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PROOF PROCESSES OF NOVICE MATHEMATICS PROOF WRITERS

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Proof Processes of Novice Mathematics Proof Writers

Chairperson: Dr. Libby Knott

Previous studies have shown that a large portion of undergraduate mathematics students have difficulties constructing, understanding, and validating proofs (Martin and Harel, 1989; Coe and Ruthven, 1994; Moore, 1994; Baker, 1996; Mingus and Grassl, 1999; Knuth, 2002; Weber, 2001, 2003). However, proofs are the foundation of mathematics; it is therefore essential that every university mathematics student be able to step through the proof writing process. Research has sought to describe the strategies involved in the process of mathematical problem solving (Baker, 1996; Bell, 1979; Carlson and Bloom, 2005; McGivney and DeFranco, 1995; Pape and Wang, 2003; Polya, 1973; Pugalee, 2001; Schoenfeld, 1985; Yerushalmy, 2000).

This study was designed to describe the detailed processes and strategies used during the proof-writing process in order to more completely understand this process.

Specifically, this study was designed to answer the questions:

- What are the proof-writing strategies of novice mathematics proof writers?
- What strategies are in use during a successful proof writing attempt?
- In what specific ways do novice mathematics proof writers use heuristics or strategies when working through a proof, which go beyond the application of standard problem-solving heuristics?
- Do the strategies used by individuals remain static across multiple questions or do questions have an effect on the choice of strategies?

In this study, 18 novice mathematics proof writers engaged in individual task-based interviews, in which each was asked to think aloud while proving results which were unfamiliar to him or her. Results indicate that each participant had his or her own set of strategies that remained, for the most part, static across all questions. In particular, three categories of strategies emerged in frequent use, but with mixed levels of success. These categories were use of examples, use of equations, and use of other visualizations. A fourth category, the use of self-regulation strategies, was found to be overall successful, when in use with proper content knowledge and without computational errors.

DEDICATION

I would like to dedicate this work to my husband, Ken, and daughter, Hannah, who have been by my side and compromised in so many ways while I finished my dissertation. Thank you for your unconditional love and support, your laughter and distractions, and most of all for allowing me the privilege of being a wife and mother.

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CHAPTER 1

INTRODUCTION

> "Proof not only provides the foundation upon which mathematical ideas are built, but also the way for each generation to learn about and extend what has already been accomplished." (Mingus & Grassl, 1999, p. 438)

Background

Proofs are indeed the cornerstone of mathematics. It is therefore essential that every university mathematics student be able to step through the proof writing process. This emphasis has been nationally recognized as necessary in grades K-12. NCTM (2000) emphasized in their standards for mathematics teaching that students should be able to "produce logical arguments and present formal proofs that effectively explain their reasoning" (p. 344). Of particular importance to this researcher is the preparation of teachers who can understand and construct mathematical proofs, in order to educate the next generation.

Martin and Harel (1989), Moore (1994), and Epp (2003) found a large portion of undergraduate mathematics students have difficulties constructing, understanding, and validating proofs (cf. Coe and Ruthven, 1994; Baker, 1996; Mingus and Grassl, 1999; Knuth, 2002; Weber, 2001, 2003). Martin and Harel (1989) found that 52% of the undergraduate math students in their study accepted an incorrect argument as a proof of an unfamiliar statement. Moore (1994) observed that, "several students [in this study] in the transition course had previously taken upper-level courses requiring proofs. All of them said they had relied on memorizing proofs because they had not understood what a proof is nor how to write one" (p. 264). This lack of ability is not a new phenomenon. Epp (2003) discusses a course she taught in the late 1970s. She states that she was "almost overwhelmed by the poor quality of their proof-writing attempts" (p. 886).

However, all mathematicians were once students themselves and at some point learned how to construct a valid proof. So, for those who are successful proof writers, what are their methods? What makes them successful?

Since proofs form the underlying structure of mathematics, this lack of ability presents a major problem. But, what is being done about it? Arguments have been made that we are not teaching students in the way that mathematicians actually construct a proof, that we give them no insight into the processes and errors that occur along the way (Alibert and Thomas, 1991; Almeida, 2003). Almeida (2003) notes that the sequence for understanding math [intuition, trial, error, speculation, conjecture, proof] is much different than how we traditionally present mathematics, which is theorem, proof, and then examples. "To reiterate the obvious, in the current tradition, a mathematical proof is a pure thought experiment divorced from context" (p. 479).

Past and current research has sought to describe the strategies involved in the process of mathematical problem solving (Polya, 1973; Schoenfeld, 1985; Yerushalmy, 2000; Pugalee, 2001; Pape and Wang, 2003; Carlson and Bloom, 2005). This research has sought to expand on the overall description of the process of problem solving based on the work of Polya (1973): read the problem, devise a plan, implement the plan, and verify the results.

"Problem solving skills are used to informally discover a proof that is later demonstrated formally" (Baker, 1996, p. 2). Work has been done in the area of problem solving to develop and refine this set of heuristics from Polya and to describe the ways in which successful problem solvers go about dissecting and solving a given mathematical problem (Bell, 1979; Schoenfeld, 1985; McGivney and DeFranco, 1995; Baker, 1996; Carlson and Bloom, 2005).

These heuristics are also related to proof writing. Studies have shown that some students are only able to progress in proof writing to a certain level. Van Dormolen (1977) defined three levels of proving as follows: the *ground level*, where proof is based on arguing from a single case, the *first level*, where proof is based on a class of cases, and the *second level*, where proof is based on reasoning about reason. Balacheff (1988) also defined similar levels of proving: *naïve empiricism*, *crucial experiment*, *generic example*, and *thought experiment*. Harel and Sowder (2003), Knuth (2002), Jones (2000), Martin and Harel (1989), Almeida (2000) have all looked at students' conceptions of proof and their ability levels along spectra similar to those defined here.

While problem-solving heuristics are similar to those for proof writing, there are some distinctions as well. "Proof is distinguished from the other aspects of mathematical activity…by the fact that it belongs mainly to the verification stage of investigation"

(Bell, 1979, p. 372). In this same light, Bell (1976) also says; "Some teachers have said that proof, for a pupil, is what brings him conviction…. Proof is an essentially public activity which follows the reaching of conviction, though it may be conducted internally, against an imaginary potential doubter" (p. 24). While constructing a proof, a student is not just looking for an answer, but seeking to convince others of the accuracy of that answer. Convince yourself. Convince a friend. Convince an enemy (Mason, Burton, and Stacey, 1982).

This study seeks to describe those strategies that are involved specifically in the process of proof writing. The framework for this study was based on the larger categories of proof-writing processes defined by Weber (2004) that classified proof processes into three categories: procedural, syntactic, and semantic processes. Procedural proof productions include constructing a proof by applying a specific known procedure. Syntactic proof productions include the manipulation of definitions, facts, theorems, and logical symbols to form a proof. The last category, semantic proof productions, includes examining appropriate representations of the statement, and objects therein, to develop the structure and ideas that build a formal proof.

This study expands on these definitions by adding to the description of the individual strategies involved in each of these types of proof productions. That is, the researcher seeks to describe the detailed processes and strategies used during the proof-writing process in order to more completely understand this process.

Research Problem and Purpose

Current work on this topic deals mostly with the final product, the accuracy of the proof itself, or the level of understanding of proofs attained by the student. This researcher is more interested in the process rather than the product. What processes are students involved in during the actual construction of the proof? What do students do when engaged in a proof-writing activity? What steps do they take? What paths do they use to reach their final product?

This study encompasses mostly undergraduate participants, with a small contingent of graduate students from other disciplines outside of mathematics, as well as two graduate students in mathematics. However, both graduate students in mathematics were taking the transition-to-proof course for the first time during this study, and therefore were inexperienced in proof writing. For these reasons, all participants are being considered novices in mathematics proof writing.

The issue of expert versus novice is not trivial in the literature. Some researchers feel that the notion of expert is determined by the skills possessed and that students can gain experience that increases their expertise (Hart, 1994; Schoenfeld and Herrmann, 1982). Solomon (2006) denotes a distinction between degree-level and research-level mathematics. For the purposes of this study, this researcher will define expert to include experienced graduate students in mathematics and, of course, mathematicians, i.e., those specifically engaged in research-level mathematics. The population of interest when defining novice is then all those who have not yet experienced a significant amount of upper division mathematics courses requiring higher levels of proof-writing ability, i.e., those engaged in degree-level mathematics. The participants from the Math History course represented in this study are certainly more experienced that those in the transition-to-proof course, possibly further on the continuum towards expert, but have not yet fully developed the notions of proof writing.

The original questions of interest for this study were modified slightly with new terms to clarify their meaning. One question originally posed was found not to be of interest during the study. With this scope of interest in mind, this study was designed to answer the following questions:

- What are the proof writing strategies of a novice mathematics proof writer?
    - What strategies are in use during a successful proof writing attempt?
    - In what specific ways do novice mathematics proof writers use heuristics or strategies when working through a proof, which go beyond the application of standard problem-solving heuristics?

The following question emerged during the study and is addressed in this dissertation as well.

- Do the strategies used by individuals remain constant across multiple questions or do the questions affect the choice of strategies?

Definition of Terms

The following terms are defined for the purposes of this study.

Proof-writing strategies refer to the specific processes students use during the construction of a proof. Some examples of such strategies are: writing out definitions, looking back to previous related work for ideas, constructing examples, and working forward/backward through the proof.

A *novice proof writer*, as discussed above, will be defined as an undergraduate student, or a graduate student lacking sufficient experience to be considered an expert, i.e., those engaged in degree-level mathematics. These are students who are still engaged in the process of learning the underlying structure of mathematics, rather than those individuals who have advanced past this stage and moved on to researching mathematics.

A *successful proof writer* will be defined for each question as a participant who achieves an acceptable level of completion of that question. The acceptability of participant proof depends on containment of proper reasoning, basic structure, and thorough explanations. Since participants were not required to write out proofs, this basis is made from the verbal proofs students constructed. The researcher determined these ratings, with agreement confirmed on a portion of the interviews by an independent outside person at a separate university from the study for validity. For a participant rated as unsuccessful on a particular proof there is an explanation of the factors contributing to this rating in the analysis by student section. A participant who had found the main ideas of the proof, but was never aware of this was considered unsuccessful. Overall, for the purposes of the final discussion, participants were rated as successful in general if they received a successful rating on at least half of the proofs they attempted.

CHAPTER 2

LITERATURE REVIEW

This study is focused on the processes involved in proof writing. To analyze the data, one must define the notion of proof, which is a complicated undertaking. This chapter outlines the literature detailing both the historic and current meanings of proof and proof writing, as well as the motivation for this study in the current known difficulties of students in this endeavor. The literature also addresses, and this chapter will summarize, several heuristics that could be expected from the process of problem solving. The similarities and differences between problem solving and proof writing will be addressed in the Theoretical Framework. The perceived levels of proof found in the literature will be outlined here as well. Lastly, a key element that arose later in the study was that of self-regulation, thus a section has been added to detail the current findings in self-regulation and its effect on problem-solving and proof-writing performance.

History of Proof

"The notion of proof is not absolute. Mathematicians' views of what constitutes an acceptable proof have evolved" (Kleiner, 1991, p. 291).

The purpose of this section is to briefly outline the changing view of proof throughout mathematical history; beginning in roughly 2000 b.c. and spanning to today. However, such a span could cover an entire thesis in itself, therefore this researcher will merely highlight major shifts and trends and leave further inquiry to the reader. This period began with the Egyptians and Babylonians. During these times, evidence of the truth of a statement was considered sufficient (Hanna and Barbeau, 2002; Kleiner, 1991; Kline, 1973). "There are no general statements in Babylonian mathematics and there is no attempt at deduction, or even at reasonable explanation, of the validity of the results" (Kleiner, 1991, pp. 291-92).

This view of proof changed with the Greeks. "Proof as deduction from explicitly stated postulates was, of course, conceived by the Greeks" (Kleiner, 1991, p. 293). The Greeks' study of axioms and resulting theorems led, through much toil and many years, to the development of Euclid's *Elements* (Kline, 1973). Euclid designed a logical structure based on a set of axioms, which were by definition not assumable by any other axioms, and set out to describe a geometric system as a whole based on these axioms, along with common notions and definitions. However, "[this] very rigorous period in mathematics brought in its wake a long period of mathematical activity with little attention paid to rigor. Too much rigor may lead to rigor mortis" (Kleiner, 1991, p. 294).

Hanna and Barbeau (2002) describe how modern mathematics evolved from "arguments that were quite *heuristic*, that is, imprecise but illuminating" (p. 36), which were used in the formulation of the theory of complex functions by Riemann, Weierstrauss, and Cauchy. Heuristic proofs explained what was happening mathematically, but did not necessarily follow the standards set by the Greeks. However, the limitations of such methods were soon evident, and the tables turned once again as Weierstrauss and Cauchy laid the foundations of the calculus of their day (Hanna and Barbeau, 2002; Kleiner, 1991).

During this time, it became clear that Euclid's proofs in the *Elements* actually contained arguments based on unstated assumptions. "As a result, mathematicians such as David Hilbert (1862-1943) undertook to make all the required assumptions explicit and thus to set geometry on a firmer foundation" (Hanna and Barbeau, 2002, p. 45). Hilbert's work set the stage for axiomatic mathematics.

In the early 1930s, other mathematicians were also moving towards a more formal view of proof. Concern was raised by a group of French mathematicians that "the standard text [of the day]...they found wanting in many ways" (Borel, 1998, p. 373). This group began work to revamp the texts used. They formulated a plan to redevelop school textbooks, starting over, and proposed to divide "[the] basic material into six 'books', each consisting possibly of several volumes, namely: I *Set Theory,* II *Algebra,* III *Topology,* IV *Functions of One Real Variable,* V *Topological Vector Spaces,* [and] VI *Integration* (Borel, 1998, p. 374). The group wrote under the pen name of Nicolas Bourbaki for many years, with members changing over the years, but not the overall goal. The books reshaped the mathematical world. The style of writing was "very dry…without any concession to the reader, the apparent striving for the utmost generality, the inflexible system of internal references and the total absence of outside ones" (Borel, 1998, p. 374). Armand Borel, as a member of Bourbaki for 20 years, experienced first hand the style of the writing, the method of writing, and the influence these books had in the mathematical community. He said, "sometimes a heuristic remark, to help the reader, would find its way into a draft…and then, almost invariably, [it would be] thrown out" (p. 376).

In this same time period, the early part of the 20th century, there arose three distinct philosophies in mathematics, *formalism, logicism,* and *intuitionism* (Hanna, 1995, 1991; Kleiner, 1991; Lakatos, 1976; Tall, 1991). Hilbert is considered by many to be a typical example of a formalist. In this school of thought, "mathematics is [seen as] a science of formal system… the validity of any mathematical proposition rests upon the ability to demonstrate its truth through rigorous proof within an appropriate formal system" (Hanna, 1991, p. 55). A logicist, such as Russell, would have a similar view that mathematics was based on solid principles, but the specific types of proof would differ. This view considers "mathematical concepts [to be] expressible in terms of logical concepts; mathematical theorems are tautologies" (Kleiner, 1991, p. 306). The intuitionist view, such as that of Kronecker and Brouwer, "claimed no formal analysis of axiomatic systems is necessary" (Kleiner, 1991, p. 307). Theirs would be the view that "mathematics and mathematical language are two separate entities… mathematical activity then consists of 'introspective constructions', rather than axioms and theorems" (Hanna, 1991, p. 55).

More recent views, in the 1960s, brought a shift in mathematics curricula in the United States, sparking the first of three major curriculum reforms (Hanna, 1995; Schoenfeld, 1992). The first movement, called the *new math,* held the view that "the secondary-school mathematics curriculum better reflects mathematics when it stresses formal logic and rigorous proof" (Hanna, 1995, p.42). Ultimately, this movement was deemed a failure in the students' apparent loss of basic mathematics skills.

The response was a complete shift in strategy, the *back-to-basics* movement of the 1970s. The curricula of this period stressed procedural skills for basic mathematics. However, this too seemed to fail our students and the next swing to problem solving and critical thinking began in the late 1970s and early 1980s.

Lakatos (1976) advocated for a new form of mathematical discovery during this time (this work was actually compiled and finished after his death by colleagues). He used a fictitious classroom to engage in debating and refuting as a means of developing mathematics and proof. Discussions occurred amongst students and between students and the teacher. It is a style much apart from that of Bourbaki and the straightforward one-directional proof writing style applied there. His concepts shifted the norm away from formal mathematical proof to a concept of proof as explaining and discovery. "The purpose of these essays is to approach some problems of the *methodology of mathematics*. I use the word 'methodology' in a sense akin to Polya's and Bernays' 'heurisitic' and Popper's 'logic of discovery' or 'situational logic'" (Lakatos, 1976, p. 3).

The National Council of Teachers of Mathematics (NCTM) (1989, 2000) has stressed the presence of mathematical proof in the classroom as well. These documents also reflect the shift to problem solving and critical thinking. In NCTM (1989), the use of these aspects within proof was stressed, but it also noted that the more complicated proof constructions be left for only those who were "college-intended".

> In grades 9-12, the mathematics curriculum should include numerous and varied experiences that reinforce and extend logical reasoning skills so that all students can -
>
> a) make and test conjectures;
> b) formulate counterexamples;
> c) follow logical arguments;
> d) judge the validity of arguments;
> e) construct simple valid arguments; and so that, in addition, college-intending students can -
> f) construct proof for mathematical assertions, including indirect proofs and proofs by mathematical induction. (NCTM, 1989, p. 143)

In NCTM (2000), the scope of proof writing expanded to all education levels, pre-K through 12. In addition, a change was made so that all students at the high school level should experience the more difficult proof constructions. At the same time, their definition of proof shifted away from strictly rigorous arguments.

> Instructional programs from prekindergarten through grade 12 should enable all students to-
>
> - recognize reasoning and proof as fundamental aspects of mathematics;
> - make and investigate mathematical conjectures;
> - develop and evaluate mathematical arguments and proofs;
> - select and use various types of reasoning and methods of proof. (p. 341)

It was also stated that

> [students] should be able to produce logical arguments and present formal proofs that effectively explain their reasoning, whether in paragraph, two-column, or

> some other form of proof. Because conjectures in some situations are not conducive to direct means of verification, students should also have some experience with indirect proofs. (p. 344)

The introduction of computers has also played a role in what might constitute a proof. According to Hanna (1995), there are several new ideas of proof, among these are the *zero-knowledge proof* and the *holographic proof*. The zero-knowledge proof "enables the prover to provide to the verifier convincing evidence that a proof exists, without disclosing any information about the proof itself" (Hanna, 1995, p. 43). This proof could be completed with or without the use of a computer. The "holographic proof … consists of transforming a proof into a so-called transparent form that is verified by spot checks, rather than by checking every line" (Hanna, 1995, p. 43).

One of the most famous uses of a computer in mathematical proving is in the proof of the four-color theorem, by Appel and Haken in 1976 (Hanna, 1995; Hanna and Barbeau, 2002; Kleiner, 1991; Tall, 1991). A computer was used to check the many possibilities that were formerly too cumbersome to check by hand. However, some mathematicians doubt that we should trust computers to do anything quite this important. However, the issue of the acceptance of proof in general is not trivial. Such a proof as this would be impossible to check by hand and so it is acceptance is not always agreed upon. It is not the goal here, however, to discuss the issue of social acceptance, but rather to point out that the use of computers opens a door previously unavailable due to the overwhelming amount of time necessary for some proofs, even beyond what one person could perform by hand in a lifetime.

Michael Rabin, also in 1976, proposed yet another type of proof technique, a probabilistic proof. "He found a quick way to determine, with a very small probabilty of error (say one in a billion), whether or not an arbitrarily chosen large number is a prime" (Kleiner, 1991, p. 312). This showed that the large number was a prime "for all practical purposes", perhaps not what the scholars of old had in mind. Again, the acceptance of such a proof should not be considered universal, but merely points out a shift in the thoughts surrounding proof given the use of computers and new theories of proof writing.

Clearly, the math world has encountered many schools of thought throughout history. Therefore, one should not view proof writing and its acceptance as a static issue, but rather one that ebbs and flows with the changing world around it.

Proof Today

"A mathematical proof, by definition, can take a set of explicit givens (such as axioms, accepted principles or previously proven results), and use them, applying the principles of logic, to create a valid deductive argument" (Hanna et al, 2004, p. 82). This is perhaps the most simple and straightforward response to the question "what is a proof?" that can be found in the literature to date. Hanna and Barbeau (2002) previously stated (not unlike the above) that a proof is "a finite number of logical steps from what is known to a conclusion using accepted rules of inference" (p. 38). Similarly, Weber (2005) states,

> Proof construction is a mathematical task in which the prover is provided with some initial information (e.g. assumptions, axioms, definitions) and is asked to apply rules of inferences (e.g. recall previously established facts, apply theorems) until a desired conclusion is deduced. (p. 352).

Mingus and Grassl (1999) defined proof as "a collection of true statements linked together in a logical manner that serves as a convincing argument for the truth of a mathematical statement" (p. 441). All of these definitions point to the same school of thought, that a proof is a logical, deductive argument.

There are, however, other views of what forms an acceptable proof. These are based on what one views as the goal of proving. The following is a list of the functions, or goals, of proof and proving as seen in the literature: (Almeida, 2003; Bell, 1976; Hanna and Barbeau, 2002; Hanna, 2000; Mingus and Grassl, 1999)

- Verification or Justification: validating the truth of a statement
- Explanation or Illumination: asserting why a statement is true
- Conviction: removing doubt
- Systematization: the organization of results into a deductive system of axioms, major concepts and theorems, and minor results stemming from these
- Communication: transmitting mathematical knowledge and reasoning to others
- Discovery or Construction: inventing new results
- Enjoyment: meeting an intellectual challenge elegantly

Proofs as Convincing and Explaining

"[A] proof is an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing" (Hanna, 1991, p. 56). Hersh (1993) addressed the different roles of proofs as both convincing and explaining. "Mathematical proof can convince, and it can explain. In mathematical research, its primary role is convincing. At the high-school and undergraduate level its primary role is explaining" (p. 398).

Some believe that not only is this view of proof as explaining sufficient in the classroom, but also that any more than this would detract from learning (Knuth and Elliot, 1998). Kline (1973) argued the following:

> In no case should one start with the deductive approach, even after students have come to know what this means. The deductive proof is the final step... [The student] should be allowed to accept and use any facts that are so obvious to him that he does not realize he is using them... Proofs of whatever nature should be invoked only where the students think they are required. The proof is meaningful when it answers the student's doubts, when it proves what is not obvious. (p. 195)

In this researcher's experience, this view of proof is not always the norm in the mathematics classroom. In the transition-to-proof course observed for this study, the class norm was to accept and use those facts that arose in the basic algebraic properties of the real numbers, and nothing more. This researcher agrees with the opinion that a proof is not meaningful to a student if it proves the obvious. This idea was evident in this study as well, when participants were unable to form proofs once they felt that a statement was obvious. However, there is no one set definition of 'obvious' for all students, nor is there one set of accepted rules, properties, theorems, etc., that are considered assumable in all classrooms.

"Every class finds it has to omit some proofs, either for lack of time, because they are too difficult for students at that level, or just because some proofs are tedious and unenlightening" (Hersh, 1993, p. 396). Hersh believes there are two views held by teachers on the role of proof in the classroom, within the context of proof as explaining. The first type of teacher he terms an *Absolutist*, such a teacher "sees mathematics as a system of *absolute* truths… Ideally, the Absolutist teacher tells the student nothing except what he will prove (or assign to the student to prove)" (p. 396). The second type of teacher is a *Humanist*. "To the Humanist, mathematics is ours, our tool and plaything, to use and enjoy as we see fit. Proofs are not obligatory rituals" (p. 397).

At the extreme, Hoyles (1997) raises concern that "the meaning of 'to prove' has been replaced by social argumentation (which could mean simply giving some examples)… and proof is labelled [*sic*] as inaccessible to the majority" (p. 9). Martin and Harel (1989) agree that such a view will not be beneficial to our students. "If teachers lead their students to believe that a few well-chosen examples constitute proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students" (pp. 41-42).

Social Acceptance of a Proof

When is a proof considered to be valid and a result considered absolute truth? This question does not have a straightforward answer, and the social process of approval and acceptance of a proof can be complicated.

> Researchers spend a great deal of time trying to construct proofs that are not only correct, but also appealing, insightful, and easily grasped in their entirety. … the acceptance of most proofs, however valid, depends on the care and experience of the prover and the readers. (Hanna and Barbeau, 2002, p. 47)

Hanna (1991) states, "Clearly the acceptance of a theorem by practicing mathematicians is a social process which is more a function of understanding and significance than of rigorous proof" (p. 58). Tall (1991) says, "many mathematicians demand that a proof should not only be logical, but that there should be some over-riding principle that explains why the proof works" (p. 16).

In this same light, Bell (1976) argues, "some teachers have said that proof, for a pupil, is what brings him conviction. … Proof is an essentially public activity which follows the reaching of conviction, though it may be conducted internally, against an imaginary potential doubter" (p. 24). Raman (2003) states, "Proof involves both public and private arguments" and defines these terms as follows: a private argument is "an argument which engenders understanding" and a public argument is "an argument with sufficient rigor for a particular mathematical community" (p. 320).

Student Views of Proof

With all of the available definitions of proof today, it is no wonder that students have difficulty defining and understanding proofs and proof writing. In a study of student understanding involving university mathematics education majors, Portney, Grundmeier, and Graham (2006) found that "students in this study expressed views of the role of proof in mathematics that included proof as explanation and verification" (p. 203). Moore (1994) also studied a group of university students and found,

> Although the students probably could not give a definition of proof, they had concept images of proof. The concept image of some students was that of proof as explanation, whereas for others proof was a procedure, a sequence of steps that one performs. (p. 264)

The following are examples of student definitions of proof collected by Hodgson and Morandi (1996) prior to a transition course specifically addressing the teaching of proof:

- "A proof is verification of a process or processes showing that the end result is achieved by a logical flow through given laws."
- "A proof is a logical sequence of steps to solve or explain an equation. The proof is based on a set of rules and regulations. There may be more than one way to prove each problem." (p. 51)

Students gave the remaining quotes after the transition course:

- "A proof is an argument (either visual or symbolic) that demonstrates effectively to an audience that the statement is either true or false."
- "A proof is an explanation of why something is done. It tries to show the reader or readers that a statement is true by showing them why it is true."
- "A proof is an argument that attempts to convince the audience that a statement is true or if it's false. There are many types of proof and the right proof to use is the one that convinces the audience." (p. 55)

Here, students shifted from views of proof as formal logic to that of proof as a convincing argument. This was expected given the intent of instruction in the course. "To facilitate the development of proof-writing skills, we encourage the students in our introductory proof courses… to think of proofs as convincing arguments" (Hodgson and Morandi, 1996, p. 52). Clearly, the instruction in this particular course had an effect on student views, as we should expect all instruction to do.

It is the belief of this researcher that proof need not be defined in only one way, but that proof could encompass multiple meanings. In the right context and setting, each of the definitions outlined here could be appropriate. However, for the purposes of this study, I will define proof and the construction of proof as Weber (2005) has:

> Proof construction is a mathematical task in which the prover is provided with some initial information (e.g. assumptions, axioms, definitions) and is asked to

> apply rules of inferences (e.g. recall previously established facts, apply theorems) until a desired conclusion is deduced. (p. 352).

<u>Student Difficulty with Proof</u>

While there is not agreement on what a proof is, there seems to be a consensus that students do not understand it. This section describes studies that indicate difficulties understanding and producing valid proofs among high school students, university students, and even teachers.

Moore (1994) studied the difficulties experienced by undergraduate students in proof writing. The study was conducted by observations in a transition-to-proof course at the undergraduate level, in which 16 students were enrolled. Two undergraduate mathematics majors and three undergraduate mathematics education majors were chosen as key participants in the study. Moore found seven main sources of student difficulty in proof writing among these students.

> D1. The students did not know the definitions, that is, they were unable to state the definitions.
> D2. The students had little intuitive understanding of the concepts.
> D3. The students' concept images were inadequate for doing the proofs.
> D4. The students were unable, or unwilling, to generate and use their own examples.
> D5. The students did not know how to use definitions to obtain the overall structure of proofs.
> D6. The students were unable to understand and use mathematical language and notation.
> D7. The students did not know how to begin proofs. (pp. 251-2)

Moore found that it was not always the case that difficulties stemmed from lack of content knowledge. "In some instances students knew a definition and could explain it informally but could not use the definition to write a proof (D5)" (p. 261). Students observed in the study were often stuck when beginning a proof, which Moore believed to be symptomatic of several other problems. "The sources of those difficulties included deficiencies in all three aspects of concept understanding [definition, image, and usage], a lack of knowledge of logic and methods of proof, and linguistic and notational barriers" (p. 263). He also found that students focused more on procedures than on content. "Several students in the transition course had previously taken upper-level courses requiring proofs. All of them said they had relied on memorizing proofs because they had not understood what a proof is nor how to write one" (p. 264).

In a study of 40 high school and 13 college students, Baker (1996) found evidence to support the belief that students focus on procedures rather than content. Participants were given one proof-writing task and four proof-analysis tasks, among other data collected on a questionnaire, involving proofs by mathematical induction. Students focused more on form than on understanding the concepts involved in the proofs. "Only 4 college and 5 high school students made any reference in their written proof to what was being shown by the proof. … A focus on the form of a proof over its substance was clearly in evidence" (p. 10). The evidence indicated that students were focusing "their cognitive attention on procedures rather than on concepts or applications" (p. 13).

However, Baker also observed that the high-school students had a greater difficulty with the proof-writing task than did the college students, indicated a lack of procedural knowledge in the subject area. "No high school student completed the proof-writing task… in contrast, half of the college students successfully wrote a proof" (p. 13). Difficulties were observed among all participants throughout the study in understanding the technique of proof by induction, both procedurally and conceptually. "A primary source of difficulty was attributable to a lack of mathematical content knowledge" (p. 15).

In a study of 30 pre-service elementary teachers and 12 secondary mathematics teachers, Mingus and Grassl (1999) found that students expressed that lack of knowledge or experience is not the only problem, but that repeated exposure to gain confidence is crucial. A survey was distributed to the participants to determine their beliefs about proof, their experience in proof, and their beliefs about the role of proofs in mathematics and teaching. The following are responses from two both pre-service secondary education teachers. The first had experience in both linear algebra and modern geometry, but had limited experience with proof in those courses: "I still feel very uncomfortable doing proofs and am very unsure of what I need to do to do the proofs correctly" (p. 440). The second had extensive exposure to proof in a college geometry course, but none prior to this, said, "I struggled with the proof process and as a result, we did all of our proofs in groups of two or more students" (p. 440).

In all, 69% of the pre-service secondary education teachers in this study felt that proofs should be introduced early, even prior to taking 10^{th}-grade geometry. One of the elementary pre-service teachers said, "From personal experience, I can honestly say I have no recollection of using proofs in grades K-8. Which could be one reason why I had such a problem with them in high school [and now in college]" (p. 441). This shows that students' limited experience in proof writing led to lack of confidence in constructing proofs, which they had said hurt their ability to produce proofs.

Weber (2001) discussed the specific strategic knowledge that he observed was lacking in undergraduate students. Weber conducted a study of four undergraduate students who had just completed an introductory abstract algebra course, and four doctoral students completing dissertations in an algebraic topic. Each participant was asked to prove seven propositions regarding homomorphisms. The first two propositions were not difficult and both groups performed well. The last five, however, were more difficult but within the scope of what all participants should have been able to prove.

Among the work on these more difficult propositions, there was a large difference in performance. "Doctoral students achieved near perfect performance, collectively proving 95% (19 out of 20) of the propositions. Undergraduates, on the other [hand], only proved 30% (6 out of 20) of the propositions" (p. 107). These counts are based on the five propositions viewed by four students, for a total of 20 attempts at a difficult proposition in each group. Specifically, Weber identified four types of strategic knowledge that the undergraduates appeared to lack, but were apparent in the doctoral students, all encompassing the "knowledge of how to choose which facts and theorems to apply" (p. 101).

This lack of ability is not a new phenomenon. Epp (2003) discusses a course she taught in the late 1970s, which was designed as a transition course to bridge the gap into higher-level undergraduate mathematics.

> Indeed, I was almost overwhelmed by the poor quality of their proof-writing attempts. Often their efforts consisted of little more than a few disconnected calculations and imprecisely or incorrectly used words and phrases that did not even advance the substance of their cases. (p. 886)

Williams (1980) studied 225 grade 11 students in Alberta, Canada who were enrolled in a mathematics program specifically designed as preparation for studying post-secondary mathematics, and so had been exposed to proof techniques. A total of 12 items were constructed to measure the high-school students' understanding of concepts related to proof. These items were administered to 255 students in nine different schools. Results showed that "less than 20% of the students understood the method of indirect proof" and that "at least 70% of the students sampled did not distinguish between inductive and deductive reasoning and hence did not realize that induction is inadequate to support mathematical generalizations" (p. 166).

In a study of 1520 students in 74 high school geometry classes, Senk (1985) tested students' abilities to fill in missing pieces of proofs and to construct full proofs. Three test forms were given, each including six items, all of equal difficulty as standard textbook problems for high school geometry. Senk found that

> If achievement is averaged across the three test forms, we see that at the end of a full-year course in geometry in which proof writing is studied, about 25 percent of the students have virtually no competence in writing proofs; another 25 percent can do only trivial proofs; about 20 percent can do some proofs of greater complexity; and only 30 percent master proofs similar to the theorems and exercises in standard textbooks. (pp. 453-4)

In pointing to the conclusions of importance to her, Senk addressed the difficulty students had in beginning a proof. With a criteria for grading proofs on a scale of 0 to 4, 0 meaning that the student wrote nothing, or only wrote invalid or useless deductions, and 4 meaning a valid proof, "many students in the sample scored 0 on the proofs, a finding that indicates that we need to pay special attention to teaching students to *start* a chain of deductive reasoning" (p. 455). Senk also found evidence that students did not address the context of the proof, or understand what they were proving. "Many students cited the theorem to be proved in their proofs. This finding suggests that we should place greater emphasis on the meaning of proof than we do currently" (p. 455).

Chazan (1993) focused on the role that computer-aided empirical verification has in geometry classes. A study of 17 high school geometry students was designed with two sets of student beliefs in mind. The first was that "evidence is proof", for example, measurement can provide justification from a single example. The second belief is that "deductive proof is simply evidence", where students view deductive proof as only applying to the diagram pictured with the proof. All students in the study had finished a unit designed to specifically illustrate the differences between examples and deductive proof. During the study, students were interviewed and asked to compare and contrast two methods of argumentation, a two-column deductive proof from their textbook and an inductive argument via examples of a separate theorem.

A few striking occurrences were found. The first was the notion that a counterexample could possible be found to a deductively proven theorem. "Some interviewees mentioned one behavior that runs contrary to accepted behavior in many mathematics classrooms… looking for counterexamples to a completed deductive proof" (p. 381). Also, "some interviewees were not sure that the basic theorems proven in their class held generally" (p. 381). Both of these observations support the notion of deductive proof as only evidence and not necessarily true in general.

Almeida (2001) addressed the proof practices of high school students as well. In a study of 10th grade students in the UK, 19 students were observed while they attempted content appropriate proof activities and were further interviewed after these activities. "The evidence reveals that pupils' views of proof are generally empirical. Apart from a few, most pupils view justification as verifying by empirical evidence" (p. 59). Furthermore, Almeida found that students lacked motivation to continue proofs beyond their own understanding. "Some pupils may have reached certain conjectures but were not motivated to explain or justify them until the interviewer teased out their often original arguments through encouraging and challenging questions" (p. 59).

One difficulty found among high school students by Williams (1980) and Almeida (2001) was also found among college students, specifically the view of inductive reasoning as a valid proof technique (Martin and Harel, 1989; Coe and Ruthven, 1994; Weber, 2003).

Martin and Harel (1989) looked at students' views of the validity of inductive and deductive verifications of statements. This subjects of this study consisted of 101 pre-service elementary school teachers enrolled in a required sophomore-level university mathematics course. These pre-service teachers had already experienced proof writing in high school geometry, and the university mathematics course also included specific attention to proof throughout the course. Martin and Harel believed that studying future teachers' perspectives was important; "If teachers lead their students to believe that a few well-chosen examples constitute proof, it is natural to expect that the idea of proof in high school geometry and other courses will be difficult for the students" (pp. 41-42).

The pre-service teachers in Martin and Harel's study were asked to judge proofs of both a familiar and an unfamiliar mathematical theorem. Both inductive and deductive arguments were given as proofs for the theorems. The inductive arguments included four types: examples (with small numbers), patterns, big number (where a large number was used as a valid example), and an example and non-example (given together). The three types of deductive arguments were a correct general proof, a false proof, and a particular proof (where correct steps were given, but with specific numbers instead of variables).

It was found that participants in this study accepted both the inductive and deductive arguments.

> Acceptance of inductive and deductive arguments as mathematical proofs was not found to be mutually exclusive. This suggests that the inductive frame, which is constructed at an earlier stage then the deductive frame, is not deleted from memory when students acquire the deductive frame. … Thus, as our results indicate, inductive and deductive frames exist simultaneously in many students. (p. 49)

Furthermore, 52% of those in the study accepted an incorrect argument as a proof of an unfamiliar statement. "Many students who correctly accepted a general-proof verification did not reject a false-proof verification" (p. 49). Martin and Harel point to the surface-level evaluation of proofs as explanation of some of this difficulty. The participants were judging the validity of the proof according to its common appearance with known valid proofs.

Coe and Ruthven (1994) studied students' understanding and use of mathematical proof in the UK. This study was conducted with students at the end of their first year of a sixth-form college, which is the first year of advanced-level schooling after mandatory education up to grade 11. Students in the study followed a reform-based curriculum in which they studied a module on problem solving. Course work was collected and evaluated for the highest level of proof used by the student; 60 pieces of coursework were collected in total. Additionally, seven students were interviewed to gain insight into their individual views and conceptions of proof.

In general, "students' proof strategies were primarily and predominantly empirical, with a very low incidence of strategies that could be described as deductive" (p. 52). Coe and Ruthven also found that "numeric data generated from a starting point quickly became the object of investigation, with the original situation being abandoned" (p. 52).

In a study of undergraduate students in a real analysis course, Weber (2003) found that students sometimes verified the validity of a proof by comparison to a known valid proof on a surface level, similar to the findings of Martin and Harel (1989). Six students participated in this study and met with the researcher individually every other week during the semester, discussing their attitudes towards proof and mathematics, their understandings of concepts and proof techniques, and were also asked to construct proofs. Weber found that "when validating their own proofs, these undergraduates would often compare how similar their proof was in form to previous arguments that they had observed or constructed" (p. 400).

Knuth (2002) found similar results among 16 in-service secondary school (grades 9-12) mathematics teachers. During this study, teachers participated in two interviews focused on their conceptions of proof in the discipline of mathematics and on their conceptions of proof in the context of secondary school mathematics. The teachers were asked to give ratings to arguments presented for five sets of statements, each statement included 3 to 5 arguments as justifications. A total of 13 arguments were valid proofs, and 8 were not valid. Teachers were asked to rate each proof as proof or non-proof on a scale of 1 to 4, 1 being a non-proof and 4 being a proof. Knuth found that if proofs were presented in a structured manner, resembling in form or style a valid proof, the teachers in this study willingly accepted the proofs, even when they were not valid proofs.

> Overall, a third of the ratings that the teachers gave to the nonproofs were ratings as proofs. In fact, every teacher rated at least one of the eight nonproofs as a proof, and 11 teachers rated more than one as a proof. (Knuth, 2002, p. 391)

Additionally, some teachers were not convinced of the truth of all possible cases of a proven statement.

> These teachers either believed that it might be possible to find some form of contradictory evidence to refute a proof or they expressed doubt regarding the conclusion of an argument even though they believed the argument to be a proof. (p. 389)

All of these difficulties, lack of content knowledge, inability to apply content knowledge, focus on procedures versus context, lack of strategic knowledge, desire to use inductive reasoning, and a lack of understanding of the universal validity of a proven result, point to a need to research and evaluate the actual proof strategies of students in an effort to further our ability to teach these students in ways that will ultimate help them to be successful proof writers.

Heuristics in Problem Solving

In studying the proof strategies of students, we need to first analyze the basic outline of the proof-writing process. This section outlines the process involved in problem solving, as identified in the literature, and describes the details involved in the individual portions of this process (see Theoretical Framework chapter for further discussion of similarities and differences between problem solving and proof writing).

"Expert and successful problem solvers transform the problem text to form a mental model or a cognitive representation of the problem that corresponds to the problem elements and their relationship" (Pape and Wang, 2003, p. 419). This mental model helps guide the students through the solving process.

It has been thought by some (Polya, 1973; Schoenfeld, 1985) that general problem-solving strategies could be summed up in a list of heuristics, which could be taught to students to enhance their problem-solving abilities. The most famous of these lists of heuristics is certainly due to Polya (1973), which states that to solve a problem one must: read the problem (understand the problem), devise a plan, carry out the plan, and then verify your work (look back).

Work has been done to refine this set of heuristics to describe the way successful problem solvers go about dissecting and solving mathematical problems (Carlson and Bloom, 2005; Pape and Wang, 2003; Pugalee, 2001; Schoenfeld, 1985; Yerushalmy, 2000), but the basic ideas remain the same. The following is a compilation of the main heuristics suggested in the literature.

- Reading/Orientation/Analysis – read and get a feel for the problem, also called "sense-making"
- Planning/Organization/Goal-setting – identify goals and sub-goals, make a plan, get information, draw diagrams/organize information
- Implementation/Execution – perform goals and sub-goals, monitor progress, seek assistance, redirect if necessary
- Verification/Review – evaluate results and decisions, review records

Carlson and Bloom (2005) further state that, while problem solving, mathematicians also work through one of two sub-processes, or cycles, of "plan-execute-check" or "conjecture-imagine-verify" (p. 53). These cycles will repeat throughout their work as often as is necessary.

There are many details within the second and third heuristics above (implementation and verification) that are not fully described by this list. Examples of individual strategies used within these categories include the following (Baker, 1996; Bell, 1979; Carlson and Bloom, 2005; McGivney and DeFranco, 1995; Schoenfeld, 1985):

- Defining unknowns
- Sketching a graph/figure
- Constructing a table
- Checking all cases and identifying a key fact
- Generating examples
- Exploiting analogies
- Work forward from the given information
- Work backward from the conclusion back to the given
- Means-end analysis
- Decomposing and recombining – break down the problem into smaller parts
- Identifying goals and sub-goals

This is by no means an exhaustive list, but it provides an overview of commonly used strategies in problem solving, as seen in actual practice. When approaching this study, the above formed an initial idea of expected strategies in proof writing. This study seeks to expand on the list and describe how these strategies are used in proof writing.

Levels of Proof

This section gives categories of proof products as defined in the literature, to be used as a comparison for defining successful proof writers in this study.

Balacheff (1988) developed four types, or levels, of proof; *naïve empiricism,* the *crucial experiment,* the *generic example,* and the *thought experiment*. *Naïve empiricism* includes verification of a statement through viewing several cases, without generalizing. The *crucial experiment* involves a test of a specific case chosen to be typical and non-trivial. The *generic example* includes verification of a statement based on the operations and aspects of a generic example, representing a class of objects. The *thought experiment* gives verification by the operations and properties of mathematics, without use of an example.

Almeida (2003) uses this list, along with the levels described by Van Dormolen (1977), which are based on van Hiele's levels of development in geometric thought. The *ground level* is the first of three levels in proofs, consisting of thinking only of special objects, or single cases (akin to naïve empiricism). The next level is the *first level of thinking*, where proof is based on a certain class of objects, rather than one example (the generic example). The final level what Van Dormolen calls the *second level of thinking* involves reasoning about reason and connecting the arguments made during the first level of thinking (the thought experiment).

For the purposes of categorizing proof-writing attempts as successful or unsuccessful, the last two levels of Balacheff's model will be considered proof, which encompass the last two levels of Van Dormolen's model as well. Any proof attempt could involve aspects of multiple levels of proof, but the final reasoning must be generalized to be considered proof. Furthermore, any generic example or crucial experiment could be included with the intent of proving cases and also be considered proof using appropriate reasoning. A participant need not move to the stage of thought experiment to establish a valid proof in the questions in this study. The final product of proof writing is of concern to this study only to describe the strategies used by each group, successful and unsuccessful participants, within each question and overall.

Self-Regulation

A significant idea emerged during this study, that self-regulation was actually at the heart of students' ability to use strategies appropriately. This section discusses the definition and categories of self-regulation, results of studies in the literature in regards to self-regulated learning use by experts, and results of novices in mathematics.

Definition and Categories

"Self-regulation, or monitoring and control, is one of three broad arenas encompassed under the umbrella term *metacognition*. … In brief, the issue is one of resource allocation during cognitive activity and problem solving" (Schoenfeld, 1992, p. 354). Metacognition is dealt with in more detail in the Theoretical Framework chapter. Here, we focus specifically on the issue of self-regulation. Most research in this area deals with problem-solving behaviors, and so the remainder of this section is laid out with those terms, with the view of proof writing as a problem-solving process.

"Self-regulated students are active learners who are able to select from a repertoire of strategies and to monitor their progress in using selected strategies toward a goal" (Pape and Smith, 2002, p. 94). Even with the proper content knowledge and heuristic strategy knowledge within a subject area, one can still fail to complete a solution due to the inability to monitor progress and strategy use (Schoenfeld, 1982).

> Poor managerial decisions may preclude one from problem-solving success. 'Wild goose chases' may lead a problem solver away from useful approaches, never to return; or the dogged pursuit of ultimately correct but difficult approaches may reduce efficiency to the point where students run out of time before they manage to solve problems. (Schoenfeld, 1982, p. 33).

Zimmerman and Martinez-Pons (1986) laid a framework of 14 categories of self-regulated learning. In a study of 80 high-school 10^{th} grade students, Zimmerman and Martinez-Pons developed a structured interview method for measuring these categories and sought to determine the relationship between the use of these categories and student achievement. The participants in this study included 40 students in an advanced achievement track and 40 students from lower achievement tracks, as assigned by the school based on test scores, GPA, and teacher and counselor recommendations prior to high school.

The 14 categories were defined as: self-evaluation, organizing and transforming, goal-setting and planning, seeking information, keeping records and monitoring, environmental structuring, self-consequences, rehearsing and memorizing, seeking assistance from peers, seeking assistance from teachers, seeking assistance from other adults, reviewing tests, reviewing notes, and reviewing textbooks. An additional category of non-self-regulated behavior was added, labeled "other", to denote those behaviors not fitting in any of the other categories to complete the data. Definitions for each of these categories can be found in Zimmerman and Martinez-Pons (1986, p. 618). They were able to use this structure of 14 categories to show that the behaviors of the 80 students in their study were closely linked with achievement groups.

> Of the 14 categories of self-regulated learning strategies that were studied, the high achievement group of students reported significantly greater use than [the] low achievement group for 13 of these categories. High achievers also reported significantly less use of a single category of non-self-regulated response than low achievers. (p. 624)

Furthermore, they were also able to reverse this analysis and accurately predict the achievement group based on the strategies used; "93% of the students could be correctly classified into their appropriate achievement track group through knowledge of their self-regulation practices" (p. 625).

Zimmerman and Martinez-Pons (2002) also viewed the processes in self-regulation in a larger context, the three phases that occur before, during, and after each effort to solve a problem. The *forethought phase* occurs in the planning stages of the problem-solving process. The *performance phase* occurs during work and the *self-reflection phase* occurs after a particular effort at a solution is completed. This process can be repeated multiple times before a valid solution emerges, and is cyclical in nature.

<u>Results in Experts</u>

Zimmerman and Martinez-Pons (2002) further note that the processes used by novices differ from those used by experts. "The self-regulation profile of novices is very distinctive from that of experts. Novices fail to engage in high-quality forethought and instead attempt to self-regulate their learning reactively" (p. 69).

Schoenfeld (1987c) reports on his recorded observations of a mathematician attempting to solve a difficult geometry problem. The mathematician was not a geometry expert, and had not worked with geometry proofs in some time, but could be considered an expert in the realm of mathematics in general, as compared to students. The mathematician "started off on a wild goose chase, but – and this is absolutely critical – he curtailed it quickly" (p. 195). Furthermore, Schoenfeld points out that, "with the efficient use of self-monitoring and self-regulation, [the mathematician] solved a problem that many students – who knew a lot more geometry than he did – failed to solve" (p. 195).

Carlson and Bloom (2005) completed a study involving 12 mathematicians. These experts were asked to complete four mathematical tasks within the scope of task-based interviews in which they were asked to "think-aloud" about their attempts. The resulting behaviors were recorded. It was found that, "in addition to making decisions about their solution approaches, the mathematicians regularly engaged in metacognitive behaviors that involved reflecting on the effectiveness and efficiency of their decisions and actions" (p. 64).

Results in Novices

The research in this area spans from elementary level up through university students. In a study of 126 students ages 8-11, Panaoura and Philippou (2007) aimed to model the development of students' metacognitive abilities in mathematics. Data was collected from the students over a period of 3-4 months. During this time, students completed a self-report questionnaire to judge their metacognitive ability. Their mathematical ability was judged by performance on four mathematical tasks, four analogical, four verbal tasks, and four matrices (to measure spatial ability). There were also instruments used to measure the students' information processing efficiency and working memory capacity. "Pupils with high self-regulatory ability are pupils with high self-image and pupils who do not use different metacognitive strategies are those who have negative self-image, as well" (p. 163). They add that self-image depends on mathematical performance.

"The results of the present study indicate that changes on thinking and metacognitive performance might be associated with processing efficiency and working memory, even during the primary school years, at the specific domain of mathematics" (p. 163). Furthermore, Panaoura and Philippou state that, "successful academic performance depends on cognitive as well as metacognitive abilities" (p. 163).

Pape and Wang (2003) developed a study involving 80 sixth- and seventh-grade students using a strategy questionnaire based on the 14 categories of self-regulatory behavior from Zimmerman and Martinez-Pons (1986). Students were first individually videotaped while solving 16 mathematics word problems, in which they were asked to "think-aloud" as they worked. Students then completed the strategy questionnaire on a separate day. "Students who reported more categories of strategic behavior solved significantly more problems correctly" (p. 435). Pape and Wang also report that, "high achieving mathematics students reported more different strategies than their low achieving counterparts. In turn, these variables were associated with problem-solving behavior and outcomes" (p. 438).

Success among high-school students engaging in self-regulatory behavior has been reported by Santos (1995) and by Nota, Soresi, and Zimmerman (2004). In a study involving 13 ninth-grade students, Santos (1995) documented the types of strategies and difficulties shown during the students' work on three problems in mathematics. Students were interviewed individually and were asked to "think-aloud" as they worked on the problems. The quality of responses was reported and distinctions were found.

> These distinctions include: (a) The use of representation as a means to work the data (table, list) and to show the result, (b) Connections in which some students linked the common features among the problems, (c) Flexibility in trying to graph and explain extensions of the problems (accumulative graph), and (d) Confidence shown by some students when they compared the responses of the problems. (p. 5)

These distinctions set apart the best quality responses. They show a tendency to employ metacognitive behaviors, including comparing work on multiple problems, and being flexible in strategy use.

Nota, Soresi, and Zimmerman (2004) studied high-school students in their fifth and final year of high school in Italy. During the first phase of this study, 81 participants were interviewed using the survey developed by Zimmerman and Martinez-Pons (1986) to determine their self-regulation methods. Later, some of these participants who went on to a particular university, 49 in all, were documented in terms of final high school grades, enrollment in the university, and GPA in the first two years at the university. "The regression analyses revealed that school grades for courses in Italian, mathematics, and technical subjects were significantly predicted by students' strategy of organizing and transforming information during self-directed efforts to learn" (p. 209).

Smith (2006) reported on the perceptions and approaches to mathematical proof of undergraduates. The study consisted of two semi-structured task-based interviews with 5 students enrolled in an undergraduate number theory course. Three of the students were taught using the "modified Moore method" (MMM), while "employs mathematical discourse among students … [through] a *problem-based* approach to teaching mathematics" (p. 74). The other two students were taught in a traditional lecture format. While Smith's structure in the study was different than those previously discussed, she also found the use of self-regulation strategies to be a valuable addition to the approaches a student takes to proof writing.

> When presented with a statement to prove, MMM students typically began by trying to make sense of the statement. The traditional students, however, frequently began by doing two things: "throwing" proof strategies at the statement in an attempt to construct a proof, and "searching" their minds for information related to the topic and for potentially similar proofs seen in class or in the text. (p. 87)

Smith also states that she found,

> MMM students also spent more time choosing a proof strategy and then trying to make their chosen strategy work than did the traditional students, who were more likely to try several strategies in quick succession without considering each very closely, or even considering *why* the strategies were not successful. (p. 87)

While Smith did not directly study the link to self-regulation, her results show that the students she classified as having a higher quality approach to proof were actually approaching proof with metacognitive strategies to monitor their cognitive actions.

All of the studies mentioned show that the use of self-regulation strategies is beneficial in problem solving, is positively linked to mathematics ability, and has the potential for promising results in proof writing as well. This study seeks to build on this work and to specifically examine the self-regulation strategies used by novice proof writers, and the success found with the use of such strategies.

CHAPTER 3

THEORETICAL FRAMEWORK

This chapter introduces the theoretical framework upon which this study was built. The purpose is to lay the foundation from which this researcher views the study and its resulting data. Specifically, it describes the lens through which this researcher is analyzing the data, and the framework that informs my thoughts, beliefs and ideas.

Procedural, Syntactic, and Semantic Proof Processes

As described in the literature review, the importance of research in proof writing at the university level is clear. Much of the research in this area, however, has focused on the products of proof writing and the validity of the actual proofs themselves. While this is a worthwhile endeavor, it is just as important, if not more so, to investigate the ways in which educators can improve the proof-writing process. In doing so, we must first investigate what occurs in the process of proof writing to fully understand how we can improve this process.

Weber (2004b) offered a framework that can be used to describe these processes, by describing categories that he termed *procedural*, *syntactic*, and *semantic* processes. This framework is based on several studies by Weber (2001, 2002, 2003, 2004a), in which he observed the processes of undergraduate students while constructing proofs. These observations consisted of task-based semi-structured interviews, during which participants were asked to "think aloud" as much as possible during their attempts at a proof. This is a well-known approach and is described in more detail, with further examples, in the next chapter.

Weber defined three categories, as mentioned above. "In a *procedural proof production*, one attempts to construct a proof by applying a procedure, i.e., a prescribed set of specific steps, that he or she believes will yield a valid proof" (2004b, p. 426). For example, if a student has observed the teacher proving via contradiction that $\sqrt{2}$ is irrational, he or she could then reproduce a similar proof that $\sqrt{3}$ is also irrational by only copying the work of the teacher. This may or may not include an understanding of the proof itself, or being convinced of its validity.

In the second category, *syntactic proof production*, "one attempts to write a proof by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (Weber, 2004b, p. 428). A student could, for example, convert a statement into its logical equivalent in symbols and complete the proof by manipulating those symbols to his or her desired result. Many mathematicians could claim to have produced such proofs, and they are prevalent in our textbooks as well.

The third category is *semantic proof production*. In such a process, "one first attempts to understand why a statement is true by examining representations (e.g. diagrams) of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof" (Weber, 2004b, p. 429). These representation types could vary based on the specific question asked, but would represent a very different approach than just symbol pushing or rote procedure. Here a student would seek to understand the proof and the statement through examining the structure itself and possible ideas for the proof would arise during this examination. For example, in this study, a participant could first view Question 1, involving sums around the sides of a pentagon, by drawing a pentagon and examining the properties and relationship to these sums. He or she could then move to an understanding of the overall sum and be able to develop equations for a generic pentagon to minimize the sum. This formal argument would be based on the ideas gained from the diagram, but divorced from any one example or picture.

The main difference that exists among these three ideas is in the way in which a student would first approach a proof, not necessarily in the final product. All three processes could lead to the same proof, but via different paths, so the interest lies in the path taken and how a student viewed the question in the initial stages of proof writing. Additionally, each student's approach may not fall into just one category. That is, within each proof attempt, students may vary their approach among more than one of the processes illustrated above, and this tendency could change from question to question. Alternatively, an individual student may always use just one approach.

Weber (2004b) also notes that semantic proof productions often reflect a greater understanding of the mathematical structures involved in a statement. If a student only engages in procedural and syntactic proof productions, he or she runs the risk of never fully understanding the process of mathematics or the development of formal theory. However, even mathematicians engage in procedural and syntactic proof activities at times and they can be timesaving and advantageous. Each proof production has its own qualities and appropriate times of use.

This study seeks to expand on the three categories described by specifically addressing the individual components of each type of proof production process. These categories represent a general view of proof writing, but clearly involve multiple individual types of approaches. For example, in a semantic proof production, a student could view diagrams, specific examples, or general classes of examples to understand the structure. He or she could develop notation to describe the situation or use physical objects to reflect his or her understanding of the statement. These are just a few of the particular ways in which this process could be manifested.

Also, any individual study done in the detail necessary to describe and categorize results as Weber has done must, by shear necessity, involve only a small number of participants. It is therefore difficult to generalize the results without support from other studies on other subjects, in other locations, or using other questions. It is for these purposes that this study was designed and implemented, as an exploration into the details of these strategies, and their respective successes in attaining valid proofs.

Other Frameworks

Similar models for describing the overall processes used by students have been investigated in the areas of both proof writing and problem solving. The following section describes three such models and their relationship to this study.

Sowder and Harel (1998) classified their students' *proof schemes* into three main categories: *externally based, empirical, and analytical proof schemes*. They define a *proof scheme* to consist of "whatever constitutes ascertaining and persuading for that person" (p. 670). In *externally based proof schemes*, "both what convinces the students and what the student would offer to persuade others reside in some outside source" (p. 671). This includes the three sub-categories of authoritarian, ritual, and symbolic proof schemes. An authoritarian proof scheme is one in which the student relies only on an authority, such as the textbook or teacher, as a basis for the validity of a proof. Ritual proof schemes are instead based on form alone; for example, arranging a proof in two-column format, or in the particular structure of an induction proof. The symbolic proof scheme involves only symbol pushing, or manipulation, for a proof, which could be bad or good depending on how it is used.

The second main category introduced by Sowder and Harel (1989), *empirical proof schemes*, includes the perceptual and examples-based proof schemes, which are both based on evidence as proof. In a perceptual proof scheme, this evidence would take the form of a drawing, where the examples-based proof scheme would be a numeric example. The last main category, *analytical proof schemes*, includes transformational and axiomatic proof schemes. These are concerned with direct reasoning towards the conclusion of the general statement using careful organization. The transformational proof scheme would involve reasoning from generic examples or categories, where the axiomatic proof scheme would involve reasoning form definitions, axioms, and known theorems. This framework is similar in structure to that of Weber (2004).

Cifarelli and Cai (2005) discuss the processes used in solving open-ended problems. Specifically, they identify a model similar to that of Polya and several other researchers. The main steps involved in the process include *sense-making* or initial reflections on the problem, *formulating goals* or problem posing, and *achieving goals* or problem solving, which is classified as carrying out the solution and reflecting on the results. Cifarelli and Cai also believe this to be a process through which one could cycle repeatedly, with results from one problem potentially becoming another problem to be solved. These ideas could be represented as components of the semantic proof production, and are representative of student action during the solution of an unknown problem. This is different from a question posed that is similar to a previous result, which could instead be classified as merely an exercise.

Gholamazad, Liljedahl, and Zazkis (2003) summarize the results of a study of one-line proofs. They outline the general process used to complete such a proof, and each step they include represents an area of potential error and difficulty if a student is unable to complete that particular step correctly. The steps they identify in writing a valid proof are: recognizing the need for proof, recognizing need for representation, choosing correct and useful representation, manipulating representations correctly, and interpreting manipulation correctly.

The outline of Gholamazad, et al, addresses several of the same issues as in the three steps given by Cifarelli and Cai. Both encompass an initial understanding stage, moving on to a planning stage, and finishing with an execution and reflection stage. Again, it is within these categories that this study seeks to expand the results and describe the individual actions of students and the success achieved, or not achieved, by these actions.

All four studies mentioned, especially the work of Weber, provide a framework from which proof processes can be viewed and provide a springboard for viewing the expected strategies and difficulties of participants in this study.

Personal Biography

This section is included to give the educational background of the researcher, her personal motivation for this study, as well as her pre-analysis thoughts and what changes developed in these thoughts and notions as the analysis was completed.

This researcher's educational background includes an undergraduate degree in mathematics with emphasis in secondary education and minor in computer science. At the end of undergraduate work, the researcher had the opportunity to enroll in many upper division mathematics courses, as well as independent study opportunities beyond the courses offered. Additionally, she was able to be a part of a Research Experience for Undergraduates (REU) funded by the National Science Foundation, researching the effects of symmetry on bifurcation diagrams. This opportunity to research and explore new mathematical topics allowed me to develop proof-writing skills and understand the need for such skills as a mathematician and researcher in the future.

Upon entering graduate school, there was a noticeable distinction between those fellow students who had similar experiences and those who had not. Those with experience in upper level mathematics courses, or similar REU programs, were more likely to be confident in their proof-writing ability and to succeed in their first few graduate courses. Those without these experiences struggled in their proof-writing attempts and were cautious and guarded in their first courses in this respect. While these were merely anecdotal observations and not yet valid study results, this researcher began to view her world through a new lens, and began to ask how others might be taught these skills.

The ideas for this dissertation began with a search for a "toolbox" of sorts, which included all the tools needed to complete proofs. This matches the view in cognitive science of information processing, which will be described later in this section. If this toolbox could be found in general for all successful proof writers, then the question of how to teach the tools to others could be investigated further. Of particular importance to me was the development of this toolbox in order to further the educational goals of instructors of a transition-to-proof course. It is this desire to find the toolbox of proof-writing skills that motivated this study. The notions prior to conducting the study, those the researcher felt would be found through this study, included the following: (those of particular importance in this researcher's mind are noted with an asterisk *)

- Reword the question to understand it
- Define all terms*
- Look for similar proofs from other areas, homework, or in the book
- Work backwards*
- Work forwards
- Convert words to algebra (if appropriate)
- Look for parts that are familiar*
- Use other results to build on
- Try small examples
- Build from smaller cases
- Work with others or work alone
- Draw pictures or visualize
- Specific form – ex. Work from known to what needs to be shown *
- Convert question to area of math that is familiar
- Start a strategy like direct proof working forwards and if it doesn't work turn to another like induction, or moving to contradiction, etc.
- Use proof technique used recently

However, these strategies took on a surprising difference after the transcripts were completed and analysis began. There was a distinct shift to several main categories of strategies, with each category including more specific details related to the individual question. While a common set of strategies was expected that would encompass all questions, this set or "toolbox" seemed to be different for each question and unique to each participant. The final analysis was shaped based on a new, much more detailed list of strategies that were observed during the interviews, and refined in the initial stages of analysis. This list is included in Appendix A. It is therefore from this perspective that the analysis was developed and continued as described in the Methodology section of this work.

Cognition

This researcher considers herself to be working from the viewpoint of a cognitive scientist, that is, interested in what students think while they are working in mathematics, and more importantly, while working on mathematical proofs. Schoenfeld (1987b) stated,

> A basic assumption underlying work in cognitive science is that mental structures and cognitive processes (loosely speaking, 'the things that take place in your head') are extremely rich and complex–but that such structures can be understood, and understanding them will yield significant insights into the ways that thinking and learning take place. (p. 2)

Specifically, and important to this study, cognitive research is focused more on the *process* versus the *product* (Mayer, 1985; Schoenfeld, 1987b). In other words, through this lens, we view the *steps* involved in proof writing, more so than the actual product as being valid or not valid.

As stated previously, this study began with the idea of a "toolbox" that could be filled with strategies for use in proof writing. This viewpoint is consistent with cognitive scientists investigating *information processing (IP)*. Silver (1987) defines terms involved in cognitive science theory and research related to IP. He explained the idea of *working memory* where knowledge is transferred from the senses to long-term memory storage. "Working memory maintains an internal representation of the current state of cognitive activity" (pg 38). IP is the activity of bringing items back and forth between working memory, long-term memory, and the senses, "recognizing, comparing, and manipulating symbols in working memory; and storing information in LTM [long-term memory]" (p. 39).

The study of information processing in cognitive research involves the desire to break down human thoughts and processes into small sets of activities. Often IP also links these processes in a form that could potentially be modeled by a computer program for further study. For example, if we were able to program the main knowledge items necessary for a problem, the ways in which these can be manipulated and related to one another, and an initial problem into a computer, we would test to see if the program could use just these skills to successfully solve a problem. This would indicate that no other knowledge was necessary to solve such problems. This was the underlying idea behind the toolbox. If those items needed to process a proof could be summarized, such as knowledge of individual types of proofs, as well as how one might choose each type, a student could use this to prove many theorems by choosing the appropriate "tool".

Weber (2006) used this type of theoretical perspective to investigate the question, "How can one describe a set of cognitive processes that an undergraduate can use to prove statements about group homomorphisms?" (p. 198). He stressed the link between these ideas and the heuristics and metacognitive behaviors in problem solving outlined by Schoenfeld (1985). Ultimately, one of Weber's goals in his study was to help educators "understand the decision-making processes used in constructing proofs" (Weber, 2006, p. 224).

However, as the interview process for this study progressed, the shift to a different perspective formed. This shift was to a more constructivist viewpoint within the frame of cognitive theory. "According to the constructivist perspective, we all build our own interpretive frameworks for making sense of the world, and we then see the world in the light of these frameworks" (Schoenfeld, 1987b, p. 22). With respect to the toolbox idea, this means that each student develops his or her own personal toolbox, and views proofs with this toolbox in hand, not my tools or someone else's, but his or her own personally developed tools and the conceptions (or possibly misconceptions) associated with each tool (Silver, 1987). "Learning is dependent upon existing knowledge… individuals are likely to construct differing knowledge in response to a given experience" (Beswick, 2005, p. 43). Barkatsas and Malone (2005) found two existing orientations of secondary mathematics teachers' beliefs in their study of the beliefs in regards to teaching and learning mathematics, the *contemporary-constructivist orientation* and the *traditional-transmission-information processing orientation*. Specifically, the contemporary-constructivist orientation involves beliefs that students should reflect on and evaluate their own work, be prepared to be critical thinkers, and that knowledge in mathematics often develops within a social learning environment.

Schoenfeld (1987a) describes a very similar process to the shift that occurred during this study, but in the area of problem solving. He first “set out to develop prescriptive models of heuristic problem solving” but later turned to ideas in “metacognition, belief systems, and ‘culture as the growth medium for cognition’” (p. 30). After research in the area of heuristics, Schoenfeld discovered that “knowing the strategies isn’t enough. You’ve got to know when to use which strategies” (1987a, p. 32). He called this a *managerial strategy*. His goal was “to understand what the person did, why he or she did it, and how those actions contributed to his or her success or failure at solving the problem” (p. 34). This managerial strategy is also known more commonly as metacognition.

Metacognition

The frameworks described above, as well as the individual strategies that were expected, involve both cognitive and metacognitive behaviors. Metacognition refers to one’s thinking about thinking. “One way of viewing the relationship between [these behaviors] is that cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done” (Garofalo and Lester, 1985, p. 164). This idea plays a major role in educational practice. We, as educators, desire our students not only to perform the tasks we assign, but also to understand what they are doing and why, and to be aware of the process by which they are completing the task. “Metacognition as a component of mathematics instruction involves active learning to help students become aware of, reflect upon, and consciously direct their thinking and problem-solving efforts” (Gray, 1991, p. 24).

Research has found that there is a distinct advantage in problem solving for those who are able to monitor their actions and thoughts; in other words, those who engage in metacognitive behaviors (Artzt and Armour-Thomas, 1992; Cai, 1994; Oladunni, 1998; Pugalee, 2001; Swanson, 1990). A student who is able to monitor his or her own thoughts and overall plan of action is better able to solve a problem and can better deal with new problem situations than a student who seems to have no inclination how to monitor these processes. In a study of 27 students working in 6 groups in a seventh-grade math class, Artzt and Armour-Thomas (1992) found that "the only group that did not solve the problem was the group with the lowest percentage of episodes at the metacognitive level and the highest percentage of episodes at the cognitive level" (p. 161). In fact, Swanson (1990) reports findings that "high-metacognitive individuals outperformed lower metacognitive individuals in problem solving regardless of their overall aptitude level" (p. 312).

Successful problem solvers seem to be able to monitor the strategies they are using and switch strategies when one is not successful for them. They are better able to organize the material and their thoughts and seem to work more smoothly through a problem. Pugalee (2001) found that "the writing [from students in this study] shows how an interplay of cognitive and metacognitive actions was necessary to monitor the progress of computations needed to solve the problem" (p. 241).

Metacognition has also been studied as it applies to proof writing (Blanton and Stylianou, 2003; Weber, 2006), which is of primary concern for this study. In particular, these studies addressed the issue of teaching students to think more critically about proof writing and to monitor and question their work as they proceed through a proof. Blanton and Stylianou (2003) report that their results "suggest that students who engage in whole-class discussions that include metacognitive acts as well as transactive discussions about metacognitive acts make gains in their ability to construct mathematical proofs" (p. 119). For more discussion on metacognitive behavior in proof writing, see the Self-Regulation section in the Literature Review.

<u>Problem Solving versus Proof</u>

The initial ideas of the researcher were met with a question of where problem solving ended and proof writing began, since many notions of problem-solving strategies were currently in existence. The following is a response to this question. The first subsection gives a personal viewpoint on the differences expected and found. The second subsection addresses the viewpoints in the literature, giving both the perspectives of the differences as well as the similarities between problem solving and proof writing.

Personal Perspective

The key difference between problem solving and proof may be in the particular wording of a question being posed. A question worded in one way may evoke problem-solving responses, while a different wording may evoke proof-writing responses. For example, in Question 1 in this study, the original statement of the question as it was posed to a participant in the pilot study, asked for the minimum sum attainable on a pentagon. However, this question caused the participant to work for several minutes on finding this sum, with no reference to a proof at all during this process. It was only after the process was completed that the participant even considered how to prove that this was the minimum sum. In other words, the question could have asked to show what the minimum sum would be, which would need no formal proof. The question was changed to include this detail, that the smallest sum would be 14, in a conscious effort to bring participants to the work of proving the statement sooner. Finding a pentagon whose sum was 14 was not even necessary, and would not be expected as part of the proof itself. (see Methodology chapter for full details on the pilot study and questions that were asked).

The researcher felt prior to the interviews, and later after the interviews as well, that the work attempted by participants would differ based on how they viewed the question, as one to be proven, or one to be solved. This was seen in the study, as participants did not feel satisfied just giving the picture for sums of 14, but rather knew that more was needed for a proof. Several participants said as much, as they monitored their progress towards an actual proof. Another observation that was made was in the individual tendencies of participants to use a particular style of work and proof because it was what they deemed to be an acceptable form of proof. For example, Jon was focused on equations since he felt the most confident in the validity of a proof that used equations.

Other participants even went beyond their proofs, which they recognized as having proven the statement, in search of something more in tune with their preconceptions of what a proof should include. Those who did not seek a specific style of proof, or did not understand what would constitute a proof, did not find valid proofs to some questions. Instead, they were willing to stop their work once they were convinced that a statement was true, or stopped with a specific example, rather than connecting this to generic notation. This discussion is addressed in detail in the analysis section of this work, but is included briefly here to clarify the distinction between problem solving and proof writing.

<u>Literature on Differences and Similarities</u>

"Problem-solving skills are used to informally discover a proof that is later demonstrated formally" (Baker, 1996, p. 2). Proof writing goes beyond normal justification of a solution; it aims to remove all doubt. The attitude a student possesses while constructing a proof is not just one of a student looking for an answer to a question, but one who is looking to convince others of the accuracy of that answer. In advising students how to answer the "why" of a conjecture, Mason, Burton, and Stacey (1982) recommend the three stages, "convince yourself, convince a friend, [and then] convince an enemy" (p. 106). They elaborate that convincing a friend pushes you beyond your own understanding of a conjecture, to externalize the justification. However, the last step, convincing an enemy, forces you to prove yourself to a doubter, one who will potentially not be so easily convinced as a friend.

Communicating a proof to others in mathematics also has a language of its own, apart from that of problem solving.

> What is evident from [the] responses is that the students connect the requirement to prove with the investigations part of the curriculum where they have learned a format and a language of presentation. They have appropriated some structures to help them make sense of the situation and to assist in developing a language for proof. (Hoyles, 1997, p. 13)

"Proof is distinguished from the other aspects of mathematical activity … by the fact that it belongs mainly to the verification stage of investigation, the stage at which ideas become articulated and visibly expressed" (Bell, 1979, p. 372).

Some researchers, however, believe that proof writing is essentially a problem solving activity. "Viewed as a problem-solving activity, we see that proof is actually the final stage of activity in which ideas are made precise" (Tall, 1991, p. 16). This idea parallels the words expressed above by Bell (1979), but links to problem solving specifically. Weber (2001) agrees that such a link can be made. "To examine the process that the undergraduates and doctoral students [in this study] used to construct proofs, I view proof construction as a problem-solving task" (p. 110). He further clarifies this relationship as follows:

> In problem-solving tasks, the problem solver typically is presented with an initial state and is asked to perform a sequence of actions that will transform the initial state into a desired goal state. In constructing the proof of a statement, the prover is given an initial set of assumptions and is asked to derive a sequence of inferences (e.g. Recall definitions and apply theorems) which conclude with the statement being proven. (pp. 110-111)

McGivney and DeFranco (1995) discuss this relationship particularly in geometry.

> Proofs in geometry can be classified as "problems to prove" because they typically include given information; such operations as axioms, postulates, and theorems; and a specified goal. … Understanding the problem-solving strategies associated with information processing, for example, working backward and establishing subgoals, can help students identify various solutions paths… (p. 553)

For the purposes of this study, proof writing and problem solving are viewed as similar, but distinct processes. This researcher agrees with both the differences and the similarities noted in the literature, and sees this study as a method of viewing problem-solving strategies through the new lens of proof-writing activities.

Summary

Based on the framework developed and later adapted by Alcock and Weber (in press), this study seeks to expand on the ideas of syntactic and semantic, or *referential*, approaches to proof writing. This will include imposing a fine grain analysis of those particular strategies and processes that make up each of these approaches, as well as the success or lack thereof, of each strategy within the scope of the questions asked in this study. In this way, this researcher will develop a richly detailed qualitative view of the processes used by students during unfamiliar proof-writing scenarios.

CHAPTER 4
METHODOLOGY

The purpose of this study is to identify and describe the strategies used by novice proof writers in the initial stages of proof writing using task-based interviews and group observations, compare these strategies to the traditional problem-solving strategies and heuristics, and consider the effect of individual learning preferences on these strategy choices. This chapter describes the setting in which these interviews occurred, pilot study participants and results, research participants, data collection procedures, methods used to collect and analyze data, and the potential limitations of this study.

Study Description

Both the pilot and research studies were conducted in the spring semester of 2006 at The University of Montana in Missoula, Montana. Missoula had a population of 57,053 in 2000, according to the U.S. Census Bureau. It is a town largely made up (93.6%) of white, non-Hispanic people, with a small population of American Indian, Asian, African American, and other ethnicities. The median income in 1999 was $30,366 with a median house price of $132,500. The University offers a variety of majors across the spectrum. Fall 2006 enrollment, according to university records, was 13,961 students. Many students are Montana residents; 68% of fall 2006 enrollment consisted of in-state students. According to the mathematics department web site, as of the summer of 2007, the mathematics department had approximately 20 tenure and tenure-track faculty, 100 undergraduate mathematics majors, and 35 graduate students.

The participants of the research study were recruited from two classes offered to undergraduate and graduate students in the spring term. The first class selected was Introduction to Abstract Mathematics (MATH 305). MATH 305 is a required course for undergraduate mathematics majors and, for most students, is their first experience in proof writing. The course sets the stage for further mathematics courses in which students would be required to write proofs. In this course, students were taught explicitly about logical arguments and proof skills, and were beginning to form the techniques and strategies needed to write a mathematical proof. The researcher was familiarized with the content of the course by sitting in on the class throughout the semester, and held weekly office hours to assist students with homework. In this way, the students became comfortable working with the researcher. This was done to help reduce any anxiety felt during the study by participants who may have been fairly inexperienced in proof writing or in working through a new proof in front of the researcher.

The second course from which participants were recruited was History of Mathematics (MATH 406). MATH 305, or the equivalent at another institution, was a prerequisite for this course. The course itself did not deal directly with proof writing, however the variety of prior completed math courses and variety of experiences in MATH 305 for students in the course, along with the ease of access to the particular class, as it was taught by the researcher's advisor, made it a top choice for recruitment. Most students in this course had also already taken other courses beyond MATH 305 in which they were required to write proofs, such as an abstract or linear algebra course.

The research study was carefully designed, following the pilot study, to elicit free response proof writing that could be used to infer, directly or indirectly, the strategies and tools used by the participants in forming the initial ideas for a mathematical proof. The main component of the research study consisted of a semi-structured task-based interview, during which participants worked through a task introduced by the researcher within a specified environment. Goldin (1999) distinguishes structured interviews from unstructured, "where no substantial assistance that would facilitate a solution is given by the clinician to the subject" (p. 519).

The interviews conducted for this research study were semi-structured, meaning that some assistance was given to participants when gross errors in calculations occurred, or when participants needed more prompting to continue past stumbling blocks from which they could not recover. This assistance was partially scripted, to the extent that could be predicted from the pilot study, as outlined in the interview protocol, which can be found in Appendix B. However, when necessary, the researcher deviated slightly from the wording in this protocol as was appropriate for interaction with each individual participant.

This design was chosen to give an accurate and thorough picture of the processes used in proof writing. Task-based interviewing has been used by many researchers in mathematics education and is considered a primary means of collecting data on cognitive and metacognitive behaviors in both problem solving and proof writing (Alcock and

Simpson, 2002; Alcock and Weber, in press; Carlson and Bloom, 2005; Kantowski, 1977; Smith, 2006; Sriraman, 2004; Yerushalmy, 2000). The results obtained from such a study are qualitative and descriptive in nature. For such a study, "the interim products are a set of complex, detailed, qualitative reports" (Goldin, 1999, p. 523). This allows for an analysis of participant work that will be without pre-conceived notions of the results. The reports speak for themselves and reveal details both expected and unexpected. This is not to say that the researcher will come to the study without pre-conceived ideas of the results, but rather that the process of collecting this data and the reports generated from it will enable free interpretation and analysis by any trained researcher under the same lens and scope as this researcher, to produce similar discussion and results. The questions chosen for such a study must be chosen with care, to be accessible to the participants, to allow for flexible responses and representations, to encourage free response, and to allow for reflection afterwards (Goldin, 1999).

Every effort was made to control, whenever possible, several variables at play during these interviews. Variables under the control of the researcher are the tasks assigned, the setting of the interviews, the subjects of the study, the physical materials available to participants, and the time allotted for each interview. Also to be considered is the extent to which the researcher's hints and suggestions can also be controlled. While a precise script, which is never deviated from, would allow for reproducible interview conditions in the future beyond that of an unstructured interview, this would not allow for as much flexibility in response to participant work. For these reasons, the interviews were conducted in a semi-structured manner.

The selection of participants reflects an effort to control the prior knowledge of the subjects, however this was only controlled here to the extent that the prerequisites for each course allowed. Beyond this, participants had various previous instructors and other courses that could be reflected in their knowledge of individual definitions as well as proof writing in general. However, the variety of participant prior knowledge was a conscious choice on the part of the researcher to yield multiple perspectives on the proofs, rather than to have a sample in which the previous instruction could sway the general results. The purpose of this study was to discover the varieties of thought processes that exist among many different students and to compare both those in the same proof-writing course, as well as those who had different instructors and designs to their proof-writing courses.

During any study, a researcher must choose what aspects of the raw data to observe and later analyze. This researcher chose to observe spoken words, writings, drawings, and physical actions, which were all observable via audio and videotaping. Other aspects, such as thinking, reasoning, cognitive processes, and meanings can only be inferred from the observed aspects and reflective questioning of the participants. This list of observable and inferred aspects is modified from the writings of Goldin (1999, p. 526).

Participants

Participants from both courses were asked to volunteer for this study and were given incentives to participate. All students who volunteered for each portion of the study were allowed to participate.

This study was broken into three phases. The first phase consisted of a questionnaire on multiple intelligences given to all students in both courses. A total of 14 students from MATH 305 and 19 students from MATH 406 participated in this portion of the study by choosing to complete and return the questionnaire. Participants received five extra credit points in their respective courses for completing the questionnaire within a given time frame, as an incentive to encourage participation within a reasonable time period. In the interest of fairness, and as required by the Institutional Review Board, any student who did not wish to participate was also given an extra credit opportunity in the form of an extra homework problem to be scored by their instructor. Students were told that there would be a second portion of the study in the form of task-based interviews in which they could participate only if they filled out the questionnaire, but that they were in no way obligated to continue their participation in the study.

During the second phase of this study, respondents to the initial questionnaire were asked to participate in one-hour task-based interviews with the researcher. Some participants from MATH 305 were asked to return for a second interview to finish as many remaining questions as possible. A total of 10 participants from Math 305 and 8 from Math 406 agreed to participate in the initial interview session. Participants were given a $5 gift card to a local store as an incentive.

These 18 participants were all enrolled in their respective courses during the time of the interviews, though one participant did withdraw from MATH 305 later in the semester. From MATH 305, 7 participants were undergraduate students, 4 of whom were mathematics majors. The remaining 3 participants from this course were graduate students, of which 2 were from the mathematics department. From MATH 406, 6 participants were undergraduate students, 5 of whom were mathematics majors. The remaining 2 participants were graduate students from the department of curriculum and instruction. For confidentiality purposes, pseudonyms will be used throughout this work to identify each participant, where the gender of the name matches that of the participant. There were 12 females and 6 males in the main group of 18 participants.

The third portion of the study consisted of two group study sessions during which participants from MATH 305 who had been involved in the task-based interviews, were asked to work together on proof-writing tasks aloud in groups. Participants who volunteered for this portion of the study were given two meeting times to choose from and were separated into two groups based only on their preferred times. A total of 8 participants were broken into two groups of 4 each. Groups were given the same questions, which they saw for the first time at the beginning of the study session. In the interest of fairness, all students in the class were given the review sheet during the class period following the group sessions. Participants received a pizza dinner at the beginning of the study session as an incentive to participate and were also allowed to stay after the hour for additional help on the questions from the researcher.

Pilot Study

A pilot study was conducted at the beginning of the spring semester, 2006. The intent and purpose of this pilot study was to test and refine both the tasks for the research study interviews, as well as the interview protocol.

Participants in the pilot study consisted of 4 graduate students and 1 undergraduate student from the mathematics department at The University of Montana. These students volunteered to participate in the pilot study between late January and early February 2006. As an incentive, all participants were entered into a drawing for a $20 gift certificate to a local store. All 5 participants had taken MATH 305, or its equivalent elsewhere, and other upper division mathematics courses. Also, they all were female, though this was not a purposeful choice on the part of the researcher, but rather just coincidence of those who volunteered.

Each volunteer was asked to participate in a one-hour task-based interview in a quiet room in the mathematics building. Individual interview times were arranged with participants to fit their schedule, and only the researcher conducted the interviews. Prior to the interviews, a protocol was developed with specific directions for the participants to be given at the start of the interview, and choices of prompts to be used during the interviews. A list of reflective questions was also developed at this time. Each participant was given as many questions to work through as time allowed, with only Questions 1 through 3 being tested. Participants were provided with paper and a calculator during the interview, with most participants choosing to use their own writing instruments. The original form of the statement of each question can be found below, followed by a description of how each question was changed as a result of the pilot study and why the changes were made.

<u>Question 1</u>

The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16. Find, with proof, the smallest possible value for a sum and give an example of an arrangement with that sum. (see Figure 1, p. 55).

Sue, a graduate student, was the first participant to attempt Question 1 during the pilot study. She spent most of her work on this question focused on finding the lowest possible sum, which led to her discovery of the ideas for the proof, but the proof itself seemed to be second in priority for Sue to the finding of the lowest number. She found the maximum possible sum through her work as well. Due to her long search to find the value of 14 as the lowest possible sum, the question was changed to include this information in an effort to help participants focus on the proof, rather than the problem-solving portion of the question.

Callie, a graduate student, and Nancy, the only undergraduate student, also attempted this question in later interviews. The new statement of the question that they encountered was as follows.

The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16. Prove that the smallest possible value for the sum is 14.

This is also the final form of the question for the research study and the same diagram as can be found there followed the question. Callie was able to move straight to attempting to find an arrangement with sums of 14, and proved that arrangements of less than 14 were not possible. The new statement of the question helped to guide her work more quickly to the goal instead of a random search for the minimum sum of 14 first.

Nancy began by understanding the sums involved using the given pentagon, and was able to accurately produce an equation to prove that the smallest possible sum would be 14. She then went on to show that 14 was possible. Like Callie, Nancy was able to move quickly through her proof, partially due to the fact that she did not have to first find the value of 14 for the minimum. This value being given did not seem to detract from her ability to understand the question and develop a proof, but instead positively affected her work.

<u>Question 2</u>

*We call a positive integer N a 4-flip if 4*N has the same digits as N but in reverse order.*

a) *Find all 4-flips.*
b) *Prove that you have found all 4-flips.*

Jackie, a graduate student, was the first participant to work on Question 2. The original form of this question was in two parts, as shown above, given at the same time. During the first interview, the researcher discovered that this question was too broadly worded and that a refinement could assist participants in stepping through their work from a small scale up to larger numbers of digits. After a struggle with larger numbers of digits, Jackie was able to go back and prove that there were no two-digit 4-flips, and to limit the three-digit cases, but was unable to complete the proof of the question as it was stated.

Rachel, another graduate student, also attempted Question 2 but in a new form. Here she was asked in one question to prove that there were no two-digit or three-digit 4-flips. With the new statement of the question, Rachel was able to find a proof, using equations, to both parts presented to her. Since there was enough time to continue to just one more part of this question, the researcher asked her to consider whether there were any four-digit 4-flips. Rachel attempted this portion using equations as well, but was unable to finish a proof. This led to the addition of part c to Question 2 in the form it was used in the research study.

Sue and Callie were also given Question 2 in its new form; both were only able to attempt part of the question due to a lack of time. Each portion of the question was given separately. Sue was able to form a proof for part a using equations, where Callie instead used brute force. Sue began part b using the same idea as she had in part a, but ran out of time before she was able to complete her proof. Callie was not able to attempt part b because of lack of time as well, but the new statement of the question seemed to help both participants work from the smaller scale up and better understand the definition presented.

Question 3

A traditional chessboard consists of 64 squares (8x8). Suppose dominoes are constructed so that each domino exactly covers two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes is covering every square of the chessboard without overlapping any of the dominoes.

Consider a generic chessboard of size m x n.

Prove that the generic chessboard of size m x n has a perfect cover if and only if at least one of m or n is even.

Both Callie and Nancy attempted this question during the pilot study. Neither appeared to have trouble understanding the definitions given or the statement of the question. Nancy was unable to finish due to time constraints. Callie was able to understand the need to prove the statement in both directions, since it was a bi-conditional statement. No changes were made to the format of this question.

All participants in the pilot study were able to discuss their thoughts and work aloud with the researcher. They readily shared ideas that came into their minds and there were no periods of long silence in any of the five interviews. The interview protocol appeared to be appropriate and was kept the same for the research study. However, this ease of sharing ideas could have been partially due to the familiarity of the participants with the researcher and their comfort level with proof writing. It later became apparent during the research study that not all participants were as able to express their ideas verbally and this could be seen as a potential limitation in the study.

Other ideas suggested by this pilot study included the addition of the questionnaire after the interviews. This arose later as the researcher considered the strategies used by each participant and noticed already the tendencies of some participants to be more visual and draw pictures, and others to be more algebraic. It was therefore important to the researcher to understand the background and the tendencies of each participant in the research study, as self-reported by these participants. It was also important to further understand their previous experience and comfort level with proof writing. This was in an effort to understand how this played a role in their ability to handle the proofs during the interviews, since not all participants were at the same level of mathematical maturity and experience.

Research Study Design

The research study comprises three sections. All sections occurred in the spring semester of 2006, after the pilot study had been completed. All participants were volunteers from the two classes specified. The following describes each phase of the study in detail, along with the reason that each section was chosen for the study.

Phase 1

All students in MATH 305 and MATH 406 were given an initial questionnaire to assess their primary intelligences from the following list of multiple intelligences: verbal/linguistic, logical/mathematical, visual/spatial, bodily/kinesthetic, interpersonal, intrapersonal. The purpose of this questionnaire was to ensure that participants in the study varied in terms of these intelligences in order to lessen the impact of this variable on the results of the study. As stated, this phase was included in the study to measure the possible types of work that would result from each participant; for example a logical minded person might stress the use of equations and logical manipulations, where as a person strong in visual intelligence may instead first choose to view pictures and other visual representations of a question.

Phase 2

After data was compiled from Questionnaire 1, respondents were asked to participate in a task-based interview in which they would be given several proofs to work on. Interviews were held in April 2006, which was the last full month of the spring semester, to ensure that Math 305 students would have seen most styles of proof prior to the interview.

Interviews were approximately one hour in length, during which time students worked through as many of the five questions as possible. Participants were asked to work through proofs aloud and wrote what they felt was necessary. A description as well as discussion of the reason for choosing each question will be given below.

The task-based interviews are the true heart of the data collected for this study. The purpose of the interviews was to observe participants in the act of proving new statements. The goal was to capture the beginning stages of proof writing in an unfamiliar question to attempt to bring about the maximum amount of strategies used, and to analyze how participants dealt with new terms and their own success and failure in first working with these terms.

The setting of the task-based interview allowed for some interaction on the part of the researcher to bring forth those aspects of work that were not directly observable, such as meaning or otherwise unspoken thoughts. This allowed the researcher to ask participants to explain their ideas during the interview, and to help guide participants to correct minor errors that may have otherwise prevented their finishing a proof. These suggestions, hints, and questions were consciously kept to a minimum to allow participants to work as freely as possible without interruption of their thought processes. However, as described later in the limitations of this study, these promptings were not all consistent between participants.

After each question was completed, participants were asked to reflect back on their work and their strategy use. They were also asked questions about whether the question resembled any they had seen before, or where the ideas they had used had originally been learned. This allowed more unobservable aspects to come forth, and was a valuable part of the data collected in many cases.

At the end of the initial interview, before leaving the interview room, participants were given a second questionnaire designed to gather information on their coursework background, self-reported grades in mathematics courses, and learning style preferences. This questionnaire can be found in Appendix D.

MATH 305 participants were also asked to return for a second session to gather more information on the questions they were not able to work on during the initial session. All who were willing to come back for another interview were allowed to do so. A total of 6 participants chose to return for a second interview session, varying in length from one-half to one hour. These interviews were conducted in the same manner as the initial interviews.

Phase 3

The last phase of the study consisted of a group study session. During this session, Math 305 participants were asked to work together on an assigned review sheet from their instructor. Designed by the researcher and the instructor, the questions were intended to review for the participants' upcoming exam in the class, as well as contribute to this study. They were all proofs to be completed and were similar in style to those used in the textbook for the course and those proofs done in class by the instructor. All were considered by the instructor to be within the capacity of the participants to complete. A copy of this review sheet was given to all students in MATH 305 the day after the group sessions were completed. The participants saw the review sheet for the first time when they arrived for the group study session.

In order to facilitate group discussion, an informal version of these sessions was conducted throughout the semester in the form of group time to work on homework outside of class. These informal sessions were not part of the data for this study; rather, they were conducted in an effort to make participants comfortable with each other and with the researcher, and in working aloud together on proof-writing activities. The researcher was present during the informal sessions and answered questions as they arose. During the actual group sessions for the research study, however, the researcher only observed the group activity and did not participate or interact with the participants in any way.

To complete the data for the Math 305 participants, the researcher also observed the classroom daily for several weeks. During this time, notes were taken on what processes the instructor was specifically teaching the students to use during proof writing. Additionally, 5 participants allowed the researcher to collect and photocopy their course notes for a complete picture of the course.

Instruments

As stated, Questionnaire 2, as well as the interview protocol can be viewed in the appendixes of this document. Permission was granted to use the questions found on Questionnaire 1 in this study, which was reproduced from a portion of an online survey (Gay, 2002). However, reproduction of the questions in publication was forbidden. The full version of the survey can be found online.

The purpose of the remainder of this section is to describe the questions used for the task-based interviews in detail, as well as to give an understanding of the reasons why each was chosen and why each was appropriate for this study. Since only the first four questions that were designed for use in this study were actually attempted by participants, they will be described in detail, but Question 5 will be omitted from this discussion.

The questions selected for use in the task-based interviews were adapted from textbooks as well as on-line sources; each source is listed with the corresponding question below. Questions were chosen to be challenging, yet accessible to all students interviewed. Each question had been pilot tested, as described previously, for difficulty level, wording and understandability, as well as for the appropriateness of the mathematical content. All questions were written at a level appropriate for the participants, needing only previous knowledge in basic mathematics, such as an understanding of basic multiplication, equations, and shapes. In addition, questions were chosen so that all could be proved using direct proofs, to eliminate as a factor any confusion caused by the need for indirect proofs. These questions offered the potential for a variety of proof techniques and possible avenues of thought, for example, Question 1 could be proved using only equations, or could be proved using pictures and logical arguments without the use of any equations. Questions 1, 2, and 3 were to be used by as many participants as possible. However, Questions 4 and 5 were intended for use only if a participant completed the other questions quickly and needed more challenging material or simply other material to work on. No participant reached the point of attempting Question 5 during either session, and only 5 participants attempted Question 4. Again, for this reason, Question 5 is omitted from further discussion in this chapter.

<u>Question 1</u>

The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16. Prove that the smallest possible value for the sum is 14.

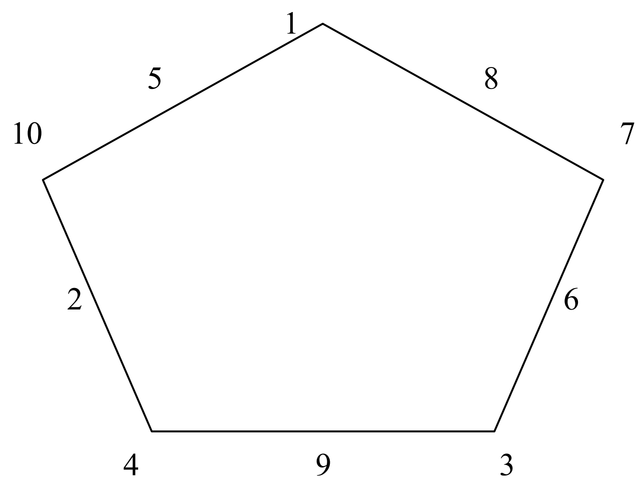

Figure 1: Pentagon given to participants in Question 1

Question 1 involved a new concept that was designed to offer a visual picture included with the question, which was unique to this question alone. This question was posed in the 2004-2005 USA Mathematical Talent Search, but has been modified slightly from the original question. It presented an example of an arrangement and provided the end result of 14 as the smallest possible sum, but left open for the participant to determine the best way to prove this result. This question was rewritten during the pilot study to help focus the work of participants and get more directly to the data that this study was designed to observe.

This question was included specifically for its use of visual imagery, as well as for the open choices for a valid proof. Participants could use several different approaches to find a valid proof, including pictures, logical arguments, and equations, all of which were seen in use during the study. There is at play both a basic shape with a low-level proof using pictures and argumentation for choices, as well as a higher-level optimization approach that could be used in conjunction with logical argumentation or with specific equation manipulation, where no pictures would be necessary in any way. This facilitates an interaction between this question and the prior experiences of the participants. Those who had taken an optimization course in the past were much more quick to approach this question in that fashion than those who did not have such a background, or who were not inclined to think and reason in this manner.

Question 2

*We call a positive integer N a 4-flip if 4*N has the same digits as N but in reverse order.*

a) *Prove that there are no two-digit 4-flips.*
b) *Prove or disprove the following statement: There are no three-digit 4-flips.*
c) *Prove that N = 2178 is the only four-digit 4-flip.*

Question 2 was designed by the researcher as a complement to a question on the existence of a similar idea, a 9-flip number, posed to her during a prior comprehensive exam (see Gardiner, 1987). Each portion of this question was on a separate page given one at a time to the participants, and all were revised after the pilot study. Not all participants who attempted part a were asked to also attempt part b, and similarly for parts b and c. This was due to either time constraints or a decision by the researcher that no new results or strategies would arise because of confusion and lack of success on the previous portion of the question. Some participants were given the option to continue to other parts of the question but choose on their own not to do so.

Question 2 provided a definition of a new term, unfamiliar to participants. Unlike Question 1, though, no example was given in the beginning of the question from which the participants could work. This was done intentionally to study how participants would go about finding an example, if they did indeed try to find one at all. The question was broken up into multiple steps to draw out various aspects of the proof. Additionally, the second portion of the question left to the participant the responsibility of deciding whether to prove or disprove the statement. This tested participant's ability to verify the statement itself. In the final part of the question, participants were given an example of a 4-flip and asked to prove that it was the only four-digit 4-flip. This portion was added to the question to try to elicit participant work differing from that used during the beginning of the question.

The goal of this question is to address a completely new definition with no references to known objects, such as the pentagon in Question 1, or the chessboard in Question 3. This allowed for observations of participant-created understandings and written interpretations of the definition. It also challenged the participants to think about numbers in a whole new way, and to use ideas they may have learned previously to work with this definition in some concrete way. The extensions to parts b and c, in particular, added a depth to the mathematical structures at play in this question. Rather than being able to perform a brute force search of the possibilities, participants had to move to a generic understanding and possibly a grouping of the remaining options. There were several possible proofs, including the use of equations, or breaking the interval into smaller parts for closer analysis. This question also allowed insight into the internal understandings of numbers and number manipulation existing in each participant.

Question 3

A traditional chessboard consists of 64 squares (8 x 8). Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A perfect cover of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes.

Consider a generic chessboard of size m x n.

Prove that the generic chessboard of size m x n has a perfect cover if and only if at least one of m or n is even.

Question 3 was inspired by a question in a textbook written by Diane Schwartz (1990, pp. 5-6). The wording for the definitions and the set-up of the question are taken from this text. However, the desired proof itself has been slightly modified to address only one specific aspect of the proof and to specifically ask only for a proof, and not the development of the theorem.

This question allowed the researcher to view how participants dealt with a two-directional statement, which is the only example of such a proof in the questions. The proof itself could be fairly straightforward and could lend itself to either a study of cases or a view into the details of the two-directional aspect of the statement, by actually proving the statement in both directions. Along with a new definition and a potential visual image, this question also brought forth deeper understandings of even and odd numbers and their interactions. It also allowed participants to connect other notions they had previously learned and stored internally to be used for some aspects of the proof.

Question 4

Let x and y be two integers.

We say that x divides y if there is an integer k such that y = k x. Consider three integers a, b, and c. Prove the following: If a divides b and b divides c, then a divides c.

Question 4 was not found specifically in another source, but rather is a common question in any introductory proof course, or number theory course. In this question, participants were essentially asked to prove that the relation defined here is transitive, however the terms relation and transitive were not used since most participants would not have seen such ideas prior to this study. This meant that Question 4 was another question that posed an unfamiliar definition, allowing for observation of participants' initial inquires, and understandings of the statement.

This question encompassed ideas from the previous questions as well, but in a different setting. It provided the possibility of trying examples, since none were given, as well as giving a straightforward result, which could yield an elegant proof. Question 4 is unique in its required use of equations, which allowed a view of the manipulation of equations from some participants who were otherwise reluctant to do so. Through viewing the responses and interactions with this question, participants' ability to deal with a definition not only unknown, but also counter to their pre-conceived notions could be tested. Participants' abilities to reason on a larger scale were seen through this question as well as their ability to take skills that seem unrelated and apply them to a new situation, such as extracting information from a definition or carefully defining new variables.

Data Collection Procedures

Data collection proceeded in three phases.

Phase 1

During this phase, all students from each of the two classes were asked to participate. The questionnaires were distributed at the end of one class period in mid-March of 2006, with the permission of the instructor for each course, and were completed and returned directly to the researcher at the end of the following class period.

Participation was voluntary, with the incentive of extra credit, along with an opportunity for extra credit for those who did not wish to participate in the study. A consent form was also given to each student and those who wished to participate in the study returned the completed consent form with their questionnaire. A copy of this consent form is included in Appendix C.

Phase 2

Interviews were conducted after the completion of Phase 1, during the month of April, 2006. Interviewing was done on multiple days throughout the semester and every effort was made to complete interviews with MATH 305 participants during a common time frame to eliminate differences due to the material they were learning in their course. However, this was not always possible as participant schedules did not all allow for interviewing during the first two weeks of interviews. All interviews were complete by the end of April.

Interviews were held in one of two rooms in the mathematics building, which were quiet, private, and without disruption from others, and specifically reserved for this study during the normal school hours for the researcher's use. There was only one exception to this, in that Andy was unable to meet during normal school hours and so that interview was done in a separate room in the mathematics building on the weekend, but was still quiet and without interruptions. The interviews were all done one-on-one with the researcher.

Participants were aware of both audio and videotaping occurring in the room, which were both set up and run by the researcher. The audiotape-recorder was placed on the table in view of the participants and the video camera was set up to the side of the room and recorded only the participants' hands and the papers on which they were writing. Participants were provided with paper, calculator, and writing utensil, but many chose to use their own personal writing utensil. All interviews were scheduled for one hour and most were completed within 5 minutes of this time limit. The interviews began with the researcher reading a set of instructions, which can be found in the interview protocol, asking the participant to work aloud as much as possible and reminding them of the purpose of the study.

Since the purpose of the study was to understand the proof processes used by the participants, the researcher corrected basic algebra and computational mistakes as they arose in order to continue progress on the question. The participants defined the pace of the interviews. They were allowed to work on a question as long as they were making progress, had not become stuck with no new ideas, or had completed what they felt was a proof for the question. After each question, the researcher asked the participant to reflect on the work they had completed and the strategies they had used. These questions were also scripted in the interview protocol.

The researcher varied the order in which the questions were given to the participants, again in an effort to reduce the impact of such a variable on the results of the study. Participants completed between 1 and 4 questions during these sessions, some participants returned for a second session and completed the remaining questions at that time.

Phase 3

The group sessions were conducted at the end of April, after all initial interviews had been conducted for the MATH 305 participants. The researcher gathered the participants in a study room in the mathematics building on a weekend, to eliminate disruptions from others. Instructions were given to work together and aloud as much as possible, and participants were given the review sheet and asked to decide together which question to work on first. The researcher then sat down at the back of the room and observed only, with no other input to the participants. There were two video cameras used to capture the work done on the white board in the room, as well as two audiotape recorders used to capture the spoken words of the group. The researcher operated all tapings, and no other persons were present other than the participants and the researcher.

Participants were given paper, markers for the white board, and the review sheet. They were allowed to use their own writing utensils and calculators, as well as their textbooks, and any other resources they had available at the time. The sessions were recorded for one hour, after which time the researcher allowed participants to ask questions and work together outside of the scope of this study. Both group sessions were done in one weekend, two class periods prior to a test in MATH 305, for which the review sheet was designed.

Treatment of Data

Data from Questionnaires 1 and 2 were entered into a spreadsheet for ease of analysis. The first questionnaire was used to ensure that participants varied in terms of their self-identified intelligences. All participants who volunteered for Phase 2 were allowed to participate because they varied in terms of their intelligences and there was no one tendency that was prevalent among the volunteers. Data from Questionnaire 2 were used in the analyses to describe the prior experiences of each participant and for comparison purposes.

Audio and videotapes were copied to secure the information onto a second medium. The videotapes were transferred to DVD form for ease of viewing on computer formats. The researcher made all copies and the original audiotapes were used for transcription and accuracy checks. All written materials from the interviews were collected and scanned into the computer for storage, and for insertions into the analysis. The originals were used for comparison purposes and to add details to the transcripts.

All audiotapes were transcribed by the researcher, or by hired assistants. For clarification, the researcher then listened to any inaudible words or portions of the interviews again. Remaining inaudible portions were noted in the transcripts, but were few. The researcher reviewed transcripts completely a second time, with the audiotapes, for accuracy. After all transcripts were complete and accurate, the researcher then used the DVD recordings to add notations, noted in brackets in the transcripts, which referenced the participants' writing during the interview and connected the written documents with the verbalization of their work. An additional pass was made by the researcher with the DVD recordings to add notations of the timing of individual pauses of silence, and for an additional accuracy check of the transcripts.

Prior to the study, the researcher constructed a list of expected strategies. However, this list was only a rough idea, and the actual strategies that arose were much more detailed than expected. After the transcripts were completed, the researcher revised this list into a more thorough representation of the actual types of proofs that occurred in each question, as well as the detailed strategies that were seen in use. Still, however, this list was not complete. To clarify and add all remaining portions, the researcher began to analyze the data itself.

The analysis began with a summary view of each individual participant. Using the transcripts, the researcher first summarized the overall ideas of each participants' work. The second pass at this analysis by participant consisted of a detailed explanation of each step that each participant took, and what ideas, strategies, and difficulties each had along the way. This was broken into discussions for each question that the participant attempted and later general summaries were added to link all questions together for each participant. The last addition to this analysis was a link to the group work and other observations of the participants from MATH 305 by the researcher to give a full picture of whether the behavior observed during the interviews was typical for the participant. All of these ideas make up the first portion of the analysis chapter.

The second stage of analysis was to view each question across all participants. This began with a view of each question individually, looking at the summaries of each participant who attempted that question. The list giving general strategies used in each question was updated and revised at this time. An overall grid was then made for each question in a spreadsheet. This included a list of all potential strategies, broken into broad categories and then further listed by individual details specific to the particular question. These specifics were designed to be observable or inferable attributes that could be pulled from the transcripts of each participant as they arose.

After the list was made and refined, each transcript was read again and the frequency of each strategy, key idea, and difficulty was documented. Each question was viewed independently. After all participant work had been documented for a particular question, a summary was written of the overall ideas for each participant, and the analysis by participant was checked for accuracy against the overall grid. Any discrepancies between the two were checked via the transcript and, when necessary, against the DVD recordings and were corrected. Each question was completed individually and the analysis by question was written as each grid was completed. This included a separation of the participants into those who successfully found a proof to the particular question, and those who did not, with a comparison of their work among each group and between each group. Portions of the overall grid are included in the analysis by question to view the actual data on frequency and total count for particular strategies of interest.

Lastly, the final portion of analysis was completed with the overall findings chapter, which includes a view of each individual strategy use across all questions and all participants. This describes how each broad category of strategy was used, both by successful and unsuccessful participants on particular questions, as well as whether the use of the strategy was helpful to the participants, or hindered their progress. An overall conclusion is given for each category of strategy for its general success in use.

To establish reliability and validity in data analysis, a second outside person was asked to randomly analyze transcripts and complete the overall grid for these participants. The researcher described the overall grid and what items were observed and inferred from the transcripts, and gave an example with one transcript for clarity. The outside person then randomly chose five transcripts and one question on each of these transcripts to complete. She coded the transcripts via the categories on the overall grid and additional labeled each participant as successful or unsuccessful in the particular question she coded. Any discrepancies in the coding, which were few, were discussed and resolved.

The group work portion of the data was video and audio taped from two vantage point with each type of device. These tapings were not transcribed, but were used as an source of comparison for each individual participant's strategies and tendencies. Additionally, the researcher's personal observations in and out of the classroom in MATH 305 provided another source of comparison.

Limitations of Study

The act of observation can inherently change the behavior being observed. In this way, one limitation of the study could be found in the actual collection of data. Participants could possibly use different strategies given that they were being observed and may have felt uncomfortable attempting to express ideas or work that they were unsure of.

There is always a potential for researcher bias in any descriptive study, since the results are not a quantity to be measured and analyzed statistically. This researcher addressed this issue by first making clear the preconceptions that were brought to the study, rather than assuming that none existed. Furthermore, the researcher allowed the data to guide this list of expectations further and was open during the interviews to whatever data arose. A last check for bias was evident in the use of a secondary person to analyze the data at random to check for any discrepancies. While bias cannot be fully removed from any study, every effort was made to minimize the effect during this study.

During the collection of data for the interviews, there were time limits imposed on the participants. These were necessary for the participants' busy schedules as well as to keep the interviews consistent. However, this prevented several participants from finishing proofs, and was expressed as a concern to some in feeling restricted for time.

The setting itself could have had an effect on the participants. The interviews were all completed in the mathematics building, which could have made the participants feel as if they were being tested, or made them feel as if this data had a bearing on their grades in the course. The researcher made every effort to address this issue with participants, to reassure them that data was confidential and their instructors would have no access to the data during the semester, or in any way that would personally identify the participant. The central location was used because it was convenient for the researcher and all participants and there was the availability of a reserved room assured to be quiet and without disruptions.

Some limitations that could be addressed in further studies include requiring participants to formally write out proofs, to address the difficulties in defining successful participants based on their oral and partially written proofs. Additionally, the interview protocol could be refined further to include more options for prompting participants who do not readily express their ideas aloud, and to address the inconsistencies which occurred when errors were corrected or participants were allowed to proceed. The researcher acknowledges that the promptings were not consistent across all interviews, particularly in the first four interviews, and would suggest further refinement of such ideas for any further study.

Lastly, this study included 18 participants, in one university, during one semester. In this way, some unobserved aspects could have influenced the data, such as the particular population or the common instructors. Any qualitative in-depth study will encounter such limitations. Further research is needed to confirm these results among other populations, with additional participants, to properly generalize the results.

CHAPTER 5
ANALYSIS OF RESULTS

Analysis by Student

Lisa

Lisa was a MATH 305 student. She was a senior majoring in mathematics. Her previous coursework included calculus I and II, and statistics. Lisa participated in two interviews. During the first, which was 1 hour in length, Lisa worked on Questions 1 and 2, including all three parts of Question 2. However, not all parts were completely correctly; that is, she did not have a complete argument for part b. Participants were not instructed specifically to write down formal arguments. Instead, they were asked during the interview to describe verbally what would be needed to complete a proof for an instructor. Lisa chose to complete her work verbally. However, some students did choose to write their arguments down.

During the second interview, which was only 16 minutes in length, Lisa worked on Questions 3 and 4. Her work was organized and mostly complete. She kept track of her goals and her thoughts well, not stumbling or forgetting her path along the way. Unlike the first interview, her proof for Question 4 was completely written down. Lisa produced it very quickly (in 1½ minutes) without any discussion aloud or hesitation.

Question 1. Lisa clearly understood the question from the beginning. She read the question to herself quickly and spent very little time going over instructions. Instead, she dove into her work right away. Lisa started by redrawing the given picture of the pentagon. She looked for sums of 14, trying to find numbers that were in close proximity on the pentagon and added to 14. Seeing that this did not seem to be working, she moved on quickly.

Next, Lisa searched for sums of 14 in general, writing possible combinations in a list. At first, her list appeared random, just guessing and checking what could work, but she did develop a system for her choices later. Lisa paused during her search and checked to ensure she understood the question. Lisa then drew a new pentagon, placed one combination on it, and continued to search for sums of 14. At this point, she discovered the key to placing the numbers on the pentagon, that large numbers must be placed on the sides rather than the vertices. Her reasoning included the argument that smaller numbers were useful in many more combinations than the larger numbers. Though she did not make a formal argument, she did continue her work using this discovery and, as a result, later found the solution for 14.

Lisa made a list of all the combinations with sum 14, organized by the highest number in each combination. She worked from highest to lowest (i.e., 10 down). She also made sure to line up the common numbers in the combinations she was choosing to use. When Lisa slowed down, the researcher encouraged her to continue this line of thought.

Lisa finished the list of combinations necessary to fill in a pentagon correctly. She was able to monitor her progress through the lists along the way, and was more systematic after her discovery mentioned previously. She wrote down only short notes and did not readily express her ideas aloud. But, she clearly had a system that she was working through. Lisa did not seem to gain much benefit from the drawing, instead preferring to visualize the connections in a list form, where vertices could be though of as connecting the two ends of her rows.

The researcher clarified what had been found thus far, but it was clear that Lisa already knew that she had found what she was searching for. Once Lisa had the solution in list form, she placed the appropriate numbers on a pentagon. However, again, it was clear that she had been convinced of her result without needing to draw the pentagon. She may have been visualizing the pentagon in her head, but did not ever mention that aloud. When asked if she was finished, Lisa recognized that she was not yet done with the proof. The researcher followed by asking, “Okay, so how would you go about showing that you couldn’t get anything less than 14?” (Transcript 1, lines 114-115). Such a question could have been too leading, but the researcher felt that Lisa had already known what she needed to further show and so was merely prompting her to continue.

Lisa recognized that 13 was the smallest possible sum to be checked. The argument that those less than 13 were not possible was very quickly resolved in her head and she only mentioned it when prompted. She worked on finding combinations of three numbers with sum 13, following the same system of writing combinations in a list. Starting with the combinations with larger numbers first, she worked backwards and, in a few steps, was able to see the proof that 13 was not possible. The researcher asked her to clarify her statements, which unfortunately led the student astray. So, the researcher then recapped Lisa's work and verified that she had been correct.

Lisa said, when asked, that she did not know how to write up her proof formally. She seemed convinced that her work was accurate, but still did not know how to formulate a proof in writing. The researcher went over Lisa's work again, and indicated that she had the makings of a valid proof. Lisa then recapped her own work but was clearly unsure of the validity of her proof that 13 was not possible. She seemed to desire some other way of writing it up, as if her method would not be convincing enough, or was not rigorous enough. However, she did eventually agree that she was finished with the question.

Overall, Lisa was somewhat successful at finding a proof for this question, though she did not complete all of her justifications and was not sure of the final details. Her success had come from carefully monitoring her attempts, organizing her work, and having some clear insights into the overall structure of the question. Lisa was able to make a plan and follow through with that plan. She was also able to keep her work tidy and follow her reasoning to the result. Her only struggle was being uncertain that she had reached a valid proof, particularly in regards to clearly showing that 13 was not possible. This may have been due to the format of her proof, which was unlike most formal proofs she had seen in MATH 305.

Question 2. After the researcher read the question, Lisa spent a great deal more time understanding this question than she had the first. She did not understand the definition of a 4-flip. The researcher reworded the definition for clarification. Lisa said that she understood, but did not seem sure of this. She then looked at an example of a three-digit number, which the researcher corrected as not being part of the question being asked. Lisa seemed to understand that the question was dealing with two-digit numbers but said, "Okay, well I was just trying to see how it flips it" (Transcript 1, line 212). As requested, though, she moved to a two-digit number, writing that 34 x 4 = 144, but expressed that she still was not sure of the definition. The researcher asked Lisa what she would expect of a 4-flip and it was then finally clear that Lisa understood the definition as she said that for $N = 36$, she would have wanted $4N = 63$.

Lisa then moved on to writing the equation shown in Figure-Lisa. 1.

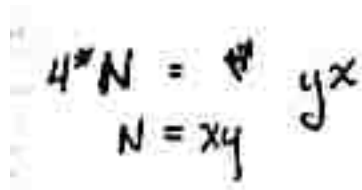

Figure-Lisa. 1: Representation of flipped digits in Question 2

She was apparently trying to represent the digits of *N* and how they would be flipped. However, once she seemed to understand what she was looking for, she abandoned the equation and moved on to limiting the possibilities. Lisa soon saw that *N* must be less than 25, recognizing that *N* was limited since *4N* would need to remain a two-digit number. She checked the example *N* = 24 and explained why it was not a 4-flip. The researcher recapped, unintentionally stating information that Lisa may not have been aware of previously, that the only remaining numbers were 10 through 24. However it was not clear whether or not Lisa had already reached this conclusion on her own.

Lisa said that a proof could consist of checking all of the possibilities through exhaustion. She began to examine the cases 10 through 24 in her head, making a list to organize her work. Her list included the numbers 10 through 24, along with the calculation of the number times 4, and what she would want that calculation to be, if it were to be a 4-flip. Lisa completed the proof by exhaustion, making it clear that she knew she had finished the proof by saying, "So, it wouldn't work for any of them" (Transcript 1, line 281). The researcher gave a synopsis of Lisa's work before moving on the next portion of the question.

On part b of Question 2, Lisa understood the question immediately after reading it to herself. She attempted to tie in her work from part a to begin her proof, but made an error in her calculations, writing that $N \leq 599$. When asked, Lisa said that this conclusion had resulted from her idea that 4 times 600 would equal 1000. The researcher helped her correct this error and Lisa was then able to correctly limit the possibilities to $N \leq 249$. The researcher asked if Lisa thought she would want to prove or disprove the statement, but Lisa was not sure which she was planned to do at that point. After the researcher summarized that *N* could only be the numbers 100 through 249, Lisa wrote the equation $4*xyz = zyx$, similar to one she had written in part a. However, that seemed to be just a way to keep track of the goal, since she never directly worked with the equation later in her proof.

Unlike her work in part a, where she had checked all the cases remaining after this initial limitation, here Lisa instead decided to eliminate large portions of the remaining numbers first. She tried to narrow down the options for *N* by looking at the possibilities for each digit. Again, she did most of her work in her head, writing little down except notes to keep track of what she had tried and what she had left to do. Lisa first expressed the idea that if the middle digit of *N* were 0, then *4N* would have a middle digit of 0 as well. The researcher suggested that other numbers might work, and that there could be carry over from the last digit, *z*, of *N* as well. Lisa agreed and decided to limit the last digit to $z \leq 2$ and the middle digit to $y = 0$.

Lisa organized her work, stopping to write down what she had done so far. She then described her thinking, showing evidence that she was monitoring her progress. The researcher asked questions to clarify that Lisa was working on the case where there would not be any carry-over from the last digit and trying to limit the options within that case. Lisa again stated that the middle digit must be 0, because with no carry over it was the only number that remained the same when multiplied by 4. Limiting herself to only this case led her to the conclusion that the first and last digits must be the same, for which she only needed to try 101 and 202 (see Figure-Lisa. 2).

Even though Lisa had not completed a valid proof, it seemed to the researcher that she felt she had checked all possibilities. However, Lisa did not explicitly say that she was finished and the researcher wrapped up this portion of the question without Lisa having the opportunity for such a statement. Lisa again was not sure how she would write up her arguments formally. She never recognized her earlier error in only considering the scenario of no carry over from the last digit, and did not even go back to consider other options after looking over part c. She seemed to abandon it once it was decided that part b was complete and there was an apparent disconnect between parts b and c, perhaps because she did not have to consider the middle digit being the same in both *N* and *4N* for part c.

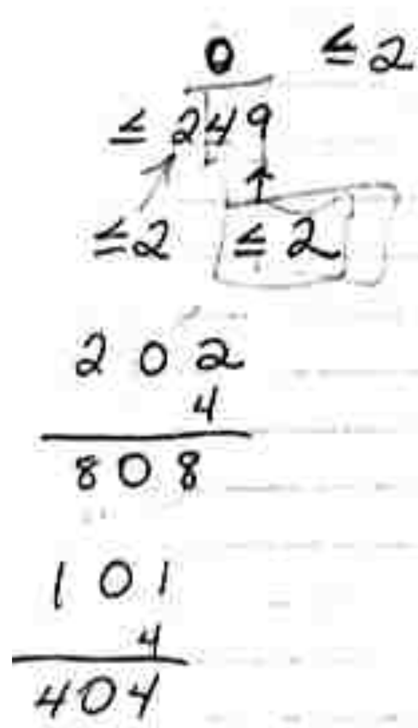

Figure-Lisa. 2: Final limitation of digits and check of 101 and 202 in Question 2

On part c of Question 2, Lisa first wrote the equation 4**xyzm* = *mzyx*, as she had done in both the previous parts of the question. She also verified that 2178 was a 4-flip. She proceeded directly to limiting the possibilities to $N \leq 2499$, using the same logic as in parts a and b. From there, she did not seem to have a clear plan of attack as to how to proceed. She looked at two examples, the two extremes of her range, and then said that she was thinking of the entire process of multiplying "and the carry-overs and everything else" (Transcript 1, line 486). She circled the last digit, 9, of 2499 and further noticed that for the range 2000 through 2499, the last digit could be 8 so that $4N$ would end in the digit 2. After checking all ending digits 0 through nine, she concluded that only 3 and 8 would give 2 as the last digit of $4N$. Considering the other portion of her original interval, she correctly identified that 1000 through 1999 could not be 4-flips, since $4N$ could not end in 1. Lisa then wrote notation representing her two cases for ending digits in the interval 2000 through 2499, 2 *y z* 8 and 2 *y z* 3. She next considered the second digit, *y*, of each possibility and noted that the digit could be the numbers 1 through 4, based on the limitation that $N \leq 2499$, forgetting to include 0 in her list as well (see Figure-Lisa. 3).

Figure-Lisa. 3: List of four-digit options with third digit unknown in Question 2

She tried to connect the number N to its flip, but lost some of the rules in the process, just as she had in part b. However, she recovered, not ever mentioning her error in part b, but moving past it anyway.

Lisa began a list of cases to be checked. Systematically, she went through her list of choices for the second and third digits and eliminated options as she went. She further refined her lists, organizing her work and crossing off the options she had ruled out. Lisa seemed to be able to keep track of her work and stay organized fairly well. She was also able to connect earlier stated rules at later times, helping her to further limit her cases as she went. Once she had narrowed down the options to a small list, she checked those remaining by hand.

Lisa made a note of the one option that she knew would work, 2178. She explained that she had also made notes to herself to remind her of the carry-over in the multiplication by 4. Lisa continued working through her list. During that process, she seemed to be developing equations in her head that she was solving. Evidence of this was shown when she wrote equations involving the missing digits that she was trying to find in each of her cases (see Figure-Lisa. 4).

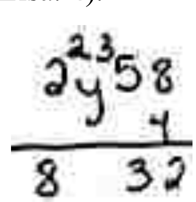

Figure-Lisa. 4: Example of calculation in Question 2

When considering the particular calculation in Figure-Lisa. 4, she said, "you can't multiply a whole number by 4 and add 2 to get 5" (Transcript 1, line 616). While she worked, she noted that the overall plan was to eliminate all the remaining choices. She finished checking all the options in her list and indicated that she was done with the proof.

When asked, she was able to recap her work and follow all of her steps. She never got lost during her descriptions and recalled all of her arguments along the way. While summarizing what she had done, Lisa mentioned trying to understand the question, limiting her cases systematically, and keeping track of her thoughts along the way. She mentioned that in Question 1, she had worked down from the larger numbers, where in Question 2, she had worked with the smaller numbers first. She also said that she used lots of examples because "it helps get an idea of what you are looking for" (Transcript 1, line 725). This indicated that she knew examples did not constitute a proof, but could be used to guide a proof.

Overall, while Lisa did have difficulties first understanding the question, she overcame those difficulties to form a clear proof of two of the three parts of this question. She made use of examples, equations, looking for patterns in the numbers, proof by exhaustion, and splitting the question into smaller intervals that she could prove. Lisa looked back at her work from the previous parts of the question and was able to recall her reasoning throughout. Her only struggle came in part b of the question, where a few errors caused her to leave the proof incomplete. She never realized this, but was able to move past it to find a proof for part c. Lisa was always organized, kept track of her work in lists, and monitored her progress well throughout this question, as well as the entire interview.

Question 3. The second interview began with the researcher reading the question aloud. After only a moment pause, Lisa drew an 8-by-8 chessboard and drew in dominoes to form a perfect cover. When asked, she said that she was thinking of the chessboard and the dimensions of the domino. She also said that she understood why the statement was true and could see for herself that at least one dimension had to be even. Lisa illustrated this by saying,

> And so, I can see where the proof would be correct, because if one was even, then it'd be divisible by 2, so say *m* is even. It would equal 2*k* [writes $m = 2k$], which is some number, which would work, because the dominoes are two long and only one wide. So, *n* could be whatever number, and that would just equal the number of dominoes, [writes *n* = # dominoes, pause] that are side-by-side. (Transcript 23, lines 24-28)

She had written that *n* would equal the number of dominoes, though the researcher believed that she was referring to the number of dominoes laid across the chessboard in one of the dimensions, as she had pictured them laid vertically, rather than the total number of dominoes on the entire chessboard.

Lisa said that she did not know how to go about proving the statement. She rewrote the statement in notation, including recognition that it was a bi-conditional. Her work is shown in Figure-Lisa. 5.

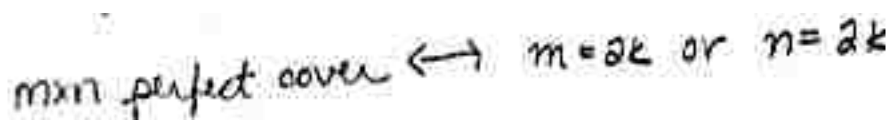

Figure-Lisa. 5: Indication of knowledge of bi-conditional statement in Question 3

After drawing a picture of a domino and noting its dimensions, she wrote that *n* x *m* would be *1n* x *2k*, if *m* were even. She compared the ratio of the dimensions of the domino with that of the chessboard. After a lengthy pause (40 seconds), she said that she was still unsure how to go about proving the statement. Lisa drew another example, this time of a 3-by-5 chessboard. She filled in the dominoes and noted that one square would be left uncovered, saying that if both dimensions were odd, the cover would not work. When Lisa said that she could not think of anything else to try, the researcher asked her to continue with the idea that both dimensions being odd meant that there was no cover, and to explain why that was true. Lisa said this was because there would always be one leftover square, and there would need to be two remaining squares to place another domino. When asked if that was true for all odd-by-odd, not just her example, Lisa wrote out the equations $(2k + 1)(2l + 1) = 4kl + 2k + 2l + 1$ and $(2k)(2k + 1) = 4k^2 + 2k = 2(k)(k+1)$. She noted that the product was divisible by 2 if one or both of the dimensions were even, and said, "since the domino fills up two squares, that's the reason why it works" (Transcript 23, line 91). In the odd-by-odd case, she repeated that there would always be a remainder of 1 and so a perfect cover was not possible.

The researcher asked her what she would need to do to complete the proof. Lisa said that she would write up what she had just found, both why it works in the even-by-even and even-by-odd cases, and why it did not work in the odd-by-odd case. She did not specifically note that this would cover both directions of the bi-conditional statement, just that this would form her proof. When asked if she had seen anything like this question previously, Lisa said that she had worked, in MATH 305, with proving that a number was divisible by 24, which seemed similar because it dealt with cases and parities. Lisa then discussed her strategies with the researcher, pointing out that she had drawn the picture of an 8-by-8 chessboard to find a pattern and to see why the statement would be true.

The proof of this question was partially completed, lacking a demonstration of a generic perfect cover. It is not clear that Lisa understood that the use of cases could cover both directions needed for the bi-conditional statement, though she did feel that her proof was complete. Her work, again, was done mostly in writing, with very little audible input from Lisa, except where requested by the researcher. Her strategies included drawing a picture, looking for patterns, proving the statement for herself, writing what was known and what needed to be shown, looking at examples in various cases, using a proper proof technique, forming equations and notation, and monitoring her progress. Lisa did not make an overall plan for the proof until she looked at the different cases towards the end of her work.

Question 4. After the researcher read the question, Lisa again jumped directly into her work with just a brief pause. She first rewrote the definition of x divides y with her own notation, saying that y/x implies that $y = kx$. Then, she went straight through the proof, taking care to make sure her notation was correct along the way. Lisa did not say anything aloud while she was working and it took her only 1½ minutes to complete a proof. Her work is shown in Figure-Lisa. 6.

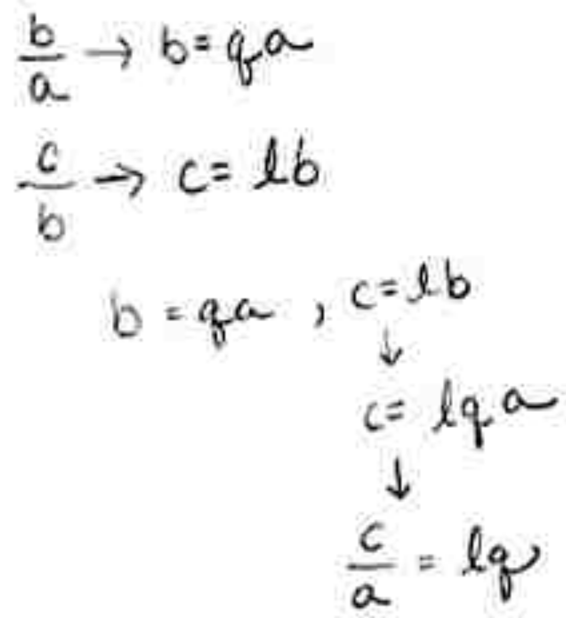

Figure-Lisa. 6: Proof for Question 4

She had clearly written what was known, in her own shorthand notation, and then converted that into equations. She worked from one equation, substituting into the other and finding that *c* could be divided by *a*. Her careful use of different constants in the equations and the ability to properly interpret the definition were both key to her success in this proof.

When asked, Lisa described her work aloud, recapping her steps. She first said that she looked at the example of the definition including *x* and *y*. Then, she stated her proof verbally, saying,

> And I used that to show that if *a* can divide *b*, then *b* equals something times *a*. And if *b* can divide *c*, then *c* will equal something times *b*. And then, with the two equations, I can plug in *b*, so I get *c* equals something times *a*, making *c* divisible by *a*. (Transcript 23, lines 155-158)

While she did not add to her written proof to describe why she could take each step, she expressed the main proof verbally. Lisa said that she might have seen something like this question before, but did not recall having seen it in MATH 305. When the researcher noted how carefully Lisa had been in her notation, using separate constants for each of the equations for the given information, Lisa agreed and said, “Right, because then it just causes confusion if they were the same thing” (Transcript 23, line 180). She said that she was always careful about things like this, and added, "Because, if these are the same thing, then that's the same thing as saying that *a* would be equal to *c*, I think. And that's not necessarily the case." (Transcript 23, lines 188-189). Lisa clearly had a very good grasp on the question and realized what impact her notation would have on the proof and on the situation in general. The interview ended with that explanation.

Lisa had been able to very quickly develop a valid proof for this question. Her ability to do so relied on understanding the definition and using careful notation. This was a simple proof for those that could unpack the definition properly without imposing their own preconceived notions of division onto it. Lisa was able to do this successfully. Again, she worked silently and only in writing until asked to do otherwise. While her written work did not include all explanations, she was able to verbally express her reasoning throughout. Her strategies included understanding the question, unpacking the definition, writing what was known, forming equations, and recognizing a valid proof. She also seemed to have an overall plan for the proof, but did not specifically express this aloud. However, her work was done so quickly it was difficult to be sure of this. Other issues of self-monitoring also did not arise due to the extremely brief nature of her solution time.

<u>Summary.</u> Overall, Lisa was a quiet student who worked well with proofs involving careful notation and attention to detail. She was able to think quickly and understand most questions that were presented to her. Her proofs were concise and accurate, for the most part, as well as being organized. Lisa did not often stumble in her reasoning, nor did she forget the path of her thoughts midway through a question. While an overall plan was not always evident, her work showed a clear understanding of what was needed to form a proof.

In both interviews, Lisa showed a desire to first understand the questions before moving to working through a proof. She looked at examples in only Questions 2 and 3, when her understanding of the statements was not immediate. However, in all cases, she eventually moved to generic notation and understood that examples would not constitute a proof. She made lists often, organizing her work, and referred back to these lists to keep track of her progress through the proof. Her work was structured and remained, for the most part, focused without trailing into unrelated computations or ideas. Lisa was able to redirect her thoughts quickly after trying any ideas that did not result in progress toward her proof. She was able to understand the necessary parts to a proof and to valid proofs, except in the case of Question 2 part b. Unfortunately, Lisa was often unable to recognize the validity of her proof. She seemed to have a strong desire to construct proofs that resembled those she had seen and used in MATH 305, even when she believed that other methods were done in a correct manner and convinced her of a result.

Lisa was one of several students who worked together with the researcher during informal study sessions for MATH 305 throughout the semester. During this time, she worked through homework questions while seated with other students in the undergraduate study lounge. However, Lisa did not often interact with others until she had first worked through a proof on her own. After she had come to her own conclusions, worked through the proof herself, or found that she was unable to find a proof on her own, she would then turn to other students and compare work. Her seclusion during those times went so far as to close out the discussions happening around her, to the point of not picking up clues from other students that may have helped in her own proof. She would then not have even heard the discussions of others until after she finished her own private thoughts and turned back to the group.

This manner in which she worked was also evidenced during the interview with the quiet, intense way that Lisa would work through a question. Most verbal comments that she made were responses to the researcher or were made after she had found the proof and then engaged in discussion about her results. Her meticulous proof-writing techniques were also in evidence during both the interviews and during informal observations, where Lisa would work hard to put together a proof in a certain structure that matched those used in class. Even when her understanding of a question occurred outside of this structure, she would then conform her thoughts to fit into a mold that she had seen in use or used herself previously. Unlike others, however, this tendency only strengthened her proofs instead of taking away from them. The researcher believed this to be a result of Lisa's overall understanding of the structures she was attempting to use, instead of blindly fitting into a particular mold with little understanding of how it would prove a statement.

Ellen

Ellen was a MATH 305 student. She was a graduate student in the math department who was taking MATH 305 to prepare for other upper division math courses; however, her previous degree was in engineering, not mathematics. Her previous coursework included calculus I and II, differential equations, dynamical systems, applied math, statistics, probability, partial differential equations, and real analysis. During the interview, which was just over one hour in length, Ellen worked on Questions 1, 3, and 2 part a. Time did not allow for work on the other parts of Question 2. Ellen wrote out proofs for Question 1 and 3 in somewhat full detail, any details lacking were expressed verbally.

Question 1. After the researcher read the question, Ellen read the statement to be proven aloud. Then, she checked to make sure that she understood the question, clarifying some details. While other students required a good deal of prompting to work aloud, Ellen did so immediately and described her thoughts in detail throughout the interview. She began her work by saying, "I would imagine then the smallest possible sum would be sort of some optimization of the largest with the smallest numbers" (Transcript 2, lines 16-17). She seemed to be relating the question to others she had seen before, tying it to optimization. This was also the first place that the main idea for her proof had surfaced.

Ellen outlined a plan for the first part of her proof, that she would attempt to exhibit an arrangement with sums of 14. Ellen stated that she would use the smallest numbers as vertices on her pentagon, immediately writing the numbers 1 through 5 on the vertices. This was a main key to finding a proof of the statement, though she did not prove her conclusion at that time. However, the numbers she had written on the pentagon were not written in the correct places to be able to complete the arrangement. As Ellen began to place more numbers on the new pentagon, she quickly decided that she would need to place 10 first, which forced her to change the placement of two of the vertices. Ellen stopped her search to keep the overall goal in mind. She said that she needed to find a minimum. She also said that she had never done a proof like this before, indicating that she was also searching for similar proofs from which to gather ideas.

Ellen turned once again to finding a pentagon with sums of 14. When she saw that her attempts at filling in numbers were continuing to result in failure, Ellen tried a new technique, showing an ability to monitor her progress and redirect her work as needed. This time, she began to systematically list the possible combinations of numbers whose sum was 14, working from those including 10 down. From this list, she was able to see her choices more clearly and began to correctly fill in a new pentagon. Ellen stated that she could sometimes be sloppy in her proofs, since she was used to working with the computer program MatLab, "where you don't have to have it exact. You can kind of think you have the right idea and plug and test it and see if it works", she said (Transcript 2, lines 62-63). As Ellen went back to placing her numbers, the researcher summarized that the combination 10, 1, 3 was forced to be on the pentagon somewhere. Ellen replied,

> Right. And 10 is not going to be on a vertex because that will be more towards the maximum end of things… because it was shared. I would imagine, to minimize, you would want the low numbers to be the shared ones. (Transcript 2, lines 88-94)

Though she still did not further explain the justification of this conclusion. She also noted a pattern in the number of possible combinations for each of the largest digits, however this pattern was not helpful to the proof. She filled in the pentagon correctly, keeping in mind her requirement that the lowest numbers should be on the vertices. Ellen was able to recognize that she did not yet have a proof. She also recognized that a key to the proof might lie in the placement of the lowest numbers on the vertices.

On a new sheet of paper, Ellen began to write out the rest of her proof. She developed equations for the sum of each side, *sum = 2 vertices + side*, and an incorrect equation for finding the minimum, $5(2v + s) = min$, though she was able to determine that her second equation was not valid. She was also able to state that her overall goal would be to minimize the sum of all of the sides together, and found the equation that was a core element to her proof, $2\ (v_1 + v_2 + v_3 + v_4 + v_5) + (s_1 + s_2 + s_3 + s_4 + s_5) =$ Total sum. However, Ellen expressed that she did not know how to continue to write up her proof formally, and writing that the vertices must be the numbers 1 through 5 and the sides the numbers 6 through 10. She justified this statement by noting that the smallest numbers should be those to be doubled in order to get the minimum sum. She made a calculation error, but the researcher corrected her.

Ellen stopped her work in an effort to regroup, saying that she was being sloppy. She was not satisfied with her explanation, and attempted to rewrite it in another way. She made another arithmetic error, which the researcher hinted at and Ellen corrected herself. Eventually, Ellen was able to determine that the sum, with the vertices being 1 through 5 and the sides being 6 through 10, was 70 (Transcript 2, lines 178-205). Ellen said that proofs that are obvious to her are often difficult to write out. The researcher asked Ellen to justify that 14 was the lowest sum and helped her to look back at her equation and continue. Since there were five sides, she divided 70 by 5 and found that it equaled 14. Even though she never proceeded further with her explanation of why the vertices needed to be 1 through 5, she was now satisfied with her solution since the numbers worked out. The researcher and Ellen then discussed what other students had done after Ellen inquired about this (Transcript 2, lines 258-285).

When asked again if she had seen anything like this before, Ellen responded that she had not. However, she mentioned that her experience with optimization in her engineering education might have had an influence on her proof. She stated that she had never had a mathematics course in optimization, but had taken an engineering course in the topic. She felt her background had given her strong spatial skills as well.

Ellen's progress on this question was made through two key discoveries, that the vertices would be the numbers 1 through 5 and finding the equation for the total sum. These discoveries were made, in part, because she looked back to her previous experiences and recognized the link between this question and optimization. However, these two main ingredients alone would not have made a proof without Ellen's ability to self-monitor. She was able to keep track of her attempts while finding the correct pentagon with sum 14. Ellen had begun by setting up an outline for how to proceed with her proof, and also kept the overall goal in mind and recognized that just finding the pentagon was not enough for the proof. Being able to find the equation had been possible because she kept in mind the goal to minimize the total sum. She also recalled that to do this she would have to place the lower numbers on the vertices, which allowed her to complete the proof. The only detail she lacked was justifying that placement of the vertices, though it could be considered clear once the equation was in place.

Question 3. The researcher next read Question 3 aloud. Ellen immediately recognized that the statement was a bi-conditional statement that she would need to prove in both directions. She chose to start with the forward direction and wrote out what she knew and what she wanted to show. She clearly understood why one dimension would have to be even and so was having difficulty forming a proof, which was a common difficulty experienced by many students on various questions. She also struggled to articulate what she was thinking. Finally she said, "Each domino covers a 1 by 2 generic chessboard.… Kind of a mini-chessboard … some sort of like an integer divisible thing" (Transcript 2, lines 349-351). Here, she was both breaking down the question and also trying to make a connection with previous material that she had seen in MATH 305.

Ellen was careful to use correct notation and stated that since the board was perfectly covered, there existed an integer k such that $k (1 x 2) = mn$. She then made a few errors going further with her equation, writing that $k (2k) = mn$ and furthermore that $m = k$ and $n = 2k$. The researcher helped her correct the error of the relationship to m and n by discussing specific examples. Ellen finished the forward direction by saying that since the total was even, then at least one of m or n had to be even. She clearly had an understanding of the question, but again was unable to form a proof of what she felt was obvious. While she did understand that she needed to say more, she was not sure how to do so.

After being asked to consider what would happen if the conclusion were not true, Ellen decided that she should use a proof by contradiction. She said that the odd-by-odd case would give an odd number of total squares, which would not be coverable, but she did not ever actually prove why it would not be coverable. When asked, she knew that she was not finished with the proof because she still needed to prove the other direction, but she wanted to "tidy up this one" first (Transcript 2, line 428), referring to the forward direction. Ellen wrote out her conclusion for the forward direction, but still did not prove why it would not be coverable. She said that if she had written this up for a class, she would have rewritten it from the beginning as a proof by contradiction and "neaten it up" (Transcript 2, line 446), but that for now she would just move on to the other direction. She had maintained the overall goal in her mind and was monitoring her progress well.

Ellen now proceeded with the reverse direction. She again was careful to write out what she knew, that she had a generic chessboard of size *m*-by-*n*, where *n* was even. She considered proof by contradiction again, but opted to continue without using this technique. Her overall idea for this direction of the proof was that since one of the dimensions was even, the total number of squares would also be even, and therefore the board could be perfectly covered. Crossing off her assumption that $n = 2k$, Ellen stated that one of the dimensions would be even, showing a desire to stay as generic as possible. She then said, "what I'm thinking and what I'd like to do is show that if we have 2*k* squares and we successively subtract two squares we would get to zero eventually" (Transcript 2, lines 468-470). She and the researcher discussed how this would then demonstrate a perfect cover; the researcher told her that this is what would be termed an existence proof.

The researcher expressed concern that Ellen's method may not work for a proof and challenged her to create a new idea. Ellen understood that the researcher wanted her to say more and so formed a new plan. She went on to both describe and draw a generic construction for the placement of the dominoes on an *m*-by-*n* chessboard (see Figure-Ellen. 1).

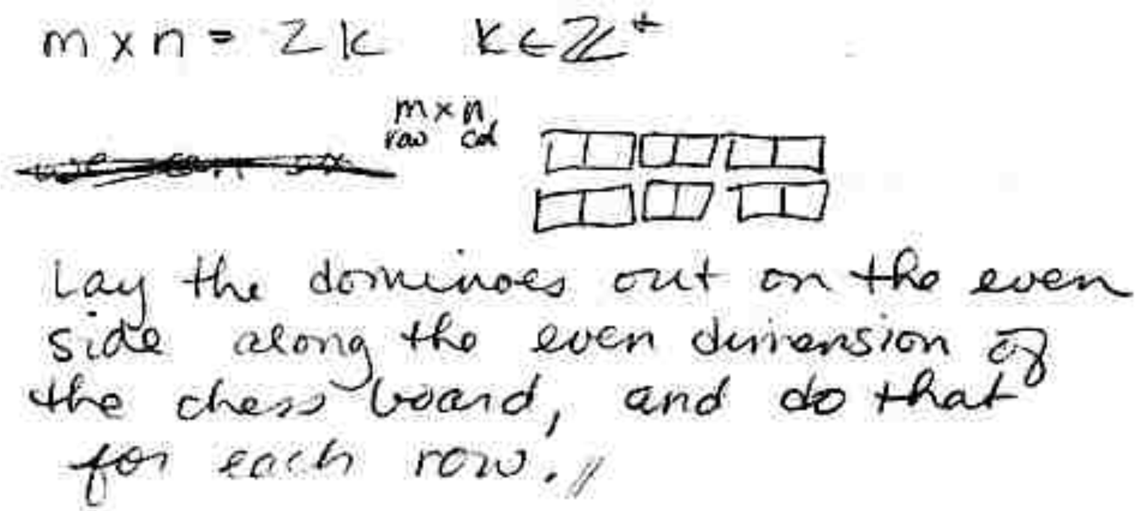

Figure-Ellen. 1: Generic pattern for domino placement in Question 3

In this way, Ellen had shown that if at least one of the sides was even, then the dominoes would fit and form a perfect cover. When asked if she was finished with the proof, Ellen thought that she was but was hesitant. After rereading the question, she concluded that if she had rewritten the forward direction, she would be happy with her proof. After a brief summary by the researcher, Ellen expressed that she still did not feel entirely satisfied, but that she had experienced this same feeling in MATH 305 when doing proofs by contradiction.

When reflecting on the strategies used, Ellen said that she had recalled the idea of working with cases and the if-and-only-if proof structure from MATH 305. She also said that being familiar with dominoes and checkers helped her to have a good visual representation, and that talking a proof out with someone else was a common thing strategy she used when writing a proof. At first she had not known what to do fro the proof, but sometimes "I can just be completely stuck and then one little thing changes and boom it's a flood," she said (Transcript 2, lines 645-646). Lastly, she noted that, "I see a lot of pictures in my head. … I do a lot of visualizations like with the dominoes with little white dots and the chessboard" (Transcript 2, lines 654-659).

Ellen had addressed visualizing the situation as helping her in her work, both in this question and on Question 1. As she said, she formed a great mental picture of the situation, which allowed her to fully understand what was happening. However, it also hindered her from being able to move past that understanding. Ellen was prompted several times during this proof to continue past her mental roadblocks. She had some difficulties forming a proof because it was very clear to her that the statement must be true.

One of the key elements to Ellen's success in this question was that she recognized the statement as a bi-conditional, knowing she would need to prove both directions. She was also careful in her notation, writing out what she knew and what she needed to show, and took care to use proper proof language throughout the proof. Ellen also used several different strategies that helped her overall, including making a plan, making goals and sub-goals, monitoring her progress, and recognizing the validity of her work.

After being prompted, Ellen had been able to see that she could use proof by contradiction for the forward direction. This allowed her to finish that direction of the proof, though she did leave out part of the explanation. Again, however, it was a piece that could be considered obvious or trivial. Ellen also received prompting in the proof of the reverse direction, but was able to finish from there. She could have benefited from being able to redirect herself and also being more satisfied with her proof by contradiction.

<u>Question 2.</u> The interview continued with part a of Question 2, though time ran out before the question could be completed. After the question was read, Ellen seemed to understand the definition of a 4-flip quickly. She demonstrated her understanding with an example, $N = 12$ with $4(12) = 48$, but said she would have wanted $4N = 21$ if $N = 12$ were to be a 4-flip. She represented the digits of N and clarified the constraints on them. First, she expressed N as $d_{10}d_1$ where the d's represented a digit between 0 and 9. She wrote the equation $4d_{10}d_1 = d_1d_{10}$. Ellen told the researcher that she wanted to express the number in terms of its digits, but was not sure how to do that since they were grouped together, not multiplied as her notation suggested. She looked at a few examples of numbers and their flip, but was unable to think of a way to generalize from the examples, and was unsuccessful in representing what she needed to show in equations. Her final equation dealt with the consideration of how the numbers would divide each other (see Figure-Ellen. 2).

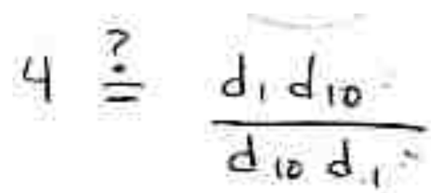

Figure-Ellen. 2: Equation representing 4*N* divided by *N* in Question 2

This equation led Ellen to try two more examples before beginning a list of all the two-digit numbers divisible by 4. Next, she worked to find constraints on N that could shorten her list. After some thinking, Ellen decided that she could eliminate all the numbers in her list where d_1 was less than d_{10}, since $4N$ must be bigger than N, therefore the first digit of $4N$ must be bigger than the first digit of N. She expressed that she would have continued to eliminate numbers from her list, looking for other constraints, if she had more time. She briefly considered whether N should be even or odd, but realized that it would not help her. Time was running short, so the researcher stopped her work, and

Ellen repeated that she would have continued to look through her list and eliminate more numbers until she had more of an insight. She stated that there were not really any new strategies that she used in this proof, just trying to represent the digits, but she did not have experience with that and so did not know how to do it. She identified her representation of the digits as a way to understand the question and to play with it.

In this question, Ellen looked at examples both to clarify the definition and to gather ideas for the proof. She also attempted to use equations to aid her proof, but was unable to move further with those. Given more time, Ellen would most likely have been able to finish the proof through her list by exhausting all the possibilities. However, it is impossible to know what other ideas may have surfaced during this work. Unlike the previous questions, here Ellen did not seem to develop a mental picture of the scenario, perhaps because there were no geometric or physical objects involved. She also struggled to develop an outline for her proof until after looking at examples. This could indicate that her mental images aid in her development of the structure for a proof, and in the understanding of the statement in general. Her previous background as an engineer would certainly point to a strong use of visualization in her thought processes.

<u>Summary</u>. A main key to Ellen's success was the ability to stop herself when she began work that was not productive. At different points in her work, she recognized the path she was on as not being beneficial to the proof and was able to redirect her efforts, both with and without input from the researcher. Redirecting allowed her to spend time on work that was helpful instead of chasing other work to a dead-end.

Overall, Ellen had complete proofs to Questions 1 and 3 and expressed her strategies well. She was able to develop a skeleton proof, or an outline for a proof, for most questions before even beginning to delve into her work. This allowed her to think about the big picture and kept her from getting lost in the details of her work. Ellen monitored her progress fairly well and kept track of her overall goals and the outline of the proof while working. She understood the language necessary to complete the proofs, and demonstrated the use of tools that she was learning in MATH 305, including comparing questions to known proofs and searching for clues from related results. A very important aspect of Ellen's work was seen throughout the interview; she was able to recognize and acknowledge the validity of her own proofs. This was a skill that was lacking in many other students but was prominent here.

Ellen worked occasionally with other students from MATH 305 during informal group work sessions throughout the semester. During this time, and in the classroom as well, she demonstrated a high level of mathematical maturity in her ideas and conceptual ability. Her mathematical training and background outweighed many other students in the course, and this was evident through her ideas and her ability to bring in other concepts and ideas to aid in proof writing. Ellen was able to work both alone or in groups, silently or aloud, though she seemed to prefer working aloud. Her flexibility in work environments was also rare among the students observed from this course. In these ways, both her everyday proof writing and her work done during the interview was more closely linked to those students participating in the pilot study, who were also mathematics graduate students, than her fellow classmates during this study. Both her mathematics background and her level of academic maturity seemed to contribute to her success during this interview.

Shelly

Shelly was a MATH 305 student. She was a graduate student outside of the math department. The previous math courses she listed as having taken were linear programming, biometrics, statistics, and calculus. Shelly withdrew from the MATH 305 course partway through the semester, but had participated in the study prior to that time. During the interview, Shelly worked on Questions 1, 3, and 2 parts a and b. The interview was 1 hour and 6 minutes in duration.

Question 1. The interview began with the researcher reading Question 1 aloud. Shelly read the question and verified that the pentagon in the example did contain all 10 numbers. She noted that the highest and lowest numbers were at the vertices. Right away, she was searching for patterns, saying that she was looking to see if there was a pattern in the placement of numbers on the vertices versus on the sides of the example pentagon given. The search for patterns turned out to be the key to her discoveries in this question; however the first particular pattern she observed did not turn out to be useful for her. After checking all the numbers, Shelly organized the information she had gathered on which numbers were on vertices and which were on sides of the pentagon;

doing a good job of self-monitoring at that point. Her list included the numbers on the vertices (1, 3, 4, 7, and 10) and those on the sides (2, 5, 6, 8, and 9). She checked her understanding of the question; that three numbers made up each total edge, and then verified that all sides in the example did add to 16.

Shelly stated that she needed to show that the smallest possible value for the sum was 14. While looking for a pattern, she noted that she needed to take combinations of 3 of the 10 numbers, and that vertices were used in two combinations, indicating her attempt to understand the question and use the example to make conclusions for her proof. Shelly said that the combination of 1, 2, and 3 would be the smallest combination, resulting in a sum of 6. She looked for values that could be used with 10 to make a sum of 14, finding those values to be 1 and 3. Shelly then examined the given pentagon with sums of 16, again searching for patterns. Looking at the placement of larger and smaller numbers on the example, she considered where numbers should go on a pentagon with sums of 14. She noticed that the numbers 10, 1 and 2 were all vertices on the example pentagon, but was able to see that they could go elsewhere in a pentagon with sums of 14.

The key to a successful proof was discovered next. Shelly decided, upon examination of the example pentagon and searching for sums involving the number 10, that the larger numbers must be placed on the sides. She continued by drawing a new pentagon and trying to place the numbers to form sums of 14, saying that in finding the pentagon, she may find what it would take to prove the question. First, Shelly placed 10 on a vertex, but then reconsidered and moved 10 to a side instead. She felt that in making this switch, she might be able to keep the value of the sum down, since 10 would only need to be used once. By chance, she then wrote all five of the highest numbers on the sides in order, and proceeded to fill in the entire pentagon on the first try. Seeing the pattern that the vertices should be labeled with lower numbers and the side labeled with higher numbers had allowed Shelly to find the pentagon quickly. However, she was unable to see that the main idea of a correct proof had emerged.

Shelly went back to read the question and felt that she still needed to prove that 14 was the smallest possible sum. She stressed the use of proper proof language throughout the interview. Here, she said she could disprove the statement using a counterexample, indicating that she was thinking of different proof techniques to try. Unfortunately, she was failing to make the proper connections between rote procedures and her actual understanding of the question.

Shelly tried to find a pentagon with sums of 13, eventually deciding that she was convinced that 13 was the smallest possible sum to be considered. She drew a new pentagon and first placed the combination 1, 10, and 2 along one side. She attempted to fill in the numbers to find a sum of 13, stating, as mentioned, that she felt 13 would be the absolute minimum to have to check, though she did not explain her reasoning for that idea. Later, however, there were several moments that she doubted this discovery and had to prove it to herself all over again. One example of this error, showing that she lacked the ability to monitor her progress, occurred soon after her discovery was made. She was unable to keep in mind the reason she had decided that 13 was the smallest sum to consider, having to go back through all choices to verify her original thought before again being convinced.

Shelly continued trying to find a pentagon with sums of 13. However, she struggled, as she was not organized in how she proceeded through her choices. She lacked the organization and monitoring of her attempts to actually prove that 13 was not possible. As she placed numbers on the pentagon, she did keep a list to the side of those that she had already used, but it did not seem to help her organize her work. When her attempt failed, she indicated that she was convinced that 13 was not possible and that 14 would be the lowest possible sum, though she offered no proof of her conclusions.

When the researcher asked her to prove that 13 was not possible, Shelly again questioned her assumption that 13 was the smallest sum to check. She was unable to keep her previous arguments in mind and had to convince herself that no number less than 13 would work, repeating her argument for the third time. She labeled her work by cases, 14 and 13, marking that 13 was not possible, and finally organized her work. Even though she felt 13 would not be possible, she did not realize that it was this argument that could prove 14 was the minimum sum.

After organizing her work, Shelly read the question again and expressed that her stumbling block was proving what she had already convinced herself was true. In other words, now that she was convinced 14 was the smallest sum, she was having difficulties thinking of how to actually prove that it was the smallest. She said, “Because once I know it, that's just as far as my mind wants to go. I could care less about proving it to somebody else" (Transcript 3, lines 127-128). What followed was a discussion of what could motivate Shelly to continue the proof. She said that she would pretend that she cared about proving it to someone else and continued, deciding that to prove it to others she would just go through the reasoning that she had used to prove it to herself. Attempting to encourage her to continue, the researcher suggested Shelly imagine that the question was given to her as a homework assignment for MATH 305, but Shelly insisted that she would be more motivated assuming that her father had disagreed and challenged her to prove that 13 was not possible. She had to find an external motivating source in someone important to her in order to move past her reluctance to attempt a proof.

Shelly then resumed her work on the proof. Again, as evidence of her lack of ability to recall earlier notations she had made, Shelly had to restate that it took three numbers to make the sum of 14 for the question. She followed this idea to develop the equation $A + B + C = 13$, which did not actually help her proof then or later. Shelly left that equation temporarily, while again forgetting that she had already proven that sums less than 13 were not possible. She had to be told by the researcher that she had already shown this, who reminding her that she had made a statement about why sums less than 13 were not possible, i.e., since 10 had to go with at the least 1 and 2. The researcher then guided her to move past that thought.

Shelly continued trying to write a proof and she expressed that she was trying to use proper proof language. Returning to her equation, she wrote the words "let's suppose" prior to it (Transcript 3, lines 173-174). Her desire to stress this proper language, rather than address the larger picture further showed that she focused on the procedures of proof writing instead of using a proof to make connections between her ideas. However, as previously stated, the equation was not useful to her and even she herself stated that she did not know if it was useful. No further generic notation was mentioned; instead Shelly used the equation only solve for a specific combination. She let A and B be the smallest possible, 1 and 2. That gave her the equation $1 + 2 + x = 13$, which she solved to find $x = 10$. She said that she would continue in that pattern, using her equation with specific numbers in place, and "I would just do algebraically what I did pictorially", she said (Transcript 3, lines 180-181).

The researcher recapped the work that had been done, but Shelly contributed little to the discussion (Transcript 3, lines 194-211). It was not clear that she even understood she had a valid proof in the making. The researcher then asked her to discuss her strategies. Shelly reported that she had been looking for patterns and went on to describe more specifically what she had been searching for. The types of patterns she had considered included whether all the odd numbers were on the vertices (or all the even numbers), the difference between consecutive numbers on the pentagon – whether it was always equal or had perhaps either an ascending or descending pattern, and if there was a pattern in which numbers were on the vertices versus on the sides. Shelly said that, though she had been looking for patterns in the example for 16, she had not found any that would carry over to 14. She then described how she had gotten stuck in one idea and had forced herself past it by starting a new page of work.

When asked if she had seen the ideas that she had used in her work prior to this session, Shelly told the researcher that she had learned to look for patterns while teaching math for elementary school teachers. The particular patterns she had looked for were a result of that experience. She also stated that some ideas had come from math games. Shelly said that she preferred pictures to words and that she would have drawn a picture of the pentagon right away if it had not been given to her. She also said that she was glad the question was not all words, saying "words confound me" (Transcript 3, lines 281-282).

Overall, Shelly was unable to recognize the valid portions of her work on this question. She lacked a clear plan for developing a proof. While there were times that she did monitor her progress and made notes to herself as reminders, the majority of her work did not show evidence of self-monitoring. She never seemed sure of what would be needed to constitute a valid proof. As previously mentioned, Shelly did eventually withdraw from the proof-writing course. Throughout this interview, there was evidence that she struggled as a proof writer, not clearly understanding what was necessary for a complete proof. Her progress on this question was made mostly through searching for patterns, searching in a manner that suggested she was merely looking for a solution to the question, not a proof. Her lack of motivation to form a proof once she was convinced of the validity of the statement is further evidence that she viewed the question in a problem-solving realm, not one of proof writing.

Question 3. The researcher collected papers from Question 1 and moved on to Question 3. After the question was read aloud, Shelly recalled similar material from her previous experience, teaching math for elementary teachers, but said that she did not assign this particular question. She read the question herself and developed a picture in her mind of the dominoes on the chessboard. Shelly noted that the traditional chessboard portion of the question was extraneous information, since it did not pertain to a generic chessboard.

Shelly said that she was working with an *m*-by-*n* chessboard and drew a picture of a one-by-two chessboard, breaking the question down into the smallest possible example. She also noted that her example would have a perfect cover and that one dimension was even. She consulted the question again, clarifying what needed to be shown and underlining portions of the question for understanding purposes. Then, she looked back at her example. Shelly stated that she was looking at the different cases, and that the first case was where at least one dimension was even. Making a plan to look at an example of a chessboard with both dimensions being odd, she said that this was the negation of at least one dimension being even, and wrote the notation for this statement (see Figure-Shelly. 1).

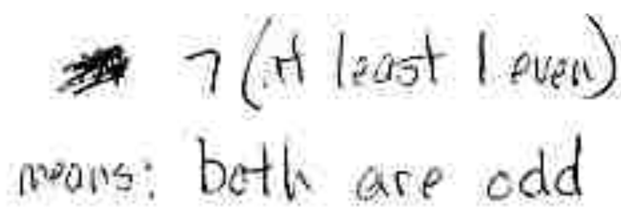

Figure-Shelly. 1: Notation for negation of at least one dimension even in Question 3

This showed again her desire to use proper notation and follow specific proof techniques. Shelly drew a 5-by-3 chessboard, though she labeled it and called it 3-by-5. She noted that she was looking at the negation of the statement that she was trying to prove, and that she needed to show her example could not be perfectly covered. However, she gave no indication of needing or wanting to discuss a more generic picture rather than a specific example.

Shelly decided to start covering her example chessboard with dominoes, searching for a proof that no perfect cover existed. Beginning by placing two dominoes on the chessboard, she realized that it would not be possible to find a perfect cover since 3 times 5 was 15, which was an odd number. She noted that the area of the chessboard was 15 squares, but the area of a domino was two squares, so there would be always one square not covered. At this point, she had discovered the heart of the proof that the odd-by-odd case was not possible, but failed to generalize this idea to form the proof. Even though she seemed to recognize the underlying generic argument, Shelly did not mention any other possibilities for the dimensions, or prove that this case would not work in general.

Shelly had, unfortunately, used improper notation to describe that 15 was odd, writing that $15/2 = k + 1$. She saw that her notation was incorrect after the researcher paused to ask for clarification. However, she was not able to correct it, instead she wrote that $k = 2L$ and so was even. The researcher clarified that Shelly was trying to describe that she would always have one remaining uncovered square when dividing the 15 total squares into sets of two to form dominoes. When Shelly was asked to prove this in general, she did not respond. So, the researcher asked Shelly if she believed that no matter what her choice of m and n, the case of both being odd would not be possible, giving a few examples of odd-by-odd dimensions. Shelly still did not directly respond to the question, instead she stated that the only case she had not tested was both dimensions being even.

Shelly described that when both dimensions were even, the product would be even and so the board would be able to be perfectly covered. The researcher asked Shelly how she would go about proving what she had found to someone who was unconvinced, again having to prompt her to follow through with a proof of the statements that she was making. Shelly stated that she would show “a bunch of numbers” (Transcript 3, line 423) to prove to someone that her conclusions were correct, apparently not understanding that examples, regardless of quantity, would not constitute a proof. The researcher recapped Shelly’s work and told her that she had, for the most part, proven one case, that of odd-by-odd.

The researcher recapped more of Shelly's work then asked her to show that she could find a perfect cover for the two cases, in general. Shelly had been unable to see that she had not finished a true proof of either case, but was looking only at specific examples. Once prompted, Shelly did say that she may have been misled by her example of a 1-by-2 chessboard and it could be a special case, which students were warned to be wary of in MATH 305. Shelly then looked into the general case of even-by-odd and described that the product would be even and so divisible by 2. The researcher asked if she felt she had a satisfactory demonstration of a perfect cover. Shelly responded that she thought perhaps she should also prove that they could be arranged correctly on the chessboard as well, not just that there were the appropriate number of squares in total.

Shelly decided that, regardless of the board size, if one dimension was even, then she could rearrange the dominoes to be one domino wide by as tall as was needed. When asked to show a picture of this, it became clear that she was not considering the pattern over the entire board, but actually wanting to represent the entire board in one column of dominoes. As evidence of this, she wrote 3 x 4 = 12, then drew a 2-by-6 chessboard representing six dominoes laid in a column to cover 12 squares. She said that it was not the arrangement of the 3-by-4 chessboard, but that it would 12 squares covered perfectly. Recognizing that her drawing had not been entirely correct, but not actually acknowledging the error, she did move on and made a new argument that was correct but incomplete. This argument began as she drew a 3-by-4 chessboard and said that one side was even and so could always be broken down into a number of dominoes laid on their sides. The researcher asked her to clarify how the entire pattern would be laid out on the chessboard. Shelly responded by saying that the rest of the dominoes would follow in the same way and cover the board, restating her argument in a bit more detail, still referring to a specific example, but speaking in general terms. Shelly needed to be told by the researcher that she was finished with the proof, though she did not formally write up any results. She indicated that she still did not see her justifications as proof, but did not add any thoughts on what could complete the proof. "This is fun. I like this. If I don't have to prove it –", she said (Transcript 3, line 533). This indicated that she was seeing the question as a problem to be solved, but that her process of working through it was not viewed as part of proof writing in her mind. There was a disconnection, as stated earlier, between proof writing and convincing herself that the statement was true. This disconnect could also have been part of the reason that Shelly was so reluctant to consider how to generically state or prove the conclusions she was making.

The progress Shelly was able to make on this question was made primarily through the use of examples as well as splitting the proof into cases. She was able to develop arguments that, if fully fleshed out, could have resulted in a complete proof. However, she was again unable to recognize her valid arguments, and there was no indication that she understood that there was a bi-conditional statement to be proven or that a proof by cases would address both directions of the statement. Shelly also continued to struggle due to of her lack of self-monitoring. She was unable to keep the larger goal in mind or to monitor her progress throughout her work. She was able to form a plan to begin the proof, but did not always follow through with this plan.

There was significant evidence that Shelly was unable to complete a proof once she clearly understood the argument herself. When she had proven to herself that each case would either have a perfect cover, or be impossible, she did not proceed to generalize her example to the extent of a full proof. In some cases, though, she had mentioned a sketch of how a proof could proceed, but only after several prompts from the researcher to do so.

Overall, success in this question for Shelly would have required the ability to self-monitor, the understanding of the validity of her arguments, and the desire to move beyond convincing herself to convincing others. Her key discoveries were made due to her attention to detail (which did tend to lead her astray at times, as well), her visualization of the question, and indirectly from her search for patterns.

Question 2. The researcher next asked Shelly to move on to Question 2, knowing time would not allow for the entire question, but that some of it could be addressed. After the definition of a 4-flip was read aloud, Shelly reworded this statement to understand it. She also stated that the number N must be at least two-digits, not yet having read the remainder of the statement to be proven. She looked at an example, $N = 12$, to clarify the definition. The researcher asked her to clarify what would be necessary for the N to be a 4-flip. Shelly demonstrated that she understood the definition.

The researcher directed Shelly back to the question, who had forgotten to even consider the question. Already, she was so focused on the definition and the small details that she was having difficulty keeping the overall goal in mind. Shelly reread the question, and stated that she would need to show that no number from 10 to 99 would be a 4-flip. She also stated that she could prove the question by exhaustion of all cases.

Shelly made a plan to begin looking at the list of numbers, saying that a pattern might emerge. She began with $N = 10$. As she considered this, Shelly decided that instead of looking at just the values of *N*, she would also examine the values of *4N*, saying that she wanted to work from both ends in hopes of reducing her workload. She continued with $4N = 98$ and divided by 4, seeing that it was not a 4-flip, but she was unable to make further progress on this idea.

Abandoning the *4N* example, Shelly began to break the question into smaller parts, considering the range of numbers 10 through 19. She started to say that all of the 10s would have a leading digit in *4N* of 4, so she would check only 14, but she realized that at some point the leading digit would increase. She did notice, though, that no number times 4 would end in a last digit of 1. She had found the correct reasoning for the interval to be impossible, however did not recognize this. The researcher asked her to continue thinking about that interval and her conclusion, asking her to restate what she had said. Shelly was able to conclude that no number beginning with 1 could possibly be a 4-flip, recognizing that this meant that the numbers 10 through 19 were impossible.

Shelly followed this same strategy for the 20s, saying that only 3 and 8 times 4 would end in a 2. However, she stumbled in her attempts to continue. In the middle of considering what the ending digit meant for the 20s, Shelly pointed out that 25 times 4 was equal to 100. It was not clear if this was a continuation of her thought that at some point in the interval, the first digit was increase of *4N* would increase, as she had tried to address in the interval 10 through 19, or if she was recognizing that 25 would give a three-digit number. Whichever the case, this thought had interrupted her ideas and work, but she did not pursue it further. Instead, she went back to check 23 and 28, finding *4N* to be 92 and 112, respectively.

Shelly was unable to monitor her progress and ideas through this interval. While she had actually found the reasoning that no number in the interval 20 through 29 would be a 4-flip, she was not satisfied with her justifications. She said, "I'm not convinced that that process of elimination eliminates all the 20s" (Transcript 3, line 603). When the researcher attempted to clarify her reasoning, Shelly was unable to follow her own work, not recalling immediately how reached some of her conclusions. However, she eventually recovered her ideas and proceeded to explain her reasoning. She crossed out 28, saying that 92 was the largest that the range from 20 to 29 would produce, meaning that *N* flipped would be no larger than 92 for that range. She said that 23 was already at 92 and so all those above it would have gone too far, referring to *4N*. Unable to recognize that she had in essence proven that the range from 20 through 29 was not possible a second time, Shelly decided to give up trying to find a proof for this interval, and moved on. She would look at the 30s next, she said, and would search for a pattern.

Shelly stopped to organize her work, writing down why each interval was not possible and stating that she was finding contradictions. She made an error while trying to recall her reasoning for the interval 10 through 19, but was able to recover from this. She then wrote that the numbers 10 through 19 were ruled out because no number multiplied by 4 would end in 1. She continued to try to organize her work and recall her arguments.

When trying to recall the rule for the numbers 20 through 29, Shelly again expressed that she felt she had left something out. She reconsidered her reasoning, adding that *4N* would be in the range 80 to 80 + 36, but that she could only go up to 99. Shelly was unable to monitor her work or to recall her previous arguments, and so during the process of re-explaining why this range would not be possible, she ended up moving to a different idea. Remembering that after 23, *4N* went over 92 anyway, she concluded a second time that she would only have to consider the numbers 20 through 23.

Shelly next considered the range 30 through 39, having not been able to make the connection that *4N* was a three-digit number for all numbers above 25. Instead of remembering that 25 times 4 was 100, she said that numbers in the 30s when flipped could end in 3, and so could potentially go to 93, instead of just to 92 in the 20s range. So, she felt that she needed to consider the 30s independently of her previous arguments. After looking back over her work and checking an example, *N* = 30 giving *4N* = 120, Shelly was able to see that nothing above the 20s would work. The researcher attempted to organize Shelly's thoughts, saying that she had ruled out the 10s and everything above 23. Again, Shelly expressed doubt of her conclusions and had to repeat her argument for herself. She looked at $N = 24$, saying that it would still be two digits for *4N*, but that *4N* would not end in two, as it needed to.

The researcher prompted Shelly to explain her thoughts and organize her work. Shelly was still unable to recall her previous arguments without restating them completely. She checked 24 and 25 and said that nothing above 24 would work because *4N* would exceed two digits. The researcher recapped what she had said, though it was not clear that Shelly could have done so herself. When the researcher said that Shelly had already ruled out 20 through 23 for the reason that it would not end in 2, and she had checked 23, Shelly again had to recheck 23 to make sure it did not work. The researcher stopped Shelly's work at this point, though it was not clear that Shelly knew that she was done with the proof.

The researcher asked Shelly to try one more part of Question 2, part b. Shelly agreed to stay for a few extra minutes to work on that. The question was read aloud and Shelly wrote that she needed to consider the numbers 100 through 999. She used ideas from the previous part of the question but was unable to make a complete connection, saying that there would be a point where *4N* would exceed three digits and set out to find that point, using long division. She refused the offer of a calculator, saying that doing work by hand helped her work through a proof. She later also stated that the offer had distracted her. Shelly did successfully find the upper limit to be 249, but then asked herself why she had done the calculation at all, showing her need to consciously monitor during even this brief period to keep on track. However, she was able to recall her short-term goal and organize her work.

Shelly stated that she could use the techniques from the previous portion of the question to move through this proof, but would prefer to find a different method. This was most likely due to the fact that she had not felt satisfied about her reasoning in part a of the question. Shelly, however, continued to use the very reasoning that she was trying to avoid, indicating that she was having difficulties monitoring her progress and work. She said that 100 through 199 would be ruled out since no number times 4 would end in 1. Shelly organized her work into a neatly compiled list of intervals already proven. She had broken the question down into smaller portions, looking at intervals of numbers and trying to make general arguments to eliminate them. Shelly made a possibly erroneous statement regarding the interval 200 through 249. However, it was unclear what she was trying to say. The researcher tried to clarify, but possibly gave Shelly more information than she had actually recalled.

Shelly wrote that the range from 200 through 249 would have *4N* ending in 2. Shelly identified that she did not know how to proceed, and made an error considering how her previous argument for ending digits related to the new question in three-digit numbers. She expressed that she "wanted to diverge from that test that we did with two digits, because I'm not clear what happens with this middle digit" (Transcript 3, lines 867-868). She was again unable to recall the actually reasoning behind the result. Shelly said that by this point that she was distracted because she wanted to go eat, so the researcher let her stop her work there.

Shelly's only comment about her strategies was that doing calculations long hand was somehow important to the way she approached the question. She said that the offer of a calculator had distracted her, and that she was fighting inside with her instinct to do it by hand versus the urge to go faster with the calculator. Using her own words, Shelly reported that she was unable to self-monitor at times. She said,

> Even though, getting bogged down in a procedure, I could sometimes lose track of why I headed there. Which is why I make these notes and stuff, so that I could go back and say, oh, where was I and why did I think about doing this? (Transcript 3, lines 911-913)

Shelly's progress on this question was made primarily by breaking it down into smaller proofs. She did make a plan of attack for the proof and followed through with that plan. She also organized her work to keep track of her progress on the question. Unlike the previous two questions, she was able to monitor the overall goal. However, she was still lacking in the ability to recall her own arguments. Again, she had to repeat justifications multiple times throughout her work on this question. Her view of the overall goal, to eliminate each number, could have contributed to her lack of confidence in the individual smaller proofs that she was producing along the way, as if the proofs could not meet this goal since she did not actually check each number. Unable to bring the entire proof together at one time, she did not see the parts of the proof that she had completed as legitimate proofs.

Summary. Shelly may have made more progress throughout all of the proofs in this interview had she been taught to keep careful track of the overall goal and plan for the proof. If she could have defined sub-goals for herself, been able to complete those goals recognizing their completion, and come back to the overall goal, she could have formed valid proofs in all cases. Her ideas and examples contained all the necessary components for the proofs, which other students in some cases were unable to develop, but in the end she was unable to complete the proofs in their entirety. In sharp contrast to another student, Ellen who was also a graduate student but in mathematics and with significantly more mathematical background, Shelly was not able to tie all of her ideas together or to monitor her progress towards a structure of a proof. She was also not as organized in her ideas and patterns of thought as Ellen was, in addition to being easily distracted from her thoughts and unable to recapture her ideas and the path that she was traveling in her work.

Shelly rarely attended the informal group study sessions for MATH 305 while she was enrolled in the course. When she did attend, the researcher observed both her confusion and frustration in working with the other students. Her thoughts were often derailed by other students' ideas and she had difficulty communicating her ideas to them as well. While her thoughts and ideas were often correct and could have led to a proof, even at times in a more efficient way than other students' work, Shelly had difficulty recognizing the validity of her own arguments and therefore experienced difficulties following through and reasoning them out with others as well. She would even talk herself out of a valid argument at times and be left where she had started her thought process in the first place.

Shelly did contribute ideas for examples and the visualizations to aid in understanding the question, her difficulties usually resided in following these understandings with a valid proof. As seen in this interview, Shelly also struggled a great deal with the motivation for writing a proof. Once she understood the basic idea and even the underlying reasoning for a proof, she lacked the desire and intrinsic motivation to produce a proof with sufficient argumentation and attention to detail.

Jon

Jon was a MATH 305 student. He was a graduate student in mathematics, however his previous mathematics experience was limited to only calculus I and II. He was also concurrently enrolled in linear algebra. Jon participated in two interviews. During the first, which was just under one hour in length, Jon worked on Question 1, and parts a and b of Question 2. His work was characterized by a search for equations to manipulate as well heavy emphasis on organizing his thoughts and looking at examples.

During the second interview, Jon worked on Questions 3 and 4. It was a short interview, only 23 minutes long. His work for Question 3 was well organized, he was able to keep track of his progress, and provided a somewhat complete proof of the question, though he was not satisfied with the results. For Question 4, however, Jon was not able to reach a proof in the end. He was still organized in his work and was able to keep his overall goal in mind, but an error in notation may have caused the inability to complete the proof.

Question 1. The first interview started with the researcher reading Question 1 aloud. Jon began his work by clarifying that the numbers were to be placed on the vertices and the middle of the sides of the pentagon. He drew a new pentagon and paused while considering the question. Then, he attempted to find combinations of three numbers with sum 14, and began a list of those combinations. Jon seemed to have a good understanding of the question right from the start and did not have difficulty with the rules of placing the numbers on the pentagon, as some other students had. However, he expressed concern that even if he could, in fact, find the appropriate pentagon with sums of 14, he would only be showing that 14 was possible, but not proving the statement. He said, "but I don't know if I'm gonna be able to prove it, rather than just show it" (Transcript 4, lines 31-32). Jon decided to continue his search for the pentagon after a moment to consider this, saying, "I'm not so sure if I'm going to be able to prove it, necessarily, so I want to just prove it to myself visually, more than anything" (Transcript 4, lines 42-43).

Jon went back to finding combinations, still placing them in a list. He said again that he was not treating the question as a proof, but more like a puzzle. During his search for combinations, he made a small arithmetic error, which the researcher corrected. After placing several numbers, on the pentagon, including the combination 9, 2, 3 with 9 and 3 on the vertices, Jon reconsidered this particular combination and opted to switch the positions of 9 and 2. He was beginning to understand the key element to the proof, that the lowest numbers must be placed on the vertices to achieve the lowest sum. After his first attempt failed, Jon stopped his work to state that there were too many combinations to just list them all and treat it like a puzzle. He started searching for another way to approach the question.

Jon's next idea was to consider the combinations in a broader sense. He said, "I'm trying to think of how I can have five sets of three – I'm contemplating how I can throw that into an equation right now" (Transcript 4, lines 97-101). He wrote the equation $14 = x + y + z$, but realized that this equation was not helping likely to be helpful. Jon was drawn to the idea of using an equation, he said, because he was thinking that 14 would have to equal a sum of three numbers, indicating that an equation might be useful. He continued to think of how to form an equation, but wanted to keep it in the simplest form possible.

Jon continued to search for an idea to begin his proof. He restated that the goal was to prove that the smallest possible value was 14, and decided to find the sum of the numbers 1 through 10, which he found to be 55. However, Jon stated that some numbers were used twice, and so it was not clear to him how to relate the sum of 55 to the overall question. He continued his thought of numbers being used twice and added together the numbers 1 through 5, finding a sum of 15, and stated that these numbers would be his intersection points, or vertices. Still, he was unsure how to fit the sums of 55 and 15 together for a solution. Jon had discovered a key point to the proof, but was not certain how to articulate his thoughts or how his ideas could form a proof.

In his next steps, Jon further developed the idea of using the smallest numbers on the vertices, saying there would be two small numbers and one large number in each sum around the pentagon. He then took the average of the larger numbers, which was 8, and 2 times the average of the smaller numbers, for a total of 14. He saw that this was the sum he desired, and was convinced of the validity of the statement to be proven, but he stated that what he had formed was not a proof. Jon struggled to find what he considered to be correct proof language to accompany his result, and said that he felt stuck at that point.

He also said that what he had done had proven the statement to himself, but he knew that he had not yet written a formal proof for others.

In an effort to continue his search for such a proof, Jon redirected his work, looking back at combinations to be placed on the pentagon. Keeping a list, organized by the largest number in the sum, he worked systematically through the options. Choosing from his list, he was able to fill in a pentagon very quickly. Jon moved on to attempt to form a proof that no sum lower than 14 was possible, but struggled with this. Observing that the current sums were 14, he first thought that 13 would not be possible, because he felt he would need to lower the combination 10, 1, 3 to include a 0 by reducing 1 to 0. He stopped himself, though, saying that the example pentagon had sums of 16, so his thought to just reduce the given combinations would not be true since it had not occurred in his pentagon with sums of 14. Jon then found that the combination 10, 1, and 2 would give a sum of 13 and observed that this was the lowest possibility for a combination including 10, but was still unsure how to prove that a pentagon with sums of 13 could not be possible. Deciding at that point to proceed with his former ideas, he began systematically writing out all combinations with sums of 13 and making an organized list. Jon drew a new pentagon and attempted to fill in those combinations, but his attempt was unsuccessful.

Jon wanted to look back to his thoughts on taking the average value of the two sets. He again recognized that his drawing of the pentagon with sums of 14 did not prove the question, but was unsure how to continue. The researcher allowed him to stop work on this question when he seemed to reach a dead-end trying to form the words of a more formal proof.

Jon's work on this question was often unproductive. He jumped from idea to idea. However, because he was able to identify work that was not beneficial, he did not get stuck in any one place. He was able to redirect his thoughts and attempts several times, allowing him to form at least half of the proof.

Jon had many good strategies that helped his efforts; such as understanding and clarifying the question, making drawings, keeping organized lists, and searching for equations. He also discovered the key to the proof; that the smallest five numbers must be on the vertices. He developed a clever way to see that 14 was indeed the smallest sum possible through finding averages of the sums of all the numbers. However, he was unable to clearly explain his result, and was not convinced that he had indeed found a proof. This could have been due to the fact that Jon stressed the use of proper language and proof techniques, and this proof did not look like any he had seen before. He also said that he felt he was approaching it as a puzzle rather than as a proof. Perhaps, if he had accepted his searches and ideas earlier in the proof, he would have seen the validity of his later arguments.

Question 2. Jon proceeded with Question 2, part a. After reading the question, he looked at an example, $N = 124$, to verify that he understood the definition. Jon then came to the conclusion that no two-digit number with repeated digits, such as 22, 33, etc., would work. He considered $N = 14$, but that also did not work. Jon stated that he was thinking of assuming that there was a two-digit 4-flip and finding a contradiction, although he incorrectly called this the converse of the statement. He struggled, trying to find an equation that expressed how the digits of N would flip when multiplied by 4. His equations are shown in Figure-Jon. 1.

$$xy = 4 \cdot N \qquad 4 \cdot xy = yx \qquad 4 \cdot (x+y) = yx$$

Figure-Jon. 1: Equations used to represent flipped digits in Question 2

He said that he would like to find a 4-flip that existed, but was assuming that none existed, so obviously that was not possible.

The researcher encouraged Jon to go further with his examination of the digits. Jon imposed restrictions on the digits of N, x and y. He said that x would have to be greater than 10 and less than 25, showing an error in his view of the digits, but he proceeded anyway. He restricted y to be between 1 and 10. He then stated that the only other thing he could think to do was go through all the choices. Jon seemed to correct his earlier mistake, saying that he only needed to check the values 10 through 25 for N, since after that 4 times the number would result in a three-digit number. The researcher encouraged him to go ahead with that idea. He proceeded to list all the numbers, N, as well as $4N$, and proved that none existed, by exhaustion of all the options. The researcher told him that he had a proof, but it was not clear that he knew that himself. Jon then said that this method was what he had thought first to do, but had discarded the idea in search of something else, such as the equations he was looking for to express the digits of *N*.

When Jon said that in any other situation, he would have sought out others to find a way to express the digits, the researcher offered to be that resource, in the interest of seeing where he could go with the information. She told him that the digits could be represented as $10x + y$. Jon was then able to represent the question with the equation $4(10x + y) = 10y + x$. He worked with that equation to try to solve for one of the variables. He simplified until he reached $x = \left(\frac{6}{39}\right)y$ or $y = \left(\frac{39}{6}\right)x$; neither of which, he stated, could ever be an integer. When he reached the point where he had only fractions, the researcher prompted him to consider whether any values of *x* and *y* would produce something that was not a fraction. Jon said, "Well – [pause] the only way, the only numbers I can multiply that to get it to be an integer would be – another fraction. Or a, or a um, or a common multiple" (Transcript 4, lines 463-465).

Seeing that Jon had no other ideas immediately forming, the researcher asked him to try the next part of the question. Jon immediately wrote a similar equation to that in part a, $4(100x + 10y + z) = 100z + 10y + x$, again trying to solve for the individual digits in an attempt to prove that no 4-flips were possible. He simplified his equation and solved for *y*, finding the equation $y = 3.2z – 13.3x$, and said that he was trying to think of specific numbers for *x* and *z* in the equations that would keep all the variables as integers. Jon said that he would start by guessing and checking different options for each of the digits. He began with *x* being 0, but recognized that this would cause *N* to no longer be a three-digit number. He described again, for clarification, the reason that *x* could not equal 0. Still focusing on potential values for *x*, Jon decided to try $x = 2$, since he wanted to make the decimal an even number in hopes of canceling with the other decimal value in the equation. Jon saw quickly that $x = 2$ would cause *z* to equal 9, to ensure that *y* would remain positive. When he substituted those values in, the result was $y = 2.2$. He then determined that the decimal values would actually cancel if $z = 3$, but that would result in a negative *y* value. The researcher recapped his work, and asked Jon to continue looking at values of *x*.

When prompted to take this result further, Jon stated that *x* could not be greater than 2, otherwise *z* would have to be greater than 9 to ensure that *y* remained positive. Since *z* was only a single-digit, this was not an option. He also said that he still needed to check that $x = 1$ was not possible. Jon then noted that the coefficient for *x* was a problem since it ended with a decimal value of 3, which caused him to reconsider and second-guess the necessity of each of *x*, *y*, and *z* being an integer. He was lead astray by thoughts that *y* could possibly be 2.2. The researcher attempted to correct the error in thought that the values could be non-integers; however, when asked to restate what *x*, *y*, and *z* represented, Jon still did not correct the problem. The researcher then allowed Jon to pursue the value $y = 2.2$. Jon stated that his values for *x*, *y*, and *z* resulted in the number $N = 924$, which he could not multiply by 4 to get 429, or vice versa; he later corrected this statement, saying that $N = 294$, but it still would not be a 4-flip. However, he realized that he had forgotten to include other limitations into the equations, and said that *N* must be less than 250. Jon was attempting to go further with his equations, but did not realize that he had actually found a proof that *x* could not equal 2 since his solution would violate the limitation that $N > 250$. He resolved in the end, though, that having $y = 2.2$ would not work since it would affect *N* and the flip differently.

When asked, Jon stated that he was still not sure that he had proven anything, but that he was certain that no three-digit 4-flips existed. He said that he was more comfortable with the proof of part a because, even though he had used equations after the fact, he had also been able to actually list out all the options and see that none worked. Since he could not try every number in this case, he was not convinced that his equations were valid.

Jon and the researcher discussed the steps that he had taken, and Jon elaborated on his own strategies (Transcript 4, lines 674-769). When discussing his strategies, Jon stated that he would normally first look for reference materials if he was stuck. He said that even in his math classes, he often found himself looking back through previous material covered in the course as a resource. The researcher said that she noticed his first step was to try to find an equation. He agreed that this definitely was the case and that when he could not do that, he then turned to looking at examples. Specifically, Jon said that he would look at examples to prove to himself that the statement was true first, prior to finding a way to prove it to others.

Jon used similar strategies in Question 2 as he had in the first, and these strategies were seen again in the second interview. There was again a clear desire to use equations, which, as mentioned, he was aware of and explained. He also used examples to gain an understanding of the definition. He made a plan to write a proof by contradiction, but struggled with carrying out that plan.

The use of equations both helped and hindered Jon. He was not able to fully develop the equations on his own and so resorted to further examination of the digits. This led to a major discovery in part a, that N must be less than 25. Jon was able to proceed with a proof by exhaustion, but his desire for an equation had at first trumped the idea that, in the end, resulted in a correct proof.

However, Jon still desired to work with equations to form a proof. After the researcher gave Jon the information he needed, it consumed his work in part b. While the equations did result in a proof, Jon was so wrapped up in them that he was unable to see his proof. It also caused him to forget his main assumptions and make errors that led him astray. In the end, he had been convinced by his work that no three-digit 4-flip existed, but not that he had a valid proof. Overall, he lacked the ability in this question to redirect his attempts, and also a lack of understanding of the variety of valid proof techniques available.

Question 3. The second interview began with the researcher reading the question aloud, pausing to allow Jon to read it himself midway through. Jon jumped right in, drawing a rectangle and clarifying the definition of adjacent, for which he did not at first have a correct idea. He stated that he already knew, in his own mind, that one of the dimensions must be even, since the dominoes were even. He drew an example chessboard of size 2-by-4, and then added another column to the chessboard to illustrate his understanding. He also drew a chessboard of size 3-by-3, covered with dominoes, and noted that there would be one square that remained uncovered. Jon then said, “I guess, again, just like last time, I’m I am wanting to put it into an equation, but I, I can’t” (Transcript 21, lines 37-38). He reread the question and proceeded to find equations to work with, writing out that if n was even he could say that $n = 2x$.

Jon said, "Well, I guess I'm going to try to prove the contradiction. So, try to show that it can work, when they're both odd" (Transcript 21, lines 40-42). However, he had made an error in this statement, as this would not be a proof by contradiction, and would actually not prove the statement if he could prove this to be true. It was not clear that this idea was fully formed in Jon's mind, and he seemed to move on without noting his error in assumptions. Still desiring to form equations, he represented both dimensions being odd in the equation for their product, shown in Figure-Jon. 2.

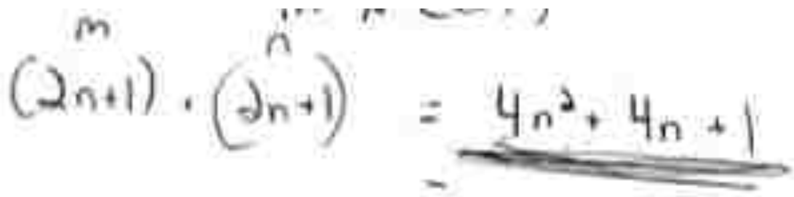

Figure-Jon. 2: Equation for product of two odd numbers, with errors, in Question 3

However, his notation was not correct, since he used the variable *n* improperly and in both dimensions. While this still had the potential to help him in the proof, it may have also led him astray when working further. Jon substituted $n = 1$ and $n = 2$ into his equation before he realized that the result would always be odd. He did not seem to understand why this would always be true, again this could have been a result of not being able to link the actual dimensions of the chessboard to the total number of squares. He had become so focused on the equation that he seemed to forget what it actually represented.

Jon then stated that to have a perfect cover, the product, *m* times *n,* would have to be divisible by an even number. He wrote this as shown in Figure-Jon. 3.

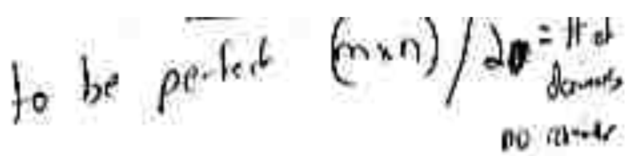

Figure-Jon. 3: Equation to represent divisibility of a perfect cover by 2 in Question 3

Here he noted that dividing the product by 2 would result in finding the number of dominoes with no remainder. After writing this, he said that he did not know how to show that it was true. He stated that he had been trying to show the proof by contradiction, but was unable to form the proof. He said, “But I don’t – but I’m not really writing a proof” (Transcript 21, lines 63-64).

The researcher encouraged him to proceed, asking him to describe what he would like to be able to show. Jon pointed to his result, $4n^2 + 4n + 1$, and said that since this was not divisible by 2, the perfect cover would not exist; but, he still did not feel that he had proved this. Instead of pursuing that further, he decided to try a proof by cases. Writing out all four cases for the parities of m and n, he said that only the cases where at least one dimension was even would result in an even product. He again did not know what to do to finish his proof, saying, “I think this is what I want to do, but I guess I don’t. [pause] I don’t really know how to finish it I guess” (Transcript 21, lines 88-89). It was not clear that he recognized the beginnings of his proof as fulfilling the bi-conditional statement, nor did he even mention the need to show both directions or that his cases would cover both directions in the end. The researcher said that he could move on if he wanted to do so, and Jon decided to move to the next question.

Jon later stated that he agreed he had gone straight to drawing a picture in Question 3 to try to understand the question. He said he then tried to think of what would happen when both dimensions were odd and when he was unable to complete that process, had turned to a proof by cases. As he worked on the proof, though, Jon remained unsure of his work, saying, “I don’t know if that’s necessarily right” (Transcript 21, line 181). When asked if he had seen anything that had helped him in his work, Jon stated that he had previously seen proofs by cases. He also said that drawing pictures helped to visualize the question and see the pattern and how it worked and that it led him to trying such a proof.

As previously stated, Jon had a strong desire to use equations in his proof, as he had in his first interview. This often led him to focus too much on the details of the equations, and not on the overall picture of the proof. While he was able to find the general outline of the proof, he did not recognize this, nor did he elaborate further on his proof by cases. He also never mentioned the statement as being a bi-conditional, or having two directions to prove, in any way. His strategies included reading and understanding the question, drawing pictures, looking at examples (including recognizing the need to look at examples of different parities), proving to himself that the statement was true, identifying and using proper proof techniques, developing equations, making a plan, redirecting after failed attempts, and monitoring his attempts. The main difficulties that arose for Jon were the inability to see his work as progress towards a proof, and also the inability to generalize appropriately from examples. He also struggled due to his overwhelming desire to find equations, and the repeated idea that a proof without symbols or equations was not a real proof. This idea had surfaced during his first interview; although Jon did not specifically mention it during this interview, he also did not see his work on this question as the beginnings of a valid proof.

Question 4. After the researcher read the next question, Jon again jumped into equations immediately. He recognized that the first two parts, that *a* divides *b* and *b* divides *c*, were given, and the third was to be proven, but was not able to make the connection between them. Unfortunately, he began by using the same constant in both the equation for *a* divides *b* and for *b* divides *c*. He also made the common mistake of switching the order of the variables (see Figure-Jon. 4).

Figure-Jon. 4: Equations of assumptions with errors in Question 4

Jon continued by solving for *k* in one equation and substituting into the other, which only confused him further. During a discussion of what he was trying to find, Jon realized that his equations were written with the variables backwards. He rewrote the equations, noting which portion was known and which needed to be shown (see Figure-Jon. 5).

Figure-Jon. 5: Equations with one error corrected in Question 4

While Jon now knew the general idea of what he desired, he still used the constant, *k*, in all equations, leaving him unable to see how to prove the statement, and causing him to work in circles. The error continued throughout his work, and he was never able to recover from it. After further work with his equations, he determined that $b = a$, which was clearly not correct in general but he was unable to identify this error. He concluded that since *b* divides *c*, and $b = a$, then *a* must also divide *c*. While he did not specifically address his error, Jon did seem to be aware that something was wrong and was not sure that he had reached an appropriate result. He said, "Maybe? I don't know – I'm uncertain, but this is probably what I would hand in, I guess" (Transcript 21, lines 137-141).

The researcher asked him to go back and discuss his strategies. Jon stated that he had not seen either question prior to the interview, but that Question 3 seemed more familiar. He then described his work on Question 3, summarizing his end result. Jon further agreed that in Question 4, he had focused on manipulating the equations and that he preferred that technique. He commented on his work in question 4, saying,

> Well, at least I can start writing stuff instead of just brainstorming. – So, that's why I just prefer to try to just get right into it – it's like an if then. So, then I was just saying that this is already proven, or we already know it, [circles $b = ak$ and $c = bk$ at bottom of page] so that's why I was thinking we might as well insert the *bk* for *c* and find that *b* equals *a*. (Transcript 21, lines 211-215)

While Jon did carefully note the portions of the question to be assumed and those to be shown, his errors in writing out what those meant caused him to be unable to complete the proof. Unlike other questions he attempted, in this question he never drew any pictures, nor did he try any examples. This may have been because he was able to immediately work with equations. As he stated, he was able to start writing things down right away, without having to brainstorm. Unfortunately, the process of brainstorming that he had skipped may have allowed him to understand the relationships between the variables, but without this, he could not see past the equations. When he reached the point where he concluded that $b = a$, Jon was so engrossed in the equations that he did not recognize the error in this result. He was able to complete what seemed like a proof with only equations, but he still was unsure of the end result, perhaps due to his error. His strategies included only forming equations, making a plan, and noting what was known and what needed to be shown. He was able to correct one error while considering his work, but it was not enough to complete the proof in the end.

Summary. Jon had a tendency to search for equations to manipulate when asked to prove something, since that was what he viewed as a proof in MATH 305. He described the idea of proving during the first interview, saying, "what I think of, when I think of proofs is, is some equation, not English words in a sentence" (Transcript 4, lines 768-769). This statement explained his reluctance to accept his proof of Question 1 as well as his proof in part b of Question 2, since there was not a clear contradiction from an equation in his mind.

As mentioned, there were times that the use of equations helped Jon and other times that it hindered him, depending on the particular question. There does exist a valid proof for each of the questions that Jon attempted which involve a heavy use of equations, however, Jon in most cases was not able to find the appropriate equations or to use them for his proof in the end. His search for equations was often times random and disconnected from the overall question. In particular, after finding a basic equation, Jon would manipulate and attempt to form a proof with it, but without keeping in mind the main goal or purpose of the equation as it had originally been used. This resulted in an inability to form a full proof for many of the questions.

Overall, Jon had many good ideas and discovered some of the key points to several of the questions. However, his inability to keep the main picture and goals in mind, in addition to his lack of understanding of what would constitute a valid proof, prevented him from finishing the questions.

Jon was a MATH 305 student that was observed throughout the semester during informal student homework sessions. During these times, his use of equations was seen often and would lead him astray many times. He was commonly unable to articulate his goals to others, and they were unable to follow his reasoning as he manipulated equations. There were times that his proof via equations was correct, but even then, if other students disagreed, Jon was unable to defend his result, as he was sometimes unsure of its validity himself. Jon was a bright individual who clearly understood the mathematics involved in new concepts the majority of the time, but had difficulties in understanding and recognizing the validity of a proof. He struggled to accept the proofs of others as well as to form a clear outline of a proof before beginning his work. These difficulties were also seen during the two interviews and the researcher therefore believed the interviews to be an accurate picture of Jon's work and normal tendencies during proof-writing tasks.

Beth

Beth was a MATH 305 student who participated in two interviews. She was a sophomore majoring in mathematics. Her previous coursework included calculus I through III, differential equations, and linear algebra (which she was taking concurrently with MATH 305). Beth worked on Question 1 during her first interview, which was 55 minutes in length. Beth's work on this question involved drawings and lists, which she kept fairly well organized as she went. Her organization was also present in labeling the pages she worked on and desiring to rewrite her pentagons so they were neat and tidy. Throughout the first interview, she was very good at expressing what she was doing and thinking aloud. Beth was able to monitor her progress and think about multiple tasks at one time.

In the second interview, her work and thoughts were not as clearly organized. Beth jumped from page to page in her work, and did not seem to be monitoring her progress as well as she had on the previous question. While she did keep a running list to help organize her work, the rest of her pages were more scattered. During this interview, which was approximately 52 minutes in length, she was able to form verbal proofs of both parts a and b of Question 2, but did not attempt to express them further in writing.

Question 1. After the researcher read Question 1 aloud, Beth asked if all numbers were listed only once on the pentagon. When that was answered, she said that she understood the question, and restated it aloud. Beth made a plan to look at the combinations of the numbers 1 through 10, clarifying that she wanted combinations of three numbers. Her first remark in her actual work was that the smallest sum would be 6, with the combination of 1, 2, and 3. However, given that the question asked to prove that 14 was the smallest, Beth assumed that the statement was implying that sums of 6 through 13 would not be possible. Seeing that, she decided to try to find a pentagon with sums of 14 as a starting point to proving the statement. She stated that she was trying to understand the question, citing her MATH 305 instructor's advice to do so before proceeding with the question.

Beth started to build her pentagon, beginning with 10 and looking back at the given example. She immediately thought the smaller numbers should be placed near 10 on the pentagon. An important aspect of Beth's work was her ability to monitor her progress and her capacity to think about multiple tasks at the same time. At this point in the interview, Beth was looking at the example pentagon and her new pentagon, attending to thoughts of putting the small numbers around 10, as well as forming the ways in which numbers could sum to 14. Amazingly, she seemed to stay on top of all of those thought processes at once and was able to modify her overall plan as appropriate. She asked and answered her own questions about the statement of the question and about her next directions. While she only completed one question during the hour, she did so in a clear, organized process that resulted in a valid argument.

Beth stopped her search and regrouped. She noted that if she could find the pentagon for 14, she would still need to prove that sums of 13 or less were not possible. Wanting to find anything she could about the 14 case that would help her with 13, she continued to examine the case of 14. She looked back at the example pentagon given, and again struggled trying to place numbers on her pentagon. Since only one combination with a sum of 14 included 10, Beth noted that 10 must go on a side instead of a vertex.

> I guess these have to go here [puts 3 and 1 on her pentagon with 10, pause, (10 sec)]. Can we use 0? No [pause (15 sec)]. Okay, if that doesn't work, maybe that means that 10 doesn't go right there. [draws new pentagon with 10 on a side] Maybe we can share that big of a number. (Transcript 5, lines 64-67)

Again, Beth looked back at the example pentagon, searching for patterns, but was unable to find any useful ones. She then tried to check more options in her pentagon, and wrote a list of what numbers she had left to place (Transcript 5, lines 73-92). Beth was systematically going through her options and making some good and some bad choices. She stopped to say that the question reminded her of a puzzle, specifically the popular logic game, Sudoku. Then, she continued searching for the correct pentagon. When her first attempt failed, she decided that 14 might not be possible. However, during the process of placing the numbers, Beth had made an error on one of the entries in her pentagon, which the researcher corrected by asking her to look it over again. When she went back to check the numbers, Beth asked herself what her reasoning had been when she placed the numbers there, checking her logic along the way.

Beth made another attempt at placing the numbers on the pentagon (Transcript 5, lines 111-116). When the pentagon she was trying to fill became too messy, she drew a new one, noting that she liked things to be neat and that she considered herself to be a visual learner. Moving on, she made yet another attempt at the pentagon (Transcript 5, lines 125-131). When she expressed her frustration, the researcher encouraged her to continue her attempt, which she did. Beth looked again for a pattern in the example pentagon with sum 16. If she had been working on a homework question, she said, she would normally stop at this point and seek help on the question from others. She stated that usually she does homework with someone else and would ask what he or she had come up with. Beth drew a new pentagon and resumed her search. Considering her earlier assumption that the smallest sum could be 6, she now realized that a sum of 6 would not be possible, because the other numbers on the pentagon could not also be that small.

> This is the first thing that I wrote down [pointing to where she wrote 1,2,3 – 6], um I know isn't possible cause, the numbers left 4,5,and 6 [writes 4, 5,6] – there's no way to make 6 with that since 6 is going to be on one of the sides, there's no way to get less. (Transcript 5, lines 169-171)

She was then able to see that the smallest combination would be 10 with 1 and 2, resulting in a sum of 13.

After realizing she only had one other sum to check, Beth decided to make a new plan to find out why 13 was not possible. She began a new sheet of paper, and wrote down the combination 10, 1, and 2, and listed the unused numbers to the side. She continued by listing the possible combinations with sums of 13. While doing so, she noticed that she could use numbers twice, and decided to draw a picture to keep track of her choices. Beth paused to say that she felt that she really needed to show 14 was possible before showing that 13 was not, incorrectly calling this an induction proof. However, she went back to finding 13, saying that "I'm just going to try to see if this can jog anything" (Transcript 5, lines 221-222). She rediscovered that 10 would need to be placed on a side instead of on a vertex; a discovery that had taken much longer for Beth to link between the cases than it had for most other students.

Seeming to be frustrated that she could not think of another way to go about the question, Beth stated that it seemed "like a lot of guessing and checking" (Transcript 5, line 236). She felt foolish that she could not find a proof, but the researcher reassured her that the questions were meant to be challenging. Accepting this, Beth proceeded to place numbers on the pentagon for 13, again, but said she was just randomly trying numbers. She stopped to say that she knew the pentagon would not work from the statement of the question, but then expressed doubt that she should assume the statement to be proven was correct. However, she decided to assume that it was and proceed, confident that 13 would not work, but was not sure how to prove this.

Beth decided that guessing and checking was not a good way to approach the proof. The researcher encouraged her to look for any other forced choices, such as the combination with 10, 1, and 2. Beth went back and made a list of all possible combinations, starting with 10 and working down, listing them according to the largest number included in each. She kept track of which choices were forced, and decided to finish the entire list, hoping to see a pattern. Once the list was made, Beth was able to quickly determine that she could show 13 was not possible since the combinations were limited and she would be left with no way to make a combination including 7 without using one of the numbers a third time.

When she finished the proof for a sum of 13, Beth decided to try this process for 14 as well, instead of looking only at a picture. She proceeded to list all possible combinations with sums of 14, starting again with those including 10 and working down. She kept careful track and was organized, grouping the combinations by number as she worked, though she did make a few small arithmetic errors that the researcher helped correct. Beth had written the combinations for 10 through 8 when she had an idea about how she might have been able to proceed with her proof, but decided against pursuing that idea. Continuing her list, she wrote the possibilities first to the side, going through each option for the first number, then looking at each possibility for the second number within these groups. She realized there were repeats within each group (e.g. 9, 4, 1, and

9, 1, 4 both in the 9 group), but said that she had listed both to be systematic. She eliminated these repeated combinations before adding to the main list. However, she was still listing some repeated entries, not realizing that once she had reached the lower numbers as the first number in the combination, everything had already been listed as an earlier combination (e.g. 9, 4, 1, and 4, 1, 9 were both listed). At that point in the interview, Beth's ability to maintain vision of her overall plan started to fade. When she reached the combinations involving 3, she thought that she had been forgetting 10 in all the lists. However, the researcher reassured her that she had not and that 10 was not in any combination in the groups 9 through 4. Beth completed her list, finding all combinations in groups 10 down to 1. However, she had been too wrapped up in the systematic process to see the reasoning and logic of her choices, as evidenced by the previously mentioned errors.

Once she completed her list, Beth seemed better able to cope with the overall goals of the question. She said that she needed to choose combinations from this list for the pentagon, and decided to first consider the possibilities for 8 and 6.

> So, now comes the part I guess that's kind of fun. You know that this is the only way [puts box around 1,10,3 on previous page] and now maybe what I can do is I see that a lot of them have 1s in them. And I wonder if let's take 8, if we look at 8 and 6, these have the least amount of options, they can either have the one with a 1 in it or the other one. (Transcript 5, lines 458-462)

She knew that 10 must also include 1 in its combination, and so she could only choose one of either 8 or 6 to also be with 1. She picked one choice and started to consider what would follow from there. Beth incorrectly stated that she would have to choose 10 combinations, one from each group in her list. As she considered this task, Beth wondered about an existence proof, though she did not use this term, asking herself if she really needed to find the correct pentagon or if she could somehow prove that the choices could potentially be made without actually explicitly stating which combinations would be chosen.

The researcher asked her how she would prove that 14 was possible. Beth said that she would just need to find one way to make five pairs, correcting her earlier mistake. She tried to find an appropriate set of five combinations, beginning by listing her choices so that numbers in common were lined up vertically (see Figure-Beth. 1).

Figure-Beth. 1: Organized partial list of combinations summing to 14 in Question 1

She circled those numbers that would become vertices and then began placing her combinations on a pentagon (Transcript 5, lines 489-506). Beth suddenly stopped her work, saying that she was working like a robot, forgetting the bigger picture. Then, she proceeded to list out what numbers she still needed, in an effort to organize her thoughts. The researcher pointed out that each combination occurred in her original list three times, which corrected the earlier error. While Beth was unable to find the correct pentagon on this attempt, she did step back to consider where to go next and what she should be thinking about if she tried again.

The researcher prompted Beth several times to reconsider her previous work and see what could help her now (Transcript 5, lines 573-602). Eventually, Beth was able to recall that when she first began her work, she switched 10 to be on a side instead of a vertex because it had caused difficulties. She then used that thought and switched 8 onto a side as well, which allowed her to continue and to find the correct pentagon. She rewrote the pentagon and summarized her conclusions for both 14 and 13.

Beth said that she would now have to show sums of 12 down were not possible. "So, I can't use 10 to go with 12, because 10, 1, and 2, the lowest combination is 13 so I'd have to start with 9" (Transcript 5, lines 656-657). Forgetting that she had assumed earlier that anything below 13 could not work at all, Beth thought that if she just started building with 9 instead, she might find what she needed. The researcher encouraged her to continue her proof by examining 12 with the same format as she had done 13 and 14. She reminded Beth that she started both the 13 and the 14 case with finding what had to go with 10. Through this idea, Beth discovered again that 12 was not possible, but did not recognize that her arguments would actually prove this. The researcher prompted her to reconsider her thoughts. Beth was then able to recover her proof that none of the sums below 13 would be possible. She summarized what she had done for the proof. She also thought about looking at 15, but realized that it would not be necessary for the proof.

The researcher asked Beth to reflect on her strategies and they discussed together how she had proceeded through the question. Recapping, Beth noted that at the beginning of the question, she was just guessing and checking to find a picture of a pentagon with sum 14. She also said that disproving what she was not including, namely 13 and below, was another technique. When asked, Beth agreed that at some point she had to convince herself that the statement itself had been true, and that she had looked for patterns. In the end, she still seemed to desire a more efficient way to prove the question, but was convinced that her proof was valid regardless.

Beth's work on this question was visually well organized and she was able to develop a complete proof. However, she stumbled several times as she became engrossed in her work and forgot the bigger picture. She had a clear plan, which began with fully understanding the question. Beth had even said that she began in this way based on instruction in MATH 305. Her overall goals were to show 14 was possible by finding a correct pentagon, and also to show that sums of 6 through 13 were not possible.

Beth was able to monitor her trials well and also organized the information she knew. While she did have to make several attempts at finding the correct pentagon for 14, she also stopped herself before becoming too overwhelmed and tried new ideas when appropriate. She searched for patterns in the example pentagon and also moved on to look for a pentagon with sums of 13 to gain more information. In this process, Beth was able to discover the key elements to her proof, as well as to find ideas necessary to complete her pentagon for 14. She did have one issue with recalling her earlier reasoning for sums of 6 through 12, which could have been due to the fact that the idea had been merely an afterthought when it was originally stated rather than a key point in her mind. She clearly knew when she had finished her proof and was able to recap all of her work and summarize the proof aloud.

Beth made use of many good ideas and strategies during the first interview including: understanding and restating the question, making a plan with goals and sub-goals, breaking the question into smaller parts, looking at examples, drawing pictures and tables or lists, looking for patterns, self-monitoring her trials and organizing her work, redirecting herself after failed attempts, and recognizing her proof when it was complete. While she did complete a proof, it was certainly not sophisticated, done only with brute force and trial and error, which took Beth a great deal of extra time and effort. Key strategies, of those listed above, that seemed to lead to her success were making a plan, breaking the question into smaller parts, and monitoring her trials. Beth made lists and kept careful track of her work. She did get lost at some points and wrapped up in the process, but her ability to bring herself back on track and keep in mind her plan and the original question allowed her to eventually find a proof.

Question 2. The second interview began with the researcher reading part a of Question 2. Beth seemed to understand the definition of a 4-flip immediately, reading the question aloud, then looking at an example, $N = 1234$. Writing that she was looking at N times 4, she tried to write a backwards N to show that she would want it reversed. This showed her attempt to visualize the situation (see Figure-Beth. 2).

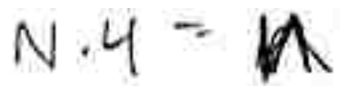

Figure-Beth. 2: Representation of flipped *N* in Question 2

Beth stated what she needed to show, then looked at another example, this time of a two-digit value, $N = 24$. Again wanting to have some way to represent the digits in reverse, she now wrote that she wanted to find a number X that went to a new number, Y, which would be the reverse of X. She again visualized this, as shown in Figure-Beth. 3.

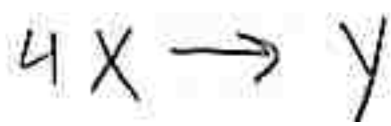

Figure-Beth. 3: Second representation of flipped digits in Question 2

She identified the possible numbers to check as those integers in the interval 10 through 99. She seemed to limit this interval further, saying that the smallest possibility was actually 40, but she looked next at the example, $N = 22$, implying that this limit of 40 actually applied to the values of $4N$ not N. Her desire to represent the digits continued as she wrote $4N = M$ and $4(ab) = ba$. She looked back at the example, $N = 24$ and noted that she wanted $4N = 42$, but stated that this clearly did not occur.

Next, Beth decided to look at three-digit numbers, assuming that a 4-flip would exist in this case. She wrote the incorrect equation $4(cde) = ecd$ and paused. Beth said that she was trying to find an example of a 4-flip to look at, but it was difficult to do so. When asked, she described that she was thinking of how the digits would multiply by 4 and where the resulting digits would be in $4N$. She considered another example, $N = 234$, saying that it also was not a 4-flip. She temporarily made an error in what she would have wanted to see for $4N$, but the researcher quickly corrected this. Beth looked back at the individual digits, saying that the 4 of 234 multiplied to get a result of 6, so the original N would have had to begin with 6. She paused and stated, "Gosh, these, these problems make me feel dumb" (Transcript 16, line 77). The researcher reassured her that the questions were meant to be challenging and to make her think, but Beth responded that she did not know where to even start. These feelings had surfaced during the first interview, and did so again later in this interview. The researcher suggested she look back to the two-digit case, and reminded her of the earlier statement that she would test the numbers in the interval 10 through 99.

Beth said that if she had all day, she could check each number 10 through 99 and show that they did not work, but she did not want to do that. Instead, she decided to pursue looking at the first few, stating that she would try to find a pattern, but not yet realizing the upper limit was $N = 24$. She looked at N equal to 10, 11, 12, and 13, and then jumped to 20 and 21. When she stopped there, the researcher encouraged her to continue her attempts, saying that she may find something if she continued. Beth checked 22 and 23, but then jumped to the 30s. After Beth had checked up to 33, the researcher asked if she had any ideas forming. Beth noted that the largest that the flip of N could be was 99, and so decided to move on to trying the 90s. She also noted that $4N$ could not be smaller than 40, however she did not see any ideas for the proof yet.

Beth went on to write out the examples 90, 91, and 92, before pausing to consider her work. It was not until then that she was able to see that $N = 30$ was too large, because the result of $4N$ would be three-digits instead of two. When prompted, she recognized that this would also hold for all values above 30. At this time, she began a list of those numbers that she had been able to eliminate, for discussion purposes it will be referred to as the *not possible list*, writing the interval 30 through 99 first. This began her attempt to organize her work and was the beginning of her proof. Again, forming a list and keeping careful track of her attempts would lead to her success. Beth noted that she now needed to look only at the first sets of numbers, the 10s and 20s. She looked back to the 10s list, observing that the largest value of *4N* in this set was 76. She said that since the values of *N* flipped went from 31 to 91 and the values of *4N* went from 52 to 76 in those she had not yet checked, she felt that it might be possible for a 4-flip to exist, though she did not complete her search in this set at the time.

Moving to the list of 20s, Beth felt that there could be such a point in this list as well, observing that the largest value of *4N* would be 116. At this time, she double-checked her previous calculations for the 10s list and verified that they were correct. She then noticed that 116 was a three-digit number and so she could expand her range of eliminated numbers. On her not possible list, she changed the interval 30 through 99 to the newly found interval 25 through 99.

Beth stopped to ask herself why the values would not match up in the interval 10 through 19. Deciding to look at the multiples of 4, she listed the values 4, 8, 12, etc. up to 40 and checked for any that might end in 1, since 10 through 19 flipped would end in 1. She momentarily wanted to check the multiples of 4 from 4 times 10 through 4 times 19, but caught herself and said that they would still end in the same digits as those she had already written down. She said, "I mean, I'm not saying it eloquently, but since – how do you say it? Sort of like divisibility by 2" (Transcript 16, lines 274-275). She was not satisfied with her explanation, but clearly knew that she had done what she had intended in that interval. On her not possible list, she added the interval 10 through 19, saying that this left only the numbers 20 through 24 to check. Since she had already looked at all those except for 24, she quickly did this calculation and said that she had seen by brute force that this interval did not contain 4-flips. Like several other participants, however, Beth continued to search for a different way to prove this, seemingly unsatisfied with her justification. It was not clear whether or not she believed she already had a valid proof. She again turned to the individual digits and noted that she needed the result, *4N*, to end in 2. As she struggled with this, the researcher stopped her and summarized what she had done so far. Afterwards, she asked Beth if she had proven that there were no two-digit 4-flips. Beth said that she had.

When asked to recap her strategies, Beth noted that she had first made sure she understood the question and the definition of a 4-flip; then, she had tried to find an example of a 4-flip. Not finding any such two-digit numbers, and realizing that she would not find any since she was trying to prove that none existed, she next tried to find an example in the three-digit numbers. When this also did not work, she began to list out the two-digit numbers in hopes of finding a pattern. She said that she divided the possibilities up into smaller groups of 10 and proved that no 4-flip existed in each group.

When the researcher moved on to part b of Question 2, Beth noted that it was a good thing that she had not continued searching for a three-digit 4-flip, since this portion of the question suggested that there might not be such a number. Beth then reread the question and noted that the statement could be true or it could be false, since it said prove or disprove. She proceeded by narrowing down the choices with identical procedures from part a, noting that she was using the same techniques as she had in that portion of the question. Beth first said that her possible three-digit numbers were 100 through 999, and that she would want to use reasoning to rule out intervals rather than use brute force since there were so many numbers. She also wanted to divide up the interval into nine groups of 100 numbers each. Knowing that there would be a point were $4N$ would change to a four-digit number, she thought that she might list out the first few numbers in each group to determine where this point would occur. When the researcher asked how she might find the actual point, Beth decided to divide 1000 by 4, finding the cut-off to be $N = 250$. She noted the connection between this and the number 25 from part a of the question. Beth again wrote a not possible list to the side of her work, beginning by writing that the interval 250 through 999 would not contain a 4-flip because the numbers were too large (see Figure-Beth. 4).

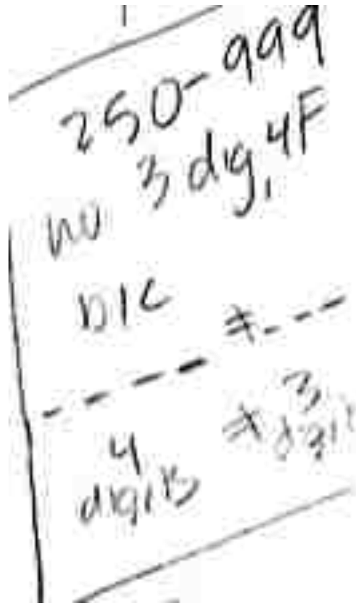

Figure-Beth. 4: Portion of not-possible list in Question 2 part b

Looking back at her work for part a, Beth decided next to look at the interval 100 through 199, and used the same reasoning as she had in part a to eliminate the entire interval. She added this interval to her not possible list, and summarized that she only had the interval 200 through 249 remaining. From there, Beth began investigating by looking at the first few remaining numbers and the last one as well. Since she had not found a pattern in part a for the corresponding interval there, but rather had just proven them by brute force, she had no plan for how to proceed. Beth recognized that the values of *N* flipped would always end in 2, and she again examined how the individual digits would be used. She believed that there would be a cut-off point where the value of *N* flipped would be below 800, and since all values of *4N* would be 800 or more, these could then be eliminated. However, Beth soon saw that this would be a small window, only up to 207.

Since she had not found a pattern or larger rule for eliminating intervals of numbers, Beth felt that she might just look at all 50 remaining numbers using brute force. She noted that she was trying to disprove the statement; however, the researcher believed that she meant that she would be showing that no three-digit 4-flip would exist (thereby actually proving the original statement). The researcher summarized what Beth had done so far, and Beth wrote the interval 200 through 207 on her not possible list. She was not sure where to go next, but decided she might need to break the remaining numbers into smaller intervals of 10. The researcher asked her to go back to the idea that *4N* must end in 2 for the set of numbers she had left.

Beth continued by breaking the numbers 200 through 249 into smaller intervals, 200 through 209, 210 through 219, etc. She then asked herself whether ending digits of 12 were options for *4N*, corresponding to the interval 210 through 219, and she concluded that they were. She moved on to the ending digits of 22, saying that *N* could end in 3 to give the appropriate last digit of *4N*. Looking back at her chart for values of *4N*, developed in part a, she noted that she would have a carry-over of 1 when multiplying by 3, which would therefore never give 2 as a middle digit for *4N*. She said, "4 times what plus 1 will end as a 2 and I don't think anything will" (Transcript 16, lines 590-591). The researcher asked if there was any other option for the ending digit of *N*. Beth said that 8 would also work, again giving a carry-over of an odd number. She concluded that the middle digit must therefore be odd, ruling out the interval 220 through 229, and adding this to her not possible list. She examined the interval 230 through 239, and found that this interval would still be possible. She then ruled out the interval 240 through 249 using the same reasoning, again adding this group to her not possible list. However, she did have to examine the multiplication for 3 and 8 each time in order to complete the argument.

Keeping track of what still needed to be completed, Beth wrote down the intervals 208 through 219, and 230 through 239 as the remaining possibilities to be checked. She again stated that she would disprove the statement, adding, "Which makes me wonder if there is such a thing as a 4-flip" (Transcript 16, line 666). The researcher told her that 4-flips did exist. Beth wondered how she might find one, saying that there must be a formula and it would exclude three-digit numbers; but she then was able to direct herself back to her current work on the question. She started a new page and wrote that 208 through 219 and 230 through 239 were the remaining possibilities to check.

Beth chose to tackle the interval 230 through 239 next, writing out the calculations. Thinking back to her reasoning for eliminating 200 through 207, she stated that the current interval resulted in values of *4N* greater than 900, so she only needed to examine the value of *N* flipped that began with 9, i.e., $N = 239$. She was momentarily sidetracked with the thought that she may have found a 4-flip, since 932 ended in 32, which was a multiple of 4. However, she was able to explain why it still was not a 4-flip and direct herself back to her work again. She added the interval 230 through 239 to her not possible list. She was now left with only the numbers 208 through 219, she said, and decided to write out all those numbers. There was a discussion that followed where Beth mentioned again that she felt silly in the way she was proving the question, but the researcher reassured her that it was her own particular techniques that were of interest and that they did not make her any less smart than anyone else (Transcript 16, lines 717-741). Beth seemed encouraged by that.

Back on track, Beth looked again at the numbers 208 through 219. After examining both ends of the interval, she noted that there might be a chance for a 4-flip to exist since the range of *4N* would intersect the range of *N* flipped within the interval. She wrote out a list of the values of *N* flipped and noted that her values of *4N* ranged from 832 to 876. Therefore, she only needed to check *N* = 218, since it would be the only other possibility to have a flip beginning with 8 (see Figure-Beth. 5).

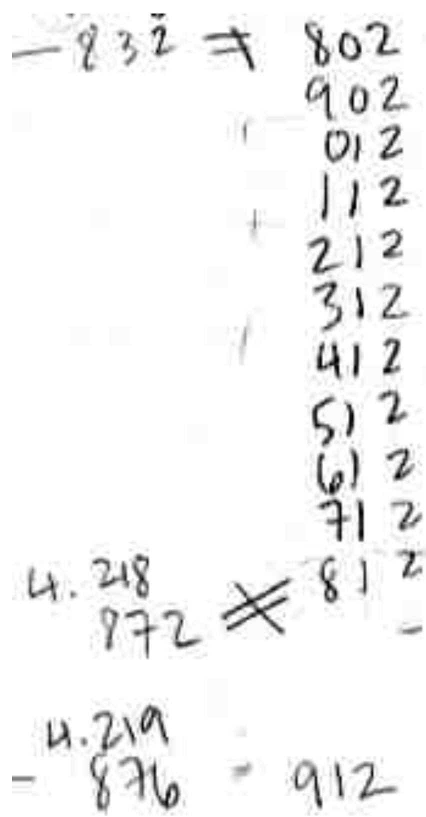

Figure-Beth. 5: List of *N* from 208 to 219 flipped in Question 2 part b

The researcher and Beth then recapped her work. Beth finished by adding 208 through 219 to the not possible list. When asked, she stated that she had not seen a question like this before, but that in her abstract math class, MATH 305, she had learned to consider integer divisibility. She felt that this had been key to her finding a proof to the question.

Beth was able to complete a proof for this question verbally, though it had not been well organized throughout. She confirmed that she finished a proof for part a, but never clearly stated the same for part b. Her work relied on much visualization of the numbers and the concept of reversing the digits. Beth struggled to find representations for the digits, and instead choose to look at the possibilities for each, rather than considering them as a whole in some sort of an equation. However, she expressed several times that she had wanted to find such an equation that would have allowed her to find an example of a 4-flip.

Her strategies during this question included reading and understanding the question, looking for examples, drawing pictures, searching for equations, breaking the question into smaller parts, looking for a pattern, looking systematically through the choices for digits, making a plan, monitoring her choices and progress in a list, writing what was known and what needed to be shown, and recognizing her proof, including the proof by brute force. While she desired to find a proof that did not rely on brute force, she also recognized that her proof had been valid on its own. The keys to her success seemed to lie in looking at examples, understanding the definition, and systematically working through the choices, keeping careful track of what she had concluded in a list. The use of a list appeared in both questions. Beth's struggles arose as she became too involved in calculations and lost sight of her goals, and in a search for equations when one was not necessary since she was heading toward a valid proof. However, she was able to overcome these issues with her use of lists, which set her apart from many other participants in the study.

<u>Summary</u>. Beth's progress during the first interview was slow but steady. She could have benefited from redirecting her work sooner during the trials for finding 14, avoiding repetitions by realizing that her list of all possible combinations could have ended at the group for 6, and being able to recall her work in the case already mentioned.

Beth also could have benefited from more attention to patterns, in particular, the ability to see that the larger numbers must go on the sides and the lower numbers on the vertices, sooner in her work. However, her abilities to organize her work and keep herself attentive to the overall goal saved her from failure and were also the keys to her ultimate success.

During the second interview, Beth's work was more scattered than in the first, not proceeding through the choices in the most direct manner. Again though, she was able to keep track of her work and made lists to organize the smaller portions that she had already completed. She did not summarize her own work or recap her proof, but did seem satisfied that she had covered all parts necessarily to have provided a complete proof.

The researcher had the opportunity to work with Beth individually as well as in groups throughout the semester during office hours and help sessions for MATH 305. During that time, she learned that Beth was a meticulous student, not satisfied until she fully understood a concept, but often struggling with new ideas in proof writing. She worked frequently in groups, contributing ideas and also learning from others. Beth was seen almost daily in the math building working on her math homework, and put in a great deal of time to understand the various proof-writing techniques that were introduced in MATH 305. She was a stickler for following the format of a particular technique that was laid out in class, sometimes to the extent of using the same wording and order of arguments as she had written in her notes, or had been written in the textbook.

Setting up a proof, writing out the definitions of all terms involved, and working through the strategies given in class were common occurrences for Beth. However, she would wrap herself up in the details of the proof, sometimes losing track of the original goal in the process. This was also seen during the interviews. Beth seemed to recognize this tendency and directed herself back to the original questions whenever she found herself veering too far off the path she intended to take. The researcher felt that Beth's understanding of the underlying mathematics used in MATH 305 was excellent, but her struggles with forming the actual proof and including the proper amount of detail resulted in an overall difficulty with the course.

Beth's difficulties and her successes were reflected in the interviews as well. During both interviews, Beth tried to first thoroughly understand a question and any given definitions before moving forward. She stressed the use of a plan, and redirected her work back towards her goals whenever she became lost in the calculations or details of her work, which occurred several times. The researcher felt that the unfamiliarity of the questions caused Beth added difficulties since she generally liked to model her proofs after others she had previously worked with or those that had been presented in class. This could have caused Beth to work through both questions using brute force, rather than any more elegant proof-writing techniques, because she was unable to make a connection to the techniques she had learned. However, Beth's tenacity and unwillingness to give up prevailed in the end, and she was able to sort through the details to construct a proof. Her attention to issues of organization was seen in the extensive use of lists, which became key to Beth's ability to refocus her work and stay on track towards the overall goal. Overall, Beth had found strategies that fit her personally and helped her to overcome the inherent difficulties she experienced in proof writing. She was able to recognize her struggles, and to monitor her work to overcome them.

Lily

Lily was a MATH 305 student who was a senior majoring in anthropology. Her previous coursework included calculus I and II, and statistics. She was also concurrently enrolled in linear algebra. Lily participated in two interviews. During the first interview, which was 56 minutes in length, Lily worked on Questions 1, 3, and 4. She worked silently often and needed to be prompted to express herself verbally.

During the second interview, which was very brief, just over 16 minutes in length, Lily worked on Question 2, parts a and b. Her work was very organized and she remembered what she had attempted throughout the interview. While she was able to prove part a using proof by exhaustion, she was unable to prove part b. The interview ended when Lily decided she was stuck and did not want to pursue the question further.

Question 1. The first interview began with the researcher reading the question. After thinking and rereading the question, Lily wrote that she knew a pentagon had five sides. She also formed an equation for the sum of three numbers being 14, and summarized her thoughts about the pentagon and her equation. Lily continued by clarifying that the numbers on the pentagon would be whole numbers 1 through 10, still making sure she had a clear understanding of the question. She said that she viewed the question like a puzzle; trying to fill in the numbers where they belong along the sides. She also stated that a number could only be used once, still declaring what she knew about the question.

After the researcher asked her what she was thinking in her head, Lily responded that she was just going around the pentagon and seeing where the numbers fit, as the example had done. She said it helped her to visualize the question. Lily noticed that the numbers on the corners were used in the sum for both sides. As she drew a new pentagon, she concluded that the large numbers would have to go in the middle of the sides, finding the first key fact to her proof, but not truly understanding its significance. Lily justified her idea by saying that, in a sum of 14, 10 would have to go with 1 and 3. She compared that with the example pentagon, saying that she would only have 2 left for the other side, which she would not be able to use with 10 to make 14. She had realized that 10 would only have one combination in which it could be involved, so it would need to be moved to an edge and had hypothesized that other larger numbers should also go in the middles.

Lily expressed doubt about the statement of the question, asking if it was actually true, that 14 was in fact the smallest possible sum. The researcher confirmed that 14 was the smallest and asked how Lily would go about convincing herself. Lily proceeded by considering what would have to be done on the pentagon if 14 was possible. After verifying that the numbers could only be used once, she reaffirmed that 10 would have to be on an edge, not a vertex. She began to fill in numbers on her pentagon, starting with 10, 1, and 3. She continued by considering what would be left in the sum of each of the edges near 10, needing remaining sums of 11 and 13 to complete those sides. Starting with 11, she systematically wrote down what combinations of two numbers would have that sum.

As she worked, Lily commented that it was going slowly because she was just guessing and checking. She continued working to find the correct pentagon, making choices as she went by eliminating possibilities and keeping track of her previous attempts (Transcript 6, lines 110-124). When her first attempt failed, Lily felt that her initial choice for the remaining numbers on the side with vertex 3, namely 9 and 2, was an incorrect choice. However, she reconsidered, saying that the numbers could change positions. Forgetting her earlier idea of keeping the higher numbers on the edges, she chose to switch 9 and 2 and try again. After her second attempt also failed, Lily concluded that 9 and 2 could not be a correct choice.

Lily continued to work with the side with vertex 3, her next attempt having 7 and 4 on that side as well. After a bit of work looking through her possibilities, she determined that 4 and 7 were also not the correct choices (Transcript 6, lines 141-153). Beginning again, Lily drew a new pentagon, this time using 5 and 6 on the side with vertex 3. After systematically checking her choices for the other sides of the pentagon, she decided to switch the 5 and 6 around. She was eventually able to find the correct pentagon (Transcript 6, lines 173-180). Each time Lily had started a new attempt, she had drawn a new pentagon. Her work had been organized and systematic, the only issue was that she had started with the wrong choice and so it had taken time to work through each option. Lily stated that she had found her pentagon through elimination, starting with the largest number first.

After being asked, Lily said that she had not yet finished the proof. She was able to see that the smallest option would be 13 since 10 would have to be in a combination with something.

> I guess, you know that you are going to have 10 on one side for sure, and the next smallest number you could have 1, plus I guess you could have 2, 13 and so I guess that you would need to prove that you wouldn't be able to find 13 – on all of the sides. (Transcript 6, lines 203-210)

She stated that she would next try to use the same method to find a pentagon with sums of 13. Starting by placing the number 10, which must be in a combination with 1 and 2, she began again going systematically through her choices for one of the sides. Lily was eventually able to show that 13 was not possible, through a series of trials and organized attempts (Transcript 6, lines 225-262). She never actually said that she was finished with the proof, but agreed when the researcher said that with a formal write-up, she would have a proof. As Lily reflected on her strategies, she again said that she had used a guess and check method and elimination to find the examples she had been looking for. She also said that she had never done anything like Question 1 before, but that it reminded her of Sudoku puzzles.

Lily's success on this question occurred using good strategies including reading and understanding the question, visualizing and drawing pictures, finding a pattern for the placement of the larger numbers, working systematically through the choices, organizing her work, redirecting after a failed attempt, and understanding when she had more of the proof to consider. The use of equations early in her work was not a helpful aspect to Lily, though it did not appear to hinder her either.

The guess-and-check method was certainly not the most efficient way to go about finding the pentagon for 14. However, organization and the ability to monitor her trials allowed Lily eventually find what she desired. She had noticed the limited choices of combinations for the larger numbers, helping her to decide to work from the largest number down. The placement of one forced entry, as well as the realization that 10 must go on an edge, were key to her progress. Unfortunately, Lily was unable to remember her conclusion, or to understand the significance, that the largest numbers should go on the edges. If she had been able to recall that fact, her success on the trials for 14 would, most likely, have been reached sooner.

Question 3. After the researcher read the question, and Lily read it to herself as well, she felt that she understood the question. She said that she would first prove to herself that one dimension did have to be even. Lily began by considering the first odd-by-odd case, one-by-one, and said that was obviously not coverable. She moved to a 3-by-3 chessboard, drawing out the chessboard and shading in where dominoes could be placed. She said that she had proven it to herself because, with her pattern, she would always be left with one uncovered square in the middle of the board, and she would need two squares to add another domino. Lily said that the smallest case where a perfect cover would exist was a 1-by-2 board. She said that she understood how it worked for her examples, but she was stumped moving beyond that. "But I guess the hard part for me is kind of like proving it for the generic, just in *m*-by-*n* matrices. – Because it's impossible to prove like all the [laughter] combinations" (Transcript 6, lines 363-368).

After a long pause (40 seconds), Lily said that she had been thinking about the different possibilities for *m* and *n*: the possible cases included both dimensions being odd, one odd and one even, or both even. She said that she was thinking of small examples of each of these possibilities, “just because it’s easier to visualize” (Transcript 6, line 384). She had also thought of different proof techniques that she could try, such as proving it directly, or using proof by contradiction. Lily then said that she did not know where to go next. The researcher prompted her to explain why the odd-by-odd case would not work. Lily explained her reasoning using the example that she had drawn, 3-by-3, saying that there were a total of nine squares and so the board was not able to be covered because the total number of squares was not divisible by 2. Then, she said that the chessboard would always have to have at least one even dimension because the dominoes have an even number, but did not justify this statement further.

The researcher then asked what else was needed to prove the statement. Lily was not clear about what the researcher had asked, so the researcher recapped her work so far, and then asked if there was anything left to do to prove the statement. Lily decided that she still needed to show the other direction of the bi-conditional statement, that if one of the dimensions was even, then it could have a perfect cover. “I mean cause, since this is an if and only if statement, we have to prove it both ways” (Transcript 6, line 474). She explained that the dominoes would always cover an even number of squares, the product of two even numbers would be even, and the product of an even and an odd number would also be even. Lily decided she needed to divide the proof into cases. She made two cases; the first with one odd and one even dimension, and the other with both dimensions being even. She wrote out small proofs, with notation and equations, that each of these cases resulted in an even product, or an even total number of squares. However, she did not say that there would be a third case where both dimensions would be odd, but the researcher suspected that was because she had already ruled out the odd-by-odd case, and so was considering only one direction of the original statement.

The researcher asked Lily what picture she had in her head of how the dominoes could be laid out in a perfect cover. Lily described a layout that would work for either of her cases, drawing dominoes laid horizontally. She then said that she could put as many dominoes as she needed in this way to make up the even dimension, and that she could add as many rows as was needed for the other dimension. She said that she now felt she had proven the statement, but had problems finding the wording to write it up formally.

Lily identified her strategies for Question 3 as first proving the question to herself, then drawing a picture and visualizing, and thinking about the question in relationship to her pictures. While she had not seen the question before, she said that MATH 305 had given her the language for writing out equations with even and odd integers and the idea to try a proof by cases.

Lily's work on this question was fairly quick. She had indeed first proven the question to herself, just as she had planned to do. She then looked at examples and drew pictures. She understood that she needed to go further to prove it to others, but had difficulties considering how to do that. The researcher asked her to state a proof that the odd-by-odd case would not be possible. However, it was not clear that Lily would have done so without being prompted. She discussed how her example showed what would happen, but did not explain the proof for the generic case in detail.

Also unclear was whether Lily felt she was done with her proof at that point. When prompted to consider this, she had continued her work, saying she needed to prove the other direction of the bi-conditional statement. She was again identifying a specific proof technique that she wanted to use due to the statement of the question. Lily split the proof into cases, and thought of small examples of each case. This also showed her desire to follow a specific proof type. Lily had visualized chessboards in her head throughout this question, and stated so herself several times. Again, she showed a desire to use equations somewhere in her proof, to fully understand the question, and to visualize.

Unlike her work on the first question, however, Lily was unaware that more of the proof needed to be considered for this question. This showed her lack of a plan. She did understand the need to go beyond proving to herself, but was not able to make a plan to do so. Overall, her proof to this question was left incomplete for lack of detailed explanations. Most likely, it would have been even less complete if not for the questions and promptings from the researcher.

Question 4. Since there was not enough time in the interview to go back to Question 2, the interview continued instead with Question 4. After the researcher read the question, Lily paused for 10 seconds and then said that she understood the question. She began by writing what she would assume to be true, with notation of what it meant (see Figure-Lily. 1). She was able to properly interpret the definition and write the statements accurately.

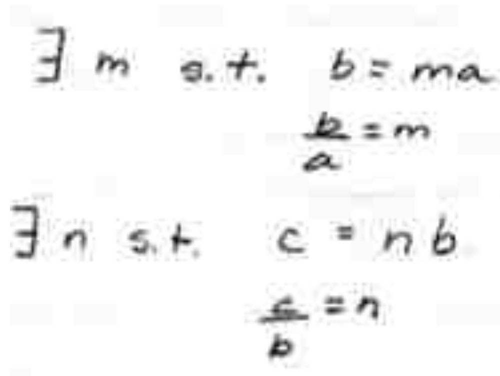

Figure-Lily. 1: Written assumptions with no errors in Question 4

Lily stopped and said that she was trying to look at what would happen if she put in actual integers. As she tried an example, it was clear that she understood the definition. The researcher asked her to write out what she was thinking, so Lily substituted $a = 3$, $b = 12$, and $c = 24$ into her equations. Then, she said she was trying to think of how to prove the generic forms, saying that obviously she could not prove the statement by just going through every possible number. She suggested that she could work with prime factorizations of the numbers, but did not pursue the idea further. Lily explained why it seemed obvious to her that the question would be true, reiterating that her example would work, since 3 went into 12, and 12 went into 24, then 3 would obviously go into 24. Again, having proven the statement to herself, she found it difficult to prove the generic statement for others. Lily related her example back to the equations she had written. She also wrote out what she wanted to show (see Figure-Lily. 2).

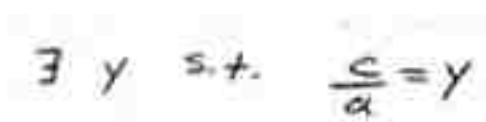

Figure-Lily. 2: Notation for desired conclusion with no errors in Question 4

She struggled with ways to substitute what she knew into what she wanted to show, having written out all the equations correctly and knowing that she needed to find an appropriate y value.

However, at that point, Lily made a few errors in her reasoning, though she was nearly able to come up with a proof of the question. She tried a substitution and actually found what she would have needed for a proof, that $y = mn$ (see Figure-Lily. 3).

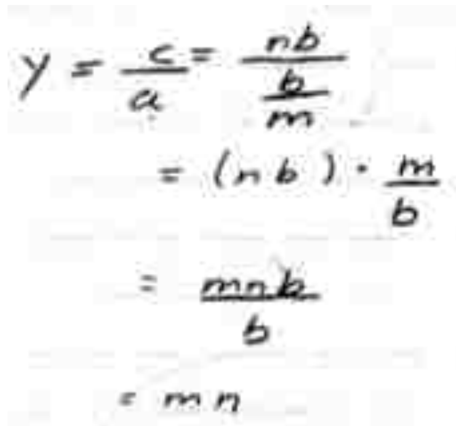

Figure-Lily. 3: Equations that could have led to a proof in Question 4

From there, she could have explained her reasoning in reverse for the proof. However, she became confused about what she was actually finding and was unable to see her result. The researcher prompted her to consider what her conclusion, that $y = mn$, told her or what she could conclude from it. Lily said that she was stumped, and instead of recognizing her result, she substituted m and n back into her equation and solved, finding that $y = c/a$, which meant that she had circled back to the original equation (see Figure-Lily. 4).

Figure-Lily. 4: Equations showing circular reasoning in Question 4

Unable to recognize the makings of a proof, she said that she was convinced that the statement was true, but still did not know how to prove it. The interview ended as Lily agreed that her strategies on Question 4 included looking at an example, convincing herself that the statement was true, proving that something existed by substituting in what she knew, and working with the definitions.

Unlike the first two questions, Lily ended her work on this question without forming what she felt was a proof. The use of equations could finally have been helpful to her, however she was not able to follow the equations to the conclusion. Lily clearly understood the question, unpacking the definitions carefully and successfully. She looked at an example, in which she correctly interpreted the question, but did not fully see the connection to the conclusion of what she was trying to prove. Again, while she was able to understand and prove the question to herself, she was unable to prove it to others. She had worked backwards from what she wanted to show to what was known, but did not see the significance of her discovery that $y = mn$.

If Lily had been able to make that last connection, her success would have been due, in part, to careful use of notation and definitions, the ability to translate the question into equations, organized work, and the understanding that one example did not prove the statement. She had written out what was known and what she wanted to show, but otherwise did not have a clear overall plan to connect the two. With more time, Lily may have been able to resolve her issues, and possibly could have found a full proof.

Question 2. The second interview began with the researcher reading part a aloud. Lily reread the question herself and then paused for approximately one minute. When asked, she said that she was thinking about the definition and rereading it to herself because it was confusing to her. She restated the definition in her own words, and the researcher helped to clarify with some further explanation. Lily looked at the example, $N = 100$, and stated that she thought the definition was saying that it would be a 4-flip because $4N$ had the same number of digits as N. The researcher corrected this error. Lily said that she now understood the definition and was trying to think of possible two-digit 4-flips. When asked to write down what she was considering, Lily wrote the example $N = 10$, and correctly stated what would be needed for this to be a 4-flip. She next tried $N = 25$, found that it was also not a 4-flip and further noticed that $4N$ would no longer be a two-digit number. Even though she had assumed earlier that $4N$ would need to have the same number of digits as N, she asked the researcher to confirm this. Lily was showing difficulties in truly grasping the larger picture, this particular doubt, though, could have surfaced because of the earlier correction made during her interpretation of the definition.

Lily stated that all numbers over 25 could not be 4-flips. After asking the researcher if she needed to write her conclusion down, Lily listed the interval and the reason that these numbers would not be possible. She summarized that she had the numbers 10 through 24 to further consider. When asked where she would go next, Lily said that she was trying to think of a way to prove the statement in general, noting that she could just list out all the remaining possibilities to show they were not 4-flips. The researcher told her that she could pursue that idea, if she wished. Lily proceeded to do so, taking approximately 2 minutes to finish the list, which is shown in Figure-Lily. 5.

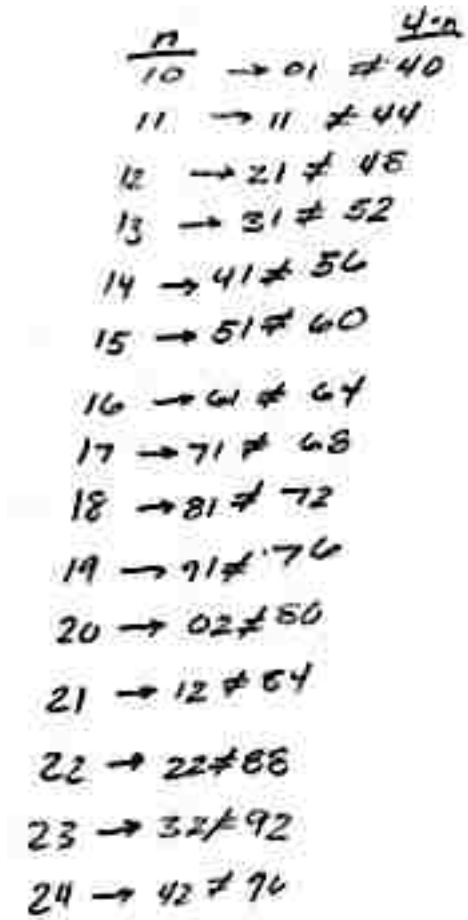

Figure-Lily. 5: Brute force proof for the interval 10 through 24 in Question 2 part a

Lily wrapped up by saying, “So, I guess that proves that none of those are equal” (Transcript 24, line 116). However, she did not actually say that she was satisfied with this as a proof for the statement overall. The researcher asked her to go on to the second portion of the question.

After the researcher read the question, Lily again paused for approximately one minute. There was a great deal of noise in the room at this time due to a printer being used. During that time, Lily was mostly silent, waiting for the printer to stop; this lasted for almost one minute. When the printing had finished, Lily said she was thinking of what number would be the maximum possibility for three-digit 4-flips, as 25 was for two-digits. She was able to determine that this value could be found by dividing 1000 by 4. Lily momentarily thought that this value would be 500, before correcting herself and saying that it was actually 250. She summarized that she was now looking at the numbers 100 through 250, but stated that there would be too many numbers to check them all through brute force, as she had done in part a.

The researcher asked what else she might try. Lily said that she was looking back at her work from part a to see if there was a pattern that she could find to help. She said, “I mean, basically, kind of what I’m looking for is like the numbers that are closest to each other. – But, I don’t know if that really makes a difference anyways” (Transcript 24, lines 160-166), thought she did not explicitly state what she meant by the numbers being close together. She again looked back to her work from part a, saying that it seemed to her that the numbers in the middle were the closest together. The researcher believed this to mean that the difference between the desired and actual values for *4N* was the least in the middle of her range in part a.

Lily said that she would examine a few three-digit numbers towards the middle of her range, “just to see what happens” (Transcript 24, line 177). She first tried $N = 175$, then moved on to 176 and 177. When asked, she said that she was again looking for a pattern. She found that the entries in the *4N* column would always be exactly 4 away from the previous entry, but the values of *N* flipped were changing by 100 each step. She said that she was still seeking a pattern, but that she did not know how to move on without checking all the numbers in her range of possibilities. When asked, she said that she did not have any other ideas to try and so the researcher allowed her to stop there. Lily said that she had never seen a question like this before, and that she felt this was the reason she had gotten stuck, because the definition was new to her and "kind of just threw me off from the beginning" (Transcript 24, line 215-216).

The researcher began to recap Lily's work, saying that she had first picked apart the definition. Lily said that she had been "figuring out what the definition meant and then kind of then trying to figure out what the question meant for sure, too" (Transcript 24, lines 224-225). The researcher added that Lily had moved to looking at some examples. Commenting on this, Lily said, "that's kind of how I figured I've always kind of started stuff, just finding examples just to see how it works out. – Whether it proves it or disproves it, just so I know in my mind how it's flowing together" (Transcript 24, lines 229-235). As the researcher continued discussing strategies, Lily agreed that she had used brute force on part a, then had looked back to that work during part b. She had been searching for patterns and any way to use what she had done on the first part to inform her work on the second.

Lily was able to form a proof for part a of this question, and she stated before making her list of possibilities that this would prove the statement. Unlike other students, she left her work there and did not pursue another proof technique. She had started her work by understanding the question and clarifying the definition of a 4-flip, then looked at examples, before being able to limit the possible numbers down to only a few remaining to be checked. She made a plan, followed through with the plan, and recognized her proof as valid, though the researcher moved on to part b immediately, which could have prevented Lily from doubting her proof. However, Lily did not express any further doubt during her work on part b.

In the second portion of the question, Lily understood the question, and immediately connected her work from part a to limit the possibilities. However, with too many options to check by hand, she no longer knew how to proceed with the proof. She struggled to find patterns that would assist her, looking back at part a and at some examples of three-digit numbers. She was unable to form any new ideas and had to end her work there. Overall, her work was organized and neat, and she monitored her attempts well. However, Lily was unable to redirect her thoughts and move past the ideas from part a, and so she could not form a proof for part b. She also felt that some of her difficulties had surfaced from having a new definition to work with, showing a reliance on previous material to form a proof. Her search for patterns resulted in no new ideas, and she was unable or perhaps unwilling to move to any other strategies in her attempts to find the proof.

Summary. Overall, Lily used many good strategies throughout both interviews. She was also able to work through her different strategies and redirect her work when necessary. She was not overwhelmed by notation and was able to understand each question and its potential for proof. However, what was lacking in the first interview, particularly in Questions 3 and 4, was an overall plan or a set of goals and sub-goals to be attained, as well as the ability to monitor her progress towards that goal.

The interviews differed in a few aspects. The first was the clear plan that was followed through on Question 2 during the second interview. Even though Lily was unable to form a complete proof of part b of the question, she had the goal in mind and did not stray from that during her work. She linked her work to that in part a but saw the differences between the two parts as well. This distinction between the interviews can be partially viewed through the lens of another difference. In the first interview, Lily attempted to fit equations to each question, but was never successful in doing so. However, in the second interview, she did not make such an attempt. From the start of Question 2, her thoughts were more directed and a plan was formed quickly to examine the numbers. This lack of trial and error with equations could have contributed to a better sense of the question in general and the ability to keep her mind focused.

Since the second interview was very brief, it is not clear that Lily would have continued to choose proper goals and keep her focused on the questions. This ability may have lessened as she continued to work, as it had during the first interview. However, it is not possible to make conclusions either way with any certainty.

As a MATH 305 student who regularly participated in group study sessions, Lily was observed by the researcher outside of the bounds of the interviews as well. During these times, Lily tended to be quite shy and reserved in her views. She worked quietly and only occasionally had a definite plan for a proof that she was willing to defend to others. However, her mathematical knowledge was never in question and she often would find a correct proof before others. Lily struggled with notation and was very particular to use proper wording in her proofs, which may have contributed to her difficulty in forming proofs without access to her notes or her textbook. She could be found poring through her textbook and notes in search of a similar proof, which can also be seen in her immediate reference to part a during her work on part b of Question 2. In general, Lily was a hard-working student with a moderately good grasp on the material for MATH 305. Her success and her struggles during the interviews were not unusual for Lily, and can be partially explained by her work outside of the interviews as well as the manner in which she commonly worked on proofs.

Sam

Sam was a student in MATH 406. He was a graduate student in curriculum and instruction. In Questionnaire 2, Sam indicated that he had taken calculus I and II, linear algebra, MATH 305, discrete optimization, and number theory. During the interview, which was 58 minutes in length, Sam worked on Question 2, all parts, and Question 1. Sam's proofs were oral, not written, but most were complete with reasoning as he went through the questions.

Question 2. The interview began with the researcher reading part a of Question 2, after which Sam read it again silently. Sam asked the researcher to tell him to write things down as he worked, indicating that it was not his tendency to do so on his own. When considering the definition, he first looked at the two-digit numbers that were divisible by 4. He stated that there were 25 possibilities, and said he might just start looking at all 25. He realized that some of those possibilities would not be valid, because *N* also needed to be two-digits. Since 10 was the first two-digit number, Sam stated that 40 would be the first potential 4-flip. The researcher corrected Sam by saying that *N* was actually referred to as the 4-flip, not *4N*. He then understood that in his example, 10 would be the 4-flip, not 40.

Once he understood this, Sam said that he would start to look at the numbers because there would be even fewer than he originally thought, and he tried a few examples. Instead of translating his earlier comment that there would only be 25 numbers to his new understanding of the question, he now said he would have to look at all the numbers 10 through 99. Sam decided to look for another method because he thought there would be too many numbers to check. After looking at the calculations for the two ends of his interval, he saw that 4 times 99 was a three-digit number. He then realized he would only have 25 options to check, but did not link this to his previous findings that there would only be 25 numbers that were multiples of 4. Knowing now that there were so few of them, he said he would just start checking each one.

Sam began to list out all of the numbers 10 through 29. After looking at the first few, he realized that none of the numbers in the interval 10 through 19 would be 4-flips because *4N* could never be odd. He went on to say that 30 through 39 would not work by the same reasoning, but realized that he had already excluded those numbers because *4N* would be too large. He briefly second-guessed his justification for eliminating 10 through 19, but was able to resolve this quickly. He then went on to check the remaining five numbers. Sam concluded that there were no two-digit 4-flips, clearly indicating that he understood that he had finished his proof.

The researcher and Sam recapped his work (Transcript 7, lines 128-152). Sam said that he had originally misunderstood the wording of the question. He noted that he originally tried to minimize the number of choices for the 4-flip, so that he would have a reasonable number of possibilities to check by brute force. He had thought he would use ideas from number theory to eliminate choices, but said, "I'm not very strong on the theory part of stuff, so when that breaks down, if it's within reason to try by brute force, I'll do it" (Transcript 7, lines 208-209). Sam said that he had not seen anything like this question before, perhaps something similar though. Understanding the question, he said, was the key to being able to prove it. From MATH 305 and other math courses in college, he had learned that he could try to simplify the question to understand it. He said that he had not been exposed to proof writing in high school, or in his undergraduate courses.

The researcher then proceeded to part b of Question 2. Sam read the question aloud and to himself. From the beginning, he noted that he would approach part b as he had part a. First he found the maximum number, 249, past which *4N* would be four-digits. Once again, he eliminated the numbers 100 through 199, linking the reasoning to part a as well. Sam then eliminated those numbers that ended in zero since the flip would not be a three-digit number. He said that the remainder of the proof could be done by brute force, but he was going to try to find another way so that he could avoid all the work that this would entail. He next considered *N* values of 200 through 224, and noted that *4N* would begin with 8. Sam could then say that the only numbers to be checked in that interval would be 208 and 218. He performed the calculations for those two numbers to show that they were not 4-flips.

Similarly, Sam correctly stated that in the interval 225 to 250, since *4N* would begin with 9, *N* would have to end in 9. However he made an error, saying that the number flipped would then be odd, so the entire interval could not be possible. He had confused his earlier logic, involving the 100 range and the first digit, with the choice of the ending digit here. After further consideration, he was able to see his error. Sam went on to correctly state that in the interval 225 through 250, he would only have to check 229, 239, and 249. He performed the calculations for those numbers to show that they were not 4-flips.

Sam ended by saying, “So, that’s how I would do it” (Transcript 7, line 322). He again had a clear understanding that he had finished the proof. As he mentioned, he was not inclined to write he thoughts down, his work showed only some notes to keep track of the information he had found. When asked, he recapped his work, and afterwards said that he had not used any new strategies than he had in the first part of the question, he was just still trying to eliminate possibilities (Transcript 7, lines 324-363).

Sam went on to part c of Question 2. When the researcher read the question, Sam jumped right into working. He first wanted to check that 2178 was truly a 4-flip. After confirming this, he said that he would continue working as he had in the previous two parts of the question. He narrowed down the options to a maximum of 2499, then eliminated the interval 1000 through 1999. He stopped to write out his reasoning for this range, starting a list of conclusions for each range of numbers. Following his own example from part b, he added to his list of conclusions the range 2000 through 2249, saying that the only numbers to check would have to end in 8. The actual list to this point is shown in Figure-Sam. 1, the conclusion just mentioned is faintly written below the second line in the list.

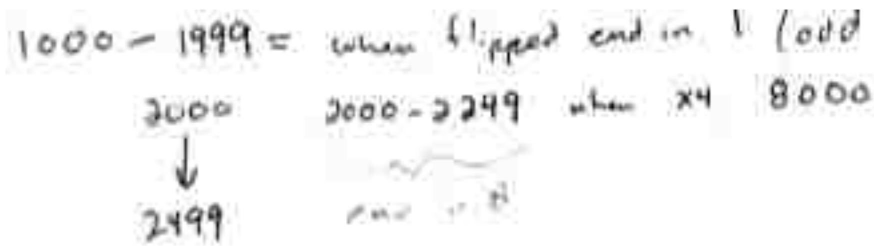

Figure-Sam. 1: Justification for intervals in Question 2 part c

He also added that in the range 2250 through 2499, the numbers would have to end in 9 for consideration as a 4-flip. He noted that he had many more to try than he had in the three-digit case, so he continued searching for other ways to limit his choices.

Sam paused briefly before saying that in the entire range 2000 through 2499, *N* flipped would end in 2. He continued by saying that this would imply that *4N* would have to end in 2. Therefore, the ending digits for *N* must be limited to those that when multiplied by 4, end in 2. He found that the only ending digits for *N* that would achieve this goal would be 3 and 8. Using this, he was able to eliminate the range 2250 through 2499, since he had already said numbers to consider in this range would only be those ending in 9.

Sam noted that he could then use brute force to check all those in the range 2000 through 2249 that ended in 8, but again that was too many to do by hand. He decided to begin looking through the remaining possibilities to see what they would look like when multiplied by 4. After checking several examples, Sam incorrectly stated that none of the numbers 2000 through 2098 would work because he had checked them until the result of *4N* was over 8100. From then on, he said, *4N* no longer had the desired zero. However, he had been looking at the wrong digit for the zero to occur. He said that his next step would be to look at the range 2100 through 2198.

Sam attempted to further restrict the maximum *N* value. By taking the largest possible *N* in his range, 2498, flipping it to be 8942, and dividing by four, he declared the largest possible number for a 4-flip to be 2235. He continued to struggle with limiting his possibilities, looking now for the minimum *N* value. He looked at the smallest *N* in his remaining range, 2098, multiplied by 4, getting 8392. He then said that since *4N* would continue to increase, his smallest *N* value would be 2938, which was outside of his range. He concluded that he had finished the proof, but it would require being written up a bit more neatly. His errors here had continued from the previous error, not identifying the appropriate digit to work with. He was unable to see, for example, that as 8392 increased, the flip could actually decrease (e.g. 8412 is a multiple of 4 larger than 8392, but its flip is 2148, which is less than 2938).

The researcher asked Sam to recap his work. While restating his reasoning, he recognized his error for the range 2000 through 2098. He said that he had been looking at the zero in the 100s place of *4N*, but he actually needed to look at the 10s place. Seeing that error, he said that he would probably start the search in that range again. He would begin at 2008 and check all the numbers that ended in 8 until he had an answer that went outside of the range he was looking for. Sam decided to pursue these thoughts, and went back to look at this range again. However, he was still unable to keep track of the way the digits would change in *N* versus *4N*. He argued that the largest *N* in his range was 2248, so the largest *4N* he could have would be 8422. He did not see that some *N* in his range prior to this would produce a larger flip (e.g. 2198 gives 8912). Sam then went on to observe that 2108 times 4 would be 8432, exceeding his limit of 8422. He concluded that he would only need to check those numbers ending in 8 between 2008 and 2108 and that would complete his proof. Even though he had verified previously that 2178 was a 4-flip, he did not seem to recall this at the time, and did not notice that his reasoning would exclude this as a potential 4-flip.

The researcher asked Sam if he had used any different strategies than he had in the previous two parts of the question. After describing several ways in which the four-digit portion was different than the others, Sam stated that he was sure there was a better way to prove the statement, possibly more efficient, but he did not know what that would be.

> I would be searching for some sort of um, [pause] check. Some sort of theorem, some sort of postulate that I could come up with that I, that I could just um, prove on its own, separate from the numbers that would tell me that that was true. … Which I'm not very comfortable with, which is why I do things the way I do. (Transcript 7, lines 615-621)

Sam started this question with solid reasoning and a good strategic approach. In part a, he read the question and realized that he would have only 25 possibilities. While he did make an error in the use of the definition and it cause him to momentarily get off-track, he was able to recover. He then was able to discover a few key points; that *N* must be in the interval 10 through 24, and that the numbers 10 through 19 were not possible since they began with an odd number.

Sam looked at examples, redirected his work appropriately, was organized, developed an overall plan, and was able to recognize when he had developed a proof. All of these strategies carried through into part b as well. There he also split the question into smaller parts and was able to systematically eliminate all possibilities. He used general reasoning to start, and switched to showing the remaining numbers by brute force when it was appropriate. He also recalled and used his previous work in his new proofs.

In part c, however, Sam struggled. He began his work by verifying that 2178 was a 4-flip, and used the ideas he had developed from parts a and b. Narrowing down the range of options to 2000 through 2249, he knew that only those numbers ending in 8 in this interval were possibilities. It was at that point that his proof began to fall apart. Sam was unable to keep track of the placement of digits in *N* versus *4N*. His reasoning included many errors. Sam had stopped considering examples, did not check his work and forgot the overall goal. Perhaps if he had continued his search through the possible examples, he would have been able to see that his reasoning would fail. In the end, Sam had supposedly shown that no four-digit 4-flip existed, even though he had verified himself that 2178 was a 4-flip. He was unable to correct most of his errors and did not complete a valid proof for part c.

Question 1. After the researcher read the question, Sam immediately said that he should know how to do the proof based on the courses he was taking at the time. He stopped to read the question himself for 10 second. After this, he noted that there were three numbers in each sum, and that there were 10 total numbers, verifying this information. He developed the equation $a+b+c = 14$ to denote the sum of one side and said that the question was asking to prove that the smallest value for that sum would be 14. He paused for another 10 seconds to read the question again.

Sam said that he would probably need to prove the statement by contradiction, assuming that the sum could be 13 or less and finding a contradiction, and decided to proceed with this proof technique. He felt that the key to the proof would be to use the correct equation to find the contradiction, and decided that he had five different sums involved in the question. Calculating five times 13, and finding that to be 65, Sam said that he would want to show that the total sum of all the sides would be greater than 65. He noted that the total sum would include the vertices counted twice and the sides counted once. In order to minimize that sum, he said that he would want the smallest numbers to be counted twice and the largest to only count once. He had, at this point, discovered the two key points for his proof. Sam wrote out the equation for what he was describing and found the sum to be 70 (see Figure-Sam. 2,the zero of his number 10 had been written off the edge of the page).

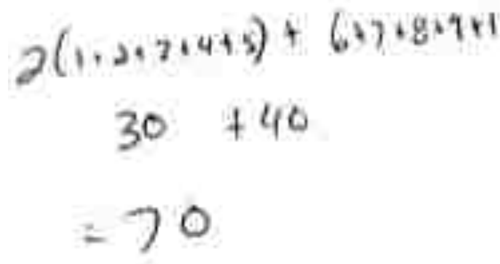

Figure-Sam. 2: Equation for proof in Question 1

He concluded that the minimum total sum was 70, and therefore it was not possible to find a sum of 13 or less.

Sam said the total sum argument showed that it would be potentially possible to get a sum of 14 per side, but he had not yet shown that a pentagon with sums of 14 could be found. The researcher asked him to prove that 14 could happen, but it did appear that Sam understood it was not actually necessary for the proof. Sam said that he would have to find an arrangement that worked for 14 to show this. He questioned whether it was valid to say which numbers should be chosen for the vertices and for the edges, as he had done for the total sum. If that was valid, he would then have to arrange those on a pentagon, he said. Sam understood that the numbers 1 through 5 should be on the vertices, but that he needed to determine on which vertex to place each number. He decided to first try 10 in combination with one, though he did not know why he wanted to do that. He then placed the next largest number, 9, with one as well. He was able to proceed systematically through his choices and find the correct pentagon (Transcript 7, lines 760-767).

Sam agreed that he was finished with the proof. He explicitly noted now that the statement had asked only to prove that 14 was the smallest possible sum, and he had not needed to find the arrangement for 14 to have a valid proof. Sam felt that he had gotten lucky on finding the pentagon on his first attempt. When asked if he had seen a question similar before, Sam stated that he had seen questions involving the pigeon-hole principle in discrete optimization lately and that reminded him of this question.

During his work on this question, Sam returned to having solid reasoning and was able to develop a complete proof. His success on this proof was, for the most part, due to making an overall plan, recognizing the appropriate equation, and seeing that the smallest numbers would need to be placed on the vertices where they would count twice in the total sum.

Sam's strategies included relating the question to other material from his courses, carefully reading the question, making an overall plan including an appropriate proof technique, and developing an equation for the total sum. His full understanding of the question and the situation allowed him to see that vertices would count in two of the individual side sums, but edges would only count once. Sam was then able to find the contradiction he had been looking for. He also recognized his proof immediately.

When asked to find the pentagon with sums of 14, Sam did so on his first attempt. The ability to find the pentagon for 14 so quickly was due, in large part, to his previous observations about the placement of the small numbers, the systematic approach to placing them, and also a bit of luck.

Summary. Overall, Sam was efficient in his work and use of strategies. For the most part, he was able to keep the overall goal in mind, monitor his progress, and use his previous work to assist him later. Though there was an element of luck in some of his success, it was also his ability to recognize appropriate trials and paths that had opened up the opportunity for this luck. Sam did not write very much down during his proofs, choosing instead to only jot himself notes along the way. As he recapped his work, he was able to recall his reasoning and even correct some of his errors. Sam's main key to success was his ability to self-monitor; including making a plan, monitoring his attempts, redirecting when appropriate, organizing his work, and recognizing his completed proofs.

Julie

Julie was a MATH 305 student. She was a junior majoring in Spanish. Her previous coursework included calculus I through III. During this interview, which was 51 minutes in length, Julie worked on all parts of Questions 2 and 3. She did not write out formal arguments for any questions. There was evidence during her time spent on Question 3 that she was drawn to certain strategies or ideas because this was labeled as something to prove, versus a problem to solve; however, this was not the case in Question 2. Julie was quick to begin writing ideas on both questions. She did not spend very much time thinking prior to writing; instead she seemed to prefer thinking while writing and experimenting with different ideas concretely.

Question 2. After the researcher read the question aloud, Julie read the question again silently. She jotted a few notes to herself and looked at an example to verify that she understood the question. She noted that there were 99 possible numbers to look at, and then paused to look back at the question. Julie began looking through the possibilities, choosing a few scattered examples. She commented later that she tried a few examples "just to organize my work, to see where I was gonna start" (Transcript 8, line 66). Within a few moments, Julie had found that only the numbers 10 through 24 would be potential two-digit 4-flips. She paused to look over her work. Julie then started to say that 10 through 19 could be eliminated, but stopped before explaining what she had been thinking. Moving on, but not reconsidering her thoughts on this interval, she showed that 10 was not a 4-flip. Then, she asked if she could just go through and test all of them. Julie began an organized list of all 25 numbers, indicating that the numbers multiplied by 4 were increasing, which she referred to later in parts b and c of the question. She wrote the values of N, $4N$, and N flipped for each possibility, and in this way had shown that there were no two-digit 4-flips. However, she did not specifically indicate that she felt she was finished with the proof. When asked her to describe her strategies, Julie said that she had begun by looking at examples, narrowing down the possibilities, and then "I just did it the long way" (Transcript 8, line 72). Julie stated that she had never seen anything like this question before. She also mentioned that guessing and checking was a strategy that she has used previously in proof writing. Papers were collected from Julie between each part of Question 2, but were accessible if she wanted them. However, she never looked back at previous work, though she did refer to it specifically in other portions of the question.

The interview continued with part b of Question 2. After reading the question silently, Julie jumped right in. She began with the incorrect conclusion that she could limit the possibilities to 100 through 333 (similar to the two-digit limit of 10 through 24), but she never corrected her error. Julie began by again trying a few examples. She determined that she could eliminate everything that ended in 5 because "when they are multiplied by 4 it's gonna end in a zero, which would make it only two digits when we flip it" (Transcript 8, lines 103-104). She started a list to organize the numbers that had been eliminated. She added that multiples of 10 were also not possible for *N*, for the same reason (see Figure-Julie. 1).

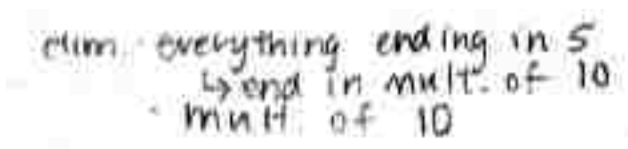

Figure-Julie. 1: Partial list of eliminated numbers in Question 2 part b

Julie continued looking at various examples, stating that she was looking for a pattern. She paused for 50 seconds to look over her work, and then stated that she was considering how the last digit of *N* would affect the digits of *4N*. Julie proceeded to list out the possible ending digits of *N* and their affect on *4N*. She marked an *X* by those that would not be allowed because *4N* flipped would be outside of the previously determined range of *N*. She marked those that would still be possibilities, and crossed off 5 and 0 entirely because she had already eliminated those options. A portion of this list is shown in Figure-Julie. 2.

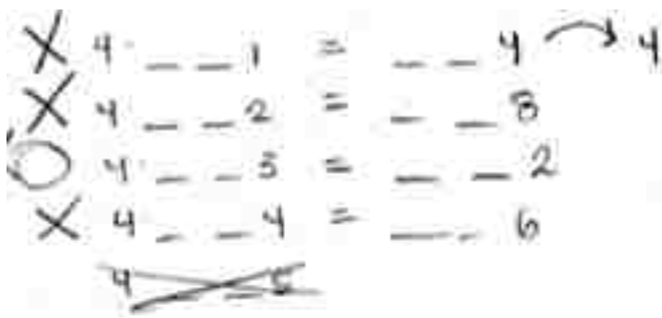

Figure-Julie. 2: Portion of list eliminating possible ending digits in Question 2 part b

Julie discovered that the only remaining ending digits were 3 and 8, and began to systematically examine these possibilities. She tested the first four ending in 3, saying that she was looking for a pattern. Seeing that the values of *4N* in the range 100 through 199 were 400 or more, Julie concluded that there could not be a 4-flip ending in 3. She added that the same argument would hold for all *N* values ending in 3 in the interval 200 through 299, and further summarized that an ending digit of 3 would never be possible.

Julie attempted to repeat this process with numbers ending in 8. When she saw that the values of *4N* were less than 800 in her examples, she felt that it might be possible to find a three-digit 4-flip. When she reached $N = 208$, she found that $4N = 832$, but she would have wanted it to equal 802 to be a 4-flip. At this point, Julie stated that *4N* was larger than *N* flipped, where previously *4N* had been less than *N* flipped. Therefore, she concluded that she had passed the point where it would have been possible to find a 4-flip, and felt that a three-digit 4-flip would not exist. After the researcher and Julie summarized her work, Julie agreed that she felt she was finished with this portion of the question.

On part c of Question 2, Julie immediately started searching for the upper limit for *N*, past which *4N* would no longer remain a four-digit number. However, she could not think of what this limit might be. After the researcher clarified that Julie was looking for the point when *4N* was more than a four-digit number, Julie was able to find the upper limit to be 2499. Julie again eliminated those numbers that were multiples of 5 or 10, linking this to the previous two parts of the question. She investigated which ending digits were possible for *N*, finding again that the only possibilities were 3 and 8. Julie looked at two examples and quickly said, “once again, anything that ends in 3 is going to be too, when you multiply it by 4, it's going to be too big to make a 4-flip" (Transcript 8, lines 298-299).

Julie felt that there would only be one 4-flip in the entire range 1000 through 2499 because there would be only one point where *N* flipped would equal *4N*. The term *meeting point* will be used to describe such a point. Prior to the meeting point, *4N* would be greater; past this point, *N* flipped would be greater. She indicated that she felt that *N* flipped would be increasing at a faster rate than *4N*, so there would be only one meeting point. The visual representation of her ideas is shown in Figure-Julie. 3.

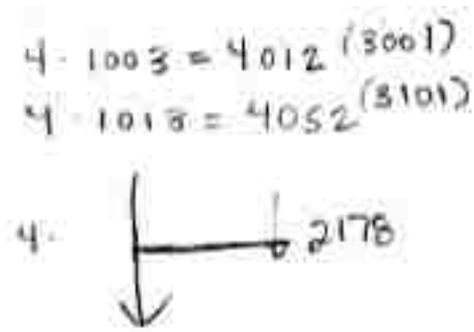

Figure-Julie. 3: Visualization of meeting point idea in Question 2 part c

The researcher then encouraged Julie to further examine this conclusion. Julie looked at the numbers 2168, 2178, and 2188. She wrote *N*, *4N*, and *N* flipped for each, and verified that they followed the pattern she had expected.

After being prompted by the researcher to test her theory on the number 2218, Julie discovered that *4N* was then larger than the flip 8122. Therefore, her theory did not hold for all of the four-digit numbers. Julie revised her theory, deciding she would have to break the interval 1000 through 2499 into 25 sets of 100 numbers each and finding the meeting point in each set. She never justified this, nor did she try to prove why her new theory would work, where her last theory had failed. She referred to each set by the first two digits of the numbers in the set (i.e., 2000 through 2099 were the 20s). Julie stated that since she knew that 2178 was a 4-flip, that would be the meeting point for the 21s.

After checking a few examples in the 10s, Julie decided that the numbers in this interval would not be large enough. Similar to her reasoning in the last case, here *4N* began in the 4000s, and the only possibilities for the first digit of *4N* would be 8. She determined that the same reasoning would apply to all the groups up to those beginning with 20, and then continued by looking at the example $N = 2008$. At this time, the researcher summarized Julie's work by saying that she only had the 20s, 21s, 22s, 23s, and 24s remaining. Julie agreed that she had narrowed down the options to 2000 through 2499. She continued by looking at the next possibility in the 20s, $N = 2018$, and indicated that she had passed her meeting point between $N = 2008$ and $N = 2018$. Therefore, she concluded that no 4-flips existed in that interval.

Julie moved on to the 22s. After looking at $N = 2208$ and $N = 2218$, she said that $4N$ was already too big to work. She also said that the same would be true for the 23s and 24s. Julie concluded that at 2308, $4N$ was in the 9000s, and so everything above that would not be possible since it could not have $4N$ beginning with an 8. The researcher summarized Julie's work (Transcript 8, lines 439-469). When asked, Julie said that her solution would be complete if fully written up.

Between the three parts of Question 2, Julie lost track of information that could have potentially sped up her progress. For example, in part a, Julie quickly limited the choices to 10 through 24, but in part b she incorrectly limited the choices to 100 through 333. Then, in part c, she initially could not think of what number would be the limit and only continued her search for it after being prompted to do so. She also rewrote a new list in part c representing the different possibilities for ending digits of N, rather than recalling that only 3 and 8 would work, as discovered in part b.

Julie made several errors that slowed her progress significantly and, as stated, was unable to make crucial links between different parts of the question. She did not appear to be able to, or perhaps did not feel it necessary to, check her solution or her logic in finding a solution. Nor did she give a proof, verbal or otherwise, of why the meeting point she was looking for would indeed be the only possible place that a 4-flip could occur. Even after being confronted on the failure of her first theory in the four-digit case, Julie still did not check or verify her revised theory. She appeared to want to work quickly through the question and was satisfied with her answer as soon as she was convinced of it, not desiring to prove it with any certainty to an outside observer.

The strategies that allowed Julie to be successful in the first part of the question included reading the question, looking at examples, limiting her possibilities, and making a plan. Once she was able to determine that brute force on the remaining possibilities would be an acceptable proof, she finished her work quickly. She had clearly understood the definition and the overall question to be answered.

In parts b and c of the question, Julie was unable to connect the limits to those in part a. However, she was able to move past this error to form some ideas for her proof. If she had correctly limited her possibilities, her arguments could have formed a complete proof, with a bit more explanation needed for the interval 200 through 249. The idea of a meeting point in part c was not proven and the justifications for eliminating groups of numbers were also not complete.

The conclusions Julie was able to find were due, in part, to looking at examples, organizing her work, systematically checking all options for the end digits, and being able to keep track of those intervals and numbers that she had already considered. There seemed to be an overall plan to work through all possible cases in an organized fashion, but Julie had not made any sub-goals or ways to follow through with this plan. Furthermore, her systematic process of elimination was slowed by errors she made as she progressed through the choices, errors that may have been caused by Julie's lack of self-monitoring during this process.

Question 3. After the researcher read the question, Julie paused to read it again and began by drawing dominoes in a pattern. She drew the dominoes in rows, not drawing the chessboard itself, but stating that this was how the dominoes would cover the chessboard. After rereading the question, she decided to split the proof into cases. In Case 1, m would be even and n odd. In Case 2, m and n would both be even. She did not make note of the third possible case, which would be when m and n were both odd.

Julie said that one dimension had to be even because the domino covered two spaces. She struggled to find words for what she wanted to say because she saw that the statement must be true and was unclear how to prove it to be true. She reread the question again. The researcher asked her to describe what she was thinking. Julie said that a domino had to cover two spaces. If a row of dominoes was laying horizontally, "and if you want the dominoes to fit perfectly then there has to be an even number of spaces" (Transcript 8, lines 512-513). She said that it would not matter whether the other dimension was even or odd, because it was covered with the short sides of the dominoes, which were only covering one space. She expressed that she did not know how to explain why it had to be true, however.

The researcher encouraged Julie to think of another way to prove what she was trying to say, asking her to describe her thinking (Transcript 8, lines 527-569). After several attempts, Julie finally formed an argument that the total number of spaces must be even in order to be covered by dominoes. She wrote that odd-by-even and even-by-even gave the total number of spaces being even, but that odd-by-odd gave the total number of spaces being odd (see Figure-Julie. 4).

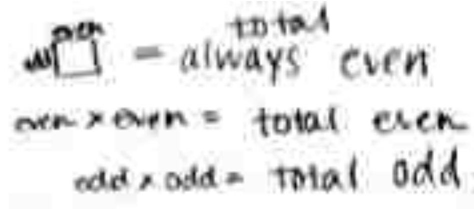

Figure-Julie. 4: Argument that only odd-by-odd gave total being odd in Question 3

The researcher asked Julie to explain why the total being odd was bad. Julie described that if the total was odd, this number could not be divided by an even number. She went on to say that a total being even, however, could be divided by an even number. The researcher reread the question and asked Julie if she was satisfied with her argument; to which she responded yes, that she had done all the cases. When asked if the question reminded her of anything else, Julie stated that she had been working with proofs involving even and odd numbers in MATH 305, and this was where her idea for doing cases had come from, including the idea for her notation for even and odd numbers.

Again in Question 3, Julie showed evidence of her difficulty in forming a proof once she was convinced that the statement to be proven was true. She quickly moved to writing down examples and working with the definitions given, but did not seem to keep the overall picture in mind, nor did she express any type of planning in her solution attempts.

Julie had read and understood the question, drawn pictures, indicated her understanding of a pattern for placing the dominoes, and had split the proof into cases. She had all the makings of a proof, but did not connect all the pieces. She stated several times that she did not know how to express in words what she had been thinking. Julie did not take note of the bi-conditional statement to be proven, nor did she originally form the complete list of cases to consider. While her ideas showed a clear picture of what the proof could be, she did not fully explain or expand those ideas into a proof. In the end, Julie indicated that since she had considered all cases, she had proven the statement, but it is not clear whether she understood this to address both directions necessary for the proof.

Summary. Overall, Julie had a good understanding of both questions. She approached both by making sure she understood them and looking at examples. She drew pictures when appropriate, looked for patterns, and broke the questions into smaller parts. However, Julie did not make an overall plan or have an understanding of what it would take to form a proof of the questions. While she was organized and able to follow her own work successfully, in the end Julie was not able to recognize her proofs as being incomplete.

Maggie

Maggie was a MATH 406 student. She was a senior majoring in mathematics. Her previous coursework included calculus I through III, MATH 305, linear algebra, statistics, geometry, and math with technology for teachers. She also listed a course number (426) that was not a valid course number, but the researcher believed it to mean number theory (326). During the interview, which was 52 minutes in length, Maggie worked on Questions 1, 3, and 2 parts a and b. Her strategies included drawing pictures, thinking of previous work, writing equations, and keeping her work organized.

Question 1. After the researcher read the question aloud, Maggie verified her understanding by asking if all the numbers 1 through 10 needed to be used on the pentagon. When she was told that they were all to be used, Maggie began to think about how to begin a proof. She summarized that the pentagon had three sides, there were 10 numbers in all, and there were three numbers on each side. After drawing a pentagon. Maggie again confirmed that there were three numbers per side and each side would equal 14. She clarified that she wanted to prove that the smallest sum was 14. However, she did not know how to start a proof.

When asked, Maggie said that she was thinking about a number line, and wrote the numbers 1 through 10 in a line. She indicated the beginnings of an optimization approach, that the sides should involve the highest and lowest numbers, 10 with 1, 9 with 2, etc. She paused and then shifted ideas, restating that she was trying to prove that 14 was the smallest sum. She noted that 10 would have to go with 1 and 3, when summing to 14, but questioned why she should use 3 in this combination since she was trying to find the lowest possible sum. Maggie checked with the researcher to verify that she could not duplicate the numbers. She then realized that 1 could go on a vertex, so that it could be used with both 10 and with 9, or some other number. Since 10 had only one possible combination in which it could be included, she concluded that it would have to be on an edge, not a vertex.

Maggie drew a new pentagon. She then asked, “Um, in order to prove this, can I just show? … Do you want an actual formal proof, I mean to say that the 14 is the best?” (Transcript 9, lines 65-73), indicating that she did not feel that finding the desired pentagon would be a formal proof. The researcher left this decision to Maggie, who decided to pursue finding a pentagon with sums of 14. First placing the combination 10, 1, and 3, and then 9 and 4 with 1 on the pentagon, she concluded that she should put all of the larger numbers on edges, and paused to consider her pentagon. Maggie felt that it might not work as drawn, but continued to attempt it anyway. On her first attempt, she was able to work through the possibilities and found a correct pentagon with sums of 14. Maggie then asked herself why 14 was the smallest value she could get. She said she would argue that the larger numbers were not being used twice, since they were not on the vertices, so this would be the smallest possible value. She pointed out that the example with sums of 16 had higher numbers on the vertices, making the pentagon have higher sums.

Maggie asked if she had to write her argument down. The researcher told her that a verbal argument was okay, but asked her to further explain why a different arrangement of the vertices would not produce a smaller sum. Maggie tried to explain further that the choices were limited because only certain numbers would go with the larger numbers. The researcher prompted her to consider whether she could get a sum less than 14. Maggie thought about that for a moment, and said that the smallest possible sum would occur with the combination of 10, 1, and 2. She then said that the next largest number, 9, was already with the smallest number. To get something different from the current picture, she thought that perhaps she could use 9 with 2 instead, but then the next number that had not been used would be 3, resulting in a sum of 14.

As Maggie tried to explain her argument again, she realized that 9 could be in a combination with 1 and 3. She drew a new pentagon and filled in the combinations for 10 and 9. The next number considered was 8, but Maggie explained that there was no place for 8 to go to make 13. She said that the only combination would be 8 with 2 and 3, which could not be done on the pentagon. When asked, Maggie repeated her argument that 13 would not be possible (Transcript 9, lines 169-208). The researcher stated that she would have a complete proof if it was written up formally, and Maggie agreed. When asked if she had seen anything like the question before, she said it reminded her of Sudoku puzzles, rearranging numbers in a puzzle. The researcher then summarized some of Maggie's strategies (Transcript 9, lines 237-250).

Maggie had struggled to begin this proof. She was unable to make an overall plan until well into her work, and even then the plan was incomplete. Her strategies included understanding the question, writing out what was known and what needed to be shown, drawing pictures, using the given example, looking for patterns, and systematically proceeding through her choices.

Her difficulties in finding a proof stemmed from a lack of understanding of what would constitute a proof of the statement. Maggie had great insights into the placement of the numbers on the pentagon, namely that numbers on the vertices would be included in two sums and that the largest numbers should be placed on the edges. Once she decided to find the pentagon with sums of 14, she used her insights to quickly develop the correct pentagon. She seemed to have been previously searching for some other way to prove the statement without finding the specific pentagon. However, she resorted to doing so after she was unable to find another proof.

Maggie did not appear to recognize the need to further prove her conclusion that her placement of numbers resulted in the lowest possible sum. She was convinced that it was true, and had to be asked to further explain her reasoning. The researcher had to specifically ask her to examine sums less than 14 before Maggie considered them at all. She understood and could explain all the aspects of the proof, but was unable to make a plan to put them all together.

Question 3. Maggie said that she understood Question 3 after it was read aloud. She wrote out what she knew, that a domino covered two squares, and what she wanted to show, that at least one dimension was even, similar to her work in Question 1. She also drew a rectangle representing an *m*-by-*n* chessboard. Maggie said that she was picturing dominoes covering the chessboard in her head, and drew a picture of what she was thinking. She indicated that the pattern she created could be extended upward, and that the number of rows would have to be even. Since the dominoes were only one square across, she said that the number of columns could potentially be odd (see Figure-Maggie. 1).

Figure-Maggie. 1: Generic pattern for domino placement in Question 3

However, she was unsure of how to say this formally. The researcher then summarized what Maggie had done so far for clarification.

When asked what, if anything, was left to prove, Maggie felt that nothing remained to be proven. The researcher prompted her to consider why the odd-by-odd case was a problem. Maggie struggled to explain her reasoning again. She said that she would need one side to be even to be a rectangle. The researcher pointed out that rectangles, and even squares, could be odd-by-odd. Maggie went on to accurately describe that an odd-by-odd chessboard would have one square uncovered, and she would need two squares to add another domino. She had been looking at an example, so the researcher asked her to prove her statement in general. Maggie responded by saying that an odd number times an odd number would result in another odd number. Since dominoes would come in sets of two squares each, they could only cover an even area. While Maggie did address this case appropriately, she again had not originally realized that she needed to do so.

When asked if she had anything left to prove, Maggie said that she had proven odd-by-odd could not work. She then noted that the statement was an if-and-only-if statement. She said that there were four cases, even-by-odd, even-by-even, odd-by-even, and odd-by-odd. Since she had covered all cases, she felt that she was done with the proof except for writing it up. She compared the question to number theory problems dealing with odd and even numbers. Maggie said she had used lots of visuals to prove the statement.

In this question, Maggie again lacked the ability to form a plan for the proof as she began her work. She understood the question, wrote out what was known and what was to be shown, drew pictures, found the appropriate pattern, but was unable to see the missing pieces to her proof. She also was unable to form her idea for the partial proof into something more formal in words.

After she had found the pattern for laying out the dominoes and had explained her ideas of why it would always work, Maggie said that she was done with the proof. The researcher had to push her to consider more, and it was clear that Maggie did not recognize the statement as a bi-conditional at first. Once asked to consider the odd-by-odd case, and after being asked for a generic proof, Maggie was able to develop the outline of a correct proof. She then recognized the if-and-only-if statement and concluded that four cases needed to be considered. While she did not further flesh out the details of those cases, she demonstrated an understanding of what would be needed for each one. Her difficulty had resulted from the inability to pick out the necessary components to a proof and make an overall plan before beginning her work, or even after some ideas had come forth.

<u>Question 2</u>. The researcher moved on to Question 2, part a, reading it aloud. Maggie read the question to herself and said she understood the definition of a 4-flip. She wrote down what she wanted to prove, that there were no two-digit 4-flips. Then, she made a representation of a two-digit number as two spaces _ _. She went on to label these spots with the numbers 1 and 2 and visualize how they would switch when multiplied by 4, shown in Figure-Maggie. 2.

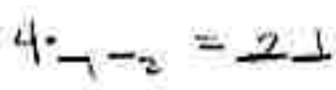

Figure-Maggie. 2: Visualization of flipped digits in Question 2 part a

Maggie said that she was not sure where to start. She was searching for a way to represent the digits of *N* to be able to show them in reverse order. Her first idea was to append a zero to digit 1, to represent that it was in the 10s place. The researcher asked her if there was any other way to represent attaching zero to a number. Maggie quickly determined that she could write it as the number times 10. She then was able to rewrite her equation as $\_{}_1 \text{ x } 10 \text{ x } 4 + \_{}_2 \text{ x } 4 = \_{}_2 \text{ x } 10 + \_{}_1$. However, she did not feel that her equation was helping her see the proof.

The researcher asked Maggie if she could go further with these ideas, or if she felt she should start new ideas. Maggie reiterated that she felt that the equations were not getting her anywhere. She said that she would want to start with the left hand side and show that she could get the right hand side, but did not know how to do that. The researcher told her that it was okay to give up this line of thinking and start fresh if she wanted to do so. Maggie said that she did not know where to start. She stated that she was having issues with the question because she knew that it was not possible to find a two-digit 4-flip and yet she was trying to represent what one would look like if it did exist. She did not know how to prove that it was impossible.

Maggie went back to her equation and said that she could keep going with it to represent a generic 4-flip of any length. She wrote out her new equation, but noted that not every number would fit, only special numbers. Maggie observed that since *4N* would be larger than *N*, the last digit of *N* must be larger than the first digit, making her first general rule. The researcher made note that she was only dealing with two-digit numbers at the moment. Maggie reworded her idea, saying that digit 2 would have to be larger than digit 1, and further considered whether it would actually be 4 times as large.

After confirming that she was only working with positive integers, Maggie said that the first digit must then be either 1 or 2 because any larger would give a three-digit result for *4N*. She next specifically considered the case where the first digit was 1, and noted that multiplying by 4 would give 40. She conjectured, incorrectly, that the second digit would also have to be a 1 or a 2, since otherwise there would be carryover into the 10s place. However, she said she had become confused at that point. The researcher suggested she go back to the point when she had determined that the first digit must either be 1 or 2, and clarified what Maggie's reasoning had been for saying this.

Maggie was able to regroup, and decided to proceed with a proof by cases. In the first case, digit 1 of N was equal to 1. She determined that there was no way to have 1 in the second digit of $4N$, since the number would always be even, therefore the first case was ruled out. In the second case, digit 1 of N was equal to 2. Maggie said she would go through the choices for the second digit to see which would result in an ending digit of 2 when multiplied by 4. She found that 3 could work, but $N = 23$ would result in $4N = 92$, which meant that it would not be a 4-flip. She further determined that 8 was the only other option, but $N = 28$ would result in a three-digit number for $4N$. Maggie concluded by saying that she had shown that there were no two-digit 4-flips because she had exhausted her possibilities.

When asked, Maggie recapped her solution (Transcript 9, lines 668-684). She said that she had not seen a question like this before, but that it reminded her of number theory when she wrote the equation for an n-digit 4-flip. She said that writing down what the digits would do when flipped had helped her to see what was happening. Maggie noted that she had wanted to look at a more general form when she first started because she knew that there were 10 options for each digit of N, and she had not wanted to go through all of them. She also said that she had not thought of any specific numbers until she looked at the options for the first digit of N.

The interview continued with Question 2, part b. Maggie immediately wrote out the representation of a three-digit number as three spaces, _ _ _. She then said that she knew the first digit had to be 1 or 2, using her results from part a of the question. After verifying that N was still being multiplied by 4, she said that 1 would again not be possible, by the reasoning from part a. Maggie also said that the last digit of N must be 3 or 8, using the same reasoning as in part a, as well. She split the question into two cases, based on whether the last digit was 3 or 8. After writing that $N = 2$ _ 3 and that she would want $4N = 3$ _ 2, Maggie was able to eliminate this case, because $4N$ would be at least 800.

Maggie moved on to the case that the last digit was 8. She summarized what she had found up to that point by saying that she knew $N = 2 \_ 8$ and would want to have $4N = 8 \_ 2$, where the blanks indicated digits that she could not yet determine. She felt that the middle digit would be tricky to figure out. After examining the multiplication by 4, Maggie commented that the middle digit must remain the same to be a 4-flip, so it would be impossible to find one that would work. When asked to prove this, she determined that the middle digit, x, would have to satisfy the equation $4x + 3 = x$. When she solved the equation, Maggie found that $x = -1$. Since all the digits must be positive, this implied that this case was also impossible. She knew that she had now proven that there were no three-digit 4-flips. When asked, she recapped her solution (Transcript 9, lines 858-879). Maggie said that her main strategy for part b of the question was to look back at her work from part a, apply the same ideas, and work through a proof by cases.

In part a of this question, Maggie again wrote out what was known and what needed to be shown. She developed an equation to represent the digits of the 4-flip. After prompting from the researcher, she was able to expand this equation into a form that could have been useful to her. She expressed frustration at trying to work with something that she knew was impossible. When encouraged to continue, Maggie went a bit off-track with more equations. However, she did make observations that helped bring her to considering the individual digits of N. The researcher reminded her that N was only a two-digit number, which brought her away from the generic forms. Maggie was able to use her ideas about the individual digits to determine that digit 1 of N must be either 1 or 2. She was then able to divide the proof into cases and sub-cases and complete her arguments.

For part b, Maggie knew exactly what she wanted to do from the beginning. She used her ideas from part a to limit her digits and again consider cases and sub-cases. She continued quickly to a complete proof of part b of the question, only needing to be asked to prove one of her statements in more detail.

Summary. Maggie's main difficulty through all three proofs was the lack of an overall plan. When that plan did emerge partially in some questions, she was able to follow through to a partial proof. In part b of question 2, the plan was fully developed in Maggie's mind and so the proof went very smoothly. She had great insights and a basic set of strategies that she implemented in every question, with some additional strategies that were only used for some of the questions, such as drawing a picture, or working with equations.

Maggie was able to monitor her attempts, did not follow unproductive ideas very far, was able to acknowledge a need for redirection, but was sometimes unable to actually choose a new path to pursue. This again points to her lack of a plan as she was unclear what ideas to try to ultimately find a proof.

Paul

Paul was a MATH 406 student. He was a senior majoring in mathematics. His previous coursework included calculus I and II, MATH 305, linear algebra, number theory, abstract algebra, complex analysis, and ordinary differential equations. During the interview, which was 59 minutes in length, Paul worked on questions 1, 3 and 2 part a. His work on these questions included many equations that he wanted to manipulate to find a proof. Additionally, he brainstormed ideas before beginning the proofs as well as during the solution process, including bringing in ideas from previous coursework.

Question 1. After the researcher read the question, Paul reread it himself twice while considering where he would start. He decided to first find the sum of the numbers 1 through 10. Knowing there was a formula to find this sum, since it was just series, he said that the total sum would be 10(11)/2. He used a smaller example to verify that his formula was correct. Paul determined the sum of 1 through 10 to be 55, and then stated that the sum along each side must be 11, since there were five sides. He had not yet realized that the vertices would count twice in the overall sum and his sum only included each number listed once. He paused, then expressed that he was having trouble deciding what to do next. He quickly realized that he had been wrong and decided to find an equation to represent the overall sum, since the vertices needed to be counted twice.

Paul labeled the vertices v_1 through v_5. He then was unsure how to find an equation from there. He stated that he could use the brute force technique instead of using an equation. He wanted to assume that the smallest value was less than 14 and proceed with the proof from there, indicating a desire to use proof by contradiction, but not using that term. He thought he would have to break the proof into two cases, where the smallest value was less than 14 and where the smallest value was greater than 14.

However, he did not pursue this thought further. He paused to look over the question again.

At this point, Paul was unsure where to go next. The researcher prompted him to describe what he had been considering when he labeled the vertices and asked him to pursue this idea further, if possible. Paul said that 2 times every vertex plus 1 times every side would have to equal 55. He still had not realized that a sum of 55 would not include any of the vertices listed twice. He wrote out the equation $2(v_1+v_2+v_3+v_4+v_5)+s_1+s_2+s_3+s_4+s_5 = 55$, and stated that he wanted to use his equation to find expressions for each of the sides. As he attempted to do this, he realized the equation was incorrect and rewrote it without the multiplication by 2. However, he saw a connection between the incorrect equation and the correct one. Paul said that adding the vertices again to both sides would result in the left hand side being that of the expression he had originally been thinking of, and that it counted up the sum of all the edges on the pentagon.

Paul's goal was to find a lower bound on the left hand side of this new equation. Reading the question again to make sure that the vertices were counted twice, he then rewrote his equation as $E_1+E_2+E_3+E_4+E_5 = 55+(v_1+v_2+v_3+v_4+v_5)$, where E_i represented the sum of the edge i. The researcher asked him to go further with this idea. Paul found that 14 times 5 was equal to 70, but did not yet relate that to his equation. He also said that the smallest possible values for the vertices would be the numbers 1 through 5. He made two mistakes in pursuing this idea further. The first mistake was a simple computational error, finding that 55 plus 1 through 5 would be equal to 65 rather than the correct value of 70. The researcher asked him to explain where his number had come from. Paul described again how he had determined that value, and was able to see his error and change the sum to 70. Once he saw that this was the same as 5 times 14, he knew that he was getting closer to the proof.

The second mistake was misreading the question. The researcher had asked him to go through his thoughts again for clarification. Paul said that the average of all the individual side sums would be 14, not understanding that each side sum would have to be the same as the others. The researcher asked him to consider what would happen if 14 was not the minimum sum. Paul said that if the combination 1, 2, and 6 were on one side, the sum would only be 9. After being asked to pursue that example further, he drew a new pentagon and placed the numbers 1 through 5 on the vertices, and added 6 between 1 and 2. He placed more numbers on the new pentagon and found that two sides of his example would have sums less than 14 and the rest would have sums greater than or equal to 14.

The researcher then asked Paul to consider whether his pentagon had the same set-up as the given pentagon with sums of 16. After comparing the two pentagons, Paul realized that he had misread the question. He said that he now knew that all sides had to have the same sum. With that information, he knew that he had found his solution. He rewrote his equation as $5E = 70$ and concluded that $E = 14$, so that must be the minimum value. When asked to restate why he could say this, Paul seemed to second-guess himself. He then said that he was just going to say it was true and leave it at that because he thought his reasoning was a “little fuzzy” (Transcript 10, line 262). The researcher reread the question aloud and asked if he was satisfied with his solution. He said that he was, though he was ashamed that he had misread the question.

While discussing his strategies, Paul stated that he often liked to work with equations and did not like looking at the big picture. He said that he was strong in algebra and weak in geometry, which was why he liked to work with equations first and the big picture last.

Paul began this question by making sure he fully understood the question. He had a strong desire to develop equations from the start, a tendency that continued throughout the interview. He first found the sum of the numbers 1 through 10, using a formula he had learned previously and checked his formula with a small example. After labeling the vertices, Paul decided to approach the question using proof by contradiction, though he said he could probably use brute force as well.

Still wanting to use equations, Paul worked with his total sum value and the idea that vertices would count twice to develop a large equation. Though he struggled and made an error at first, eventually he was able to find the correct expression for the total sum. He then made another error in misinterpreting the question, which led him astray momentarily. The researcher challenged him to examine his ideas further, and Paul was able to see his error after a few minutes. He then knew immediately that he had found the proof. Though he was not able to go back and explain his reasoning, he was certain that his work would produce a valid proof. He also acknowledged his desire to use equations and said that he considered it his strong point.

Question 3. Paul first reread the question twice. After this, he clarified the meaning of adjacent. He looked at the question a third time, then said that it was a bi-conditional statement and so he would need to prove both directions. Paul looked back at the question again and chose to first consider the forward direction, assuming that there was an *m*-by-*n* chessboard with a perfect cover and wanting to show that one dimension had to be even. He decided to work with the area of the chessboard, saying that this was equal to the product of *m* and *n*. He then struggled to explain his thoughts. He said that "the area of the chessboard, uh, is a certain amount of horizontal pieces and a certain amount of vertical pieces, and each piece counts for two, two units" (Transcript 10, lines 383-385). The researcher asked if the pieces he referred to were dominoes, and he said that they were. He tried to formulate an equation to represent this, but was unsure how to do so.

After stopping to consider the question for over a minute, Paul said that he was thinking about showing that it was impossible for both dimensions to be odd. He said that if both were odd, then the area would also be an odd number and so could not be divided into sections of 2, contradicting the assumption that the chessboard had a perfect cover. The researcher asked him to prove that odd-by-odd resulted in an odd number of squares. Paul proved that, using appropriate notation. He repeated that the odd result would not be divisible by 2, and then said that this would complete the first half of the proof.

Paul then moved to the reverse direction, assuming that at least one dimension was even. He said that he would let *m* be the even dimension, without loss of generality, and so the product would be an even number. Since the total number of squares would be even, you would have "an even number of compartments of 2 … so you can perfectly divide it up," he said (Transcript 10, line 486-490). He then said that he was satisfied with his proof.

When asked, Paul stated that he had seen other questions that he felt were similar to this, referring to working with chess pieces, dominoes, and trominoes (blocks with three pieces each) in discrete optimization. However, those questions had required induction proofs, but in the interview question he had not felt that he could use such a proof technique. Paul said that he had not understood how to prove when he took MATH 305 because he felt that he did not have a very good instructor for that course. Rather, he had learned how to prove in other 300 and 400 level math courses. He said that he had used proof by contradiction for the first direction of this proof.

When asked what he was thinking and doing in his head while working on this question, Paul said that he was just looking at equations, not pictures. He also said that he had been brainstorming at the beginning to feel out the question. When asked what happened while he was brainstorming, he stated that he first tried to reread the question, then he tried to go through the different proof techniques, such as induction and contradiction (which he stated were the "fancier ones" (Transcript 10, line 589)) or brute force through algebra. He said he chose a technique by deciding which one fit the question and how hard or easy it would be for him to use the particular technique. He finished his comments on the question by saying, "I guess, [I] just kind of get out my playbook… and see what uh, what I could use" (Transcript 10, lines 600-604).

Again, Paul first read over the question several times. He recognized the statement as being a bi-conditional and so he would need to prove both directions. He organized his thoughts, stating what was being assumed and what needed to be shown. He clearly had developed an overall plan and structure for the proof, and proceeded to fill in the details. His proof revolved around the total area of the chessboard. Within his work, he stopped to make some goals and plans for how to continue. He still desired to use equations, but was unable to do so until he proved that an odd times and odd was also odd. Paul did not lose sight of his planned structure for the proof. When he recognized that he had completed one half of the proof, he moved on to the other. His reasoning for the second half was not as clear as the first, but he decided on his own that he had completed the proof.

Paul then compared this question to others that were similar, and restated that he had used proof by contradiction. He referred to his thought process when beginning the question as brainstorming. During this process, he said that he reads and understands the question, and then goes over a list of proof techniques in his head before determining which to use. He called this list or part of the process his *playbook*.

Paul successfully proceeded through this question with a clear overall plan. He understood the various proof techniques and chose an appropriate option. He was then able to produce a proof, though it did lack in some details. He exhibited self-monitoring abilities such as monitoring his overall goal, creating and completely sub-goals, and recognizing his completed proof.

<u>Question 2</u>. The last question Paul attempted was part a of Question 2. Time did not allow him to attempt the other parts of Question 2. The researcher read the question aloud, after which Paul reread it to himself. He stated that the question was "fairly easy to understand" (Transcript 10, line 624). After a brief pause, Paul said the most obvious choice for the proof would be to assume that two-digit 4-flip existed, and try to find a contradiction from there. He said that thinking about the digits might be tricky, but he was quickly able to form an equation to represent them. For a two-digit number, ab, he could represent the digits with the equation $10(a) + b$. He then decided to use the fact that he assumed it was a 4-flip to write $4(10(a) + b) = 10(b) + a$. Using his equation, he tried to form some ideas about the relationship between the individual digits. He first said that $b = 4a$ and $a = 4b$, thinking of the 10's and 1's places in his equation being equal. But, he crossed those equations out, saying that he would not know that they were equal because there could be carry-over. He brainstormed, considering congruence classes modulo 10, but decided that this did not "seem very fruitful" (Transcript 10, lines 672-673).

Paul paused for 35 seconds, looking at his work and the question. He still desired to work with equations, trying to find bounds for both a and b. He then proceeded by writing inequalities based on the fact that a and b were single digits, i.e., between 0 and 9, and combining terms from his previous equations. He continued until he was able to determine that, since the 10's place could not have carry-over into a three-digit number, he knew that $4a – b \leq 0$ and solved to find that $b \geq 4a$. He said that he did not think there would be a solution that would work for his inequality. However, after further consideration, he realized that a being 1 would work for sure. He then said that, given more time, he felt that he could figure out the proof in full.

After being prompted by the researcher, he determined that the value of *a* could only be 1 or 2. Paul stated that he would then plug 1 and 2 back into the equations he had, test the cases, and solve from there. He mentioned again that he was searching for a contradiction, but felt that he was going in circles. The researcher told him they would stop there, in the interest of time, and proceeded to have Paul reflect on his work. Paul said he was still "cranking equations" (Transcript 10, line 825) as he had in the previous questions. He then paused and said that he could have just considered the numbers between 10 and 25. He asked if he could pursue his new line of thinking, which he was allowed to do.

Paul used his equation, $b \geq 4a$, and the fact that the value of *a* could only be 1 or 2 to say that 10 through 13 could not work. He then wrote out the possible cases. In the first case, $a = 0$ and *b* could equal 1 or 2, though this was not actually a viable case. In the second case, $a = 1$ and *b* could equal 4 through 9. Lastly, in the third case, $a = 2$ which gave *b* equal to 8 or more. Paul stated that the third case was not possible since *ab* could not be greater than 25. He concluded that he would only need to check the numbers 14 through 19 and then he would be finished with the proof.

When asked if he had seen a question like this before, Paul said he had not, but that he had learned in number theory how to represent the digits of a number in an equation, as he had. He said that when he had done proofs in his number theory course, they had taken things like digits or remainders, put them in different forms, such as equations, and thought about congruence classes. He stated that this was the only example he could think of where he had used this type of strategy previously.

As he had throughout the interview, Paul stressed the use of equations in this question. He read the question to himself before making a plan for the proof, again choosing to proceed using proof by contradiction. He then developed an equation and used this equation to put constraints on the digits of *N*. He said that at that point he could break the proof into cases based on the value of *a* and solve, using the equations, from there.

Upon reflecting on his work, Paul suddenly realized that he could also have just examined the values 10-25 and check for 4-flips. He then asked to go back and work further. He was able to finish his proof nearly to completion and explained the remaining steps needed to complete it. He had used information from other courses as a guide for this proof as well as his own desire to use equations. He kept his work generic and did not consider the actual possibilities for the digits until the very end.

Summary. Unlike other students, Paul did not turn to examples to understand the questions in this interview. In fact, considering specific numbers seemed to be his last resort. His equations were, for the most part, valid and contributed greatly to the positive outcome of his proofs. However, his lack of willingness to look at examples or try options other than writing equations may have slowed him down.

Overall, Paul had a very specific approach that was consistent throughout all three questions. He read and understood the questions, made a plan, brainstormed different proof techniques, broke the question into parts, and searched for appropriate equations. He was able to monitor his overall plan and attempts, organize his work, and always wrote down what was known and what was to be shown. The strategies that were pointedly not included in his work included looking at examples, drawing pictures, and looking for patterns. However, the lack of these tools did not seem to greatly hinder Paul's progress. Overall, his techniques seemed to work well for him.

Sandy

Sandy was a student in MATH 406. She was a senior majoring in mathematics. Her previous coursework included calculus I through III, linear algebra, MATH 305, and statistics. During the interview, which was approximately 52 minutes in length, Sandy worked on Questions 1, 3, 2 part a, and Question 4. She completed a verbal proof of Question 1, but in Questions 2 and 3, she could not continue past specific examples. Even when prompted to move into more generic terms, she was clearly rooted in the examples and was unable to develop the notation for a complete proof. Once convinced that a statement was true, she seemed unable to move to proving it in general in an effort to convince others.

Question 1. After the researcher read Question 1 aloud, Sandy read it silently. She then verified that she needed to find an arrangement where the sums of the sides were 14 instead of the 16 in the example. She read the question again, and wrote the numbers 1 through 10 under the given pentagon. Sandy also wrote that she needed to find five ways to get a total sum of 14. Her first attempt began by drawing a new pentagon and placing the same numbers on the vertices as they were placed on the example pentagon, that is, she put the numbers 10, 1, 7, 3, and 4 on the same vertices as the example pentagon. However, she crossed out the entire pentagon after considered it for a moment. The researcher summarized what she had seen to clarify. Sandy agreed that she had looked at the vertices 3 and 4, noticed that she would need 7 to get a sum of 14, but 7 was another vertex, so she knew that the pentagon would not work as she had drawn it.

Sandy decided to draw a new pentagon and start over. Without placing any numbers on the pentagon, she decided that she would write out combinations that made 14, finding five different ways that she could then place on the pentagon. She said that it would not work with five totally different sums, though, because the vertices would each be in two of the sums. However, she realized that she might be able to carefully find the five combinations she was looking for.

After talking this through, Sandy proceeded with her idea and began to build a list of combinations, starting with 10, then moving on to sums for 9, 8, 7, and 6. She used one number from the previous sum for each new combination, marking the repeated number as a vertex, though she incorrectly used the word vertice [*sic*]. Her list of combinations is shown in Figure-Sandy. 1.

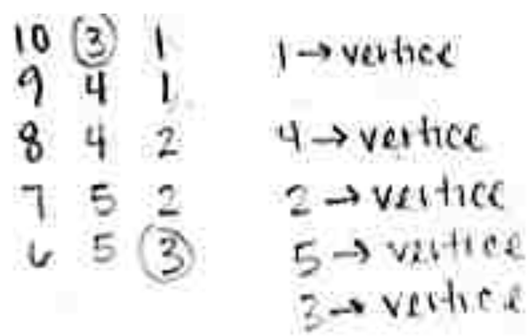

Figure-Sandy. 1: List of combinations summing to 14 in Question 1

She then crossed off the numbers representing vertices from her list of 1 through 10.

When asked, Sandy explained that she was finding sums of 14 using three numbers and when she saw that a number was used twice, marked it as a vertex. She also noticed that she had the numbers 1 through 5 as vertices, and then began filling in the pentagon. Using the list, her first attempt resulted in the correct pentagon with sum 14.

The researcher asked Sandy what, if anything, was left to be done to finish the proof. Sandy suggested that she might try to prove that 13 would not be possible, but was unsure of this. The researcher asked her what she would do next if she were able to prove that 13 was not possible. Sandy said that the sum could not be less than 13 because it had to involve combinations of three numbers and 10 must be used. So, the smallest possible combination with 10 would be 10, 1, and 2, forming a sum of 13. The researcher prompted her to pursue the idea of proving that a sum of 13 was not possible.

Sandy used the same technique as she had for 14. Starting with 10 and working down, she listed the combinations, noting which numbers should be used as vertices. She explained that when she reached 8, she could not use 4 and 1, since the 1 was already a vertex. So, she would have to use 8, 3 and 2, making the vertices 1, 2, and 3. Once she rewrote the list of the numbers 1 through 10 and crossed off her vertices, Sandy realized that finding the pentagon would be impossible. She pointed out that there was no way to find a sum involving 7 because all the possible combinations with 7 involved a number that was already used twice elsewhere. She thought that she might be able to make a different choice for 9 that would change the result, but quickly realized that there was no other combination. She concluded that a pentagon with sums of 13 could not be found.

The researcher summarized what had been done so far, and then asked if Sandy had proven that the smallest possible value for the sum was 14. Sandy said that she had. She asked if she needed to write up her proof, to which the researcher responded that she did not have to. Sandy stated that she had not seen a question like this before. When asked to reflect on her strategies, Sandy summarized her own work, saying that she had worked to find combinations from 10 down, marking those numbers that were used twice as vertices. She kept track of which numbers were used on a list of 10.

Sandy was able to complete a proof for this question. However, she did need prompting several times before continuing with her ideas. Her work included several productive strategies including reading and understanding the question, writing what needed to be shown, making a plan, looking for patterns, drawing pictures, redirecting when appropriate, organizing work, making lists, using previous work for new parts of the proof, and recognizing a complete proof. She had been able to keep track of the overall goal and make sub-goals along the way. Overall, her work was clear, concise, and well organized.

<u>Question 3</u>. After the researcher read Question 3, Sandy reread it as well and stated that she really did not understand. The researcher finished reading the rest of the question, and Sandy looked over it again. She wrote that when *m* was even, representing it as $2m$, then *n* would be odd, writing it as $2n+1$. Sandy continued with similar poor notation throughout the question, never converting to a more accurate representation of odd and even with different variables (e.g. she could have written $m = 2j$ and $n = 2k+1$). She then suggested that this could be an 8-by-7 chessboard, for example.

Sandy read the question again and clarified that one domino covered two squares. She wrote this assumption down, but went back and crossed off her previously writing about *m* and *n*. She said that she had been thinking of abstract algebra, and was not sure why she had written those statements down. Sandy then drew out a picture of an 8-by-7 chessboard, stating that it had 56 squares and so she would need 23 dominoes to have a perfect cover. She said that she was not sure if this was right or not, but seemed unable to move on. When asked where she would go next, Sandy responded “Isn’t that all it’s asking?” (Transcript 11, line 312). She clearly did not see a need for more work, nor did she see that her one example did not constitute a proof to the generic question.

The researcher asked if she had proved what the question asked for, then read the question aloud as well. Sandy responded by saying that “anytime you use an even or an odd value for *m* and *n*, you get, if the outcome is an even number then you know that you’d be able to cover the chessboard perfectly” (Transcript 11, lines 317-319). The researcher asked her when she would not get even, because she had said *if* the outcome was an even number. Sandy tried two more examples, 2-by-1 and 4-by-3, only noting that they had an even product. She then said “you’ll always get an even, won’t you?” (Transcript 11, lines 331-332).

The researcher agreed that Sandy had been using even-by-odd examples and so her result had always been even, but asked Sandy to prove that her result would always happen. Sandy turned to writing her statements slightly more generically. She formed the equations, $m = 2m$ and $n = 2m+1$, representing that *m* would be even and *n* would be odd. Her notation was again poor, using *m* in each when she should have used new variables. Sandy then considered the product of the two, concluding that it would be even. Her work is shown in Figure-Sandy. 2.

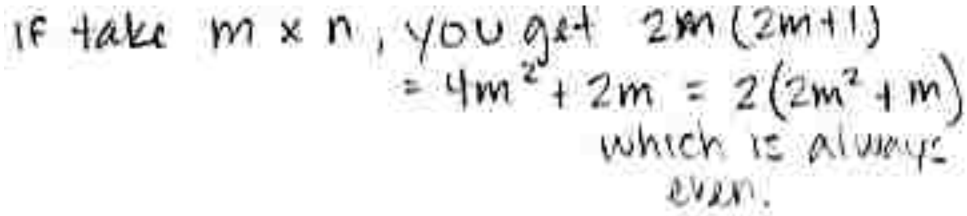

Figure-Sandy. 2: Argument that odd times even equals even in Question 3

She did not examine any other possibilities. When asked if there was anything left to prove, Sandy said that she could also prove the result when *m* was odd and *n* was even, but said that other than that, she felt good about what she had. Sandy then recapped her work, stating that the ideas for notation and splitting the options into even and odd cases had come from her abstract math course, referring to MATH 305.

Sandy's work on this question showed a lack of understanding of what constituted a proof. This was not seen in Question 1, most likely because her proof there had dealt with specific numbers only, rather than generic notation. Sandy began this question by reading and understanding the question. She attempted to use generic notation briefly before discarding it and looking at a specific example. Once she understood what a perfect cover would consist of on her example, she was convinced of the truth of the statement and ended her work there. When challenged to consider whether she had found a proof, Sandy clearly did not see a need to move past her example. When asked again, she finally moved to a generic explanatory statement, but still without proof. She never considered or acknowledged the case of both dimensions being odd, nor did she prove that a perfect cover could exist in any other case. She only proved that the area of a chessboard with at least one even dimension would also be an even number. However, she did not mention how dominoes could make a perfect cover, even in her example.

Sandy explained that at some point she had made a plan to look at cases, though her work showed that she actually had not considered all possibilities. Overall, she was unable to move past her understanding of a single example to proving the statement in general. It was clear that she did not understand what needed to be proven, including the fact that the question was a bi-conditional statement.

Question 2. Seeing that Sandy was unlikely to proceed further with her proof for Question 3, the researcher moved on to Question 2. Sandy reread the question after it was read to her, but said that she still did not understand the definition of a 4-flip. The researcher reworded the definition, however Sandy then thought that N would have to be a four-digit number. The researcher clarified further. Once that thought had been corrected, Sandy proceeded to look at an example, $N = 12$, to see if she understood. Again showing confusion on the definition, she thought she would want 4 times N to equal 4 times the reverse of N. The researcher explained what she would have wanted to find when multiplying by 4. For her example, when $N = 12$, she would want $4N = 21$.

Sandy paused and looked at her work and the question again. She then wrote that 4 times 10 would be 40, but she would want to see 01 instead, showing that she now understood the definition. She felt that there would clearly be no way for such a number to exist, but did not know how to explain it. It seemed that since she was convinced it would be impossible, Sandy was struggling to begin to describe what one may look like or formulate an argument of why it could not exist. The researcher encouraged her to continue by asking how she might prove that it could never happen and what she might try next. Sandy still struggled however, so the researcher asked her to consider what she would do if this had been assigned as a homework question.

Sandy looked back at her example, but still was unable to find any new ideas. The researcher suggested that she was having trouble representing the number, and she agreed. Sandy tried again to look at examples, saying that when $4N = 21$, she would want $N = 12$. She then divided $4N$ by 4, trying to show that the result (in this case $N = 5$ ¼) could never actually be N. Her work is shown in Figure-Sandy. 3.

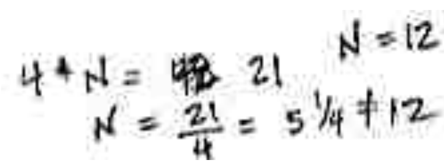

Figure-Sandy. 3: Calculations with example $N = 12$ in Question 2 part a

She had to complete the work to see that the N she solved for and the original N, made from the reverse of $4N$, were not equal; seemingly unable to find another way to describe her thoughts.

Sandy started to look at the question in reverse, saying that the *4N* could be the numbers 10 through 99, and trying to solve backwards to find *N*. When she did that, she could see that the resulting numbers were not what would be needed for a 4-flip. However, she did not seem to have a good grasp of a way to explain what she was doing or why. The researcher continued to challenge her to find a proof, asking if her process could work for other two-digit numbers. Sandy then tried $4N = 48$, saying that $N = 84$, and repeated the process. She said that it could never happen, but she did not know how to explain it. She felt that dividing *4N* by 4 would always result in a number that was less than the *N* value necessary for a 4-flip. The researcher asked her if that would always be true and challenged her to think of a value for *4N* such that her statement was not true. Sandy used the same process with $4N = 10$, where she knew she would be looking for $N = 01$. She saw that her theory failed because the result was $N = 2.5$, which was larger than 01.

When the researcher asked her again what she might do next, Sandy tried another example, but then said that she did not know where to go next. She still contended that her theory was correct, and the one example where her process resulted in a number greater than she had wanted would be the only exception. Since $N = 01$ was not a two-digit number, she stated that the case did not even count. When asked again to prove her statement, Sandy was unable to do so, but remained convinced that it was the correct way to go about the question.

Sandy stated that she had never seen anything like this before. She described her strategy as looking for a pattern by looking at examples. Seeing that Sandy was not making any further progress, the researcher chose to move on to another question, feeling that this would be more fruitful than looking at parts b and c.

Sandy had not made much progress on this question. It appeared that even after several explanations, she did not truly understand the question nor did she have any idea what it might take to prove it. While she did look at several examples, no overall idea for a proof ever emerged. Again, she stated that she was sure that the statement was correct. After she was convinced of that, Sandy was unable to proceed with a proof. Part of her difficulties may have arisen from working backwards from *4N* to *N*, rather than forwards where she may have seen some of the limitations on *N*. Instead, she never considered any limitations beyond that *4N* would be in the interval 10 through 99. Since there were so many to look through, Sandy never attempted to write out a proof by brute force, either. There was no evidence of strategies beyond looking at examples and looking for patterns.

Question 4. Problem 4 was not given to many participants, since it was designed for use only if necessary to gather more information from a participant who worked through all they could in the first three questions. Sandy again was unable to understand the new definition at first. Instead of using the notation given, she immediately wrote out incorrect interpretations of the statement, writing $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{a}$. Sandy paused and seemed to consider where to go next in trying to understand the question further.

Sandy looked at the definition of divides as it was written in the question, writing that $\frac{x}{y}$ if $y = kx$. She said that she did not understand how x could divide y if the expression given were true. She noted that solving for k would result in $k = y/x$, instead of what she expected x/y. The researcher suggested that the term was actually new to Sandy, and that it may mean something counter to what she wanted to write for the definition. The researcher then reiterated that the definition meant if you took y and divided by x, you would get k. Sandy seemed to accept the definition now and wrote out the equations $b = ka$, $c = kb$, and $c= ka$, crossing out what she had written for x and y. However, she had incorrectly used the same constant, k, for all three equations. Similar to Question 3, she was unable to see that the integers in each of these three equations must be different. She asked if her other interpretation had been wrong, and the researcher said that it was incorrect and may have led her astray.

The researcher reread the statement to be proven and asked where Sandy would go next. While working with the equations, Sandy showed that she considered the k values to be equal, saying that the three equations implied that $b = c$. She had made another error here by using all there equations simultaneously. When she paused and did not seem to know where to go next, the researcher asked her what else she might try. Sandy then went on to say that because of her previous result, she also knew that $ka = kb$.

While verbalizing her thoughts to the researcher, Sandy finally realized that there was an implication statement to be proven, not just a list of three equations to work with. She said that she actually should be proving that the first two equations implied the third.

Sandy wrote *if* prior to the first two equations and *then* prior to the third. However, she incorrectly wondered if she could use the consequent to prove the antecedent, rather than the other way around. Again, with improper notation, she was unclear how these equations could imply each other. Sandy went back to her result that $b = c$ and tried to say more from there, but did not know what else to do. She said that there was nothing else she could think of to try. Sandy then discussed her work with the researcher. When prompted, Sandy agreed that she was having a difficult time visualizing the situation, and that was proving to be a roadblock for her. Time did not permit any further work on Question 4.

Unlike the previous questions, Sandy never looked at any examples during her work on this question. It was not clear that she ever truly understood the definition. After it was explained to her again, Sandy was able to move past her desire to express *a* divides *b* as *a divided by b* instead. However, her notation, similar to work in other questions, was not careful and she failed to use separate variables in the different equations. She then went on with these errors, giving the incorrect result that $b = c$. Not having a clear view of the overall picture, this did not seem to make her second-guess her work at all.

After some time, Sandy did realize that she was not approaching the proof correctly because it was an implication statement. However, she was not able to break out of her previous thoughts nor was she able to form any part of a proof. Her strategies included only reading the question, forming equations, and looking for, but not finding, proper notation. As stated previously, it was never clear that Sandy truly understood the statement to be proven for this question.

<u>Summary</u>. Overall, Sandy's strategies were not consistent and lacked organization and planning. With the exception of the first question, she was unable to form proofs and did not seem to understand what would be needed for a complete proof. There was little evidence of self-monitoring, goal setting, or attention to the overall question to be proven in any of her work. Her strategies in the last three questions were quite limited and she had been unable to redirect her work after a failed attempt. She was not even able to recover after prompting from the researcher. Sandy may have made more progress overall if she had been able to approach all the proofs as she had Question

1. In that question, she used many different strategies, made an overall, plan, was able to monitor her progress, and redirected her attempts with promptings from the researcher. This was the only time during the interview that there was evidence of self-monitoring and true understanding of the components necessary for a valid proof.

Rick

Rick was a student in MATH 305. He was a senior majoring in chemistry. His previous coursework included calculus I through III, discrete math, linear algebra and statistics. He participated in two interviews. During the first, which was 56 minutes in length, Rick worked only on Question 1. While Rick was able to formulate a verbal explanation of Question 1, it did take some prompting for him to be able to verbalize and to finish the question. He struggled to find direction, as well as to keep track of his plan and what he had already tried. The researcher had worked often with Rick during office hours; therefore, Rick was very comfortable expressing himself to her and letting her know where he was struggling.

During the second interview, which was 47 minutes in length, Rick worked on Question 2 part a, and Question 3. Like his first interview, Rick's work here was scattered. He had many good ideas, but was unable to monitor his progress through the questions. When looking back on his work, he could neither recall his reasoning nor follow the order in which he had proceeded through the question.

Question 1. As stated, the first interview consisted of work on Question 1. After the researcher read the question aloud, Rick read it to himself as well. The word prove in the question seemed to catch him off guard and he immediately became worried at the thought of having to do a proof. Throughout the interview, Rick erased often, which he remembered he was asked not to do. Since this seemed to be distracting for Rick, the researcher eventually allowed him to erase. However, this did not appear to eliminate the distraction entirely, as Rick continued to mention his erasing even after this. He also mentioned being video taped, perhaps indicating that this may have been distracting for him as well. Additionally, Rick questioned his proof-writing and mathematical abilities several times throughout the interview, which clearly indicated a lack of confidence and a high level of frustration with his difficulties in this area.

Rick proceeded by reading the question. He verified that the sum of every side was 16 in the example. After rereading the question again, he noted that he had to use three consecutive numbers around the pentagon in his sum. However, the researcher misunderstood his comment and a discussion followed on his actual meaning. Rick thought he could take sums of any three numbers in a row on the pentagon, including going around corners. The researcher corrected this. Rick verified again that the sum along each side was 16, using the new information about which numbers should add to that sum. The researcher noted that the numbers could be moved around the pentagon to forms sums of 14. She also reiterated that the question asked to prove that the smallest value for the sum was 14. Rick verified that all sides were required to have equivalent sums. He then multiplied 16 by 5, since there were five sides, and found that the total sum was equal to 80 on the example pentagon. After asking the researcher again if each side had to equal 14, he divided 80 by 5, which he saw resulted in 16, not seeming to connect that he had found 80 using 16 times 5. He then asked if the pentagon was possible and if anyone else had proven it. The researcher said that it was possible and others had found a proof.

The researcher directed Rick back to his work, asking what he was thinking about the question. He was stuck at that time in the thought that the total sum was 80, not understanding how changing the placement of numbers on the pentagon could change the total sum and therefore produce an individual sum of 14 on all sides. When Rick asked, the researcher verified that the current picture contained a sum of 80 distributed to all sides. He also asked if the figure still had to remain a pentagon, which the researcher said it did. The idea that the total sum was equal to 80 still bothered Rick. He noted that, to get a pentagon for 14, 10 must be taken away from the 80, making a total sum of 70. This continued to confuse him throughout the interview and led him astray in his work.

Rick checked to make sure he could not subtract numbers, and that he could only use the numbers 1 through 10. The researcher restated that the numbers 1 through 10 needed to be arranged around the sides, and also restated the rules for where numbers had to be placed. She encouraged him to continue looking for a proof that 14 was the smallest sum. Rick again verified that all five sides had to have the same sum, although this had been addressed several times. Rick seemed to enjoy working through the question and even said so several times, though he also doubted his ability to get to an answer, even within the entire hour time period of the interview.

Rick said that his main issue was that he was stuck on the idea of the total sum being 80 in the example.

> Okay, my main block right now is you have the sum equal to 80. You divide it by 5, you're going to get 16 no matter what. But somehow, you have to divide that by 5 to equal 14. – I mean 7 divided by, 70 divided by 5 is 14, right? Yeah. [pause] So, you have to lose 10 somehow. (Transcript 12, lines 153-156)

He asked again if he could use negative numbers. The researcher said that he could not and asked what he might try next. Rick said that he was still thinking of a way to reduce the total sum to 70. He stated again that he was not allowed to use negative numbers, seeming to be stuck in this thought. He was unable to redirect his work and struggled to find new ideas.

Rick suddenly moved past these thoughts momentarily, saying that the sum of the numbers 10, 1, and 3 would be 14. When he seemed to abandon this statement, the researcher asked him to write it down. Rick proceeded to list combinations whose sum was 14, however he did not mention use the combination with 10 previously mentioned (see Figure-Rick. 1).

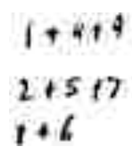

Figure-Rick. 1: Partial list of combinations summing to 14 in Question 1

Through this process, Rick was able to pull away from his roadblock and continue working. He indicated that he was looking at the pattern with the first and second numbers both increasing in the combinations. However, he saw that he could not continue his list because his only remaining digits were 8 and 10, having not realized at this point that there would be repeated numbers in the combinations. As he turned to drawing a pentagon with sum 14, Rick called his method of guessing and checking the "blood, sweat, and tears approach" (Transcript 12, line 220). He started a new list and again tried to find five different sums of 14. After writing two new combinations, he realized that he could repeat numbers. The list, as he left it, is shown in Figure-Rick. 2.

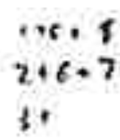

Figure-Rick. 2: Second partial list of combinations summing to 14 in Question 1

Rick had drawn a pentagon previously, but now turned to placing numbers on it for the first time. His method did not seem to have any organized system of choices, just a random search for patterns as he tried to find more combinations. He mentioned the use of the video camera for the first time in the interview, indicating that it could have been a distraction for him. Rick then erased all numbers from the pentagon and placed one combination of numbers around a corner, but erased those as well. He continued by placing the combinations 1, 9, 4, and 2, 5, 7 on the pentagon. Seeing no numbers in common, he placed them on opposite sides. He decided that he would change the numbers around and put 7 on a vertex. He then noticed that the difference between 1 and 7 was also the number needed to complete that side, 6. He seemed to be continuing to look for patterns. His pentagon at this point is shown in Figure-Rick. 3.

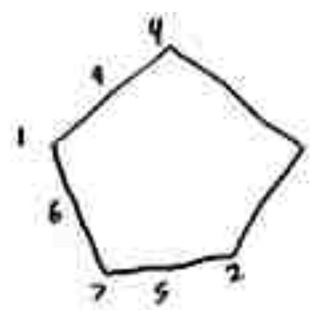

Figure-Rick. 3: Pentagon partially completed for sums of 14 in Question 1

Rick realized that 10 would have to be placed on the pentagon somewhere, and recalled that it would have to be in combination with 1 and 3. Therefore, he started over on a previously drawn, but until now blank, pentagon. He first placed 1, 10, and 3 along one side, and then continued his attempts at finding the pentagon, but again the placement of his numbers seemed very random.

Rick had been erasing often, and now asked if others had done the proof without erasing. The researcher suggested that he could avoid erasing by redrawing the pentagon when necessary. The researcher added that it might be a good idea for Rick to not erase so that he could remember what he had tried. Rick drew a new pentagon and prepared to start again. He said that, based on his previous attempts, he thought he should put 7, 8, and 9 in the middle of the sides because they were causing problems on vertices. He wrote the numbers 6 through 10 on the pentagon, placing them on the middle of the edges of each side and in numerical order. He placed 1 and 3 with 10, and by chance did so in the correct places. He proceeded to fill in the rest of the pentagon, making a small arithmetic error at the very last placement, which was corrected by the researcher. He then had a complete pentagon with sums of 14.

When asked if this proved that the smallest possible sum would be 14 and if he was done with the question, Rick again seemed to be distracted by the word prove. He stressed that the question had said to prove and so he was not finished. The researcher recapped that Rick had proved that 14 was possible, and asked if there was anything left to be done. "Here comes the 305 now, the class I'm not doing so well in", he said (Transcript 12, line 337). He was referring to having to prove the statement versus just showing the pentagon for 14. As he began to search for what he thought may give a proof, he seemed to discount the pentagon he constructed as not having proved any portion of the question at all.

Rick then looked over his pentagon and started to note the differences between successive terms (e.g. 5 and 7 were on the pentagon and he placed the number 2 above them). When prompted, he said that he was looking for patterns. Rick looked at forming an equation for the sum of each side, writing that $a + b + c = x$. He noted that the smallest sum of three numbers was 6, using 1, 2, and 3. He then said the largest sum was 27, using 10, 9, and 8, establishing a range for x.

"I'm trying to think of any other mathematical tools in my very small head, my very small library of mathematical knowledge", he said (Transcript 12, lines 361-362). Here, his lack of confidence had surfaced as he joked about his difficulties in this area. He suggested that he might graph the equation, but did not know how he could do that. Rick again noted that he would not finish the question within the allowed time, continuing to have a very poor image of his proof-writing ability, and at that point appeared to be giving up on being able to find a proof. The researcher reassured Rick that he had not thought himself able to find a pentagon with sum 14, but had indeed found one by continuing his work and so he should not give up now, either. Rick said that he did not know where to start, mentioning again the idea of graphing. When prompted, he reiterated that he had found a range for *x*, the values 6 to 27. The researcher reminded him of what the question asked to prove. Rick now decided that he would be able to handle the proof, and continued forward. He said that he could take the long approach and show that the numbers below 14 in his range would not work, starting with 13. He felt that this could be a good idea because he might see a pattern. The researcher encouraged him to pursue that idea.

Rick next decided to assume that 13 was the smallest possible value. He posed the question to himself of why 14 could then work, but decided that it was not the right way to approach the proof. He went back to the idea that the total sum was 80, and that somehow he had subtracted 10 to get a total sum of 70. When asked to relate this to his work, he incorrectly labeled the pentagon with sums of 14 as having a total sum of 80 as well. The researcher asked him to check the total sum for his pentagon. He first indicated that he would have the sum of the numbers 1 through 10, since addition was commutative. Instead of adding up the numbers directly off of the pentagon, Rick decided to write them out in an equation by grouping them in their combinations. It was then that he finally realized that some numbers were used twice in the total sum. He thought that perhaps every other number was used twice, not yet connecting this with the picture and those numbers being vertices. After some further thought, he noted that there were five groups, each having a sum of 14, with every other number used twice.

Making note of the time, Rick said that it could be another hour before he was done, continuing to show his lack of confidence. He also said that he was slow, but did not mind doing the proof and being in the interview. The researcher directed him back to his work, recapping that he had five groups and each added to 14. Rick thought the proof might use the fact that some numbers were used twice. However, he abandoned this idea and noted that there were 15 numbers total, which was only one away from the desired sum. Clearly, his search for patterns was not grounded in what he knew about the question and was leading him to look at totally unrelated occurrences.

The researcher asked him to go further with finding the total sum. He did and noticed that it was 70, not the 80 he had thought it would be. Rick redrew the example pentagon and repeated the equation for it, finding again that the sum was 80. He stated that he was still looking for patterns. The two equations for the different pentagons are shown in Figure-Rick. 4.

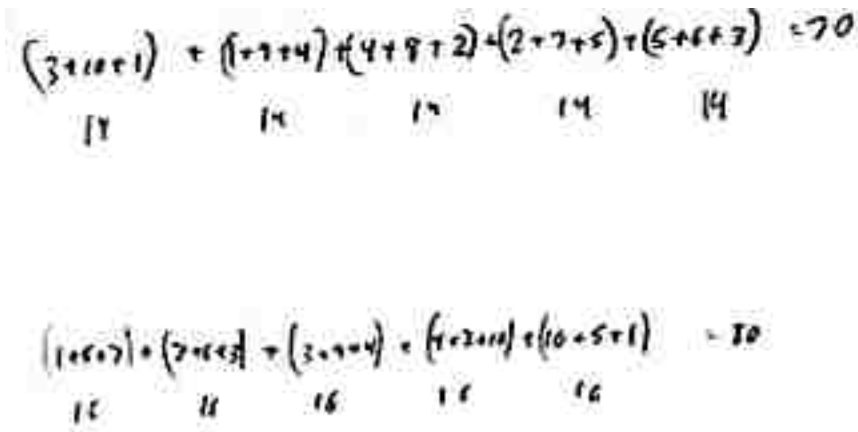

Figure-Rick. 4: Equations for overall sums in Question 1

Rick was distracted momentarily by the idea that 90 may be the highest possible sum, but the researcher asked him to continue the original question for now. Rick again stated that he was looking for patterns, and said that he would like to try to find a pentagon with sums of 15 next. When the researcher suggested that this might take a bit of time, Rick decided against that direction, even though he was told that he could pursue it if he wanted to.

After Rick became more distracted with discussions of coming back for a second interview, the researcher prompted him to look back at the question and at why sums of 13, 12, and 11 would not work. Rick surprisingly found that to be a hint, apparently not remembering that he had already started to work on these sums, only recalling this after being reminded. He also did not remember that he had said his minimum possible value would be 6 and needed to remind himself of the justification of that statement. After recalling his previous reasoning, Rick said that he would go back and consider 13, trying to prove that it was not possible. However, he hit a roadblock in how to start the proof that 13 was impossible. He attempted to find a pentagon with sums of 13 using the method of listing the combinations. After writing and then crossing out the combination 1, 2, and 10, Rick started over with a new list. The combinations he had written to this point were organized by lowest number, one sum for each low number, but did not include a combination involving 10. He stopped to say that it was frustrating because he knew that it would not be possible to find the pentagon. The researcher asked him to further pursue whether his list of combinations would lead to the result that the pentagon was impossible to find.

Rick started another new list, going back to combinations with sums of 14, trying to look for patterns. He observed that the lowest five numbers were used twice, i.e., they were on the vertices. He recapped the rules for placing numbers; and then said that to get the smallest possible value, the smallest numbers should be those used twice. Rick had just discovered a key component to the proof. However, he did not know how to write this idea formally, nor did he seem to recognize the significance of it for the proof. The researcher told him that verbal arguments were sufficient. At that point, Rick restarted his attempt at 13. He crossed off his previous list of combinations with sums of 13 and began a new list. In these combinations, he again organized by the lowest number in the combination, but instead of only one combination for each low number, he wrote two. The updated lists, as well as the previous lists, are shown in Figure-Rick. 5.

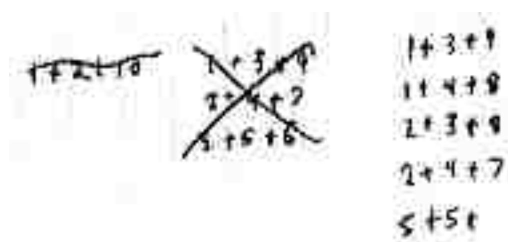

Figure-Rick. 5: Lists of combinations summing to 13 in Question 1

As he wrote the last entry, Rick said that he could not list a combination beginning with 3 or 4, because they were both already used twice. Therefore, the next possible first term would be 5, but he could not complete this combination because it would require one of the lower numbers as well. He realized that he had not yet used 10 in any combination, and then concluded that 13 was not possible. The researcher asked him to go back over the fact that 10 was not yet used. He said that 10 had to go with 1 and 2. The researcher asked him to start the process of looking at combinations again, with the thought in mind that the combination 1, 2, and 10 must be used. Rick said that he would go backwards now, starting with 10 then moving to 9, instead of looking at the lower numbers first.

When Rick started to become frustrated with this method, the researcher encouraged him to continue his search as he had been. He started his list again. However, this time he listed the numbers 10 down to 6 in rows, saying that he had established that each could only be used once, through this had only been an idea and had not actually been proven in any way. He wrote the combinations 10, 1, 2, and also 9, 1, 3. When considering the combination for 8, he noted that 1 was already used twice, so he would need to use the combination with 2 and 3. After this, he said that he could no longer use 1, 2, or 3; so the remaining combinations had to be made with only 4 and 5, which would not be possible. He recapped his work, saying that he had shown that 14 was possible, and that 13 was not, but did not know how to put that in words.

In this last attempt, Rick had been able to verbally express a correct proof that 13 was not possible. He concluded that it would never be possible because his choices were forced and other than the order in which the numbers were written within the combinations, there were no other options. However, a great deal of encouragement and hints had been necessary to reach this point. Next, the researcher asked him if a sum of 12 would work. Rick said that he could use the same process for 12 and all the numbers below it to show that they were not possible, either. Rick expressed that he felt rushed because he had wanted to be able to start another question, so the researcher told him that they would only finish this question during the time that day. Rick also expressed frustration that he had not yet even finished one proof.

When asked to describe the proof for 12, Rick followed his process of looking at the larger numbers first and began with 10. Unfortunately, he made an error in using a number twice on the pentagon, stating that 10 could be in a combination with one and one. That error, he said later, stemmed from the correct assumption that a number could be used twice in his list, implying that it would be a vertex. However, he had forgotten that a number could not be used twice within the same combination, since that would mean it would occur twice on the pentagon.

After also saying that 9 would have to go with 1 and 2, Rick placed the two combinations thus far in a new pentagon. Even after being asked twice if the combination 10, 1, 1 was legitimate and if it followed the rules established in the question, Rick did not correct his error. The error propagated itself during the search for a proof. He filled out more of the pentagon, using both 1 and 2 twice (see Figure-Rick. 6).

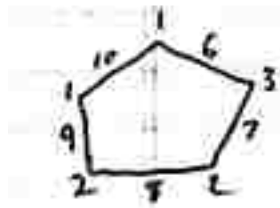

Figure-Rick. 6: Pentagon for sums of 12, with errors, in Question 1

Rick stated that the pentagon had worked for every side except the side with 1, 6, and 3, because that sum was only 10. The researcher asked where 4 and 5 were. Rick finally realized his error, saying that he could only use numbers twice if they were placed correctly, i.e., in different rows, signifying that they would be vertices on the pentagon. He had become so lost in the process and rules that he created, that he had forgotten about the overall question and what the picture should actually include. Rick was further able to conclude that no sum less than 13 would be possible. When asked, he said that there was nothing left to prove for this question.

The interview concluded with a discussion of strategies that Rick had used. He recapped his work, saying that he had looked for patterns, and repeated that he had used the "blood, sweat, and tears" approach (Transcript 12, line 880). He said that switching to a new piece of paper had helped him. In trying to recap what he had done, Rick got lost in his work a bit. He also mentioned again the use of the video camera. Rick summarized his work in the order he had originally gone through it, adding that verifying that the statement was true and understanding the question were his first steps. He agreed that once he understood the question, he had then looked for examples and patterns.

Rick had been unable to redirect his work throughout this question. He used several potentially good strategies such as reading and understanding the question, looking at examples, and looking for patterns; but he did not recognize unproductive lines of thought. His main source of difficulty was his continual desire to use the total value of the sum. This could have been a productive idea, but Rick had not been able to understand how that sum could be reduced. He was only able to proceed with the proof once he understood the calculation of this total sum and the idea that using lower numbers twice would result in a lower sum. He also struggled with an inability to form the wording for a valid proof, and the apparent lack of confidence in any proof-writing ability he may have had.

Rick required a great deal of prompting to continue his proof and redirect his work. He developed an overall plan only with prompting but did not keep that plan in mind as he worked. He did not monitor his attempts well, nor was he able to recall his reasoning throughout the interview. His continual search for patterns often confused him further and led him away from the actual proof.

Several main ideas came to light during his work, including that the larger numbers needed to be on the edges instead of the vertices, which allowed him to complete the proof. Again, however, it should be noted that he did not do so on his own. Overall, Rick was unable to self-monitor. He completed the proof due, in large part, to the attempts from the researcher to redirect his work and keep him focused on the question.

Question 2. The second interview began with the researcher reading Question 2 aloud, after which Rick read the definition aloud again himself. Rick gave an example of a number and what it would be in reverse order, namely 12 and 21. The researcher clarified that to be a 4-flip, 12 times 4 would need to equal 21. Rick followed with another example of a number and its flip, but stated, "Well, I don't know what I'm doing" (Transcript 20, lines 13-14). He paused before restating the example of $N = 12$. He said that when multiplied by 2, the result would be 24, which gave the correct number of digits, but not in reverse order. The researcher corrected Rick's error of multiplying by 2, saying that a number would be two-digits, but if it was a 4-flip it would be multiplied by 4 to get the reverse order. Rick's confusion only increased as he looked at another example, 10. He stated that he would let $N = 2$ and so $4N = 8$. He was now not even looking at a two-digit number, and it was not clear where he had connected 10 with 2 in his thoughts. He stated that he was confused. When the researcher asked if he had any questions that she could answer, Rick restated what needed to be shown, and then said that he would try a three-digit number instead. He further said that he was still trying to understand the definition.

Since Rick knew there were no two-digit 4-flips, he had decided to look at three-digit numbers to hopefully find an example of a 4-flip. The example he considered first for three-digit numbers was $N = 123$, finding that $4N = 492$, which was not the reverse of N. He again said that he was trying to find an example of a 4-flip, but that he may just spend the entire hour trying to find such a number. As in the first interview, here his uncertainties in his abilities surface. In his next example, $N = 100$, Rick correctly stated the result of $4N$ as well as what would have been required for N to be a 4-flip. After another example, $N = 222$, Rick restated what he was trying to show and added that he still felt there would be a three-digit 4-flip, and that he wanted to find such a number to show that the definition could actually hold for some number. But instead, he again restated that he needed to show that no two-digit 4-flips existed, and also indicated that he may have been headed towards showing that no three-digit 4-flips existed.

The researcher encouraged Rick to continue, asking how he could prove the original statement. After over 30 seconds, Rick said that he felt 5 would be the lowest possible number to check, though he did not justify this statement. He realized quickly that one-digit numbers could never be 4-flips. Rick began to look at the two-digit numbers at that point, eliminating those numbers with repeated digits, i.e., 22, 33, 44, etc., saying he was considering these particular numbers because the digits were the same forward and backward, but then further explained why they would not be 4-flips. He looked specifically at $N = 11$ when asked to explain what he was thinking, but he saw that $4N$ obviously had different digits and so gave up the idea. He said again that he liked concrete examples, but that he thought the question may take more abstract thinking.

When asked to continue, Rick wrote down the pairs 15, 51 and 24, 42, saying, “I’m obviously just getting examples of reverse order. [pause (25 sec)] It’s hard to imagine that there are anything, that it works for anything. I’m sure it does.” (Transcript 20, lines 120-122). He moved back to a larger number of digits, this time looking at $N = 12345$ with its reverse 54321. Rick was still in search of a 4-flip, however he did not seem to have a clear idea of how to find such a number, nor did his attempts look anything but random. He tried a few more ideas before saying that he had hit a roadblock. The researcher again encouraged him to continue.

Rick switched to a new piece of paper and began his search again. With this attempt, he first noted that the number must remain two-digits when multiplied by 4, representing this with two x marks to denote the two digits (see Figure-Rick. 7).

Figure-Rick. 7: Representation of flipped digits in Question 2 part a

Unlike previous attempts, he was now able to find upper and lower limits of 10 and 24.999. He became sidetracked when thinking further of the lower limit and made an error, writing $4x = 10$ and solving as if 10 was the result of *4N*, finding $N = 2.5$. The decimal seemed to confuse Rick, and he said that he had lost his train of thought. The researcher reminded Rick that he had last been discussing that 4 times 25 was equal to 100. Rick then repeated that the biggest number he could use for *N* would be 24.999. When he stopped to consider whether the number had to be a whole number, he became confused again, and he said that he was not sure if he should restrict the numbers to only whole numbers. He decided that he would make that assumption, but apparently then abandoned his upper bound just discussed since it was not a whole number.

Rick worked on a calculator for a few moments, before stating that he thought 88 would be the largest value of *4N* he could think of that was a two-digit number, giving $N = 22$. At this point, he had lost track of the larger picture completely and was not able to see that his limit of 24.999 would translate to the whole number 24 being the largest possibility. He wrote down his limits of 10 and 22 in a list, and then wrote the values of *4N* beside these for each. In this list, Rick was able to continue past 22 to look at $N = 23$. He found that $4N = 92$, but did not note the increase in his previous so-called maximum. He termed this search the "exhaustive method" (Transcript 20, line 175), to "just plug and chug and chug" (Transcript 20, line 179).

Rick now said that he needed to determine his goal in using this method before he went any further. Ultimately, he planned to just try every two-digit number within his limits, he said, and he was still trying to find his upper limit. He then noted that 24 would give a two-digit result for *4N* but 25 would not, however he still did not clearly state that 24 would be his upper bound. The researcher asked Rick if he could keep going with these thoughts, to which Rick said he could, but that he wanted to think about the question and what he was trying to prove. Repeating the definition and what needed to be shown again, Rick reiterated once again that his lower limit was 10. He never revisited his list, though he had been on the right track and obviously understood the definition of a 4-flip, at least in terms of his calculations. Unfortunately, he was not able to see his overall goal or to see that his path was beginning to be fruitful. Rick said that the question was difficult because "you're multiplying a number *N* by 4. Yet, when you reverse the order of the digits, you're going to get a smaller number" (Transcript 20, lines 215-216). Rick still did not understand how to prove the question. Even though he had stated that he could exhaustively search the numbers between his upper and lower limits, it was not clear that he understood that this method would result in a valid proof. His thoughts were scattered and he was not able to pursue any one idea fully.

Rick decided to organize his work now, writing at the top of his page that he was eliminating two-digit numbers, then starting a second page, which he titled "Trying to find one" (Transcript 20, line 220). He said that he had divided his work into two things, and that originally he had started to try to find a 4-flip. Now, he was thinking about what that meant. He went back to his five-digit example, 12345, and also wrote down other examples *N* = 123, 1234, 444, and 144, in that order. He admitted that he was just guessing and checking, still hoping to find an example of a 4-flip, saying, "I'm trying to get – trying to find one that feels like I might be heading the right way. And I'm not really feeling it yet" (Transcript 20, lines 249-250).

The researcher offered Rick the opportunity to start a new question and Rick decided to do so. He struggled with the decision to move on, still desiring a valid proof for this question. He did finally decide to move on and was given Question 3.

On Question 2, Rick was unable to monitor his progress or keep track of his thoughts. He first read and reread the question, but did not fully understand the definition of a 4-flp. He tried several examples, through which it was clear that he had no overall plan nor did he truly comprehend the complete picture of what needed to be shown.

Throughout his work, Rick repeated that he wanted to find an example of a 4-flip in order to build from this to be able to show why no two-digit 4-flips would exist. However, his search was quite random and without a clear direction. It would have been pure chance and luck had he been able to stumble upon a 4-flip using his search methods.

Rick felt frustrated and confused several times during the interview. The researcher often encouraged him to continue working towards the proof, but Rick's work ran in circles and prevented him from finding a valid proof. One main idea surfaced during the work, which was the idea of upper and lower limits on the possible numbers and a brute force search through those that remained. However, Rick was unable to follow this idea through, even showing a lack of understanding and connection of the whole picture as he struggled to nail down the upper limit. It was also not clear that he would have been convinced of the validity of this method as a proof, even if he had been able to complete the work. Rick had other ideas of eliminating possible numbers, but again did not follow these ideas through to their completion. His ideas turned to more abstract thoughts and a desire for overall rules and ideas for the 4-flip numbers, but he was unable to connect these ideas with the concrete examples that he had attempted.

Overall, Rick's work on this question was unfocused. He made attempts to organize the work and make a plan, however he was unable to stay on track during any one idea long enough to discover how it could possibly aid in his proof. His strategies included reading and attempting to understand the question, looking at examples, and writing what was known and what needed to be shown. In the end, he was not able to use any of these strategies to actually further his proof.

Question 3. The researcher read the first portion of the question aloud, including the definition of a perfect cover, pausing to allow Rick to consider the definition. Rick drew an 8-by-8 chessboard, taking great care in doing so. While he drew the chessboard, he made the comment that the work on these questions was tiring, or that maybe he was just tired from lack of sleep, which could have influenced his performance during the interview. He looked back at the question, reading aloud that a domino would cover two squares. The researcher then read the remainder of the question.

Rick labeled his chessboard with the dimensions *m* and *n*, and drew a domino to the side, complete with dots. He mimicked drawing one domino lengthwise on his chessboard and stated that it would take 4 times 8, or 32 dominoes to cover his chessboard. This gave insight into how he envisioned covering the chessboard with four dominoes lengthwise in each row, though he never explicitly stated how he would form his covering. He commented that the question made perfect sense to him, seeming surprised, and added that the current work in MATH 305 would also "make perfect sense in my mind, but now I have to write it down, I have to prove it" (Transcript 20, lines 319-320). Clearly, now that the statement was understandable to him, he was having difficulty thinking of how to prove that it was indeed true.

When asked to do so, Rick went back to describe his thoughts, saying that since the dominoes would cover two adjacent squares, that would mean that m or n must be even. He found the notation for this, that the squares covered would be equal to $2k$, but was not sure what to do with this thought. A specific proof technique, proof by contradiction, came to mind, but he did not elaborate on how he would proceed with such a proof. He also stated that a number could only be even or odd, and that m and n must be natural numbers, however he did not seem to know where to go from there. The audiotape needed to be turned to the other side at that time, giving Rick a chance to break away from his thoughts. As he began to look back at this work, he stated that the question was a MATH 305 type of question, and shared that he was not doing well in that course. Even though he was struggling with the question, he said that it seemed simple, which showed his frustration at not being able to develop a proof.

Finally able to go further with one of his ideas, Rick now stated that the product of the dimensions, mn, must be equal to $2k$, where he also stated that a domino, d, was equal to $2k$. It was not clear whether he meant that the total amount of dominoes covered $2k$, which was the area of the chessboard, or if he was unknowingly using poor notation to represent that both quantities were even. Apparently, he was working under the assumption that there would be a perfect cover, but never made his assumptions or the direction of his proof clear. Moving on to describing m and n individually, Rick stated that $m = 2k$ and $n = 2m + 1$, incorrectly using m in the second equation. He continued his use of this notation, without seeing his error, and considered the product mn. Repeating that he wanted to use proof by contradiction, he now indicated that he would let one of the dimensions be odd and see if the product was still even. He stated, but did not prove, that an even number times an odd number would be even, and said that this would be "the case where it works" (Transcript 20, lines 358-359), appearing to see that this was not what was needed for the proof by contradiction.

Rick moved to the case where both dimensions were odd, saying that he wanted to show that the product would not be even. After calculating the generic product and showing that it would be odd, (see his work in Figure-Rick. 8), Rick had a difficult time explaining what he had shown.

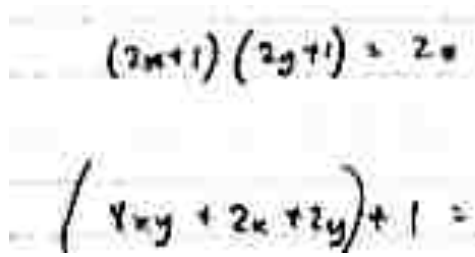

Figure-Rick. 8: Equations showing that odd times odd equals odd in Question 3

He knew that he had reached a contradiction, but was not sure how to express the words for the proof. He continued to struggle with the words, trying to compare the area of the domino to the area of the chessboard.

After being prompted several times by the researcher, and Rick saying that he did not know where to go or what he had been doing, he finally went back to the beginning of his work and explained what he had done step-by-step. Even in that process, he was unsure of the order in which he had originally gone through his steps and missed some reasoning along the way. He said that he had first drawn a picture and visualized dominoes on a chessboard, and then looked back at the question and determined that he needed to find a proof by contradiction. Assuming that both *m* and *n* were odd, he was able to find that their product would also be odd, and therefore dominoes could not cover the chessboard. He did not tie in his original thoughts that at least one of them being even would work, nor did he mention or address the bi-conditional nature of the statement to be proven. While he had formed the ideas for one direction of the proof, his reasoning was not clearly laid out, and he did not discuss the other direction of the proof at all.

The researcher ended Rick's work there, after which Rick said that he had only seen questions similar to this in MATH 305 dealing with even and odd integers. He agreed that he had used strategies such as drawing a picture and thinking about previous coursework to work on the proof. The interview ended as Rick talked about his struggles in MATH 305 and the personal issues he had at the beginning of the semester. For privacy reasons, the researcher did not record that portion of the interview.

Unlike other questions, Rick was able to find the major idea for part of this proof, and to verbally explain the reasoning in most of its detail. However, he struggled to achieve this, not following his work or reasoning well as he reviewed what he had done. This question seemed to be one that he understood better than the others, and as such, Rick had an idea of why the statement was true from the beginning. He did not need to look at other examples, but rather was able to gather information from just one drawing. He made several notational errors, but they did not seem to slow his progress further, nor did they become a problem at any point for Rick's understanding of the question. He again mentioned again one specific proof type, and after an error in what was to be assumed, was able to find the general idea for the proof of one direction of the original statement.

It should again be mentioned that Rick had to be prompted several times in this question, as well as throughout the interview, to continue past his roadblocks and frustrations in order to proceed with his work. However, unlike work done on other questions, Rick's work for Question 3 resulted in a partial proof, the validity of which he seemed to understand. He did not address the second direction of the proof, or indicate that he understood that there would be more of the proof to be completed, even though his picture in the beginning of his work would have shown the other direction sufficiently.

Summary. Overall, Rick tried to read the questions and carefully understand the definition given and what needed to be shown. He attempted to form a plan for the proof of each question, but often became sidetracked or would lose his way during the attempt at one particular type of proof or idea. He became frustrated and unable to move past major stumbling blocks many times during the two interviews, but the researcher was able to persuade him to move on in most cases. Rick had many good strategies, but his inability to redirect his work or to follow through on his plans caused great difficulties for him in every question.

Rick was often found in the student study lounge throughout the semester in which this study was conducted. The researcher had numerous opportunities to work with him and to observe him working with others. Rick's difficulties during this study were an accurate reflection of the difficulties he faced with proof writing in general. He tended to get lost in the details of a proof and not be able to find his own way out. Rick also had trouble focusing on any one particular aspect of a question, or proof, and so was often unable to complete a proof without input from others. He typed all of his homework so that he could make revisions and updates as he thought and worked more on the questions, which he spent a great deal of time doing. Working hard was certainly not one of his weaknesses, and in fact, he could be found several hours every weekday working in the student lounge on MATH 305. Rick definitely paid close attention to details, attempted to mimic proofs done in class, and was able to construct proofs regularly after a good deal of effort and attempts had been made. The lack of time to revisit questions or to work through this process most likely contributed a great deal to the difficulties that Rick experienced during these interviews.

Amy

Amy was a student in MATH 406. She was a senior majoring in French. Her previous coursework included: calculus I through III; discrete optimization; and applied, ordinary, and partial differential equations. During the interview, which was 54 minutes in length, Amy worked on Questions 1, 3, and 2 part a. Time did not permit any further work on Question 2. Several strategies and tendencies surfaced during the interview, including a large emphasis on pictures, imagery both in her head and drawn on paper, as well as the need to visualize the question in order to understand it. Furthermore, she had a preference to use physical materials to work through a question. Amy had a hard time keeping track of her work, which she seemed to realize since she highlighted the important portions of the statement of the questions. Later in the interview, Amy told the researcher that she was diagnosed with dyscalculia, which is a form of dyslexia specifically associated with numbers.

Question 1. Amy began by reading Question 1 aloud herself before the researcher even did so. Her highlighter was already out and she began to highlight parts of the question right away. She read the question again silently, and then clarified which numbers would be included in each sum. She moved from the understanding of the question straight to drawing a pentagon and trying to place numbers on it that added to 14. When asked, Amy said that she was trying to arrange the numbers in her head. She also said that "I'm figuring that it could be solved with a combinatorics proof but that isn't gonna work [laughter] for me. So, I'd just rather try to solve it in a spatial form as much as possible" (Transcript 13, lines 23-25). She wrote the numbers 1 through 10 and found the sum of those numbers to be 55. However, she never used this result again in her work. Amy said that she had the idea of adding the numbers to see if it could further her proof, but that she really did not know what to do next. At that point, there had been quite a bit of distracting noise from chair movement in the classroom above the interview room. This noise continued throughout the first portion of the interview.

The researcher asked Amy to pursue the idea of placing numbers on the pentagon. Amy said that even if she could find the arrangement for 14, it would not prove the statement. "I mean to me a proof would be, you know, something that's numerical in some form…" (Transcript 13, lines 48-49). She said that she was looking at the numbers and "just hoping something would come out" (Transcript 13, line 56). Amy wrote the numbers 1 through 10 again, and began to place them on the pentagon. Even though she had been asked not to erase in the pre-interview introduction, she erased most of her work as she went. Unfortunately, her hair fell in the way of the video, so the individual numbers she tried were not able to be seen. She scratched out numbers and replaced them multiple times, working for over two minutes to try to find the pentagon. She noted along the way, "part of the problem is I can't remember exactly what I've written" (Transcript 13, lines 67-68). She made several arithmetic errors, which she corrected herself, but that seemed to really distract her from her main purpose.

Amy continued her attempt at the pentagon, representing another nearly three minutes of silent work. The researcher allowed her to work quietly as long as she was writing down the numbers she was trying. Throughout the interview, anytime Amy paused for more than a brief moment, the researcher prompted her to explain what she was thinking and doing.

Amy then said that she felt there had to be an easier way to find the proof. As mentioned previously, she had made several arithmetic errors during her attempts, and now commented that this had become a real problem for her. She continued her attempts for another 3 minutes. Amy made lists of the remaining numbers as she worked, attempting to keep track of what was left to be used. However, she did not seem to erase from this list as she changed numbers on the pentagon, so it was not always a current representation of the numbers in use. She commented that she wished she could use negative numbers, and then returned to trying to find the pentagon for an additional nearly 3 minutes. The researcher asked if she could think of another method to fill in the pentagon, but Amy could not. Seeing that Amy was not able to proceed further with her work on this question, the researcher asked if she would like to move on. Amy opted to end her work on this question, rather than continue guessing and checking. There was no evident pattern in her guessing, just what looked to be a random search through possible combinations.

Amy worked a good portion of the time on this question in her head, only expressing her reasoning when prompted to do so. At the times when she did say what she had been working through, it was clear that there was a lot of work she was not writing down or verbalizing. Her entire work on Question 1 was spent trying to find the pentagon with sum 14, which she did not succeed in finding.

The strategies Amy utilized in this question included reading and understanding the question, drawing pictures, and looking at an example. She did not have an overall plan for the proof, and noted that her attempts to find the correct pentagon did not represent even a portion of the proof to her; so, there was no clear evidence of whether or not she knew what would constitute a valid proof. Amy was unable to monitor her attempts and, as a result, was unable to find the pentagon with sums of 14. While she did stop her work several times to comment and answer questions, she never attempted to find a different direction for the proof. She tried to make lists of remaining numbers as she worked, but forgot to keep them updated with new attempts.

Amy was obviously aware of her inability to remember and monitor her attempts, commenting on it as was mentioned previously. She later explained why this monitoring was so difficult for her, saying that she was dyscalculaic, though the researcher had been unaware of this prior to the end of the interview. Her attempts were, therefore, highly inefficient and she was unable to complete a proof.

Question 3. After the researcher read Question 3 aloud, Amy reread it to herself. She immediately began to draw an 8-by-8 chessboard and also a single domino off to the side, complete with dots. When asked what she was picturing in her head, Amy said that she knew what a domino looked like. She then noted that her example chessboard had 64 squares. She also said that she remembered working on a question similar to this previously; however, she was unable to recall the details of the proof. Later, she stated that thinking of her previous work had not helped her, because she could not remember her work exactly and she had trouble remembering how the question had been different.

With her fingers, Amy outlined where the dominoes would be placed on her example chessboard. She seemed to be relying heavily on the mental pictures she had of both the domino and the chessboard. Amy then drew in the dominoes on her chessboard, finding a perfect cover for it. She commented that 64 was an even number of squares, and that it could be divided by another even number. She also drew dominoes in a different direction on the chessboard, representing another cover.

Amy moved to the generic case, saying that she would assume m was even and n was odd. She stated that in this case, with the arrangement she had drawn out, the last row would "be off the board" (Transcript 13, line 226). She also stated that the same would be true in the case of n being even and m being odd; she indicated with her highlighter being halfway off of the table, that the dominoes would be hanging halfway off of the chessboard. She seemed to be saying that that would be a perfect cover of the chessboard, but the researcher corrected her by saying that it would not be a perfect cover since it was too big for the chessboard. Amy understood and corrected her mistake saying that the last row would actually have to turn the other way and lay horizontally. Again, she indicated turning the dominoes by physically holding up her highlighter and turning it on its side. She indicated that her pattern would work for any dimension, as long as at least one side was even.

When asked to describe verbally how she would write her thoughts up for a proof, Amy said that she felt she had already done that. This was an indication that she felt her proof was already complete and did not need any other details explained. The researcher recapped Amy's work briefly and then reaffirmed that she had seen a similar question previously. Amy commented that this had been easier than the other proof, because in this question she was able to "actually pick things up and move" (Transcript 13, lines 279-280), referring to her highlighter being used to show a domino that she could physically touch. She also said that it helped her to know what a chessboard looked like in her mind, and she was picturing the different colors of the chessboard as well, which also aided in keeping track of where the dominoes should be placed. She felt there were no numbers involved in the proof, just the physical representation, which also helped her.

Again, Amy approached the question by reading and drawing pictures. She did not outwardly express an overall plan and she was unable to see what would be required for the proof. She did not take note of the bi-conditional nature of the statement, nor did she even look at the case of m and n both being odd. Her explanation began with justification using one example in the case of an even-by-even chessboard; however, she did not prove this case in general. While she did move on to more generic notation using m and n, Amy only considered the case of one of those dimensions being even. She briefly explained her pattern, after a correction for the researcher on her understanding of a perfect cover, but she barely noted that it would work for any size chessboard where exactly one dimension was even, not explaining why it would work.

Amy did not appear to consider the case of both dimensions being odd. She also did not mention that the statement was a bi-conditional statement. Given that she declared her proof finished without addressing the other direction of the proof, it was clear that she did not notice the other direction or the need to prove it. While she was able to monitor her progress in this question, Amy failed to recognize that her proof was incomplete.

<u>Question 2</u>. As soon as the researcher handed her the paper, Amy read the question to herself and showed that she clearly thought that it was going to be difficult. The researcher pointed out that this question was probably unlike others she had seen before, and Amy agreed. The researcher read the question, using some rewording to

make sure Amy understood the definition. It seemed that Amy understood the definition, after reading the question again. She said that she had thought through the question in her head, using an example that she called a palindrome (e.g. 11), but that she did not get the original number back. She also stated that she could prove the statement by listing all the two-digit numbers and checking them, but that she did not want to do that.

Amy said that she was unsure of how to explain her ideas in "formal math" (Transcript 13, line 354), and that she was also not sure why one would think there was a 4-flip. The researcher suggested that it might help to know that there were 4-flips, but not in the two-digit case. She also asked Amy to consider what other thoughts she would have for the proof, other than listing all the numbers and trying them. Amy said that she could start off by looking only at the multiples of 4, in order to reduce the workload. The researcher verified that Amy was thinking of the end result, *4N*, and looking back to the reverse of that number. She asked if Amy had any desire to begin listing the numbers. Amy said that she was still thinking.

Amy wrote that her options were the numbers 10 through 99, and looked at one example. She asked for a reminder of which number in her example, 51 or 15, she was supposed to multiply by 4. She had already started to lose track of the definition. She wrote out this example, then started to list the numbers 11 through 14, but then erased the list. When asked what she had been considering, she rewrote the list again. However, Amy was still confused about which number should be multiplied by 4. She had taken the flip of *N* and multiplied that by 4. This technique could have produced the same results if followed through all the time, but she was switching back and forth between ideas throughout her work and so could not proceed in either direction.

When the researcher attempted to correct this error, Amy said that she was very confused. A discussion followed concerning the definition and the numbers that should be multiplied by 4 (Transcript 13, lines 395-418). Amy tried some of her examples again, this time multiplying appropriately. Her work is shown in Figure-Amy. 1.

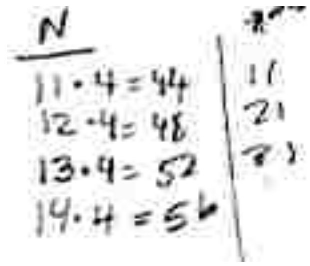

Figure-Amy. 1: Partial list of examples viewed in Question 2 part a

She then asked the researcher to read the question aloud again, putting down her own pencil and covering her eyes to concentrate. The researcher read the question aloud. Amy asked for a further explanation. She then verified her understanding with an example.

At that point, Amy seemed to head in a little better direction as she wrote down all two-digit multiples of four see Figure-Amy. 2.

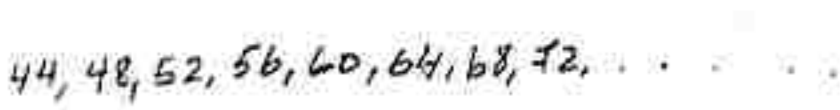

Figure-Amy. 2: List of two-digit multiples of four starting at 44 in Question 2 part a

She made a separate list of what appeared to be values of *N* flipped, starting with 11 through 19, but then switching to multiples of 10, see Figure-Amy. 3.

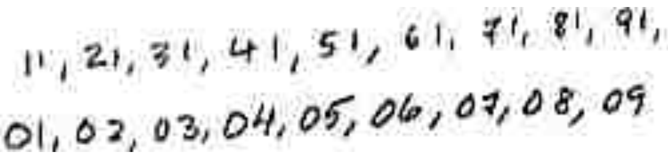

Figure-Amy. 3: Partial list of values of *N* flipped in Question 2 part a

She was also trying to remember a rule for what multiples of 4 would look like. As she continued her second list with flips of the 20s, the researcher verified that the numbers she was writing were values of *N* flipped.

Amy stopped her list, put her head in her hands, and sighed. She said that she had a thought about what to do but then lost it. "Well, sometimes it's really hard because if I start off doing something, I lose it… 10 minutes into it…" (Transcript 13, lines 476-477). As the researcher recapped her work, Amy was able to go back to the idea of eliminating some of the possibilities of the flips. The 10s flipped resulted in odd numbers, i.e., ending in 1, and so they could not be multiples of 4, she said. Furthermore, she said,

> I'm thinking that it couldn't be anything below a certain point, because if you flip it, the multiple, when you multiply by 4 then it's got, the flip would be less and you can't have that. (Transcript 13, lines 513-515)

She gave the example that 11 was less than 44. She also said that beyond a certain point, *4N* would become a three-digit number, and guessed that point to be at $N = 50$. The researcher summarized more of her work. While Amy did not further pursue finding these cutoff values, she maintained that once found, these values and her other rules would eliminate almost every number.

Amy again had reread this question for understanding as she first began her work. Next, she employed another of her common strategies by visualizing a number being a 4-flip in her mind. Unlike other questions, she was able to determine a plan that would have resulted in a proof. However, she was not willing to check all possible numbers and pursue this path. Instead, she decided to attempt reducing her workload by eliminating some possibilities. This also could have resulted in a proof, had there been enough time for her to fully follow through with the plan.

Summary. Amy's work was organized and she carefully kept track of her trials. She also looked at examples throughout the interview and stopped to ensure her understanding of the question. Though she was unable to complete the proof, and several times misunderstood the definition, the potential cause of these issues is explained in the following final portion of the interview.

The researcher asked Amy to go over some strategies that she had used and whether it was common for her to use them while working through a proof. Amy said that she often thought back to previous work as a strategy. She also said,

> The things that I've generally used, use in math or physics is that if I can draw it and I can figure it out using tactile or visual methods, I'll do it or auditory. Because I don't trust myself in numbers. (Transcript 13, lines 605-607)

It was at that point that Amy mentioned she was dyscalculaic and said that she had really struggled with flipping the numbers in Question 2, since it actually required her to flip around numbers on purpose. She said it was particularly difficult for her to do that question because she may just flip a number on her own, not necessarily on purpose.

> It's very difficult like with this flip, I can switch em [laughter]. And so if, if it's numerical then it'll be much, much, much harder…. But, if it's something that I can easily visualize then I have an additional way of solving a problem. I can get around the numbers. (Transcript 13, lines 615-621)

The difficulty she had with the first question would also make sense, as well as trying to keep track of her work, since she was not able to clearly remember exactly which numbers she had used and which order she had put them in. While Amy may not have been successful at finding proofs of these particular questions, it may not be a good indicator of her ability to reason and to write other mathematical proofs.

Vicki

Vicki was a student in MATH 406. She was a senior majoring in mathematics. Her previous coursework included calculus I and II, linear algebra, statistics, and MATH 305. She was also concurrently enrolled in calculus III. During the interview, which was approximately 62 minutes in length, Vicki worked on Questions 3, 2 (parts a and b), 1 and 4. She was very confident in her solutions to each question and used productive strategies during her work. However, only Questions 2 and 3 were actually finished with a complete proof. Overall, Vicki was very organized in her work, kept track of her goals along the way, and seemed to monitor her progress well.

Question 3. The researcher began by reading the question aloud. In the next 4 minutes, Vicki developed her proof as described here, working silently throughout this time. She began by reading the question herself, and then drew an example chessboard of size 8-by-7. She made a note that the board had 56 squares, still without speaking. She proceeded with her proof, writing the following:

> If m is even and n is either even or odd, the number of squares in the chessboard is going to be even. Since the number of chessboard squares are even the dominoes can cover all of the squares of the chessboard. (Transcript 14, lines14-16)

When she had finished her proof, Vicki explained her reasoning aloud. She stated that with the assumption that at least one of the dimensions was even, the total number of squares on the chessboard would be even and so would be able to be covered with dominoes. She referred back to her example, tracing out where dominoes could be placed on the chessboard. The researcher asked if there was anything left to prove, and

Vicki responded that she felt she was finished with the proof. However, she did not mention or prove that if both dimensions were odd, there would not be a perfect cover for the chessboard.

When asked to reflect on her strategies for the question, Vicki said that she had first drawn a picture to understand what the question was asking. She commented that "I guess somewhat of the proof is what I've learned from, um, abstract math, to writing a proof" (lines 59-60). She also said that she had seen questions similar to this in abstract math, referring to MATH 305, that involved the consideration of even and odd numbers and their products.

Vicki's proof for this question was clearly incomplete. She did not prove her generic statement, that dominoes could cover an even number of squares. There was no evidence that she recognized the bi-conditional nature of the statement, nor did she address the reverse direction of the proof in any way. Her strategies were few, including only looking at an example before proceeding with the proof. She did draw a picture and relate it back to her reasoning, but only after completing her written work. There was no outward evidence of an overall plan, but her work was done silently, so the lack of such a plan is unclear.

Question 2. After the researcher read Question 2 aloud, Vicki read it again silently. She looked at an example, $N = 362$, finding that $4N = 1448$. The researcher asked her if she understood what was needed for her example to be a 4-flip. When Vicki said that she did not, the researcher explained that she would have wanted $4N = 263$. Vicki said that she understood this, and then noted that the question was asking her to prove that there were no two-digit 4-flips. She also said that her idea for the proof would be to take all possible two-digit numbers and check them. The researcher asked her to proceed with her proof.

After looking at another example, $N = 12$, Vicki noticed that $4N$ was too large for this example to be a 4-flip. She said, "any two-digit number that is multiplied by 4 is gonna be um larger than any number that I can put in the 10 spot" (Transcript 14, lines 126-127). She thought further about this, and concluded that the second digit would need to be 4 times as large as the first digit. However, as she looked at another example, $N =$

28, she also noticed that she might have carry-over from the multiplication of the 1s place, so her original idea had not been exactly accurate.

Vicki said "the next plan of attack" would be to consider the limitations of *4N* having to remain a two-digit number (Transcript 14, line 144). She struggled to describe what she was thinking, but ultimately said that any number larger than 2 in the first digit of *N* would result in a three-digit number for *4N*. She also concluded that the largest possible value for *N* would be 25, by the same reasoning. She tried to consider possible values for the second digit of *N*, but stopped and said, "Can I just say it doesn't work? [laughter]" (Transcript 14, line 167). She had become frustrated with the process.

Vicki summarized that the only possible values for *N* would be in the interval 10 through 25. She wrote out the list of the numbers 10 through 24, also writing *4N* for each of these below them, showing that none were 4-flips. She noted that her list would prove that there did not exist a two-digit 4-flip. However, she still desired another proof, saying, "well this just literally proves it that there isn't one. Um, but to do some more of a math, mathematical proof, I guess you could say…" (Transcript 14, lines 180-182). In searching for such a proof, she noted that the first digit could only be 1 or 2, so the second digit of *4N* would have to be 1 or 2 as well. In her list of *4N*, she pointed out that the only values satisfying this requirement were 52, 72, and 92. However, none of those values would work because the first digit was too large in each case. She again expressed a desire for a different way to show the proof, saying, "I guess I can't come up with a way of actually writing a real proof of what, of why we can't do that. Um, other than that I just took the 15…" (Transcript 14, lines 191-193).

When asked if there was anything left to do for the proof, Vicki stated "Um, the only thing would be if there was an actual proof to where I wouldn't have had to write out these 15 to figure it out, yeah, that'd be the only thing" (Transcript 14, lines 204-209). Again, she expressed that she did not feel what she had done was an *actual* or *real* proof, though she clearly had stated earlier that her work had proven the statement.

When describing her strategies, Vicki said that she had first looked at a random number for an example, to understand what the question was asking. She described her work, and then also said that she had tried a few more examples later to try to find a pattern. Seeing none, in the end she instead had just written out all the possibilities.

In Question 2 part b, after briefly reading the question, Vicki felt sure that there were no three-digit 4-flips. She incorrectly stated that the smallest three-digit number was 300. She explained that this already gave a four-digit result for *4N*, and so there could not be any three-digit 4-flips. The researcher questioned her lowest value of 300. Vicki saw her error and corrected it, saying that the smallest three-digit number was actually 100. She then limited the possibilities to the interval 100 through 250, linking this to part a of the question. She looked at one random example in this interval, $N = 125$. Not finding any hints from this example, she said,

> In the previous one I didn't find a proof… But, we're dealing with more numbers between 100 and 250, so the 150 numbers, I'm not gonna write 150, so there's a way of proving it without using, you know, writing out all 150 of em [*sic*]. (Transcript 14, lines 249-252).

She again wavered between identifying her previous work as a valid proof or not, but either way it was not what she was looking for. Clearly, she desired a proof that would have some sort of logically reasoning, other than just proving by brute force.

Vicki began to examine the individual digits of *N*. First, she observed that the last digit could not be less than 4, because *4N* would have to be at least 400. She took note of this observation on her work, writing her three-digit number as the digits *abc* and that $c > 4$. However, she had made an error at this point, since she could not yet truly rule out $c = 4$. She then wrote out the numbers 0 through 9, and crossed off 0 through 4. She quickly saw her error and corrected herself, saying that *c* could be equal to 4. Continuing, Vicki also said that *c* could not be 5, since the result would end in 0 and so would not flip to be a three-digit number. Next, she found that *c* could not be 6 because *4N* would then end in 4, resulting in a flip that starting with 4, which was outside of the range for *N*. With the same reasoning, she eliminated 7, but further eliminated 8 and 9 without checking their products. When asked to check these again, Vicki realized that 8 was actually a possibility, but not 9 or 4. She summarized that a three-digit 4-flip must satisfy the multiplication shown in Figure-Vicki. 1.

Figure-Vicki. 1: Visualization to solve for missing middle digit in Question 2 part c

Vicki took care to note that the carry-over from the multiplication of 8 times 4 was 3, shown above the empty spot for *N*. She checked the numbers *N = 208, 218, 228*, seeing that none were 4-flips and also noting that *N = 228* was actually too large since *4N* would then begin with 9 instead of the desired 8. She recapped her work and concluded that there were no three-digit 4-flips. When asked to describe her strategies, she went back through her work, forgetting her reasoning once, but otherwise able to recall her justifications (Transcript 14, lines 314-341). She stated that she had not seen anything like this question before, but had done others by trial and error in the past. The researcher skipped part c of the question, in the interest of time and believing that working on a different question instead could reveal more strategies.

In this question, Vicki was able to construct a valid proof for both parts a and b, with some minor error corrections along the way. She again began her work by looking at an example to understand the question. Unlike Question 3, however, she then made an overall plan to check all two-digit numbers. Although this would have been a lengthy process without her further observations, in using this plan she was able to see the key elements for limiting the possibilities. After a few more examples, she was able to complete a proof by brute force. As noted previously, Vicki seemed dissatisfied with such a proof. At points she called this proof valid, but she also mentioned that she felt it was not a "real proof" (Transcript 14, line 192). She indicated a desire to find a proof through argumentation that would exclude all possible numbers without actually having to check many numbers. Her strategies included understanding the question, looking at examples, looking at the choices for each digit, searching for patterns, making a plan, monitoring her work and keeping track of her eliminated choices. However, she was unsure that brute force was a legitimate proof technique.

In part b of the question, Vicki first made an error, which was corrected by the researcher. She was able to move past this error, though, and connected her work from the previous part of the question to this new proof. She was able to move further in her reasoning, building off of ideas developed in part a. She again was able to move systematically through her choices, keeping track of her work and monitoring her progress appropriately. New strategies that were used included using earlier results to guide the new proof, developing equations, visualizing the placeholders of the numbers, and recognizing her completed proof.

Question 1. Vicki reread Question 1 after the researcher read it aloud. Her next statement seemed to suggest that she believed that only two numbers made up each sum, though her real meaning was not clear and the work that followed did not match up to this statement. "… the first thing I would look at is that in order to use all 10 of the numbers, two of them have to be um, put together" (Transcript 14, lines 364-365). She began to look at sums of 13, correctly using three numbers, and was able to list the combination 10, 2, and 1 correctly. However, she then produced sums of 14 when looking at those including 9. The researcher asked if she had intended to move to sums of 14, but Vicki said that she had not. She corrected her error, saying that the two possible combinations involving 9 were: 9, 2, and 2; and 9, 1, and 3. She said that the first could not be possible since a number could not be used twice, and since the combination with 10 already used a 1, the second would also use a number twice. The researcher asked Vicki to look back at the example pentagon and confirm that the number 1 could not be used twice. Vicki realized that numbers could appear twice in her list and, after being asked, explained how that could occur. She said that she could use a combination for 10 and 9, both using the number 1.

Vicki proceeded to draw a pentagon. She noted that in the example pentagon, 10 was used in two combinations, but here she only had one option. When the researcher asked if there was any way to draw the pentagon so that 10 did not have to be used in two combinations, Vicki saw that 10 could be written on an edge instead of a vertex. She further explained that the number 1 had to be on a corner because it was used twice. Since 9 could also only have one combination, she placed it on an edge as well. She continued her list of sums for 13 by looking at those containing the number 8. She determined that 8 was contained in only one valid sum (the other used the number 1, which was already used twice). The remaining combination was 8, 3, and 2. Since 3 and 2 were both already placed on the pentagon on opposite vertices, the pentagon was not possible.

Seeing that 10 would have to be in a combination with 2 other numbers for any sum, Vicki concluded that sums less than 13 would also not be possible. Therefore, she concluded that the smallest possible sum was 14. When asked if there was anything left to be done for the proof, Vicki stated that she was finished. When asked to describe her strategies, she said that since the question had asked to prove that the smallest possible value was 14, she had proceeded by proving that nothing smaller would work. She commented that she might want to also prove that 14 was possible, saying that she could show this by repeating the process that she had used for 13 to find the correct pentagon with sums of 14. For the proof that 13 was not possible, Vicki said that she had started with the highest numbers and worked down. She compared the question to brainteasers she had seen in the past, but said that none were specifically the same or close to it.

Vicki formed a complete proof to this question. Her strategies included reading and understanding the question, making a plan, forming organized lists of possible combinations, drawing pictures, recognizing that the largest numbers needed to used on edges not on vertices, monitoring her progress, keeping track of her choices, and recognizing her completed proof. She did make several errors that were corrected with help from the researcher. Her success in this question seemed to again be linked to her ability to make an overall plan and monitor her progress, as well as clearly understanding what was needed to form a complete proof. The stumbling block for her might have been working too quickly through the proof, which caused some errors to occur.

Question 4. Since there was still more time in the interview, and seeing that Vicki had gone quickly through the other questions, the researcher asked her to try Question 4. The interview did end up running a bit longer than the others because of that, but was still completed in just over one hour. Question 4 posed difficulties for Vicki that she had not encountered in the other questions.

After the researcher read the question aloud, Vicki read it to herself silently. From the start, she assumed that the word divides should be thought of as divided by.

She had rewritten the definition of divides that was given as is pictured in Figure-Vicki. 2.

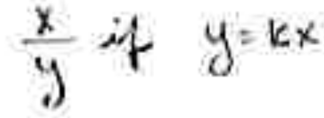

Figure-Vicki. 2: Rewritten definition of divides, with error, in Question 4

She then stated that she would consider *x* divided by *y* to be an integer *z*. Even though she had correctly rewritten the equation $y = kx$, she now instead switched to $\frac{x}{y} = z$. She simplified this to be $x = zy$. When asked if this was similar to the definition given, Vicki said that it was not. She noted that *z* was on the opposite side of the equation. Instead of seeing her error, she proceeded to let $k = 1/z$ and rewrote her equation as $kx = y$. When the researcher pointed out that *k* was supposed to be an integer, Vicki concluded that *z* would not be an integer but did not resolve the issue further at that time.

Vicki's next step was to choose specific values for *a*, *b*, and *c*. She chose the values $a = 1$, $b = 2$, and $c = 3$. This showed that she did not understand the definition after all, since *b* and *c* divide each other neither in the true sense of the definition nor in Vicki's interpretation. She computed the values of a/b, b/c, and a/c. Noticing that Vicki was rereading the question but struggling with something, the researcher offered to reread the question aloud and discuss what Vicki was focusing on. Vicki then went back to her equations for *x* and *y*. She concluded that *z* could not be an integer, but rather was the inverse of an integer, i.e., $z = 1/\,integer$. She struggled with this, checking that her division of fractions was correct with an example. She continued by showing that *k* was an integer, her work is shown in Figure-Vicki. 3.

Figure-Vicki. 3: Equations to show that constant was an integer in Question 4

Vicki said that she was now trying to relate what she had written with the *a, b*, and *c* in the question. The researcher asked her to go back to her example considering the knowledge that her variable *z* was the inverse of an integer. Vicki said that she could say that *a/b* was equal to *z,* because it was equal to *1/ integer*. However, the value of *b/c* was not the same. She concluded, incorrectly, that *b/c* could only be equal to *1/ integer* if *b = 1*. This would, in turn, mean that both *a* and *c* would also have to be equal to 1. While it was true that these values of *a*, *b*, and *c* were one valid example, they were certainly not the only choices. The researcher asked if she could think of a way to change *c* to make the value of *b/c = 1/ integer*. When Vicki said that *c* would have to be a fraction, the researcher repeated that she would want 2 divided by something to actually be 1 over something. It was then that Vicki understood and said that *c* could equal 4. In general, she said, *a = 1, b = 2*, and *c = 2b*.

Due to time constraints, Vicki's work had to end there. When asked to consider her strategies, Vicki said that first she looked at the equation given for the definition and had to decide what to do with it. She had "I used a – a different value, *z*, just to not get *k* and *z* confused to where we can figure out what that is" (Transcript 14, lines 627-628). She said that if she were to keep going, she would figure out other values for *a* and corresponding values for the other two variables. She ended by stating that she had not seen anything like this question before.

It is not clear that Vicki would have been able to complete the proof, even if she had been given more time. Her current line of work was not quickly producing any sort of valid proof, so there is no indication that she would have been able to move past this. Her strategies included reading the question, rewriting definitions, looking at equations, looking at an example, keeping track of her attempts, and using careful notation. However, in this question Vicki struggled to understand the definition, and never seemed to truly grasp it. She also did not make an overall plan, nor did she seem to have an idea for what would constitute a valid proof. She made several errors, was not able to keep track of her variables or the meaning of the question, and was not able to correct many of her errors during her work.

While Vicki had identified the other questions as being similar to others she had seen in the past, this question was completely new to her. This may have caused some of the confusion and the lack of confidence as compared to her work on the other questions. Using previous ideas and work seemed to be a guide for Vicki in her proof-writing attempts, and the inability to associate this question with anything else left her without a plan of where to start or what was needed for the proof.

Summary. Overall, Vicki was able to make goals, monitor her work, and provide proofs for two of the four questions she attempted. Her work was well organized, but she had required prompting at times to continue her thoughts and her work. In the first two questions Vicki attempted, Questions 2 and 3, Vicki was able to understand the questions and had a plan for what was needed for the proof. However, in Questions 1 and 4, she was unable to determine what was needed for a proof and made several errors that she was unable to recover from.

Shaun

Shaun was a MATH 305 student. He was a junior, double majoring in physics and mathematics. His previous coursework included calculus I through III, linear algebra, and ordinary and partial differential equations. He participated in two interviews. During the first interview, which was 50 minutes in length, Shaun worked on Question 3. His work was characterized by writing out statements for understanding and to organize his thoughts, as well as a desire to write statements in logical notation.

During the second interview, which was 48 minutes in length, Shaun worked on Question 2 part a, and Question 1. Again, Shaun wrote his work down completely, being very thorough about including everything he was thinking. He explained his thoughts and worked through the process both verbally and in writing. Shaun's work was very organized, but later in the interview he became confused in his arguments, remembering his overall goal, but not able to recall his previous steps or the sub-goals he was working through.

Question 3. The first interview began with the researcher reading Question 3 aloud. Shaun started his work by drawing a chessboard. However, he stopped and redrew the chessboard on lined paper. His desire for organized and neat work surfaced several times throughout the interview. He drew an 8-by-8 chessboard and a domino to the side, and then reread the question. After doing so, he began drawing dominoes on his example chessboard. He stated that he was "trying to just get kind of a visual of what's going on" (Transcript 15, lines 22-23). He again seemed to have a desire to be very organized, even getting a second pen of a different color out to draw dominoes onto his chessboard. He reread the question and continued filling in dominoes on his chessboard. Throughout both interviews, Shaun marked important portions of the question and rewrote them himself. In this question, he drew a box around the words *perfect cover* and wrote "Perfect Cover: No overlapping dominoes" (Transcript 15, lines 27-29). His placement of dominoes seemed random, and he was led astray momentarily as he noticed that the dominoes had finished placement in the opposite direction that he had begun to place them (i.e., vertical versus horizontal), but this indicated that he clearly had a pattern in mind. However, Shaun did note that this was not necessarily the only way to draw in the dominoes, and paused to consider how he might prove that a perfect cover would exist. He said that his example was an 8-by-8 board and that he needed to consider a generic *m*-by-*n* chessboard.

Shaun again read the question and wrote down what he wanted to prove, showing that he understood that the question was a bi-conditional statement. He wrote that he needed to prove the generic chessboard had a perfect cover "$\leftrightarrow \exists m,n\!: m=2l$ or $n=2p$" (Transcript 15, line 44). He paused and looked back at the question several times. When asked what he was thinking, Shaun stated that he had first drawn a chessboard to visualize the question. Then, he had noticed the key definition to be used, and rewrote what was needed "just to cement that in my head" (Transcript 15, line 61). He said he would proceed by looking at another example to prove to himself that the statement was correct, next considering a chessboard with one odd and one even dimension. Shaun drew out a new example, a chessboard of size 8-by-5, and found a covering of dominoes for that chessboard. While placing the dominoes, he made an error in his understanding of the word adjacent, thinking that it referred to the dominoes all facing the same direction. The researcher corrected the error and Shaun finished filling in his chessboard. He then said that this new example proved to him that the statement was true.

Shaun labeled his chessboard as an example and wrote a label for a non-example to the side, saying he would next look for a non-example, i.e., a chessboard that did not have a perfect cover. He decided to consider a 3-by-5 chessboard. Briefly, he thought that his choices for dimensions were incorrect, but was able to recover and proceed with the original choice. Drawing in dominoes on his chessboard, he noted that one square would be left uncovered. He again mentioned that the statement was a bi-conditional, saying, “it’s an if and only if, so I need to prove it forwards and backwards. Which is going to be – easier? – Well, let’s do it forward” (Transcript 15, lines 153-155). He had therefore shown that he understood the statement and how to proceed with a proof. After looking at examples and clarifing the question, he was now proceeding with a plan for the proof.

Shaun went on to say that he thought that proving the forward direction with a direct proof was not the way he wanted to continue; instead, he said that he was thinking of trying a proof by contradiction. Shaun carefully proceeded with finding the correct logical notation for a proof by contradiction. He labeled his rewritten statement of the question as the hypothesis (*P*) and the conclusion (*Q*). Starting by assuming *P* and not *Q*, he began to form the proper wording for the negation of the statement *Q*. However, he had difficulty with this portion of his proof. He verified that his original notation for *Q* was correct before proceeding, and then began to rewrite the statement *Q* as “$Q: \exists mn, m = 2l \vee$” (Transcript 15, line 213). The researcher noted that he might not need to include the *there exists* in *Q* as it was written. Shaun agreed and finished putting in the notation for *n* being even. However, he felt that he still needed to include wording for the variables *l* and *p*, which represented the integer coefficients in the notation for *m* and *n* being even. His resulting logical statement is shown in Figure-Shaun. 1.

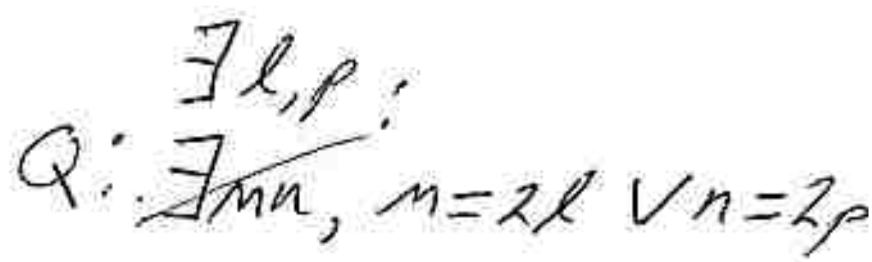

Figure-Shaun. 1: Logical notation for at least one dimension even in Question 3

Shaun was able to correctly find the notation he desired for *Q*, but made two errors when taking the negation. He incorrectly formed the negation of this statement as “$\neg Q: \forall l, p: m = 2l \wedge n = 2p$” (Transcript 15, line 246). One error was the change from *there exists* to *for all*, since Shaun was still dealing with the definitions of even and odd, this should have remained as *there exists*. The other error was in negating the *or* portion of the statement. The statement *Q* said that *m* was even or *n* was even. In his negation, Shaun wrote that *m* was even and *n* was even, not negating the individual portions, i.e., not changing even to odd. Apparently not yet aware of his errors, and even though he had not been sure of his notation of the statement *Q*, Shaun did not hesitate in his notation for the negation of *Q*. He went on to say that for the proof by contradiction, he now needed to show that the statements together would not work.

Shaun quickly recognized that something was wrong as he looked back over his statements, because a chessboard with both *m* and *n* being even would have a perfect cover. However, he did not see the actual error in his logical notation; assuming his negation was correct, he believed that a proof by contradiction was not the appropriate technique. The researcher prompted Shaun to state the negation in words and encouraged him to think outside of the symbols to clarify. To the request, Shaun responded by repeating his incorrect idea, but it did cause him to pause and reconsider. He broke this pattern by stopping to ask himself what the statement *Q* meant and when it would hold. He then proceeded to discuss with the researcher the difficulties of the use of *or* in mathematics versus in the English language, as was discussed in MATH 305. Shaun then returned to asking himself when the statement would not hold. He correctly determined that it did not hold when both *m* and *n* were odd.

Shaun decided that a proof by contradiction was not necessary and what he would really be proving was the contrapositive, i.e., not *Q* implies not *P*. He went back to his notation for the negation of *Q* and said that it was causing him problems. Looking over his writings about when the statement did not hold, he said that there were cases to consider. When asked, he agreed that the cases he was thinking of were those he had mentioned of the parities of *m* and *n*. He said then that

> I think sometimes when I'm doing proofs, uh, it's – you know after I've written a bunch of stuff and I don't see anything, I'll sit there and try and put it, put it all together in my head. – So, then I'm almost too scared to put anything else down until I feel like I've mastered what I've put down already. (Transcript 15, lines 374-381)

He also stated that he was "notation oriented when it comes to proof" (Transcript 15, lines 395-396) and does well on proofs requiring manipulation of symbols and lots of notation, such as proofs that functions are one-to-one and onto, for example. Shaun stated that this could also be his downfall at times when notation was not given, or was not readily apparent in the question. He said he had a harder time when the question was in words and trying to translate those words into notation. "Cause I flip things around a lot in my head. With some, if it was, if more symbols were there, I would have a little bit easier time maybe" (Transcript 15, lines 435-436).

The researcher offered Shaun the opportunity to move on to another question, but he stated that he was "kind of stubborn, too" (Transcript 15, line 445), and wanted to pursue this question further. He went back to his work and said that he had an example of a case where the statement held and when it did not hold. He then became sidetracked with a discussion of when the implication, *P* implies *Q,* would be false, and drew a truth table to recall the idea. His truth table was incorrect, however, and so the researcher engaged Shaun in a discussion meant to help him to fix his error (Transcript 15, lines 468-516). After the correction of the truth table, Shaun said that he had forgotten where he had been going with that idea. However, he was able to quickly retrace his thoughts and get back on track. He determined that what he had written was correct, the statement *Q* did not hold when both *m* and *n* were odd, and again went back to using notation to describe this case. However, he did not seem to know where to go from there. Shaun stated that he had felt fairly confident starting the proof by contradiction, but had moved to a proof by contrapositive. However, he was now considering what the converse could tell him about the actual proof. Thinking about this for a moment, he decided that it would not be fruitful for him to pursue this.

After being asked to describe where he wanted to go next, Shaun stated that he would like to be able to use the example and non-example to help find a proof, since his use of notation did not seem to be helpful. He also said that he was having troubles with the question because he was not able to link it to another course that he knew of in mathematics, as he normally could with many of the proofs in MATH 305. "It's like I don't have really, I can't relate this to any specific subject" (Transcript 15, line 603). He stated that he could see patterns in his examples, but was not sure what else to do.

The researcher encouraged Shaun to look back at his non-example and go into more detail about why there was no perfect cover in that case. Shaun immediately noticed the relationship between the chessboard having one uncovered square and both *m* and *n* having a remainder of 1 when divided by 2. He noted, "each of these has a remainder 1 and so does, so does this, has a remainder 1" (Transcript 15, line 630). He then stated that a domino was even because it covered two spaces on the chessboard. Again, as he started to discover something in the picture, he wanted to turn to notation and it led him away from an understanding of the question and how to proceed with the proof.

Shaun said that when he looked back at the picture of the non-example, he was reminded of his MATH 305 instructor working with cases, and looking at questions involving the division algorithm and remainders. He went back to his chessboard and counted seven whole dominoes and a reminder of one uncovered square. He said that he was trying to think about the chessboard in terms of units, and so there were 14 + 1 units. Considering the dimensions of the chessboard, and noting that *m* was 5 and *n* was 3, he then found the values of $l = 2$ and $p = 1$ in his notation, where $m = 2l + 1$ and $n = 2p + 1$. He incorrectly linked the total number of squares on the chessboard to these values, saying that $15 = 7l + p$. While this was a valid equation, it held only by coincidence in this particular example, which Shaun did not recognize. Seeming frustrated, he then said that he would like to move on to another question.

Seeing that there was not time for another question, the researcher instead asked Shaun to reflect back on this question. Shaun walked the researcher through his work, step-by-step, noting as he did so that he had never proven the second direction of the statement. He also noted that he had stressed notation and had written statements along the way to clarify and help guide his work, such as "when does the statement hold" (Transcript 15, lines 294-295). He was not always able to follow the order in which he had originally done his work. Shaun said,

> Sometimes it happens to me that I'm doing a problem and I get so into the specifics of the notation and stuff that I lose sight of the general picture. – I kind of think that's what happened here. As I got more and more confused in the ramblings in my head about what was going on, I lost sight of the original. (Transcript 15, lines 821-828)

He said that at this point, he usually would go back and reread the question to recall what needed to be done. He ended by saying that he had not seen anything like Question 3 in his past experiences.

Shaun was not able to complete a proof for this question, and chose to stop his work on it when he could not think of any further ideas. While he was able to recognize the statement as a bi-conditional and seemed to understand what would be needed for a proof, he was unable to complete his ideas for a full proof. His use of notation could have been helpful, but instead seemed to be a barrier for him as he was not able to link the pictures and main ideas of the proof with the specifics of the equations and logical notation that he had developed. He did have an overall plan and had many good ideas that could have been keys to the proof. However, he was unable to monitor his work and quickly became sidetracked with other ideas along the way. Shaun's strategies included reading and understanding the question, drawing pictures, organizing his work, visualizing the chessboard and dominoes, taking note of important parts of the question, looking for patterns, looking at examples, identifying proper proof techniques, developing equations and notation, proving the question to himself, unpacking definitions, redirecting his work, trying to recall similar questions, and writing what was known and what needed to be shown.

This is quite an extensive list, and Shaun worked through many different ideas and proof techniques during his search for a proof. His main difficulties arose from the inability to develop the proper logical statement for the negation of *Q*, and the inability to link his thoughts from the pictures and examples to more generic statements and proofs. While he did have an overall plan and maintained direction towards this goal, he was unable to make the switch to generic notation and ideas that were necessary to finding the proof. Shaun also seemed to feel that the validity of a proof depended on the ability to turn words into symbolic, logical notation to be manipulated. It was not clear that he understood that there could be other methods of proving the statement.

Question 2. The second interview began when the researcher read Question 2 aloud. Shaun wrote out the first few natural numbers to remind himself what they were, "just so I can see a visual" (Transcript 22, line 22), and repeated the definition of a 4-flip aloud. He then realized that the question was asking for positive integers, but said that 0 would not count anyway, so his natural numbers would work as well. He wrote down the definition of a 4-flip, taking almost 1½ minutes to make sure his written definition was correct, and then made a plan to look at examples. He identified this as his plan, even writing it down as such, and was clear about the organization of his work. He struggled to explain which numbers he would examine first, noting that one-digit numbers would not qualify as a 4-flip. He began to say that the question stated that there were no two-digit 4-flips and seemed to indicate that he would start searching for an example of three-digits, but abandoned this thought temporarily.

Shaun looked at the example $N = 13$. He stated that he made this choice for N because he wanted a number without a 2 in it, so that he would not confuse the individual digits with the fact that it was a two-digit number. He found that $4N = 52$, and stopped to look at this result. When asked if he needed any clarifications on the question, Shaun said that he had just gotten off track in his head and turned back to his example again. Shaun regained his thoughts, and it was at that moment that he recalled he had wanted to look for a three-digit number instead of a two-digit, since he knew by the statement of the question that no two-digit 4-flips existed. However, he decided to continue with his current example for the time being. He organized his work by asking questions to himself in writing, such as "Question: Is 13 a 4-flip number?" (Transcript 22, line 117). He then noted the two conditions needed for a 4-flip: that $4N$ would have the same number of digits as N, and that the digits would be reversed. The current example, Shaun said, satisfied the first condition but not the second. He felt he would need to see an example of a number that was actually a 4-flip to be able to find the proof that no two-digit 4-flips existed, still assuming that there would be a three-digit 4-flip.

Again organizing his work, Shaun labeled his first example as one in which the proof would be true. He switched to a new piece of paper and labeled the new space as an example in which the definition would be true. Shaun said he would try to hurry up his work, but the researcher reassured him that it was the process he was going through that was of interest so there was no need to rush. For the example in which the definition would be true, Shaun tried $N = 123$ and found $4N = 492$, after which he was sidetracked by unrelated thoughts. When he brought himself back to the question, Shaun decided to again detail the parts of the definition to determine if this number was a 4-flip, asking the questions: i) Does *4N* have the same amount of digits as *N*? and ii) Is *4N* the same as *N* in reverse? Shaun noted that in his example, *N* = 123, *4N* had the same number of digits as *N*, but he became confused during the process of considering whether *4N* was *N* in reverse order. He first flipped *4N*, but not recalling what he needed to find, he stopped this idea before comparing the result to *N*. Instead, he decided to go back and divide *4N* by 4 to find *N*. When he found that *N* =123, he thought that he had found an example where the definition had worked, not seeing the error in his reasoning.

When asked about his thought process, Shaun said that he had tried to find an example where the definition held, saying "After all, it is a definition so it should hold. But, I needed – I needed to see it" (Transcript 22, line 203). It was clear to the researcher that Shaun felt he had found a three-digit 4-flip in his last example. Shaun conjectured that he would be able to find a 4-flip for every amount of digits greater than 2. He continued his explanation, saying that he had moved to identifying the parts of the definition that needed to be satisfied. He stated that both examples satisfied the first requirement that *4N* remain the correct number of digits, and further felt that this would be "pretty hard not to satisfy" (Transcript 22, line 225). He stopped to reconsider this statement, and when prompted, tried an example in the calculator to check. He quickly saw that the previous statement would not be true, seeming embarrassed for having thought it in the first place.

After more thought, Shaun decided that maintaining the number of digits would be dependent on what single-digit number you multiplied by. He concluded that for any multiplication, there would be a bound, past which the results would increase in digits. The researcher asked Shaun to go back to the original question, to prove that there were no two-digit 4-flips. Shaun opted to rewrite the definition of a 4-flip as an if-then statement. He again was turning to logical manipulation to attempt a proof. Examples could not yield a proof, he said, because there would be infinitely many to check.

Beginning what he termed the *new plan*, Shaun wrote an if-then statement on a new piece of paper, later identifying the hypothesis and conclusion of the definition as *P* and *Q*, respectively (see Figure-Shaun. 2).

Figure-Shaun. 2: Rewritten statement as if-then statement in Question 3

He said that the definition had been given in the reverse order of an if-then statement, where the conclusion had been written first and the hypothesis second. However, he did not appear to recognize that all definitions could be written as if and only if statements. As he wrote the if-then statement of the definition, Shaun also said that having the word reverse in the question “gets me reversed in my head” (Transcript 22, line 324). He was bothered by the fact that there were not any symbols to manipulate in the statement and not much notation to work with. After again becoming distracted by an off-task discussion (Transcript 22, lines 338-372), Shaun brought himself back to the question, first not making any coherent statements aloud. He acknowledged that he had been thinking in his head, and had not expressed his thoughts very well aloud. Shaun explained that he wanted to proceed by assuming *P* and not *Q*, working with a proof by contradiction, though he did not identify the proof type by name or say what he intended to prove when making these assumptions. At that point, Shaun felt confused, comparing his work to the previous interview. He recognized that he was getting lost in the notation, and summarized, saying that he wanted his assumptions to lead to a contradiction.

Still searching for a proper technique and a way to use symbols, Shaun compared this question to one he had seen in MATH 305, where it was proven that **N**, the set of natural numbers, was countable. He said, “I guess it’s kind of like twisting the words around in a different manner to make logic out of that” (Transcript 22, lines 404-405). He expressed his confusion and again related this to previous work he had done, saying that he would get so involved in finding the contradiction and twisting the definition that he would no longer be sure of what was going on. He also added that he was not good at keeping track of words, which was why he preferred symbol pushing. Seeing that he was not going any farther with his thoughts or the proof, the researcher asked Shaun to move on to another question, to which he was agreeable.

Shaun had made a plan for his work in Question 2, but did not seem to know what he needed to do to form a proof of the statement. He looked at only a few examples, and was not able to fully grasp the main ideas necessary to understand the definition and complete the proof. Even after he felt he had an example of a 4-flip, he did not pursue the main ideas further. Instead, he became lost in the technical details of writing logical statements and using them for a proof. His work on this question resembled closely what was done in the previous interview. He had many good ideas and strategies, but became so involved in the symbols and logical notation, that he completely lost track of the overall goal and even his original understanding of the question. His reliance on notation and symbols suggested that he did not fully accept a proof, or could not develop a proof, without use of logical quantifiers and symbol manipulation.

In other words, it again appeared that Shaun identified a valid proof as one involving a direct connection to logical notation, and one in which the proof itself could exist apart from the meaning of the question. Shaun clearly made use of techniques and strategies learned in his MATH 305 course, but his continuing struggle to move past symbols and to accept a proof using words had apparently become a major factor in his inability to form a proof for this question.

Question 1. After the researcher read the question aloud, Shaun said that the questions he had been working on were interesting and also said, “This is really cool like that people think of doing stuff like this” (Transcript 22, line 450). He read the question himself and repeated some of the statement aloud. He then wrote that a pentagon had five sides and five vertices. Checking the sums around the example pentagon, Shaun verified that all were currently 16. He decided to find the pentagon with sums of 14, adding that it could take more than one try to accomplish this, and drawing a pentagon to begin his search. However, rather than diving in and placing numbers on his pentagon, Shaun instead looked for patterns in the example pentagon, first checking the differences between each pair of adjacent numbers. He said that he did not want to just put numbers on the pentagon in hopes of getting 14, but rather would work from the one that was given to him as an example and try to find patterns that would help his search. He noticed, in the first numbers that he looked at, that the difference also happened to be the number beside the pair he was checking. However, this pattern failed quickly when looking at other pairs around the pentagon.

As Shaun searched for further ideas, he wondered what the largest possible value for the sum would be. The researcher said that this value could be found, and that Question 1 could have been written to ask to prove that that number would be the largest possible sum. Shaun felt that the argument would be a combinatorial proof, and the researcher agreed that there would be several ways to approach the question, adding that the choice of questions stemmed from this ability to use more than one method to find a proof. Directing Shaun back to the original question, the researcher asked him what else he would do, but Shaun did not have any further ideas. He felt that he would just move on by attempting to place numbers on the pentagon, which was what he had been trying to avoid when he had chosen to look for patterns.

As soon as Shaun began his attempt to place numbers, though, he stopped and went back to searching for patterns. He noticed that in the example, the largest number, 10, and the smallest number, 1, were together in a combination to add to 16. When he moved to the next largest number, 9, however, his pattern failed when 9 was not in a combination with the next smallest number, 2. The next pattern he considered was in the positions that the numbers were placed, whether on a side or on a vertex, thinking that perhaps when listed in numerical order, this would make a pattern. Even when this too failed, Shaun remained undeterred. His next attempt at a pattern involved noting the difference between the current sum on the example and the desired sum that he was looking for.

> Okay, here's my silly idea. Let's take the numbers below – from the example up here [referring to the example pentagon with sums of 16], take the numbers that are less than 5 and add 2 more. Take the numbers that are 5 and greater and subtract. (Transcript 22, lines 604-606)

After stating this idea but not pursuing it, he said, “that’s probably not going to do anything, but that’s all I have” (Transcript 22, lines 614-615).

Since Shaun did not have any other ideas for the proof, the researcher stopped him in the interest of time and asked him to reflect on the strategies that he had used. Shaun said his strategy with this question was just to try to find a pattern, but that he would need more time to develop this idea. He also said that he had never seen a question like either Question 1 or 2 before, but that working with the divisibility of a number in MATH 305 seemed similar to Question 2. The researcher noted the use of an example and a non-example in first understanding the definition of a 4-flip, and Shaun agreed that this idea had also come from MATH 305. He still thought that the proofs seemed to stem from number theory or combinatorics, but he did not have experience with those courses, so he felt that had posed a difficulty for him.

Unlike his work in the previous questions, Shaun did not seek symbols or logical notation in his search for a proof of Question 1. Instead, he worked exclusively with a search for patterns. He read and understood the question before beginning his work, but did not make a plan, or mention a proof method at any time, again showing a drastic difference from the other questions he attempted. He was not able to form a proof, nor was he able to find any of the key ideas that could have aided his attempts.

Summary. Despite the differences in his work on Question 1 versus the others, one portion remained similar, Shaun's tenacity and desire to use one idea or strategy without consideration of any others. Even when he did decide to try a new approach to a proof, such as he did in Question 1 by deciding to just place the numbers on the pentagon in an attempt to find sums of 14, he was not fully committed to the new idea and reverted back to the old quickly. Shaun's thoughts and strategies again reflected those used in MATH 305, however he was not able to redirect his thoughts when his ideas were not producing results. He made an overall goal and several smaller goals in some of the questions, but did not seem to know how to tie his thoughts together or to be able to flesh out those goals into actual justifications to form a valid proof.

Shaun obviously memorized several methods or proof techniques, and in many cases, chose the appropriate technique for the proof. However, it was in the details of completing these methods for specific questions that Shaun failed. He was working at a procedural level, not really understanding what he was doing or why, but rather just pushing symbols and following a format learned from examples of proofs in MATH 305 for a particular proof type. He also followed a type of format when choosing strategies to approach a proof, first trying to understand the statement, then looking at examples and non-examples, and later choosing a particular proof technique to use. This was part of an outline for beginning a proof that was given in MATH 305. However, Shaun seemed unable to move past the motions of this outline to truly understanding what he learned from each step that could aid in his proof. For example, in Question 3, Shaun drew, and even labeled as such, an example of a chessboard satisfying the statement, as well a non-example that did not satisfy the statement. It was not until the researcher asked Shaun to consider what the non-example could tell him about the overall reason that the odd-by-odd case would not have a perfect cover that he had looked into this thought. After he had done so, Shaun was further able to develop ideas for the proof, though those were not fleshed out entirely to their final product of a proof.

Throughout the semester during which this study took place, Shaun was one of the MATH 305 students that took part in informal study sessions with other students in the course. During these times, Shaun was able to aid other students in their use of notation and symbolic representation of questions, however it was often again seen to be a determent to his ability to write a proof at times. He often would focus on setting up a proof, but was unable to complete the details. While he could accept ideas from others and move past this dependence on notation, he did not always do so on his own. He worked well with others most of the time as he would bring to the group the clarity and desire to follow a rigorous format for a proof. However, it was not uncommon for him to arrive with many questions started, even with several pages of ideas written, but having been unable to complete them needed help from other students. These tendencies were also seen throughout this interview, even though the outside influence of other students was not present.

<u>Jill</u>

Jill was a MATH 406 student who was a graduate student in curriculum and instruction. Her previous coursework included linear algebra, MATH 305, math with technology for teachers, number theory, discrete optimization, and abstract algebra. She did not list calculus on her second questionnaire, however calculus I and II are prerequisites for some of the other courses she had taken. During the interview, which was approximately 57 minutes in length, Jill worked on parts a and b of Question 2. At the beginning of the interview, Jill was talking aloud and describing her thoughts and her steps well. She was keeping track of her work and making a plan for finding a proof.

However, as the interview went on, Jill seemed to get frustrated and lost in the question. Her thoughts drifted and she had to bring herself back to the question several times.

Question 2. The interview began with Jill reading the question to herself, then the researcher reading it aloud. Jill said that the first thing she had done was reread the sentence herself slowly to make sure she understood what it was saying. She knew what a positive integer was, she said, but wanted to find an example to fully understand the definition of a 4-flip. Jill opted to begin looking for an example of a three-digit 4-flip, trying the random example, $N = 312$, and finding that $4N = 1248$. She explained what would have been needed for N to be a 4-flip, showing her understanding of the definition. Jill added that $N = 300$ would give a four-digit result for $4N$, so she proceeded to try another number lower than 300, $N = 105$, and saw that it also did not work. Jill moved past guessing and checking to look at the place values of each of the numbers, N and $4N$. She said that she needed the last digit of N to be equal to the first digit of $4N$, again showing her understanding of the question.

Jill went on to say that she really did not know how to pick the right number for N. She began to look at the possibilities for the individual digits, first saying that she would try the value one in the first digit of N. This would give a 4 in the first digit of $4N$, she said, which in turn would mean that the last digit of N must also be 4. She looked at the example $N = 104$ and found that it would not be a 4-flip. Jill began to wonder if there would actually be any three-digit 4-flips, and explained that she was searching for a three-digit number because she knew that no two-digit 4-flips would exist, since this was what she was trying to prove. She said that she would try a few more things in the three-digits before she gave up.

Returning to her examination of the digits, Jill again looked at the possibility of having 1 in the first digit of N. She now said that this would give a 1 in the last digit of $4N$ and noted that this would not be possible. Therefore, she ruled out all of the numbers in the interval 100 through 199. However, she did not recall this reasoning later, in part b of the question, nor did she use the same logic in the two-digit case.

Jill moved on to the 200 range, looking at one example that did not work. She said that she really hoped she would be able to find a 4-flip,

> Because I'm assuming that if I can figure out why it works or what it takes to

> work in a three-digit, then maybe I can see why it wouldn't work in a two-digit? - I don't know if that will work. (Transcript 17, lines 112-118)

She added that she really did not know what a 4-flip number was, and wanted to be able to find one.

Jill returned to her work with the individual digits, still in the three-digit case, this time choosing the first digit of *N* to be 2. Switching tactics, she now said that this would give a first digit in *4N* of 8 or 9. However, she changed her mind and said that the last digit in *4N* must be 8. Losing her train of thought, she mentioned that it was difficult to write down everything she noticed as she punched numbers into the calculator. She suddenly switched to the 300 range and noted that she had ruled those numbers out, showing the example *N* = 312, which resulted in a four-digit value for *4N*. She then said that she wanted to find two numbers for the first and last digits of *N* that were "closer together [circles the 2 and 8 she wrote down for *N*]. Because if one of 'em's larger, much larger than the other, then you're gonna have a um – it carries over." (Transcript 17, lines 140-142). At this point, she decided to abandon her search for a three-digit 4-flip, saying that it was not getting her anywhere so she would return to the two-digit case. Her work up to this point was quite scattered. While she did keep the overall goal in mind and had potentially good ideas to work with, she redirected her thoughts too quickly without fully considering any one direction.

Jill restated Question 2 and represented a two-digit number as two blank spaces, then called these spots x_1 and x_2, and showed the desired result when this number was multiplied by 4 (see Figure-Jill. 1). She said that she still did not have a clear direction of where to go next, but that she might try proof by contradiction.

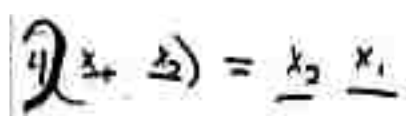

Figure-Jill. 1: Representation of flipped digits in Question 2 part a

When Jill asked if she was eventually supposed to be able to find the proof, the researcher responded by saying that either way would be fine; the point of the interview was the process she was going through to get there. Jill continued her work on a proof by contradiction by stating that she would assume to the contrary that a two-digit 4-flip did exist. However, she did not want to write out her proof until she actually reached the end of it. She jotted down notation for N as the digits xy and wrote that $4(xy) = yx$. She said that she wanted to check the definition to make sure she was using it correctly, which she determined that she was. Here she was showing good organization in her work, keeping in mind the overall goal and trying to form notation to fit the definition. However, it was clear that she did not have a plan for how to proceed. Jill had an idea to divide by 4, noting that the right hand side must remain an integer after this division. Saying that she did not know if this would get her anywhere, she chose to leave the thought and possibly come back to it later.

Jill decided instead to find the maximum allowable value for N that would maintain $4N$ as a two-digit number, which was an idea that began in the three-digit case. She was quickly able to determine that this cutoff would be $N = 25$. Jill said that if she was unable to find another proof, she could always use brute force on the remaining values. This was a technique that she had seen in use in number theory, when possible numbers were limited to a manageable amount, but said that she did not yet want to use this technique for this question. While she did recognize brute force as potentially yielding a valid proof, Jill desired another technique. At this point, she was unaware that there would be another part to the question, and so there was no reason for her to want to find a more generic proof that could be used for larger numbers. Like other students, however, her desire for a different argument was strong. Instead of using brute force, Jill thought she might move on using cases, though she did not say what cases she was considering. She added that N must be greater than or equal to 10, and restated that her parameters for N were the following, $10 \leq xy < 25$. Seeing that the cases were now even further limited, she again said that she could use brute force on the remaining values. Jill instead decided to keep trying to find another proof, though she said that she sometimes did give up on other methods and would resort to using brute force. She gave no further explanation at any time during the interview as to why this technique was her last choice for a proof.

Jill tried a few examples, $N = 10$ and 20, noting that they were not 4-flips. She said, “there’s gotta be an easy, easier way to say this, with the digits” (Transcript 17, line 240). Representing the digits as blank spaces again, she considered each one individually, starting with *y*, the second digit of *N*. She went through the first few choices for *y*, 1 through 4, and wrote down what the corresponding value would be for the second digit of *4N*. She said that she was almost using brute force by doing this, but continued anyway. She took note that every result was even, since 4 was even. Jill incorrectly used this to eliminate all the odd numbers from her list of *N*, and further said that her list actually only needed to include the multiples of 4. However, after looking at this idea a second time, she realized that she had been incorrect, and looked at one example of an odd number being multiplied by 4, showing that it was valid to use such numbers.

Jill turned back to looking at examples, and eventually wrote down all remaining values and their products with 4 (Transcript 17, lines 274-286). Her work is shown in Figure-Jill. 2.

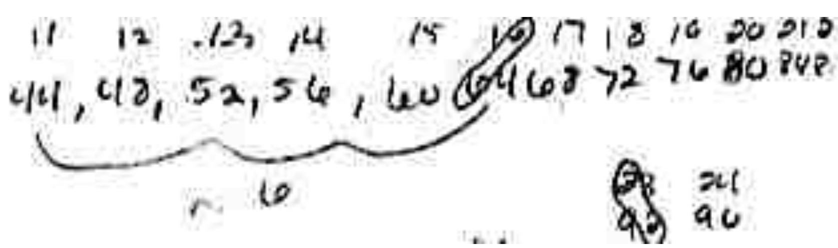

Figure-Jill. 2: Brute force proof for interval 10 through 24 in Question 2 part a

When she was finished, Jill said that she had listed all the numbers and had ended up using brute force after all. However, she said that she still wanted to find a different proof. She examined the only two combinations in her list that had any of the desired matching digits, 16 with 64 and 23 with 92, saying that $N = 23$ was "partially right" (Transcript 17, line 289), since one digit was correct. She began to think of other ideas to check, but second-guessed her arithmetic. After looking back over that again, she stated that she was tired of looking at the question and wanted to move on.

The researcher asked her to work on part b of the question. Jill immediately took back her work from part a, since she had already started to work on the case of three-digits. Seeing the difference in the wording of the question, Jill felt that she may be able to find a three-digit 4-flip because, unlike part a, this portion said to either prove or disprove. She stated that to disprove the statement, she would need to show one counterexample, and said, “So – if I had a better reasoning of why they didn’t work for the two-digit, it might be easier to come up with a three-digit. But, I’m moving past that still” (Transcript 17, lines 331-333). The researcher offered her the opportunity to move to another question, which Jill seemed hesitant to do. So, the researcher asked her instead to look back at what she had done with three-digits in part a of the question and get back into her thoughts.

Jill did this, and recalled that she had limited her values to those less than 300, since *4N* needed to remain a three-digit number. She checked $N = 299$ and realized that she needed to further limit her possibilities. She said that she did not think she would be able to complete part b using brute force, so she needed to find the correct parameters. Jill did not want to think too hard about what the upper limit was, but instead decided to just try some numbers. At this point, she had not yet linked this search with her work in part a. She indicated that she might be able to use a ceiling function of some sort to find the maximum, but felt that this would take too much time. After using the calculator to try a few more numbers, Jill easily saw that her upper limit was actually 249. She then saw that this value could have been found by considering $\left\lfloor \frac{1000}{4} \right\rfloor$, which would actually have been easy to find. She rewrote her parameters as $100 \leq xyz \leq 249$, and also wrote that she would want $4(xyz) = zyx$. She again turned to viewing each individual digit and wanted to look at the possibilities. When she started to examine this, though, Jill still did not feel that she was making any progress, saying, “I’m still not coming up with this. Something’s still not clicking like I feel it should be” (Transcript 17, lines 390-391). Thus far, she had been able to keep the overall goal and picture in mind, but was having difficulties in the implementation of her plan to find a 4-flip. While she did recall her work, she forgot some details that could have helped to speed up her search. She also again jumped from idea to idea without fully expanding her thoughts in any one area. Jill then said that she had been considering the parities of the digits, but that had not seemed to be helpful, either.

Jill was trying to find patterns, and continued to look at the options for the last digit of *N*. She began by choosing the last digit to be 2, but decided that she would instead want to look at numbers that would give carry-over into the middle digit, though she did not state her reasoning for this choice. She stated that 6 and 3 “actually kinda seemed to work” (Transcript 17, line 430), but did not further explain her reasoning for this statement either. She continued by looking at the case where the last digit of *N* was 6, noting that this would give the last digit of *4N* being 4. However, her thoughts jumped around at this point. She saw that the middle digit of *N* needed to remain the same in *4N*, but was unsure how this could occur. She again wanted to divide by 4, but was not sure that would help either.

Again, Jill’s thoughts were very difficult to follow. She said that she needed to go back to simple algebra to solve this, indicating that she would first multiply by 4 and then divide by 4 to solve. Not knowing how to continue with this thought, she decided that it might be easier to work from the values of *4N* backwards to *N*. It was obvious to Jill that the values of *4N* must be divisible by 4, and she desired an abstract way to represent that. After Jill had looked at a few more examples, the researcher asked if she wanted to leave this question. Jill said that she just wanted to start “plugging away with the numbers and seeing what happens to ‘em” (Transcript 17, lines 484-485). She said that this was all that she was thinking about, adding, “I’m not really thinking about the problem as much” (Transcript 17, line 490). This was an indication that she had gotten wrapped up in the details, but recognized that she was no longer thinking about the bigger picture.

Jill directed herself back to her work with the ending digits, asking herself what she had been trying to find when she had gotten sidetracked. She recalled that she was looking for the middle digit to remain the same. She looked back over her work from part a, looking at the list she had written. Jill stopped to ask the researcher if she should continue her work on the question, saying that if this had been a homework question, she would have moved on by then. Seeing the remaining time was short, the researcher asked her to continue this question for another 5 minutes before they would conclude the interview.

Jill asked if the researcher was starting to feel bad for her, indicating her unease at not being able to find the proof. The researcher reassured her and said that Jill had many good ideas and just needed to connect them together. She encouraged Jill to look back at what she was doing in part a with the three-digit case, saying that she had eliminated numbers there that she had not yet done in part b. Jill said that she wanted to break the question up into cases, not saying what these cases would be. She first eliminated the numbers $N = 111$ and 222, since they repeated, and so obviously could not be 4-flips. She further eliminated all numbers ending in 5 or 0, since their product would result in the last digit of 0 for $4N$, making the flip only a two-digit number.

Jill then skipped to the last digit of N being 2, saying that the last digit of $4N$ would need to be 8, so the flip would need to begin with 8, which was outside of her parameters for N. For the same reason, she eliminated 4 as an option. However, as she further considered the cases, she moved to looking at 7, and at that point lost her train of thought. She now said that the last digit being 7 would make the first digit of $4N$ a 7, which she said was not possible, though she did not explain why. The researcher tried to help get her back on track, asking her to compare her previous work and reasoning to what she was examining now, but Jill was unable to recover. Jill switched to checking the last digit of $4N$, seemingly unaware that this was not what she had previously been working on. The researcher again suggested that she had reversed her thought process, but Jill was unable to move back to her previous line of thinking. Jill concluded that the only remaining possibilities were 1 and 2, giving values of 1 and 2 for the first digit of N. While this was a correct limitation, the reasoning that led her to this conclusion had not been complete or correctly followed through.

When asked what she was thinking about when determining this limitation, Jill stumbled and reconsidered her thoughts. She was able to see that her conclusion was correct, but that her justifications were incorrect and fixed this error. She determined that the first digit of N could not be 1, since the result of $4N$ could never end in 1. Since the first digit of N needed to then be 2, she said that this would imply that the last digit of N must be 3 in order to get a 2 in the last digit of $4N$. The researcher corrected her, saying that there was one other possibility, which Jill was then able to find to be 8.

Jill switched to a new piece of paper, saying that she would look back at what she had found. She recapped her conclusions and said that she would have to spend more time looking over it to make any further conclusions. The researcher stopped her work at this point because of time constraints. The researcher went back through Jill's work with her. Jill agreed that she had first reread the question to understand it. She then had looked for a three-digit 4-flip, which she now said was "just to see what the properties would be" (Transcript 17, line 723). As the researcher continued to recap, Jill agreed that she continued to search for a proof other than brute force, saying, "You know there's always little tricks – And I know there has to be one" (Transcript 17, lines 742-744). She finished by saying that the ideas she had used had come from number theory and her MATH 305 course.

Summary. In her work, Jill had many difficulties. She was unable to maintain and work with an idea to its completion and became confused and turned around in her reasoning several times. While she did seem to understand what would constitute a proof of the question, she was unsatisfied with a proof by brute force in part a, and unable to complete any proof in part b. The strategies that helped her included reading and understanding the question, looking at examples, making and keeping in mind an overall goal, choosing proper proof techniques, breaking the question into smaller parts, writing what was known and what needed to be shown, using previous work, and recognizing her proof as being valid. However, other strategies that she used actually detracted from her success, including looking for patterns, searching for a representation of the digits, redirecting her thoughts, and looking through the choices for each digit (since she was not able to properly monitor this work). She was also unable to monitor her progress and her ideas were not organized.

Jill's biggest hurdle seemed to be the inability to properly monitor and redirect her thoughts. She did not allow herself the opportunity to work out most thoughts and did not carefully keep track of what ideas she was using and had already considered. Jill had many good ideas, which could have been pivotal in forming a proof, but in part b, was unable to follow through and connect these ideas.

Andy

Andy was a MATH 305 student who was a junior majoring in mathematics. His previous coursework included calculus I and II. He was also concurrently enrolled in linear algebra. During the interview, which was approximately 45 minutes in length, he worked on Questions 3 and 1. While Andy did discover the main idea for the proof of Question 3, he did not realize that he had done so and was unable to complete a written or verbal proof. He did not mention or address the bi-conditional nature of the statement in any way. However, he was able to complete a verbal proof of Question 1, though he did not seem satisfied with his arguments. Andy was very quiet, did not write much down, and had to be pressed to explain his ideas. It was obvious that he was doing work in his head and that he was visualizing what he was working on. He kept track of his progress well and did not get confused or stray from the questions at all.

Question 3. The interview began with the researcher reading Question 3 aloud. Andy paused for one minute, and then said that he was thinking of the different proof techniques from MATH 305. Specifically, he was thinking of a proof by cases, where he was considering two cases based on whether *m* and *n* satisfied the second portion of the statement or not. He stated that he was trying to "formulate in my mind how I'm gonna do that" (Transcript 18, line 24). After a 30 second pause, Andy said that he was trying to put his mind in abstract mode. Most pauses Andy made throughout the interview were quite long as he thought, but the researcher tried not to interrupt the process.

When prompted to discuss what he was thinking, Andy said that he was trying to think of the generic case. He noted that he had an example given, the 8-by-8 chessboard, which was of the form *m*-by-*m*. After another pause, he said that he was unsure how to start the proof. He further noted,

> In my mind, I can see what I'm trying to prove, I know what I want to do… I'm picturing the traditional chessboard and how it's covered with each, uh two dominoes takes, or a domino takes up two squares. – And um, I'm seeing in my mind how it obviously has to be even, because of that fact. (Transcript 18, lines 36-48)

When asked to draw what he was picturing, Andy drew a 6-by-7 chessboard and proceeded to explain how he could cover this board with dominoes. He expressed again that it was already clear to him that one of the dimension must be even so that no overlap would occur. He also said that it did not matter which dimension was even, since the direction of the dominoes could be changed appropriately.

After a pause of nearly one minute, Andy said that he was thinking about the definition of even and odd integers, and trying to find a formula to use for the proof. He also thought that he might have to "come up with a generic matrix" (Transcript 18, line 79). The researcher believed this to mean a generic chessboard. Andy moved on to writing out the two cases he had previously mentioned. His work is shown in Figure-Andy. 1.

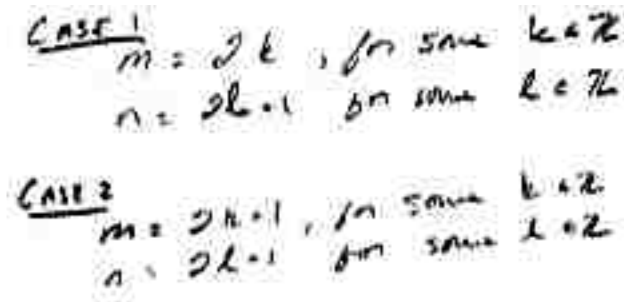

Figure-Andy. 1: Proof of two cases in Question 3

He said that he needed to consider the case where both dimensions were even, such as in the traditional chessboard, though he did not add this to his list of cases.

Andy then wanted to look at the contrapositive of the statement; "– if *m* and *n* are both odd, then it does not have a perfect covering" (Transcript 18, lines 108-109). He did not indicate that there would be a second direction to be proven or how his cases might address the bi-conditional statement. He said that a proof by contrapositive was just another idea, but he did not know what he wanted to do. The researcher asked him to flesh out the argument for the contrapositive. Andy said in response,

> I always want to quantify things. That's where I – that's why I had troubles in this class in the first place anyways. – I'd like to see equations and always end up doing, wanting to do, ah, chasing arguments – proving existence. – And it's not necessary all the time. (Transcript 18, lines 114-128)

He further said that he was struggling to find equations or something to quantify in the definition. When asked, Andy confirmed that he now wanted to show the existence of a perfect cover and that he wanted an argument to show that Case 2 would not have a perfect cover.

Drawing a 3-by-9 chessboard and filling in dominoes, Andy said that he did not know how to proceed, and that his previous ideas did not seem to apply here. The researcher clarified that he was thinking of element chasing arguments. Andy also said, "I just don't know how to do, go about this like mathematically other than just like a picture, you know, writing it in short, concise sentences" (Transcript 18, lines 176-178). This statement suggested that he did not believe a proof by picture to be a valid proof. He was referring to his picture of the 3-by-9 chessboard, where the dominoes he had placed would overlap and so would not be considered a perfect cover. The researcher asked Andy to explain what his picture was saying, and he described the overlapping dominoes. He further explained his placement, when asked to do so, and redid his placement to show that at least one square would have two overlapping dominoes covering it.

The researcher asked Andy to pursue this idea in general. After a pause of almost one minute, Andy said that there needed to be an even number of squares on the board, as there was in the 8-by-8 case. He went on to say that in the even-by-odd case, the total number would also be even. The researcher prompted him to consider what this would mean in Case 2. Andy said that there would be an odd number of squares in this case, and said that this would mean there was a remainder. Again the researcher asked him to explain his thoughts, and Andy restated that a domino covered two squares and so he needed an even number of squares to be covered. He further said that if there were an odd number of squares, there would be an overlap of dominoes, because a domino could not cover just one square.

The researcher asked him if there was anything further he wanted to say, do, or pursue for the question. Andy said he was not sure, but that he did still desire an equation to work with. When asked, he said that this question reminded him of matrices, which could have been due to his concurrent enrollment in linear algebra. The researcher asked him to walk her through the steps he had taken in his work, describing what he was thinking and working on in his head as he had gone through them. Andy said that he had first tried to visualize the traditional chessboard being covered with dominoes, and had then tried to think more generically of a chessboard where the dimensions were not equal. He also had wanted to consider how he might prove the generic case. He said that this was where he had run into a “brick wall” (Transcript 18, line 327). Andy had begun to consider different proof techniques from MATH 305, such as proving using the contrapositive, induction, or cases. The researcher asked if this was a strategy that he often used, to look through the different options and consider which would work for the question at hand. Andy said that he did usually do this, adding, “if I think something might work, then I’ll ah try it out a little bit and then if nothing comes about then, maybe I’ll try something else” (Transcript 18, lines 338-339).

Andy's work on this question was well organized and he did not get sidetracked as he worked. He was able to discover the main idea for the proof, that the total number of squares on the chessboard must be even, but did not seem to understand that developing this idea would finish the proof. He even used the technique of proof by cases, which could also have addressed the bi-conditional statement to be proven. However, Andy did not seem to recognize that there would be two directions to prove, nor did he acknowledge his use of cases as covering these directions. He had many good strategies, including reading and understanding the question, looking at examples, choosing a proper proof technique, making and keeping in mind an overall goal, monitoring his progress, and redirecting his work. The lack of understanding of the dual directions needed for the proof and that his work was forming a valid proof resulted in Andy being unable to finish this question.

Question 1. After the researcher read the question aloud, Andy paused for approximately 40 seconds. When asked, he said that he understood what was being asked. Andy verified that he needed to show that any arrangement of the numbers on the pentagon would form a minimum sum of 14. The first thing that came to his mind was to do trial and error, making one side equal to 14, and working from there to rearrange the rest of the numbers.

Considering this, Andy said he could see that 10 would not be able to be used on a corner because it could not be used in two different sums to make 14, instead it must be placed in the middle of a side. After a 40 second pause, Andy agreed that he was going though numbers in his head. He conjectured that 8, 9, and 10 would all need to be placed in the middles, and that he would build around these, since there were so few combinations with sum 14 containing those numbers. The researcher asked him to write down some of what he was trying in his head. Drawing a new pentagon, Andy placed 8, 9, and 10 on edges. He continued by writing out the two combinations involving 9 and the one combination involving 10. He noted that 9 would have to be on the edge adjacent to the edge 10 was placed on, since they would have to share one number.

Andy made a plan to "write the different ways to get 14, using the larger numbers first" (Transcript 18, lines 404-405). He spent the next 1½ minutes writing the combinations including 8, 7, and 6. He stopped there and said that he could continue this process, but that he was not thinking about how to prove that the smallest sum was 14. Feeling that he had gotten engrossed in finding these combinations, Andy showed that he was keeping the overall goal in mind. He redirected his work to think of the proof in general instead of further pursuing the pentagon for 14, even though he was given the opportunity to continue his search.

Andy now considered the smallest possible combination involving 10, and determined that to be the combination 10, 1, and 2, with a sum of 13. He said he would then think about what combinations the other large numbers would have to have to make a sum of 13. He took 45 seconds to list the combinations for 9 and 8. Noticing that there was only one combination for both 9 and 10, with both including the number 1, Andy knew they would again have to share a corner. He placed these numbers on the pentagon, and filled in the combinations they required. After a brief pause looking at the pentagon, Andy said that it now fell apart for 8. When asked, he described that 8 only had two combinations, and since the 1 was already used twice, and the 2 and 3 were on opposite sides, there would be no way to place 8 on the pentagon. He said that he could then conclude that a pentagon with sums of 13 was not possible.

Andy went back to finding a pentagon with sums of 14, using the ideas that he learned from trying 13. Placing the combination 1, 10, and 3 along one side, he said that he would proceed from there using trial and error. He finished his previous list of combinations with the only remaining combination for 6, and then stated that 6 would either have to be placed with 1 or with 3. In other words, it would need to be on one of the sides adjacent to the side including 10. He chose to place the 9 and 4 on the pentagon next, and was able to fill in the remaining numbers to form the appropriate pentagon.

When asked, Andy summed up his work. The researcher asked if he was satisfied with his proof, and Andy said "Pictorially, yeah" (Transcript 18, line 507), indicating that he might not be convinced that his arguments were enough to prove the statement. He said that he could not recall having seen a question like this before, and went through his work again when asked about his strategies.

Andy was able to complete the proof to this question, though it was not clear that he fully recognized his work as a proof. His success in the proof was due to his systematic treatment of the placement of the numbers, as well as the recognition to consider the larger numbers first, since they had fewer combinations to try. He was quickly able to see the main points of the proof, and followed his ideas through without getting sidetracked. He was able to monitor his thoughts and make a plan for approaching the proof. Even when he did go astray without a plan at first, he was able to see this and redirect his work back to the question and keep in mind the overall picture. His strategies included reading and understanding the question, systematic choices, breaking the question into smaller parts, monitoring his attempts, organizing work, redirecting, making a plan, monitoring progress towards the goal, drawing pictures, and staying on track towards the goal. He was very quiet while working and needed to be prompted to explain his thoughts and reasoning. He clearly understood the question and his reasoning, but may not have recognized his work as a valid proof.

Summary. Andy was very quiet and worked often in his head throughout the interview. He had many good ideas, however, and was able to make good progress in both questions. The biggest issue Andy had was not truly being convinced of the validity of his own proofs. It seemed that he had some form of proof in mind and since his arguments did not fit this form, he had difficulty accepting that they could constitute a proof.

Throughout daily observations of Andy's work with others, the researcher noticed that he often was able to make a plan for proofs. Andy had good ideas and insights to share with others and often made positive contributions to the other's work. In the group work time as well, Andy was able to offer good suggestions and worked well with the other participants. His major issue of not understanding the validity of his proofs was not a major difficulty in the group scenario, since he had several other people to confirm or correct his work.

Overall, Andy was fairly successful in his work. He clearly was able to monitor his thoughts and did not get wrapped up in wild good chases that would lead him astray. He seemed confident but critical of his abilities, and this shows in his lack of certainty with his proofs.

Katy

Katy was a student in MATH 406. She was a junior majoring in mathematics. Her previous coursework included calculus I and II, linear algebra, number theory, statistics, and MATH 305. During the interview, which was approximately 51 minutes in length, Katy worked on Question 2 part a, and Questions 3 and 1. While she did start Question 2 with a clear direction for the proof, her arguments did not fully pan out and as a result she ended up working in circles, unable to form a valid proof. Work on all questions was filled with good ideas and thoughtful processes, however she was unable to complete the proofs. She indicated that being convinced that a statement was true was key in proof writing; however, once she passed that step she did not seem motivated to further explore a proof.

Question 2. After the researcher read the question aloud, Katy read it to herself but misinterpreted the definition. The definition was clarified and corrected by the researcher, after which Katy showed that she understood the definition correctly using the notation $N = abcd$. At first saying that she wanted to proceed by using a counterexample, Katy then corrected herself, saying that she meant to say that she wanted to use proof by contradiction. She wrote out that she would assume a two-digit 4-flip existed. Again using variables for the digits, she further developed what this would mean. Katy made an error however, saying that $2ab = ba$, instead of multiplying by 4. The researcher stopped her and corrected the error.

Katy developed a set of equations relating the individual digits of N, also trying to incorporate the potential carry-over when multiplying the second digit by 4. When asked, she described what she had written. She said that if $N = ab$, she would want $4b$ to equal a or $c + a$, indicating that c represented the carry-over. Under these options, she wrote $4a+0$ or $4a+c = b$, though she did not prove or explain these equations. She was trying to express the relationship between the digits of N and $4N$; they would be the reverse of one another if N was a 4-flip. However, she was not sure that her equations would actually help her proof. Next, Katy said that she wanted to find an example to look at. The example she chose was $N = 82$, where she had first chosen $b = 2$, which gave $a = 8$. As she examined the calculation of $4N$, Katy noticed that the result had been a three-digit number instead of a two-digit number, though the ending digit of 8 did match what she would have wanted to find. She next tried $b = 1$, giving $a = 4$, and $N = 41$. She further saw that these examples could be flipped and would still fit one of her equations, so she now considered both $N = 14$, and $N = 28$ as well. She noted that 28 and

41 would be too large to multiply by 4 and remain a two-digit number. She performed the calculation of *4N* for 14, finding that it was not a 4-flip. Doubting her equations now, she continued to consider the resulting *4N* that were three-digit numbers instead of two.

When asked to describe her thoughts further, Katy second-guessed her conclusion that $N = 28$ would give a three-digit value for *4N*. She performed this calculation and confirmed that she had been correct. However, she seemed to now have lost her train of thought. She went back to her equations to examine them further. Katy began to manipulate them, writing that $4b = a$ implied $a = c+a$, which in turn implied $2a = c$. All of these conclusions were incorrect but Katy did not recognize that, nor did she seem to be connecting the equations back to the actual 4-flip any longer. They seemed to stand alone as she lost track of the purpose of her work.

Katy went on to write $4a+c = b$. Using that equation along with the previous result and simplifying, she found that $a = 1/6\ b$. Katy finally considered what her results would mean for the actual 4-flip and stated that she did not think that she could work with the equations, because her result would not make sense. The statements that she made following this indicated that she was not understanding the overall picture or goal. She said that her results would not tell her anything since *N* could be written as *ab*, or *ba*. She did not consider that her variables were generic and so their order was arbitrary. After trying to explain her reasoning again, she decided that the equations did not seem right and so abandoned that effort. Katy did not recall that equations leading to no possible solution was the goal, she had lost track of the purpose of her work.

Katy next considered going backwards with these same ideas, meaning that she would work with *4N* instead of *N*. She stated that 4 divided *ba*, using proper notation for this concept. This would result in the equation $4k = ba$, for some *k* value. She also stated that she would want *k* to equal *ab*, meaning the two-digit number, rather than the product of *a* and *b*. Thinking back to her proof by contradiction, she said, "Can we then show that that's wrong?" (Transcript 19, lines 132-133). After a pause, she doubted that her thoughts were a good way to approach the question. The researcher asked her to consider what else she might have tried, if this had been assigned as a homework question. Katy would have found another person to work with and talk over the question, she said. She noted that she normally worked on homework in a group, not by herself, and each member of the group contributed something to the proof. The researcher asked Katy if she would like to try part b of the question. Katy looked it over and said that she did not think she would be able to work any further on that portion since she was not able to complete the first part of the question. Given the opportunity, Katy said that she would like to move on to another question.

The researcher asked her to reflect on the strategies that she had used in Question 2, and Katy said that she had first thought to do the proof using contradiction since she needed to show that there were not any two-digit 4-flips. She added, "and then, well then I just wanted to work with equations because that seemed right. I don't know, it seems easier to see equations for me I guess. Like to show that things aren't equal…" (Transcript 19, lines 167-169). She said that she had not seen a question like this before.

Katy was not able to complete or even start a proof for this question. She did understand the definition, and determined an appropriate proof technique to use. However, she was not able to implement this technique nor to move past her equations to a different idea. She did not look at many examples, nor was she able to gain an understanding of the general idea of what would be needed for a 4-flip. Her strategies included reading and understanding the question, forming equations, looking at examples, selecting an appropriate proof technique, and making an overall plan for the proof. She did not follow through with her plan or the appropriate proof technique, nor was she able to discover some of the main points needed for the proof. Katy was not able to redirect her thoughts away from working with equations, even when she judged them to not be helpful to her. The idea of proving something did not exist seemed to cause Katy difficulties. Even with a good start, she lacked the monitoring needed to recall her goal. Her normal working environment was in a group situation, according to Katy, and this may have contributed to her difficulty thinking through the question on her own.

Question 3. The researcher read the question aloud. Katy first drew two squares of a chessboard, asking if that was how the domino covered squares on the chessboard. The researcher confirmed that she was correct. Katy restated what needed to be shown, again asking for verification that she understood what was being asked. She then said, "I just need a picture" (Transcript 19, line 199), at which point she drew an 8-by-8 chessboard, labeling the dimensions m and n. She wrote that one domino "covers $2/m$ of the row and 1/*n* of the column" (Transcript 19, line 206). She paused because she was unsure whether to assume that a domino would cover exactly two squares, or if she needed to know the actual measurement of the length of the domino to know that this would occur. When Katy further asked if she could say that a domino covered 2 divided by *m* of the row, the researcher asked if she meant that two out of the *m* squares were covered. Katy hesitantly agreed, then realized that to say that the length of the chessboard equaled *m* meant that there were *m* squares in a row, not that the physical measurement was *m*. She said, "that's what happens to me when I'm writing, when I'm writing proofs… I'm like I don't know what I meant by that. Then, I just get confused and give up" (Transcript 19, lines 236-238).

Katy had lost track of what she had been thinking and why she was considering it in the first place. However, she said that the question now made sense to her, and it was clear that *m* would have to be even.

> So, then obviously if it covers two of these squares, then *m*, the number of squares in this row has to be even and it doesn't matter if *n* is even or odd, because you can have as many dominoes as you want, stacked on top of each other. (Transcript 19, lines 239-242)

She also added that this would be true if the pattern was repeated in the other direction, giving that *n* would be even and *m* could be either even or odd. She asked if she needed to write down what she had said, and the researcher verified that Katy felt she had described aloud what she would have wanted to include in a proof. Katy agreed and explained her reasoning again aloud. Since it was still unclear, the researcher asked again if Katy was satisfied with her proof, and Katy said that she would write up two cases, and began to indicate that these cases would be based on which variable would have to be even.

The researcher then asked about the strategies that had been used. She prompted Katy to begin this discussion by saying that Katy had first read the question and drew an example to make sure she understood the question. Katy then added that she had needed to see the domino visually. She said that she had realized throughout her coursework that year that "it's hard to write a proof if you don't believe it" (Transcript 19, line 301). Thinking through the question and convincing herself it was true was a common strategy that Katy said she used in proof writing. She also said that she looked at small cases to see that a statement would be true there, as well, saying that was why she drew pictures. Katy said that she had not seen a question with a chessboard before, but that she had done proofs with cases, and considering even and odd numbers.

In this question, Katy became stuck after fully understanding the question and why the statement would be true. At that point, she was unable to further her thoughts and attempts at the proof to the statement. While she did continue her work, she was never able to move past what she felt was an obvious explanation. Her reasoning was not fully developed into a proof. However, the ideas that had surfaced were key to what could have become a proof of the forward direction of the statement. She did not acknowledge, nor seem to be aware of, the bi-conditional nature of the statement and her ideas in no way covered the reverse direction. She identified a proper proof technique, proof by cases, but did not properly identify the cases that could have formed the basis for a valid proof. Her strategies on this question included reading and understanding the question, drawing pictures, convincing herself that the statement was true, looking at examples, and identifying a proper proof technique.

Question 1. As soon as Katy was handed the statement of the next question, she remarked, “Oh, geometry, shoot” (Transcript 19, line 342). This gave the impression that she felt that the question may pose difficulties because she it contained a geometric shape and therefore, she identified it as a geometry question. The researcher read the question aloud, after which Katy paused to read it herself for approximately 20 seconds. Katy clarified that each number would be used only once on the pentagon, and circled the number 10 on the example. After a pause of almost 1 minute, the researcher prompted Katy to describe what she was thinking. Katy said that she was considering starting with 10, since that was the largest number to be used and would affect how small the sum could be on a side. She noticed that the smallest possible sum involving 10 would be a total of 13, using the combination 10, 1, and 2. However, she said that this must not be a valid sum, since the question asked her to prove that 14 was the smallest possible sum. She added that 10 would have to be placed with 1 and 3 for a sum of 14. Seeing 10 on a vertex and being used in two combinations on the example pentagon, Katy felt that it made sense that 13 would not be valid because there would not be a second combination for 10. At that time, she did not pursue any further thoughts about the sum of 13, instead she decided to try to find a pentagon with sum 14

Going back to using 10 on a corner, Katy noticed that she only had one combination in which 10 could be used and so she was confused. She wondered if she was missing a detail, or just was not adding properly. She clarified that she would need to use three numbers on a side, and that all numbers needed to be positive. The researcher agreed and further said that only the numbers 1 through 10 could be used, no zeros or negative numbers. She asked if there was anything else that Katy could think of to try. Not knowing what else to do with the 10, Katy said that maybe she did not need to start with it. She then stopped to consider the larger picture, saying, "But – I should probably try to think about how to write it as a proof, so I'm not just sitting here forever" (Transcript 19, lines 407-408).

After a pause, Katy began to erase the numbers she had written on the pentagon. She said that she was still thinking of how to find the correct sums for 14 on the pentagon, noting that she was not able to move past this idea. It was at that point that she realized 10 could go somewhere other than the corner where it had been in the example. She was clearly frustrated with herself for not noticing this sooner, adding that the nice weather was distracting her with thoughts of going outside instead of working through the proof. Katy now understood the question, but this hang-up had prevented her from being able to do any more on the proof. She also added that an error like this could make people give up on a proof, herself included, when an important detail was missed. In another situation, she would have given up, she said, and looked for help from someone else. Instead, here she restated what needed to be proven, then crossed off the previous attempt on the pentagon.

When the researcher asked what was going through her head, Katy said that she was thinking about odd and even numbers. She went on to explain that she would need one even and two odd numbers to get a sum that was even. Her thoughts went back to the case of 13, saying that a similar idea would work to show 13 was not possible. She said that an odd sum would require two even and one odd number, or three odd numbers in the combination. Noting that there were five sides, she considered what would happen if all five sides included two even and one odd number. There were not 10 even numbers available from 1 to 10, she said, and added that this was why 13 would not work. She noted that the combination 10, 1, and 2 would have to be used on one side, and so there could not be three odd numbers on every side. She said,

> But, to have the smallest number of odds, which there's 5 odds in 1 through 10, you put them all here, but then that'd mean you have to have 10 other even numbers to go along, but you only have 10 numbers. So, that's why 13 doesn't work. (Transcript 19, lines 464-467)

Satisfied with this explanation, Katy moved on to a sum of 14. She stated that she could have three even numbers, or two odd and one even number. Suddenly, she realized that numbers were shared on vertices and so her argument for 13 was incorrect. However, she moved on, saying, "I'll just accept that 13 doesn't work" (Transcript 19, line 471). Katy was again thinking of how to write her thoughts in a proof, and began to write equations, as she had in the previous question. The equations she found were: $2l + 2k + 2m = 14$ and $(2a + 1) + (2b + 1) + 2c = 14$. She said that she was trying to think of the sums equaling 14, going back to her ideas of the different possibilities for even and odd numbers in the combinations. She did not indicate any plan here, and did not mention that she had just ruled out her reasoning as being invalid, instead she continued with these equations. She noted that the variables *l, k, m, a, b,* and *c* must all have values of at most 5, since they were multiplied by 2 to get numbers on the pentagon. When she struggled to continue, the researcher asked her if she had any other ideas. Katy said she felt that she might be able to find something using her equations and she would probably continue in that pursuit, not having any other ideas at that moment. To continue with her equations, she would try plugging in different values for the variables and see what she could find.

When Katy said that she did not have any other ideas and seemed stuck, the researcher finished up the interview by asking her about her strategies on the last question. Katy and the researcher recapped her work (Transcript 19, lines 512-553). During this time, Katy agreed that she had reread the question and again had searched for equations. This was a common idea for her, she said, and proof by induction was her favorite type of proof because she could set up really nice equations to work with. Katy had not seen a question like this before, but again did have experience with other proofs involving the consideration of even and odd integers.

Like her work on previous questions, Katy struggled to form an overall plan for this proof. While she did understand the structure of what would be needed for the proof, she was unable to fill in the details. She became confused several times in her work and had trouble bringing herself back to her original ideas. When she began the question, Katy had an understanding of the overall picture, even choosing to start with placing 10 and recognizing the lowest option for a sum as being 13. However, the error in not understanding that 10 could move to a different place on the pentagon, rather than on a vertex, caused Katy a great deal of difficulty. Even after she corrected this mistake, her concentration and train of thought seemed broken and she was unable to ever fully recover. From that point on, she was not able to keep in mind the overall picture. In her confusion, Katy went back to the idea of parities and made more errors. Again, she did discover her errors, but did not keep this in mind as she continued to use the same thoughts again very shortly later. In the end, Katy resorted to searching for equations. This search was related to her idea of parities and she was again unable to discover anything of use for the proof. It is not clear that any extra time would have yielded a proof to this question for Katy, and her frustration level had come to a peak near the end of the interview.

Summary. Overall, Katy made use of several potentially good techniques, and did not seem to have difficulty understanding the general idea of each question and knowing how to start her work. However, she was unable to connect her ideas, to redirect after failed attempts, to make an overall plan, or to keep track of her progress and monitor her efforts. Since she was not able to reach a proof of any of the questions, there was no evidence of whether she would have recognized her proofs as valid or not. Her main difficulties seemed to result from the inability to self-monitor during all questions, and the high occurrence of sidetracked thoughts from which she was not able to fully recover.

While she did seem to attempt to keep track of her work, Katy was unable to keep her overall goal in mind and monitor her progress towards this goal. This may have been a result of not truly understanding what was needed for the proof, or of not being able to see how her ideas could be related to form a proof.

Analysis by Question

Question 1

The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram below shows an arrangement with sum 16. Prove that the smallest possible value for the sum is 14.

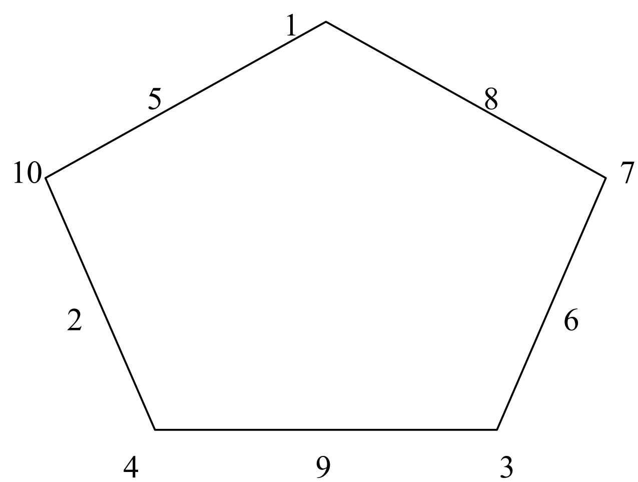

Figure 2: Pentagon given to participants for Question 1

Question 1 was very visual in nature. Unlike the other questions, it included an example already drawn for the participants. This helped to clarify the definitions quickly for most participants and avoided possible confusion. Many participants exhibited a pentagon with sums of 14 on each side. However, exhibiting the pentagon with sums of 14 is not necessary for the proof that 14 is the lowest possible sum, since the question did not specify that such a sum could be attained. The proof for this question requires only that one prove sums of 13 or less are not possible in this set-up.

A proof for this question could have been completed in several different ways. One such way could be accomplished through equations involving the total overall sum of the pentagon, with the insight that the vertices count twice in this sum and that the smallest numbers should therefore be used on the vertices to minimize the overall sum. Through this reasoning, it can be shown that the lowest potential sum is 14. Participants could also argue that sums of 12 or less are not possible since 10 must be placed somewhere on the pentagon within a sum with two other distinct numbers. This forces the lowest potential to be 13, involving the sum of 10 with 1 and 2. From there, one could prove, via diagrams or tables, that 13 is not attainable because of the choices that are forced to be made in the combinations.

A total of 16 participants attempted this question, 9 from MATH 305 and 7 from MATH 406. Of those who attempted this question, 11 were successful: Lisa, Ellen, Beth, Lily, Rick, and Andy from MATH 305; and Sam, Maggie, Paul, Sandy, and Vicki from MATH 406. However, one student from each course did not complete an accurate proof but was unaware that their arguments were not accurate and felt that they had done what was necessary for a proof.

The successful participants used a variety of ideas to finish the proof of this question. Constructing a proof via diagrams was the most popular technique among these participants. Most of those who used this idea did so by first demonstrating that 14 could be completed, and then moved on to show, via an argument for placing the combinations of sum 13 on a pentagon, that 13 was not possible. They also concluded that any sum less than 13 was not possible since 10 needed to be placed with two other numbers, and the minimum sum from this would be 13, from the combination 10, 1, 2. Lisa, Lily, Maggie, Sandy, and Rick were able to complete the majority of the argument in this manner.

Lisa first searched for sums of 14 on the given example pentagon. When this was not productive, she quickly switched to listing sets of three numbers whose sum was 14. Her work was organized and she rewrote her list and lined up the numbers in common between the sets. In this way, she had actually built the pentagon for 14, but without the drawing. Lisa then placed the combinations on a pentagon to complete this portion of her proof. She recognized that she was not finished and continued, noting that 13 was the smallest possible sum to be checked due to the lowest combination for 10 being 10, 1, and 2. Using the same system as she had for 14, Lisa showed that 13 could not be completed, however she did not seem completely satisfied with her proof.

Lily clarified several portions of the question before beginning her observations that numbers on the vertices would be used in the sum for both sides and so the larger numbers should be placed on the edges rather than the vertices. She continued to search for other patterns on the example given, and then began to place numbers on a pentagon for 14. She systematically worked through many attempts, changing positions of numbers and other options along the way. After finding the pentagon for 14, Lily knew that she was not yet finished with the proof. Like Lisa, she described why 13 was the only other sum that needed to be checked and used the same process that she had for 14 to show that 13 was not possible.

Maggie also had difficulty starting her work, first verifying several ideas before being prompted to describe her thoughts. Similar to Lily, Maggie decided that 10 should be placed on an edge before beginning her work on the pentagon itself. She then worked on the pentagon and found the correct arrangement on her first attempt. She had noticed along the way, as well, that all the larger numbers should be placed on the edges, rather than the vertices. When Maggie struggled to describe her proof, the researcher prompted her to consider whether she could attain sums less than 14. Maggie then described that 13 was the lowest to be checked and began to prove that 13 was not possible. She also used the same process as she had for 14 and quickly finished the remainder of her proof.

Like others described previously, Sandy also struggled to understand the statement of the question when she first began her work. She then attempted to use the example pentagon to find her pentagon for 14, first using the same placement of vertices as was given. However, she quickly discovered that this was not feasible and started over on another pentagon. Like Lisa, Sandy worked to write out sets of three numbers whose sum was 14 separately on her paper. She also lined up the common numbers in each set and knew that these would represent vertices. She then transferred her results onto a pentagon. Sandy, like others, described that 13 was the lowest number to be checked and used the same ideas as she had for 14 to prove that 13 was not possible.

Unlike the other participants already mentioned, Rick struggled for the entire first interview to prove Question 1. He spent time verifying several ideas as well, and began his work originally by considering the total sum of the entire pentagon. He became stuck in this idea and worked for several minutes trying to determine how the overall sum could be reduced to produce individual sums of 14 on each side. Finally, he broke free of this line of thought and began to examine the combinations of numbers giving sums of 14. He worked independently of a pentagon, like Lisa and Sandy, and listed his combinations to the side. However, Rick worked from combinations involving 1 up to larger numbers. He was the only one to do so. This caused him difficulties since the number of combinations for these is not as limited as they are for the larger numbers. After several random attempts, and also some searching for patterns as he went, Rick was finally able to complete his list and fill in a pentagon with sums of 14. He knew that he was not finished. The remainder of the interview, Rick struggled to find patterns and relate equations to his pentagons. He eventually was able to determine that 13 was the lowest sum to be checked and was also able to prove that 13 could not be possible, though this did take the remainder of the interview to complete.

Ellen also began by exhibiting a pentagon with sums of 14. She made a plan to place the smallest numbers on the vertices immediately and linked her idea to that of optimizing her choices. She then worked systematically through the choices of numbers and found the pentagon for 14. Ellen described her work, but then turned to equations to prove that all sums of 13 and below were not possible and did not return to diagrams to complete the remainder of her proof.

Beth and Andy began their search by looking for a pentagon for 14, but in the process found the proof that 13 was not possible before completing the arrangement for 14. Beth used the entire initial interview to work through this proof. She first attempted to find sums of 14 on the pentagon, searching for patterns in the example given as well as discovering the idea that 10 must be placed on an edge rather than a vertex. She drew several pentagons, making a separate attempt on each, until she became frustrated. At that point, she saw that 13 would be the only other sum to be checked and redirected her work to searching for 13. Eventually, Beth was able to make the argument that 13 was not possible and find the arrangement for 14, but only after a long, yet organized, search through all possible combinations for each sum.

Andy, however, immediately realized that all the larger numbers should be placed on the edges. He worked through the possible sets of three numbers summing to 14, but did so much more quickly than Beth had. He stopped this search when he felt that he had gotten too far away from the question being asked, which was when he decided that the smallest possible sum to be checked would actually be 13. Andy was able to work through the sets for sums of 13 and the argument that 13 would not be possible. He then returned to his attempts at finding 14 and quickly finished that pentagon as well.

Vickie, unlike the other participants already mentioned, did not draw a diagram for 14 at all. Instead, she first proved that 13 was not possible by examining a pentagon, and then showed that all sums less than 13 were not possible by the argument that 10 must be used in some combination. Vickie was able to determine that she was finished without exhibiting a pentagon with sums of 14. However, she did make the comment upon reflecting on her work that she may also have wanted to exhibit such an arrangement.

The remaining successful participants did not rely on a pentagon for the main portion of their proof. Instead, at some point, they constructed equations involving the total sum, and found a proof by optimizing this equation using the lowest five numbers as vertices. As mentioned, Ellen used such equations in her proof, but only after finding a pentagon with sums of 14. Sam completed a similar proof with equations as Ellen had, and was able to do so prior to finding the arrangement for 14. Sam's proof was complete before he had exhibited the existence of 14, which he was aware of. At the request of the researcher, though, he produced the pentagon for 14 quickly. Both Ellen and Sam mentioned that the ideas they used in this question were similar to those in an optimization course, which could have led to the immediate observations by both that the lowest five numbers should be placed at the vertices in order to minimize the sum.

Paul also desired to use only equations, similar to those used by both Sam and Ellen. However, after producing the equation for the overall sum, he momentarily misunderstood the idea that all sides would have equivalent sums. The researcher asked him to find a pentagon for which the sums would be less than 14. After Paul had drawn such a pentagon, with sides having unequal sums, the researcher helped him to understand this error by viewing the example given. At that point, Paul was able to see that he had actually found the proof and finished his arguments via equations.

Looking at the individual strategies that were used, there was no strategy that was a distinct marker for success. In other words, no strategy was used by all successful participants, but not used by any of the unsuccessful participants. However, there were two main points related to the actual proof that were unique to only the successful participants and used by at least 10 of those participants, which were being able to recognize their own valid proof, and proving that a pentagon with sums of 13 could not exist.

The tables used throughout this section contain a list of various strategies, key ideas, and difficulties experienced by the participants. Each column represents one participant and each entry contains the frequency of use of each strategy for that participant. For example, the first entry in Table 1 under Lisa's column, in the row marked *Read the question*, represents that Lisa read the question once during her work on Question 1. Paul, however, read the question aloud or silently six times during his work, denoted by the number 6 in the first row in Paul's column. Table 1 lists those strategies used and difficulties experienced by at least 10 of the successful participants. Also of interest are those strategies, etc., used by most of the successful participants. Table 2 lists strategies used and difficulties experienced at least half of these participants, but have not already been listed in the previous table. Table 3 lists strategies, etc., used by only 3, 4, or 5 of the successful participants. Lastly, Table 4 lists strategies used and difficulties experienced by only successful participants.

Table 1

Question 1: Strategies used by at least 10 successful participants

Strategies	**Lisa**	**Ellen**	**Beth**	**Lily**	**Sam**	**Maggie**	**Paul**	**Sandy**	**Rick**	**Vicki**	**Andy**
Read the question	1	1	1	1	2		6	2	2	1	1
Work backwards from 10 down	2	1	3	2	1	2		2	1	1	3
Draw pentagon	2	1	5	6	1	3	1	3	6	1	2
Fill in numbers on pentagon	3	1	5	6	1	2	1	2	6	1	3
Make a plan	1	1	6		2	1	2	1	1	1	1
Make subgoals	1	1	7	1	3	1	4	1	4	2	4
Monitor work	3	6	15	4	2	3	3	3	9	1	4
Verbalize ideas	1	1	7	2	2	2	2	1	2	1	1
Recognize valid proof	1		2	1	2	1	2	1	1	1	2

Note. The values represent frequency of use of each strategy per participant.

Table 2

Question 1: Strategies and difficulties of 6-9 successful participants

	Lisa	Ellen	Beth	Lily	Sam	Maggie	Paul	Sandy	Rick	Vicki	Andy
Strategies											
Verify information	1	3	1	2		3		1	11		1
Look at given,16	1	1		1			2	1	4		1
Look for 14 pentagon	1	1	3	4	1	1		1	2		2
Look for 13 pentagon	1		2	1		1		1	1	1	1
Make table/list	2	2	9	2			1	3	1	1	
List numbers adding to 14	3	1	3	1				1	2		3
List numbers adding to 13	1		2					1	4	1	1
Recognize potential guess and check proof		1	4	2			1		2		1
Use systematic check to prove	2		2	1		2		1	2	1	2
Organize work	1		4	1				2	2	1	1
Label figures		1				1	3	2	1	1	1
Write out ideas		1	2	2	2		2	2	3	1	
Acknowledge known / to show		1		2	1	1			1		1
Correct a previous error	2	1	1			1	2	1	2	2	
Redirect		1	2	3			1		1		1
Difficulties											
Computational errors	1	1	1				1		2	1	
Receive help	1	1	7			1	1		7	2	
Difficulty writing formal proof	1	1	1			1	1		3		
Prompted by Researcher		1	10			3	7	1	11	1	

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 3

Question 1: Strategies and difficulties of 3-5 successful participants

	Lisa	Ellen	Beth	Lily	Sam	Maggie	Paul	Sandy	Rick	Vicki	Andy
Strategies											
Look for patterns	1	1	4						6		
Develop equations		2		1	3		4		3		
Work with equations					2		4		2		
Recognize potential proof by systematic checking			2	1					1	1	1
Use guess and check to prove			3	1					1		
Prove with equations		1			1		1				
Recognize lack of proof	1	1		1					1		
Link to other parts of the question			2	1				1	1	1	
Difficulties											
Unable to recall previous work			3				1		4		
Error in understanding the question			1			1	1		4	1	
Compare to puzzle			2	2		2					
Opinion of own abilities low		1	2				2		14		

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 4

Question 1: Strategies and difficulties of only successful participants

	Lisa	Ellen	Beth	Lily	Sam	Maggie	Paul	Sandy	Rick	Vicki	Andy
Strategies											
Understand the question			3								1
Look, in general, for less than 14							1				
Look for less than 13									1		
Work forwards from 1 up									2		
Visualize				1							
Recognize potential proof by systematic checking			2	1					1	1	1
Recognize potential proof by contradiction					1		1				
Recognize potential proof with equations							1				
Recognize valid proof	1		2	1	2	1	2	1	1	1	2
Link to other parts of the question			2	1				1	1	1	
Difficulties											
Viewed as problem not proof							1		1		
Over-emphasis on procedure			2						2		

Note. The values represent frequency of use of each strategy or difficulty per participant.

All 5 unsuccessful participants began by at least attempting to find a pentagon with sums of 14. Shelly was able to find this pentagon, and knew that she needed to then show that 13 was not possible, but had difficulty completing that portion of the proof. Jon attempted to find the pentagon for 14, but without success. He then essentially found the entire proof for the question using equations and considering the average sum, but he was unable to recognize this proof. He went on to examine 13, but was not successful in proving that it would not be possible. Katy also attempted to find the pentagon for 14, and also included work with equations involving the parities of numbers within the individual sums, but she was unable to form a proof in this way. Amy and Shaun both spent all of their time on this question searching for a pentagon for 14, but were unable to find such a pentagon or to move on to other ideas for the proof.

The strategies used by the unsuccessful participants overlapped with those used by successful participants to some degree. However, there were some strategies, and difficulties, that were unique to this group, see Table 5.

Table 5

Question 1: Strategies and difficulties of only unsuccessful participants

	Shelly	Jon	Amy	Shaun	Katy
Strategies					
Desire to prove to oneself		1			
Convinced that the statement is true	1	1			
Difficulties					
Cannot find pentagon with sums of 14			1	1	1
Unorganized/random search			2	1	1
Stuck with no new ideas - unable to move on	1	1	1	1	2
Convinced finished when not actually finished	1				
Over-emphasis on proof as a certain structure		1			
Over-emphasis on finding 14			1	1	
Work stopped by participant		1			1

Note. The values represent frequency of use of each strategy or difficulty per participant.

As can be seen in other questions as well, participants had difficulties proving a statement after they were convinced that it was true. This happened in two instances for Question 1. It can also be seen from these strategies that one vital element to finding the pentagon for 14 was the use of a systematic technique in searching through the possible numbers. Amy and Shaun searched for a pentagon for 14, but struggled when their searches were unorganized and too random to keep track of. They were unable to continue past this search in their work. Katy was also not able to find the correct pentagon, however she did understand that 13 was the lowest option to check. Unfortunately, she was unable to complete the argument that 13 would not be possible. Even though Shelly and Jon were able to find a pentagon for 14, and recognized that 13 was the lowest sum to be checked, they were also unable to complete all ideas necessary for the entire proof of Question 1.

There were many strategies used and difficulties experienced in common among at least half of the unsuccessful participants. Table 6 contains those items experienced by 3, 4, or 5 of the unsuccessful participants. Of the strategies, all were also used by some of the successful participants. This indicates that none were clearly cumbersome to the process of proof writing, but they also cannot be considered indicators of success. Unsuccessful participants used strategies that led to some of the proof, but they were unable to complete the rest of the proof that was needed. While there are many strategies in common, as mentioned above, the participants still struggled with different portions of the proof. As an additional note, those strategies, etc., experienced by only 1 or 2 of the unsuccessful participants are given in Table 7.

Table 6

Question 1: Strategies and difficulties of at least half of unsuccessful participants

	Shelly	Jon	Amy	Shaun	Katy
Strategies					
Read the question	2		3	3	1
Verify information	3	1	2	2	3
Look at given, 16	2			3	2
Look for 14 pentagon	1	1	2	1	1
Work backwards from 10 down	2	1			1
Make table or list	2	2	4		
Draw pentagon	1	3	3		
Fill in numbers on pentagon	2	3	6		2
Look for patterns	4			5	1
Develop equations	1	1	1		2
Work with equations	1	1	1		2
Use guess and check to prove	1		2		1
Make a plan	1			2	1
Make subgoals	3	1		1	1
Monitor work	3	3	3	4	6
Label figures	2			2	2
Write out ideas	1	2		1	1
Verbalize ideas	1	2		1	2
Acknowledge known / to be shown	1	1	1	2	
Correct a previous error		1	1		2
Redirect	1	1			1
Recognize lack of proof	1	1		1	1
Link to other known ideas	1		1	3	1
Difficulties					
Cannot find pentagon with sums of 14			1	1	1
Unorganized/random search			2	1	1
Stuck with no new ideas - unable to move on	1	1	1	1	2
Unsure of where to go next			1	2	1
Difficulty writing formal proof	1	1			1
Prompted by Researcher	4		3	2	2
Opinion of own abilities low		1	1	1	4

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 7

Question 1: Strategies and difficulties of 1 or 2 unsuccessful participants

	Shelly	Jon	Amy	Shaun	Katy
Strategies					
Desire to prove to oneself		1			
Look for 13 pentagon	2	1			
List numbers adding to 14		1			
List numbers adding to 13		2			
Recognize potential guess and check proof				1	
Use systematic check to prove	1	1			
Use equations to prove		1			
Organize work	2				
Link to MATH 305		1		1	
Convinced that the statement is true	1	1			
Difficulties					
Unsure where to start			1	1	
Believe that 6 is lowest possible sum	1				
Unable to recall previous work	3		3		
Error in understanding the question		1			3
Computational errors		1	2		
Receive help		1			
Stuck in search for pattern				1	
Stuck in search for equations					1
Once result is obvious, struggle for proof	2	1			
Convinced finished when not actually finished	1				
Over-emphasis on proof structure		1			
Over-emphasis on finding 14			1	1	
Unable to recognize proof as valid		2			
Desire another proof even with valid proof		1			
Unaware of what would constitute a proof			1		
Compare to puzzle		1			
Appear distracted by the interview situation					1
Work stopped by researcher			1	1	
Work stopped by participant		1			1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Question 1: Individual Strategy Use

With the statement of Question 1, it is clear that many participants would choose to draw pentagons and search for the requested sums (see Table 8). As mentioned above, all successful participants except Paul and Vicki at some point constructed a pentagon with sums of 14. For most of these participants, this observation came before the remainder of their proof and seemed to show them the key ideas necessary to develop the remainder of their arguments.

Table 8

Question 1: Strategies representing participant work on pentagons

Strategies	Total Frequency	Total Count	Total Count Successful
Look for 14 pentagon	22	14	9
Look for 13 pentagon	12	10	8
Look, in general, for less than 14	1	1	1
Look for less than 13	1	1	1
Work backwards from 10 down	22	13	10
Work forwards from 1 up	2	1	1
Visualize	1	1	1
Make table or list	29	11	8
List numbers adding to 14	15	8	7
List numbers adding to 13	12	7	6
Draw pentagon	38	14	11
Fill in numbers on pentagon	44	15	11
Work systematically through the choices	23	12	10

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Two unsuccessful participants, Shelly and Jon, also found this pentagon. However, Shelly did so with only one attempt after a key observation and therefore did not need to work through other choices. However, both she and Jon were unable to fully develop the ideas needed to prove that a pentagon with sums of 13 would not be possible. All participants mentioned here were able to find a correct pentagon through a systematic search of all of the choices from 10 down to 1. However, Rick was the only participant to find the pentagon for 14 by working from 1 up to 10. He struggled a great deal due to this since the choices were not as limited and much more difficult to work through.

An additional participant, Katy, did work from 10 down to 1, but was not systematic in her choices within this structure and was therefore unable to show that 14 existed or to finish her proof. Also, Katy was the only participant who worked with pentagons but did not draw new pictures to organize her work. Instead, she chose to work on the pentagon that was given.

Several participants also developed equations during their work (see Table 9). As mentioned, only three participants used these equations to reach a successful complete proof. However, the use of equations did not seem to derail thinking as it did in some of the other questions. Here, only one student, Rick, truly seemed to focus so much on manipulating the equations that he struggled to move to new ideas. For the unsuccessful participants, the use of equations did not aid in their proof, nor did it distract them.

Table 9

Question 1: Strategies representing participant use of equations

	Total Frequency	**Total Count**	**Total Count Successful**
Strategies			
Develop equations	18	9	5
Work with equations	13	7	3
Develop equations with vertices counted twice	5	5	4

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Other proof techniques that arose during this question were guessing and checking to find the pentagons desired, systematic checking of all possible combinations of numbers to find the desired sums and pentagons, proof by contradiction, and as mentioned, proof using equations. As can be seen in Table 10, many participants were able to use these methods to develop at least a portion of the proof, but did not directly address a plan to do so ahead of time. Several commented on the inefficiency of guessing and checking, and made a specific goal to be organized and systematic in their further attempts to find a particular pentagon. These participants were more focused and able to stay on task than others who did not make such statements and seemed to only stumble upon the correct pentagon or a method to find such a pentagon.

Table 10

Question 1: Strategies representing participant recognition and use of proof techniques

Strategies	Total Frequency	Total Count	Total Count Successful
Recognize potential guess and check proof	12	7	6
Recognize potential proof by systematic checking	6	5	5
Recognize potential proof by contradiction	2	2	2
Recognize potential proof with equations	1	1	1
Use guess and check to prove	9	6	3
Use systematic checking to prove	15	10	8
Use equations to prove	4	4	3

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

With no particular strategy giving a guarantee of success, it is of particular interest to note the self-regulation strategies found in Table 11. As mentioned previously, many strategies were only truly helpful when participants were able to plan ahead to use these ideas and to keep in mind their overall goals, such as in the case of using equations and systematically checking potential combinations for pentagons. Those participants who were able to stay focused on the question itself and recognize not only their successful work, but also the remaining work to be done were better able to make further plans to finish their proof. Also previously mentioned is the organization that led to discoveries, such as drawing new pentagons to keep information clear and easy to read later when needed. This aided in the ability to correct errors as they arose, rather than those errors leading to an inability to finish the remainder of the proof.

Table 11

Question 1: Self-regulation strategies in use by participants

Strategies	**Total Frequency**	**Total Count**	**Total Count Successful**
Make a plan	21	13	10
Make subgoals	35	15	11
Monitor work	72	16	11
Organize work	14	8	7
Label figures	16	10	7
Acknowledge known / to be shown	12	10	6
Correct a previous error	16	11	8
Redirect	12	9	6
Recognize valid proof	14	10	10
Recognize lack of proof	8	8	4

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Most participants, 15, were able to make a plan for their work at some point during Question 1, but not all were able to follow through with this plan as some were unable to finish the proof. Of these participants, all but one was able to monitor their work to some extent towards the goal or plan. However, only 3 of the 5 who did not complete the question were able to redirect their work when it was necessary to do so to finish the proof.

One key idea specific to Question 1 that emerged frequently during the search for a pentagon was finding that the larger numbers should be placed on the sides with vertices labeled as the lowest five numbers, which 10 students were able to notice. Most successful participants made this observation and were able to use it to construct a correct pentagon. However, Lisa, Beth, and Vicki were aware that 10 needed to be placed on an edge, but did not generalize this conclusion. Shelly and Jon also noticed the generalization, but, as mentioned, were unable to form a proof. Paul did not exhibit or attempt to exhibit a pentagon with sums of 13 or 14 at all, but in his proof via equations noticed that the smaller numbers were those you would like to be doubled and so observed that they should be placed on the vertices.

The main difficulties encountered for Question 1 included the previously mentioned unorganized or random search, an inability to recall previous work, and a struggle to formally write a proof beyond the verbal or partially written argument. Of the 5 participants who were unable to complete a proof for this question, Shelly, Jon, Amy, Shaun, and Katy, 3 were unable to find a pentagon with sums of 14 due to a random or unorganized search, 2 were unable to recall previous work, all 5 became stuck with no new ideas, and 2 had great ideas at some point but were unable to express them formally.

Question 2: Part a

*We call a positive integer N a 4-flip if 4*N has the same digits as N but in reverse order.*

a) Prove that there are no two-digit 4-flips.

Question 2 required participants to work with a completely unknown definition. It challenged them in different ways than the other questions because it did not have an obvious visualization nor did it include an example to work from. This question further challenged some participants due to the need to prove that something did not exist, which again pointed out the issue of proving something that seems obvious or trivial as well as something in the negative.

A proof of Question 2 could have taken the form of an exhaustive search through all possible options, either with or without a preliminary restriction to only the numbers 10 through 24. Another valid proof could be a systematic check of the possibilities for each digit of the number, with restrictions based on the definition of a 4-flip and the characteristics of integers. A third option would be the development of equations assuming a 4-flip did exist, solving for the individual digits, and reaching a contradiction. Finally, a proof could also consist of breaking the larger interval into smaller intervals and proving there exists no 4-flip within each smaller interval, which were typically intervals of 10 numbers each for this portion of the question. All four of these options were used by participants in this study, each to varying degrees of success. Some participants combined these methods; for example, using some restrictions on the possibilities within intervals and then checking the remaining numbers by hand.

A total of 17 students attempted this portion of the question, 9 from MATH 305, and 8 from MATH 406. Of these, 11 students were able to complete a valid proof, Lisa, Shelly, Jon, Beth, Lily, Sam, Julie, Maggie, Paul, Vicki, and Jill. A proof by exhaustion after limiting the options to 10 through 24 was the most common proof for this portion of the question. Lisa, Jon, Lily, Julie, Vicki, and Jill all completed their proof in this way.

Lisa first looked at an example of both a three-digit and a two-digit number, she then drew a visual with equations to represent flipping the number when multiplied by 4. She did not work further with this equation, but rather understood immediately that she could limit the possibilities to numbers less than 25. She was aware that a proof by exhaustion would be valid and completed this proof by listing all of the possibilities in an organized manner and checking each.

Jon also first looked at a three-digit example. He then eliminated all the repeating digit two-digit numbers, such as 22, 33, etc. He looked at a two-digit example and, like Lisa, wrote equations to represent the number being flipped when multiplied by 4. Jon was able to limit his possibilities just as Lisa had, and understood that a proof by exhaustion would be valid. He listed all the numbers and showed that none were 4-flips. However, unlike Lisa, he was not completely satisfied with his proof. Jon tried to further develop equations to form a proof. The researcher helped him to develop these, but Jon was unable to work through them to a second proof.

Lily looked at both a three-digit and then a two-digit example when starting her work as well. She immediately limited her possibilities to those less than 25, and identified a potential proof by exhaustion. Listing the possibilities and checking each option, Lily showed that there were no two-digit 4-flips. She had expressed earlier in her work that she was searching for another way to prove this portion of the question without the use of exhaustion, but seemed to understand when she had finished her list that she had finished the proof.

Like the other participants already mentioned, Julie also looked at both three-digit and two-digit examples before moving on in her work. She then limited her possibilities, just as the others had, to those less than 25. She made a list of those numbers and showed that none were 4-flips. Like Lily, Julie did not form any visual representation of the flipping of the digits during her proof of this portion of the question.

Vicki was another participant who looked at both a three-digit and two-digit example when beginning her work. After she understood the definition, she considered working with the individual digits before moving to a view of the number as a whole. Vicki then determined that the largest value for N would be 24. She listed the possibilities and showed that none were 4-flips. Similar to others, though, Vicki was not completely comfortable with her proof and desired another proof without listing all the options. However, she was unable to find an alternative proof of this portion of the question.

Jill's work was different than most others who used a proof by exhaustion. She began her work by trying to find an example of a 4-flip that would exist. Knowing that there would not be a two-digit 4-flip, she focused her search on three-digit numbers. She tried several examples and then tried to examine the individual digits and impose restrictions on these. Jill was able to rule out all numbers in the range 100-199 since $4N$ would need to end in 1, but obviously could not since it would be an even number. She then moved to numbers in the 200s, finding that the last digit must be 8 or 9. Eventually, she became lost in her search and opted to abandon this effort. Instead, Jill decided to represent the digits of a two-digit number in an equation to note the flip. When this effort also did not pan out, she switched to examining limitations on the numbers as a whole. However, she was unable to tie in her work in the three-digit case to help her here. She was able to discover that the upper limit was 24 and knew that a proof by exhaustion would work. Like other participants, though, Jill was uncomfortable with this as a proof and so sought other ideas. As she looked at several examples in this range, searching for a pattern or further idea, she eventually wrote out all possible options and showed that none were 4-flips. Jill recognized her proof, but still desired another. However, she was unable to find any new ideas. Jill's difficulty with this question extended into part b, and she was unable to work on any other questions during her interview.

Shelly and Beth also first looked at several examples before beginning their work. However, both of these participants ended their work by breaking the numbers into smaller intervals of 10 numbers each and proving that each interval was not possible.

They each also discovered the limit to less than 25 during the search through the smaller intervals.

Shelly began with several two-digit examples, and worked to clarify her understanding of the question. She then discovered that the range from 10 to 19 would not be possible since 4*N* was even, just as Jill had found in the three-digit case. Seeing the breakdown into groups of 10, Shelly moved on to the 20s. Unfortunately, she was unable to see the correct justifications that she found during her work in this interval. Shelly had stated that only ending digits of 3 or 8 would result in 4*N* ending in 2, and also that 25 times 4 was equal to 100. She checked the numbers 23 and 28, but was not convinced that this eliminated the interval from consideration. Shelly struggled several times to recover her thoughts within this interval, but was never fully convinced of her arguments. Eventually, Shelly was able to prove that the 30s were not possible, limit the possibilities to less than 25, and test the remaining numbers. She remained unsure of her proof, and it was not clear that she understood she had finished the proof overall.

Beth was another participant that spent an entire interview on only Question 2, just as Jill had done. During this time, she began with a four-digit example, $N = 1234$. She then drew several visuals and equations representing the number being flipped. She looked at two examples of two-digit numbers before being work in three-digits. Just as Jill had done, she was attempting to find an actual 4-flip, and assumed she would be able to find such a number within the three-digit options. Unlike Jill, however, Beth switched her line of thinking relatively quickly and focused back on two-digit numbers. She began to look through examples, starting in the 10s and checking only the first few numbers in each set of 10. She moved from the 30s directly to the 90s before realizing that these values of *N* were too large. Beth was eventually able to conclude that her upper limit was 25. After writing down several multiples of 4, she concluded that the interval 10-19 was impossible. Beth finished her proof by checking the remaining five numbers and showing that there were also not 4-flips. However, like many other participants, she was not completely satisfied with her proof but was unable to find another proof.

Sam also worked with smaller intervals of numbers in order to prove Question 2. Unlike Shelly and Beth, though, he was able to first determine that he could limit the possibilities to only those less than 25. This sped his progress through the question a great deal compared to Shelly and Beth. Sam's work originally started by examining the two-digit numbers divisible by 4, but he had misunderstood that *N* would be called the 4-flip, rather than 4*N*. Once he understood this idea, Sam restarted his work, looked at a few examples, and determined the upper limit of 25. He soon realized that the entire interval 10-19 was impossible, with the same reasoning as already mentioned, checked the remaining numbers, 20-24, and showed that they were not 4-flips. He understood that this constituted a proof and seemed to be satisfied with this result.

Not all participants looked first at examples or specific numbers. Paul skipped this idea and instead moved directly to forming an equation. He used these equations to consider bounds on the individual digits. After finding an equation he felt he could work further with, Paul decided that he would plug in specific options for each digit and solve to find a contradiction. As time was running short, the researcher attempted to end Paul's work. It was at that moment that Paul realized that only the options 10 through 24 were available and asked to continue his work. He used his equation to eliminate several of these numbers and said he would only need to check the remainder to prove that there were no two-digit 4-flips. Paul's use of equations was similar to his work in other questions and it was not a surprise to see the use here without visuals or examples.

Maggie also dealt first with equations in her work. However, her ideas were not as clearly defined and she struggled to understand the question and discover a valid proof. Her work began with a visual representation of the digits and how they would interchange places. She was then able to develop the same equation as Paul had, involving the multiplication by 4 and representing the placement of the first digit of the number by multiplication by 10. Maggie struggled, though, to interpret what these equations could mean for her proof and how to use them further. She further wrote equations in the same form for any generic length potential 4-flip, but was still unable to relate these to a proof. Maggie used this view of the numbers to consider the individual values of the digits. She was then able to determine that the first digit could only be 1 or 2, similar to others limiting the possibilities to less than 25. Her work concluded with a proof by cases, where she considered each of these options and ruled them out in turn.

Unlike Question 1, there were very few common strategies. This is not due to a lack of commonality among successful participant work, but rather to a lack of variety in strategies in any individual participant's work. That is, the first several participants mentioned above used very few strategies when approaching this question, as they were able to find a proof quickly using exhaustion of all possibilities. Those strategies common to at least 9 of the successful participants are listed in Table 12. Some participants with frequencies listed in the *Use proof by exhaustion* category did not complete the entire proof in this manner, but at least some portion of their proof involved a brute force check of the remaining options. A listing of the strategies common among 6, 7, or 8 successful participants can be found in Table 13. The final table for successful participants, Table 14, contains the strategies seen only among successful participants. This last table is not a surprising list, given that most items contained there led directly to the proof itself and so clearly would not have been experienced by the unsuccessful participants.

Table 12

Question 2 (part a): Strategies used by at least 9 successful participants

	Lisa	**Shelly**	**Jon**	**Beth**	**Lily**	**Sam**	**Julie**	**Maggie**	**Paul**	**Vicki**	**Jill**
Strategies											
Read the question		2	1	1	1	1	2	1	1	2	5
Look at examples - two-digit	2	2	1	5	2	1	2			2	2
Recognize potential proof by exhaustion		1	1	2	1	3	1	1		2	2
Use proof by exhaustion	1		1	1	1	1	1	1		1	1
Make a plan			1	3	1	1	1	1	1	2	2

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 13

Question 2 (part a): Strategies used by 6-8 successful participants

	Lisa	**Shelly**	**Jon**	**Beth**	**Lily**	**Sam**	**Julie**	**Maggie**	**Paul**	**Vicki**	**Jill**
Strategies											
Look at examples - three-digit	1		1	3	1		1			1	2
List 10-24	1		1		1		1			1	1
Look at individual digits		1	1	2					2	2	4
Develop equations to represent digits	1		4	3				1	2		2
Constraints possibilities as a whole		1		2	1	1	1		1		2
Monitor work			1	5			1	2	2	1	4
Organize work				2		1	1	1		1	1
Recognize valid proof	1			1	1	1		1	1	1	1
Link to other known ideas			1	1		1		1	2		1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 14

Question 2 (part a): Strategies used by only successful participants

	Lisa	**Shelly**	**Jon**	**Beth**	**Lily**	**Sam**	**Julie**	**Maggie**	**Paul**	**Vicki**	**Jill**
Strategies											
List 10-24	1		1		1		1			1	1
Recognize potential proof by cases								1	1		1
Use proof by cases									1		
Recognize valid proof	1			1	1	1		1	1	1	1
Link to other parts of question/proof/work				1							

Note. The values represent frequency of use of each strategy or difficulty per participant.

The work of unsuccessful participants varied a great deal more than the work of the successful participants. Those strategies and difficulties experienced by only the unsuccessful participants are listed in Table 15. None of these are unexpected or surprising in any way. However, what is unique in this question is that the work of the unsuccessful participants was so diverse. In other questions, participants had commonalities in their difficulties and there were generally a few key points that were left unaddressed, leading to their difficulties.

Table 15

Question 2 (part a): Strategies and difficulties of only unsuccessful participants

	Ellen	Sandy	Rick	Amy	Shaun	Katy
Strategies						
List of numbers tried			1	2		
Recognize lack of proof			1	1	1	1
Difficulties						
Difficulty with lack of visuals				1		
Unable to move past examples		1	1	1		
Unorganized/random search		1	1			
Stuck with no new ideas - unable to move on		1	1	1	1	1
Convinced finished when not finished		1				
Unaware of what would constitute a proof		1	1		1	

Note. The values represent frequency of use of each strategy or difficulty per participant.

The only true commonality between all the unsuccessful participants was looking at examples (see Table 16). This is also not surprising since it was seen among most of the successful participants as well. Many wanted to find an actual 4-flip in order to be able to determine how they could prove that no two-digit 4-flip existed. Ellen, Sandy, Rick, Amy, and Shaun all viewed examples first in their work. Ellen and Sandy both went on to develop equations to try to determine a proof. Ellen then listed all numbers divisible by 4, was able to constrain the individual digits, and may have found a proof if she had more time to do so. Unfortunately, time ran short before she could complete her work. Sandy, however, struggled with random examples and an inability to link these examples to the equations she had developed. In this way, she was unable to see the key ideas necessary for the proof.

Table 16

Question 2 (part a): Strategies and difficulties of at least 5 unsuccessful participants

	Ellen	Sandy	Rick	Amy	Shaun	Katy
Strategies						
Verify information		3	1	1	1	2
Look at examples - two-digit	3	4	5	2	1	2
Work from N	1	1		1	1	1
Work from 4N	1	1		1	1	1
Make a plan	2		2	1	3	1
Monitor work	7		4	1	3	1
Redirect	2		3	1	6	2
Difficulties						
Unaware of limit to < 25		1	1	1	1	1
Stuck with no new ideas - unable to move on		1	1	1	1	1
Unsure of where to go next		4	1	1	1	2
Prompted by Researcher		5	2	1	1	1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Rick also struggled with random examples. He was essentially able to determine the upper limit of 24, but did not clearly see that he had done so. His work was marked with moments of forgetting his previous statements and arguments, as well as getting lost in the trial and error process he began.

Amy was also unable to recall her previous work, leading to a lack of proof on her part as well. She listed all possible choices, but was unable to determine the upper limit of 25, and so did not understand that there were only a small number of options remaining to be checked. The interview time ended when Amy was unable to see any other ideas to choose or to determine that the direction she was headed could lead to a proof. She revealed then that she had a learning disability, which made this question particularly challenging.

Shaun was able to recall his previous work, but spent too much time and effort trying to decipher logical notation to prove the statement. This was a common difficulty for Shaun, and he struggled to convince himself that any other type of proof could be valid. He often made his work too complicated and was unable to see the simple brute force proof available for this question.

Katy was the only unsuccessful participant to develop equations first, prior to looking at examples. However, she was still unable to link these ideas together. Katy ended her work when she was stuck and had no further ideas for the proof.

As can be seen in Table 17, Katy struggled in an unorganized manner and so could not find a proof. This table also lists the other strategies used and difficulties experienced by 3 or 4 of the unsuccessful participants. Similar to other questions, participants here had difficulty moving past their concrete examples to a generic idea for the proof, even unable to see the larger picture and limit the choices to only 10 through 25. This caused a great deal of trials. Unsuccessful participants were unable to both consider the details as well as keep the overall goal in mind.

Table 17

Question 2 (part a): Strategies and difficulties of 3 or 4 unsuccessful participants

	Ellen	**Sandy**	**Rick**	**Amy**	**Shaun**	**Katy**
Strategies						
Read the question		1	3	4	2	
Visualize digits and flip	1				1	1
Develop equations to solve for N or 4N	2	2				3
Make subgoals	1		2		2	
Organize work	1		4	1	5	
Recognize lack of proof			1	1	1	1
Difficulties						
Difficulty proving something does not exist		1	1	1		
Unsure of how to start		1	1	1		
Unable to move past examples		1	1	1		
Error in understanding the question		1	2	2		1
Doubt previous work	1	1	1			1
Unaware of what would constitute a proof		1	1		1	

Note. The values represent frequency of use of each strategy or difficulty per participant.

<u>Question 2 Part a: Individual Strategy Use</u>

As already discussed, the use of examples was a very prevalent strategy among all of the participants (see Table 18). For the successful participants, viewing examples led to a greater understanding of the question and the idea of limiting the possibilities to make the proof easier. However, several unsuccessful participants were unable to be organized in their examples and to move past the examples to this larger understanding of the question as a whole.

Table 18

Question 2 (part a): Strategies representing participant use of examples

	Total Frequency	Total Count	Total Count Successful
Examples			
Look at examples - two-digit	36	15	9
Look at examples- three-digit	16	9	7
Look at examples- more than three-digits	5	3	2
Search for actual 4-flip	8	5	3
Generalize from examples	6	6	5

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Another useful strategy was to develop an organized list to help keep track of work already done (see Table 19). For those who used this idea, it was often successful. However, it was not essential to the proof, as several successful participants were able to find a proof without such a listing. The unsuccessful participants who were unable to keep track of their work or be organized in their choices for examples could have benefited from such a list to help focus their thinking and allow them to recall previous work.

Many successful participants also, at some point, developed equations to represent the digits of N and how they could be flipped (see Table 20). This added to their understanding of the question and was usually left there without further work with equations. However, as mentioned, Paul and Maggie were able to find a proof using equations in at least some form.

Table 19

Question 2 (part a): Strategies representing participant use of lists and other visuals

	Total Frequency	**Total Count**	**Total Count Successful**
Make list of those not possible	6	4	3
List 10-24	6	6	6
List of numbers tried	3	2	0
List multiples of 4	3	3	2
Visualize digits and flip	11	8	5

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Table 20

Question 2 (part a): Strategies representing participant use of equations

	Total Frequency	**Total Count**	**Total Count Successful**
Develop Equations			
Develop equations to represent digits	16	8	6
Develop equations- solve for N/4N	11	6	3
Work with N and 4N			
Work from N	10	10	5
Work from 4N	7	7	2
Relate N to 4N	3	3	1

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

For the unsuccessful participants, the use of equations did not pan out as well. Ellen seemed to understand the question better as a result of her equations, as well, and may have had success with this technique if given more time. However, Sandy’s use of equations did not assist her since she was unable to focus and be organized in her examples and was therefore unable to connect the two ideas. Katy also was unable to follow through with her equations to a proof. Both Sandy and Katy made errors in their understanding during work with equations and became stuck in their search here, leading to a lack of proof as they were unable to move past the equations and there misguided ideas stemming from them. Overall, the use of equations seemed to have a positive effect on most participants' work, though it did not always lead to a proof.

Participants who were able to find a proof did so in one of two general ways, as mentioned previously (see Table 21). Either a proof by constraining the possibilities to only 10 through 24 and either checking those in full or in intervals, or by limiting the individual digits and either checking those or making smaller proofs to develop the larger proof.

Table 21

Question 2 (part a): Strategies representing participant use of constraints

	Total Frequency	Total Count	Total Count Successful
Constrain possibilities			
Constraints possibilities as a whole	14	11	9
Constrain individual digits	13	5	4

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Some of the unsuccessful participants were also able to make some limitations to the possibilities, but not enough to narrow down the window appropriately. They were also unable to see the larger picture and understand what they had found. For Question 2, this strategy had extremely good results overall, with much success occurring. This was, in large part, due to the nature of the question and the natural tendency of the participants to reduce the potential numbers of options to be viewed. Those who were unsuccessful were often instead in search of an actual 4-flip and so were unable to understand what could constitute a proof and go in search of this proof by brute force.

The ability to recognize and especially to use proper proof techniques is, of course, essential to the process of developing a valid proof. Table 22 summarizes the abilities of participants to utilize such tools.

Table 22

Question 2 (part a): Strategies representing participants recognizing and using proper proof techniques

	Total Frequency	Total Count	Total Count Successful
Recognizing proper proof techniques			
Recognize potential proof by exhaustion	17	11	9
Recognize potential proof by contradiction	6	6	4
Recognize potential proof by cases	3	3	3
Using proper proof techniques			
Use proof by exhaustion	10	10	9
Use proof by contradiction	3	3	2
Use proof by cases	1	1	1

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

It was again seen in Question 2 that the ability to keep the larger goal in mind and keep careful track of work was absolutely critical to success (see Table 23). This is not to say that all who did so were able to complete a proof. Some participants were only able to do this a portion of the time, and so could not follow through with their ideas and complete a proof.

These are also not always tools that are readily apparent during work. Some participants were able to find a proof and develop their ideas so quickly that it was not clear where such organization and planning could come into play. Still others were unable to verbalize these ideas and therefore it is difficult to judge when planning and monitoring was actually occurring. For those times when the researcher was able to see such strategies, they were extremely helpful to the participants for keeping all the possible examples and notation in view and readily accessible in order for other ideas to bloom.

Table 23

Question 2 (part a): Strategies representing participant use of self-regulation

	Total Frequency	Total Count	Total Count Successful
Make a plan	22	14	9
Make sub-goals	7	5	2
Monitor work	32	12	7
Organize work	18	10	6
Acknowledge known/to be shown	5	4	3
Redirect	23	10	5
Recognize valid proof	8	8	8
Recognize lack of proof	4	4	0

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Question 2, as all questions did, had specific ideas that were key in finding a proof. A summary of the discovery of such key ideas is found in Table 24. Here it again can be seen that the ability to work systematically and in an organized fashion was a very helpful and positive part of participants' work.

Through the use of examples, many participants were able to find the first three key ideas, involving constraining the digits and finding the overall proof for the interval 10 through 19. Finally, being aware of what would constitute a proof was a key for many successful participants that led them to attempting to find constraints on the digits. As mentioned, though, this proof was not always given with confidence and was often accompanied by a desire for another type of proof. This could very well be a product of the type of proofs a student is exposed to. Those participants who referenced number theory were more likely to accept such a proof. Where those in MATH 305, who would not have taken such a course, were more likely to desire a proof with more notation and a general idea rather than brute force proof.

Table 24

Question 2 (part a): Key ideas known or proven

Key Ideas Known/Proven	Total Frequency	Total Count	Total Count Successful
Recognize < 25 due to three-digit limit	13	12	11
Recognize 10-24 as the only options	9	8	8
Prove 10-19 not possible since 4N is even	6	6	5
Know that proving 10-24 will be proof	7	7	7
Eliminate options based on rules	8	7	5
Work systematically through the choices	14	14	11

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Question 2: Part b

Still thinking of 4-flips as we just defined:

b) Prove or disprove the following statement: There are no three-digit 4-flips.

The first extension of Question 2 was to move to the three-digit numbers. Here participants had to choose for themselves whether to prove or disprove this statement. Many participants had assumed in part a that there would be a three-digit 4-flip, but now were unsure when faced with this new question. A proper proof of this portion of the question would be to prove that there are no three-digit 4-flips. The wording confused some participants, but only in whether to say they were proving or disproving. All participants who attempted part b eventually understood that there were no three-digit 4-flips.

To develop a valid proof, one could use equations to solve for individual digits and work their way to a contradiction. A key observation would occur again here, that the possibilities could be limited to 100 through 249. A proof could then consist of breaking this interval into smaller pieces and proving each separately. These separate proofs could take different forms. These will be described throughout this section as each individual participant is discussed. Lastly, a proof could also consist of constraints on the individual digits without equations first, showing that each is restricted to the point that no three-digit 4-flip could exist.

A total of 10 participants attempted Question 2 part b, 6 from MATH 305 and 4 from MATH 406. Of these, Jon, Beth, Sam, Maggie, and Vicki were successful in finding a proof. Each did so in a slightly different way, and all but Vickie did so in a very similar fashion as their work from part a.

Jon ended his work from part a by attempting to solve equations for the individual digits of the potential 4-flip to find a contradiction. While he was not successful in completing his proof in this way, he still began part b with the same ideas. He developed equations representing the digits multiplied by 10 and 100 as placeholders and solved for the middle digit, which he labeled *y*. When considering the other digits, *x* and *z*, Jon eventually found only one solution, $x = 2, y = 2.2,$ and $z = 9.$ He had made in error in believing that the middle digit containing a decimal could be allowed. When he further examined this situation, he decided that this was not possible, but was not certain that he had formed a proof.

Beth also used similar work to that in part a, however not with equations. Her proof consisted of an examination of smaller intervals within the larger restriction of 100 through 249. She first eliminated the interval 100 through 199 since *4N* would need to end in 1, but clearly cannot since it will be even. Again, as in part a, Beth kept careful track of those intervals that she had shown to be impossible through the use of a list to the side of her paper. After checking the numbers 200 through 207, Beth decided to break the remaining numbers into groups of 10. Using her ideas from part a, she determined that *N* must end in either 3 or 8. She was then able to prove that the intervals 220 through 229 and 240 through 249 were not possible by showing that *4N* could not end in two even numbers. Eventually, she was also able to form the arguments that the remaining numbers were not possible.

Sam worked with intervals of numbers after limiting the possibilities to 100 through 249 as well. His work here was similar to that in part a. He also eliminated the interval 100 through 199 with the same reasoning as Beth had. However, he chose to work next with groups of 25 instead of only 10. Sam did so with the observation that the numbers 200 through 224 would be in the 800s when multiplied by 4, where the numbers

225 through 249 would be in the 900s. He therefore checked the numbers 208 and 218, 229, 239, and 249 to complete the proof. Sam's work was completed much more quickly than Beth's had been, however his reasoning ended with brute force, where Beth was able to form a separate argument for each sub-interval.

Maggie and Vicki both used arguments about the individual digits to prove this portion of the question. Maggie began with linking her observations from part a to the appropriate extensions here. She said that the first digit of *N* must be either 1 or 2, and that 1 could actually be eliminated since *4N* could not be odd. Furthermore, the last digit must be 3 or 8 since *4N* must end in 2. Maggie then split her work into two cases, with each representing one of these ending digits. She was able to eliminate the case with ending digit of 3. The remainder of her work consisted of equations used to prove that the second case was also impossible. She did so by completing the multiplication of the digits and considering that the middle digit must remain the same.

Vicki had completed part a using brute force and so needed to develop new ideas to prove this portion of the question. After limiting the possibilities to the interval 100 through 249, she turned to an examination of the individual digits. Like Maggie, Vicki was able to determine that the ending digit of *N* could only be 8. She concluded that the first digit must then be 2. After checking the numbers 208, 218, and 228, Vicki determined that at 228, the first digit of *4N* would be 9 and so she had already reached numbers that were too large. She finished her proof by describing this work in detail.

Again, in this portion of the question it was crucial to limit the possibilities to only those in the range 100 through 249, which all successful participants were able to do (see Table 25). They all examined the individual digits to constrain them, however each did so in a slightly different way. There was also a strong link to the work done in part a, in Vicki's case only to the limit to the proper range, but in the other cases the majority of the work was created by comparison to part a.

Those strategies used and difficulties experienced by 3 or 4 of the successful participants are shown in Table 26. As previously stated, each participant did form a unique proof for this question, however there were many similarities in portions of their work. They were often organized, made a plan of attack for the proof, and were able to systematically work through the options and eliminate as they went. However, many of these strategies were also seen among unsuccessful participants. Very few strategies were unique to successful participants (see Table 27), but the main advantage the successful participants had was the ability to combine the individual strategies and ideas into one larger proof and to maintain an organized method of searching through the options.

Table 25

Question 2 (part b): Strategies used and key ideas known by only successful participants

	Jon	Beth	Sam	Maggie	Vicki
Strategies					
Look at individual digits	1	1	1	1	2
Constrain individual digits	2	1	2	3	2
Monitor work	1	5	2	1	1
Link to other parts of question/proof/work	2	4	2	2	2
Use same method as part a	2	3	2	2	2
Key Ideas Known/Proven					
Recognize < 250 due to three-digit limit	1	1	1	1	1
Recognize 100-249 only options	1	1	1	1	1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 26

Question 2 (part b): Strategies and difficulties of 3 or 4 successful participants

	Jon	Beth	Sam	Maggie	Vicki
Strategies					
Read the question		1	1		1
Look at examples - three-digit	2	6			2
List of possible numbers remaining		3	4	1	
Develop equations to represent digits	2	2		1	
Relate N to 4N	1			1	1
Use portion of proof by exhaustion		1	2		1
Use proof by cases		1		1	1
Make a plan	1	1		1	
Organize work		4	2	1	1
Recognize valid proof		1	1	1	1
Difficulties					
Prompted by Researcher	1	2		2	

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 27

Question 2 (part b): Strategies and difficulties of only successful participants

	Jon	Beth	Sam	Maggie	Vicki
Strategies					
Understand the question		1			
Verify information		1			
Break into smaller intervals		1	1		
List multiples of 4		1			
Use proof by contradiction	1				
Recognize valid proof		1	1	1	1
Difficulties					
Over-emphasis on procedure/details	1				
Over-emphasis on proof structure	2				

Note. The values represent frequency of use of each strategy or difficulty per participant.

There were also several strategies and difficulties unique to the unsuccessful participants (see Table 28). However, none of these were seen among all unsuccessful participants. Just as the successful participants work varied, the work among Lisa, Shelly, Lily, Julie, and Jill varied as well.

Table 28

Question 2 (part b): Strategies and difficulties of only unsuccessful participants

	Lisa	Shelly	Lily	Julie	Jill
Strategies					
Reword question					1
Search for actual 4-flip					1
List of numbers tried			1		
Recognize lack of proof			1		1
Difficulties					
Meeting point idea				1	
Unable to move past examples			1		
Unable to recall previous work					2
Stuck in search for pattern			1		1
Stuck with no new ideas - unable to move on			1		
Convinced finished when not finished	1			1	
Appear distracted by the interview situation		1			
Ran out of time		1			1

Note. The values represent frequency of use of each strategy or difficulty per participant.

All 5 unsuccessful participants attempted to link new work with that in part a, starting with limiting the possibilities. All, except Julie, were able to find the interval to be 100 through 249. Julie incorrectly stated that her interval would be 100 through 333. Shelly and Jill eventually were able to find the right line of thinking, but ran out of time before they were able to finish. Shelly eliminated the interval 100 through 199 and wanted to use a different method than she had for part a to continue. She opted to examine the last digit of *N* and, with work, found that it had to be either 3 or 8. Time ended there, however, and her work could not continue.

Jill had finished the proof to part a using brute force, as did Lisa, Lily, and Julie, and so was forced to find another method of proof for part b. She also turned to examining the individual digits of *N*. After many promptings and other work, she was able to find that the first digit must be 2 and the last must be 3 or 8. Time also ran out for her at this point and she was unable to finish her proof.

Lisa attempted to constrain the individual digits as well. However, she made two errors that blocked any possibility of success. The first error was in believing that the middle digit of *N* must be zero to eliminate problems caused by multiplication carrying over to the first digit. The second error occurred when she incorrectly determined that the first digit must equal the last. This gave the only remaining options being 101 and 202. After trying these, Lisa felt that she had finished the proof, though she clearly had not done so.

Lily was unable to move past the observation that the options were limited to 100 through 249. After finding this limit, she looked at several examples but had no further ideas for the proof and chose to end her work there.

Julie, as mentioned, was the only participant unable to find the correct upper limit of 249. Instead, she believed that limit to be 333. This did not seem to cause her difficulties, though. She next examined the individual digits, as many others had, attempting to further limit the possibilities. Through a search of the ending digits by viewing examples and making general observations, she found the only options to be 3 and 8. She was also able to determine that ending digits of 3 would not be possible. In searching for ending digits of 8, however, Julie incorrectly stated that at the number $N = 208$, she had found a point where $4N$ was, for the first time, larger than the value of *N* flipped. She concluded that this finished the proof. The idea of a meeting point continued in her work on part c of the question.

It is not unexpected that these participants reached many of the same key ideas and used the same strategies as the successful participants had used, since they themselves were successful in part a of the question. However, a combination of errors, misunderstandings, and lack of time contributed to their inability to form a proof in the end. As seen in Table 29, many were organized, had good ideas, and were able to work through at least a portion of the proof. For Shelly and Jill, their ultimate problem was running out of time. Jill had spent too much time on other ideas and was not able to reach her conclusions quickly enough to finish the question. Shelly may have successfully found a proof in time had she not been seeking a new type of proof, and instead had tried to work more from her previous proof. Lisa and Julie made errors that caused them to reach false conclusions, which they believed to be true. Lily was ultimately not able to move past the concrete examples to a generic proof and so could not find any new ideas for this portion of the question.

Table 29

Question 2 (part b): Strategies used by at least half of unsuccessful participants

	Lisa	**Shelly**	**Lily**	**Julie**	**Jill**
Strategies					
Read the question	1			1	1
Look at examples - three-digit			1	3	2
Make list of those shown not possible		1		1	1
Look for Patterns			3	2	1
Look at individual digits	1	1		2	3
List of possible numbers remaining	1	3	1	1	1
Constraints possibilities as a whole		2	1	1	2
Constrain individual digits	2	1		2	1
Use proof by cases	1	1		1	1
Make subgoals	1	1			2
Monitor work	1	1	1		4
Organize work	1	2	1	1	
Link to other parts of question/proof/work	1	2	2	1	2
Use same method as part a	1	2	1	1	2

Note. The values represent frequency of use of each strategy or difficulty per participant.

Question 2 Part b: Individual Strategies

While the use of examples was heavily prevalent in part a of this question, it did not occur often in part b (see Table 30). The only participants who attempted part b were those who had successfully completed a proof for part a, therefore they already had a good idea and plan of what direction to take and did not need to view examples to better understand the definition. Some participants did look at specific examples, within particular intervals or to gain further ideas past their brute force proof from the previous portion of the question, but most used the same type of work and only looked at examples to complete the last few numbers needed for the proof.

Table 30

Question 2 (part b): Strategies representing participant use of examples

Strategies	Total Frequency	Total Count	Total Count Successful
Look at examples - three-digit	16	6	3
Look at examples - >three-digit	0	0	0
Search for actual 4-flip	1	1	0
Generalize from examples	5	3	1
Break into smaller intervals	2	2	2

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

The use of lists and other organizational tools to keep track of intervals did not occur as often for this portion of the question (see Table 31). Unlike part a, a proof by brute force was not reasonable here. Half of the participants used equations for their work, diverging from a check of digits or a need for listing the possibilities. Many of the remaining participants who used lists were able to eliminate intervals in large chunks. Therefore, they did not need as detailed of a list as the previous portion of the question. Furthermore, the immediate understanding that occurred in part b can be seen in the decreased amount of participants who used visualizations of the digits being flipped during their work. Even those who did use this tool seemed to already understand what would occur and were using this merely to link their ideas back to those in part a.

Table 31

Question 2 (part b): Strategies representing participant use of lists and other visuals

	Total Frequency	Total Count	Total Count Successful
Strategies			
Make list of those shown not possible	11	5	2
List of numbers tried	1	1	0
List multiples of 4	1	1	1
Visualize digits and flip	5	4	2

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

As stated, half of the participants developed equations during this portion of the question (see Table 32). Not all did so for their final proof, but the use did not seem to hinder any participant in their work. Jon's focus on the equations was consistent with the other questions he worked on, and there was some confusion for him on whether he was finished, but ultimately he had attained a proof via equations. Maggie was the only other participant to use equations for the final proof, and she did so with ease and understanding.

Table 32

Question 2 (part b): Strategies representing participant use of equations

	Total Frequency	Total Count	Total Count Successful
Strategies			
Develop equations to represent digits	7	5	3
Develop equations to solve for N or 4N	5	3	2

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

The most commonly used strategy occurring in valid proofs was the constraining of digits to a smaller amount of possible options (see Table 33). Most participants, 9 out of 10, at some point at least considered the options of the individual digits and limited these options through logical arguments. Only Jon and Maggie, who used equations to limit their possibilities, did not do so as a whole. This strategy was therefore not essential to the proof, but it did not hinder progress either. For all participants who did limit their choices, this was a positive and productive portion of their work towards a proof.

Work on this portion of the question was very closely related to previous work done during part a. For this reason, there was very little apparent planning or recognition of particular proof techniques, as can be seen in Table 34. As stated, participants used a variety of techniques for the proofs, and even some who were unsuccessful were able to complete portions of the proof using various techniques.

Table 33

Question 2 (part b): Strategies representing participant use of constraints

	Total Frequency	**Total Count**	**Total Count Successful**
Strategies			
Look at individual digits	13	9	5
Constrain possibilities as a whole	11	8	3
Constrain individual digits	16	9	5

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Table 34

Question 2 (part b): Participant recognition and use of proof techniques

Strategies	Total Frequency	Total Count	Total Count Successful
Recognize potential proof by exhaustion	3	3	2
Use partial proof by exhaustion	5	4	3
Use proof by contradiction	1	1	1
Use proof by cases	7	7	3

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

When viewing the self-regulation strategies, Table 35 shows that all participants linked their work to part a, and used similar methods in some portion of their work. All of the successful participants, as well as all but one unsuccessful, were able to monitor their work as well. However, again, all participants attempting this portion of the question were previously successful on part a. Therefore, they understood the question and did not stray into random checking and unorganized searches for clues in the proof. The fact that all participants used prior ideas for their work here is also indicated in the lack of outward planning and the making of sub-goals, since most participants already knew where they were headed and what they were attempting to show.

This portion of Question 2 uniquely viewed previously successful participants being asked to expand their notions within a proof. This caused a few errors and misunderstandings, some of which were unable to be resolved. It also resulted in almost all participants being able to form some portion of the necessary proof. This extension forced participants to move beyond the concrete brute force notions to more generic arguments. Within this process, participants own tendencies towards certain proof styles, such as equations or organized lists, again came forward. For example, Jon felt a valid proof needed to include an argument via equations. Shelly did not want to complete the second proof using ideas from the first since she had eliminated all numbers only in intervals in part a, but without what she considered solid argumentation.

Table 35

Question 2 (part b): Strategies representing participant use of self-regulation

Strategies	**Total Frequency**	**Total Count**	**Total Count Successful**
Make a plan	4	4	3
Make sub-goals	8	5	2
Monitor work	17	9	5
Organize work	13	8	4
Redirect	6	3	2
Recognize valid proof	4	4	4
Recognize lack of proof	2	2	0
Link to other known ideas	2	2	1
Link to other parts of question/proof/work	20	10	5
Use same method as part a	18	10	5

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

<u>Question 2: Part c</u>

Again, for 4-flips as defined previously:

c) Prove that N = 2178 is the only four-digit 4-flip.

As a further extension to Question 2, part c offered participants the opportunity to challenge their former ideas and move to an even more complex task. Time restraints and difficulties on the first two portions of the question did not allow for many participants to be able to encounter this portion of the question, but for those who did, it challenged their notions and proofs from the first two portions.

A valid proof for Question 2 part c could expand on the potential proofs for part b. Equations could be used to prove, through a bit more work than needed in part b, that the only option for a four-digit 4-flip was 2178. Participants could also break the potential options into smaller intervals and use similar notions as used in part b to limit the possibilities to a reasonable number to work with. A combination of these two ideas could be used to complete the proof as well; for example, the possibilities could be limited with an examination of smaller intervals, and equations could then be used to eliminate the remaining options except for 2178. A quick check is also needed to show that 2178 is indeed a 4-flip.

Only 3 participants were able to attempt part c, Lisa, Sam and Julie. Lisa and Julie had some errors in their proof of part b that the researcher hoped could be resolved with the challenge of a larger interval. Lisa was able to overcome these difficulties and find a proof, but Julie was not. While Sam had a valid proof for part b, he was not able to expand his results to part c sufficiently. All 3 participants checked that 2178 was a 4-flip, found the limit to the interval 1000 through 2499, and were able to eliminate the interval 1000 through 1999, similar to their work from parts a and b.

Lisa first checked the validity of $N = 2178$ as a 4-flip. She immediately limited the options to the interval 1000 through 2499. In part b of the question, Lisa had been unable to move past an assumption that the middle digit must be zero, and the first and last should be equal. However, with the increase to four-digit numbers, she moved past these ideas and correctly determined that the last digit could only be 3 or 8. She then eliminated the interval 1000 through 1999. At this point, she listed all remaining options based on the second digit, which could be 1 through 4, and systematically worked through each using equations to solve for the missing third digit. Through organized and well-planned work, Lisa was able to complete her proof and show that $N = 2178$ was the only four-digit 4-flip.

Sam also began his work by checking that $N = 2178$ was a 4-flip, limited the options to 1000 through 2499, and immediately eliminated the interval 1000 through 1999. Similar to his work in part b, Sam then broke the remaining numbers into two intervals and drew conclusions. He determined that the numbers 2000 through 2249 would need to end in 8, and the numbers 2250 through 2499 would end in 9, therefore they could be eliminated since he also knew that any 4-flip must end in 3 or 8. However, his proof for part b had moved from this point to a check of the small number of remaining possibilities, but here there were still too many to check by hand quickly. Sam looked at a few examples to gain ideas, but made an error in his assumptions in the process. Ultimately, while he was able to correct this error, he was unable to finish the proof and was aware that he had not found a valid proof.

Julie's work began in a different manner. She did first limit the options to those less than 2499, but her work continued in a manner similar to that in part b, which was unlike the previous two participants. After rechecking all ending digits, Julie was able to determine that the only possibilities were 3 and 8, and she could again eliminate 3. In part b, she had described an idea of a meeting point, where the values of *N* flipped and *4N* would reverse the relationship between which was the larger of the two, and had ended her proof of part b without proof of the validity of this idea. In part c, Julie believed that the entire interval from 2000 to 2499 would only have one such meeting point, which would be at 2178. The researcher challenged this idea and asked Julie to consider other examples to verify her idea. When those examples showed Julie that she was incorrect, she opted to break the possibilities into group of 100 instead. She worked through all remaining groups and believed that she was finished with the proof, but still had not proven or attempted to justify her meeting point idea.

Similar to part b, this portion of the question illustrates that participants followed ideas that were already formed in their minds from previous work to tackle a new proof. Even when these ideas were challenged, such as in Julie's case, participants are not always willing to give up what they believed to be true. It is also again apparent that those ideas that are most obvious to students are the most difficult for them to prove. There were many similarities in the work of all 3 participants (see Table 36). Lisa, Sam and Julie had previously been successful on part a of the question and so it is not surprising that they should all work through the proof attempt with similar successful strategies. Several other strategies were used by only 1 or 2 participants (see Table 37).

The error made by Sam, and the continued misunderstanding that Julie could not correct, led to their inability to prove part c of this question. The strategies and difficulties unique to these unsuccessful participants are shown in Table 38. Both participants looked at examples and generalized from examples, where Lisa had not done so. This indicates Lisa's clear plan without having to look back to examples for ideas or validation. There is again no one strategy that absolutely made participants unable to finish, but rather a combination of errors and lack of using other strategies that contributed to their inability to find the correct proof.

Table 36

Question 2 (part c): Strategies used by all 3 participants

	Lisa	Sam	Julie
Strategies			
Look at examples four-digit	1	3	2
Make list of those shown not possible	1	1	2
Look at individual digits	2	1	2
Constraints possibilities as a whole	1	2	2
Constrain individual digits	2	2	3
Use proof by cases	1	1	1
Make subgoals	1	1	1
Monitor work	3	2	1
Organize work	3	1	1
Recognize valid proof	1	1	1
Use same method as part a	1	3	3

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 37

Question 2 (part c): Strategies and difficulties of 1 or 2 participants

	Lisa	Sam	Julie
Strategies			
Read the question			1
Generalize from examples		2	2
Break into smaller intervals	1		1
List of numbers tried	3	2	
Visualize digits and flip	2		
List of possible numbers remaining			3
Develop equations to represent digits	1		
Relate N to 4N		1	
Recognize potential proof by exhaustion		2	
Make a plan	1	1	
Link to other parts of question/proof/work	1	3	2
Check N = 2178	1	1	1
Difficulties			
Meeting point idea			2
Convinced finished when not finished		1	1
Over-emphasis on proof structure		1	1
Desire another proof even with valid proof		1	
Prompted by Researcher			1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 38

Question 2 (part c): Strategies and difficulties of only unsuccessful participants

	Sam	**Julie**
Strategies		
Read the question		1
Generalize from examples	2	2
List of possible numbers remaining		3
Relate N to 4N	1	
Recognize potential proof by exhaustion	2	
Difficulties		
Meeting point idea		2
Convinced finished when not finished		1
Over-emphasis on proof structure	1	1
Desire another proof even with valid proof	1	
Prompted by Researcher		1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Question 2 Part c: Individual Strategies

With the small number of participants attempting this portion of the question, the individual strategies used are quite unique to each participant, rather than occurring in groups. Both Sam and Julie, as stated above, used examples. The actual use of examples did not deter from their work, but rather were an indication of their search for ideas and explanations. However, all participants did check the validity of $N = 2178$, but this was not a check of examples per se, instead it was a portion of the proof itself.

Lisa was the only participant to use a visual representation of the digits and their flip. This was consistent with her work from other portions of the question. However, this was most likely just a continuation of her other work to connect the portions of the question together, rather than a necessary tool for understanding the question and the definition of a 4-flip. Lisa was also the only participant to utilize equations in her work. These led directly to her proof. While none of the participants had used equations in previous portions of the question, Sam had proven the previous portions successfully with other ideas, and Julie's unsuccessful idea continued throughout. The switch to equations was critical to Lisa's ability to move past her own error, but the lack of equations elsewhere did not necessary have a negative impact on the other participants' work.

All participants used restrictions on *N* as a whole in order to limit the possibilities in their work. Lisa and Sam were able to make further limitations, giving Lisa a proof, and Sam the beginnings of a proof. Julie, however, was not concerned with the individual digits because her idea of a meeting point did not originally need further limiting factors to get results. Her examination of more digits was only done after prompting by the researcher to correct her error and even then the restriction was only an organizational tool to examine each set of numbers.

All 3 participants were also able to monitor their work and keep track of their progress. None had difficulties that were caused by lack of organization, inability to link portions of the question together, or an inability to make plans and sub-goals. Here, all of the positive self-regulation strategies were present and it was only the actual errors and misconceptions that stopped Sam and Julie from reaching valid proofs. That the self-regulation strategies would be present is also to be expected that since all 3 participants had previously been successful in another portion of the question. Therefore, they demonstrated an ability to understand and prove this type of question and it was other difficulties that prevented their complete proof here.

<u>Question 3</u>

A traditional chessboard consists of 64 squares (8 x 8). Suppose dominoes are constructed so that each domino covers exactly two adjacent squares of the chessboard. A <u>perfect cover</u> of the chessboard with dominoes covers every square of the chessboard without overlapping any of the dominoes.

Consider a generic chessboard of size m x n. Prove that the generic chessboard of size m x n has a perfect cover if and only if at least one of m or n is even.

Unlike the previous questions, Question 3 is a bi-conditional statement. All participants had the background to identify the need to prove both directions of the implication, but few actually did so. Many students opted to approach the proof using cases based on the parities of the dimensions of the chessboard. Such a proof, when completed correctly, would be valid, however it is unclear whether students who used this particular idea were aware that they were proving a bi-conditional statement.

Ideally, a valid proof would include two smaller proofs. For the forward direction, one must prove that if a perfect cover of a chessboard exists, then at least one of the dimensions of the chessboard must be even. A proof by contrapositive would suffice for the forward direction. The reverse direction is to prove that if at least one of the dimensions of the chessboard is even, then a perfect cover exists. Showing the generic pattern to cover a chessboard would prove the reverse direction. Most participants covered only one of the two directions.

In total, there were 15 participants who attempted this question, 9 from MATH 305 and 5 from MATH 406. Of these, only 4 participants from MATH 305 (Ellen, Lily, Julie, and Andy) and 1 from MATH 406 (Maggie) were able to complete both directions of the proof. However, not all of these did so without prompting from the researcher.

Ellen was extremely organized and made clear plans for her proof, including acknowledging her assumptions and her goal. She was also able to keep the overall goal in mind and was meticulous in the details of her work. She did struggle a bit in proving what she found obvious and articulating her thoughts at that point. However, the researcher assisted in some errors in notation and prompted Ellen to consider new ideas. This led to a complete proof. Ellen noted, when she finished the proof, that she had a good visual image of the question and compared the ideas with those used in MATH 305, such as using cases and proving a bi-conditional statement. Her organization and ability to keep the overall goal in mind were similar to her work in other questions, as well as was her ability to try different ideas without losing track of her work.

Maggie also clearly wrote out her assumptions and what needed to be shown. She also expressed a visual picture of the question. While she did not immediately form a plan, she did find the generic pattern. She was prompted to move past this pattern. After describing her ideas, Maggie realized the statement was a bi-conditional. She stated that four cases would be needed and that she had already addressed them all in her work, with only a formal write-up left uncompleted.

Andy struggled to begin his work. Unlike Ellen and Maggie, he chose to look at specific examples on paper to gain an insight into the proof before beginning the rest of his work. He considered the given 8-by-8 chessboard and also drew a 6-by-7 chessboard as well. During this time, he informally described the generic pattern, though it was not clear that he knew he had done this. Like Ellen, Andy struggled to prove what he was already convinced was true. After being prompted to verbalize his thoughts, he was able to develop cases and eventually describe the argument that at least one dimension must be even, giving the total number of squares being even. Andy's work here was similar to his work on other questions in that he was organized and focused on the larger picture, with a desire to make a plan. However, he was not able to make a plan immediately, which led to his uncertainty about starting the proof. Andy did not acknowledge that there was a bi-conditional statement to be proven and it is not clear whether he understood that his proof would address both necessary directions.

Lily also examined a specific chessboard to convince herself of the validity of the statement. Unlike Andy, though, Lily looked first at an odd-by-odd chessboard. She also convinced herself of the truth of the statement and then had difficulty proving what she knew to be true. Lily considered a proof by cases and was prompted several times to continue her work in this direction. After a recap of her work, she finally stated that the question was an if-and-only-if statement and so needed to be proven in both directions. Lily divided the remaining proof into two cases, each having at least one even dimension, and proved that the product would be even. Prompting was needed for her to verbalize the pattern needed and, with this, she completed her proof. Like many others, her proof lacked a full explanation of several details, however overall the proof was mostly complete. Lily's work on this question was similar to her other work during the same interview, characterized by quality ideas and careful work, but needing prompting to continue several times.

Unlike the other participants who drew pictures of chessboards, Julie never referred to a specific size of chessboard. Instead, she began by immediately forming a pattern for domino placement (see Figure 3). Julie split the proof into only two cases, the odd-by-odd case was missing, but had difficulty forming a proof of what she was convinced of and seemed to be clearly true. She had to be prompted many times to move past this mental roadblock. Julie eventually used her picture to describe the need for at least one even dimension and then moved on to generic notation. She added that odd-by-odd would give an odd product and therefore would not be coverable. With more prompting, she further described this last case and ended her work there. Like Andy, Julie never acknowledged the bi-conditional statement or that her proof by cases would address both directions necessary for the proof.

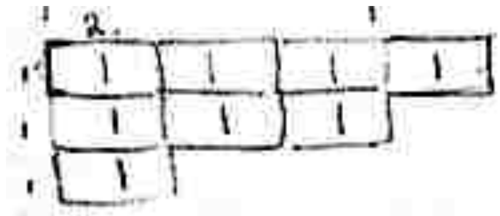

Figure 3: Julie's pattern for domino placement in Question 3

When viewing the individual strategies, no strategy was a guarantee of success. That is to say, there was no one strategy that was both exclusive to only those participants who were able to complete a valid proof and also used by all of those participants. However, there were strategies common to all 5 successful participants. These strategies are listed in Table 39, along with the difficulties experienced by all 5 (numbers listed are frequencies of occurrence). Additionally, the strategies and difficulties listed in Table 40 were common to 3 or 4 of the successful participants, though the specific participants varied among the strategies. Lastly, Table 41 contains the strategies used and difficulties experienced by only 1 or 2 of the successful participants.

Table 39

Question 3: Strategies and difficulties of all successful participants

	Ellen	Lily	Julie	Maggie	Andy
Strategies					
Draw dominoes	2	1	2	1	1
Label / write down ideas	2	3	2	2	3
Verbalize ideas	4	1	2	2	5
Recognize valid proof	2	1	1	1	1
Convinced that the statement is true	2	1	1	1	2
Difficulties					
Struggle and prompted by researcher	4	2	3	2	4

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 40

Question 3: Strategies and difficulties of 3 or 4 successful participants

	Ellen	Lily	Julie	Maggie	Andy
Strategies					
Read the question	3	1	4	1	
Visualize		1		1	2
Fill in dominoes on a chessboard	1	2		2	3
Develop equations	1		1		1
Develop even/odd notation	2	1	1		2
Work with equations or even/odd notation	1	2	1		1
Recognize bi-conditional statement	1	1		1	
Recognize potential proof by cases		2	1	1	1
Use proof by cases		1	1	1	2
Make a plan	2	1			1
Make sub-goals	3	1	1		2
Monitor work	2	3		1	5
Organize work	1			1	2
Acknowledge what is known, what needs to be shown	3			1	1
Link to ideas for matrices	1	1			1
Link to Math 305 ideas	2	2	2		3
Difficulties					
Receive help from Researcher	2	3		1	
Unsure of where to go next		2		1	1
Once result is obvious, struggle for proof	1	1	1		1
Difficulty writing formal proof		3	2	1	1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 41

Question 3: Strategies and difficulties of only 1 or 2 successful participants

	Ellen	Lily	Julie	Maggie	Andy
Strategies					
Understand the question	1				1
Reword definitions			1		
Desire to prove to oneself		1			
Look at examples (draw specific size)		4			2
Generalize from examples		1			1
Build from small examples		1			
Example of each case		1			
Draw pictures in general					2
Draw chessboard - generic		1		2	
Recognize potential proof by contradiction	1	1			
Recognize potential proof by contrapositive					1
Use proof by contradiction	1				
Use proof by contrapositive					1
Correct a previous error	2			1	
Redirect	2			1	
Tie to other known ideas				1	1
Link to other parts of the question					1
Difficulties					
Unsure where to start		1			1
View as problem not proof				1	
Error in notation	1				
Stuck in search for equations					1
Convinced finished when not finished				1	
Too much emphasis on certain proof structure					1
Desire another proof even with valid proof	1				
Unaware of what would constitute a proof			1		
Opinion of own abilities low		1			1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Of those participants who completed only one direction of the proof, there was a mixture of results. Amy and Katy only found a pattern and did not further describe any notation or ideas for the forward direction of the proof. Shelly also found the correct pattern and additionally described the ideas necessary for the other direction, but was unable to complete that proof. Lisa and Rick both proved that an odd-by-odd chessboard could not have a perfect cover, but no mention of a pattern or the need for more of a proof occurred. Paul described both the proof of odd-by-odd and the proof that at least one even dimension gave the total number of squares being even, but did not elaborate further on how a perfect cover could be found specifically for the second case.

The remaining 4 participants were unable to complete either direction of the proof. Sandy and Vicki both proved that at least one even dimension gave an even number of squares, but did not go further with any other proof. Jon and Shaun were unable to prove any portion entirely, only giving the basic ideas and outline that could have led to one direction of the proof, but not completing the proof itself.

There were several strategies used and difficulties experienced by only the unsuccessful participants, see Table 42. Further information is given in Table 43, which lists those strategies used and difficulties experienced by at least 6 of the 11 unsuccessful participants.

Table 42

Question 3: Strategies and difficulties of only unsuccessful participants

	Total Frequency	**Total Count**
Strategies		
Verify information	14	7
Make table or list	1	1
Look for patterns	3	3
Representation with physical movement	3	1
Recognize lack of proof	5	3
Difficulties		
Unable to recall previous work	7	3
Error in understanding the question	3	3
Think example is enough	3	2
Stuck with no new ideas - unable to move on	2	2
Too much logical manipulation	3	1
Unable to recognize proof as valid	1	1
Appear distracted by the interview situation	3	3

Note. The values in the total frequency and total count columns represent frequency of use of each strategy overall, and the number of participants who used each strategy, respectively

Table 43

Question 3: Strategies and difficulties of at least half of unsuccessful participants

	Lisa	Shelly	Jon	Paul	Sandy	Rick	Amy	Vicki	Shaun	Katy
Strategies										
Read the question		2	2	7	5	6	2	1	4	1
Verify information			2	1	2	1	3		1	4
Look at examples	1	4	2		2	1	2	1	4	1
Generalize from examples	1	3	1				1	1	1	1
Draw dominoes	1		2			1	1		1	
Fill in dominoes on chessboard	1	2	1			1	3	1	4	1
Develop even/odd notation	2	1	2	2	2	2			2	
Work with equations	2		1	2		1			2	
Recognize potential proof by cases	1	2	1						1	1
Make subgoals		2	1	2		1			4	1
Monitor work		2	1	1		5	1		8	1
Label / write down ideas	2	2	1	3	2	2	3	1	7	2
Verbalize ideas	1	2	1	1	2	3	1	1	5	1
Acknowledge what is known, what is to be shown		1	1	2		2	1		1	
Link to Math 305 ideas	1				2	3		1	4	1
Convinced that the statement is true	1	1	1		1	1	1		1	1
Difficulties										
Struggle for proof of obvious	1	1			1	2			1	1
Convinced finished when not	1	1		1	2		1	1		1
Struggle and prompted by researcher	4	3	1	2	3	3			5	
Unaware of what would constitute a proof	1	1			2	1	1	1		1

Note. The values represent frequency of use of each strategy or difficulty per participant.

Of the strategies listed in Table 43, only one was not also used by the successful participants, which was verifying information. This strategy was used when participants were unsure of the statement of the question and clarified details for understanding purposes. This indicates that the successful participants understood the question immediately with no need for further clarification.

In fact, all the remaining strategies in Table 43 were used by at least 2 out of the 5 successful participants as well. This suggests that these strategies are commonly used but not necessarily a predictor of success on a proof-writing task.

Question 3: Individual Strategy Use

The use of examples arose frequently in this question (see Table 44). A total of 11 participants examined a specific chessboard drawing at some point during their work. The most common drawing was of an 8-by-8 chessboard, most likely since it was the example stated in the question itself, however others were also viewed. The two successful students who utilized examples in their work were Lily, who examined a 3-by-3 chessboard in the beginning of her work, and Andy, who examined a 6-by-7 chessboard after being convinced of the validity of the statement and later also drew a 3-by-9 chessboard. Ellen, Julie, and Maggie also drew pictures, but not of a specific size of chessboard.

Table 44

Question 3: Strategies representing participant use of examples

Strategies	**Total Frequency**	**Total Count**	**Count for Unsuccessful**
Look at examples (draw specific size)	24	11	9
Generalize from examples	11	9	7
Build from small examples	1	1	0
Example of each case	4	4	3

Note. The values in the total frequency, total count, and total count unsuccessful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of unsuccessful participants who used each strategy, respectively.

Of the participants who were unable to form a complete proof, Lisa, Rick, Amy, Shaun, and Katy all looked at the 8-by-8 chessboard before beginning any other work. Lisa then filled in her chessboard with dominoes to form a perfect cover. It seemed at that point that she had an idea of a generic pattern for covering any appropriately sized chessboard, however she did not specifically address this issue. Lisa, like many others, was then convinced of the truth of the question statement, but was temporarily unsure how to move past this to find a proof. Lisa also drew a 3-by-5 chessboard later in her work, which she used to discover the ideas necessary for the proof of the odd-by-odd case.

After drawing a picture of a single domino, Rick also covered the 8-by-8 chessboard with dominoes to form a perfect cover. Like Lisa, he was then convinced that the statement was true and seemed to be concerned about being able to find a proof now that he had a clear picture and understanding in his mind of why it would be true. Rick moved on to other ideas and strategies, and did not return to looking at examples or drawing pictures again during his work.

Similar to Rick, Amy drew a single domino to the side of her chessboard and then found a perfect cover of the 8-by-8 chessboard with dominoes. Amy used this picture to describe the generic pattern needed for a perfect cover. After describing her pattern, she felt that her proof was finished and did not go on to prove the other direction of the statement.

Shaun also covered his 8-by-8 chessboard with dominoes, in a much different pattern than others had (see Figure 4). This pattern did not give Shaun a clear picture of why the statement would be true, as it did not follow the generic pattern many other participants found. He then moved to logical notation for the statement. As Shaun continued his work, he drew two more example chessboards, organized into two ideas, an example of a perfect cover, and a non-example. Returning to the manipulation of logical notation, Shaun did not consult his examples again until he became stuck in his work. He searched for a way to connect his examples to generic terms, but was unsuccessful and ended his work there.

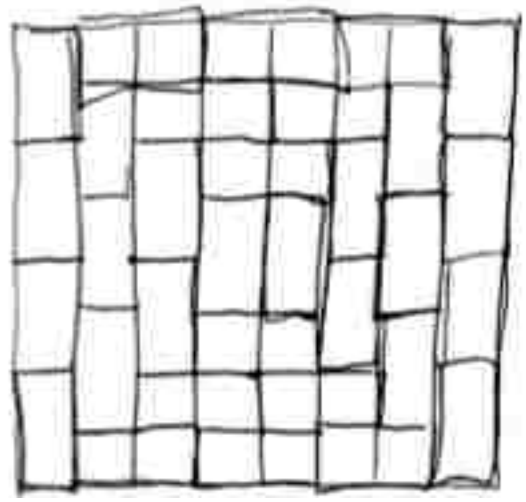

Figure 4: Shaun's pattern for covering 8-by-8 chessboard in Question 3

Unlike the other participants who drew an 8-by-8 chessboard, Katy did not draw in dominoes to cover this board. Instead, she used her picture for reference to describe in words the generic pattern that could be used to form a perfect cover. With that, Katy felt she was finished with the proof and did not prove the other direction of the statement.

Shelly and Jon both drew an even-by-even and an odd-by-odd example before proceeding with their work. Shelly began with a 1-by-2 chessboard. After deciding to work with cases, she looked at an instance of odd-by-odd, a 5-by-3 chessboard. Like others, Shelly also began to fill in her chessboard with dominoes. However, she did not need to cover the entire board to understand that the total number of spaces was odd and therefore she would be unable to cover the board. Her argument for this case ended here with no generic proof or mention that one would be needed. Again, the use of a particular example convinced Shelly that the statement was true and even allowed her to find the heart of the proof, but she was unable or unwilling to finish the necessary steps after this. Shelly later looked at other examples, but continued to use only specific numbers in her arguments, and was unable to connect her examples with the generic notation she desired.

Jon clarified several ideas before beginning his work. He then immediately drew two chessboards, a 2-by-4 and then a 3-by-3, already convinced of the truth of the statement. He also covered the boards with drawings of dominoes and took note of the uncovered square that remained in the 3-by-3 case. Unlike other participants, Jon did not seem to need the examples in order to be sure that the statement was true and he was not using the pictures to explain the proof at all. He never returned to his drawings during his work, instead focusing on finding equations to work with for a proof. In his case, the use of examples was not a factor in the rest of his work, neither positive nor negative.

Vicki, however, began with only an odd-by-even example, an 8-by-7 chessboard. After determining that the total number of squares would be even, she switched to a generic *m* and *n* and wrote that if one were even, then the total number of squares on the chessboard would be even and so would be coverable. While she did trace out where dominoes would be placed on her example, Vicki did not describe the generic pattern or mention how this would be used in the generic case. She also did not address the other direction of the proof in any way. Vicki, as many others previously mentioned, was unable to, or unaware of the need to, make generic arguments and prove her statements in full. It again appeared that the use of an example convinced the participant so fully of the truth of the statement that she was unable to understand the necessity of a more detailed and generic proof.

Sandy examined generic notation for the dimensions first, attempting to represent *m* as an even number and *n* as odd. However, she quickly abandoned this notation and moved to an example, an 8-by-7 chessboard. After determining, without any further drawing, that there were 56 squares and so 23 dominoes would be needed for a perfect cover, Sandy felt she had completed her work. When prompted, Sandy looked at two more even-by-odd examples, but still did not attempt a generic proof. The researcher prompted her several times to go further and to prove her results. Sandy then used generic notation to show that an even number times an odd number would result in another even number, and ended her work there. Again, the use of concrete examples helped the participant to understand the question but hindered her in regards to forming a more generic proof. Sandy clearly felt that her picture and single example were sufficient and had no desire on her own to move past this point.

Nearly all unsuccessful participants drew example chessboards, and many were able to develop at least half of the proof after drawing these examples. Clearly this strategy is therefore helpful to developing a proof, as viewing examples did aid many participants in gaining an understanding of the question and becoming convinced of the validity of the statement. That being said, the researcher also found that 10 of the 15 participants struggled to form a proof after first convincing themselves of the truth of the statement. The mere act of examining specific instances of the statement seemed to lead participants astray from a more generic proof-writing style into one particular example. Once in that mind-set, participants had difficulty stepping back into a general proof.

In addition to drawing the chessboards, as mentioned in the individual summaries above, participants also represented a cover of their examples with dominoes, or showed how no cover could be found on their specific example, if that was the case. Furthermore, 10 participants drew a domino to the side while examining it in particular, most noting that it had an even number of squares and making conclusions based on that observation. The use of such visual methods again was not shown to be a good predictor of proof, as some participants were able to use these to form portions of the proof, but others were unable to move past the concrete visualizations. For those who were able to form proofs, the strategy was helpful to them in their discoveries. However, again the issue of being convinced of the truth of the statement and unable to move past this arose with these visual methods. (See Table 45 for a complete list).

Table 45

Question 3: Strategies representing participant use of visualizations

	Total Frequency	**Total Count**	**Count for Unsuccessful**
Strategies			
Draw pictures in general	4	3	2
Visualize	9	7	4
Draw chessboard - generic	6	5	3
Draw dominoes	13	10	5
Fill in dominoes on chessboard	22	12	8

Note. The values in the total frequency, total count, and total count unsuccessful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of unsuccessful participants who used each strategy, respectively.

In this question, as in the others, some participants utilized equations in their work, see Table 46. The most common type of equation used was the development of notation to represent the parity of the dimensions, *m* and *n*. Of the 11 participants who were able to develop the even and odd notation, 9 were able to use their equations to form at least part of the proof. Participants used this idea for proofs by cases. For a proof by cases, this notation is crucial to fully develop each case and to prove, for example, that an even number multiplied by an odd number results in another even number.

Shelly and Shaun, however, were unable to use their notation to find a proof. Shelly formed some generic notation, but had difficulties expressing her ideas in this way. She further struggled to connect what notation she had found with her specific examples. Shaun also struggled to relate his concrete examples to the generic notation he had found, though his notation was better developed than Shelly's notation. While Shaun was able to see the products being odd as an issue, he was not able to generalize this result.

Paul and Ellen were the only two participants who developed equations without the use of examples. Paul never drew pictures of any sort during his work, but was able to complete one direction of the proof with only equations. Ellen proved one direction with only equations before using pictures in general to show the pattern needed for the second direction of the proof.

Table 46

Question 3: Strategies representing participant use of equations

	Total Frequency	Total Count	Count for Unsuccessful
Strategies			
Develop equations	6	6	3
Develop even/odd notation	19	11	7
Work with equations	13	9	5

Note. The values in the total frequency, total count, and total count unsuccessful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of unsuccessful participants who used each strategy, respectively.

No specific proof technique was a guarantee of success either, since most students were unable to recognize the need to prove two directions to complete the proof. However, many students were successful in at least half of the proof by choosing a potentially appropriate proof method. Table 47 gives the counts of the particular proof techniques recognized and used. These techniques were often recognized as participants linked the question to a known idea or proof that they had previously seen.

Table 47

Question 3: Participant recognition and use of proof techniques

	Total Frequency	Total Count	Count for Unsuccessful
Strategies			
Recognize bi-conditional statement	5	5	2
Recognize potential proof by cases	11	9	5
Recognize potential proof by contradiction	9	6	4
Recognize potential proof by contrapositive	2	2	1
Use proof by cases	7	6	2
Use proof by contradiction	5	5	4
Use proof by contrapositive	3	3	2

Note. The values in the total frequency, total count, and total count unsuccessful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of unsuccessful participants who used each strategy, respectively.

In many of the questions, the ability to keep track of work, organize material, and recall previous work were crucial to avoiding major roadblocks. Question 3, however, did not have a strong showing of such strategies (see Table 48). Most students were able to keep track of their work and, rather than getting lost, the major issue was often a lack of understanding of the parts necessary to prove the entire statement. As mentioned, another common issue was the difficulties experienced after participants fully understood the statement to be true. Once convinced of the statements validity, participants were unsure how to move on to prove what they felt was obvious. Many participants were prompted by the researcher to continue their work, but some were still unable to form new ideas and a generic proof.

Table 48

Question 3: Strategies representing participant use of self-regulation

Strategies	Total Frequency	Total Count	Count for Unsuccessful
Make a plan	9	7	4
Make subgoals	18	10	6
Monitor work	30	11	7
Organize work	10	5	2
Acknowledge what is known, what is to be shown	13	9	6
Redirect	10	6	4

Note. The values in the total frequency, total count, and total count unsuccessful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of unsuccessful participants who used each strategy, respectively.

<u>Question 4</u>

Let x and y be two integers. We say that x <u>divides</u> y if there is an integer k such that y = kx. Consider three integers a, b, and c. Prove the following:

If a divides b and b divides c, then a divides c.

Question 4 contains a common concept in an upper division algebra course, but most participants in this study would not have had exposure to this definition at the time of the interviews. The question was designed for use if participants worked quickly through others, or had difficulties that resulted in their other work being cut short. The idea of using a new definition in a proof is similar to Question 2, however here the question does not involve proving a specific number does not exist and so different strategies were expected.

Potential proofs of Question 4 include using the definition to form equations for each of the given statements and working with those equations to form the proof. Alternatively, a proof could result from looking at examples and relating these to the generic statements, resulting in a similar equation manipulation, but from a different origin.

A total of 5 participants attempted this question, of which only Lisa and Lily were successful. Table 49 lists the strategies used by both successful participants, there were no common difficulties between these 2 participants. Lily worked through the generic statements, finding correct equations for each assumption. She looked at an example, but struggled to link this to her equations. As she worked to manipulate the equation she wished to prove, she had actually come across the basis for a valid proof. However, she seemed unaware that this would constitute a proof and continued to struggle for another direction. Lily had found a valid proof, but ultimately was unable to see its validity and ended her work without finishing the remaining necessary details.

Lisa was also successful in finding the correct proof. Unlike Lily, Lisa was fully aware that she of what she had found and was able to construct her proof in 1 ½ minutes, after only a brief pause to consider the question. She stated that she had not seen this question or definition before, but still dealt with the new term with ease. Table 50 lists the strategies used and difficulties experienced by only one of Lily or Lisa.

Table 49

Question 4: Strategies of all successful participants

	Lisa	**Lily**
Strategies		
Understand the question	1	1
Use equations from the definition	1	1
Generic notation with *a, b*, and *c*	1	2
Develop equations for what is known	1	1
Work with equations	1	1
Assume hypothesis, prove conclusion	1	1
Acknowledge known / to be shown	1	2

Note. The values represent frequency of use of each strategy or difficulty per participant.

Table 50

Question 4: Strategies and difficulties of 1 successful participant

	Lisa	Lily
Strategies		
Generic *x, y* notation	1	
Develop equations for what is to be shown		1
Look at examples		2
Monitor		2
Redirect		1
Recognize your proof / lack of proof	1	
Link to other known ideas		2
Difficulties		
Unsure of where to go next		1
Unable to connect examples with generic notation		2
Unable to recognize proof as valid		2
Prompted by Researcher		4

Note. The values represent frequency of use of each strategy or difficulty per participant.

The common link in the successful participants was that they were the only 2 participants to understand the definition and use two different constants to represent the equations (see Table 51). However, Lisa worked so quickly to her correct solution that there was little opportunity to observe any self-monitoring techniques. Lily exhibited several good strategies, including making a plan, monitoring, and redirecting. Unfortunately, she was unable to see that her work had led to a valid proof.

Table 51

Question 4: Strategies and difficulties of only successful participants

	Total Frequency	Total Count
Strategies		
Understand the question	2	2
Link to other known ideas	2	1
Difficulties		
Unsure of where to go next	1	1
Unable to recognize proof as valid	2	1

Note. The values in the total frequency and total count columns represent frequency of use of each strategy overall, and the number of participants who used each strategy respectively.

Lisa noted that she had been careful to use different letters to represent the constants of multiplication, and further that she would probably have found two of *a*, *b*, and *c* to be equivalent had she not used proper notation. In fact, Lisa described what all 3 participants who could not complete a valid proof actually did find, but all were unaware of their error. This error was caused in most cases by using the same constant for both equations, *a* divides *b* and *b* divides *c*. Another common error was to reverse the meaning of the definition, instead of the correct definition of *a* divides *b*, the meaning turned into the interpretation of *a* divided by *b* (see Table 52).

Table 52

Question 4: Strategies and difficulties of only unsuccessful participants

	Total Frequency	Total Count
Strategies		
Read the question	3	1
Desire to prove to oneself	1	1
Connect examples to generic notation	1	1
Make a plan	1	1
Difficulties		
Reverse definition	3	3
Use only one constant	3	2
Use all three equations simultaneously	1	1
Determine $a = b$, or $b = c$, but unaware of error	3	3
Incorrect example	1	1
Stuck in search for pattern	1	1
Stuck in search for equations	1	1
Stuck with no new ideas - unable to move on	2	2
Too much emphasis on proof as a certain structure	1	1

Note. The values in the total frequency and total count columns represent frequency of use of each strategy overall, and the number of participants who used each strategy, respectively.

Table 53

Question 4: Strategies and difficulties of all 3 unsuccessful participants

	Jon	Sandy	Vicki
Strategies			
Generic notation with *a, b,* and *c*	1	1	1
Difficulties			
Reverse definition	1	1	1
Determine $a = b$, or $b = c$, but unaware of error	1	1	1
Prompted by Researcher	1	3	5

Note. The values represent frequency of use of each strategy or difficulty per participant.

The researcher attempted to help the participants correct the errors, but some continued with their equations and ultimately were unable to complete the proof. Sandy and Jon both began with the error of reversing the definition. While Jon was able to see this error, he continued to use the same constant in both equations. This led to the result that $a = b$ and Jon was therefore unsuccessful. He said that he would have submitted this proof if assigned to do so for a course, but was unsure about his work. Sandy was unable to correct her error and was therefore also unable to complete the proof. She found that $b = c$, though she was clearly aware that her proof was incorrect.

The third participant was could not complete a proof for this question, Vicki, struggled with a different error. She initially set up her equations in reverse and, when corrected, only partially fixed her error. The result was notation that was cumbersome and led to several minutes of work that did not help in the completion of the proof. She opted to look at a specific example, but unfortunately chose numbers that were not actually an example of the theorem. Vicki was unable to connect her generic notation with the example and was stopped at that point due to time.

As mentioned, the 3 unsuccessful participants had common errors and difficulties in dealing with the definitions. Although each was able to either make a plan, monitor their progress, redirect, or correct a previous error, they were still not successful in reaching a valid proof, see Table 54.

Table 54

Question 4: Strategies and difficulties of 1 or 2 unsuccessful participants

	Jon	Sandy	Vicki
Strategies			
Read the question			3
Desire to prove to oneself			1
Use equations from the definition	1	1	
Generic *x, y* notation		1	1
Develop equations for what is known	2	1	
Develop equations for what is to be shown		1	
Work with equations	2	1	
Look at examples			2
Connect examples to generic notation			1
Assume hypothesis, prove conclusion	1	1	
Make a plan			1
Monitor		1	
Acknowledge known / to be shown	1	1	
Redirect	1	2	
Recognize your proof / lack of proof		1	1
Difficulties			
Use only one constant	2	1	
Use all three equations simultaneously		1	
Incorrect example			1
Stuck in search for pattern			1
Stuck in search for equations	1		
Stuck with no new ideas - unable to move on	1	1	
Too much emphasis on proof as a certain structure	1		
Unable to connect examples with generic notation			2

Note. The values represent frequency of use of each strategy or difficulty per participant.

Question 4: Individual Strategies

Question 4 was unique in its required use of equations in some form for the proof. All participants at least attempted generic notation for the variables and seemed to recognize the need for such equations in the proof, see Table 55. The two successful participants, Lisa and Lily, utilized equations appropriately and were able to keep careful track of the use of variables in their equations to reach the conclusions necessary for the proof. Both Jon and Sandy made an error in using the same constant in their equations, which caused them to be unable to reach the appropriate conclusions. Vicki became confused during her search and work with equations and turned to a specific example, but was unable to choose correct numbers and therefore unable to correct the error in her equations to complete the proof.

Table 55

Question 4: Strategies representing participant use of equations

Strategies	Total Frequency	Total Count	Total Count Successful
Use equations from the definition	4	4	2
Generic *x, y* notation	3	3	1
Generic notation with *a, b*, and *c*	6	5	2
Develop equations for what is known	5	4	2
Develop equations for what is to be shown	2	2	1
Work with equations	5	4	2

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

Lily and Vicki were the only participants to look at examples during their work, see Table 56. Lily was able to find a valid example, but unable to link this to her generic notation and abandoned the idea to work instead with equations. Vicki was not able to find a valid example, which caused her to struggle in linking the example to her also incorrect notation. With both of these errors working against each other, she was unable to find a proof or recognize the errors. While the use of examples could have been fruitful to understand the definition and correct potential errors from reversing the meanings, the small number of participants to attempt this question did not show this to be the case for their work. Again, with major errors in understanding occurring, participants like Vicki were unable to use even good strategies in a positive way to reach a valid proof.

Table 56

Question 4: Strategies representing participant use of examples

Strategies	Total Frequency	Total Count	Total Count Successful
Look at examples	4	2	1
Connect examples to generic notation	1	1	0

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

The use of self-regulation strategies did not occur in both successful participants, see Table 57. However, Lisa had worked through her proof so quickly that it was impossible to see any of these strategies occurring. She also worked completely silently, so there was no outward evidence of planning and monitoring. These strategies tended to surface in other questions as participants struggled for ideas or needed to think thoroughly through their work in order to find a valid proof. Here, the strategies occurred in two unsuccessful participants as well. As mentioned previously, this indicates that even the best of strategies and plans cannot always overcome major errors in understanding the question or errors made in notation or computation along the way.

Table 57

Question 4: Strategies representing participant use of self-regulation

Strategies	Total Frequency	Total Count	Total Count Successful
Make a plan	1	1	0
Monitor	3	2	1
Acknowledge known / to be shown	5	4	2
Redirect	4	3	1
Recognize your proof / lack of proof	3	3	1
Link to other known ideas	2	1	1

Note. The values in the total frequency, total count, and total count successful columns represent frequency of use of each strategy overall, the number of participants who used each strategy, and the number of successful participants who used each strategy, respectively.

The key ideas necessary for this proof included the proper use of equations, and a complete understanding of the definition and its interpretation. It was also important for participants to understand the ideas that would contribute to a proof that they discovered along the way. Lily did not truly understand that she had finished her proof or accept her ideas as valid. Jon and Sandy both reached the conclusion that two of the variables *a, b,* and *c* would have to be equal, and both were uncertain of this result. It should have pointed out to them that there was a major error in their initial assumptions and directed them back to correct these errors. However, neither was able to make this connection and further correct the error.

Vicki also became caught up in a fruitless search for ideas. She should have been able to see that her example and her equations did not match up and therefore go back to examine both of these ideas. The inability to recognize the misunderstandings and issues that were occurring cost all three of these participants in the end. It was therefore vital to understand what would constitute a proof, and when ideas were not producing results.

Overall, this question posed issues for participants conceptually with a new definition that was similar to a definition they were already aware of. This caused confusion and the inability of participants to see the errors in their own logic. Participants did attempt to monitor their work and make goals, but without understanding the question as a whole, they were unable to complete the proof. Those strategies used and difficulties observed among at least 4 of the 5 participants are shown in Table 58.

Table 58

Question 4: Strategies and difficulties of at least 4 participants

	Lisa	Jon	Lily	Sandy	Vicki
Strategies					
Use equations from the definition	1	1	1	1	
Generic notation with *a*, *b*, and *c*	1	1	2	1	1
Develop equations for what is known	1	2	1	1	
Work with equations	1	2	1	1	
Assume hypothesis, prove conclusion	1	1	1	1	
Acknowledge known / to be shown	1	1	2	1	
Difficulties					
Prompted by researcher		1	4	3	5

Note. The values represent frequency of use of each strategy or difficulty per participant.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

Throughout all four questions, there were interesting tendencies to notice. The individual strategies used across all questions will be discussed here, as well as the difficulties that were generally experienced. Lastly, there will be a discussion of the participants identified as successful in at least half of their work, and of those who were identified as unsuccessful in most of their work. It can be seen in the discussion of cases that even the best set of strategies cannot overcome major errors in understanding, lack of organization in thoughts and work, and lack of motivation for various reasons.

Individual Strategies

Use of Examples

The use of examples was prevalent among all questions. Participants viewed examples to understand the question, to gain ideas and insights, and to actually form portions of the proofs. In Question 1, examples occurred both visually on the pentagon, as well as in table or list form. The construction of a pentagon with sums of 14 was not necessary for the proof, but was completed or attempted by most of the participants. For some participants, this was the help they needed to understand what was necessary for the proof and form ideas that led to their proof. Others, however, were unable to move past the concrete into the abstract, and were stuck attempting to go further after finding this pentagon.

In Questions 2 and 3, almost all participants looked at examples to understand the question and the new definitions. This was not surprising since the new terminology was intended to be challenging and required extra thought to plan for the proof. In Question 2, most participants who viewed examples were aided by these in at least some ideas for the proof; however not all participants were able to move past the examples to other more general ideas. For Question 3, most participants found examples to be an essential part of understanding the question and the validity of the statement they were asked to prove. Unfortunately, though, this also became a hindrance as participants attempted to prove what they then felt was obvious.

Only 2 of those who attempt Question 4 viewed examples as part of their work. For these participants, the use of examples neither helped nor hindered. They were unable to link this to generic notation and therefore were unable to move forward towards a proof using ideas gained from the examples.

In general, participants were able to recognize the need to move past examples in their work, but not all could actually do so. They used examples as a tool to understand the definitions and new ideas posed and, in some cases, to form portions of the proof, such as using brute force in Question 2 or showing a generic pattern via pictures in Question 3. While viewing specific examples was not always helpful, the examples themselves did not often hinder work. The exception to this was a common issue in Questions 2 and 3 in moving past examples to form a generic proof, as previously mentioned. Once participants were convinced from the examples they viewed, they were unable to produce the formal argument needed for a valid proof.

Use of Equations

Searching for equations and trying to manipulate those found towards a proof was an idea that occurred in all questions. Success with this strategy was mixed. Some participants were unable to link the equations to what they desired for the proof, while others found success in manipulating equations towards the ultimate goal. In some cases such as with Jon throughout his work, focusing on equations became distracting and was a factor in his inability to prove a statement. Overall, however, the use of equations was not a hindrance for most who attempted it, but seldom led to a full proof.

A proof using equations was possible for all questions, though. In Question 1, this came in the form of an equation for the total sum to be minimized and would not require any other portion of a proof. A select few participants used this proof in the end. Several other participants, though, were able to gain ideas for their proofs, or portions thereof, by examining equations for individual sums or the total sum. The equations then led to key ideas being discovered, even for those who were unsuccessful in ultimately finding a proof.

In Question 2, many participants used equations both to understand the definition of a 4-flip and to represent the digits of a number being flipped when multiplied by 4. This only resulted in a proof for 2 participants directly, but many others used the ideas gained from this visual representation in other aspects of their proofs, or portions of a proof. The use of equations did not seem to hinder any participants in this question.

For Question 3, most participants used equations in some form, particularly to prove that the products of odd and even numbers resulted in what was expected. Jon became too focused on equation use and was an example of a participant who was hindered by the drive to push his proof in this direction. Overall, however, most participants who used equations did so in a positive way that aided in their proof.

Question 4 was unique in the fact that it required the use of equations to form any sort of valid proof. All participants who attempted this question recognized this and worked to form proper equations. However, errors in notation and understanding the definition given resulted in incorrect equations and therefore a lack of proof in many cases. While equations were vital to the proof, they also caused a great deal of confusion within this particular definition.

Overall, equation use was very prevalent throughout all participant work and generally did not, at the least, have a negative impact on the progress towards proof. Some students commented that they felt a proof must use equations and manipulation of these equations. This was most likely due to the exposure to proofs involving equation manipulation during their math courses and a feeling of completion and satisfaction with such proofs when they are completed. Most participants were able to overcome this desire and seek other forms of proofs, but a few were unable to move past this in certain questions.

Visualizations

As mentioned previously, the use of examples was at times a visual clue towards the proof of a question, particularly in Question 1. Equations were also used to visually represent the questions, such as in Question 2. These visualizations helped in most cases, so long as work was organized and the goal of the statement was kept in mind. Lists and other types of visual organization were also helpful to many, especially in Questions 1 and 2, but were not necessarily essential to the process of developing a valid proof. In Question 3, drawings of chessboards helped some participants to understand the statement and the reasons that it must be valid. However, they were also a hindrance here as participants had difficulty moving past this understanding and being motivated to develop a formal proof.

Participants who used visual tools tended to do so in the same fashion across all questions. For example, Beth was careful to make lists and keep her paper visually organized throughout both questions that she attempted. This was essential for her to keep track of her work, particularly considering the length of time needed for her to complete each question. While it was not essential for all to use these tools, many others may have benefited from them, especially those whose work was disorganized and random. Just like the other strategies mentioned, the use of these tools was for the most part beneficial, but not vital to all, and was not always a direct indicator of success.

Self-Regulation

One major issue that occurred for many participants was the inability to stay on track towards the ultimate goal of a proof. Some participants were unable to recall previous work, or got lost in their computations and thoughts along the way. Others became so caught up in making their ideas fit into their preconceived notions of what constitutes a proof that they were unable to see the key ideas and larger picture necessary to complete the proof.

In Question 1, several participants struggled in a random unorganized search for a pentagon with sums of 13 or 14, such as Amy and Shaun who were never able to find such a pentagon. They did not keep track of their work, or make lists to organize their ideas as some other participants had done, and so could not recall what they had already tried and therefore could not complete their work. Other participants, like Lisa, Lily, and Beth, searched systematically through all possible options for each side and combination. In doing so, they kept track of all previous attempts and were able to work towards the pentagon they desired. While a few participants were able to find the correct pentagon with a bit of luck or well planned choices, most who did so were organized and systematic. Another key element to finding a proof in this question was to understand that finding a pentagon with sums of 14 was not a sufficient proof. To understand this, a participant must keep in mind the goal of the proof and that it is a proof, not just a demonstration of such a pentagon or a puzzle to be solved. Those who systematically worked through the proof, understood the proof needed, and were able to monitor their progress were able to complete the proof.

Question 2 had similar issues of random guessing and checking. Some participants were able to move from random examples to the proof using key observations. Those who were not able to do so struggled to link their examples to notions and equations they had also formed, and were unable to make the crucial links that were necessary. They could not keep the overall goal in mind and were unable to systematically work through choices until they could observe the key ideas that would have allowed them to form a proof. Again, other errors also posed obstacles, even for some participants who were organized but unable to understand the definition of a 4-flip. However, no participant was able to form a proof for this question without monitoring their progress towards the ultimate goal through organization, planning, or systematic work, and knowing what would be needed for the proof.

Where as ideas of self-monitoring and goal making were vital in Questions 1 and 2, the same ideas did not surface during Questions 3 or 4. Here, the major errors that occurred were lack of understanding of what was necessary for the proof, lack of understanding of the definitions given, and proving what, for the participants, seemed obvious. Many participants had the potential for success through key ideas and solid strategies, but were unable to overcome these other errors that also occurred.

Overall, strategies used to monitor work, make goals, redirect work, and keep goals in mind were always beneficial and never a detriment for participants. While other errors occurred that even the best of self-regulation strategies could not overcome, the majority of participants who were organized and aware of their own work and how their ideas were progressing were ultimately successful in finding valid proofs.

Key Ideas Recognized

The key ideas needed for each question are unique to that particular question, however no questions could be completed without some discoveries along the way. The strategies mentioned so far aid in these discoveries, but were not guarantees of success in finding the particular ideas necessary in each question. Some participants were able to stumble upon ideas even in their unorganized work without a plan at all. Still others were able to find these ideas, but unable to make the final connections necessary to complete the proof. All in all, these ideas were crucial to success and a major component in the lack of success for many participants, but not always indicators of a successful participant.

Difficulties Overall

Along with the ability to find the key ideas necessary for the proof, participants needed to avoid other errors in order to find success throughout these questions. Several major difficulties arose throughout the questions, which will be described in detail here, and some have been mentioned previously as well.

Computational and comprehension errors were the most difficult for participants to overcome. The researcher attempted to help correct some computational errors, but not all could be corrected, or led to other errors that could then not be corrected. Such errors occurred in all questions, such as errors in basic arithmetic in Questions 1 and 2. The errors in comprehension and basic understanding of the question and included definitions were much more difficult to correct and overcome. For example, Amy struggled a great deal in Question 2 to keep track of which number she should multiply by 4 and what she was comparing this to.

In Question 1, Rick made several errors during his attempts to find a pentagon with sums of 14 in understanding what numbers he was allowed to use and where combinations could be placed around the pentagon. Julie struggled in Question 2 with a major idea that she posed, but for which she did not investigate evidence to support or offer a proof. Even when confronted with a counter-example in part c, Julie only slightly modified her meeting point idea and moved forward with her proof still without making any effort to prove the idea. She was never able to overcome this notion. Lisa made assumptions without proof in part b of Question 2 as well, but was able to move past them in her work on part c and find a proof for that portion of the question.

In Question 4, a common error was the misuse of notation and misinterpretation of the definition of divides. This led to grossly incorrect results, which the participants recognized as at least troubling, but were unable to identify where the error had occurred. Even in this question, where participants were, for the most part, able to monitor their work, make goals, and use other positive strategies, they were still unable to find a proof due to the errors that had occurred.

Other common issues and difficulties that arose, as previously mentioned, were unorganized and random searches, especially in Questions 1 and 2, and in connection with this, the inability to recall previous work. This points to a lack of monitoring and organization overall, and can be seen throughout the work of several participants, for example Rick and Amy who struggled with these issues in multiple questions.

Participants were also often unaware of what would constitute a valid proof, and unable to recognize portions of a proof that they had actually found. This issue was particularly a struggle in Question 3, where many participants did not complete, and did not recognize the need to complete, both directions of the bi-conditional statement.

A related issue to this was being so focused on a particular proof technique or style of proof that participants were unaware that their ideas may have formed a proof that was already valid. Participants, such as Shaun, were so focused on the proof technique they felt most comfortable with, that even the best of organization failed to assist them towards a proof.

In Question 1, several participants struggled in writing a formal proof, and seemed to view the question as a problem to be solved, more like a puzzle than a proof to be written. Due to this, they were unable to make solid arguments past those used to construct the pentagons they desired or to write anything more than this for a formal proof. Another way that participants struggled when trying to express their ideas clearly occurred after they had convinced themselves of the validity of a statement. A common difficulty, especially in Question 3, was that participants were unable to describe their ideas verbally or in writing once they had seen evidence that the statement was accurate. A common difficulty in trying to find motivation in an introductory proof course occurs when students are asked to prove what seems trivial or obvious to them. This was seen in several questions during this study as well, but most clearly seen in Question 3.

The last difficulty that arose in several cases was simply a lack of time to complete a proof. This was a limitation of the study, since participant schedules as well as other scheduled interviews did not allow for unlimited time to work on all questions and some participants were forced to cut their work short due to this. However, in many of these cases, work was progressing in a good direction and the potential proof could be seen. In other cases, work was random and no new ideas were surfacing, so the time limit did not truly restrict the ability to finish the proof, but rather this would have occurred regardless of time constraints.

General Discussion

As can be seen in the individual analyses by participant, each person had their own personal tendencies and these tendencies occurred in all of the questions they attempted. The strategy use and difficulties experienced by each participant were fairly consistent from question to question, and the individual questions had little effect on changing the way the participant approached a proof. This is of particular interest among the MATH 305 students. In this course, students were specifically taught a set of strategies, from Polya's problem solving heuristics, which they were encouraged to use in approaching every proof.

However, this approach was only seen in use among a few of the participants from this course. Instead, what was demonstrated was a personal view of what would be necessary to examine and what would constitute a proof, which varied for each participant. For example, Jon tended to attempt to develop equations for each question, whether it was warranted or not, and even when he had already developed a different type of valid proof. Shaun, however, focused heavily on manipulation of logical notation and attempting to fit each statement into its logical equivalent for proof in this fashion.

The teaching of strategies such as reading the question and writing out examples and non-examples was seen in a few participants, Shaun and Beth for example, but not all. In fact, some MATH 305 participants rarely viewed examples at all, such as Lisa and Ellen who were both, for the most part, successful in their proof-writing attempts. It is therefore legitimate to question what effect we, as educators of these beginning proof writers, can have on the ultimate method that students will choose to use.

However, lest we fear that all of our teachings are a total loss, it is also important to note that almost all participants in this study from the MATH 305 course included some of the strategies and ideas they learned in this course during their proof-writing attempts, and were able to identify the ideas as having come from this course. Other participants from MATH 406 also linked ideas they used throughout their proofs with other proofs they had seen or work they had been taught in previous courses.

Successful and Unsuccessful

The participants can be grouped into two categories, those who were identified as being successful in at least half of the questions they attempted, and those who were not. The first group consists of Lisa, Ellen, Beth, Lily, and Andy from MATH 305 and Sam, Maggie, Paul, and Vicki from MATH 406. The second group, those who were not successful at least half of the time, includes Shelly, Jon, Rick, and Shaun from MATH 305, and Julie, Sandy, Amy, Jill, and Katy from MATH 406. The personal tendencies of each of these are summarized below.

Successful Participants Overall

Lisa was successful in three of the four questions she attempted. She was well organized, careful, and paid attention to details throughout all of her work. She was even able to identify potential problems that could have occurred had she not been careful in her notation in Question 4. The only question she was unable to complete was Question 3, where she only identified one direction of the proof that was necessary. Lisa did not encounter the same errors as others had and was able to work quickly through her own systematic process in each question.

Ellen completed proofs for two of the three questions she attempted, leaving only Question 2 unfinished due to time constraints. She was also organized in her work and was systematic in her approach. Unlike Lisa, Ellen did encounter a few errors, but was able to identify and correct these errors. Ellen was a graduate student, so she had previously experienced proof writing, but felt her skills needed improvement and therefore enrolled in MATH 305. She was able to express herself and her ideas to the researcher clearly, and stopped herself when work was not productive to redirect to new ideas.

Beth was also extremely organized in her work, with attention to details and an ability to work through every planned step of her work. She completed valid proofs to both questions she attempted, though she did take one interview for each question. She also was able to correct errors that she encountered along the way and kept her goals in mind throughout the process of proving the statements. She linked her work to the tools she learned in MATH 305 and had a definite plan for approaching each question.

Lily successfully completed proofs to Questions 1, 3, and 4, and Question 2 part a. However, she was unable to find new ideas for part b of Question 2 and so could not work through a proof there. Her work was also systematic, but often silent. She was able to redirect when necessary and understood what was needed in each question. However, she did need prompting several times throughout both interviews in which she participated to verbalize her thoughts and move past mental road blocks that she encountered.

Andy was another quiet individual during his interview. He was able to complete a valid proof for both questions that he attempted, though he only recognized the proof to one of the questions, in the other he failed to see his argument as valid. Andy seemed to also be aware of his process and even noted that he needed to consciously put himself in the correct frame of mind to view the questions as proofs instead of problems to be solved. He clearly monitored his own work, and redirected when he saw that he was going astray from the proof he intended.

Sam was also a graduate student, and his work closely resembled Ellen in many areas. He was very organized, had a clear plan for each question, and was able to express his ideas and directions verbally to the researcher. He completed proofs to Question 1 and Question 2 parts a and b, but was unable to finish part c due to some errors and a lack of time to finish correcting these errors. Sam did encounter other errors in Questions 1 and 2, but was able to correct those errors and form a proof. He worked systematically, used previous work for future portions of proofs, and was able to stay focused on the direction he needed to head.

Maggie was also clearly organized in her work. With several previous proof-writing courses in her past, she was comfortable with proof writing and had an ability to communicate with the researcher similar to that of Sam and Ellen. Maggie was able to complete proofs for all three questions that she attempted, though with some prompting needed at points to continue in her work. She was able to express a clear desire for a plan at the start of each question, monitor her work towards her goal, and even, at times, mentioned aloud that she felt her work was not leading her in the right direction towards a proof. However, she was not always able to find a main goal to begin the proofs and prompting was needed to help her search through multiple ideas to find success.

Paul also had several proof-writing courses in his background and found sophisticated proofs similar to Ellen, Sam, and Maggie. He was able to complete proofs for two of the three questions he attempted, but did not complete the proof for Question 3. He made a few errors along the way, but was able to move past these and correctly form proofs through monitoring, as well as promptings from the researcher. Paul was specifically focused on forming and manipulating equations, but was able to do so successfully, where other participants sometimes failed to form the proper equations. This could be due to a better overall understanding of the questions and what would constitute a valid proof. The only proof he was unable to complete was for Question 3, where he did not demonstrate a perfect cover, an indication of his clearly abstract mode of thinking, in which he did not see the need to develop such a concrete picture.

Vicki successfully found a proof in two of the four questions that she attempted, however she felt that she had found a proof to all four questions. Vicki made several errors in her work, but was able to overcome most. Vicki was able to work through each question recalling all her steps and working without going in circles, with the exception of some of her work in Question 4. However, she was not able to make an appropriate plan in some cases, or to recognize what would be necessary for a valid proof. Unlike the other participants in this first group, Vicki struggled to link specific examples to the computations and equations she desired to use. While she was partially successful, her work did not compare to the other successful participants already mentioned, but more closely mirrored that of the unsuccessful group.

Unsuccessful Participants Overall

The second group of participants includes both those who lacked clear direction in all of their proof writing, and those who simply could not overcome the errors in their thought processes along the way. Several of these participants tended to over-think each scenario and attempt to fit each question into a preconceived notion of what a proof should look like. Shelly is an example of such a participant. Shelly was a graduate student, but unlike the other graduate students in this study, was unable to make clear plans for her work or to keep track of her work along the way. She also failed to recognize portions of the valid proofs she actually found in her work. She was easily distracted and unable to recall her previous work and ideas as she went. Shelly clearly had an idea in mind of what a proof should look like, and what the process of finding that proof should also resemble, but was unable to fit her ideas into this picture. Even with good ideas, and discoveries of key notions, she was still unable to pull together the complete proof in the end, only able to complete one proof out of three in entirety.

Jon also had a clear concept in mind of what a proof would look like. He desired in each question to fit the information into equations, which he then would manipulate to the desired proof. However, not all questions were this straight forward. He struggled many times to develop equations, and in Question 2, even after having these equations, was unable to complete the arguments needed to use them for a proof. Jon also discovered many key ideas, and made progress towards a proof in all four questions, however he was unable to recognize potentially successful work that did not fit into his concept of a valid proof and therefore able to complete only proofs for Question 2, out of all four questions he attempted.

Shaun was also focused on proof in one light. He desired to make each question into a logical statement, which he could then prove using tools from MATH 305. However, he had difficulties with notation, and even in the cases where this could have been a successful idea if used correctly, he was unable to form a proof. In the end, he was unable to complete a proof for any of the three questions he attempted during two separate interviews. There was clear evidence of strategies that he had learned in MATH 305, including specific reference to understanding the question and writing out examples and non-examples. Shaun had used some of these ideas in a positive way during his work. His focus on logical notation, however, derailed all other ideas and did not allow him to complete any of the proofs.

Rick's work was not nearly as organized as Shaun's had been. Rick tended to work in circles, forgetting his previous ideas and having to reprove each detail numerous times. He was able to complete a proof for one of the three questions he attempted, however only after a great deal of effort spent recalling ideas and rewriting proofs. While his written work was filled with organizational notes and details, Rick was still unable to recall ideas. This occurred in all three questions, and was particularly a struggle in Question 2. Here, he was unable to see his proven limitations on the possibilities and unable to complete the small amount of work that was actually needed to finish the proof, since he did not understand what would constitute a proof. In all questions, Rick attempted to make a plan and keep track of his work, but was ultimately unable to do so. Even when noting his efforts to conform to the process used in MATH 305, Rick still could not overcome the errors he made and the inability to recall previous efforts.

Julie, on the other hand, was much more organized and understood each of the two questions that she attempted. She was able to recall her previous work and ideas, but ultimately did not known what would constitute a valid proof, and so was unable to recognize that her proofs were not complete. Julie was able to prove only part a of Question 2, but not able to form proofs for parts b or c, or for Question 3. She had made an error in Question 2 in her meeting point idea that she did not move past, since she felt that it was a legitimate proof. In Question 3, Julie had great ideas that would have resulted in a valid proof if fully explained, but like her work in Question 2, she did not prove the necessary details to be truly successful. Her organized work and other successful strategies are indicators of her abilities, but the lack of motivation and understanding of the remaining details to be proven left her without valid proofs.

Sandy, the first in the list of unsuccessful MATH 406 participants, had difficulty moving past specific examples to a generic proof in several cases. She was ultimately able to complete only one valid proof of all four questions she attempted. For Question 1, she was able to verbally describe what was needed for the proof, but this question was much more suited to viewing individual numbers and specific pentagons. When she attempted the other questions, though, Sandy needed to move past her specific numbers in order to make general arguments, but she was unable to do so. In Question 3, she even lacked the understanding that such a general proof would be necessary. When pushed to move into this realm, she did so only in a small portion and was unconvinced that it was even necessary. In Question 2, this lack of motivation to prove the question in general surfaced again. Sandy was a participant who specifically struggled with motivation and ability to formally prove a statement once she was convinced of the validity of the statement.

Katy also struggled with this lack of motivation and understanding of how to move past the notions in her head to a formal proof. She was actually able to verbalize this internal struggle and stated that she was having difficulty in doing just that in the proofs during her interview. Katy attempted three questions, but was unable to find a valid proof for any of the three. Her work on all questions contained great ideas and processes that could have resulted in valid proofs, but she was unable to complete the proofs. Her arguments tended to go around in circles, with Katy losing track of her ultimate goal during the search. Katy was able to redirect her work several times and realized that she was getting lost, but ultimately was not able to recover on any of the questions and understand what would be needed for the proof.

Jill was another graduate student, with a significant background in mathematics. Like the other graduate students previously mentioned, she was able to communicate effectively with the researcher and describe her thoughts and ideas as she worked. She kept track of her work along the way and made specific plans for each proof, which should be have had a positive result in the end. However, Jill became frustrated and distracted several times during the interview. Where other participants had viewed some questions as problems to be solved and failed in this manner, Jill recognized this tendency and struggled to turn away from it. In doing so, she abandoned all concrete notions and focused too much on the abstract. She was unable to move past this desire for generic proof in order to utilize the real examples that she was viewing to inform such a proof. In this way, she blocked the potential proofs that could have been developed and ran out of time to complete any further work.

Amy was a different case altogether. She struggled greatly throughout the interview to grasp the new definitions and recall her work. She attempted Questions 1 through 3, but was unable to find valid proofs for any of these questions. It seemed that her main goal was to visualize the questions and develop a proof through this imagery, but was unable to bring this proof to complete fruition. At the end of the interview, however, she revealed that she actually suffered from a learning disability called dyscalculia. This disability is a form of dyslexia associated specifically with numbers, where Amy would easily transpose digits in the numbers she was working with. In this way, she was unable to keep track of her work and unable to work through the proofs given to her. She did strive to keep lists and organize her work, but was not able to do so to an extent that afforded her the help that she needed to work to a proof. These questions were therefore most likely not a good judge of her proof-writing ability due to the nature of the proofs that were required.

Conclusions

The framework of this study was based on three categories of proof processes, procedural, syntactic, and semantic. All three of these processes were evident in the results of this study. The goal of the study was to further describe the details of the proof-writing strategies that are included within these categories. As discussed, several sub-categories of strategies emerged; the four of most interest are use of examples, use of equations, visualizations, and self-regulation.

The use of examples both helped and hindered participants in this study. Alcock and Weber (in press) specifically studied the use of example in proof writing, and the related purposes of use, difficulties in construction of examples, and rate of success among use. The results of the study are strikingly similar to my own. First, Alcock and Weber found that success was not dependent on the use of examples, but that the use of examples could be key to a student's proof. They point to two key students in their study, both of whom were successful overall, but with very dissimilar tendencies, Brad and Carla.

> Both were coherent and both took actions that were mathematically sensible and likely to lead to progress. Brad, however, spontaneously referred to examples in response to all of the interview tasks, while Carla never did so. We note that this occurred despite the fact that they had attended the same class and so had been exposed to the same lectures, the same homework assignments and the same earlier examination. (p. 13)

Across all students in the study, six used examples in multiple questions, while four never used examples at all. These results are consistent with my findings. It further points to the tendency of students to continue to use one means of constructing a proof, regardless of the statement being proven.

The use of equations was another common strategy in my study, but was only seen in use during procedural or syntactic proof productions. During these episodes, participants had difficulty understanding the main ideas of the statement, and the equations were often expressed that did not fully represent the situation. These participants would model equations in a procedural manner, relating statements to other notions learned prior, but without a true understanding of the question they were unable to do so in an appropriate manner. Smith (2006) found that

> The students from the traditional section tended to begin the proving process by listing everything they knew that might possibly relate to the proposition. This 'listing' of properties and choice of proof strategies appeared to be based on *surface features* of the statement to be proved, rather than on an understanding of the problem or concept. (p. 82)

This was seen in this study as well, for example when several participants wrote what they knew of the shape and properties of a pentagon on Question 1, with no reference to what these features would say about the theorem to be proved.

Furthermore, participants in this study sometimes forced the situation into logical notation, in a syntactic approach to the process. Again, without the full scope of understanding of the question, this notation was often incomplete or incorrect. Even when correct, participants, like Shaun, were unable to merge the formal nature of the notation with their informal understanding of the situation and what needed to be proved. Smith (2006) also found that students who had been introduced to the modified Moore method of instruction "tended to introduce notation in logical and natural ways in the context of making sense of the proposition to be proved" (p. 82), thereby avoiding the useless notation seen among participants in this study.

The use of both examples and equations was sometimes done for the purposes of visualization. Other visuals, such as drawings and tables, also helped participants to understand the statement of the question and verify the theorem. However, many participants then became stuck when trying to move past this understanding to form a proof. Once convinced of the validity of a statement, they were unable or unwilling to prove the result formally. Moore (1994) also found this to be the case among the participants in that study. "Concept images lack the language needed to express mathematical ideas. The students often commented that they 'understood' a proof, or a step in a proof, but did not know how to say it" (p. 257). Williams (1980) also found that "approximately half the students sampled did not see any need to prove a mathematical proposition which they considered to be intuitively obvious" (p. 166). This lack of motivation to complete a formal proof is often an issue in transition-to-proof courses.

In all cases of strategy use, those participants who used strategies with an overall goal in mind, one that was attended to both cognitively and metacognitively, were more successful than those who used these strategies only as cognitive tools. Self-regulation played a major role in the success of participants in this study. Evidence of this arose in the use of monitoring and organizational techniques, as well as participants actively questioning their own choices and paths that were taken. These tools often helped participants move from the use of rote procedures or specific examples to a view of the overall question and to make a plan for a formal proof.

A key discovery made in this study was the consistency in strategy use across questions by each individual participant. Whether participants chose to use examples, an equation, lists, pictures, or other techniques, the use was constant throughout every question they attempted, almost without exception among all participants. Even with the similar background of MATH 305 participants, having all learned the beginnings of proof writing from the same instructor, in the same semester, there were still unique tools in use for each participant. It is therefore vitally important that we consider the broad use of self-regulation as a means to cope with any set of strategies appropriately. When approaching this study, this researcher believed that teaching students to use a set of strategies would be key to their success. However, this evidence shows that instead we need to assess each student individually and attempt to change the regulatory behavior controlling the strategies rather than the strategies themselves. Alcock and Weber (in press) stress this individualized picture as well.

> We therefore wish to recognize the value in what each student is doing, and think in terms of helping the students to build upon their existing strategies and augment these with skills that they may well possess but do not often spontaneously invoke. (p. 37)

Suggestions for Future Research

Suggestions for adapting this methodology

Some of the limitations of this study could be addressed in future research. Throughout the last two years, as this researcher has analyzed the data and written this work, she has discovered several factors that could be changed in future research, which could benefit the completeness of the data collected, lessen the work involved in analysis, and provide additional conclusions to be reached. For example, it would be useful to consider what results would be found if participants were required to formally write proofs, rather than being allowed to express their proofs verbally. During analysis, this element would have been very beneficial for addressing the accuracy and completeness of participant work.

Inconsistent promptings may also be remedied in further studies. The pilot study was designed to test the questions, as well as to test the interview protocol. However, since the participants involved in the pilot study were graduate students, and one undergraduate with a high level of prior mathematical experiences, the data gathered lacked insight into how novice students would approach the questions and what struggles they would face. This researcher suggests that, in the future, pilot studies be conducted with only a few students, but from the same population as the main study. In this way, new researchers can gain needed experience in interviewing techniques, learn which cues are appropriate and how much help he or she wishes to give in the study, as well as how well the interview protocol fits actual situations which may occur. The interview protocol for this study was designed to allow for spontaneous interaction with the participants, which was helpful, however the amount of interaction and help given was inconsistent in the interviews. This difficulty should be addressed in future studies of this type.

This study encompassed only 18 individuals, and so broad generalizations cannot be made. Further studies to support these findings are needed within the scope of novice proof writers, as well as a comparison to expert proof writers with these specific ideas in mind. However, the number of participants in this study was a large number for such a study. Future studies using fewer participants may be able to place an even more fine grain analysis on the data. This researcher suggests that, now that this larger study has been completed, studies with fewer numbers of participants are needed to address the specific ideas found here in even greater detail. For those who wish to conduct exploratory research such as this study, the larger size of the study did allow for a broader picture of the data, was beneficial in reaching conclusions in this study, and is suggested for future work as well.

An extremely useful component of this study was the ongoing observation of MATH 305 students during the semester, from which the researcher could compare the strategies in use during the interviews and group sessions to determine what behaviors and strategies were normal for the participants. An obvious extension of this observation, were the study to be repeated, would be to observe the MATH 406 students as well. Additionally, group sessions involving those students would have been useful information to draw from for comparison. This researcher suggests such interaction for future studies of this type, as well as second, or even third, interviews to gain information on all questions for all participants. The comparison of participants over all questions would add a nice final component to this study and would be useful to gain a complete picture of each participant in all questions.

With regard to the specific information gathered during the follow-up questions during the interview, in hindsight this researcher would have found it helpful to have more information from the MATH 406 participants on where they had completed the equivalent of the introductory proof-writing course, when it was taken, and who had been their instructor for that course. This information could have been used to compare the experiences of these participants. It would also have been helpful to address, to a greater degree than was done during the study, the question of where all participants had learned the strategies they used. While participants may not have been able to self-report this data, any observations they made would have added to the overall picture. Furthermore, this researcher felt that a preliminary version of the overall grid of strategies could have been completed after the pilot study and the information gathered in this grid could have added to the further questions asked during the main study.

A different style of interviewing could also be used, adapting the way in which participants are prompted. A participant could be asked to think aloud with minimal input from the interviewer until the participant feels that he or she has completed as much of the question as is possible. At this point, the interviewer could interject hints and suggestions to help, and additionally could ask specific questions targeted to promote self-regulatory behaviors. An analysis could then be completed on the amount of additional work that was completed after the hints were given, and what impact the guidance in self-regulation played in that work.

Lastly, the specific data collection procedures could have been improved in two ways. The first is that a second camera aimed at the participants' faces could add a beneficial element to the study. With this additional view of the participant, clarifications would be easier to make in the transcript, and expressions could be seen to clarify intent of statements. This angle would allow the researcher to address other non-verbal cues of misunderstanding and other such issues. The second data collection procedure that could be improved is that done during the group sessions. Upon review of the tapes of these sessions, it became very difficult to understand the participants and clarify who was speaking at what time. This researcher suggests that future studies employ the use of multiple camera angles as well as possibly microphones for each participant to allow for ease in transcription. The written work of these participants was not collected during these sessions, since it was considered review for an upcoming test in the MATH 305 course. However, this researcher would suggest photocopying and returning original written work to participants so that this work could enter into the data of the study.

Suggestions for furthering this research

Weber and Alcock (2004) addresses four abilities that provers must possess in order to reach a semantic proof production.

- One should be able to instantiate relevant mathematical objects.
- These instantiations should be rich enough that they suggest inferences that one can draw.
- These instantiations should be accurate reflections of the objects and concepts that they represent.
- One should be able to connect the formal definition of the concept to the instantiations with which they reason. (p. 229)

These four items require an adequate content knowledge of the subject, an adequate basis of representations from which to draw, an ability to choose the appropriate instantiation, and the ability to relate the chosen representation to the overall scope of that which they are trying to prove. These needs could possibly require repeated exposure to proof writing before being fully attainable. Further study of the proof-writing strategies of experts would be of value to discover whether these attributes are in evidence later in research-level mathematics.

Weber (2001) stated that “one would expect that the doctoral students’ strategic knowledge was shaped over many mathematical episodes, including the wrong turns made during proof attempts, discussions with others, and learning about other mathematical domains” (p. 115). However, he further points out that students often develop poor strategies, or inappropriate uses of strategies, when left to discover these on their own. "Experience alone generally is not sufficient to remedy these deficient strategies" (p. 116). It would therefore also be of use to study what types of experiences actually further the acquisition of appropriate strategies and self-regulatory behaviors to control these strategies.

Self-regulation strategies could be of use to overcome the difficulties associated with computational and comprehension errors, as well as notational difficulties. If participants have a global view of their goals, rather than a specific view only of the individual attempt they are making, it could provide them the ability to recognize the errors they are making as outside of the bounds of what is expected. With metacognitive behaviors could come the potential to move to an understanding of the question in entirety, rather than participants being unwilling to move past the procedural processes often in use.

This study shows great promise in the area of self-regulation and its impact on proof writing in mathematics. However, further research on the impact of specific training in self-regulation strategies at the novice level in proof writing is necessary.

Other areas of interest to this researcher as a result of this study include specific ways to teach (if possible) the use of self-regulation strategies in proof writing, the impact of this training on the ability to write proofs when directly addressed, and the extent to which flexibility in strategy use plays a role in proof-writing ability.

A few other ideas surfaced during the analysis of this data that were not specifically addressed by this study. The first is the lack of motivation to complete a proof, particularly after the student understood the question and the justification was clear to him or her. This issue arose for many of the participants, however it was not an original intention to view this particular difficulty. Future research on this apparent lack of motivation, the reasons for its occurrence, as well as techniques to help motivate students in proof writing appear to be of value, based on the observations made during this study.

This study has added to the body of knowledge in proof writing and understanding the proof-writing strategies of novices. The further ideas presented in this section serve as a starting point from which to continue past this work, particularly with the purpose of understanding the behaviors that lead to success in proof writing and identifying how to help students gain the skills identified, by this and future work, as beneficial in the process of proof writing. It is the hope of this researcher that such knowledge will lead to an increase in the overall proof-writing abilities of students at both the undergraduate and graduate level.

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APPENDIXES

Appendix A: Anticipated Useful Strategies for Each Question

1. Pentagon proof:
    - Read the question carefully
    - Understand the question
        i. Break into smaller parts
    - Look at examples
        i. Look for pentagon with sums of 14
        ii. Look for pentagon with smaller sums
            1. Recognize that 13 is the only smaller option to be checked
            2. Prove that 13 is not possible use good system of choices
    - Draw pictures/ Visualize
        i. Draw pentagon
        ii. Organize final choices for labels in table format
    - Look for patterns
        i. Note that the largest 5 numbers need to go on sides instead of vertices (either from example pentagon or trial and error)
        ii. Choose to place the largest numbers first
        iii. Recall similar questions
        iv. Work systematically through the possible choices
    - Develop equations
        i. Equations for the overall sum
            1. Recognize that the vertices will be counted twice
        ii. Convert words to algebra
    - Identify and use proper proof techniques
    - Self-monitor
        i. Make a plan
        ii. Monitor trials
        iii. Monitor overall plan
        iv. Organize work
        v. Write what is known and what needs to be shown
        vi. Redirect after a failed attempt
    - Recognize you have a proof
        i. Understand that pictures and argumentation from pictures can constitute a proof
        ii. Understand that equations may not be necessary for a valid proof

2. 4-flip proof:
    - Read the question carefully
    - Understand the question
        i. Reword the question
        ii. Define all terms
        iii. Break into smaller parts
    - Look at examples
        i. Systematically look through all choices for each digit

   ii. Look at small examples to eliminate intervals of numbers
- Look for patterns
   i. Build from small cases
   ii. Use results from earlier parts of question to guide new proof
   iii. Recall similar questions/ portions of questions
- Develop equations
   i. Convert words to algebra
   ii. Recall notation for digits (i.e., the number ab can be written as 10a+b)
   iii. Visualize spots and placeholders (e.g., 4(_ _ 3) = _ _ 2)
- Identify and use proper proof techniques
- Self-monitor
   i. Make a plan
   ii. Monitor choices
   iii. Monitor examples looked at
   iv. Organize work
   v. Keep definition at forefront
   vi. Write what is known and what needs to be shown
   vii. Monitor progress, keep track of choices eliminated
- Recognize you have a proof
   i. Recognize that arguments for eliminate choices constitute proof that those choices will not work
   ii. Recognize that brute force is a legitimate proof technique

3. Dominoes on chessboard:
- Read the question carefully
- Understand the question
   i. Reword the question
   ii. Define all terms
   iii. Break into smaller parts
- Draw pictures
   i. Visualize chessboard and dominoes
- Look at examples
   i. Look at small examples
   ii. Build from small examples
   iii. Recognize and look at examples of different parities
   iv. Recall similar questions/proofs
- Identify and use proper proof techniques
   i. Notice the if and only if statement needs to be proven in both directions
   ii. Work through different proof techniques
- Develop equations
   i. Unpack the definitions of odd and even
   ii. Consider the product (or overall number of squares) in relationship to even and odd dimensions
- Self-monitor

  i. Make a plan
  ii. Write what is known and what needs to be shown
  iii. Monitor attempts
  iv. Organize work
  v. Redirect after failed attempts
  vi. Stay on course for overall goal or proof technique
- Recognize you have a proof
  i. Understand that a draw and description of a layout of dominoes in a case does constitute proof that the case has a perfect cover
  ii. Understand that one counterexample shows a case to be false
  iii. Understand that equations are not necessary to prove the statement

4. Transitivity of divides:
- Read the question carefully
- Understand the question
  i. Reword the question
  ii. Define all terms
  iii. Unpack definitions
  iv. Break into smaller parts
- Look at examples
  i. Look at small examples
  ii. Build from small examples
  iii. Work forwards (i.e., from what is known to what needs to be shown)
  iv. Work backwards (i.e., from what needs to be shown to what is known)
- Proper notation
  i. Careful use of notation (e.g., a divides b ≠ a/b)
  ii. Use definitions correctly
  iii. Label constants with different variables (e.g., $b = ak$ and $c = bl$)
- Develop equations
  i. Convert words to algebra
  ii. Unpack definitions into equations
- Identify and use proper proof techniques
- Self-monitor
  i. Make a plan
  ii. Monitor attempts
  iii. Monitor progress towards the overall goal
  iv. Organize work
  v. Write what is known and what needs to be shown
  vi. Watch for incorrect switches from work from one direction to the other (e.g., using both what is known and what needs to be shown simultaneously)
  vii. Redirect after failed attempts
- Recognize you have a proof
  i. Understand that one example does not prove a statement

Appendix B: Interview Protocol

*All instructions marked with a * are for interviewer purposes only. All other instructions are to be read to the participants. Try to limit each problem to at most ½ hour in order to get to at least two questions.

This interview will last for approximately one hour. During this hour, you will be asked to work on two to three problems, as time permits. These problems are designed to be challenging and not to be answered quickly, so do not be frustrated if the answers are not immediately obvious. The purpose of this interview is to study the effort made in formulating your answers, not in the answers themselves. In this respect, you should attempt to work aloud as much as possible. Write down whatever you feel necessary and do not erase anything, so that your progress through the problem can be documented. I will ask you questions as you work to illicit responses, which will help me to understand your thinking and the processes you are using. At any point in time, if you have any questions or need any clarification, please do not hesitate to ask. We will now begin the interview by reading the first problem.

*Give student a copy of problem one and read the problem aloud to student.

Do you have any questions about this problem? Is there anything that needs clarification? Please begin working on this problem now. I will give you a minute to collect your thoughts. Please begin working aloud as soon as you are ready.

*After one minute, ask student what they are thinking if they have not yet begun work.

Probing questions:
1. Explain in your own words what the question is asking so that I know you understand it.
2. How do you think you should begin this problem?
3. Tell me what you are thinking.

*Allow students to work uninterrupted so long as they are explaining what they are doing, working aloud, or working on the paper. Ask questions if they get stuck or are not working. Additionally, ask clarifying questions if there is anything they try that is not clear. If they work for a period of time without talking, wait until they slow down or stop working and then ask them to recap what they have just done for clarification.

Probing questions:
1. Please tell me more about ____________.
2. Please write down what you are thinking about _________________.
3. Where would you go next?
4. What is the next step?
5. What else would you try?
6. I would like to understand _______________ better. Please explain it to me.
7. You just told me about __________. Can you go any further with this?

*Once students seem to have reached the end of their work or when they need to be wrapped up ask the following questions.

Wrap-up questions:

1. Do you believe you have gone as far as you want to on this problem?
2. Have you justified your solution as much as you would like?
    a. Please add justification to _________________.
    b. How would you go about showing __________ is true?
    c. Do you need to add anything to finish this proof or is it complete?
    d. Are you satisfied with your solution?

Now, let's move on to the second problem.
*Give student a copy of the second problem and read the problem aloud to student. Again ask for any questions. Repeat the above questioning/observing procedures.
*After both questions are complete (possibly three if time), ask the following questions.

Final Questions:

1. Can you identify any strategies you used to construct these proofs?
    a. Look back at your work. Was there anything specific you were trying?
    b. If asked to summarize your work, what steps would you say you went through?
2. What are some of the ways you approach problems such as this?
3. Please tell me more about what you tried _____________________.
4. Do you recall ever being taught any of the steps/strategies that you have used today or that you use on a regular basis to construct proofs?
    a. Who, if anyone, taught you ______________ (i.e., another student, course instructor, self-taught, from book, etc.)?
    b. When do you recall first using ________________?

Appendix C: Consent form for research study

Title: Proof Processes of Undergraduate Mathematics Students

Project Directors:

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Purpose: You are being asked to take part in a research study looking into proof writing in mathematics. My goal is to better understand how students think about proof, how they approach proof, and what tools we can equip them with to better construct a proof.

Procedures: If you agree to take part in this research study, the following will occur:
Part 1: You will be given a questionnaire to evaluate your learning preferences. You will be asked to take this questionnaire home to complete and return the next class period. Extra credit will be given in the associated mathematics course you are taking for your participation in this study.

Part 2: Some participants from part 1 will be asked to participate in a task-based interview where you will be given two mathematical proofs. These will in no way be graded or evaluated beyond the scope of this study. I ask only that work on these proofs to the best of your ability. You will be asked to describe the steps you are taking and talk aloud as much as possible. The session will last for approximately one hour. At the end of the session, you will be given a final questionnaire on your prior experience in proof writing in mathematics, your comfort level in this area, and your learning preferences. You will be given a gift certificate for your participation in this interview.

Risks/Discomforts: You may experience some anxiety or frustration during this study in working through the problems. Please remember that this feeling is normal, however, also remember that the problems are in no way graded nor will your work be personally identifiable in any way to anyone other than the researcher and her faculty supervisor.

Benefits: Although you may not benefit directly from taking part in this study, your help will assist me in developing protocol for a more in-depth study to be done this spring as well as guide me in developing a list of proof writing strategies which could help students when constructing a mathematical proof. Additionally, all participants will receive a $5 gift certificate and extra credit in your math course, as previously discussed, as compensation for your time.

Confidentiality: Only the researcher and her faculty supervisor will have direct access to the information we gather here. Your identity will be kept confidential. If the results of this study are written in a scientific journal or presented at a scientific meeting, your name will not be used.

Compensation for Injury:

Although we do not foresee any risk in taking part in this study, the following liability statement is required in all University of Montana consent forms.

> In the event that you are injured as a result of this research you should individually seek appropriate medical treatment. If the injury is cause by the negligence of the University or any of its employees, you may be entitled to reimbursement or compensation pursuant to eh Comprehensive State Insurance Plan established by the Department of Administration under the authority of M.C.A., Title 2, Chapter 9. In the event of a claim for such injury, further information may be obtained from the University's Claims representative or University Legal Counsel.

Voluntary Participation/Withdrawal:

Your decision to take part in this study is entirely voluntary. You may refuse to take part in or you may withdraw from the study at any time without penalty or loss of benefits to which you are normally entitled. If you decide to withdraw, inform the researcher immediately upon your decision. You may make this decision after your participation is over. In this case, please contact the researcher with the contact information above. All materials which are associated with your participation in this study will then be destroyed. You will still receive the incentives mentioned above even if you withdraw from the study. You may be asked to leave the study if you fail to follow the Project Director's instructions or if this study is terminated.

Questions:

If you have any questions about the research now, during the study, or after your participation please contact the study director named above. If you have any questions regarding your rights as a research subject, you may contact the Chair of the IRB through the University of Montana Research Office at 243-6670.

Subject's Statement of Consent:

I have read the above description of this research study. I have been informed of the risks and benefits involved, and all my questions have been answered to my satisfaction. Furthermore, I have been assured that any future question I may have will also be answered by a member of the research team. I voluntarily agree to take part in this study. I understand that I will receive a copy of this consent form.

____

Printed Name of Subject

__________________________________ ____________

Subject's Signature Date

Appendix D: Questionnaire #2

Questionnaire

Name:

Please answer the following questions honestly as they apply to you. All answers will be used for data collection purposes only. Your identity will be kept confidential and no answers will used for purposes outside of this study. For each question, circle the answer(s) that best describe you, you may choose as many answers for each questions as you believe pertain to you.

1. I am most comfortable working on schoolwork
    a. Alone.
    b. With a small group (2-4 people).
    c. In a large group (5 or more people).

2. I consider myself an
    a. Extrovert
    b. Introvert

3. When working on math homework, I prefer to
    a. Talk out my thoughts and ideas
    b. Write down my thoughts and ideas.

4. While working I tend to
    a. Do most of my work silently in my head
    b. Write everything down
    c. Draw pictures
    d. Talk to myself
    e. Talk with others

5. I consider myself to be
    a. Not very comfortable with proof writing
    b. Somewhat comfortable with proof writing
    c. Very comfortable with proof writing

6. I prefer working with things that are
    a. Drawn out in pictures or graphs
    b. Written out in symbols
    c. Written out in words

7. What other math courses have you taken in your undergraduate career? (either here or at another university) Please list as many as you can remember

8. Overall, what would you say your average grade was in these courses?

9. Would you describe the work in these courses as
    a. Very difficult for me
    b. Somewhat difficult for me
    c. Neither difficult nor easy for me
    d. Somewhat easy for me
    e. Very easy for me