f(x_0)|}{2^n \|f\|}$ for every $n \in \mathbb{N}$. This implies that

$$|h(x)| < \sum_{n=1}^{\infty} \frac{|f(x_0)|}{4^n \|f\|} < \frac{|f(x_0)|}{\|f\|}.$$

Hence, for $x \notin U_1$, $|f(x)h(x)| < \|f\| \dfrac{|f(x_0)|}{\|f\|} = |f(x_0)|$.

For the other case, suppose $x \in U_{n-1} \setminus U_n$ for some $n > 1$. Then $x \in U_i$ for $1 \leq i \leq n-1$ and $x \notin U_i$ for every $i \geq n$. Therefore, $|h_i(x)| < \dfrac{|f(x_0)|}{2^i \|f\|}$ for every $i \geq n$. Because $x \in U_{n-1}$, we have $|f(x) - f(x_0)| < \dfrac{|f(x_0)|}{2^n}$, so

$$\begin{aligned}|f(x)h(x)| &= |f(x) - f(x_0) + f(x_0)|\,|h(x)| \leq (|f(x) - f(x_0)| + |f(x_0)|)\,|h(x)| \\ &< \left(\frac{|f(x_0)|}{2^n} + |f(x_0)|\right)\left(\sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} + \sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i}\right).\end{aligned}$$

Because each h_n is a peaking function of A, it follows that $|h_n(x)| \leq 1$ for any $x \in X$ and therefore each series above is bounded above by a convergent geometric series. We thus have

$$\sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} \leq \sum_{i=1}^{n-1} \frac{1}{2^i} = \left(1 - \frac{1}{2^{n-1}}\right)$$

and

$$\sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i} \leq \sum_{i=n}^{\infty} \frac{|f(x_0)|}{4^i \|f\|} < \frac{|f(x_0)|}{2^n \cdot 2^{n-1} \|f\|}.$$

Hence,

$$\begin{aligned}|f(x)h(x)| &< \left(\frac{|f(x_0)|}{2^n} + |f(x_0)|\right)\left(1 - \frac{1}{2^{n-1}} + \frac{|f(x_0)|}{2^n \cdot 2^{n-1} \|f\|}\right) \\ &= |f(x_0)|\left(\frac{1}{2^n} + 1\right)\left(1 - \frac{1}{2^{n-1}} + \frac{|f(x_0)|}{2^n \cdot 2^{n-1} \|f\|}\right) \\ &\leq |f(x_0)|\left(\frac{1}{2^n} + 1\right)\left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n \cdot 2^{n-1}}\right) \\ &= |f(x_0)|\left(\frac{1}{2^n} + 1\right)\left(1 - \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^n}\right)\right) \\ &< |f(x_0)|\left(1 + \frac{1}{2^n}\right)\left(1 - \frac{1}{2^n}\right) \\ &= |f(x_0)|\left(1 - \frac{1}{2^{2n}}\right) < |f(x_0)|.\end{aligned}$$

We have shown that $|(fh)(x)| < |f(x_0)|$ for every $x \notin E(h)$.

It remains to be shown that we can choose h such that its peak set is contained in any neighborhood of x_0. Let U be an open set containing x_0. Take a peaking function $h_* \in \mathcal{P}_{x_0}(A)$ such that $E(h_*) \subset U$. Then $|h_*(x)| < 1$ on $X \setminus U$ and the function hh_* satisfies (3.1) with $E(hh_*) \subset U$. □

In their paper [18] considering additive spectral conditions under which mappings between uniform algebras are isomorphisms, Rao, Tonev, and Toneva proved the following additive version of Bishop's Lemma.

Lemma 3.4.4 (Additive Version of Bishop's Lemma)**.** [18] *If $E \subset X$ is a peak set for A and $f \not\equiv 0$ on E for some $f \in A$, then there exists a function $h \in \mathcal{P}(A)$ that peaks on E and satisfies the inequality*

$$|f(x)| + N\,|h(x)| < \max_{\xi \in E} |f(\xi)| + N$$

for any $x \in X \setminus E$ and any real number $N \geq \|f\|$.

Below, we strengthen this result to be able to force the sum of the moduli of an algebra element and a peaking function to take its maximum modulus at a specific point in the Choquet boundary of the algebra instead of merely within a given peak set. Lemma 3.4.4 follows as a corollary to this stronger result. The proof of Lemma 3.4.5 is similar to the proof of Lemma 3.4.3, but we can also address the case in which $f(x_0) = 0$.

Lemma 3.4.5 (Strong Version of the Additive Bishop's Lemma)**.** *For any $f \in A$, $x_0 \in \delta A$, and real number $r > 1$, there exists an $\mathbb{R}$–peaking function $h \in r\,\|f\| \cdot \mathcal{P}_{x_0}(A)$ such that*

$$|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| \tag{3.2}$$

for every $x \notin E(h)$ and $|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$ for every $x \in E(h)$. In particular, $\||f(x)| + |h(x)|\| = |f(x_0)| + |h(x_0)|$.

If U is a neighborhood of x_0, then h can be chosen such that $E(h) \subset U$.

Proof. We first consider the case when $f(x_0) \neq 0$. As in the proof of the Strong Multiplicative Bishop's Lemma, for every $n \in \mathbb{N}$, we define the open set

$$U_n = \left\{ x \in X\colon \ |f(x) - f(x_0)| < \frac{|f(x_0)|}{2^{n+1}} \right\}.$$

Then $x \in U_n \subset U_{n-1}$ for every $n \in \mathbb{N}$. For each n, we choose a peaking function $k_n \in \mathcal{P}_{x_0}(A)$ such that $E(k_n) \subset U_n$ and let $h_n \in \mathcal{P}_{x_0}(A)$ be a large enough power of k_n such that $|h_n(x)| < \dfrac{|f(x_0)|}{2^n r \|f\|}$ on $X \setminus U_n$.

We claim that the $\mathbb{R}$-peaking function $h = r\|f\| \cdot \sum_1^{\infty} \dfrac{h_n}{2^n}$ satisfies inequality (3.2). We note that because h is clearly in $r\|f\| \cdot \mathcal{P}_{x_0}(A)$, we have that $\|h\| = r\|f\| = |h(x_0)|$. The fact that $E(h) \subset \bigcap_{n=1}^{\infty} E(h_n) \subset \bigcap_{n=1}^{\infty} U_n = f^{-1}(f(x_0))$ follows exactly as in the proof of the Strong Multiplicative Bishop's Lemma.

For any $x \in E(h)$, we have $|f(x)| + |h(x)| = |f(x_0)| + \|h\|$, and for any $x \in f^{-1}(f(x_0)) \setminus E(h)$, $|f(x)| + |h(x)| = |f(x_0)| + |h(x)| < |f(x_0)| + \|h\|$. When $x \notin f^{-1}(f(x_0)) = \bigcap_{n=1}^{\infty} U_n$, there are two possibilities.

Case 1: In the first case, $x \notin U_1$. Then $x \notin U_n$ for every $n \in \mathbb{N}$, so $|h_n(x)| < \dfrac{|f(x_0)|}{2^n r\|f\|}$ for every $n \in \mathbb{N}$. Thus, $|h(x)| < r\|f\| \cdot \sum_1^{\infty} \dfrac{|f(x_0)|}{4^n r\|f\|} < |f(x_0)|$, and, therefore, $|f(x)| + |h(x)| < r\|f\| + |f(x_0)| = |f(x_0)| + \|h\|$.

Case 2: In the second case, $x \in U_{n-1} \setminus U_n$ for some $n > 1$. Then $x \in U_i$ for $1 \leq i \leq n-1$ and $x \notin U_i$ for every $i \geq n$. Therefore, $|h_i(x)| < \dfrac{|f(x_0)|}{2^i r\|f\|}$ for every $i \geq n$. Because $x \in U_{n-1}$, we have $|f(x) - f(x_0)| < \dfrac{|f(x_0)|}{2^n}$, so

$$\begin{aligned}|f(x)| + |h(x)| &\leq |f(x_0)| + |f(x) - f(x_0)| + |h(x)| \\ &< |f(x_0)| + \frac{|f(x_0)|}{2^n} + r\,\|f\| \cdot \sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} + r\,\|f\| \cdot \sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i}.\end{aligned}$$

Because each h_n is a peaking function of A, it follows that $|h_n(x)| \leq 1$ for any $x \in X$, and therefore, each of the series above is bounded above by a convergent geometric series. We thus have

$$\sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} \leq \sum_{i=1}^{n-1} \frac{1}{2^i} = 1 - \frac{1}{2^{n-1}}$$

and

$$\sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i} < \sum_{i=n}^{\infty} \frac{|f(x_0)|}{4^i r\,\|f\|} \leq \sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{1}{3 \cdot 4^{n-1}}$$

Hence,

$$\begin{aligned}|f(x)| + |h(x)| &\leq |f(x_0)| + \frac{|f(x_0)|}{2^n} + \left(1 - \frac{1}{2^{n-1}}\right) r\,\|f\| + \frac{|f(x_0)|}{3 \cdot 4^{n-1}} \\ &< |f(x_0)| + \left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{3 \cdot 4^{n-1}}\right) r\,\|f\| \\ &= |f(x_0)| + \left(1 - \frac{1}{2^{n-1}}\left(1 - \frac{1}{2} - \frac{1}{3 \cdot 2^{n-1}}\right)\right) \|h\| < |f(x_0)| + \|h\|.\end{aligned}$$

Thus $|f(x)| + |h(x)| < |f(x_0)| + \|h\|$ for every $x \notin f^{-1}(f(x_0))$.

If $f(x_0) = 0$, we must show that $|f(x)| + |h(x)| < |h(x_0)| = \|h\|$. For any $n \in \mathbb{N}$, we define the open set

$$V_n = \left\{ x \in X\colon\ |f(x)| < \frac{(r-1)\,\|f\|}{2^{n+1}} \right\}.$$

Clearly, $V_n \subset V_{n-1}$ and $x_0 \in V_n$ for every $n \in \mathbb{N}$.

As in the case when $f(x_0) \neq 0$, for each n we choose a peaking function $k_n \in \mathcal{P}_{x_0}(A)$ such that $E(k_n) \subset V_n$ and let $h_n \in \mathcal{P}_{x_0}(A)$ be a large enough power of k_n such that $|h_n(x)| < \dfrac{r-1}{2^n r}$ on

$X \setminus V_n$. We claim that the $\mathbb{R}$-peaking function $h = r\,\|f\| \cdot \sum_{n=1}^{\infty} \frac{h_n}{2^n}$ satisfies inequality (3.2) in this case. Similarly to the previous case, one can see that $E(h) \subset f^{-1}(0) = \bigcap_{n=1}^{\infty} V_n$. We note that $\|h\| = r\,\|f\|$ because $h \in r\,\|f\| \cdot \mathcal{P}_{x_0}(A)$. It is clear that if $x \in E(h)$ then $|f(x)| + |h(x)| = \|h\|$, while $|f(x)| + |h(x)| = |h(x)| < \|h\|$ for any $x \in f^{-1}(0) \setminus E(h)$.

Now suppose that $x \notin f^{-1}(0)$. If, in addition, $x \notin V_1$, then we obtain as before that $|h(x)| < r\,\|f\| \cdot \sum_{1}^{\infty} \frac{r-1}{4^n r} < (r-1)\,\|f\|$, and, therefore,

$$|f(x)| + |h(x)| < \|f\| + (r-1)\,\|f\| = r\,\|f\| = \|h\|\,.$$

In the case where $x \in V_{n-1} \setminus V_n$ for some $n > 1$, we have that $x \in V_i$ for $1 \leq i \leq n-1$ and $x \notin V_i$ for every $i \geq n$. Therefore, $|h_i(x)| < \frac{r-1}{2^i r}$ for every $i \geq n$. Because $x \in V_{n-1}$, we see that $|f(x)| < \frac{(r-1)\,\|f\|}{2^n} < \frac{r\,\|f\|}{2^n}$, so

$$|f(x)| + |h(x)| < \frac{r\,\|f\|}{2^n} + r\,\|f\| \sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} + r\,\|f\| \sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i}.$$

Each h_n is a peaking function of A, so $|h_n(x)| \leq 1$ for every $x \in X$ and the series above are bounded above by convergent geometric series. We thus have

$$\sum_{i=1}^{n-1} \frac{|h_i(x)|}{2^i} \leq \sum_{i=1}^{n-1} \frac{1}{2^i} = 1 - \frac{1}{2^{n-1}}.$$

and

$$\sum_{i=n}^{\infty} \frac{|h_i(x)|}{2^i} < \sum_{i=n}^{\infty} \frac{(r-1)}{4^i r} < \sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{1}{3 \cdot 4^{n-1}}.$$

Therefore, we have

$$|f(x)| + |h(x)| \leq \frac{r\,\|f\|}{2^n} + \left(1 - \frac{1}{2^{n-1}}\right) r\,\|f\| + \frac{r\,\|f\|}{3\cdot 4^{n-1}}$$
$$\leq \left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{3\cdot 4^{n-1}}\right) r\,\|f\| < r\,\|f\| = \|h\|\,.$$

Hence, $|f(x)| + |h(x)| < \|h\|$ for every $x \notin f^{-1}(f(0))$.

Now let U be a neighborhood of x_0. If $h_* \in \mathcal{P}_{x_0}(A)$ is a peaking function of A with $E(h_*) \subset U$, then $|h_*(x)| < 1$ on $X \setminus U$, the function $h\,h_*$ satisfies inequality (3.2) and $E(h\,h_*) \subset U$. $\square$

We note that in the case in which $f(x_0) \neq 0$ the inequality (3.2) also holds for $r = 1$.

Not only does the Additive Bishop's Lemma follow as a corollary to this version, we also get the following stronger version of Lemma 3.4.4 directly from Lemma 3.4.5.

Corollary 3.4.6. *Let $f \in A$, E be a peak set for A, and $r > 1$ be an arbitrary real number. Then for any $x_0 \in E \cap \delta A$, there exists an $\mathbb{R}$-peaking function $h \in r\,\|f\| \cdot \mathcal{P}_{x_0}(A)$ with $E(h) \subset E$ such that $|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| = \max_{\xi \in E} |f(\xi)| + \|h\|$ for every $x \notin E$.*

The next proposition, which is also a consequence of Lemma 3.4.5, will be useful in our study of norm-additive and norm-linear mappings in Chapter 4.

Proposition 3.4.7. *Let $f \in A$, $x_0 \in \delta A$, and $\alpha = \exp\{i \arg(f(x_0))\}$. For any real number $r > 1$, there exists an $\mathbb{R}$-peaking function $h \in r\,\|f\| \cdot \mathcal{P}_{x_0}(A)$ such that $E(f + \alpha h) = E(h)$, $|f(x_0) + \alpha h(x_0)| = \|f + \alpha h\|$, and*

$$|f(x) + \alpha h(x)| < \|f + \alpha h\| \tag{3.3}$$

whenever $f(x) + \alpha h(x) \neq f(x_0) + \alpha h(x_0)$. Consequently, $f + \alpha h \in \mathbb{C} \cdot \mathcal{P}_{x_0}(A)$ and $\sigma_\pi(f + \alpha h) = \{f(x_0) + \alpha h(x_0)\}$. If U is a neighborhood of x_0, then h can be chosen such that $E(f + \alpha h) \subset U$.

Proof. Let the function h be as in Lemma 3.4.5. If $\alpha = \exp\{i \arg(f(x_0))\}$, then we have that $|f(x_0) + \alpha h(x_0)| = |f(x_0)| + |h(x_0)|$ and, therefore,

$$\begin{aligned} \|f + \alpha h\| &= \max_{\xi \in X} |f(\xi) + \alpha h(\xi)| \leq \max_{\xi \in X} (|f(\xi)| + |h(\xi)|) \\ &= |f(x_0)| + |h(x_0)| = |f(x_0) + \alpha h(x_0)| \leq \|f + \alpha h\|. \end{aligned}$$

Hence $\|f + \alpha h\| = |f(x_0)| + |h(x_0)| = |f(x_0) + \alpha h(x_0)|$, so $f(x_0) + \alpha h(x_0) \in \sigma_\pi(f + \alpha h)$. Inequality (3.2) implies that for any $x \notin E(h)$, we have $|f(x) + \alpha h(x)| \leq |f(x)| + |h(x)| < |f(x_0)| + |h(x_0)| = \|f + \alpha h\|$. Therefore $f(x) + \alpha h(x) \notin \sigma_\pi(f + \alpha h)$, so $E(f + \alpha h) \subset E(h)$. Because $E(h) \subset f^{-1}(f(x_0))$, for any $x \in E(h)$, we have $f(x) + \alpha h(x) = f(x_0) + \alpha h(x_0) \in \sigma_\pi(f + \alpha h)$, which implies that $E(h) \subset E(f + \alpha h)$. Thus, $E(h) = E(f + \alpha h)$ and $\sigma_\pi(f + \alpha h) = \{f(x_0) + \alpha h(x_0)\}$, as claimed. If U is a neighborhood of x_0, then any function h from Lemma 3.4.5 with $E(h) \subset U$ satisfies inequality (3.3). □

The final result in this chapter, which will also be useful in Chapter 4, is another corollary to Lemma 3.4.5.

Corollary 3.4.8. *Let $f \in A$, $x_0 \in \delta A$, and $r > 1$, and let the function $h_0 \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ be as in Lemma 3.4.5. Then*

$$|f(x_0)| + r\|f\| = |f(x_0)| + |h_0(x_0)| = \||f| + |h_0|\| = \inf_{\substack{h \in \mathcal{E}_{x_0}(A) \\ \|h\| = r\|f\|}} \||f| + |h|\|. \tag{3.4}$$

Proof. Let $h_0 \in r\|f\| \cdot \mathcal{P}_{x_0}(A)$ be a function that satisfies inequality (3.2). For any $h \in \mathcal{E}_{x_0}(A)$ with $\|h\| = r\|f\|$, we have that

$$\begin{aligned} \||f| + |h|\| &= \max_{\xi \in X} (|f(\xi)| + |h(\xi)|) \geq |f(x_0)| + |h(x_0)| \\ &= |f(x_0)| + |h_0(x_0)| = \max_{\xi \in X} (|f(\xi)| + |h_0(\xi)|) = \||f| + |h_0|\|. \end{aligned}$$

Thus, according to Lemma 3.4.5,

$$\inf_{\substack{h\in\mathcal{E}_{x_0}(A)\\ \|h\|=r\|f\|}} \||f|+|h|\| = \||f|+|h_0|\| = |f(x_0)|+|h_0(x_0)| = |f(x_0)|+\|h\|\,.$$

$\square$

Chapter 4

Sufficient Conditions for Uniform Algebra Isomorphisms

Throughout this chapter, $A \subset C(X)$ and $B \subset C(Y)$ will be uniform algebras on compact sets X and Y, respectively.

The following proposition gives sufficient conditions under which surjective maps are algebra isomorphisms.

Proposition 4.0.9. *If $\psi \colon Y \to X$ is a homeomorphism and if $T \colon A \to C(Y)$ is a surjection defined by $Tf = f \circ \psi$ for every $f \in A$, then T is linear, multiplicative, injective, and continuous and thus is an isometric algebra isomorphism.*

Proof. Let $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$. Then $T(\lambda f + \mu g) = (\lambda f + \mu g) \circ \psi = \lambda(f \circ \psi) + \mu(g \circ \psi) = \lambda Tf + \mu Tg$, so T is linear. Also, T is multiplicative because $T(fg) = (fg) \circ \psi = (f \circ \psi)(g \circ \psi) = TfTg$. Because ψ is a homeomorphism, it is surjective, so T is injective. Finally, the continuity of T follows from the linearity of T and the inequality $\|Tf\| = \sup_{y \in Y} |f(\psi(y))| \leq \|f\|$. $\square$

If $Tf = f \circ \psi$, then we call T a *ψ–composition operator.*

If $T\colon A \to B$ is a ψ–composition operator, then T satisfies the equation

$$\||Tf| + |Tg|\| = \||f| + |g|\| \tag{4.1}$$

for every $f, g \in A$ since the fact that ψ is a homeomorphism implies that $\||Tf| + |Tg|\| = \||f \circ \psi| + |g \circ \psi|\| = \||f| + |g|\|$. The map T also satisfies the equations

$$\|Tf + Tg\| = \|f + g\| \tag{4.2}$$

and

$$\|\lambda Tf + \mu Tg\| = \|\lambda f + \mu g\| \tag{4.3}$$

for every $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$ since $\|\lambda Tf + \mu Tg\| = \|\lambda f \circ \psi + \mu g \circ \psi\| = \|\lambda f + \mu g\|$ for any $\lambda, \mu \in \mathbb{C}$ (in particular, for $\lambda = \mu = 1$, proving that T satisfies (4.2)). We also have the following preservation of relationships among the peripheral spectra: $\sigma_\pi(Tf) = \sigma_\pi(f \circ \psi) = \sigma_\pi(f)$ and $\sigma_\pi(\lambda Tf + \mu Tg) = \sigma_\pi(\lambda f \circ \psi + \mu g \circ \psi) = \sigma_\pi(\lambda f + \mu g)$ for any $\lambda, \mu \in \mathbb{C}$ (so also $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$).

In this chapter, we show that several of these conditions are, in fact, sufficient for a mapping $T\colon A \to B$ between uniform algebras to be a ψ–composition operator and thus an algebra isomorphism.

4.1 Norm–Additive in Modulus Operators

We first show that a surjective operator $T\colon A \to B$ that satisfies certain conditions naturally induces a homeomorphism between the Choquet boundary of A and the Choquet boundary of B.

Definition 4.1.1. *An operator $T\colon A \to B$ is* norm–additive in modulus *if it satisfies* (**??**), *i.e.,* $\max_{x\in X}\{|(Tf)(x)| + |(Tg)(x)|\} = \max_{x\in X}\{|f(x)| + |g(x)|\}$, *for every* $f, g \in A$.

Example 23. The operator $T\colon A \to B$ for which $Tf = if$ is norm–additive in modulus because $\||Tf| + |Tg|\| = \||if| + |ig|\| = |i|\,\||f| + |g|\| = \||f| + |g|\|$. The operator $T\colon A \to B$ for which $Tf = -f$ is similarly norm–additive in modulus. In fact, all operators $T\colon A \to B$ such that $Tf = \alpha f$ with $\alpha \in A$ and $|\alpha(x)| = 1$ for every $x \in X$ are norm–additive in modulus since $\||Tf| + |Tg|\| = \||\alpha f| + |\alpha g|\| = |\alpha|\,\||f| + |g|\| = \||f| + |g|\|$.

Example 24. The operator $T\colon A \to B$ defined by $Tf = \|f\|$ for every $f \in A$ is also norm–additive in modulus: $\||Tf| + |Tg|\| = \|\|\|f\|\| + \|\|g\|\|\| = \||f| + |g|\|$. We note that this operator does not preserve $|f|$ unless f is a constant function.

Clearly, for any norm–additive in modulus operator, we have $T\mathit{0} = \mathit{0}$ since $0 = \||\mathit{0}| + |\mathit{0}|\| = \||T\mathit{0}| + |T\mathit{0}|\| = 2\,\||T\mathit{0}|\|$ implies that $|T\mathit{0}| = \mathit{0}$. Also, an operator that is norm–additive in modulus is norm–preserving since $\|Tf\| = \||Tf| + |T\mathit{0}|\| = \||f| + |\mathit{0}|\| = \|f\|$.

Another example of norm–additive in modulus operators is given by the next proposition.

Proposition 4.1.2. [6] *An operator $T\colon A \to B$ that satisfies $\|Tf + \alpha Tg\| = \|f + \alpha g\|$ for every $f, g \in A$ and α with $|\alpha| = 1$ is norm–additive in modulus.*

Proof. If $T\colon A \to B$ satisfies $\|Tf + \alpha Tg\| = \|f + \alpha g\|$ for every $f, g \in A$ and α with $|\alpha| = 1$, then we can choose an α with $|\alpha| = 1$ such that $\||Tf| + |Tg|\| = \||Tf| + |\alpha Tg|\| = \|Tf + \alpha Tg\| = \|f + \alpha g\| \le \||f| + |\alpha g|\| = \||f| + |g|\|$. Similarly, $\||f| + |g|\| \le \||Tf| + |Tg|\|$, so T is norm–additive in modulus. □

Definition 4.1.3. [6] *An operator $T\colon A \to B$ is* monotone increasing in modulus *if the inequality $|f(x)| \le |g(x)|$ on ∂A implies that $|(Tf)(y)| \le |(Tg)(y)|$ on ∂B for every $f, g \in A$.*

Example 25. The operators $Tf = \alpha f$ for $\alpha \in A$ with $|\alpha| = 1$ given in Example 23 as norm–additive in modulus are also monotone increasing in modulus since if $|f(x)| \le |g(x)|$, then

$|(\alpha f)(x)| = |\alpha(x)|\,|f(x)| = |f(x)| \leq |g(x)| = |\alpha(x)|\,|g(x)| = |(\alpha g)(x)|.$

The next proposition provides a connection between norm–additive in modulus operators and monotone increasing in modulus operators.

Proposition 4.1.4. [6]*A norm–additive in modulus operator is monotone increasing in modulus.*

Proof. Let $T\colon A \to B$ be a norm–additive in modulus operator. If $|f(x)| \leq |g(x)|$ on ∂A, then clearly $\||f| + |k|\| \leq \||g| + |k|\|$ for any $k \in A$. Because T is norm–additive in modulus, we have that

$$\||Tf| + |Tk|\| = \||f| + |k|\| \leq \||g| + |k|\| = \||Tg| + |Tk|\|\,. \tag{4.4}$$

Assume that there is some $y_0 \in \partial B$ such that $|(Tf)(y_0)| > |(Tg)(y_0)|$. Because δB is dense in ∂B, we may assume that $y_0 \in \delta B$. Choose a $\gamma > 0$ such that $|(Tg)(y_0)| < \gamma < |(Tf)(y_0)|$ and an open neighborhood V of y_0 in Y such that $|(Tg)(y)| < \gamma$ on V. Let r be a real number greater than 1 such that $\|Tf\|, \|Tg\| \leq r$ and let $Tk \in \mathcal{P}_{y_0}(B)$ be a peaking function for B with $E(Tk) \subset V$, so $(Tk)(y_0) = 1$ and $|(Tk)(y)| < 1$ for any $y \in \partial B \backslash V$. By replacing Tk with a sufficiently high power of Tk, we have $|(Tg)(y)| + |r(Tk)(y)| < r + \gamma$ for every $y \in \partial B \setminus V$. This inequality also holds on V because $|(Tg)(y)| < \gamma$ for every $y \in V$ and $|(Tk)(y)| \leq 1$ for all $y \in Y$. Thus we have that $|(Tg)(y)| + |r(Tk)(y)| < r + \gamma$ for every $y \in \partial B$ and

$$\begin{aligned} |(Tf)(y_0)| + r &= |(Tf)(y_0)| + r\,|(Tk)(y_0)| \\ &\leq \||Tf| + r\,|Tk|\| \leq \||Tg| + r\,|Tk|\| < r + \gamma. \end{aligned}$$

Therefore, $|(Tf)(y_0)| < \gamma$, which is a contradiction. Hence, $|(Tf)(y)| \leq |(Tg)(y)|$ for every $y \in \partial B$. ☐

Definition 4.1.5. *A mapping $T\colon A \to B$ is* $\mathbb{R}^+$–homogeneous *if $T(af) = aT(f)$ for every $a \in \mathbb{R}^+$ and $f \in A$.*

Lemma 4.1.6. *If an operator $T\colon A \to B$ is $\mathbb{R}^+$–homogeneous, norm–preserving, and monotone increasing in modulus, then for any generalized peak point $x \in \delta A$, the set*

$$E_x = \bigcap_{h \in \mathcal{E}_x(A)} E(Th) \tag{4.5}$$

is nonempty and $E_x \cap \delta B \neq \varnothing$.

Proof. Let $x \in \delta A$. We will show that the family $\{E(Th)\colon h \in \mathcal{E}_x(A)\}$ has the finite intersection property. Let $h_1, \dots, h_n \in \mathcal{E}_x(A)$ and define $g = h_1 \cdots h_n$. Then

$$\begin{aligned} \|h_1 \cdots h_n\| = \|g\| \geq |g(x)| = |(h_1 \cdots h_n)(x)| = |h_1(x) \cdots h_n(x)| \\ = |h_1(x)| \cdots |h_n(x)| = \|h_1\| \cdots \|h_n\| \geq \|h_1 \cdots h_n\|, \end{aligned}$$

so $|g(x)| = \|g\| = \prod_{j=1}^{n} \|h_j\|$ and $g \in \mathcal{E}_x(A)$. Hence for any $\xi \in \partial A$ and for any fixed $k = 1, \dots, n$, we have

$$|g(\xi)| = |h_1(\xi)| \cdots |h_n(\xi)| \leq \left(\prod_{j \neq k} \|h_j\| \right) \cdot |h_k(\xi)| = \left| \left(\prod_{j \neq k} \|h_j\| \right) \cdot h_k(\xi) \right|. \tag{4.6}$$

Because T is monotone increasing in modulus, $\mathbb{R}^+$–homogeneous, and $|(Th_k)(\eta)| \leq \|Th_k\| = \|h_k\|$ for any $\eta \in \partial B$, we obtain

$$|(Tg)(\eta)| \leq \left| T\left(\left(\prod_{j \neq k} \|h_j\| \right) \cdot h_k \right)(\eta) \right| = \left(\prod_{j \neq k} \|h_j\| \right) \cdot |(Th_k)(\eta)| \leq \prod_{j=1}^{n} \|h_j\|. \tag{4.7}$$

Because T preserves norms, there is a $y \in Y$ such that $|(Tg)(y)| = \|Tg\| = \|g\| = \|h_1 \cdots h_n\|$. Therefore, $\left(\prod_{j \neq k} \|h_j\| \right) \cdot |(Th_k)(y)| = \prod_{j=1}^{n} \|h_j\|$, which implies that $|(Th_k)(y)| = \|h_k\|$, so we have $E(Tg) \subset E(Th_k)$. This holds for every $k = 1, \dots, n$, so $E(Tg) \subset \bigcap_{j=1}^{n} E(Th_j)$. Hence, the family $\{E(Th)\colon h \in \mathcal{E}_x(A)\}$ has the finite intersection property, as claimed. Because each

$E(Th)$ is a closed subset of Y, a compact set, the family $\{E(Th) \colon h \in \mathcal{E}_x(A)\}$ must have nonempty intersection.

We observe that the set $E(Tf) = (Tf)^{-1}(\sigma_\pi(Tf))$ is a union of peak sets because $(Tf)^{-1}(u)$ is a peak set for any $u \in \sigma_\pi(Tf)$. Thus, every $y \in E_x$ belongs to an intersection $F \subset E_x$ of peak sets of B. Therefore, F meets δB (cf. [12, p. 165]), and thus $E_x \cap \delta B \supset F \cap \delta B \neq \varnothing$. $\square$

We note that Rao, Tonev, and Toneva, in [18], considered sets similar to E_x that involve peaking functions instead of $\mathbb{C}$–peaking functions but also require T to preserve the peripheral spectra of all algebra elements.

Lemma 4.1.7. *Let $T \colon A \to B$ be a norm–additive in modulus, $\mathbb{R}^+$–homogeneous, surjective operator. If $x \in \delta A$ and $y \in E_x \cap \delta B$, then $T^{-1}(\mathcal{E}_y(B)) \subset \mathcal{E}_x(A)$.*

Proof. Let $x \in \delta A$. If T is $\mathbb{R}^+$–homogeneous, surjective, and norm–additive in modulus, then T is monotone increasing in modulus and norm–preserving, as we have seen, so $E_x \neq \varnothing$ by Lemma 4.1.6. Let $y \in E_x$, fix a $k \in \mathcal{E}_y(B)$, and let $h \in T^{-1}(k)$. In order to prove that $h \in \mathcal{E}_x(A)$, we must show that $|h(x)| = \|h\|$. Let V be an open neighborhood of x and let $p \in \|h\| \cdot \mathcal{P}_x(A)$ be a $\mathbb{C}$–peaking function such that $E(p) \subset V$. Because $y \in E_x = \displaystyle\bigcap_{f \in \mathcal{E}_x(A)} E(Tf) \subset E(Tp)$, we have that $|(Tp)(y)| = \|Tp\|$, which implies that $Tp \in \mathcal{E}_y(B)$. Because T preserves norms,

$$|k(y)| = \|k\| = \|h\| = \|p\| = \|Tp\| .$$

Thus, because T is norm–additive in modulus,

$$\|h\| + \|p\| \geq \||h| + |p|\| = \||k| + |Tp|\| \geq |k(y)| + |(Tp)(y)| = \|k\| + \|Tp\| = \|h\| + \|p\| . \quad (4.8)$$

Therefore, $\||h| + |p|\| = \|h\| + \|p\|$, so there must be an $x_V \in \partial A$ such that $|h(x_V)| = \|h\|$ and $|p(x_V)| = \|p\|$. Hence, $x_V \in E(p) \subset V$ and any neighborhood V of x must contain a point x_V

with $|h(x_V)| = \|h\|$. Because h is continuous, we must have $|h(x)| = \|h\|$, which implies that $h \in \mathcal{E}_x(A)$. Thus, $T^{-1}(\mathcal{E}_y(B)) \subset \mathcal{E}_x(A)$. □

Corollary 4.1.8. *If $T\colon A \to B$ is a norm–additive in modulus, $\mathbb{R}^+$–homogeneous surjection, then the set E_x is a singleton that belongs to δB for any generalized peak point $x \in \delta B$.*

Proof. Let $y \in E_x$ and suppose there is a $z \in E_x \setminus \{y\}$. Then there is a function $k \in \mathcal{E}_y(B)$ such that $|k(z)| < \|k\|$. For any $h \in T^{-1}(k) \subset \mathcal{E}_x(A)$, we have $E(k) = E(Th) \supset E_x$, which implies that the function $|k| = |Th|$ is constant on E_x with value $\|k\|$. This is a contradiction to $|k(z)| < \|k\|$. Hence, the set E_x contains only the point y. □

We define $\tau(x)$ to be the single element in the set E_x when $T\colon A \to B$ is a norm–additive in modulus $\mathbb{R}^+$–homogeneous surjective operator, so we have

$$\{\tau(x)\} = E_x = \bigcap_{h \in \mathcal{E}_x(A)} E(Th). \tag{4.9}$$

Thus we see that T induces an associated mapping $\tau\colon \delta A \to \delta B$ such that $x \mapsto \tau(x)$. From this mapping τ we will obtain the homeomorphism that will allow us to conclude from Proposition 4.0.9 that a map T which satisfies certain of the conditions (4.1)-(4.3) is an algebra isomorphism.

We see from Lemma 4.1.7 that $\mathcal{E}_{\tau(x)}(B) = \mathcal{E}_y(B) \subset T(\mathcal{E}_x(A))$. If $h \in \mathcal{E}_x(A)$, then from (4.5) we have that $\{\tau(x)\} = E_x \subset E(Th)$. Hence,

$$|(Th)(\tau(x))| = \|Th\| = \|h\| = |h(x)| \tag{4.10}$$

for any $h \in \mathcal{E}_x(A)$.

We note that if $h \in \mathbb{C} \cdot \mathcal{P}_x(A)$ and T preserves the peripheral spectrum of h, i.e., $\sigma_\pi(Th) = \sigma_\pi(h)$, then $|(Th)(\tau(x))| = \|Th\| = \|h\| = |h(x)|$ implies that $(Th)(\tau(x)) = h(x)$ since the peripheral spectra are singletons.

We also note that, in [18], Rao, Tonev, and Toneva considered a mapping similar to τ that mapped each $x \in X$ to the singleton set $\bigcap_{h \in \mathcal{P}_x(A)} E(Th)$. In that paper, the operator $T : A \to B$ was assumed to be peripherally–additive (i.e. $\sigma_\pi(Tf + Tg) = \sigma_\pi(f+g)$ for every $f, g \in A$) and thus preserved the peripheral spectrum of every $f \in A$.

Corollary 4.1.9. *If $T\colon A \to B$ is a norm–additive in modulus, $\mathbb{R}^+$–homogeneous surjection, then $T(\mathcal{E}_x(A)) = \mathcal{E}_{\tau(x)}(B)$.*

Proof. Let $h \in \mathcal{E}_x(A)$ for some $x \in \delta A$ and let $k = Th$. Then equation (4.10) gives

$$|k(\tau(x))| = |(Th)(\tau(x))| = |h(x)| = \|h\| = \|k\|\,. \tag{4.11}$$

This implies that $k \in \mathcal{E}_{\tau(x)}(B)$, so $T(\mathcal{E}_x(A)) \subset \mathcal{E}_{\tau(x)}(B)$ and, from Lemma 4.1.7, we have that $T(\mathcal{E}_x(A)) \supset \mathcal{E}_{\tau(x)}(B)$. ☐

The next proposition shows that when T is a norm–additive in modulus, $\mathbb{R}^+$–homogeneous surjection, (4.10) holds for every $f \in A$ and $x \in \delta A$, not merely for functions that take their maximum modulus at x.

Proposition 4.1.10. *If $T\colon A \to B$ is a norm–additive in modulus, $\mathbb{R}^+$–homogeneous surjection, then the associated mapping τ that T induces is continuous and the equation*

$$|(Tf)(\tau(x))| = |f(x)| \tag{4.12}$$

holds for every $x \in \delta A$ and all $f \in A$. If, in addition, T is bijective, then τ is a homeomorphism from δA onto δB, and if $\psi\colon \delta B \to \delta A$ is the inverse mapping of τ, then

$$|(Tf)(y)| = |f(\psi(y))| \tag{4.13}$$

for every $y \in \delta B$.

Proof. We first show that $|(Tf)(\tau(x))| = |f(x)|$ for every $x \in \delta A$ and for all $f \in A$. Let $x \in \delta A$, $f \in A$ and r be a real number greater than 1. If $h_0 \in r\,\|f\| \cdot \mathcal{P}_x(A)$ is a function as in the Strong Additive Bishop's Lemma (Lemma 3.4.5), then $\|Th_0\| = \|h_0\| = r\,\|f\| = r\,\|Tf\|$, so $Th_0 \in r\,\|Tf\| \cdot \mathcal{E}_{\tau(x)}(B)$. Because T is norm–additive in modulus, the Strong Additive Bishop's Lemma, Corollary 3.4.8, and Corollary 4.1.9 imply that

$$\begin{aligned} |f(x)| + r\,\|f\| &= \inf_{\substack{h \in \mathcal{E}_x(A) \\ \|h\| = r\|f\|}} \||f| + |h|\| = \inf_{\substack{h \in \mathcal{E}_x(A) \\ \|h\| = r\|f\|}} \||Tf| + |Th|\| \\ &= \inf_{\substack{k \in \mathcal{E}_{\tau(x)}(A) \\ \|k\| = r\|f\|}} \||Tf| + |k|\| = |(Tf)(\tau(x))| + r\,\|f\|\,. \end{aligned}$$

Consequently, $|(Tf)(\tau(x))| = |f(x)|$, as claimed.

To show the continuity of τ, we let $x \in \delta A$ and $p \in (0,1)$. Choose an open neighborhood V of $\tau(x)$ in δB and a peaking function $k \in \mathcal{P}_{\tau(x)}(B)$ such that $E(k) \subset V$ and $|k(y)| < p$ on $\delta B \setminus V$. If $h \in T^{-1}(k)$, then $h \in \mathcal{E}_x(A)$, and, according to (4.10), we have

$$|h(x)| = |(Th)(\tau(x))| = |k(\tau(x))| = 1 > p. \tag{4.14}$$

Therefore, the open set $W = \{\xi \in \delta A \colon |h(\xi)| > p\}$ contains x. The first part of the proof shows that for every $\xi \in W$, we have

$$|k(\tau(\xi))| = |(Th)(\tau(\xi))| = |h(\xi)| > p, \tag{4.15}$$

which implies that $\tau(\xi) \in V$ since $|k(\eta)| < p$ for $\eta \in \delta B \setminus V$. Consequently, $\tau(W) \subset V$, so τ is continuous.

Now suppose that T is bijective. Then T^{-1} is $\mathbb{R}^+$–homogeneous, and because the equation $\||Tf| + |Tg|\| = \||f| + |g|\|$ is symmetric with respect to f and Tf, it must also hold for the operator $T^{-1}\colon B \to A$. By the remarks after Corollary 4.1.8, T^{-1} induces an associated map $\psi\colon \delta B \to \delta A$ that is continuous and satisfies $\left|(T^{-1}k)(\psi(\eta))\right| = |k(\eta)|$ for all $\eta \in \delta B$ and for any $k \in \mathcal{E}_{\psi(\eta)}(B)$. Let $x \in \delta A$ and $y = \tau(x) \in \delta B$. If $h \in \mathcal{E}_x(A)$, then $k = Th \in \mathcal{E}_y(B)$ by Corollary 4.1.9, so $|h(\psi(y))| = \left|(T^{-1}(k))(\psi(y))\right| = |k(y)| = |(Th)(y)| = |(Th)(\tau(x))| = |h(x)| = \|h\|$. Hence, $\psi(y) \in E(h)$ for any $h \in \mathcal{E}_x(A)$. Because $\bigcap\limits_{h \in \mathcal{E}_x(A)} E(h) = \{x\}$, we see that $\psi(\tau(x)) = \psi(y) = x$ for every $x \in \delta A$. Similarly, $\tau(\psi(y)) = y$ for all $y \in \delta B$. Thus, τ and ψ are both bijective and $\psi = \tau^{-1}$, so τ is a homeomorphism. Equation (4.13) follows immediately from (4.12). $\square$

In [11], Lambert, Luttman, and Tonev proved an analogue of Proposition 4.1.10 for norm–multiplicative operators.

One of Rao, Tonev, and Toneva's main results for peripherally–additive operators in [18] follows directly from Proposition 4.1.10 since every peripherally–additive, surjective operator is $\mathbb{R}$–linear (see Lemma 4.2.2). Namely,

Corollary 4.1.11. [18] *If $T\colon A \to B$ is a peripherally–additive surjection that is also norm–additive in modulus, then there exists a homeomorphism $\tau\colon \delta A \to \delta B$ such that*

$$|(Tf)(\tau(x))| = |f(x)|$$

for every $f \in A$ and $x \in \delta A$.

Without more restrictions on the operator T, we cannot omit the moduli in equations (4.12) and (4.13). If we could, then T would also be multiplicative. However, the operator $Tf = if$ satisfies the hypotheses of Proposition 4.1.10 without being multiplicative. We note that if, in addition, T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions, then, by the remark prior to Corollary 4.1.9, we have $(Th)(\tau(x)) = h(x)$ for every $x \in \delta A$ and $h \in \mathbb{C}\cdot\mathcal{P}_x(A)$. In the next section, we present sufficient conditions under which the moduli in the equations (4.12) and (4.13) can be omitted for all elements of the algebra, not merely for $\mathbb{C}$–peaking functions.

4.2 Sufficient Conditions for Algebra Isomorphisms

In this section, we give sufficient conditions under which norm–linear and norm–additive operators, which we define next, are unital isometric algebra isomorphisms.

Definition 4.2.1. *A map $T\colon A \to B$ is said to be* norm–linear *if*

$$\|\lambda Tf + \mu Tg\| = \|\lambda f + \mu g\|$$

for every $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$. T is norm–additive *if*

$$\|Tf + Tg\| = \|f + g\|$$

for every $f, g \in A$.

Example 26. Clearly, every norm–linear operator is norm–additive: take $\lambda = \mu = 1$.

Example 27. The set of operators $T\colon A \to B$ such that $Tf = \alpha f$ with $\alpha \in \mathbb{T}$ are norm–additive since $\|Tf + Tg\| = \|\alpha f + \alpha g\| = |\alpha| \, \|f + g\| = \|f + g\|$. Because $\|\lambda Tf + \mu Tg\| = \|\lambda(\alpha f) + \mu(\alpha g)\| = |\alpha| \, \|\lambda f + \mu g\| = \|\lambda f + \mu g\|$, these operators are also norm–linear.

Example 28. Every norm–preserving linear operator is norm–linear since $\|\lambda Tf + \mu Tg\| = \|T(\lambda f + \mu g)\| = \|\lambda f + \mu g\|$ and every norm–preserving additive operator is norm–additive since $\|Tf + Tg\| = \|T(f + g)\| = \|f + g\|$.

Example 29. If $T\colon A \to B$ is an operator such that $\|Tf + Tg\| = C\,\|f + g\|$ for some real number $C > 0$ and every $f, g \in A$, then the operator $\dfrac{T}{C}$ is a norm–additive operator. Similarly, if $T\colon A \to B$ is such that $\|\lambda Tf + \mu Tg\| = C\,\|\lambda f + \mu g\|$ for every $f, g \in A$ and $\lambda, \mu \in \mathbb{C}$, then $\dfrac{T}{C}$ is a norm–linear operator.

The next lemma gives several useful properties of norm–additive (and thus also of norm–linear) operators.

Lemma 4.2.2. *If $T\colon A \to B$ is a norm–additive operator and $f, g \in A$, then*

(a) $T0 = 0$,

(b) $T(-f) = -Tf$,

(c) T is norm–preserving,

(d) T preserves the distances between algebra elements: $\|Tf - Tg\| = \|f - g\|$,

(e) T is injective, and

(f) T is continuous.

If T is additionally surjective, then T is $\mathbb{R}$–linear and thus additive.

Proof. For (a), let $f = g = 0$. Then $0 = \|f + g\| = \|Tf + Tg\| = \|2T0\| = 2\,\|T0\|$, so $T0 = 0$.

For property (b), we note that $\|Tf + T(-f)\| = \|f + (-f)\| = \|0\|$. Thus, $Tf + T(-f) = 0$, so $T(-f) = -Tf$.

Using the fact that $T0 = 0$, we have $\|Tf\| = \|Tf + T0\| = \|f + 0\| = \|f\|$, so property (c) holds.

Property (d) follows from (b) because $\|f - g\| = \|Tf + T(-g)\| = \|Tf - Tg\|$.

The fact that T is injective follows from (d) since if we have $Tf = Tg$, then $\|f - g\| = \|Tf - Tg\| = 0$, which implies that $f = g$.

The continuity of T is a direct consequence of (d) as well.

If T is surjective, then the Mazur–Ulam theorem (Theorem 1.1.1) implies that T is $\mathbb{R}$–linear.

□

Definition 4.2.3. *If $T\colon A \to B$ and there exists a $\psi\colon \delta B \to \delta A$ such that $|(Tf)(y)| = |f(\psi(y))|$ for every $f \in A$ and $y \in \delta B$, then we call T a ψ–*composition operator in modulus *on δB.*

By (4.13), the bijection T in Proposition 4.1.10 is a ψ–composition operator in modulus on δB. As we noted previously, $T\colon A \to B$ being norm–additive in modulus, $\mathbb{R}^+$–homogeneous, and bijective is not sufficient to omit the moduli in (4.13) (i.e., is not sufficient for such a T to be a ψ–composition operator). The following lemma gives additional conditions that are sufficient.

Lemma 4.2.4. *Let $T\colon A \to B$ be an additive ψ–composition operator in modulus on δB. If*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A (i.e., $\sigma_\pi(Tf) = \sigma_\pi(f)$ for every $f \in \mathbb{C} \cdot \mathcal{P}_x(A)$),

then $(Tf)(y) = f(\psi(y))$ for every $f \in A$ and all $y \in \delta B$, so T is a ψ–composition operator on δB.

Before proving this lemma, we will prove the following technical lemma concerning complex numbers that will be useful to us in the proof of Lemma 4.2.4.

Lemma 4.2.5. *Let $z, w \in \mathbb{C}$.*

(a) If $|1 + z| = |1 + w|$ and $|z| = |w|$, then $z = w$ or $z = \overline{w}$.

(b) If $|i + z| = |i + \overline{z}|$, then $z = \overline{z}$.

Proof. (a) If $|z| = |w|$, then $\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2 = \mathrm{Re}\{w\}^2 + \mathrm{Im}\{w\}^2$ and if $|1 + z| = |1 + w|$, then

$$(1 + \mathrm{Re}\{z\})^2 + \mathrm{Im}\{z\}^2 = (1 + \mathrm{Re}\{w\})^2 + \mathrm{Im}\{w\}^2. \qquad (4.16)$$

Equation (4.16) gives

$$1 + 2\mathrm{Re}\{z\} + \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2 = 1 + 2\mathrm{Re}\{w\} + \mathrm{Re}\{w\}^2 + \mathrm{Im}\{w\}^2,$$

so we have $\mathrm{Re}\{z\} = \mathrm{Re}\{w\}$, which implies that $\mathrm{Im}\{z\}^2 = \mathrm{Im}\{w\}^2$. Hence, $\mathrm{Im}\{z\} = \pm\mathrm{Im}\{w\}$, so $z = w$ or $z = \overline{w}$.

(b) If $|i + z| = |i + \overline{z}|$, then we can multiply both sides by $|-i|$ to get $|1 - iz| = |1 - i\overline{z}|$. By part (a), we then have that either $iz = i\overline{z}$, which implies that $z = \overline{z}$, or $iz = -iz$, which implies that $z = -z$, so $z = 0$ and therefore $z = \overline{z}$. $\square$

We are now ready to prove Lemma 4.2.4.

Proof of Lemma 4.2.4. (a) Suppose first that $T(\mathit{1}) = \mathit{1}$ and $T(i) = i$. If $f = \mathit{0}$, then $|(T(\mathit{0}))(y)| = |\mathit{0}(\psi(y))| = 0$ for every $y \in \delta B$, so $T(\mathit{0}) = \mathit{0}$ and clearly $(T(\mathit{0}))(y) = \mathit{0}(\psi(y))$ for every $y \in \delta B$. Now fix an $f \in A$ with $f \not\equiv \mathit{0}$ and $y_0 \in \delta B$ such that $f(\psi(y_0)) \neq 0$. Because T is a ψ–composition operator in modulus, $|(T(\mathit{1} + f))(y_0)| = |(\mathit{1} + f)(\psi(y_0))| = |1 + f(\psi(y_0))|$. Because T is additive, we have that $|(T(\mathit{1} + f))(y_0)| = |1 + (Tf)(y_0)|$. Hence, $|1 + f(\psi(y_0))| = |1 + (Tf)(y_0)|$, so by Lemma 4.2.5 either $(Tf)(y_0) = f(\psi(y_0))$ or $(Tf)(y_0) = \overline{f(\psi(y_0))}$ for every $f \in A$ because $|f(\psi(y_0))| = |(Tf)(y_0)|$. We claim that $(Tf)(y_0) = f(\psi(y_0))$. It is clear that if $\mathrm{Im}\{(Tf)(y_0)\} = 0$, then $(Tf)(y_0) = f(\psi(y_0))$. In the case that $\mathrm{Im}\{(Tf)(y_0)\} \neq 0$, assume that $(Tf)(y_0) = \overline{f(\psi(y_0))}$ for our fixed f and suppose that

$(T(i+f))(y_0) = \overline{(i+f)(\psi(y_0))} = -i + \overline{f(\psi(y_0))}$. Then we have

$$\begin{aligned} |i + (Tf)(y_0)| &= |(T(i+f))(y_0)| = \left|\overline{(i+f)(\psi(y_0))}\right| = \left|-i + \overline{f(\psi(y_0))}\right| \\ &= |-i + (Tf)(y_0)| = \left|i + \overline{(Tf)(y_0)}\right|. \end{aligned}$$

Therefore, by Lemma 4.2.5, $(Tf)(y_0) = \overline{(Tf)(y_0)}$, which implies that $\operatorname{Im}\{(Tf)(y_0)\} = 0$, a contradiction.

Suppose on the other hand that $(T(i+f))(y_0) = (i+f)(\psi(y_0)) = i + f(\psi(y_0))$. Then

$$|i + (Tf)(y_0)| = |(T(i+f))(y_0)| = |(i+f)(\psi(y_0))| = |i + f(\psi(y_0))| = \left|i + \overline{(Tf)(y_0)}\right|,$$

which implies once again that $\operatorname{Im}\{(Tf)(y_0)\} = 0$, a contradiction. Thus, the possibility that $(Tf)(y_0) = \overline{f(\psi(y_0))}$ is ruled out and $(Tf)(y_0) = f(\psi(y_0))$ for every $f \in A$, as claimed.

(b) Suppose now that T is an additive ψ–composition operator in modulus on δB that preserves the peripheral spectra of $\mathbb{C}$–peaking functions. Then because $|(T\mathit{1})| = |\mathit{1}(\psi(y))| = 1$ and the constant function $\mathit{1}$ is a $\mathbb{C}$–peaking function with $\sigma_\pi(\mathit{1}) = \{1\} = \sigma_\pi(T\mathit{1})$, we must have that $T\mathit{1} = \mathit{1}$. Similarly, $|(Ti)(y)| = |i(\psi(y))| = 1$ and the constant function i is a $\mathbb{C}$–peaking function with $\sigma_\pi(i) = \{i\} = \sigma_\pi(Ti)$, so $Ti = i$. Hence, T satifies condition (a), so T is a ψ–composition operator on δB.

We also prove (b) independently of condition (a). Fix a y_0 in δB and let $f \in A$. Because T is a ψ–composition operator in modulus on δB, we need only consider the case where $f(x_0) \neq 0$, with $x_0 = \psi(y_0)$. The Strong Additive Bishop's Lemma (Lemma 3.2) and its consequence Proposition 3.4.7 imply that there is an $h \in \|f\| \cdot \mathcal{P}_{x_0}(A)$ such that $\sigma_\pi(f + \alpha h) = f(x_0) + \alpha h(x_0)$, where $\alpha = \exp\{i \arg(f(x_0))\}$. As noted after Corollary 4.1.11, because T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions, we have $(Th)(\tau(x_0)) = h(x_0)$, $(T(\alpha h))(\tau(x_0)) = \alpha h(x_0)$, and $(T(f + \alpha h))(\tau(x_0)) = f(x_0) + \alpha h(x_0)$ because h, αh, and

$f + \alpha h$ all belong to $\mathbb{C} \cdot \mathcal{P}_{x_0}(A)$. The additivity of T gives us that $(T(f + \alpha h))(\tau(x_0)) = (Tf)(\tau(x_0)) + (T(\alpha h))(\tau(x_0)) = (Tf)(\tau(x_0)) + \alpha h(x_0)$. Consequently, $f(x_0) + \alpha h(x_0) = (T(f + \alpha h))(\tau(x_0)) = (Tf)(\tau(x_0)) + \alpha h(x_0)$, so $(Tf)(\tau(x_0)) = f(x_0)$ and thus $(Tf)(y_0) = f(\tau^{-1}(x_0)) = f(\psi(y_0))$. □

Because the elements in a uniform algebra are uniquely determined by their restrictions to the Choquet boundary of the algebra, uniform algebras are isometrically and algebraically isomorphic to their restriction algebras on their Choquet boundaries. As such, a mapping $T\colon A \to B$ automatically induces an associated map $T^\dagger\colon A|_{\delta A} \to B|_{\delta B}$ between the restriction algebras on the corresponding Choquet boundaries.

Proposition 4.1.10 and Lemma 4.2.4 imply the following result concerning ψ–composition operators.

Proposition 4.2.6. *Any norm–additive in modulus, additive bijection $T\colon A \to B$ is a ψ–composition operator in modulus on δB. If, in addition, either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A,

then T is a ψ–composition operator on δB. Hence, the operator $T^\dagger\colon A|_{\delta A} \to B|_{\delta B}$ that T induces is an algebra isomorphism and the restriction algebras $A|_{\delta A}$ and $B|_{\delta B}$ are algebraically isomorphic.

Since by Lemma 4.2.2 every surjective, norm–additive operator is injective and additive, Proposition 4.1.10 and Proposition 4.2.6 give the following characterization of norm–additive operators that are also norm–additive in modulus.

Theorem 4.2.7 (Norm–Additive Operators)**.** *Any norm–additive and norm–additive in modulus surjection $T\colon A \to B$ between uniform algebras is a ψ–composition operator in modulus* *on δB. If, in addition, either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A,

then T is an isometric unital algebra isomorphism.

We note that the operator T in Theorem 4.2.7 is not assumed to be linear or continuous. The Mazur–Ulam theorem (Theorem 1.1.1) implies that any surjective operator that preserves the distances between algebra elements is $\mathbb{R}$–linear, so $\|Tf - Tg\| = \|f - g\|$ implies that $\|Tf + Tg\| = \|Tf - T(-g)\| = \|f - (-g)\| = \|f + g\|$. Thus, Theorem 4.2.7 also holds for surjective norm–additive in modulus isometries T (for which $\|Tf - Tg\| = \|f - g\|$) with $T(0) = 0$, so it extends the Banach–Stone result (Theorem 1.2.3) mentioned in Chapter 1 to the case of uniform algebras.

Theorem 4.2.7 implies the following relationship between norm–linear and norm–additive operators.

Corollary 4.2.8. *A norm–additive surjection $T\colon A \to B$ for which either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A,

is norm–linear if and only if it is norm–additive in modulus.

Proof. By Theorem 4.2.7, if T is a norm–additive surjection that is norm–additive in modulus and for which condition (a) or (b) holds, then T is a ψ–composition operator in modulus.

Thus, we have

$$\|\lambda Tf + \mu Tg\| = \sup_{y\in\delta B} |\lambda(Tf)(y) + \mu(Tg)(y)| = \sup_{y\in\delta B} |\lambda f(\psi(y)) + \mu g(\psi(y))|$$
$$= \sup_{x\in\delta A} |\lambda f(x) + \mu g(x)| = \|\lambda f + \mu g\|\,.$$

Hence, T is norm–linear.

Conversely, by Proposition 4.1.2, any norm–linear operator is norm–additive in modulus and, as we observed previously, clearly norm–additive. □

The main result of [18], in which T is assumed to be peripherally–additive (and, therefore, to preserve the peripheral spectra of all algebra elements) and to be norm–additive in modulus is generalized by the second part of Theorem 4.2.7 and so follows as a corollary. Namely,

Corollary 4.2.9. [18] *Any peripherally–additive and norm–additive in modulus surjection* $T\colon A \to B$ *is an isometric algebra isomorphism.*

We note that Theorems 1.3.4 and 1.3.5 are multiplicative analogues of Corollary 4.2.9.

As shown in Proposition 4.1.2, if T satisfies the equation $\|Tf + \alpha Tg\| = \|f + \alpha g\|$ for all $f, g \in A$ and each α with $|\alpha| = 1$, then T is norm–additive and norm–additive in modulus. Therefore, Theorem 4.2.7 implies the following:

Corollary 4.2.10. *Any surjection* $T\colon A \to B$ *that satisfies the equation* $\|Tf + \alpha Tg\| = \|f + \alpha g\|$ *for every* $f, g \in A$ *and all* $\alpha \in \mathbb{T}$ *is a* ψ*–composition operator in modulus on* δB*. If, in addition,*

(a) $T(1) = 1$ *and* $T(i) = i$ *or*

(b) T *preserves the peripheral spectra of all* $\mathbb{C}$*–peaking functions of* A*,*

then T is an isometric unital algebra isomorphism.

According to Corollary 4.2.8, every norm–linear operator is norm–additive and norm–additive in modulus, so Theorem 4.2.7 yields:

Theorem 4.2.11 (Norm–Linear Operators)**.** *Any norm–linear surjection $T\colon A \to B$ between uniform algebras is a ψ–composition operator in modulus on δB. If, in addition,*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A,

then T is an isometric unital algebra isomorphism.

We note that the operator T in Theorem 4.2.11 is not assumed *a priori* to be linear or continuous.

Both the norm–linearity and either condition (a) or (b) are necessary conditions for T to be an isomorphism in Theorem 4.2.11. For example, the operator $Tf = -f$ is norm–linear since $\|\lambda Tf + \mu Tg\| = \|\lambda(-f) + \mu(-g)\| = \|\lambda f + \mu g\|$ but does not preserve the peripheral spectra of all $\mathbb{C}$–peaking functions of A (e.g., $\sigma_\pi(1) = \{1\}$ but $\sigma_\pi(T(1)) = \sigma_\pi(-1) = \{-1\}$), nor does it satisfy condition (a) because $T(1) = -1$ and $T(i) = -i$. This operator is not an algebra isomorphism because it is not multiplicative: $T(fg) = -fg$ but $TfTg = (-f)(-g) = fg$. On the other hand, while the operator $Tf = \dfrac{f\,|f|}{\|f\|}$, $f \neq 0$, on $C(X)$ clearly preserves the peripheral spectra of all algebra elements, so it preserves the peripheral spectra of $\mathbb{C}$–peaking functions in particular, and it satisfies $T(1) = 1$ and $T(i) = i$, it is also not norm–linear. For example, on $C[0,1]$, if $f(x) = \frac{1}{2}x + 1$ and $g(x) = -x + 1$, then we have

$$\|f+g\| = \max_{x\in[0,1]} \left|-\frac{1}{2}x+2\right| = 2 \quad \text{but}$$
$$\|Tf+Tg\| = \max_{x\in[0,1]} \left|\frac{\left(\frac{1}{2}x+1\right)^2}{\frac{3}{2}} + (-x+1)^2\right| = \max_{x\in[0,1]} \left|\frac{7}{6}x^2 - \frac{4}{3}x + \frac{5}{3}\right| = \frac{5}{3},$$

so T is not norm–additive and thus not norm–linear. This operator is not an algebra isomorphism because it, too, is not multiplicative: for example, if $f(x) = x$ and $g(x) = -x + 1$ on $C[0,1]$, then $\|fg\| = \frac{1}{4}$, so

$$T(fg) = \frac{x(-x+1)\,|x(-x+1)|}{\frac{1}{4}} = 4x(-x+1)\,|x(-x+1)| = 4x\,|x|\,(-x+1)\,|-x+1| \quad \text{and}$$
$$TfTg = x\,|x|\,(-x+1)\,|-x+1|\,,$$

which are assuredly not equal.

The following corollary states that multiples of norm–linear operators are also algebra isomorphisms.

Corollary 4.2.12. *A mapping $T\colon A \to B$ that satisfies $\|\lambda Tf + \mu Tg\| = C\,\|\lambda f + \mu g\|$ for some real number $C > 0$, every $\lambda, \mu \in \mathbb{C}$, and every $f, g \in A$ is a ψ–composition operator in modulus on δB. If, in addition,*

(a) $T(1) = C$ and $T(i) = Ci$ or

(b) $\sigma_\pi(Th) = \sigma_\pi(Ch)$ for every $h \in \mathbb{C} \cdot \mathcal{P}(A)$,

then $(Tf)(y) = C(f(\psi(y)))$ for every $y \in \delta B$ and $f \in A$. Thus, $\frac{T}{C}$ is an algebra isomorphism.

Proof. Apply Theorem 4.2.11 to the operator $\frac{T}{C}\colon A \to B$. $\square$

The next corollary deals with non-unital operators.

Corollary 4.2.13. *If $T\colon A \to B$ is a norm–linear operator, then $(Tf)(y) = (T1)(f(\psi(y)))$ for every $y \in \delta B$ and $f \in A$. Thus, $\dfrac{T}{T1}$ is an algebra isomorphism.*

Proof. Apply Theorem 4.2.11 to the operator $\dfrac{T}{T1}\colon A \to B$. □

As we have noted previously, every linear operator that preserves the norms of algebra elements is norm–linear. Therefore, Theorem 4.2.11 implies the next characterization of algebra isomorphisms, which is an extension of the Gleason–Kahane–Żelazko result (Theorem 1.2.1) to uniform algebras.

Corollary 4.2.14 (Linear Operators)**.** *Any norm–preserving, $\mathbb{C}$–linear surjection $T\colon A \to B$ between two uniform algebras such that*

(a) $T(1) = 1$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A

is automatically multiplicative and, in fact, an algebra isomorphism.

Finally, we introduce two more types of operators that are norm–additive and norm–linear.

Definition 4.2.15. *An operator $T\colon A \to B$ between uniform algebras for which $\sigma_\pi(Tf + Tg) \cap \sigma_\pi(f + g) \neq \varnothing$ is called a* weakly peripherally–additive *operator, and an operator T for which $\sigma_\pi(\lambda Tf + \mu Tg) \cap \sigma_\pi(\lambda f + \mu g) \neq \varnothing$ is called a* weakly peripherally–linear operator*.*

It is clear that weakly–peripherally additive operators are norm–additive and that weakly peripherally–linear operators are norm–linear, so Theorems 4.2.7 and 4.2.11 also imply the following improvements of the major result of [18], which are in the spirit of Lambert, Luttman, and Tonev's improvement in [11] to the results of [13].

Corollary 4.2.16 (Weakly Peripherally–Additive Operators). *A surjective map $T\colon A \to B$ between uniform algebras that is weakly–peripherally additive and norm–additive in modulus is a ψ–composition operator in modulus on δB. If, in addition, either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A,

then T is an isometric algebra isomorphism.

Corollary 4.2.17 (Weakly Peripherally–Linear Operators). *Any weakly peripherally–linear surjection $T\colon A \to B$ is a ψ–composition operator in modulus on δB. If, in addition, either*

(a) $T(1) = 1$ and $T(i) = i$ or

(b) T preserves the peripheral spectra of all $\mathbb{C}$–peaking functions of A,

then T is an isometric unital algebra isomorphism.

Corollary 4.2.10 implies that the weak peripheral linearity of T in Corollary 4.2.17 can be replaced by the more relaxed property $\sigma_\pi(Tf + \alpha Tg) \cap \sigma_\pi(f + \alpha g) \neq \varnothing$ for every $f, g \in A$ and all α with $|\alpha| = 1$.

Chapter 5

Future Directions

In [11], Lambert, Luttman, and Tonev prove the following theorem for norm–multiplicative operators between uniform algebras (operators $T\colon A \to B$ for which $\|TfTg\| = \|fg\|$):

Theorem 5.0.18. [11] *A mapping $T\colon A \to B$ between uniform algebras for which $T(\mathcal{P}(A)) = \mathcal{P}(B)$ is norm–multiplicative if and only if there exists a homeomorphism $\psi\colon \delta B \to \delta A$ such that $|(Tf)(\psi(x))| = |f(x)|$ for every $f \in A$ and $x \in \delta A$.*

They also give examples to show that these conditions are not sufficient for T to be an algebra isomorphism and conclude that stronger conditions are necessary and sufficient to make T an isometric algebra isomorphism. Namely, they replace norm–multiplicativity with the weak peripheral–multiplicativity condition $\sigma_\pi(TfTg) \cap \sigma_\pi(fg) \neq \varnothing$ for every $f, g \in A$. Their proof of this theorem involves a weaker version of Bishop's Lemma than the Strong Multiplicative Bishop's Lemma (Lemma 3.4.3) we proved in Chapter 3. Because the Strong Additive Bishop's Lemma (Lemma 3.4.5) allowed us to prove that norm–additive and norm–linear operators between uniform algebras are isometric algebra isomorphisms if they also preserve the peripheral spectra of $\mathbb{C}$–peaking functions (and are norm–additive in modulus, in the case of norm–additive operators), we would like to investigate the question of whether norm–multiplicative mappings that preserve the peripheral spectra of $\mathbb{C}$–peaking functions are isometric algebra isomorphisms. We note that the counterexamples given in [11] do not preserve the peripheral spectra of $\mathbb{C}$–peaking functions.

Additionally, this and similar work have produced conditions under which mappings between *uniform* algebras are isomorphisms; a final open question we may pursue is whether these conditions are necessary and sufficient for mappings between nonuniform algebras to be isomorphisms.

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