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STUDENTS' DEVELOPMENT IN PROOF: A LONGITUDINAL STUDY

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Students' Development in Proof: A Longitudinal Study

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Despite importance of teaching proof in any undergraduate mathematics program, many students have difficulties with proof (Dreyfus, 1999; Harel & Sowder, 2003; Selden & Selden, 2003; Weber, 2004). In this qualitative case study, nine undergraduate students were each interviewed once every two weeks over the course of an academic year. During each interview, the students were asked to complete, evaluate or discuss mathematical proofs. The results of these interviews were then analyzed using two different frameworks. The first focused on *proof type*, which refers to what kind of proof is created and how it came about. The second framework addressed identifying each student's *proof scheme*, which "constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998). Using these structures as a guide, the question I sought to answer is: What, if any, identifiable paths do students go through while learning to prove?

Unfortunately, the data from this study failed to demonstrate any identifiable path that was common to all participants. In fact, only a single student made clear progress as judged by the criteria laid out at the beginning of this study. Specifically, the way she attempted proofs changed which was reflected in a greater tendency to use a particular proof type as time passed: *semantic*. Of the other students, six entered the study with a fairly mature view of proof that remained unchanged and thus had little progress to make relative to the frameworks used in the study. These students were also generally successful with the proofs they attempted and were more likely to use semantic proofs. The remaining two students were generally less successful and used semantic proofs rarely. This seems to imply that as students become more comfortable with proof, they become inclined toward the semantic proof type and this coincides with becoming more successful with proof in general.

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Chapter 1: Introduction

1.1 Background

"Proof lies at the heart of mathematics. It has traditionally separated mathematics from the empirical sciences as the indubitable method of testing knowledge which contrasts with natural induction from empirical pursuits" (Hoyles, 1997, p. 7). Clearly, proof is essential in mathematics. As seen the quote above, it can even be thought of as part of what defines the subject. Professional organizations also recognize the importance of proof. In the *Principles and Standards* (NCTM, 2000) it states that all students should "recognize reasoning and proof as fundamental aspects of mathematics" (p. 56). More recently, in the 2004 Guidelines, the MAA's Committee on the Undergraduate Program in Mathematics stated, in the first recommendation that courses designed for students majoring in the mathematical sciences "should…ensure that students progress from a procedural/computation understandings of mathematics to a broad understanding encompassing logical reasoning, generalization, abstraction and formal proof" (Baker et al., 2004, p. 44).

Unfortunately, despite the importance of learning to prove in mathematics education, many students have difficulty (Coe & Ruthven, 1994; Senk, 1985; Selden & Selden, 2003; Weber, 2004). Weber (2004) even goes on to say that "there is widespread agreement that students have serious difficulties with constructing proofs" (p. 1). Since it does present such a difficulty for students, many researchers have identified specific troubles students have and looked for the underlying reasons for them (Bell, 1976; Ernest, 1984; Moore, 1994; Dreyfus, 1999).

The act of proving, however, is not simply a task to be mastered by students. Proof is also an important tool used by those who study mathematics (Hersh, 1993; Rav, 1999; Almeida, 2003; Barbeau & Hanna, 2008). Proof can be (and has been) described in many different ways.

This is due, at least in the past, to the diverse ways in which proof is used. The most fundamental aspect of proof, however, is its ability to verify results. "A proof is a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or generally agreed facts or principles" (Bell, 1976, p. 26). When put this way, proof seems a dry and tedious chore. Almeida (2003) goes even further when he points out that "in the current tradition, a mathematical proof is a pure thought experiment divorced from context: the truth of a result or statement is deduced on the basis of internally agreed and consistent axioms" (p.479). However, verification is not the only aspect of proof in mathematics. How these alternative features of proof are manifested depends somewhat on what one wants to accomplish.

For research mathematicians, proving can lead to new insights, generalizations and even new branches of mathematics (Rav, 1999). It can also serve to systemize mathematics into a deductive system (Bell, 1976; Almeida, 2003). Proof can and is also used to convince others of a statement's validity after one has convinced him or herself (Bell, 1976). This may even have been part of the reason why the notion of proof was invented in the first place (Kleiner, 1994, p. 293).

For mathematics educators, different aspects of proof come to the surface. "What a proof should do for the student is provide insight into why the theorem is true" (Hersh, 1993, p. 396). Here, the emphasis is not the fact that a certain result is true or what new insights can be gleaned from its verification. Instead, what is important here are the underlying reasons the result holds. Proofs can also be considered "bearers of mathematical knowledge" in the sense that they "have the potential to convey to students 'methods, tools, strategies and concepts for *solving problems*'" (Barbeau & Hanna, 2008, p. 345).

Given the prevalence of proof in all aspects of mathematics, it is little wonder so much effort has gone into the teaching of the topic. What may be more surprising, then, is that this instruction has not been more successful. Many researchers have looked into the reasons for this. Some suggest this difficulty stems from the fact that the way mathematicians view proof is different from the way it is portrayed in educational settings (Almeida, 2003; Bell, 1976; Hersh, 1993). Almeida (2003) points to the fact that the order in which proofs are created (*examples* – *conjecture* (i.e., theorem) – *proof*) is different from the order presented in classrooms (*theorem* – *proof* – *examples*). This is not the only way the presentation of proof affects students' proof performance. Dreyfus (1999) says that too often instruction (from teachers and textbooks alike) is vague about when they are convincing through visual, inductive or intuitive justification and when they are displaying deductive mathematical proof. This vagueness can lead, then, to students not knowing what actually serves as mathematical proof. Bell (1976) also recognizes this difficulty. Others lay some of the blame on the fact that students do not see a need to prove and therefore give no effort to do it (Coe & Ruthven, 1994; Harel & Sowder, 2003). On top of these potential roots of student difficulty, students must deal with the problem of the complexity of the mathematics itself (Bell, 1976; Ernest, 1984; Moore, 1994).

In an effort to study the ways students learn to prove, many frameworks have been developed. One of the most basic frameworks (based on number of proof distinctions) is due to Van Dormolen (1977). Van Dormolen recognized three levels of proof (rooted in Van Hiele's levels of geometric thinking): *ground level*, *first level*, and *second level*. Van Dormolen makes the case that students must progress from one level to the next. If they are forced to work at a level they are not ready for, they will resort to tricks and memorization to get by.

Balacheff (1998) also gives a relatively concise way to characterize proofs. His, however, focuses more on the ways students become convinced about the validity of a statement than on the student's level of thinking. The classes of proof Balacheff gives are: *naïve empiricism*, *crucial experiment*, *generic example* and *thought experiment*. Weber (2004) also gives a way to classify proof attempts. The types he identifies are: *procedural* (which includes sub-types *algorithmic* and *process*), *syntactic* and *semantic*. These will be discussed more thoroughly in the theoretical framework section of the literature review.

A more elaborate description of students' ideas about proof is provided by Harel and Sowder (1998). Instead of classifying only proofs created by students, they categorize what they call students' *proof schemes*. A person's proof scheme "consists of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p.244). There are three main classes of proof schemes, each with a number of subclasses and sub-subclasses. The main classes are *external conviction*, *empirical*, and *analytical*. This framework will be discussed in more depth later.

1.2 Research problem and purpose

Aside from Von Dormolen's levels, the authors mentioned above do not explicitly give a hierarchy for the different categorizations given. Also, little attention is given to students' movement between classifications. The way in which this research fits in with existing research is not because there is evidence that students' conceptions of proof or methods of tackling them have changed. The way it fits in is the longitudinal scope of the study which has allowed for the observation of changes that occur in post Math 305 (the transition-to-proof course at The University of Montana) students' approaches toward proof as they become more proficient (or not) at proof techniques over the course of one year of taking upper division mathematics courses.

This research is of use because the students in this study were in the process of learning proof, but also (at the end of the research) had the ability to look back upon and take stock of the progression from novice to someone comfortable with proof. Typically, in the reported research, snapshots of students are taken and researchers are left to figure out why the student did or did not have success in their proof attempt. Also, it is conceivable that researchers' own memories of becoming competent provers has some bearing on the way they view students' behaviors. In this study, much of that reflection is done by the students themselves while looking back over the course of months rather than years.

1.3 Definition of terms

For the purposes of this study, the following definitions will be used.

A *proof*, in general, will be taken to be a mathematical "argument that convinces qualified judges" (Hersh, 1993, p. 391). During the course of the study, there will be occasions that a participant will think that he or she has arrived at a proof when this is not the case. When this happens, their constructions will still be referred to as *proofs* because they think they have arrived at an "argument that convinces qualified judges."

A *proof process*, or *proof type*, refers to the way in which a participant works towards a proof. At times, participants will use a variety of methods when coming to a proof. In these cases, the proof given by the student will serve to determine the classification of the student's proof process. Later, the term *process* is used to describe one of the sub-classes of *proof processes*. To avoid confusion, the term *proof type* will be used usually often in the study.

A person's *proof scheme* "consists of what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244). A *proof scheme* is more inclusive than *proof process* because it refers to both the way participants work to convince the reader of their proof and also how they become convinced themselves. This is important because most participants realize the difference between what it takes to believe a mathematical result and the standards applied to mathematical proof.

Chapter 2: Literature Review

The purpose of this chapter is to give a review of some of the relevant existing research into the study of proof. The chapter is broken up into the following sections: empirical, philosophical, sociocultural\histoical, pedagogical and theoretical framework.

2.1 Empirical Section

While many of the studies done in proof are qualitative in nature this does not mean proof research lacks empirical research. This section examines some such work. The empirical research discussed here will focus on research reporting on student performance.

It has been well documented that undergraduate mathematics students often have trouble with proof (Senk, 1985; Coe & Ruthven, 1994; Selden & Selden, 2003; Weber, 2004). In fact, "there is widespread agreement that students have serious difficulties with constructing proofs" (Weber, 2004, p. 1). Results of studies will be given here, with reasons for student difficulties to be addressed in the literature review section dealing with pedagogy. However, there is encouraging research as well. It has been shown that students can make good progress if given the right instruction (Cobb et al., 1991; Fawcett, 1938/1966; Harel & Sowder, 2003; Maher & Martino, 1986).

Along with the framework which will be discussed later, Weber (2004) also gives the results of the students' proof attempts in the studies he reviewed. In the study involving the abstract algebra courses, 56 proof attempts were made by 8 students. Of those, 46 were syntactic in nature (with 24 successful proofs provided). In the other 10 cases, no progress was made. In the studies involving analysis students, 120 proofs were examined by 6 students. Here, Weber found much greater variety in the types of proofs attempted. While no progress was made in 27 cases, 48 procedural proofs were attempted, 28 were syntactic and 17 were semantic.

Selden & Selden (2003) found that not only did students have difficulty proving results; they also had trouble judging the validity of proofs given to them (at least initially). In the study, eight students were interviewed. The students were at the time enrolled in a course meant to transition students to upper-level, proof-based mathematics. Of the eight students, only two were able to prove a statement for which all students had adequate content knowledge. Then, in subsequent parts of the interview, the students were asked to judge the validity of proofs given to them. They had multiple chances to make their judgments, and the interviewer was the professor of their transitions course. The interviews could be described as "teaching interviews," and the students' judgments improved as the sessions progressed. However, the students' initial judgments were only correct 46% of the time – near chance level.

With the poor performance of college students in making the transition to proof, changes in secondary curriculum have made to help change this. Coe & Ruthven (1994) studied a group of students who went through a British reform curriculum that emphasized student-generated mathematics. They found that the students were able to effectively explore problems they were presented. This, however, did not help them provide proofs for the conjectures they made, even though they had been asked to. Of the 60 proof opportunities the researchers observed, they classified only 2 as an "attempt(s) at clear, logical deductive argument with reasonably explicit link between starting assumptions and desired conclusion" (Coe & Ruthven, 1994, p. 44). Among the other 58, 4 showed some attempt at explaining the underlying reason for why their conjectures were true and the rest of the attempts were empirical at best.

Senk (1985) also gives the results of research that looks at the performance of secondary school students. In that study, 1520 students who were at the time enrolled in full-year geometry courses were given tests in which they were required to fill in missing parts of proofs, translate verbal statements into "an appropriate 'figure,' 'given,' and 'to prove'" (p. 449) and write full proofs. The tests were graded on a 0 – 4 scale. Senk (1985) sums up the results by saying:

> …we see that at the end of a full-year course in geometry in which proof writing is studied, about 25 percent of the students have virtually no competence in writing proofs; another 25 percent can do only trivial proofs; about 20 percent can do some proofs of greater complexity; and only 30 percent master proofs similar to the theorems and exercises in standard textbooks. (p. 453 – 454)

As mentioned above, not all the existing research on proof is negative. Encouraging results have been found at all levels of education. For example, Cobb et al. (1991) describe a study in which 10 second grade classes were oriented to comply with a socio-constructivist theory of teaching and were then compared to 8 unchanged classes in the same schools. The results of the study indicated the students in the "project" classes performed better on conceptual problems, very nearly as well on computational problems and had a more positive attitude toward mathematics when compared to the students in the "non-project" classes.

Maher & Martino (1986) discuss research in which students were (over the course of 5 years, from grades 1 to 5) given problems to work at in groups. These problems involved making claims and justifying them to a teacher and/or fellow students. In the article, one student was followed and 11 "episodes" were described showcasing her progression toward formal proof. During the progression, the student recognized patterns that she later applied to similar problems, displayed the desire to provide a "for all" type justification and used logical inference (characteristics of transformational-based proofs).

In the secondary education setting, Fawcett (1938/1966) performed a study whose purpose was "to describe classroom procedures by which geometric proof may be used as a means for cultivating critical and reflective thought and to evaluate the effect of such experiences on the thinking of pupils" (Fawcett, 1938/1966, p. 1). The procedures were carried out with an experimental class as well as a control class. Not only did the experimental class out-perform the control class (and two others from different schools, included to make a larger comparison) on a post-test covering the nature of proof, but the experimental class also performed very well on the standardized geometry test: the Ohio Every Pupil test. The median class score was 52.0 out of 80, while the statewide median was 36.5. The experimental class median score was good enough to be placed between the 80th and 90th percentile statewide.

At the college level, Harel & Sowder (2003) provide principles they used in a transition course designed to "make proofs tangible." By this, they mean to make proofs concrete, convincing and essential. They reported that this method did not have a drastic effect on students over the course of one semester. However, findings from further observations over the course of several semesters of courses suggest "that the instructional treatments we employed in our teaching experiments are potentially effective and can advance students' conceptions of proof" (Harel and Sowder, 2003, p. 266).

Besides the results mentioned here, the research into proof has also helped give an understanding of what mathematicians and students think about proof. These insights are in given the next section.

2.2 Philosophical Section

The views mathematicians have had about proof is not something that has stayed static over the history of mathematics (Kleiner, 1991; Moreno-Armella, Sriraman, & Waldegg, 2006). The ways these views have changed will be looked at in more detail in the Sociocultural/Historical section. Other aspects of the philosophy of proof will be looked in this section. In particular, what are some of the specific views that students have about proof and some conflicting philosophies of proof.

Before getting into the sub-sections, one ought to start with such a discussion with what constitutes a proof. Hersh (1993) offers three different definitions of the word "prove", based on the context in which is used. The first, the everyday definition, is to "(*t*)*est, try out, determine the true state of affairs*" (p. 391, italics in original). The next two definitions are placed with in mathematics. The first of these, what Hersh calls the "working" definition, is that a proof is "(*a*)*n argument that convinces qualified judges*" (p. 391). The last definition of proof is the "logic" definition: "*A sequence of transformations of formal sentences, carried out according to the rules of the predicate calculus*" (p. 391). An argument that is merely convincing would probably not be acceptable for publication in a research mathematics journal; however, it does tend be good enough for mathematicians in practice, as Hersh suggests (Weber & Alcock, 2004).

This distinction between working and logical definitions of proof leads to an inconsistency between how mathematicians view proof in practice and how they would describe proof to a non-mathematician, as shown by the hypothetical dialogue (Davis and Hersh, 1981, p. 39-40) between a philosophy student and "The Ideal Mathematical." When asked for a definition of proof by the student, the I.M. first gives one similar to logic definition given above. After the student notes that this does not match his or her experience with proof, the I.M.

concedes that proof really more closely matches the working definition. The I.M. ends the conversation by saying, “Everybody knows what a proof is. Read, study, and you'll catch on. Unless you don’t.” (as cited in Hersh, 1993, p. 389).

2.2.1 The proof beliefs of students

With such varied and ambiguous notions of proof, it is not surprising that students’ views on proof are inconsistent as well. Coe & Ruthven (1994) found that British secondary students had a hard time speaking explicitly about proof, and yet were able to recognize distinctions. For example (not unlike Hersh’ distinctions above), they saw a difference between “mental” proofs (being able to see why something works) and “formal” proof. This led to the students recognize the ability of proof to not only ensure certainty but also to explain.

Coe & Ruthven (1994) also found that students saw the solutions to problems as more important than the reasons for the solutions. Hoyles (1997) found a similar attitude among British students. When presented with different proofs for the same results, the students were asked which proofs they would most likely give and which they thought would receive the highest grade from there teachers. Students said that the proof they chose would receive the best score only 21% of the time (p. 12). Unfortunately, this choice was typically based on a ritualistic proof scheme, since students tended to pick formal arguments (correct or not) as being the ones most likely to be graded highest. In their opinions, the students usually felt that they chose for themselves proofs that were both explanatory and general, while they felt the proofs that yielded high scores were only general.

The students involved with the studies in preceding paragraph were British students who had taught using a curriculum that emphasized an exploratory approach to mathematics. It should not be too surprising, then, that the students did not have what one might consider the ideal perception of proof. In his book, *The Nature of Proof* (1938/1966), Fawcett describes a high school geometry class which had the analysis of proof as a central focus. As a result, the students came to understand many important aspects of proof. For example, they learned to make formal deductions, identify undefined terms and see the importance of axioms and previous results. This insight was lacking in a control class using traditional instruction. Studies have also shown that even before high school, instruction can have an impact on student's ability to solve conceptual problems (Cobb, et al., 1991) and work toward providing formal proof (Maher & Martino, 1996).

A student's ideas concerning proof are not always a function of instruction, however. Sriraman (2004) found that gifted students shared certain characteristics with professional mathematicians. Among these were the sort of argument that would or would not constitute a proof and overall methods for finding solutions to problems. On the other hand, Chazan (1993) found that high school geometry students can have a view of proof contrary to what (in the researcher's opinion) they are likely taught: that empirical evidence is proof and deductive proof is merely evidence.

2.2.2. Conflicting philosophies of proof

Differing philosophies of proof are grounded in differing philosophies of mathematics. Two main philosophies of mathematics will be discussed, along with their implications for proof. The first philosophy, social constructivism, as defined by Ernest (1991) views mathematics as a social construction. It is based on conventionalism, which acknowledges that "human language, rules and agreement play a role in establishing and justifying the truths of mathematics" (p. 42). Ernest gives three grounds for this philosophy. The first is that linguistic knowledge, conventions and rules form the basis for mathematical knowledge. The second is that interpersonal social processes are needed to turn an individual's subjective mathematical knowledge into accepted objective knowledge. The last is that objectivity is understood to be social. A key part of what separates social constructivism from other philosophies of mathematics is that it takes into account the interplay between subjective and objective knowledge. When a discovery is made by an individual, this subjective knowledge becomes accepted knowledge by the community – thus becoming objective. Then, as this knowledge is further spread to others, they internalize it and it becomes subjective again.

In her (1999), Gold offers some critiques of social constructivism as a philosophy of mathematics. The first is that this philosophy fails to account for the usefulness of mathematics in the world. Social constructivism does fine when explaining how mathematics can be created to solve practical problems. However, it does nothing to explain mathematics created long before application. Social constructivism also fails to account for cases like that of Ramanujan, who developed his results through interaction with mathematical objects and not a mathematical community.

Gold's main critique, however, is the failure of social constructivism to distinguish between mathematical knowledge and mathematics itself. Mathematical knowledge is what is socially created and/or discovered. She repeatedly draws on physics as an illustration. "(P)hysical objects either are or are not made up of atoms, and it is not the community of physicists that makes that true or false: it's the actual state of the world" (Gold, 1999, p. 377). While our knowledge of something may change over time, the reality of it does not. If mathematics is a human creation, can the same not be said for the quarks?

Hersh (1993) would say that Gold is arguing in favor of an "absolutist" philosophy of mathematics. Having this view of mathematics has implications for one's views of proof. For an absolutist, mathematics is made up of absolute truths that require complete, correct proof. Since many mathematical results have multiple proofs, an absolutist would be given preference based on brevity and generality, not on explanatory power. On the other hand, Ernest's social constructivism would be considered "humanistic" by Hersh. Under this philosophy, proof (like mathematics in general) is for us to use as we see fit. Here, the main role of proof is explanation. "Proof is complete explanation. It should be given when complete explanation is more appropriate than incomplete explanation or no explanation. … Sometimes a partial explanation suffices. Sometimes we skip the proof, if a lemma or theorem seems clear enough on its own" (Hersh, 1993, p. 397).

In his book *Proofs and Refutations*, Lakatos (1976) offers another critique of the absolutist view of mathematics and proof. There, he makes an argument that an overly formal view of mathematics (to include an absolutist view of proof) can have a negative effect on students' attitudes. When mathematics is presented as if it is infallible knowledge handed down from above, it becomes easy for students to think it inflexible. Thus, the dominance of formalism can take away the exploratory nature of mathematical practice. When students are aware of the process that occurs when mathematics is discovered (a process embraced by social constructivism), they will be more likely to see mathematics as an activity in which they can engage. When students feel mathematics is something they can interact with (as opposed to facts and rules given for memorization) they are more likely to see it as something existing in the real world in which they too live.

Not only is an overly absolutist view of mathematics detrimental to the views of students, it is also not an accurate view of the mathematics that has existed throughout history. Lakatos (1976) provides two examples (one in the main body of his book, and another in the appendix) that point out that the mathematical community has had to deal with a theorem that apparently had been proven, and yet suffered counter-examples. Hersh has a similar sentiment: "Intuition is fallible in principle; rigor is fallible only in practice" (1993, p. 395). Social constructivists would point to the fallibility of proofs as evidence that mathematics is a social construct and therefore lacks certainty. If the verification of mathematical facts can turn out to be false, then mathematical facts are subject to question as well. Gold points out, though, that proofs are among the activities that concern human knowledge. As such, they are subject to revision, as are theories in the physical sciences that mean to explain some physical phenomenon. The revision of explanatory theory, however, does not change the physical phenomenon.

The views given above no doubt have been influenced by the culture of those who held them and the events in mathematics that came before them. In the next section, some of these potential influences will be examined.

2.3 Sociocultural\Historical Section

As will be seen, there has been much interaction between mathematics and society (Grabiner, 1996; Moreno-Armella, Sriraman, & Waldegg, 2006; Siu, 2008). This interaction has led, in some cases, to society influencing mathematics and mathematics influencing society. This section will examine some of these influences, as well as some of the history of proof (particularly its evolution).

2.3.1 The interaction of mathematics and society

When people speak of the interplay of mathematics and the "real world" (at least in the context of mathematics education), they are often referring to direct applications of mathematical ideas - usually in the form of procedures used to solve particular types of problems. This is not the type of interaction that will be dealt with here. Here, what will be examined is how mathematics has been affected by the culture in which it exists and vice versa.

The connection between mathematics and society has long been recognized in mathematics education, in various forms. Perhaps the most prominent form is that alluded to above: the inclusion of "story problems" and "real world" activities in mathematics curriculum. This is not the only way the link between mathematics and society can be viewed by mathematics educators. In the book *The Nature of Proof*, Fawcett (1938/1966) recognizes the ability of mathematical training to affect change in lives of students outside mathematics. "(T)he purpose of this study (is) to describe classroom procedures by which geometric proof may be used as a means for cultivating critical and reflective thought and to evaluate the effect of such experiences on the thinking of pupils" (Fawcett, 1938/1966, p. 1). It is argued that the goals he describes are the goals the got geometry in secondary curriculum in the first place and this improvement in critical thinking should transfer beyond the geometry classroom. Despite these goals being widely shared (even among high school teachers), they are rarely reached. Many make the mistake of assuming "that since demonstrative geometry offers possibilities for the development of critical thinking, this sort of thinking is *necessarily* achieved by a study of the subject" (p. 10). This leads to a focus of geometric facts (as evidenced by examining tests administered in a typical geometry class). This new focus de-emphasizes the link that is trying to be established.

Besides the ability of proof to forge a link between mathematics and students' lives, there are other links between proof and culture. Moreno-Armella, Sriraman & Waldegg (2006) look to historical examples to see how mathematical objects are related to the contexts in which they are created and the ways in which that operate (or are operated upon). The first example is that of the concept of number held by the Pythagorean School. The Pythagoreans held that all things could be studied using integers and the ratios of integers. However, through the process of mathematical study they discovered the "incommensurability" of certain line segments (i.e. the discovery of line segments whose ratio could not be written as the ratio of relatively prime integers). Because this discovery conflicted with their foundational beliefs – and therefore created a crisis within their belief system – it was rejected. This example illustrates that their concept of number (the object) was tied to the consequences of its operations and the only ways to reconcile the inconsistency were to either alter the concept of number or ignore the inconsistency.

Another example is that of Euclid's construction of geometry. Here, the reliance on postulates necessitates a reliance on the material experiences that made the postulates "self-evident." This opens the door for mathematics to be based on material experiences that may be flawed, or at least limited. This reveals the fact that entire mathematical systems can be tied irrevocably to pre-existing ideas, or common (perhaps cultural) knowledge. This is the first example turned on its head: before knowledge was adjusted to fit pre-conceived ideas and here it is based on them.

Siu (2008) also points out ways in which mathematics, and proof in particular, has been influenced by the surrounding community. This recognition is tied to the recognition that mathematics is "part of human endeavour" (p. 355) and not a stand alone technical subject. To this end, Siu gives four examples that provide insight into how math fits into over-all human culture. The first example examines "the influence of the exploratory and venturesome spirit during the 'era of exploration' in the fifteenth and sixteenth centuries C.E. on the development of mathematical practice in Europe" (p. 356). This desire to explore inspired scientists and served as a model for discovery. Some aspects of this influence were calls for the sharing of conjectures and exploring problems considered completed. Among the new explorations was the consideration of infinitesimals.

The next example shows the influence of social environment of China during the third to sixth centuries on the work of mathematician Liu Hui. This was a period of social turmoil that saw a decrease in the influence of orthodoxy belief – which meant an increasing of free thinking. This allowed modes of thought that differed from the traditional Confucianism, including the introduction of a form of deductivism. It also led to "the predilection for rhetoric and dialectic" which could be proposed to be "conducive to the promotion of a notion of proof" (p. 359).

The third example also is set in China and looks at the influence of Daoism on mathematics of the time. The attention paid to change as a characteristic of all things in Chinese philosophy is reflected in the mathematics of China in the fourth century B.C.E.

The last example examines the influence of Euclid's *Elements* on Western culture (more on this below) compared to that in China. In this example, unlike the others, mathematical thought is shown to influence broader culture. Translated in Chinese in 1608, the book was greeted by some who recognized its ability to enlighten. However, it had little overall mathematical influence. Despite this, the *Elements* did have influence over political figures that were "main figures in the futile attempt of the 'Hundred-Day Reformation Movement' of 1898" (p. 360).

Finally, Grabiner (1998) makes a very good case for the influence of mathematics in general (not just Euclid) on Western thought. Grabiner breaks mathematics' influence in to two main categories: certainty and applicability.

The certainty aspect of mathematics is where *The Elements* is so influential. Its influence is seen in two ways. One is the Platonic idea that there is a pure, unchanging truth that exists somewhere and that our reality is an imperfect approximation of it. This idea can be supported by citing unchanging results such as the sum of the angles of a triangle must always equal π. The other way in which *The Elements* shows its influence is through its argumentation. Grabiner cites examples from other fields where thinkers start from, essentially, axioms and build arguments from there. Among the examples of this kind of argumentation are Newton's "Axioms, or Laws of Motion" ("(h)is *Principia* has a Euclidean structure, and the law of gravity appears as Book III, Theorems VII and VIII.") and in the Declaration of Independence ("We hold these truths to be self-evident…") (Grabiner, 1988, p. 221).

2.3.2 History of proof

Mathematicians' ideas of what constitutes a formal proof have evolved over time (Kleiner, 1991; Harel & Sowder, 1998; Almeida, 2003). Kleiner (1991) mentions the Babylonians as the greatest pre-Greek mathematicians, although they had no formal notion of proof. They did, however, bring mathematics to a point at which it was ready for the introduction of deduction. Then, the Greeks invented proof as deduction from explicitly stated postulates. Some of the potential reasons for this advancement include the need to reconcile inconsistent results passed down from previous mathematicians (or that developed with in Greek mathematics itself), the democratic Greek preference for discourse and argumentation and the need to get to underlying reasons for results for the purposes of teaching or philosophic inquiry (Kleiner, 1991).

This emphasis on deduction, however, did not come without a price. The Greek's axiomatic insistence prevented them from using certain ideas (e.g., irrational numbers, infinity)

(Harel & Sowder, 1998, p. 239). It also led to a long period of little rigor (Kleiner, 1991; Harel & Sowder, 1998). Kleiner shares Lakatos' opinion when he says, "Too much rigor may lead to rigor mortis" (1991, p. 294).

The next major change in the history of proof came in the introduction of symbolic notation and manipulation as methods for discovery and demonstration in the 16th through 18th centuries (Kleiner, 1991; Harel & Sowder, 1998). This allowed for the general proof instead of specific demonstration. It also brought mathematics within reach of more students. An example of symbolic notation being conducive to discovery can be seen in Leibniz's discovery of the product rule. It was also about this time that explicit formal proof was less than necessary because validity was attained through application (Harel & Sowder, 1998).

Mathematics saw the return of rigor, however, in the early 19th century. To illustrate this, Kleiner (1991) focuses on Cauchy's 1821 *Cours de'Analyse*, which provided a rigorous foundation for calculus. Kleiner notes that "most of the…basic concepts of calculus were either not recognized or not clearly delineated before Cauchy's time" (1991, p. 296). Cauchy had not yet reached the level of rigor to which mathematics is accustomed to today. For example, notions and definitions of important calculus concepts (e.g., limit, continuity and infinitesimal) were verbal in nature and there was the use of geometric intuitions in proving existence of limits. The first appendix in Lakatos' *Proofs and Refutations* (1976) describes an event in which the failure to rigorously define terms led to the proof of an incorrect theorem. It was left, then, to Weierstrass and Dedekind to lay the study of analysis on a firm foundation through the rigorous definition of the real number system and the modern definition of delta-epsilon limit – removing the use of infinitesimals.

In the late 19th and early 20th centuries, mathematics saw what Kleiner describes as the "reemergence of the axiomatic method." This was spurned on by Boole's *The Mathematical Analysis of Logic* in 1847 which served as the introduction of logic and highlighted the arbitrariness of axioms. Thus, axioms were no longer based on observations, but became tools of mathematical research. The axiomatic method developed slowly but by the early 20th century, it was established in a number of areas of mathematics.

Almeida says that "in the current tradition, a mathematical proof is a pure thought experiment divorced from context: the truth of a result or statement is deduced on the basis of internally agreed and consistent axioms." (2003, p. 479). While this seems likely to be true, strictly speaking, in the area of pure research mathematics, it is not the case in all branches of mathematics. For example, Hersh (1993) points out that applied mathematics is much less rigorous, even publishing convincing heuristics in lieu of proof. There are also other places the definition of proof is being blurred. For example, the classification of all finite simple groups was accomplished my many mathematicians. Of the proof, Daniel Gorenstein said:

> The ultimate theorem which will assert the classification of simple groups, when it is attained, will run to well over 5,000 journal pages! ... It seems beyond human capacity to present a closely reasoned several hundred page argument with absolute accuracy ... How can one guarantee that the "sieve" has not let slip a configuration which leads to yet another simple group? Unfortunately, there are no guarantees - one must live with this reality. As cited in Hersh, 1993, p. 392 - 393

Clearly, this is a departure from what is generally considered proof.

This is not the only way in which the conventional notions of proof are being challenged. So called probabilistic proofs are used to, among other things, determine whether or not a given integer is prime (Kleiner, 1991). It has even been shown that for sufficiently large integers, the probability of error when using the fast methods of probabilistic proofs is smaller than the probability of computational error in a rigorous (and longer) proof (Hersh, 1993). Proofs by computer have also spurred debate about proof. Proofs of the four-color theorem (Kleiner, 1991; Hersh, 1993) and Kepler's conjecture (Szpiro, 2003) have recently been completed by use of computers. Mathematicians give differing reasons for there resistance to accept computer based proofs. Hersh (1993) provides some of them: one can not see the inner-workings of the proof, they are not aesthetically pleasing, one does not learn from them and introduction of the computer may introduce errors of which mathematicians are not aware. To be sure, some of these criticisms can be leveled at traditional proofs as well. It seems reasonable, however, to conclude that these are more of a cause for concern when dealing with computer proofs.

Because of these new found issues with proof, Hersh (1993) concludes that it is possible eventually more proofs will contain qualifiers. For example, "by hand" or "by machine" may accompany proofs in the same way proofs today mention when and where they use things like the axiom of choice or the law of the excluded middle.

How proof fits into society and its history have important implications for how proof is taught and learned (Bell, 1976; Moreno-Armella & Waldegg, 1991; Almeida, 2003). For course, there is much more that one must consider when discussing the pedagogy of proof, as will be seen the in next section.

2.4 Pedagogical Section

While it has been seen above that students often struggle with proof, improving student performance in proof is not a lost cause. Fawcett (1938/1966) outlines methods that improve the proof abilities of high school geometry students. Other authors have also found that students' proof performance is affected by various circumstances besides explicit teaching methods, including textbooks (Coe & Ruthven, 1994) and overarching curricular goals (Hoyles, 1997). Unfortunately, these influences are not always positive (as is seen in the two previously mentioned studies). Although the following quote is referring to the lack of proof ability of incoming college students, it is appropriate here due to the connection between proof instruction and students' skills: "(U)niversity coursework must give conscious and perhaps overt attention to proof understanding, proof production, and proof appreciation as goals of instruction" (Harel & Sowder, 1998, p. 275).

This section will look at some of the pedagogical aspects of proof. These will include the uses of proof in the classroom, particular difficulties students encounter when dealing with proof and finally some research on the teaching of proof.

2.4.1 Proof's use in the classroom

Proofs have more to offer than simply the verification of results. The most obvious way this is true is in the ability of proof to explain things. "More than *whether* a conjecture is correct, mathematicians want to know *why* it is correct" (Hersh, 1993, p. 390). By including proof in classrooms, students can be shown the "why," not just the "whether." "Proof and explanation are thus interwoven in processes of understanding" (Dreyfus, 1999, p.101).

This point is further illustrated by Barbeau & Hanna (2008). There, the authors take Yehuda Rav's paper "Why do we prove theorems?" and apply it to mathematics education.

While the authors acknowledge that many proofs have the power to explain particular mathematical material, they take things a step further. "This paper aims to show that proofs can also be the bearers of mathematical knowledge in the classroom in the sense proposed by Rav: that proofs have the potential to convey to students 'methods, tools, strategies and concepts for *solving problems*'" (p. 345). The ways in which proofs are valuable beyond verification of propositions suggested by Rav include: explanation of the underlying reasons for a result holds true, invention of methods for problem solving, unexpected results and new areas of mathematics. Beyond the uses mentioned above, Barbeau & Hanna point out that proofs can be put to use pedagogically. For example, proofs can set the stage for other concepts that students may see in the future and give students access to generalizations, not just the results themselves. Also, different proofs for the same proposition bring different generalizations to light.

One needs to be careful, however, in the way proof is implemented. An over-reliance on formalism can be detrimental to students' view of and performance in mathematics (Lakatos, 1976; Kleiner, 1991). For this reason, the ways proof is used in the classroom should vary depending on the class (Polya, 1954; Dreyfus, 1999). Van Dormolen (1977) gives different levels students' understanding of proof and points out that students must progress through the levels sequentially. Without this progression, students will resort to memorization and tricks to get by. Of course, these methods only work for problems familiar to students and leaves them ill-equipped for new problems. Polya (1954) also recognizes the need to vary the types of proofs given to classes. For example, the use of delta-epsilon proofs in a freshman level calculus class would leave most of the students behind; in such a setting pictures, examples and analogy are often sufficient to convince and enlighten. However, professors of an analysis class would be doing their students a disservice by not demonstrating the delta-epsilon proofs that will be expected of them.

Because care needs to be taken when implementing proof in a class, the failure to do so can lead to student difficulties. This will be seen in the next subsection.

2.4.2 Student difficulties in proof

One reason that students find proving difficult has already been mentioned; that is, students sometimes model what they see done in their classes. Often, especially for students just beginning their collegiate mathematics courses, they have seen very little in the way of formal proof and "have had little opportunity to learn what are the characteristics of a mathematical explanation" (Dreyfus, 1999, p. 91). This in itself is not always a problem, however, as mentioned above. The problem, Dreyfus says, is that distinction between formal and intuitive arguments is rarely given. Teachers and textbook writers aren't sensitive to the need to set norms of mathematical behaviour.

> In many textbooks used at the level under consideration, more or less formal arguments are used, together with visual or intuitive justifications, generic examples, and naive induction. Even the formal arguments are often only formal in appearance. But more importantly, students are rarely if ever given any indications whether mathematics distinguishes between these forms of argumentation or whether they are all equally acceptable. (Dreyfus, 1999, p. 97)

Dreyfus goes on to say that

> explanatory discourse is more metamathematical than mathematical; it may, for example, include reasons why a certain fact is significant in mathematics, something which is

> clearly beyond the realm of a proof. … It thus appears that, at least in some measure, the task of learning and teaching mathematical justification conflicts with the pursuit of learning and teaching mathematical relationships, concepts and procedures in a flexible manner. (1999, p. 101 - 104)

No doubt, this means that teachers will often need to walk a fine line. The key is making sure that students are aware of the difference between the explanations there are given and what constitutes valid mathematical proof.

Besides the over-arching source of student difficulty mentioned by Dreyfus, other researchers have found particular troubles students have with proof. For example, Ernest (1984) lays out six specific problems students encounter when learning how to do proofs by induction. They are: the dual uses of the word "induction," the belief that you are using what is to be proven within the proof itself, a lack of understanding the roles of qualifiers and /or logical complexity of the argument itself, the view that one component of mathematical induction is not necessary (most often, the basis step), the belief that it is only good for certain types of problems (summing finite series, e.g.) and a lack of understanding the basis for the method.

Moore (1994) went through a similar cataloguing of student difficulties. He found seven major reasons students struggle, labeled D1 – D7:

D1: Students unable to state definitions

D2: Students had little intuitive understanding of the concepts

D3: The students' concept image was inadequate for doing the proof

D4: Students did not generate or use own examples

D5: Students couldn't use definitions to obtain overall structure of the proof

D6: Students couldn’t understand or use mathematical notation

D7: Students couldn’t begin the proof (Moore, 1994, p. 251 – 252)

Much of Moore’s article is framed around what he calls the concept-understanding scheme. The concept-understanding scheme is the three aspects of a concept: definition, image and usage. Concept definition is the technical, mathematical definition of a concept. Concept image is the mental pictures associated with the concept, and their properties. Concept usage refers to the ways one operates with the concept to generate or use examples and do proofs. Students with an inadequate image of a concept have trouble defining it and using it (in proofs or otherwise). Students may also have trouble distinguishing between a concept’s image and its definition. This can then result in a lack of formality in giving proofs although the student feels like he or she understands the problem. Besides a lack of formality, the failure to differentiate between image and definition can cause problems because proofs, especially in transition courses, are often structured around definitions. Without the definition to help students frame their proofs, they can experience an overload in both linguistic and conceptual difficulty. Bell (1976) also recognizes complexity as a problem for students: “It often seems that this complexity factor interacts with the knowledge factor and the grasp of a concept that is not well understood is lost” (p. 34).

Another study that looked at student difficulty with proof was completed by Bedros (2003) at The University of Montana. Specifically, that study examined post-Calculus II students’ perceptions and understandings of indirect proofs. He found that students tended to prefer using direct reasoning to indirect reasoning, that their understanding of indirect proof methods were limited to surface structure and that students generally used intuition as a guide to exploring a problem rather than viewing indirect reasoning as a tool at their disposal.

VanSpronsen (2008) also completed a dissertation at The University of Montana which examined proof. Her research sought to describe the strategies students used while completing proofs. Her results indicated that students' strategies stayed fairly consistent across different questions and that some were more successful than others. The strategies she observed were: the use of examples, the use of equations, the use of other visualizations and student self-regulation. These strategies were unique to the individual student and, as is mentioned above, remained static across question. It is not too difficult to imagine that this led to student difficulties as different problems often lend themselves to different techniques.

With all the difficulties students come upon when learning proof, it is little wonder that much research has been done regarding ways to improve proof instruction. Some of this research will be addressed in the following sub-section.

2.4.3 Teaching Proof

Dreyfus (1999) mentions that one of the obstacles students have when trying to learn proof at the college level is that they have little proficiency in proof when they get there. In *Patterns of Plausible Inference* (1954), Polya lays out the ways that people judge the plausibility of statements. By doing so, he gives a guide for teachers to show students informed ways of going about exploring a problem. "I address myself to teachers of mathematics of all grades and say: *Let us teach guessing!*" (Polya, 1954, p.158) Improved guessing on the part of the student ought to enable to students to distinguish a more reasonable guess from a less reasonable one. The instruction in judgment on the reasonableness of guesses would inherently bring to light aspects of argumentation. More informed argumentation on the part of students will leave them better able to differentiate valid proofs, an important skill for students in proof based classes (Weber & Alcock, 2004).

Polya is also influential in the instruction of proof via the problem solving steps he laid out in his book *How to Solve It* (1945). There are five steps to the plan, the first of which is "Understanding the problem." This includes verifying all terminology is understood and exploring the problem using pictures or examples. The second step is "Devising a plan." This might be done by looking over notes or a textbook or revisiting similar problem that have already been completed. After a plan has been created, the next step is "Carrying out the plan." This step is fairly straight forward and after it is completed, the final step is "Looking back." In this step, the solution is checked and alternative solutions are sought out. As mentioned above, these steps have been influential in proof instruction. In fact, the students who took part in this study used a textbook (Daepp & Gorkin, 2003) that had these steps as a focus for their Introduction to Abstract Mathematics class.

Fawcett (1938/1966) lays out some methods to improve proof instruction specific to the high school geometry level. The stated goal of the methods employed involved more than improving student proof performance; it also included a transfer of critical reasoning skills outside the classroom (Fawcett, p. 1). However, the methods did lead to students becoming better at proof.

The class began with a topic of a completely non-geometric nature (about the granting of special school awards). This discussion highlighted to the students the need for clear definitions. Then, the class clarified the distinction between definitions and accepted rules. After this, the class began to examine geometry. The students wrote their own textbooks as they went along, so the class began by defining the geometric terms they were going to use for themselves. These definitions were made and refined by the class. Once a definition was agreed upon, it would be recorded in the textbooks. These textbooks were available at all times to discourage memorization. Besides definitions, the class decided amongst themselves which terms would be used without definitions and which assumptions were to be made with out verification.

These definitions, undefined terms and assumptions came about through various class exercises in which the teacher would present the students with a diagram and ask what properties they were willing to accept after inspection of the (sometimes dynamic) diagram. During the course of stating the properties, the need to clearly identify terms and principals in use. If a term that had not been used before came up, it was either defined or placed into the undefined terms category. It should be noted that a term was called undefined if it was agreed that there could be no confusion about its meaning. Along with the terms, statements about the figures were also fodder for discussion. Through the course of the discussions, students had the chance to add new assumptions or verify “implications” (theorems). It is worth noting that the theorems were first conjectured by the students themselves.

> If the pupil is to have the opportunity ‘to reason about geometry in his own way,’ no theorem should be stated in advance; for such a statement fixes, to some extent, the direction of his thought and deprives him of discovering for himself the mathematical relations whish control a situation. (Fawcett, 1938/1966, p. 62)

Also worth noting is that during the verification of implications, various students would present proofs. Often, this led to students weighing the merits of such proofs. It is here that students had the chance to choose between inductive and deductive proofs and preference was shown for deduction.

Another activity the class engaged in was proof analysis – identifying definitions, assumptions, etc. This yielded insight into deductive proof because it forced the students to consider what would happen if an assumption turned out to be wrong. The illustration given by Fawcett described the teacher providing the students alternate notions of space which led to the questioning of their assumption of the parallel postulate.

This emphasis on student-created mathematics is not unique to Fawcett. Harel (2008) looks at the tendency for math educators to over-emphasize “ways of understanding” (e.g. results, proofs, methods) while neglecting “ways of thinking” (e.g. techniques for proof and problem-solving, beliefs about math). By allowing the students to create the mathematics they are learning, the students’ ways of thinking are automatically taken into account. It also allows the proofs to be at a level the students are comfortable with, which is also important (Coe & Ruthven, 1994; Van Dormolen, 1976).

It is also worth noting that the classroom as it is set up by Fawcett is less authoritarian and more closely related to the way research mathematics takes place. This is important because students should have the opportunity to learn about mathematics in ways similar to how it is practiced (Bell, 1976; Almeida, 2003). Also, since the students needed to convince each other, they saw a need for providing proofs. Providing a need to prove is vital for teaching proof (Van Dormolen, 1976; Harel & Sowder, 1998). When students see the need to verify proofs, they begin to focus on understanding the reasons mathematical results hold and less on simply finding solutions to problems (Coe & Ruthven, 1994).

Harel & Sowder (2003) sum up many of these goals in their “make proofs tangible” instructional principle. A tangible proof has three characteristics. First, it is concrete. Concrete proofs deal with entities the students sees as mathematical objects. Next, a tangible proof needs to be convincing; students must understand the underlying ideas of the proof, not just be able to validate from step to step. Lastly, tangible proofs are essential; students should see the need for the steps to be justified.

It should be noted that following these principles are not the only way students can learn proof. Weber (2003) describes a way in which students can learn to do proofs via what he calls a “procedural route” (p. 395). This process starts by students seeing proving as an *algorithm*, or a set of prescribed steps. At this point, “the students were generally unaware of the overall nature of the procedure that they were incorporating” (Weber, 2003, p. 396). The algorithm is only useful for very specific problems. Through repeated uses of the algorithm, the students internalized it. At this point the *algorithm* becomes *process*: a shorter list of more general, global steps (e.g., find a delta such that….vs. divide by leading coefficient…). The process is to the student still more of a way to earn class credit than a valid argument, but it is now applicable to more problems. Finally, through reflection, the process becomes an *argument* which is understood to be valid mathematically. According to Weber, this reflection is what is missing for most students who do not progress to the point at which proof becomes an argument.

2.5 Theoretical Framework

2.5.1 Main frameworks

<u>Proof processes</u>

In this section, four papers will be examined. In the first, which will be used as one of the primary frameworks through which students’ proofs will be described, Weber (2004) takes data from three previous studies on proof and uses it to build a framework that one could use to describe the processes students use to create mathematical proofs. In all, 14 undergraduate students constructed 176 proofs. Two of the studies used observed students in an abstract algebra class and one study involved students taking real analysis. Through the classification of the students' work, Weber identified the following types of proof productions: *procedural*, *syntactic* and *semantic*.

Procedural proofs are those that can be constructed by applying a procedure (such as mathematical induction). These proofs may or may not have any meaning for the student producing the proof. Even correct procedural proofs may lack meaning for the proof writer – if, for example, the proof was written by mimicking a proof completed in class or in a textbook. Procedural proofs are further divided into two sub-types: *algorithm* and *process*. Algorithmic proofs are those completed by mechanically following explicit steps. Process proofs are those that feature "a shorter list of global qualitative steps that are not highly specified manipulations, but rather involved accomplishing a general goal" (Weber, 2004, p. 2). Students employing process proofs display an understanding of the steps involved in particular types of proof, but do not necessarily know why these qualitative steps form a valid mathematical proof.

The next type of proof, syntactic, is "a proof by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (Weber, 2004, p. 4). This type of proof may be thought of as an "unpacking" of definitions and\or a "pushing" of symbols. This type of proof is a form of purely formal deduction. These proofs, although likely closer to the sort of proof professors would have their students produce, still do not necessarily foster meaningful understanding. This is especially true if students do not recognize the relationship between the symbols they are manipulating and the mathematical objects they represent. Despite their potential shortcomings syntactic and procedural proofs can form foundations which can be built upon through reflection (Weber, 2003).

The last type of proof Weber (2004) describes is semantic proofs. These proofs are those that are produced when "one first attempts to understand why a statement is true by examining representations (e.g., diagrams) of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof" (Weber, 2004, p. 5). Part of the process of constructing proofs of this type is converting the intuitive argument (once created) into a formal proof. Since these proofs are necessarily based on intuition, they are always meaningful for the student – which may or may not be the case for the first two proof types.

It is important to note here that Weber does not claim that these classifications of proof form either a hierarchy or a progression that students go through. As mentioned before, the main purpose of this research is to see if such a hierarchy or progression can be established.

Proof schemes

While Weber gives ways students' individual proof attempts can be categorized, Harel & Sowder (1998) give a more over-arching framework. They give a classification of what they call students' proof schemes. These proof schemes include ways students both attempt to prove and become convinced of the truth of a statement. This classification is based on a one semester teaching experiments in a number theory class, a college geometry class, an advanced linear algebra class, two consecutive semester teaching experiments in linear algebra and case studies involving high school juniors taking Euclidean geometry and calculus. Data were gathered by classroom observations, clinical interviews and the examination of homework.

The different proof schemes used by students are divided into three major categories: *external conviction*, *empirical* and *analytic*. External conviction proof schemes are those used to convince; those employing this scheme can be either convincing themselves or others. This type of conviction is divided into three sub-types: *ritual*, *authoritarian* and *symbolic*. Ritual conviction occurs when students' believe an argument that looks like a proof. Authoritarian proof schemes are employed when a result is believed on the basis of some authority – typically a teacher or textbook. Symbolic proof schemes are used when symbols detached from meaning are used. This is characterized by a student who "divided" by a matrix to solve a problem (Harel & Sowder, 1998, p. 251).

The second major type of proof scheme is empirical. This type is fairly self-explanatory and has two sub-types. The *inductive* empirical proof scheme is seen when students use inductive evidence to convince themselves and others of the validity of a statement. Another is referred to as the *perceptual* proof scheme. Here, properties based on how objects appear are used as justifications. For example, in a geometry class, two line segments may be taken to be congruent if they look like they are the same length. This type of proof scheme is meant to describe how students come to hold a conjecture to be true. It is worth mentioning is that students taking part in this study have already completed a transition course and generally realize that empiricism does not give mathematical validity.

The last major type of proof scheme identified by Harel & Sowder is the analytic proof scheme. These are proof schemes based on logical deductions. Again, this type of scheme has sub-types: *transformational* and *axiomatic*. Transformational proof schemes are characterized by operations on objects and the anticipation of the results of the transformations. This may include the manipulation of geometric objects or the analysis of how algebraic expressions change in order to justify inequalities. Transformational proof schemes are divided into sub-categories: *internalized*, *interiorized* and *restrictive*. Internalized proof schemes are heuristics that "renders conjectures into facts." Interiorized proof schemes can be thought of as internalized schemes employed with a deeper level of understanding.

> An interiorized proof scheme is an internalized proof scheme that has been reflected on by the person possessing it so that he or she has become aware of it. A person's awareness of a proof scheme is usually observed when the person describes it to others, compares it to other proof schemes, specifies when it can or can not be used, etc. (Harel & Sowder, p. 265).

The last sub-category of transformational proof scheme is restrictive, which describes students who have analytic proof schemes that is limited in some way. For example, a *contextual* restrictive proof scheme is held by students who interpret conjectures, and the proofs that go along with them, in a particular context. An example of this is a student who only considers results in terms of $\mathbb{R}^n$ when dealing with vector spaces in a linear algebra class. The next type of restrictive proof scheme is *generic*. Here, the student understands conjectures in more generality but is only able to give proofs in particular cases. For example, the student may understand that there exist vector spaces besides $\mathbb{R}^n$, but is only able to provide proofs for $\mathbb{R}^3$ due to an inability to generalize. Students who have the last type of restrictive proof scheme, *constructive*, are convinced by the actual construction of objects rather than simply the verification of their existence. For example, students with this proof scheme tend to be dissatisfied with proof by contradiction.

Axiomatic proof schemes are held by students who "have an awareness of an underlying formal development" (Harel & Sowder, 1998, p. 276). Students with an axiomatic proof scheme understand that "a mathematical justification must have started originally from undefined terms and axioms" (Harel & Sowder, 1998, p. 273) and understand the difference between defined and undefined terms. Axiomatic proof schemes are further divided into more specific categories by

Harel & Sowder. *Intuitive* axiomatic schemes show up when students understand the existence of axioms but can only use those that match their intuition, like those associated with the real number system. *Structural* axiomatic schemes are held by students who are able to study classes of objects based on axioms. Students in this category are able to study, for example, different types of groups based on their shared properties. Students with the final type of axiomatic proof scheme, *axiomatizing*, are able to study axiomatic systems themselves and the repercussions of altering them. Students in this category would be able to investigate non-Euclidean geometry by altering the parallel postulate.

Harel and Sowder make a point to mention that their proof schemes are not hierarchal and that students can show multiple proof schemes over a brief period of time (1998, p. 277). As with Weber's classification, part of the point of this research is to see if such a hierarchy can be established.

2.5.2 Other frameworks

The next framework that will be discussed here is due to Van Dormolen (1977). Van Dormolen gives levels of thought in proof and these levels are based on Van Hiele's levels of thinking in geometry. The first is the *ground level*, where student thought is restricted to specific examples. For instance, a student might see that a property holds for a prime number without consideration as to whether or not the property holds for all prime numbers. Once a student's thinking is less local and can begin to discuss properties shared within a class of objects (all even numbers, for example), that student as reached the *first level of thinking*. Students who move beyond this and begin to see connections between dissimilar problems have reached the *second level of thinking*. Students who reach this second level can then begin to understand the study of local arguments.

Balacheff (1988) classifies students' proof ideas from a perspective similar to Harel and Sowder. He seeks to characterize the ways students come to believe a conjecture is true. He offers four types of methods students use: *naive empiricism, crucial experiment, generic example* and *thought experiment*.

In naive empiricism, students become convinced by the examination of several cases. Crucial experiment is similar, but here the conjecture is tested by a case the student deems to be "not too special" (Balacheff, 1988, p.219). The thinking goes that if a proposition holds for a seemingly random case, then it should hold for all cases. On the other hand, a student becomes convinced by a generic example when he or she sees an argument that uses a specific object (like a particular number). The argument does not use the object as empirical evidence, but as a tool for illustration. The argument uses characteristic properties of the object that are taken to be common among the class of objects the particular object is representing. The last classification, thought experiment, is where one would find formal proof. Thought experiment is set apart because "it invokes action by internalising it and detaching itself from a particular representation" (p. 219).

Balacheff's framework is similar to Harel and Sowder's in that it gives a way to classify how students become convinced of mathematical statements. It is not as broad as that of Harel and Sowder, which is why it was not used in the study.

Chapter 3: Methodology

The purpose of this chapter is to explain and justify the way this research was completed. Included in this chapter are sections giving the overall purpose and description of the study, the background and pre-study views of the researcher, a discussion of the participants involved and an explanation of the ways data will be collected and analyzed.

3.1 Purpose and description of study

The study was designed to document the progress students make when going through the process of learning proof. The main goal for the study is to identify what, if any, stages through which students progress. Much of the existing related research identifies students' ideas about or ability with proof at a certain point in time. Little research has been done, to the researcher's knowledge, regarding how these ideas and abilities change over time. The scope of this research, which took place over the course of one academic year, is what sets it apart from the bulk of existing proof related studies. By repeatedly observing the same participants, a longitudinal study was completed which allows for a more in-depth analysis of the progress students make.

The research was completed using interviews in which the students take part in a variety of tasks and "think aloud" while doing so. In the first type of task, students were given problems they had not seen before and worked through them. This was done over the course of a long study so that the ways in which students attack new problems can be observed to see if a change occurs. The second type of task consisted of students revisiting problems they have completed in the past to see what kind of progress they had made. This past work will be in the form of exam questions the participants completed during a transition-to-proof class (MATH 305: Introduction to Abstract Mathematics). The last type of task involved students evaluating completed proofs.

This type of task was meant to directly get at how students become convinced of mathematical results. The ways these interviews will be recorded will be described in a subsequent section.

3.2 The researcher

I completed a B.S. degree in mathematics, with a South Dakota teaching certificate, in 2005 and a M.S. in mathematics two years later, both at South Dakota State University. My teaching experience at the beginning of this research was limited to the classes taught while working on my degrees. These classes include a semester student teaching in a seventh grade classroom and, at the undergraduate level, remedial algebra, pre-calculus and contemporary mathematics (a terminal class for students majoring in the humanities). While completing my graduate work toward the M.S., I took primarily pure mathematics courses. This allowed me to develop proficiency with proof that I could not have attained without entering graduate school. It was this experience that I frequently looked back on once I began coursework towards a Ph.D. This reflection intensified as I began readings that eventually led to the current research. It also led to a desire to more fully understand how one learns proof and to add to that particular area of research.

My pre-study views of the research are influenced both by research I have read and my own personal experience. I believe that students will employ many different proof techniques depending both on the situation and their particular proof scheme (as described by Harel & Sowder, 1998). I also believe that student’s proof schemes and abilities are dynamic and do not progress in a straight-forward way. I believe that students can and will give evidence for different proof schemes at different times. I think this variation will be due to both what they think is expected of them and what they see as their own limitations. I also believe that students can and will have “good days” and “bad days.” By this I mean that any given student will sometimes struggle with a problem that they may find easy at a different time, and vice versa. This is part of the reason that I believe a longitudinal study is appropriate when observing students' ability. By repeatedly observing students, one can begin to identify the good days and the bad and get a good feel for the students' true abilities and ideas. With that said, I think that as time and students progress, the good days will become more frequent and the bad days less so.

3.3 The participants

The participants of this study all took Math 305 in the Spring semester of 2009, a class in which 19 students took the final exam. I used that class as preparation for a later teaching internship and so was able to get to know the students over the course of that semester. Each Friday the class worked in groups on projects laid out in the textbook. These sessions were supervised by me. The researcher also helped grade a portion of the students' take-home midterm exam (some of which serves as fodder for student reflection in the interviews). At the end of the semester, I described the current study and asked the class to participate. All students who were willing to take part were encouraged to do so. Initially, ten students took part in the study but one withdrew due to time constraints after a month. The nine other students completed the study and the results of their work are included in Chapter 4. The students involved were at such a level that they had had some previous work to look back on, but also not so far along in their undergraduate program that they have a wealth of experience with proof.

A number of the participants took Euclidean and non-Euclidean Geometry the semester following the one in which they took Math 305, a class I took as well. The participants were offered assistance by me (in the geometry class and other classes that the researcher had taken) as an incentive to participate in the study. It should be noted that I believe the reason for participation had more to do with the students' generosity than a desire for homework help.

While the study was being conducted, the participants were enrolled in a variety of courses including Euclidean and non-Euclidean Geometry, Ordinary and Partial Differential Equations, Real Analysis, History of Mathematics and Number Theory. The individual courses the students were taking, and their particular major will be addressed later on a student-by-student basis in Chapter 4.

For the identification purposes during the study, the participants were asked to choose a two digit number to preserve confidentiality. For the purposes of this study, I felt it was necessary to use names while referring to each participant. So, I chose names based on the students' selected numbers. I used the most popular baby name for the appropriate gender in the 20^{th} century year that corresponded to each participant's number. For example, the student who chose 09 as his number is referred to as "John" because that was the most popular name given to boys in 1909. In the case of ties (for example, "Mary" was the most popular girl name in 1913 and in 1917), the second most popular name in the later year was used.

3.4 Research Design

This study employed the case study qualitative research approach as described by Creswell (2007). Creswell defines this design as "research that involves the study of an issue explored through one or more cases within a bounded system (i.e., a setting, a context)" (p. 73). Creswell lays out five steps in completing a case study.

The first step in completing this type of research is to decide whether a case study is appropriate for the problem. "A case study is a good approach when the inquirer has clearly identifiable cases with boundaries and seeks to provide an in-depth understanding of the cases or a comparison of several cases" (Creswell, 2007, p. 74). This fits nicely with my intention to document the progress individual students make in the area of proof over the course of an academic year.

The second step is to identify the cases in question. In this research, the cases are the work completed by the participants on the provided questions over the course of the study. The cases are bounded by the academic year in which the study took place (Autumn 2009 and Spring 2010, at The University of Montana). The cases are also limited to the participants' responses to the questions asked. The only exception to these bounds is that the students' answers to certain exam questions from MATH 305 are included for the sake of comparison.

The next step included in a case study according to Creswell (2007) is data collection. "Data collection in case study research is typically extensive, drawing on multiple sources of information, such as observations, interviews, documents and audiovisual materials" (p. 75). Data was collected for this research using all of these methods and the collection procedures are discussed in the next section.

The fourth step is to analyze the data collected. Again, this will be described in greater detail later in Section 3.6. However, two main types of analysis were used: *within-case analysis* and *cross-case analysis*. Within-case analysis is "a detailed description of each case" (Creswell, 2007, p. 75) and in the present study forms Chapter 4. Cross-case analysis is a "thematic analysis across cases" (Creswell, 2007, p. 75) and will be seen in Chapter 5.

The final step of case study research, according to Creswell (2007), is the interpretive phase where "the researcher reports the meaning of the case" (p. 74). This will also be completed in the fifth chapter.

3.5 Data collection

The research designed involved giving students the three types of tasks mentioned above. As a reminder, the tasks were to attempt to prove results new to the participants, re-examine work completed in the past and evaluate completed proofs. The past work came from the semester prior to the study and was in the form of questions the participants saw on exams in MATH 305. The two exams the students had taken (a take-home midterm and an in-class final) were scanned and saved for this purpose.

The tasks were completed by students in a qualitative interview setting. The interviews were audio and video recorded and then transcribed. The video record was used as a back-up for the audio recording device and was used only a few times when the audio recorder failed.

For the first two interviews with each participant, I spoke to the students as they worked in order document all they were doing (in the event that they were not “thinking out loud” enough). For example, if a student solved for a variable without stating so, I would say something like “Ok, so you’re isolating x in order to get it in terms of y.” However, I was uncomfortable with the potential that I was significantly altering how they would do on their work. So for the remaining 12 interviews, I let them work as they preferred for the first 45 minutes of the interview (unless they completed the problem or felt stuck) and the remainder of each interview was used as a reflection period. It should be noted that some students did not prefer to work silently and so I spoke with them but tried to be as vague as possible. Also, I would generally point out what I considered small mistakes if I felt like the mistake would hinder the participant’s progress on the problem.

The reflection period generally consisted of me describing to the student the steps I thought I saw them complete. During this time, I would ask the students to correct any misconceptions I had about what they did. After the participant and I were comfortable that I understood the actions that each had taken, I questioned each about the reasons for the steps taken.

After an interview was completed, I transcribed it and, as I did so, I kept notes in an electronic journal regarding that interview. This would occur either the day of the interview or the next. This way, what the student did would be fresh in my mind and the notes I kept would be as accurate as possible. After each round of interviews, I read through the notes I took for each student while I transcribed that particular interview and took notes about each question overall (or the preliminary work the students did on questions that spanned two interviews, as the case might be).

Interviews were conducted at times that suited the participants' schedules. I interviewed every student in the study once every two weeks. There were a few exceptions to this. Once, James could not make his assigned time and could not make it up before his next scheduled interview. Fortunately, the interview he missed was the second one spent discussing a problem he had completed during the first interview. Also, no interviews were conducted during the weeks of Thanksgiving vacation and spring break. The interviews were task-based and often, particularly after the student completed the proof or had given up on the problem completely, took the form of a teaching interview. This was primarily done to keep students from becoming discouraged by the problems they saw. It should be noted that every effort was made to guide the students as little as possible and only after the interviewer was convinced the student would make no further progress on their own.

Eisenhart and Howe (1992) mention that research questions should drive the techniques and analysis of a research study. This design is related to the question in that it is the best way I can think of to consistently document the methods students used to prove. The goal is to gain insight into how students' abilities in proof evolve through knowledge of students' methods (and how the methods change).

3.6 Data analysis

The data was analyzed using what Creswell (2007) calls "the data analysis spiral… The process of data collection, data analysis, and report writing are not distinct steps…they are interrelated and often go on simultaneously in a research project" (Creswell, 2007, p. 150). Creswell uses the term spiral because of this overlap. This was indeed the case for the present research as the few steps of the spiral actually began as the data was being collected.

The first of Creswell's steps is data managing where researchers "organize their data into file folders, index cards, or computer files" (p. 150). For me, this included keeping a physical folder with each student's written work and a computer folder with each participant's interview transcriptions and journal.

"Following the organizing of the data, researchers continue the analysis by getting a sense of the whole database" (Creswell, 2007, p. 150). This stage is also referred to as "reading, memoing" (p. 151). This was done by keeping notes of everything that seemed important while transcribing and then, after all students completed a particular interview, collecting these notes into a summary of what all participants did for that interview. The last part of this loop was to then go back and review these notes and re-read the interview transcriptions before beginning the next part of the analysis spiral.

The fourth loop of the data analysis spiral is "describing, classifying and interpreting" (Creswell, 2007, p. 151). "Here researchers describe in detail, develop themes or dimensions through some classification system and provide an interpretation in light of their own views or views of perspectives in the literature" (p. 151). The result of this portion of the spiral is Chapter 4. Eisenhart and Howe (1992) suggest that research should fit into existing theoretical frameworks. Creswell, however, cautions that using pre-existing coding systems (like I have)

> serve(s) to limit the analysis to the "prefigured" codes rather than opening up the codes to reflect the views of participants in a traditional qualitative way. If a "prefigured" coding scheme is used in analysis, I typically encourage the researchers to be open to additional codes emerging during the analysis. (p. 152)

This is something that I ran into while analyzing the data. Although I did not add any new codes to the frameworks of either Weber (2004) or Harel and Sowder (1998), I did adjust somewhat some of the proof scheme categories (how I adjusted them and why is discussed in the next section).

The last loop of the data analysis spiral is the presentation of data. In case study research, this means that researchers should "present [an] in-depth picture of the case (or cases) using narrative, tables, and figures" (Creswell, 2007, p. 157). Once the cases are presented, a cross-case analysis can be done from which one can make "generalizations that people can learn from the case(s) either for themselves or to apply to a population" (p. 163). For me, this is completed in both Chapters 4 and 5. Chapter 4 will hopefully provide the reader with an in-depth understanding of what each participant did during each interview. Chapter 5, on the other hand, will provide a cross-case synthesis and generalizations.

3.6.1 Validation procedures and reliability

In addition to the data analysis spiral, Creswell (2007) addresses the validation of qualitative research, which he refers to as "an attempt to assess the 'accuracy' of the findings" (p. 206). He also lays out eight "accepted strategies" (p. 207) that serve to validate qualitative research findings. Of the eight, Creswell recommends "that qualitative researchers engage in at least two of them in any given study" (p. 209). For the purposes of this study, five of the strategies were used. It is worth noting that similar suggestions are made by other authors (e.g., Eisenhart & Howe, 1992; Kirk and Miller, 1986).

The first validation strategy I used for this study is "prolonged engagement and persistent observation in the field" (Creswell, 2007, p. 207). Kirk and Miller (1986) describe a situation in which a researcher lacked crucial information until adequate time in the field was accomplished. For Creswell, the purpose of this is to gain the trust of the participants and to better make "decisions about what is salient to the study, relevant to the purpose of the study, and of interest for focus" (p. 207). As mentioned earlier, I was able to get to know the students in the study the semester before it started as I worked with them on their MATH 305 projects. Also, I took Euclidean and non-Euclidean Geometry with six of the nine participants during the first semester of the study. Despite this, the most insightful interaction with the students came from the fourteen interviews that occurred throughout the school year.

Peer review is the second validation strategy used and it "provides an external check of the research process" (Creswell, 2007, p. 208). This was accomplished by asking two fellow graduate students to each review and code participant responses for 3 questions. In both instances, there were 6 classifications to be done (one proof type and one identified proof scheme). In the case of the first peer reviewer, there was agreement on 5 of the 6 classifications and agreement was reached between the reviewer and me on the sixth. My description of one of the proof schemes led to confusion on his part, and once I described the scheme better we agreed. The second reviewer and I agreed on all six classifications.

The third of Creswell's strategies employed for this study is the clarification of researcher bias. The purpose of this is to make clear the researcher's position and any biases that may exist going into the study. In this way, the reader is aware of "past experiences, biases, prejudices, and orientations that have likely shaped the interpretation and approach to the study" (2007, p. 208). This was accomplished in Section 3.2 above.

The next validation strategy used was member checking. Member checking "involves taking data, analysis, interpretations, and conclusions back to the participants so that they can judge the accuracy and credibility of the account" (Creswell, 2007, p. 208). This was accomplished in two ways. First, the reflection period of the interviews generally started with me relating to the student what I thought I was seeing happen as they worked. In this way, they were able to correct any misconceptions I had regarding their work. Also, two of the cases studied in Chapter 4 were give to the participants for their review to make sure I did not misrepresent what they did. Both participants verified that their work had not been misrepresented in anyway.

The last validation strategy technique used in this study is thick, rich description. By this, it is meant that the data will be described in enough detail that the readers will be able "to transfer information to other settings and to determine whether the findings can be transferred" (Creswell, 2007, p. 209). These descriptions comprise Chapter 4 of this study and I hope to have provided enough detail that the reader clearly understands what happened in each interview. Included in each description of what each student did are pictures of their scratch work and final proof (where applicable) and pertinent quotes from the participants discussing what they did.

Creswell (2007) also mentions a few ways to address the reliability in qualitative research. "Reliability can be enhanced if the researcher obtains detailed field-notes by employing a good-quality tape for recording and by transcribing the tape" (p. 209). Kirk and Miller (1986) also mention field-notes as a means to increase reliability (p. 51). Transcribing all interviews myself allowed me to take notes at my own pace not long after the actual interview. This also allowed me to concentrate on what was happening during the interview without worrying about making notes.

While this helps ensure data is reliable, Creswell (2007) says that 'in qualitative research, 'reliability' often refers to the stability of responses to multiple coders of (transcript) data sets" (p. 210). As mentioned above, two other coders categorized a portion of the data, including the full transcriptions and all scratch work done by the participants.

3.7 Conceptual framework

As is alluded to above, the frameworks of Weber (2004) and Harel and Sowder (1998) were the two frameworks used to code the data. Weber's framework for proof processes, or proof types, was used to describe the actual work each student did when they attempted to complete a proof. On the other hand, proof scheme categories were used to classify how the participants became convinced of the truth of mathematical statements and how they try to convince others. By using both frameworks, I believe a more complete picture is produced and more insight is provided into the participants' ideas of proof. These frameworks complement each other in that Weber's gives a way to classify the students' final product and Harel and Sowder's gives a way to classify the conceptions of proof students display while working toward the final product.

The framework of proof processes of Weber (2004) was used without alteration. As a reminder, his categories of proof processes (how students come to a proof) are *procedural*, *syntactic* and *semantic*. Procedural proofs are those that can be completed by following a procedure and may or may not be meaningful for the proof writer. This type of proof is divided into two sub-types: *algorithm* and *process.* In algorithmic proofs, students mechanically follow specific steps laid out for them. In process proofs, students still follow steps. However, in this case, the steps proof writers follow are “a shorter list of global qualitative steps that are not highly specified manipulations, but rather (involve) accomplishing a general goal” (Weber, 2004, p. 2).

Syntactic proofs are proofs that are arrived at via formal deduction and generally involve “manipulating correctly stated definitions and other relevant facts in a logically permissible way” (Weber, 2004, p. 4). This type of proof still need not be meaningful for the participant. In fact, many proofs examined in this study are labeled syntactic because the bulk of the work completed by the student involved rearranging algebraic expressions or equations without regard to the expression’s meaning in the context of the problem.

The last type of proof process is semantic. One completes a semantic proof when “one first attempts to understand why a statement is true by examining representations of relevant mathematical objects and then uses this intuitive argument as a basis for constructing a formal proof” (Weber, 2004, p. 5).

It is important to note that two different students can give very similar proofs but have separate classifications. This is because how the student comes to the proof has great influence on the proof type. Also, it is often the case that students will display more than one proof process. When this happens the work that leads to the proof will determine the proof type. For example, a student may manipulate an equation without any success, then view the equation as a function and gain insight into the problem by examining its behavior. In these instances, if the participant gets a proof by examining the function's behavior, the proof will be labeled semantic because the proof was produced via insight.

In his paper, Weber (2004) did not label incomplete proofs. Here I did, as long as an attempt at a proof had been made. In these cases, the proof types were labeled as "proof attempts" rather than proofs.

When going over the data, I realized that the proof scheme framework of Harel and Sowder (1998) needed far more adjustment to suit my needs than did the proof process framework of Weber (2004). The first two major categories of proof scheme, *external conviction* and *empirical*, were used as described in the original paper. External conviction proof schemes are held by students when they become convinced by source outside themselves. This could be a teacher or textbook (*authoritative* external conviction), the misuse of notation (*symbolic* external conviction) or an argument that merely looks like a proof (*ritual* external conviction). Empirical proof schemes can be *inductive* (where induction evidence is used to convince) or *perceptual* (where students are convinced something is true based on the way something looks, like triangles that are drawn to appear congruent).

The last major classification of proof scheme (*analytic*) is where some adjustment was made. This is a proof scheme purely based on logical deduction. Like the others, this category also has sub-categories, *transformational* and *axiomatic*. Transformational proof schemes are characterized by operations on objects and the anticipation of the results of the transformations. Harel and Sowder (1998) further divide transformational proof schemes into three sub-sub-categories, *internalized*, *interiorized* and *restrictive*. It should be noted that, in this study, the restrictive label was not used. The reason for this was that it is used to describe deficiencies in a student's proof scheme that were not apparent over the course of the study. I believe this was the case because the sort of difficulties this category describes would be more apparent in a classroom setting where material is built up over a period of time.

The other types of transformational proof schemes were used, however. Internalized proof schemes are general heuristics that serve as proof for the participant. Interiorized proof schemes can be thought of as internalized schemes employed with a deeper level of understanding. For instance, a student who knows and can follow the steps required for a proof by mathematical induction, but does not fully understand them, would hold an internalized transformational proof scheme. A student who understands induction and is able to adjust the steps involved as needed for a particular proof has an interiorized proof scheme. For the purposes of this study, transformational proof schemes will be taken to be interiorized unless otherwise stated.

Most often, proof schemes will be labeled transformational when the participant uses some form of algebraic manipulation to complete their proof. In these cases, the heuristic the students are using involves rearranging an expression (into a form useful to them) or an equation (into one that is obviously true). Proofs by mathematical induction will generally be taken to represent a transformational proof scheme because they depend on the result of moving from one case to another (generally, n to $n + 1$). Exceptions to this may occur when a student is convinced by the structure of a proof instead of understanding the method itself. This would represent a ritualistic external conviction proof scheme. Students who display an awareness of anticipatory actions will also have their proof schemes labeled transformational. An example of this is when a student points out that a statement is made early in a proof only to be used later.

Axiomatic proof schemes are held by students who "have an awareness of an underlying formal development" (Harel & Sowder, 1998, p. 276). This sort of awareness is shown by a student who references axioms or undefined terms. To Harel and Sowder, students displaying this proof scheme are "necessarily aware of the distinction between the undefined terms…and defined terms" (1998, p. 273). Axiomatic proof schemes are further divided into more specific categories by Harel and Sowder (see Chapter 2). For the purposes of this research, however, this level of refinement is unnecessary. These sub-types refer to the ways in which students view axioms themselves, something that is not discussed in the current study.

For use in the current study, evidence for an axiomatic proof scheme will be taken to be anything that illustrates a participant's understanding of formal development of mathematics. This includes not only axiomatics but also a reliance on previously proven results or other aspects of mathematics beyond the scope of the particular problem at hand. This could also refer to explicit mention of a proof's deductions from starting assumptions. For the purposes of this study, this will be taken to be axiomatic due do the focus on deduction rather than mathematical operations.

The axiomatic proof schemes as described by Harel and Sowder (1998) would probably be more evident in a class setting, where previous results were shared between participants and could be used. Asking them isolated problems definitely hinders their use of prior results.

When completing formal proofs, students will usually display elements of both the transformational and axiomatic proof schemes. However, in most cases evidence will strongly point to one or the other. In such cases, the student's proof scheme will be labeled as the one most evidently present. Other times, the student will show strong signs of both and his or her scheme will draw both designations. In still other instances, students will demonstrate a formal understanding of proof and yet not give any strong evidence for either sub-type of analytic proof scheme. In these cases, the student's scheme will be labeled as analytic only.

3.8 Interview questions

This section will state and describe the questions used in the interviews, as well as why they were chosen and where I found the questions if I did not come up with them myself. It should be noted here that care was taken to choose problems that I found interesting in the hopes that the participants would as well. Not all interviews were task-based, however. The last interview of each semester of the study was used as a debriefing session to talk with the participants about their progression in proof.

The questions were all chosen so that the participants were aware of all background material needed to complete the problem. Specifically, all the questions could have been understood and worked on by students who completed MATH 305. That being said, the questions were also chosen so that they could not be completed easily by the students. Some of the questions took longer than the allotted time during the interview, so there were many instances when the students were asked to continue working on the problem between interviews and there were also times when two consecutive interviews were dedicated to a single question. This is why there were a total of twelve questions asked over fourteen interviews.

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

The first problem I gave the participants is referred to as the Isis problem and I heard of it during a colloquium talk given April 13, 2009 by Brian Greer of Portland State University at The University of Montana entitled: "The Isis problem as a probe for understanding students' adaptive expertise and ideas about proof." The problem was chosen because it was open-ended both in that the participants were to decide for themselves what was to be proved and also in that there are a number of ways to finish the problem. I also wanted an easily accessible problem for the first interview.

<u>Question 2a</u>

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9.

This problem and the next were both given to the participants in the second interview because I did not know how long it would take them to complete. This question was inspired by Problem 1 in an article by Pedemonte (2008, p. 391). Again, this problem was chosen because it was exploratory in nature and was able to be proved by a variety of methods.

<u>Question 2b</u>

If n is a positive integer, then $n^3 - n$ *is a multiple of 6.*

I do not recall the first place I saw this question, but I do remember that I originally saw it as an undergraduate at South Dakota State University. I chose the problem because it can be approached from multiple ways, including fairly straightforwardly via mathematical induction. Because a few students wanted to apply induction to the first problem, I thought it would be interesting to see how many participants tried to apply that method directly. Because not every student got far enough on Question 2a during Interview 2, the third interview was spent discussing this question in most cases (even though the students originally saw it in the second interview).

<u>Question 3</u>

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

Because not every participant used induction on the previous problem, I wanted to ensure that every student tried it on at least one problem. The problem came from the textbook the students used in MATH 305 (Daepp & Gorkin, 2003). It is Problem 17.7 on page 216. One of the reasons I picked the problem was that, once proven, it can be used to prove that the harmonic series diverges. I thought that perhaps the participants would find this interesting and I generally pointed this out to them once they had finished working on the problem. It was also chosen because it was considered to be rather difficult. This ended up being the case and the fourth and fifth interviews were spent working on it.

Question 4

Examine the following proof by contradiction:

Prove $\sqrt{-1} \leq 0$

Suppose $\sqrt{-1} > 0$*, then* $\sqrt{-1} \times \sqrt{-1} > 0$*. This implies* $-1 > 0$*, which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

Prove $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$*, then* $-\sqrt{-1} \geq 0$*. This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$*, so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This question was chosen because it accomplished two goals. First, it was a problem they had worked on for their MATH 305 midterm take-home exam the previous semester. Thus it served as way to see progress from the year before. Also, it was an opportunity for the students to evaluate a completed argument. This was to be the final task-based interview for the semester and I wanted to make sure that I had a question that addressed both of these goals in each semester of the study. This question was discussed during the sixth interview only.

<u>Question 5</u>

How do you feel the semester has gone in regards to proof?

Do you feel you've gotten better at proofs?

What do you think led to any improvement you saw?

What could have led to more improvement?

What do you think it takes to have a successful proof attempt? What helps but isn't necessary?

The seventh interview was not task-based and was used to debrief the participants about the first half of the study. The questions above served as a guide for the questioning, but the discussion was not limited to them.

<u>Question 6</u>

Prove that $\forall n \in \mathbb{N},\ \ n \equiv n^3$ *(mod 6).*

This question was covered in the eighth interview, the first of the second semester of the study. It is question number 28 from an abstract algebra textbook (Gallian, 2002, p. 24). I chose the question for a few reasons. One, I deemed it to be not too difficult and a good way to start the second semester. Also, it would allow me to see which students were familiar with modular arithmetic, something covered very briefly in MATH 305. I also wanted to use the question before students taking Number Theory that semester saw this or a similar problems. That it is a re-worded version of Question 2b was something I did not notice until I gave it to the first student. This was a lucky accident, however, because it gave students who recognized this fact an opportunity to display an axiomatic (as I am using the term) proof scheme. This sort of chance was relatively rare given the stand-alone nature of most of the questions.

<u>Question 7</u>

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

This question was chosen for a number of reasons. First, it is exploratory in nature; the participants were not told explicitly what to prove. Second it is a result that was not covered in MATH 305 but I thought they would find interesting. It was also a useful question because there are a number of ways to approach it. The best part about this question, however, is that it was unclear to many of the students when they had completed a proof. The students who arrived at the solution $\sum_{k=0}^{n}\binom{n}{k}$ often had a hard time "proving" that this solution worked. This challenged their idea that a proof had to be in one of the forms they had been taught (induction, contradiction, etc.). Discussion of this problem covered interviews nine and ten.

<u>Question 8</u>

Prove that the cube root of 2 is irrational using a proof by contradiction.

This question was chosen because the students first saw the problem on their take-home midterm exam in MATH 305. Also, no question to this point had explicitly asked the students to complete a proof by contradiction on their own. The participants worked on this question during Interview 11.

<u>Question 9</u>

Cantor's Diagonalization Argument

Theorem:

The set of real numbers, $\mathbb{R}$, is an uncountable set.

Proof:

Will prove by contradiction. Suppose that ℝ is countable. Then, since every subset of a countable set is countable, the open interval (0, 1) is countable as well. Then, suppose that *f*(*x*) is the 1 – 1 and onto function taking the natural numbers, ℕ, into the interval (0, 1). We'll write the outputs, *f*(*n*), where $n \in \mathbb{N}$, in decimal notation as follows:

$$\begin{aligned} f(1) &= 0.\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\cdots, \\ f(2) &= 0.\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{24}\cdots, \\ f(3) &= 0.\alpha_{31}\alpha_{32}\alpha_{33}\alpha_{34}\cdots, \\ &\vdots \\ f(j) &= 0.\alpha_{j1}\alpha_{j2}\alpha_{j3}\alpha_{j4}\cdots, \\ &\vdots \end{aligned}$$

where α_{ji} represents the i^{th} digit in the decimal expansion of *f*(*j*). Now that we have our list, define the number, *B*, as follows: Let $B = 0.\beta_1\beta_2\beta_3\beta_4\cdots$, where the digits are defined by

$$\beta_j = \begin{cases} 1 & if\ \alpha_{jj} = 2 \\ 2 & if\ \alpha_{jj} \neq 2 \end{cases}.$$

Now, since *f* is onto, $\exists\, k \in \mathbb{N}$ such that *f*(*k*) = *B*. However, $\beta_k \neq \alpha_{kk}$, by definition of β_k. This is a contradiction because the decimal expansions of *B* and *f*(*k*) should be the same if they are to be equal. Thus, the assumption that ℝ is countable led to an absurdity. Therefore ℝ is uncountable.

■

Question 9, like Question 4, asked the students to evaluate a completed proof. Also, the fact that the real numbers forms an uncountable set is something mentioned in MATH 305 but it was not proved there. For this reason, I thought the students would find it interesting. I was also curious if the participants would expect to find a flaw in the proof since it was given to them in the context of an interview, where they generally had to figure something out. This question was discussed in Interview 12.

Question 10

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

This question came out of a discussion I had with a fellow graduate student where we both wondered if the result were true. It turns out that for $0 < a < b-1$ and m an integer, the following inequality involving mixed numbers holds: $m\frac{a}{b} < m\frac{a}{b-1} < m\frac{a+1}{b}$. The problem was chosen primarily because it was open-ended and I thought the students would find it interesting. I also was curious how the students would handle finding counter-examples. This question was started in Interview 13 but since only one student finished during the first interview, it was also discussed during the final interview.

Question 11

1. How do you feel the semester has gone in regards to proof?

2. Do you feel you've gotten better at proofs? Do you think you improved more this semester or last?

3. What do you think led to any improvement you saw?

4. Have you implemented anything in your proof techniques that weren't there at the beginning of the semester? Year?

5. What role do you see examples playing in proof?

6. What could have led to more improvement?

7. What do you think it takes to have a successful proof attempt? What helps but isn't necessary?

Interview 14 was the last interview and it was used, like Interview 7, to debrief the students about the study. As can be seen from the questions, they were quite similar to the previous semester ending questions. The exception to this was the question regard the use of examples. This question was added because it was something that was discussed with most participants during Interview 7.

3.9 Limitations

Because the current research is a qualitative case study, the results found are only directly applicable to the individuals included in the study. It is hoped that enough detail has been provided to allow the reader to judge to which situations the findings might be able to be transferred. The study is quite specific in that only students who had just completed MATH 305 at The University of Montana during the Spring semester of 2009 are included. That being said, the transition from algorithmic mathematics to more proof-based mathematics is common to all undergraduate mathematics majors.

The qualitative research has other inherent limitations beyond limited generalizability. There is also a large interpretive component to the data analysis portion of such research. This allows for researcher bias to influence findings. However, attempts to minimize this effect have been taken. First, I have made all relevant biases and prior expectations clear at the outset of the study. Also, data was analyzed by two peer-reviewers and my analysis was taken back to two of the participants to check for misinterpretation on my part. Lastly, I used "member-checking" with the participants during the reflection portions of the interviews. I did this by talking through the students' work and giving them the chance to correct any misconceptions I had about it.

Another limitation with this type of research is the potential for the way data is collected to influence what happens. For instance, the presence of an interviewer may change how the participant works through a problem. This was especially the case during the first 2 rounds of interviews. Not only did I interject when I thought the student was not "thinking out loud" enough, I was also far more liberal with hints than I was later on in the study. This was partly due to my relative inexperience and partly because I wanted the first few interviews with each student to be as stress free as possible. I was very careful of this while analyzing their initial work and noted as much as possible where I helped the students. Beginning with the third interview, the students worked much more independently but the presence of someone else watching them work may have affected what they did. It is hoped that the number of interviews raised the participants comfort level to the point where this affect was lessened.

Time was another limitation in the study, as we only met every two weeks and the interviews were limited to one hour each. Because not every student finished each problem, they were often asked to work outside of the interview setting and bring back what they did. They were also asked on various occasions to continue working on the same problem two weeks later if they did not have time between interviews or did not complete the problem between interviews. This is a limitation because it allowed for some aspects of their work on a particular problem to be forgotten between when they stopped working and when they started again. To help alleviate this problem, I asked that the students bring in all scratch work that was completed outside of the interview setting. Also, if work resumed during an interview after a break, time was always spent going over the previous work in detail so the participants were comfortable with what they had done.

Lastly, limiting the cases to only nine students over the course of a one academic year at a single university allows for the possibility of unexpected limitations to affect the study. In particular, the fact that all participants took MATH 305 from the same instructor might have had some influence unique to the students specifically. This type of limitation is common across qualitative research. For greater generalizability of these results, further research with other populations is necessary.

Chapter 4: Analysis of Student Work

In this chapter, the work each student did will be examined and classified using the frameworks laid out at the beginning of this study. Recall from the previous chapter that the coding found below was validated by two peer reviewers.

4.1 John

This section looks at the progress John made over the course of the study. John is a mathematics major who is planning on becoming a secondary mathematics teacher. Over the course of the study, John took Teaching Math with Technology and Euclidean and Non-Euclidean geometry during the Fall semester and Number Theory and History of Mathematics in the Spring.

John's Proof Attempts

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

John began by setting up the equation $2a + 2b = ab$ and plugging in possible a values to find the corresponding b's. It did not take long, however, for him to look outside the context of the problem for more meaning. He looked for reasons outside of equation alone to see why $a = 1$ did not work:

> John: It's not going to work. But I'm wondering, if there's, 'cause there's got to be sort of another reason that it can't be 1. I mean, is there another reason why it can't be 1? I mean, does it just have to do with the nature of the number?

He continued looking at different *a* values until *a* = 7, he finds the solutions, checking the areas and perimeters separately as he goes. He then goes to the equation:

> John: So that…um…I'm going to try and go back and look at the original equation again and see if, um…it looks like I can re-write it so that I can just set it in terms of *b*…as 2*a* over *a* – 2. (See Figure 1)
>
> Nick: Ok.
>
> J: And …so this is telling me that's why 1 and 2 didn't work…so all we have to do is, I guess, is figure out that whenever I put any integer *a* into this form, then the result will be an integer.

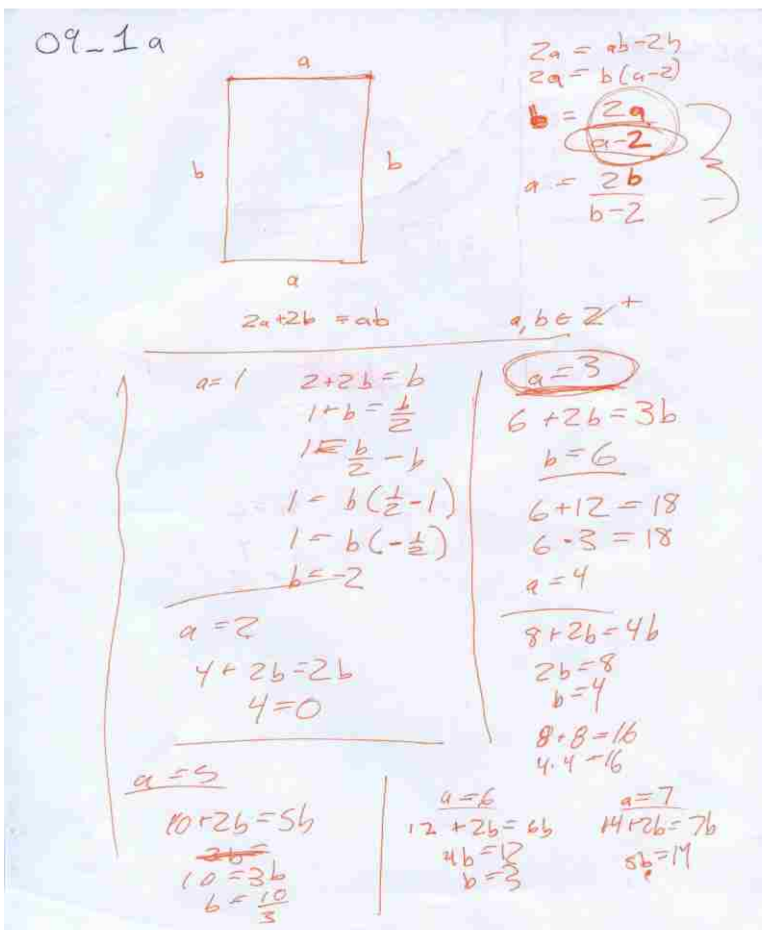


Figure 1: John's work on Question 1 (1 of 4)

At this point, he asks for guidance and I suggest looking at the graph to see many values at once. John draws the graph and labels the points as he mentions them (see Figure 2)

> J: …and we know for 3, the, uh, if *b* is 3, then *a* is going to be 6, so we have 3 and 6 somewhere in here and the 4…equals 4…and there's nothing in between those two and if we look at the 5, it's going to be on a non-integer *y* value.
>
> N: Right. We saw that with 10 thirds when you solved for it.
>
> J: So if we look at 6, if *b* is 6, then *a* is 3.
>
> N: Right
>
> J: And we go back to 3 here. And now, since the graph is decreasing…and you can not have, you can not have a value of 2 or 1, I want to say that those are the only ones, but I'm not sure.

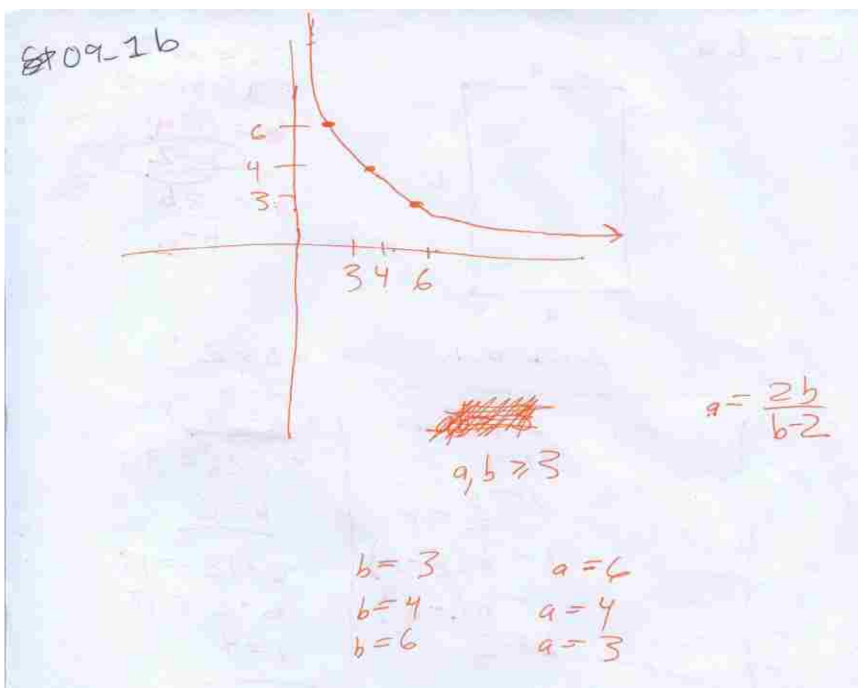
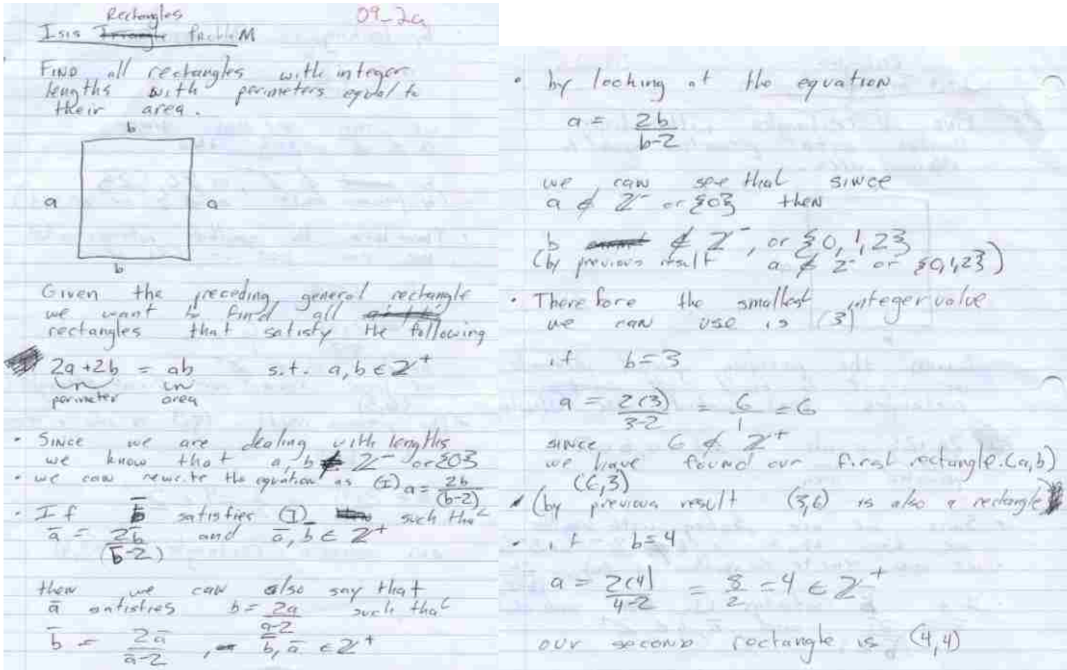


Figure 2: John's work on Question 1 (2 of 4)

We then talked through the different potential inputs for the function he graphed: 2 does not work because of denominator ($x - 2$), anything below 1 gives a negative output. Moving on to inputs greater than 3, John says:

> J: And I think that, yeah, it's definitely monotonically decreasing…so yeah, I'm going to say that these are the only values, but… Yeah. I feel fairly convinced, but yeah, I imagine that when I write it all up maybe I'll see something else that maybe I missed out on, but…

He left the interview and brought back a written up version of the argument, shown in Figures 4.09.3 and 4.09.4. In it, John makes the case that one only needs to consider integer values above 2, demonstrates that 3 and 4 work for *b* values (as does 6, by symmetry) but 5 does not. He then shows that his function $a(b)$ is decreasing by taking the derivative. His proof concludes with restating that 3 is the smallest allowable side length and that $a(b) < 3$ for all $a > 6$.

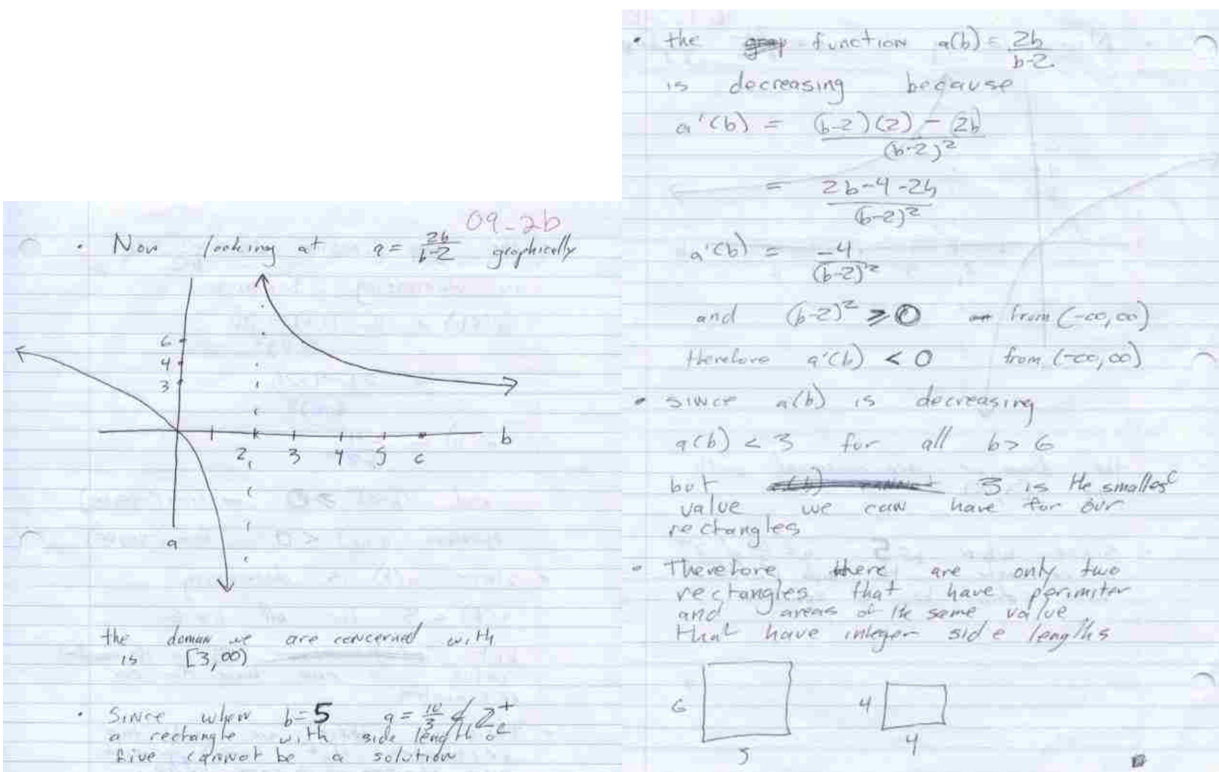


Rectangles 09_2a
Isis Triangle Problem
Find all rectangles with integer lengths with perimeters equal to their area.

Given the preceding general rectangle we want to find all rectangles that satisfy the following
$2a+2b = ab$ s.t. $a, b \in Z^+$ (perimeter, area)
• Since we are dealing with lengths we know that $a, b \notin Z^-$ or $\{0\}$
• we can rewrite the equation as (I) $a = \frac{2b}{(b-2)}$
• If $\bar{b}$ satisfies (I) such that $\bar{a} = \frac{2\bar{b}}{(\bar{b}-2)}$ and $\bar{a}, \bar{b} \in Z^+$
then we can also say that $\bar{a}$ satisfies $b = \frac{2a}{a-2}$ such that $\bar{b} = \frac{2\bar{a}}{\bar{a}-2}$, $\bar{b}, \bar{a} \in Z^+$

• by looking at the equation $a = \frac{2b}{b-2}$
we can see that since $a \notin Z^-$ or $\{0\}$ then
$b \notin Z^-$, or $\{0, 1, 2\}$
(by previous result $a \notin Z^-$ or $\{0,1,2\}$)
• Therefore the smallest integer value we can use is (3)
if $b = 3$
$a = \frac{2(3)}{3-2} = \frac{6}{1} = 6$
since $6 \in Z^+$
we have found our first rectangle (a,b) (6,3)
(by previous result (3,6) is also a rectangle)
• if $b = 4$
$a = \frac{2(4)}{4-2} = \frac{8}{2} = 4 \in Z^+$
our second rectangle is (4,4)

Figure 3: John's work on Question 1 (3 of 4)



09_2b
• Now looking at $a = \frac{2b}{b-2}$ graphically
the domain we are concerned with is $[3, \infty)$
• Since when $b = 5$, $a = \frac{10}{3} \notin Z^+$ a rectangle with side length of five cannot be a solution

• the function $a(b) = \frac{2b}{b-2}$ is decreasing because
$a'(b) = \frac{(b-2)(2) - (2b)}{(b-2)^2}$
$= \frac{2b-4-2b}{(b-2)^2}$
$a'(b) = \frac{-4}{(b-2)^2}$
and $(b-2)^2 \geq 0$ from $(-\infty, \infty)$
therefore $a'(b) < 0$ from $(-\infty, \infty)$
• since $a(b)$ is decreasing
$a(b) < 3$ for all $b > 6$
but 3 is the smallest value we can have for our rectangles
• Therefore there are only two rectangles that have perimeter and areas of the same value that have integer side lengths

Figure 4: John's work on Question 1 (4 of 4)

The proof John provides for this problem is semantic. Here, the proof is based on an examination of the equation he obtained and its graph to understand the relationship between a rectangle's side lengths under the given conditions. He then was able to see that if one side length increased, the other would have to decrease. This led him to construct a formal proof that he had found the only rectangles with the given properties.

The proof scheme exhibited is an analytic, transformational scheme. While he does rely on some previous mathematical results in his proof (e.g., decreasing functions have negative derivative), the heavy lifting of the proof is accomplished operating on objects (taking the equation and looking at it as a function) and anticipating what happens through manipulations (letting one side length increase and then observing the change in the other).

<u>Question 2a</u>

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9. (For example, if a = 4 and b = 2, then 42 – 24 = 18, a multiple of 9.)

John began by getting the terminology straight, looked at an example and then reviewed some tricks for multiplying by 9. Then, he looked at some more examples. He hit on the pattern that said if you leave *a* the same and increase *b*, the difference decreases by 9. For example, 90 – 09 = 81, 91 – 19 = 72, 92 – 29 = 63, etc. John then put that pattern into the formula $ab - ba = (a)(b + 1) - (b + 1)(a) - 9$, in Figure 4. However, he does not know how to go about proving it will always hold.

> J: I'm just having a hard time thinking about this, just notation wise. How to describe, because I think I almost understand what I need to do. Like, how [am I] …going about to show that by increasing the [*a* and *b*] by increments of 1, in both

directions…Like just showing that…I guess that's what I wrote here. That's the best way to write it; to verify it…I need to go … Right. So can we say that this is (*ab* + 1), where this is the ones place and this is (*ba* + 10)… So that would be the verification. (John rewrites *a*(*b* + 1) – (*b* + 1)*a* as (*ab* + 1) – (*ba* + 10) = *ab* - *ba* – 9, note that here the parentheses are meant to separate digits, not signify multiplication.)

John left the interview after making this realization and I asked him to write up a proof. He came back with a double induction-type proof in which he shows that if you start with a multiple of 9 and increase *a* or *b* by 1, then you still have a multiple of 9. As above, increasing *b* decreases the difference by 9. Increasing *a* has the opposite effect:

(*a* + 1)*b* – *b*(*a* + 1) = (*ab* + 10) – (*ba* + 1) = *ab* - *ba* + 9.

John then made the case that one could (starting at 10 – 01) get any two digit combination by the appropriate increase of *a* and\or *b*. I did not include the write up because the proof is quite long.

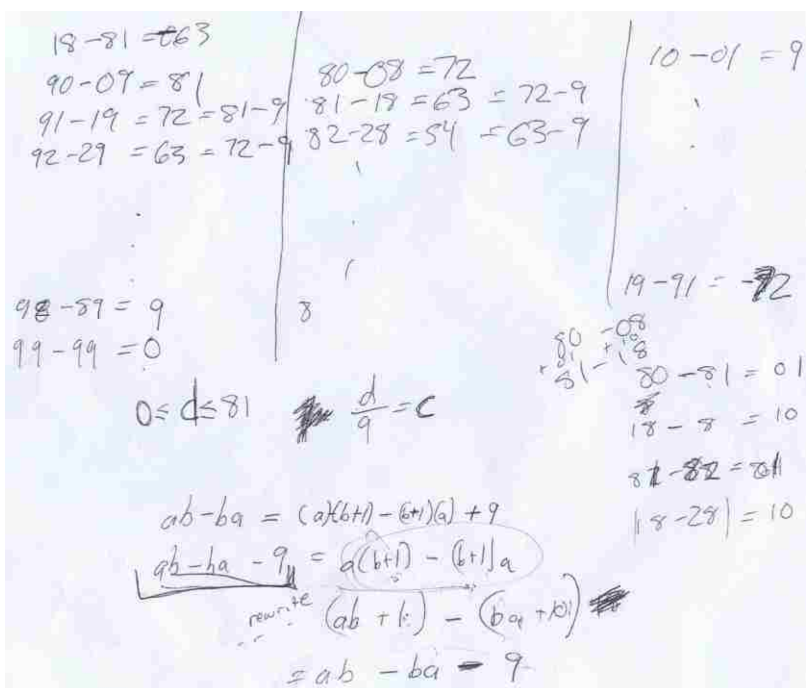


Figure 5: John's work on Question 2a

Even though John makes use of induction here (typically enough to classify a proof as procedural), this is a semantic proof attempt. In the first interview he spent time looking at examples to find a pattern. This pattern led him to an understanding of the problem that could then be turned into a formal proof. John even verbalized this at one point:

> J: So this was just a part of like what I went through in here, where I went through and did all these examples trying to see…hoping that I could recognize a pattern…kind of like rip that out of there and try to use it.

While the type of proof this qualifies as is fairly straight forward, John's proof scheme is not as straightforward. First, he realizes that a strictly empirical proof would work for this problem:

> J: I probably could have gone through and done every …
>
> N: Yeah, that's a legitimate proof technique in this case because you can go through and exhaust all combinations of *b* and *a*.
>
> J: Right.
>
> N: The question is if this was a homework assignment, you mess with it for the half hour you've messed with it, would you just say "screw it", write it out, make the chart…
>
> (laughs)
>
> J: No. I mean, if I felt like teasing the professor a little bit maybe, but I'd rather, I'd want to know why.

It is important to note, however, that this is not evidence of the empirical proof scheme. John is not becoming convinced by examples. Rather, he simply realizes that there are a finite number of cases that need to, and could be, verified.

John does display some evidence of the axiomatic proof scheme, however: "…like this was the division rule that I wrote out, hoping that it would be somewhere in there that I could, if I could prove that rule, then I could apply it to the question." Here John notes that at one point he thought the test for divisibility by 9 might be useful, but he must prove it if he decides to use it.

That being said, the strongest evidence of proof scheme comes from how he comes up with what eventually becomes his proof. During the first interview, when he considers how the difference $ab - ba$ changes when b is increased by 1, he is displaying a transformational proof scheme. He is operating on an object at hand (the expression $ab - ba$) and working with the result of that transformation. This is the hallmark of a transformational proof scheme.

<u>Question 2b</u>

If n is a positive integer, then $n^3 - n$ *is a multiple of 6.*

John began working on this problem between interviews 2 and 3. He did not, however, bring in the work he had done and we spent then end of interview 3 talking about the things he had done (written out in Figure 6). He went to induction almost immediately due to both the presence of the variable n and the work he had done on the previous problem:

> J: … just seeing the *n*'s in the problem especially…If it was a different letter, maybe I wouldn't have done an induction argument, but… But yeah, that was definitely a major part and then since I had just gotten out of doing an induction argument for the last proof

that was also sort of a similar, I mean, different notation, but a similar number theory sort of argument.

The work John described here involved a few different ways of manipulating the induction hypothesis, $(n + 1)^3 - (n + 1)$, when starting with the assumption that $n^3 - n = 6c$.

J: And I guess…basically all I was able it do, you know, assuming that this was true, then $n^3 - n$ over 6 equals c and $n^3 - n$ equals $6c$. And I was hoping that if I went in, $n + 1$ and I tried to re-write this in a bunch of different ways, so that I could substitute it in and get some sort of statement where it was just these c values…

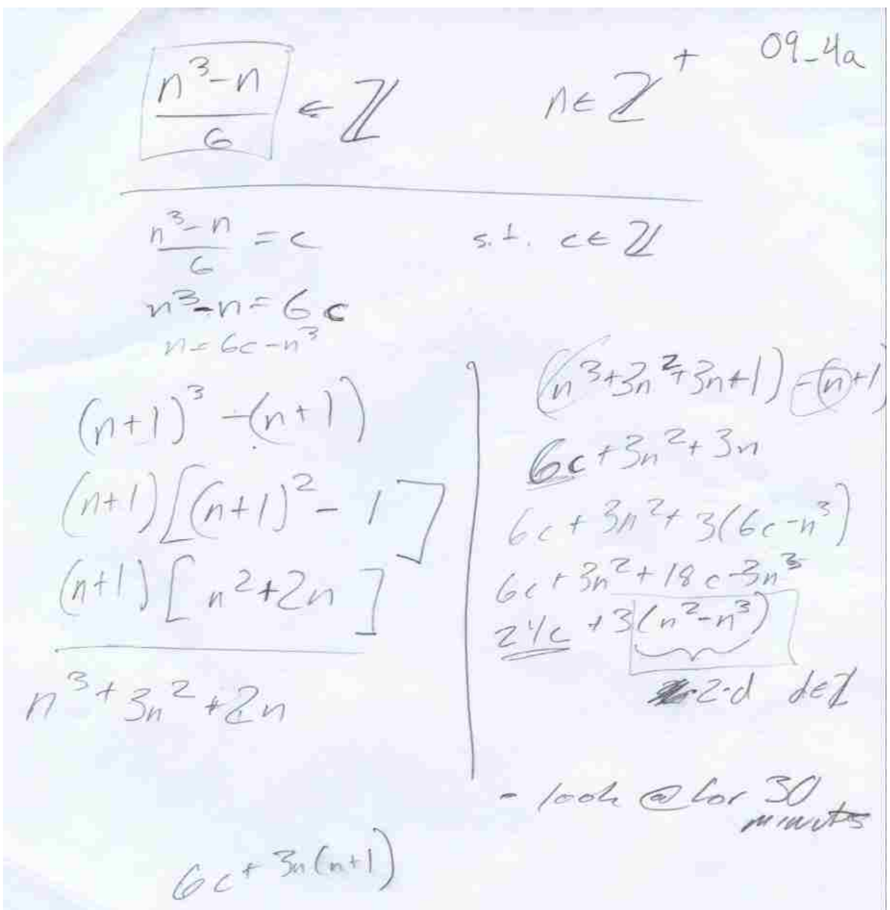


Figure 6: John's work on Question 2b (1 of 2)

N: Ok, so what sort of rearranging did you do?

J: I just factored out the $(n + 1)((n + 1)^2 - 1)$….(from $(n + 1)^3 - (n + 1)$ in the induction step)

[working, simplifies his factored form of $(n + 1)((n + 1)^2 - 1)$ into $n^3 + 3n^2 + 2n$]

N: Alright

J: I mean I played with it for a while, but I didn't …the difficult part for me was that I wasn't seeing where I could find, like, a difference…in this…

N: Right, you wanted to use the induction hypothesis and all you have is positives at this point.

J: Yeah, and then I tried just cubing this and writing it out and, you know, you just get…$n^3 + 3n$ (he meant $3n^2 + 3n$)

N: And you didn't see anything in that method either?

J: No, I don't think so.

Both of John's attempts to complete the inductive step of his proof failed at this point. In the first attempt, John simplified correctly, but not in a way that was useful. The other method ends up being useful, but he did not realize how at this point. I sent this scratch work home with John and asked that he look at the problem some more.

When John came back for the next interview, he had a complete induction argument (Figure 7). He assumed that $n^3 - n = 6c$ and went about investigating $(n + 1)^3 - (n + 1)$, making sure that he resisted the urge to simplify too much (thus removing the ability to use the induction hypothesis).

J: Yeah, instead of just going through and canceling everything out right away that I knew I could, I decided to keep this – n here and keep this $3n$ separate, so I wrote $n^3 - n + 3n^2 + 3n$. And so, re-wrote this $6c$, based on the induction hypothesis, we know that this is a number divisible by 6.

N: Right

J: And then I had to come up with some sort of argument then, that $3n(n + 1)$ is divisible by 6 as well. I mean, because this is like this (box in middle of Figure 7), 6 times a number plus 6 times another number is, you know, 6 times those two numbers added together.

N: Right

C: And went down here and looked at it, thinking that if I could re-write this here as 6 times some m, an integer, and m is $n(n+1)/2$, and since we know that this is divisible by 3.

N: Right

J: Right, so we need to see if it's divisible by 2. And then, so there's two possibilities for our n, then. If n is odd, then the $n + 1$ divided by 2 is an integer. And if n is even, then it itself divided by 2 is an integer. So either way, since it's $n(n+1)/2$, is equal to our m value, is guaranteed to be an integer because there's only two cases there.

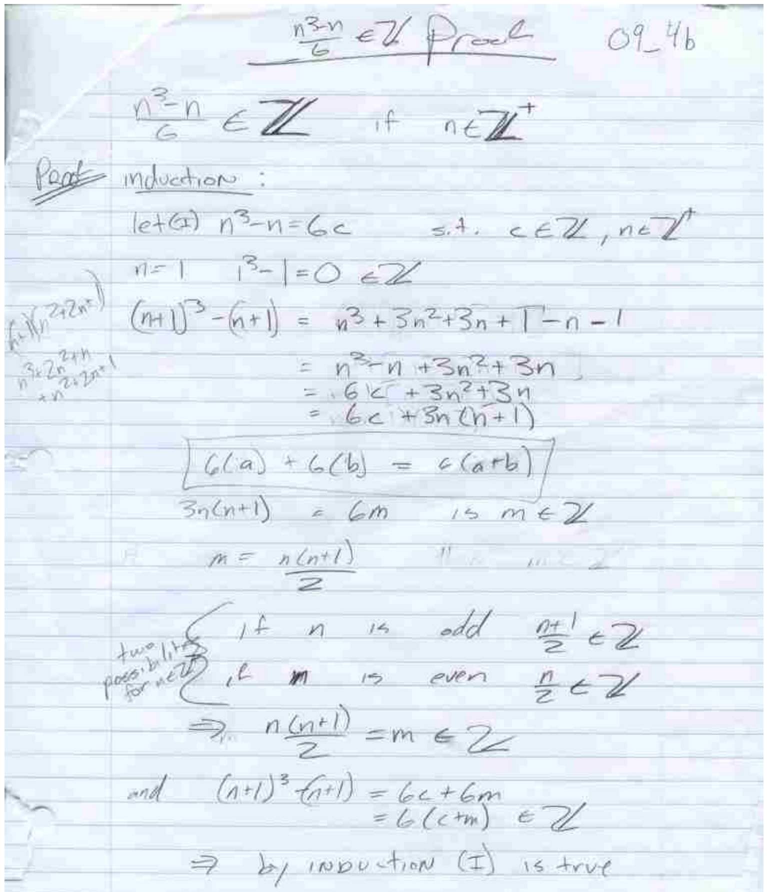


Figure 7: John's work on Question 2b (2 of 2)

John's proof was pretty straightforward once he realized he need to prove that $3n(n + 1)$ was a multiple of 6, i.e. that $n(n + 1)$ was even. The proof John provided was a process procedural proof. It involved a few global steps that needed to be accomplished but these individual steps did not have prescribed ways they needed to be completed. In particular, verifying that $3n(n + 1)$ is a multiple of 6 for all integers did not require set algebraic manipulations but rather some insight into how numbers become a multiple of 6.

Like with John's previous two proofs, the transformational proof scheme is shown here. He definitely uses valid and logical deductions from performing operations on mathematical objects. At the same time, there is no evidence that John is relying on the more formalized thinking that is typical of an axiomatic proof scheme.

Question 3

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

John began this problem by performing the steps required of an induction argument. He called the inequality stated in the problem his induction hypothesis and used n = 1 as his base case. In moving to the inductive step, he tried to create a string of inequalities that not only included what he thought to be the inequality to be verified but also the induction hypothesis (see Figure 4.09.8). This was unhelpful, however, because although he knew

$$1+\frac{n}{2} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \leq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{n+1}} \quad ,$$

he had no way of relating $1+\frac{n+1}{2}$ to any part of the above inequality other than $1+\frac{n}{2}$.

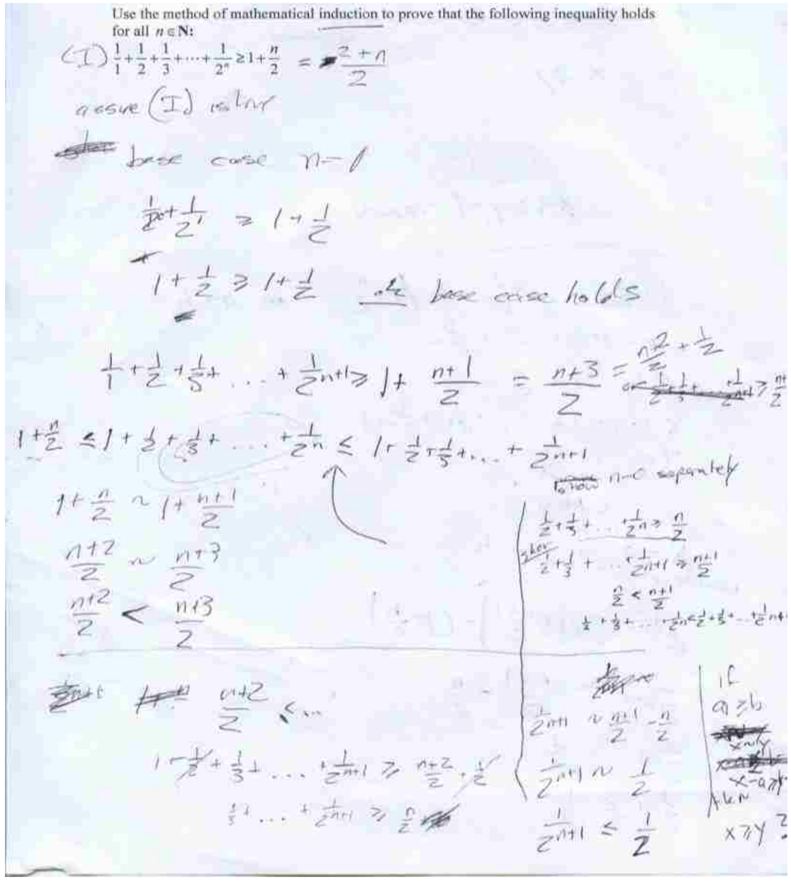


Figure 8: John's work on Question 3 (1 of 3)

This notion of comparing pieces of the inequality from the induction hypothesis to their corresponding pieces of the inductive step stayed with him as he continued. Since he could make direct comparisons between the corresponding pieces, he deduced a property that he thought would help him (see Figure 9): If $a \geq b$ and $x - a \geq y - b$, then $x \geq y$. Here, he was going to let

$$a = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}\ ,\ b = 1+\frac{n}{2},\ x = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{n+1}} \text{ and } y = 1+\frac{n+1}{2}.$$

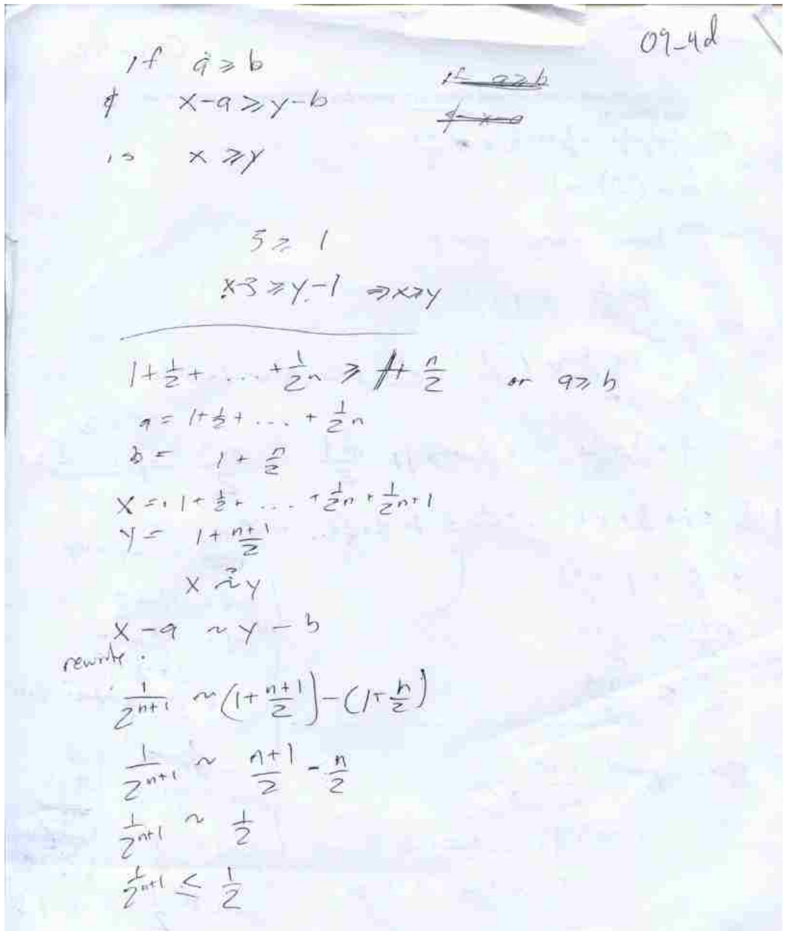


Figure 9: John's work on Question 3 (2 of 3)

However, when looking at the difference he proposed, he revealed a misconception he had regarding the problem. He claimed that $x - a$ should have been $\frac{1}{2^{n+1}}$. Leaving out the terms $\frac{1}{2^n+1}+\cdots+\frac{1}{2^{n+1}-1}$ was a fairly common mistake for the students in this study. Many of them overlooked the fact that more than one term was added to the left-hand side of the inequality when moving to the inductive step. This in turn led to trouble when looking at the difference $y - b = \left(1+\frac{n+1}{2}\right)-\left(1+\frac{n}{2}\right)=\frac{1}{2}$. In particular, John ran into the fact that $\frac{1}{2^{n+1}} \le \frac{1}{2}$ instead of the other way around as he had hoped. He was not completely discouraged, however and he left that interview agreeing to work on the problem again and try something else.

Given the difficulty John ran into at the end of the first interview, he started back at the beginning with the problem when working on it between interviews:

J: Well, I started out just going through and like, you know, re-writing it and playing around, …So, basically what I got from this was that I realized that it wasn't, for whatever reason in the session I was thinking it was $+\frac{1}{2^n}$ and there was nothing in between that and the $+\frac{1}{2^{n+1}}$.

From there, John proceeded to figure out that increasing *n* by 1 meant gaining an additional 2^n terms in the sum. It took some more playing around, but he eventually came to realize that the sum he needed to show was greater than $\frac{1}{2}$ had $\frac{1}{2^{n+1}}$ as its smallest term. Then, it was easy for him to see that $\frac{1}{2^n+1}+\cdots+\frac{1}{2^{n+1}} \geq \frac{1}{2^{n+1}}+\cdots+\frac{1}{2^{n+1}} = 2^n\left(\frac{1}{2^{n+1}}\right) = \frac{1}{2}$. His proof is in Figure 4.09.10.

J: That was it, and I was off. I wasn't sure even if it worked, I still had to write it up, and I was hoping that that would be the way to do it and it seemed like it would work out. And then I just went over, and yep, it worked out really nice and the best, I thought, was that I was able to then not just say that this was greater than some number over here and then have to relate it over again to the 1/2, but it just came out perfect.

N: It was the 1/2?

J: Yeah. That was the best part.

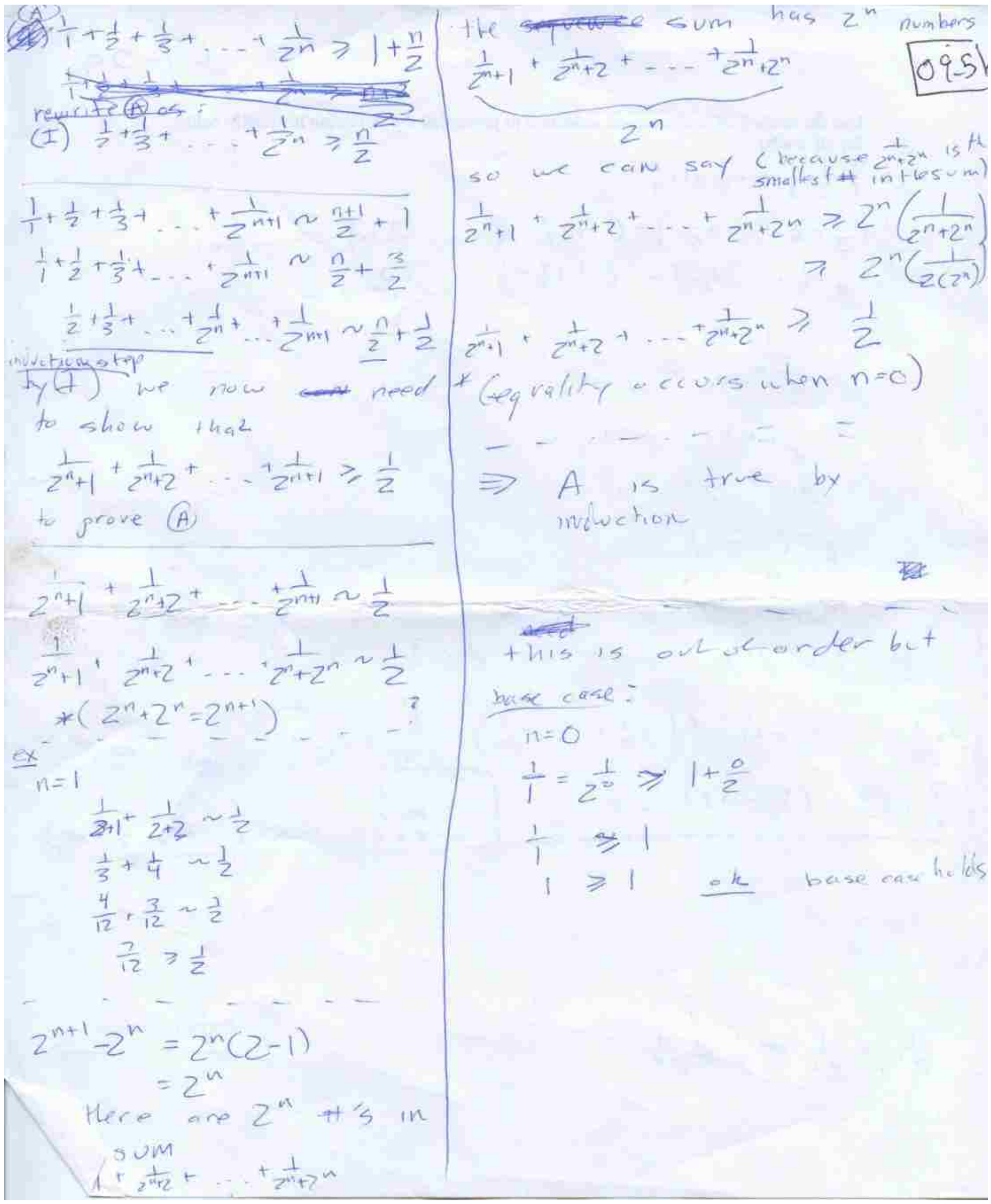


Figure 10: John's work on Question 3 (3 of 3)

As with the last proof, John provides here a process procedural proof. It was an induction proof, so there were specific steps to be completed. However, these steps were global in nature and the things that needed to be accomplished along the way were not laid out for him completely. As far as the proof scheme displayed here, most evidence points to a transformational scheme. The bulk of his work relies on the manipulation of mathematical objects within the framework of induction.

However, a subtle but important part of his work points to an external proof scheme as well. When faced with the difficulty he ran into at the end of the first interview in which he worked on the problem, John said: "I mean, I have some faith in the inequality, I'm just not completely sure yet." Due to a mistake, John reached a conclusion that was in direct conflict with what he needed to make his proof work. However, he put at least as much confidence into the fact that he was asked to prove the result as the fact that it did not seem true. Because an external authority had asked him to prove it, he did not want to throw away the result outright. In this case the decision to put some faith in the result was fruitful because if he had simply decided that it was not true, he would not have found the proof he did and he was glad for it: "…it was great, this was a really fun problem… It didn't seem that first day… Yeah, it felt good at the end, it feels good now, it's interesting."

<u>Question 4</u>

Examine the following proof by contradiction:

<u>*Prove*</u> $\sqrt{-1} \leq 0$

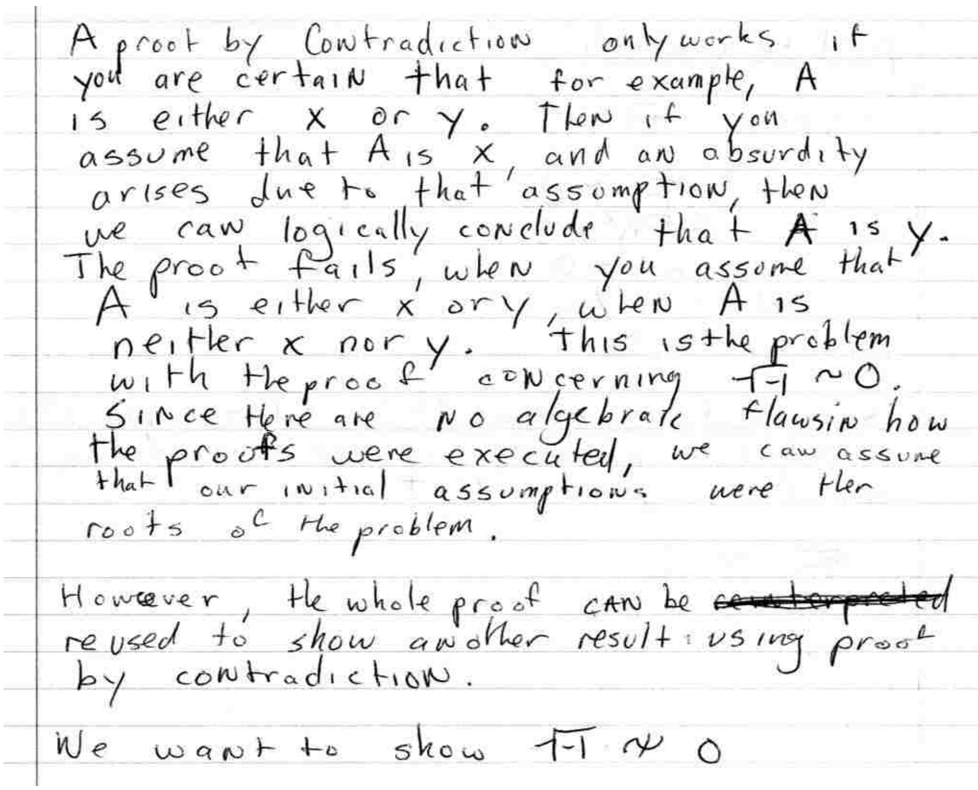
Suppose $\sqrt{-1} > 0$*, then* $\sqrt{-1} \times \sqrt{-1} > 0$*. This implies* $-1 > 0$*, which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

<u>*Prove*</u> $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$*, then* $-\sqrt{-1} \geq 0$*. This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$*, so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This was the first interview in which John was not asked to complete a proof. This problem was given to him on the take home midterm he completed while taking MATH 305. On the midterm, he explained that a proof by contradiction only works when there are only two possibilities for the object under consideration (see Figure 11). In this case, the proof is using the assumption that *i* is either less than or equal to 0 or *i* is greater than 0. The problem, then, is that there is a third option: that *i* and 0 are not comparable in the first place.


A proof by Contradiction only works if you are certain that for example, A is either x or y. Then if you assume that A is x, and an absurdity arises due to that assumption, then we can logically conclude that A is y. The proof fails, when you assume that A is either x or y, when A is neither x nor y. This is the problem with the proof concerning $\sqrt{-1} \sim 0$. Since there are no algebraic flaws in how the proofs were executed, we can assume that our initial assumptions were the roots of the problem.

However, the whole proof can be ~~counterproved~~ reused to show another result using proof by contradiction.

We want to show $\sqrt{-1} \nsim 0$

Figure 11: John's previous work on Question 4

In the interview, he remembered what he did on the midterm:

J: Well, I think what I did, if I remember, was this was part of the bigger proof that the square root of -1 is not related to 0 in any of these ways.

N: Ok

J: You know what I mean? So this is a, that itself would be a proof by contradiction.

N: The combination of these 2 sub-proofs?

J: Yeah, using these then since both of these, since this sort of contradicts each other, then you'd kind of have, then it means that it's neither greater than nor less than nor equal to…

N: Sure, ok

J: I mean, at least that's how I saw it, it's establishing none of these relations do hold…I think that, there's an algebra of complex numbers and it follows the same guidelines. So everything here seems like seems like, all this seems to work, it's just that these initial statements are untrue, or, like we were saying, you can't make that comparison.

John made a point to mention that he had not run across this sort of problem too often in mathematics:

J: And this might have been, as far as my math classes go, the first time that we did a math proof where the logic was sound but the conclusion was untrue…I'm not sure about that, but I mean, it just doesn't seem like you do very much of it…you could go through your entire math career as a student and never see something like this and come out with just as much proof skills, essentially. I mean, this is like one little issue, and these people who do only true proofs and don't see something like this, like would still have a fine understanding that if you do assume something that's untrue then the conclusion should be untrue as well, generally.

While he does not see confronting proofs like this as necessary to develop proof skills, John explicitly reveals his realization that the validity of a conclusion depends on the validity of the starting assumptions used to reach that conclusion.

It seems clear that John is exhibiting an analytic proof scheme. He points out that the soundness of a proof's relies on the starting assumptions multiple times during the interview. This sort of formality is the main property required for a proof scheme to be deemed analytic. Because he does not create a proof, it is difficult to call this evidence for a transformational scheme (he performs no operations on any mathematical objects) or axiomatic (no reference to axioms, undefined terms or previous results). As such, calling his proof analytic is as detailed as is possible in this case.

Question 5

The next interview was the last of the semester, which was used as a debriefing session with John. As such, he did not produce a proof. He did, however, explicitly state some things that reinforced the observations made in earlier interviews. In particular, he mentioned that he makes an effort to understand problems before trying to create a proof. This makes sense, given that two of the four proofs John produced were semantic in nature. When asked about the role of examples in proof, he said, "Yeah, I like to do that, but that just depends on the problem, but the number based problems, the examples always help. … I guess the examples in all would work because even in geometry, it's been helpful."

He did not explicitly say that the examples were used to gain an intuitive understanding into the problem. Instead, he said the examples were useful in seeing relations that could be used to construct proofs:

> J: I feel like, when I want to do a proof, I might have a sketch on a paper, just to have an idea, you know, have everything labeled, but I like to also pull up Geogebra and do that same sketch, that way I can just see the proper relations…so that sort of stuff is sort of the same as having this number theory problem and going through and checking all these little answers and seeing that, maybe the possible relations. So, yeah, I think that examples are helpful and probably not necessary.

When referring to Question 1, John said, "I think when we were doing the problem, you can look at it as a function and as soon as you graph it, that's when you start to understand the rest of the problem and doing it that way."

So, while they came from a line of questioning involving what was necessary and helpful for constructing proofs, with an emphasis on examples, these quotes reveal the mentality that led John to constructing semantic proofs when not using induction.

John also provides evidence of an axiomatic proof scheme, again some that showed up in earlier interviews:

> J: (I)n that geometry book, there's all these theorems where you have to say 'According to this theorem, this is how it works.' But, like, since we didn't do, we haven't been saying 'According to Euclid 3-12,' or something like that, I don't know all my theorems. And I don't know where to reference them from and stuff like that.

Here, John is referring to a difficulty he has (not citing previous results as he feels he probably should), but he does point out the fact that he realizes the reliance of what he does (especially in geometry) on previous results. This is very much indicative of an axiomatic proof scheme.

Question 6

Prove that $\forall n \in \mathbb{N},\ n \equiv n^3 (\text{mod } 6)$.

This problem was the first of the second semester of the study. When John began the problem, he thought it was referring to multiplying number by three and then converting them to base 6. After I cleared up this misconception, John tried a few examples but then moved quickly into applying the division algorithm (which he had just covered in Number Theory) to the problem.

He started this by writing n as $6(a) + b$ and cubing that expression. He then did some rearrangement which left him with $n^3 = 6[3a(12a^2 + 6ab + b^2)] + b^3$. He looked at this for a bit and then remembered something he had written above: "It took me a moment to think about that, but it went, having that little inequality ($b < 6$) there helped." He remembered that the division algorithm allows him to restrict the remainders to natural numbers less than 6. From there, he made sure he had checked all such numbers and verified that the property holds, noting "I have this n cubed in the form 6 times some c plus b cubed and then since b is less than 6, I just ran through all the cases." Although he did not check all cases (he forgot 0), he understood what it meant to have part of his new expression for n^3 multiplied by 6: "So since this is an integer times 6, when you take n^3 mod 6, essentially this is kind of crossing that out, and then this b^3 mod 6 which is b."

Here John produces a syntactic proof. It is not semantic because, although he looked at a few examples, he did not turn an intuitive understanding into a proof. In fact, when he began cubing his expression for n, he was not sure how it would turn out:

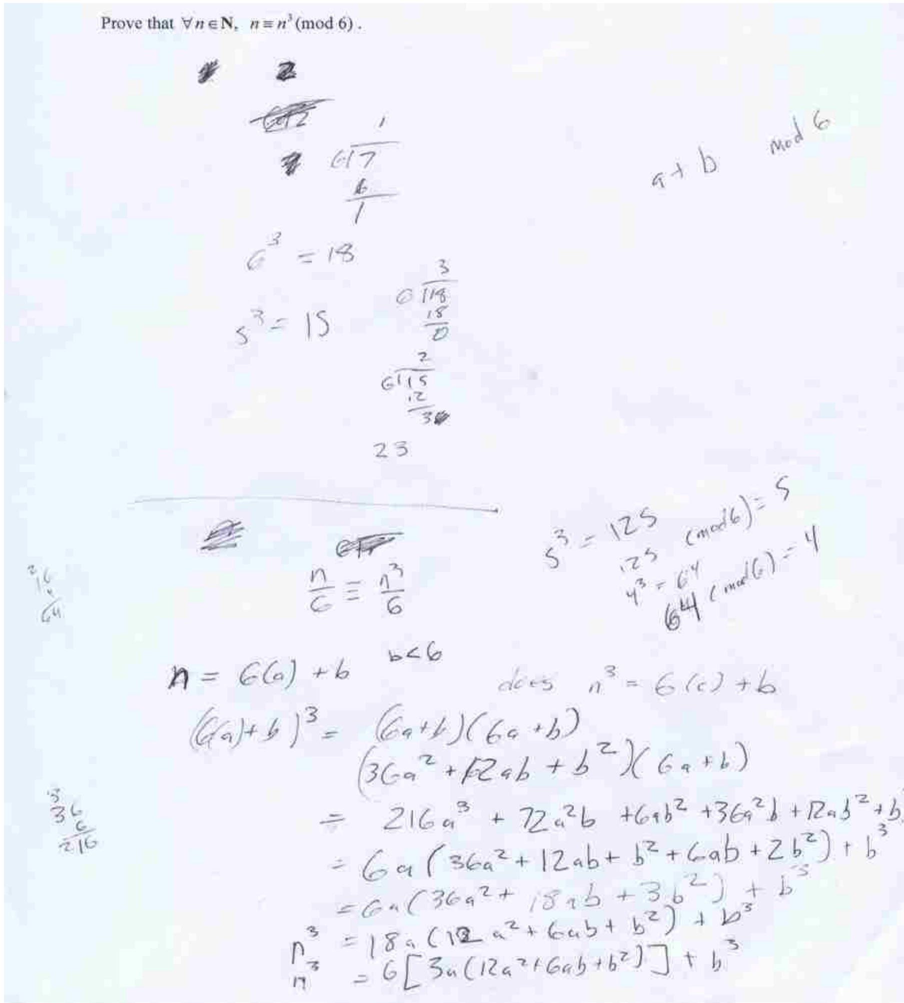


Figure 12: John's work on Queston 6

J: And once I got, I sort of, you know, I didn’t expect it to come down to this until a little while later… Once I got it all out, because I had no idea what this would end up looking like at first. I’m just not very good with the cube.

So, here John produced a syntactic proof where the objects he was working with had meaning for him.

Based on the proof, John's proof scheme is a transformational one. His proof is based on operating on objects and while he says he did not know how his operations would turn out, he does realize at the end that he had constructed a valid proof.

Question 7

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

It should be noted that I told John to only consider finite n values. He did not finish this problem in the first interview in which he worked on it, but did finish it between interviews. A second interview was devoted to the problem in which John and I discussed his solution.

John started the problem, as most students in the study, by looking at the general case in which n was not fixed. Often, students begin a problem by looking at specific examples for insight, but John did not:

> J: I just thought that maybe I could go in and see like, I didn't think of the lattice patterns right away, but doing…can I count and then get an idea of what's going on. Instead of seeing if n is 2, it would be this, if it were 3 it would be this…

Instead of using specific examples to gain an understanding, John tried to use the general case to do the same thing. He considered an n element set containing the elements 1, 2, …, n.

While he did not look at any specific n value, he did begin by looking at particular types of subsets of a set A of size n. His first work involved looking at the numbers of subsets of particular sizes. First, he found that there would be n subsets with one element, then $n - 1$ subsets containing 1 and one other element, $n - 2$ subsets with the element 2 and one other element and so on. "Yeah and then there'd be 1 set with $n - 1$ and n. And then there'd be n – n sets, so 0 sets that have n in it, following that pattern."

From there, he started looking subsets of size 3. Here, things started becoming complicated so he began limiting the subsets to those that were of size three and contained the element 1. This led to a similar pattern, only now beginning at $n - 2$ subsets containing the elements 1, 2 and something else. The pattern continued as before with $n - 3$ subsets containing 1, 3 and an additional element and so on (see the figure below).

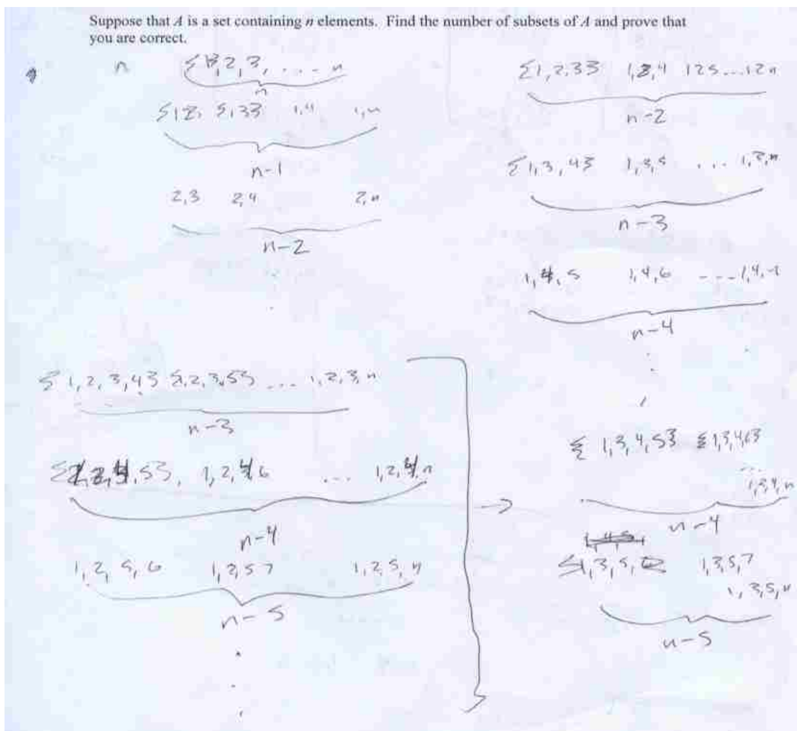


Figure 13: John's work on Question 7 (1 of 5)

John continued with this sort of analysis but eventually tried to collect his thoughts. In Figure 14, at the top, he began to try to organize the things he was seeing, but it quickly became too complicated.

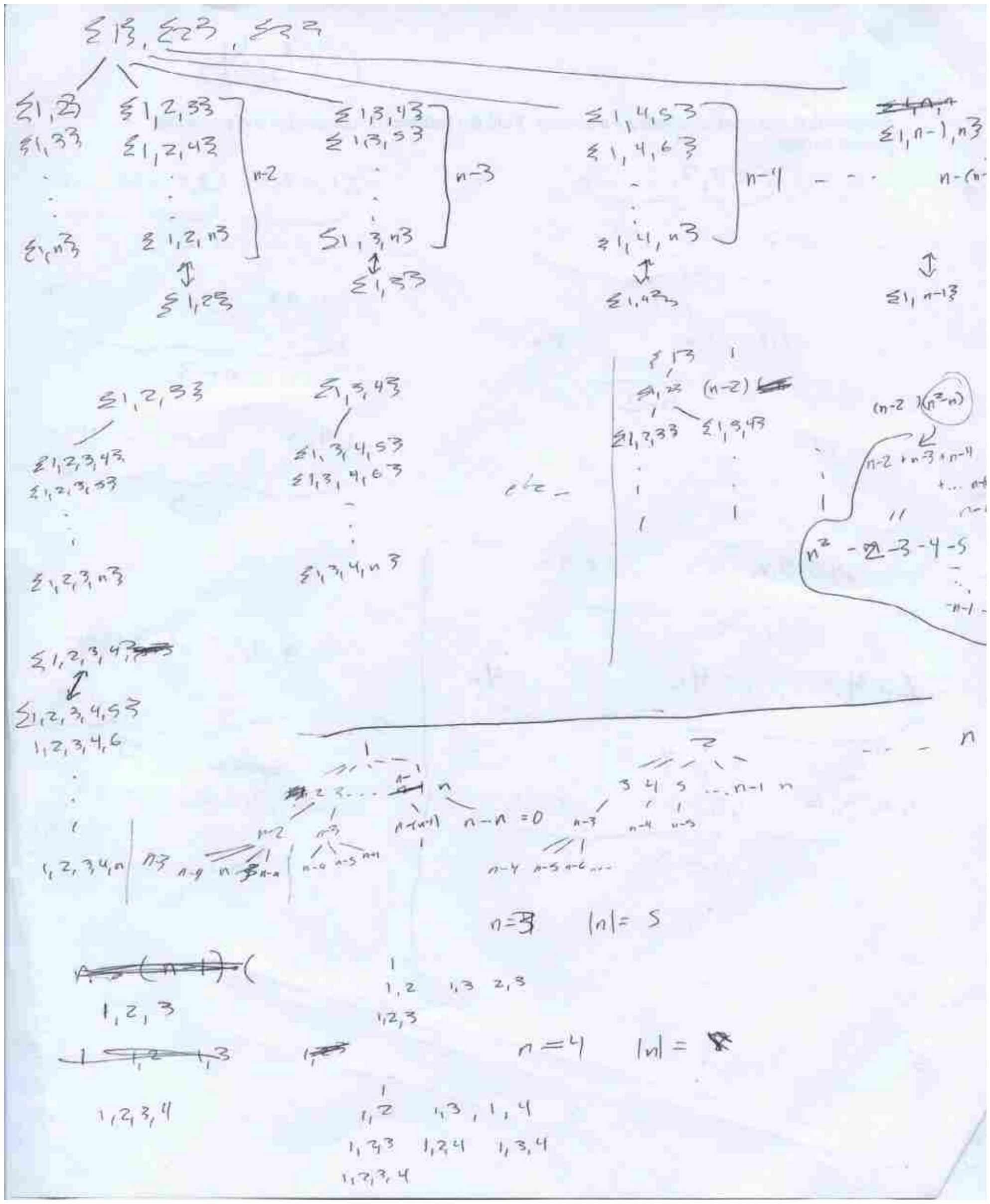


Figure 14: John's work on Question 7 (2 of 5)

J: Then I came down here and I was thinking that I was going to write not, not write out the set but right out the cardinalities…

N: Sure

J: …and see how that was working out.

N: Yeah, because it doesn’t take long for this to get really complicated.

J: Right

N: You know, a lot to really build up.

J: Yeah, it was pretty…oh, man… I'd need a bigger piece of paper, I'd imagine. So then, just doing the cardinalities with the *n*'s, and it was all kind of a dead end. I mean it helped me understand, like, I felt like I had a fairly grainy but not bad picture in my mind of how it all works… But at the same time, it wasn't until, I guess, right at the end after going through all of this, for whatever reason, thinking that 'Well, I should just start with the *n* and see this pattern and go through.' … I mean, here, I was just thinking I should go if *n* is 3 what's the cardinality…And if *n* is 4, what's the cardinality. Do some pattern recognition stuff.

John's initial attempts at finding the total number of subsets for a set *A* of sizes 3 and 4 can be seen at the bottom of Figure 14. During the reflection, John pointed out on his own that the numbers of subsets he had come up with were wrong because he was missing some subsets in his count. He was also missing the empty set in his count. This was near the end of the interview and pointed out the empty set to him. I wanted to make sure he realized this because I had the feeling that in his work between interviews, he would take the approach of looking at the total number of cardinalties: "I think if I would have done…if I would, say, just find the total cardinality, I imagine I could see what the pattern would be by just looking at the numbers for the first few."

When John went back to look at the problem between interviews, however, he did not go looking for a pattern right away in the total cardinalities. Before he got that far, wrote out the subsets of a 3 element set. In doing so, he realized that the number of subsets could be found using a more succinct sum than he had been working on in the last interview:

J: And…realizing that this was…realizing this pattern is the sum of the different chooses, so…you have *n* elements, yeah, if you have 3 elements, you want to find the ways you can take 1 out, all the ways you can take 2 out, all the ways you can take 3 out, and then add it up. So, I ended up down here with, you know, given this *n*, given the size of the *n* subsets would be *n*C1 all the way up to *n*C*n*.

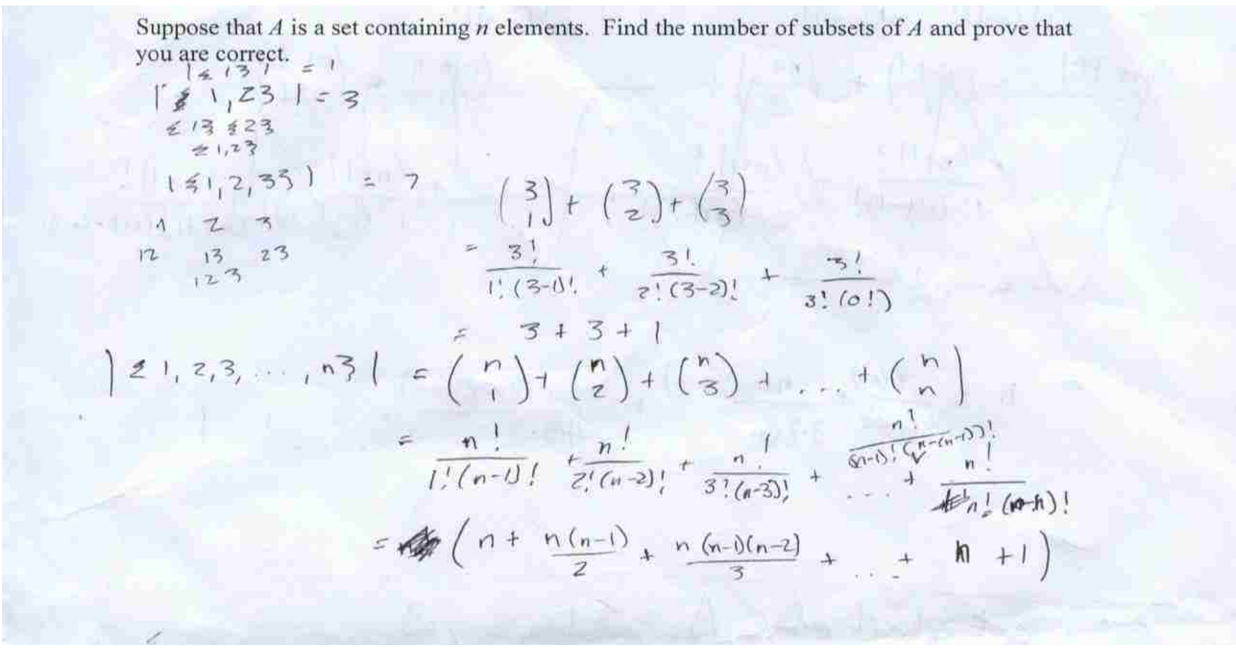


Figure 15: John's work on Question 7 (3 of 5)

This gave John something he felt like he could work with. Wanting to prove that the sum he had for $n = 3$, he tried to apply induction: "Just thinking if I could, because I was thinking that I wanted to do induction, because of the *n*." John saw the presence of the variable *n*, which is typically reserved for integers, and associated that with induction proofs.

You can see from the work in Figure 16 that John went to work on simplifying the summation in some way, because that is usually how an induction argument goes:

J: And wondering if I wrote it out in some form, if I could, you know, divide something out and get…I was thinking there would maybe be like a geometric sum or something like that…So, played with that a while and then started thinking, like, writing it out and then just, I think I just kept doing more stuff with these…this is all kind of haphazard, but just, I just kept starting over and doing all this different stuff with the fact, with the chooses and, you know, started looking at different stuff until I started to see this pattern. If you have 1, so if you double it and add 1, double and add 1, double it and add 1…

N: Right

J: This here (Figure 16) is where I found that, that the size of your set is going to be 2 times the size of the set below it plus 1.

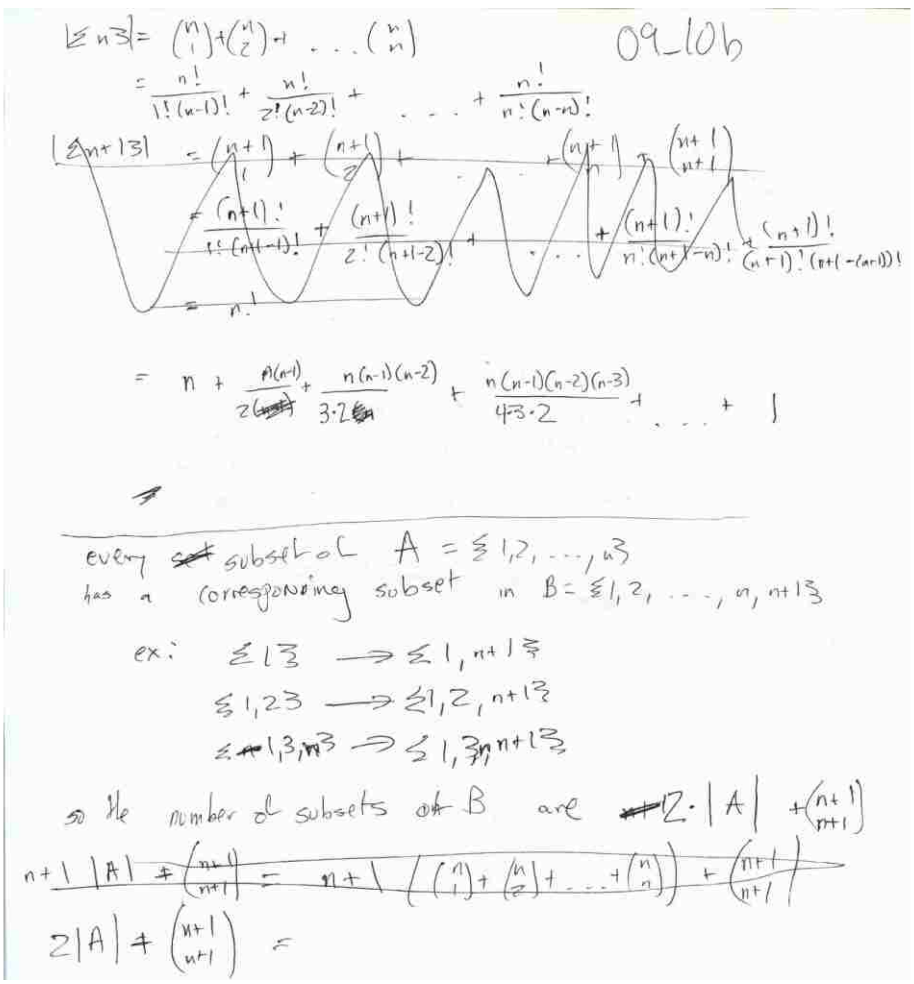


Figure 16: John's work on Question 7 (4 of 5)

John's work with the induction proof continued until he realized that the number subsets of an $n + 1$ element set was somehow related to twice the number of subsets of as n element set. This was an important realization and eventually served as the basis for a complete proof. The thinking was that if the size of a set was increased by a single element (if $A = \{1,2,\ldots,n\}$ becomes $B = \{1,2\ldots, n, n + 1\}$), then all the sets that were subsets of A would still be subsets of B. Also, each of the previous subsets of A would have a corresponding subset of B with the element $n + 1$ added to it. For example, $\{1,2\}$corresponds to $\{1, 2, n + 1\}$.

Eventually in Figure 17, John begins approaching induction with his idea explicitly in mind. Before he could finish, though, there was one more obstacle he needed to address:

> J: So playing with it like that, thinking you know like I was saying, if I could take that out, rewrite those all in, you know in their expanded form…And something, do some sort of inductive proof. But then I saw this pattern here, the 2s, the doubling it and adding 1 and thought about that for a while and what basically, you know, what happened eventually…
>
> N: So this one (09_10e) is sort of only explicitly stating that pattern, that double it and add 1.
>
> J: Yeah. And then I realized, thinking about what you told me about…because what I was thinking is that if I could figure out this pattern, I'll just add the empty set case later. Right?
>
> N: Oh, sure
>
> J: And that was like the major mistake, I guess.
>
> N: Up to this point, not seeing the empty set?

J: Yeah, well, it was nice to find this pattern of the, you know, doubling the set and adding 1, but then it but, you know, once I finally figured out that it was 2, 4, 8, 16, 32, right?

N: Right

J: And so when I saw that 2^n… And then after that I was able to write it up.

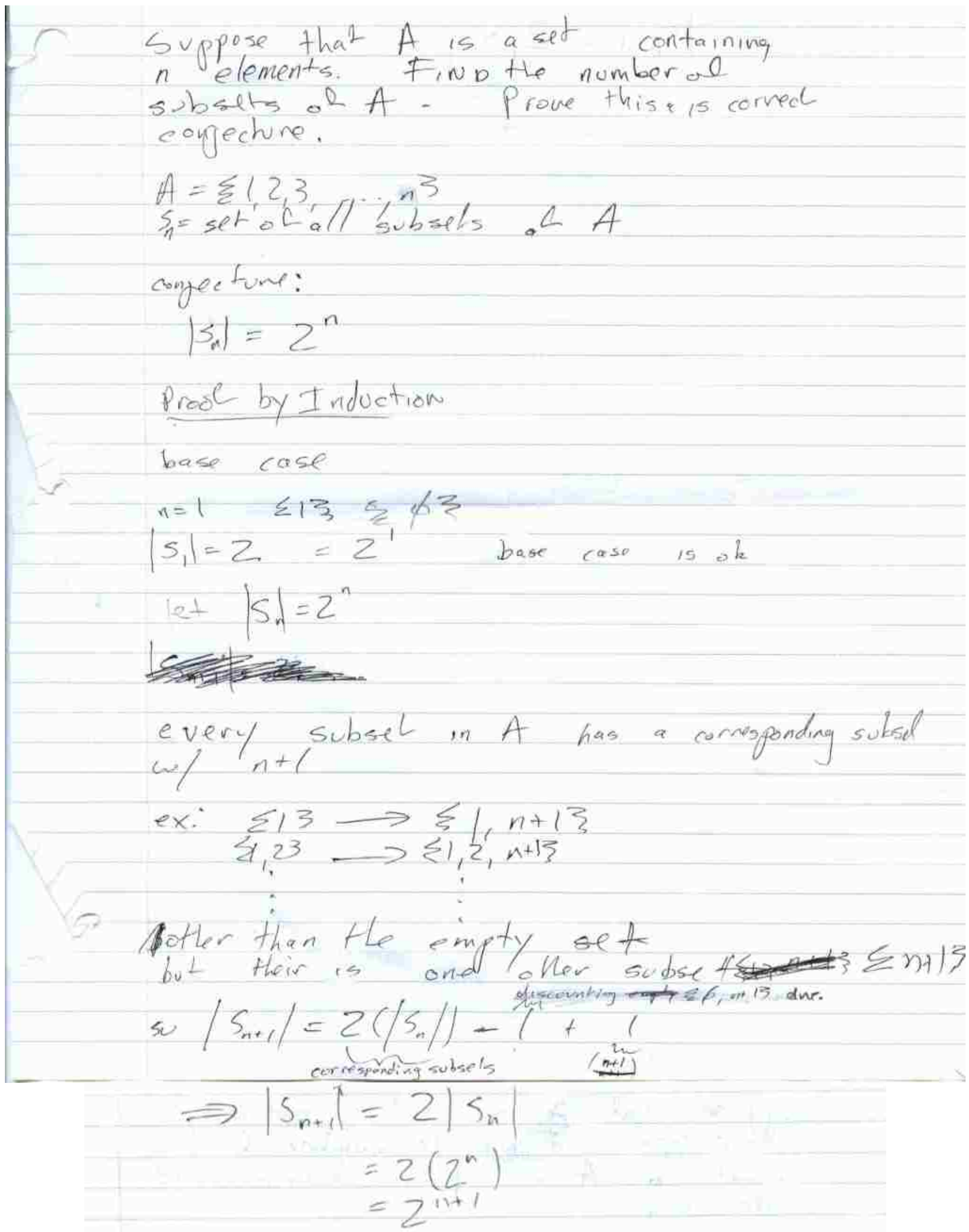


Figure 17: John's work on Question 7 (5 of 5)

John began considering the empty set a subset in its own right, and not something to simply be tacked on after the fact, which I believe was a remnant from working during the prior interview without including the empty set. Looking towards the bottom of Figure 17, you can see a couple things of interest. One is the idea that he had in Figure 16 showing up again. Another is that he seems to still have some misconceptions about the empty set. In the final formula, he subtracts 1, saying that the set $\{ n + 1, \varnothing \}$ does not exist, but then adds 1 to account for the set $\{n + 1\}$. During the discussion, he came to see $\{n + 1\}$ as the empty set with the element $n + 1$ added in.

Once we discussed the process by which John came to his solution, the main item of interest I saw was the fact that he proceeded beyond the point at which others stopped. Specifically, when he had the summation formula involving the *n*C*k*'s.

> N: Did you ever give any thought to just sort of giving a proof by construction and saying "This is where the formula comes from, this is why it works, blah blah blah" and just leaving it at that?
>
> J: No. I guess I didn't. I mean, it made sense to me but I guess I wanted it to be more like a "proof" proof, I don't know… But that might be, like, a contextual thing, maybe, you know.
>
> N: So like…?
>
> J: Since I knew I was going to come in and show you this, I mean it was just because I was stuck on the induction thing, but it just didn't seem like I should say, you know, write a little paragraph describing what you have to do to create these subsets and how that sum of the *n*C*k*'s or whatever, you know, how that relates.

John went on to later say:

> J: Right, and it felt like in this study, it's more likely that I would be doing an inductive proof, a direct proof, or a proof by contradiction, but not necessarily, you know, just a convincing argument, like just talking about it… Yeah, if I wanted to explain it to somebody, you know, get someone to believe it, I would show them that (the sum formula). But if I wanted to convince you to believe, I would show you this (the formal induction).

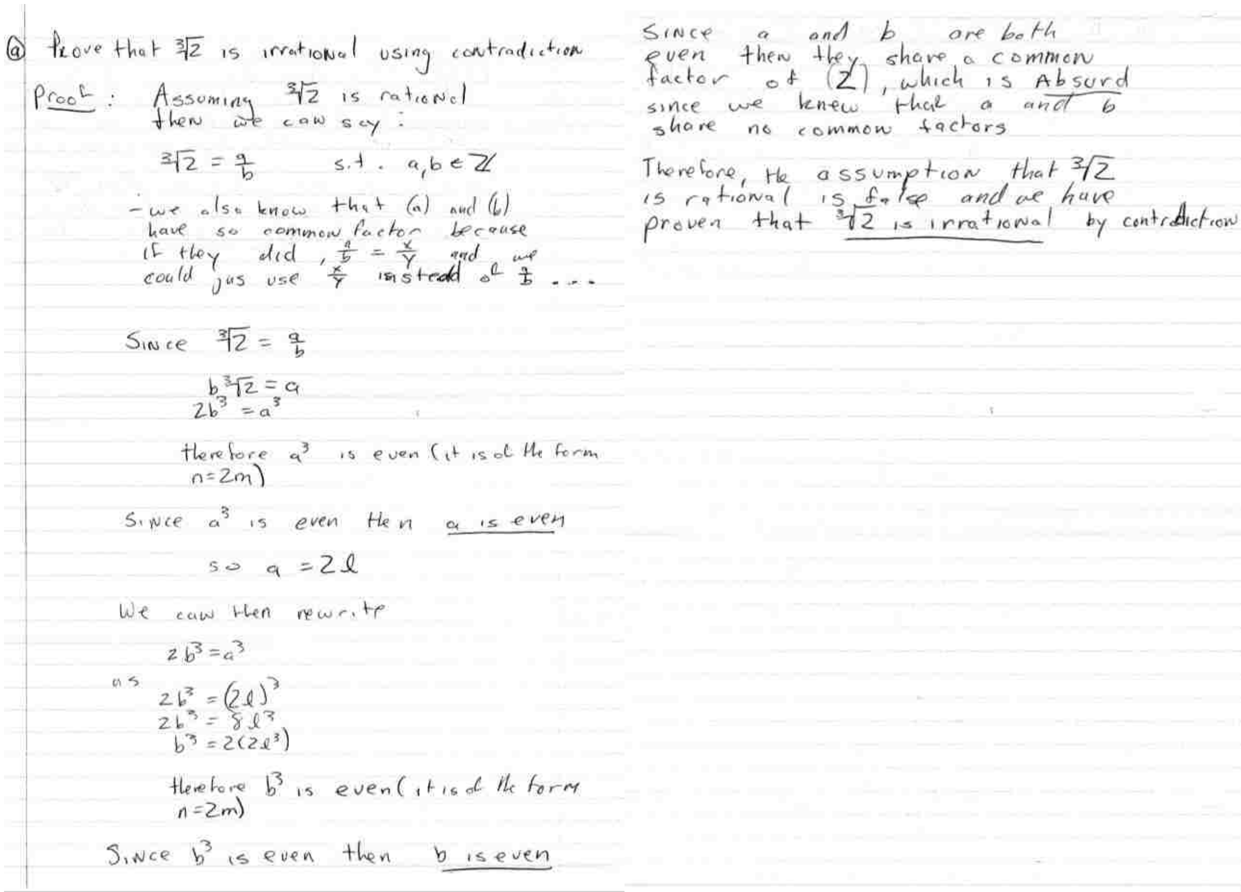
This mirrors something mentioned earlier, the fixation of using induction because of the use of the variable *n*. In both instances, an idea unrelated to the problem had an affect on John's actions while completing the proof.

In this proof attempt, John completes a semantic proof. While it is an induction argument, he begins by gaining an understanding of the mathematics involved and then turning that understanding into a formal proof. Along the way, he also displays a transformational analytic proof scheme. This is most evident in his consideration of the subsets of a set when an additional element is added to it.

<u>Question 8</u>

Prove that the cube root of 2 is irrational using a proof by contradiction.

Like Question 4, this was a problem that John had already seen on the midterm he took for MATH 305. There, John appeared to have no problem with the proof. In class, the students had seen the proof by contradiction that proves that $\sqrt{2}$ is irrational and that proof is easily adjusted for $\sqrt[3]{2}$, which is what John did on the midterm. The work he did on the midterm is in the following figure.



(a) Prove that $\sqrt[3]{2}$ is irrational using contradiction

Proof: Assuming $\sqrt[3]{2}$ is rational then we can say:

$\sqrt[3]{2} = \frac{a}{b}$ s.t. $a, b \in \mathbb{Z}$

- we also know that (a) and (b) have no common factor because if they did, $\frac{a}{b} = \frac{x}{y}$ and we could jus use $\frac{x}{y}$ instead of $\frac{a}{b}$ ...

Since $\sqrt[3]{2} = \frac{a}{b}$

$b\sqrt[3]{2} = a$

$2b^3 = a^3$

therefore a^3 is even (it is of the form $n=2m$)

Since a^3 is even then a is even

so $a = 2l$

We can then rewrite

$2b^3 = a^3$

as

$2b^3 = (2l)^3$

$2b^3 = 8l^3$

$b^3 = 2(2l^3)$

therefore b^3 is even (it is of the form $n=2m$)

Since b^3 is even then b is even

Since a and b are both even then they share a common factor of (2), which is Absurd since we knew that a and b share no common factors

Therefore, the assumption that $\sqrt[3]{2}$ is rational is false and we have proven that $\sqrt[3]{2}$ is irrational by contradiction

Figure 18: John's previous work on Question 8

During the interview, I reminded John that when he did the problem before he likely based his proof on the proof for $\sqrt{2}$. Because of that, I suggested that he try to reproduce that argument if he thought he might have trouble with this one. Initially, he worked on problem from the midterm. However, he got a bit stuck and then flipped the page over and started the $\sqrt{2}$ proof. In doing so, he quickly realized that he remembered the problem at hand and proceeded to complete it without much trouble.

What he needed to remember was to use the lemma that states that if 2 divides a number cubed, then 2 divides that number. This is an important aspect of the proof, as it allowed him to eventually contradict one of his starting assumptions: that gcd(p,q) = 1 where $\sqrt[3]{2} = \frac{p}{q}$.

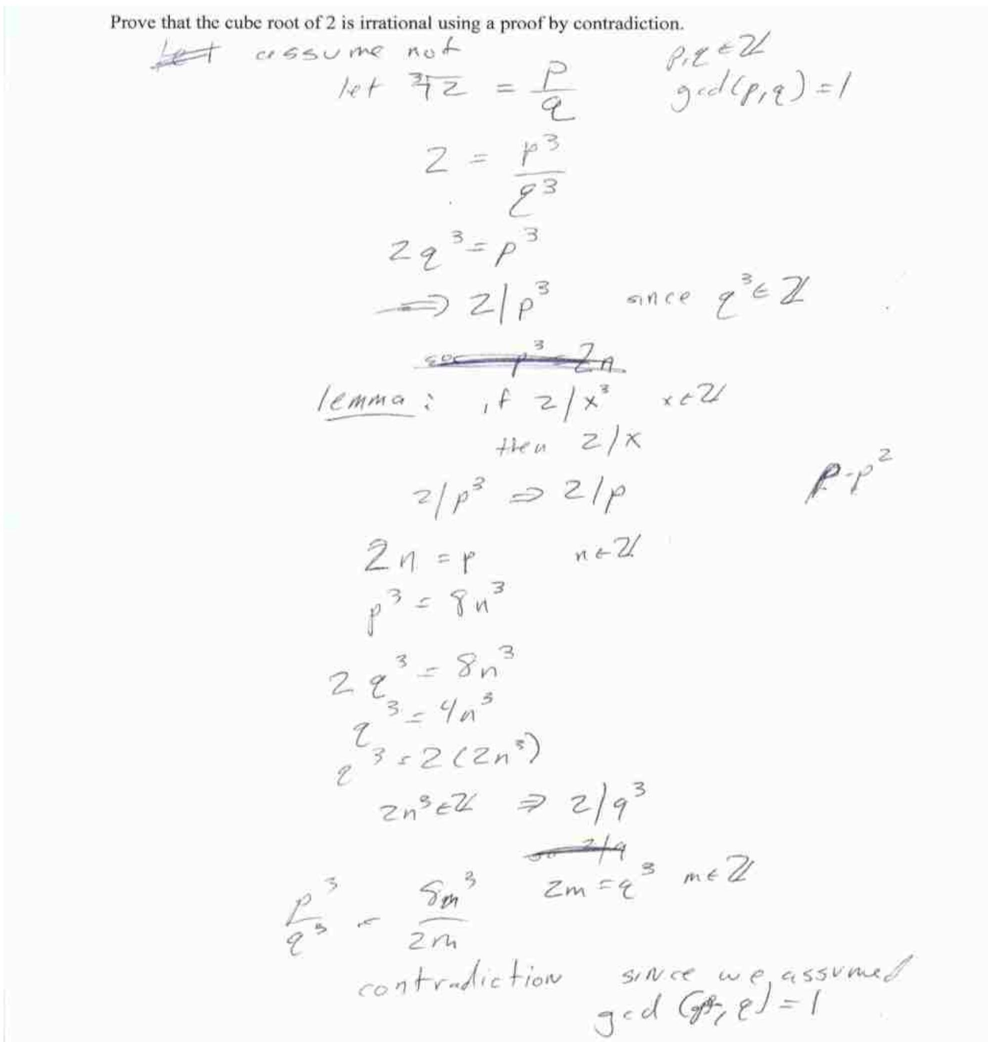


Figure 19: John's work on Question 8

This proof is a backtrack of sorts in that it is an algorithmic procedural proof. This is due to the nature of the problem, however, because he already knew that he had done the problem and that completing the proof was simply a matter of remembering the steps. However, since there were specific steps that he knew he had to complete, and he set about completing them, this qualifies as an algorithm type proof.

The fact that the proof provided is not his own and he reproduced a very similar proof would seem to indicate that this John is displaying an authoritarian external conviction proof scheme. This is not the case, however, because he made it clear in the interview that he understood the steps involved in the problem. For instance, on the midterm he took the time to prove the lemma he used but here he did not: “I don’t feel like I’d need to prove it. I think I understand what’s going on.” So, even though he is performing steps he had seen before, he is doing so because of his understanding of how the steps will turn out. This is evidence for the transformational proof scheme.

Question 9

Like Question 4, this problem involved having John read through a proof and evaluate it. The proof was a version of Cantor’s diagonalization argument and can be seen in the Chapter 3. Because John did not construct a proof for this interview, there is not a proof type associated with this problem. However, the interview did provide some information when considering his proof scheme.

First off, given that I told John it was an historical argument, he was inclined to believe it. “A lot of it probably has to do with that, going into it and knowing that this is an historical argument and feeling with it comfortable enough. You know, I took it and read it, thought about it pretty quickly.” So, he did not put complete faith in the problem because it was historically important, but he did not go looking for holes in the proof either.

This same sort of thinking showed up when discussing what might be considered a hidden lemma within the problem: the statement that every subset of a countable set is countable.

> J: I thought this (underlines the subsets of countable sets statement) was cool… To be able to say that, so, you know, if you assume that assume that the whole thing is countable then you can just talk about the interval and set it up where you're talking about every decimal expansion in between 0 and 1. …
>
> N: So you're sort of ok with this statement, then, that every subset of a countable set is countable, then you can narrow our focus down to the unit interval?
>
> J: Yeah. Because it's just an assumption that you're making. Right? Like, that's just part of the contradiction argument. … Oh, yeah, I'm fine with that, I mean that makes sense.

Here, John gives the lemma thought and considers it believable enough that it did not require further proof. He did, however, realize that the proof depended on that statement but was not concerned with it because it made sense to him.

The fact that John had some faith in the proof because it was historical may be taken as evidence that he had an external conviction scheme while reading this. This is not the case, however, as this only served to help direct his attention while reading the proof: "I went into it not looking for a mistake, just looking to understand it…" Instead, John showed a transformational proof scheme. Here, he uses his knowledge of what it means for a set to be countable and is able to see the contradiction that arises when you assume that the set of real numbers is countable. He says, "if you just say that this function works, show that it doesn't work, that's the contradiction, that's enough…to say that it's uncountable." In this proof, John sees the act of creating the new number *B* and that this creates a contradiction as the crux of the proof. This is indicative of a transformational proof scheme because it shows that he is comfortable with using anticipatory actions (here, the creation of a new number) and using the results of that operation to achieve a proof.

Question 10

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair. Example, 6/9 < 5/7 < 7/9.

John began working on this problem by putting it into general terms, working with $\frac{a}{b}$ and $\frac{a+1}{b}$ as his rational numbers and looking for something that goes in between. Not long into the process, he thought of the counter-example of $\frac{0}{2}$ and $\frac{1}{2}$. These do not work because the only possible denominators less than 2 where 0 and 1 (I had told him to disregard rational numbers with a negative denominator). This did not stop his work on the problem, though: "I don't know, I mean, it felt like this being like the low end of it, I don't know if this means that this is the only counter-example or not."

John tried a few more examples, but quickly abandoned them, returning to the general case he had set up. When I asked why he did that, he said:

> J: I didn't think about too many specific examples, I thought I could just go through it. I didn't think that it would help too much because it would be…a not very, it's sounded pretty difficult to be honest. … Like, having to come up with, just like doing that test of

all these little fractions, because I mean, for a couple small numbers it wouldn't be bad…but even past…6 or 7 it would start to be too difficult. A little much.

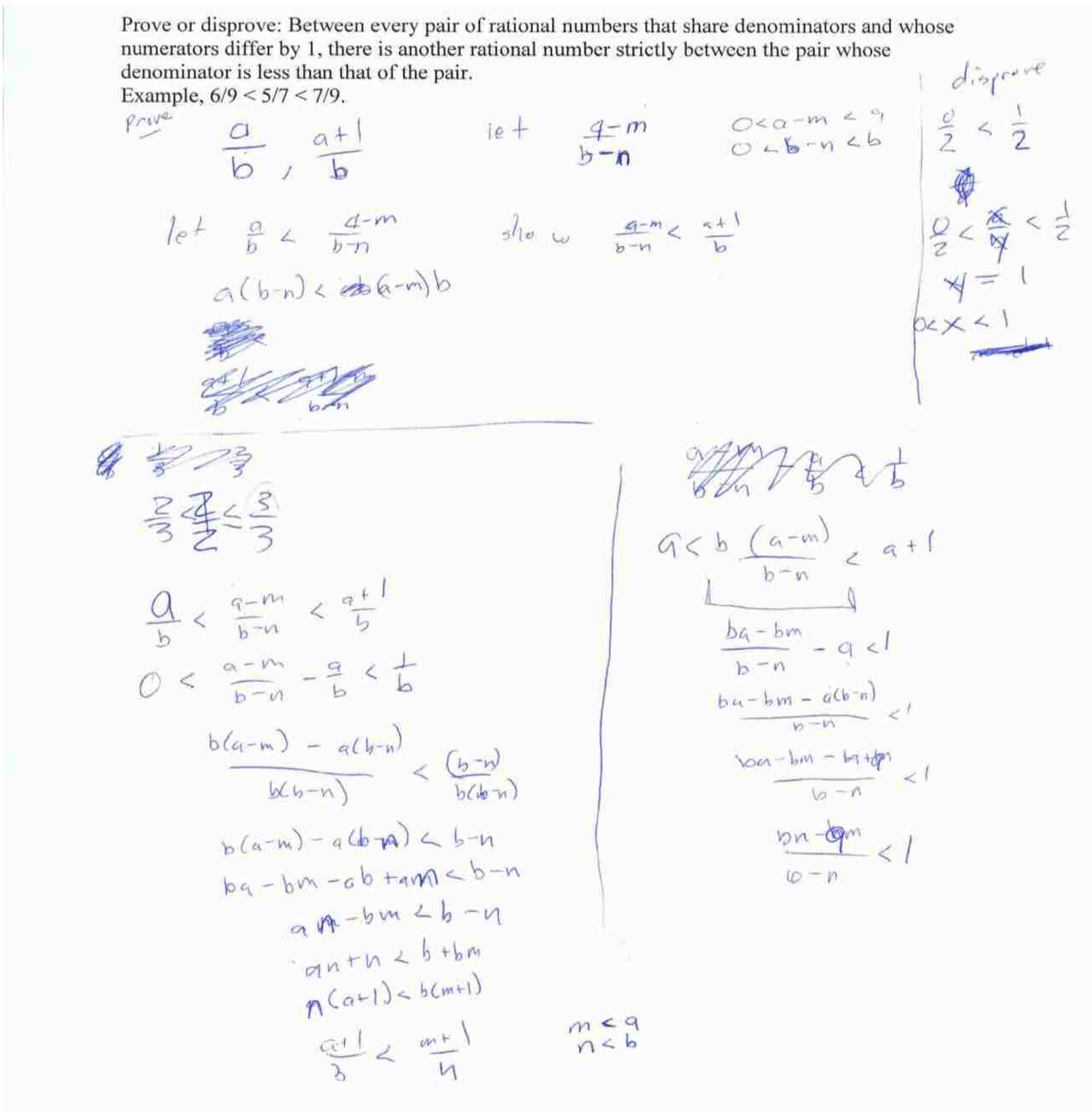


Figure 20: John's work on Question 10 (1 of 2)

Here, John is still trying to come up with a way to gain insight into the problem. However, he is not doing so by looking at examples or some other representation of the problem. Instead, he looks at the equations he set up for insight:

J: Yeah, I just wanted to see if there was…like if there was, like if I could somehow understand it in a different, like think about it in a different way and just try to work through it like…you know, see what I could do with it.

Despite these set backs, John left the interview thinking he could make it work with the proper restrictions: “Well, I mean, it seems like there should be enough rational numbers…that it would work out.” I asked that he continue working on the problem and he said that he would.

John’s between interview work looked much like his work in the first interview. He started by looking at a general case and then moved onto examining counter-examples. From there, he did not know what to do and we continued from there during the next interview.

One thing that did come out of his between sessions work was a method for deciding if the property would hold for a pair of fractions. What he did was, for example, start with an inequality such as this: $\frac{4}{4} < \frac{x}{3} \leq \frac{5}{4}$. At the time, John was considering adjusting the upper restriction to allow for the fractions to be equal. This was due to the $\frac{1}{2}, \frac{2}{2}$ counter-example. Starting with the inequality above, John simplified to see that for the property to hold there needed to be an integer strictly between 3 ad 3.9. This let him know that he had found a counter-example and he did the same thing for the pair $\frac{5}{5}$ and $\frac{6}{5}$ (see Figure 4.09.21).

Although John did not complete a proof here, it is clear to me that he was trying to work towards a semantic proof. Although he did not go to various alternative representations of the problem to gain understanding, he did make an effort to get that understanding before going to the proof. He mentions that the only reason he went straight to a general case for understanding was because he thought the examples would get too tedious. Also, once he saw the pattern that

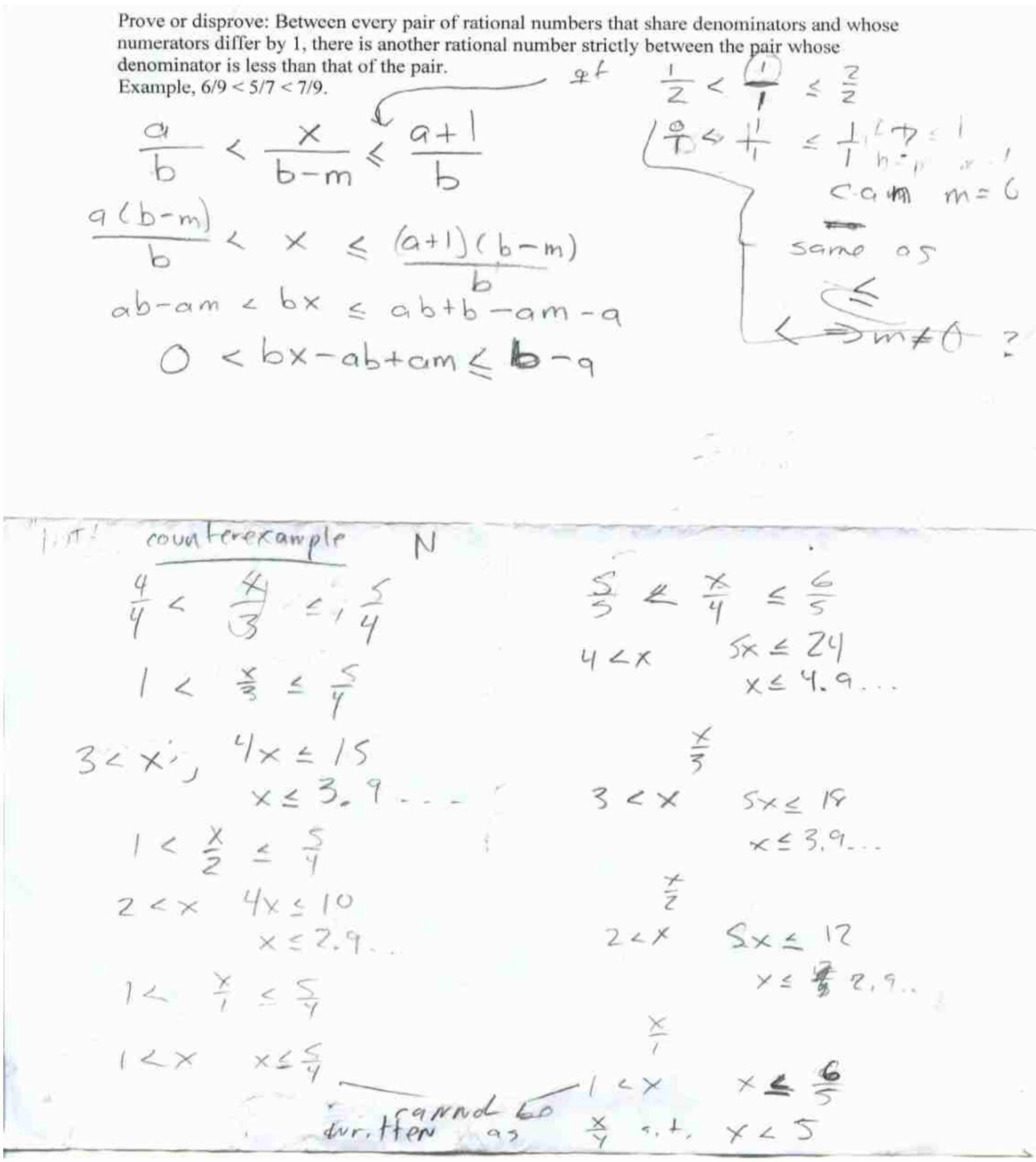


Figure 21: John's work on Question 10 (2 of 2)

had to be proved, his first instinct was to go to a number line or some other picture before going to the algebra for the proof itself.

Over the course of working on this problem, John displayed a transformational proof scheme. The majority of the work that John involved operating on the inequalities he wrote, both in general and when examining specific examples. Considering what happens when operating on mathematical objects is the defining characteristic of a transformational proof scheme.

Question 11

The second interview in which John worked on Question 10 was the last interview of the semester, so it also served as a debriefing session like last interview of the first semester of the study (Question 5). As such, there was no proof attempt by John and so there will be no proof-type classification for this question.

John did reinforce, though, some of the things described above. For example, in referring to the last problem he worked on, John said:

> J: Yeah, and looking back now, I mean, maybe if I had a little bit, just had done a few examples maybe I would have been able to see those, like the differences in there and why those restrictions need to be in place. …But I was hoping to find that stuff just from doing the algebra and I should have done some examples.

Here, John is referring to his failed attempt at gaining an intuitive understanding into the problem via algebra. This line of thought was also evident later in the interview when I asked him what helps when trying to complete a proof: “Diagrams, like pictures, different, depending on the different type of problem. Examples, stuff like that, knowing…like picking a method that I’d want to use, like deciding if I’d want to just do a straight deduction…”

John also alludes to the two types of proof schemes that showed up most often over the course of the study. When I asked him what he thought led to any improvement in proof he saw over the course of the study, John said: “…just all the, you know, whatever info, like new little pieces of information I get it always helps because it's just another way to think about it. So yeah, I've got, I've gotten, like, an improved tool kit." I think that John is referring to different methods of handling varied situations that may come up, in other words different methods of operating on mathematical objects: evidence of a transformational proof scheme.

John's proof scheme is not limited to this, however. He sees these new methods as taking him only so far.

> J: And it's just stuff like that where you kind of…I don't know, I feel like it's good for that class, but as far as, I mean all the information, it certainly does build on itself and it kind of, it fits all together, but sometimes certain things like that don't always add new…methods that I can take to every aspect of the math class I'm taking.

When he refers to information building on itself, John presents evidence of an axiomatic proof scheme. This is even more evident in the following quote. When asked what is necessary to complete a proof, John said, "A sound, a valid logical argument with true assumptions, I guess, or assumptions that are as true as to be, or are accepted or something like that." Here, John talks about the fact that in order to have a correct proof, it has to be based on some "true" and "accepted" assumptions. That John does not stop at 'true" likely alludes to the fact that some things can not be verified; only agreed upon. This understanding is a sure sign of an axiomatic proof scheme.

John's progression

Below is a chart of each question and the type of proof John used and the proof scheme displayed:

Question	Type of proof	Proof scheme
1	Semantic	Transformational
2a	Semantic	Transformational, Axiomatic
2b	Process	Transformational
3	Process	Transformational, Authoritarian
4	N\A	Analytic
5	N\A	Axiomatic
6	Syntactic	Transformational
7	Semantic	Transformational
8	Algorithm	Transformational
9	N\A	Transformational
10	Semantic (attempt only)	Transformational
11	N\A	Transformational, Axiomatic

Table 1: Summary of John's work

When looking at the chart it does not seem like John had made much progress at all. In fact, this is true. The reason for this is that John did not have much progress to make. The proofs he provided were generally the sort that one would hope their students provide. The types of proofs he produced were usually semantic unless the structure of the problem dictated it be otherwise.

Also, his proof scheme was very consistent. In every interview, he provided evidence for an analytic proof scheme. As can be seen, he showed signs of either a transformational or axiomatic proof scheme, or both.

With that said, John still feels like he has more improvement to make. He feels like he could practice more, but also keep building his tool box: "I mean, because I sort, I still feel like I'm pretty limited in…really it feels like there's only a couple different ways I can go about trying to prove something."

4.2 Mary

This section looks at the progression made by Mary over the two semesters of the study. Mary was a mathematics major and her future plans include being a secondary mathematics teacher. During the first semester of the study her only math class was MATH 301, Teaching Mathematics with Technology and during the second semester, she took Number Theory and History of Mathematics.

<u>Mary's proof attempts</u>

<u>Question 1</u>

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Mary began this problem by drawing a rectangle and writing out the properties she knew rectangles to have. Her original labeling was erased, but she initially labeled the rectangle as a general quadrilateral with side lengths *A*, *B*, *C* and *D*. When she wrote out the equations for area and perimeter, I suggested she change *B* to *A* and *D* to *C* for ease of calculation. After she had the basics down, she set the equations for area and perimeter equal to each other so that, as she said, "so we can work with them being equal to each other." Her work can be seen in the figure below, moving down the left hand column. After she did this, she said "I got $-\frac{-2A}{2-A}=C$. So I'm redrawing the rectangle and instead of doing, I'll do it all in A."

At this point, Mary got stuck and asked for a suggestion of how to continue the problem. I gave her the following hint:

Nick: And, um, you know that if [you] pick a variable for *A*, you know, you'll…things would work out because you've set the equation to work out. So I'd try a few values of *A* and see what you come up with.

Mary: So I'm going to do *A* = 2 first because there's a lot of 2's in the problem.

(working)

M: -4 over 0…undefined

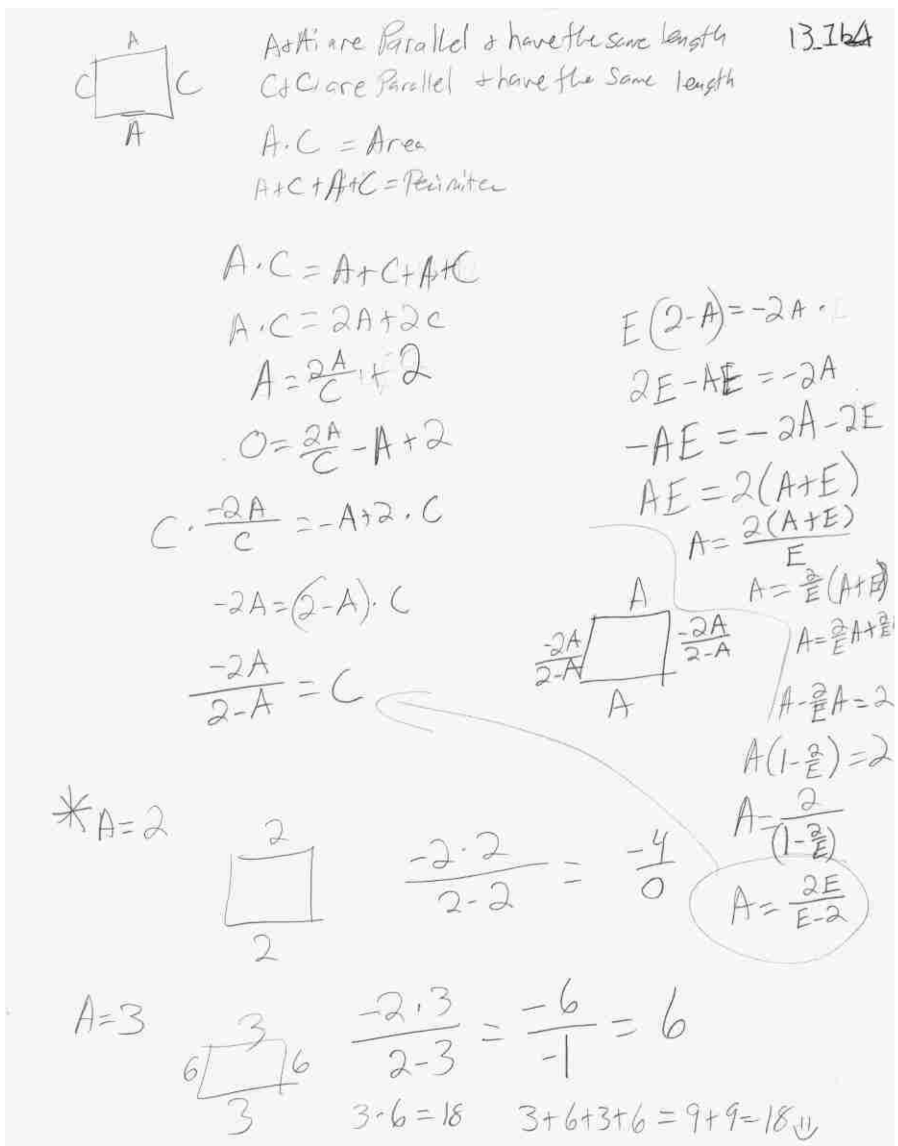


Figure 22: Mary's work on Question 1 (1 of 3)

Mary then continued on with the examples, tying A = 3 and A = 1.

> M: ... 6, so two of the sides would be 6 and the other two would be 3…And 3 times 6 is 18 and 3 + 6 + 3 + 6 is also 18. So now I'll try with A = 1. -2 over 1 which equals -2…
>
> N: So you tried 1..
>
> M: I tried 1 and I got a negative number which would be really difficult to have a negative length…So that doesn't work as well.

Mary then proceeded to try A = 8 because "it's not close to the lower numbers but still small enough to make the multiplication [easy to handle]…". She gets a corresponding C value of $\frac{8}{3}$ and needed to be reminded that the problem asked for integer side lengths only.

Mary continued her systematic checking of examples with A = 6, reasoning that "3 worked, then would it be multiples of 3 that worked? …[I'll] try to find the pattern that way." She saw that 6 worked and then tried 5 to see if being odd had anything to do with why 3 worked. It did not and then she moved on to A = 9 to see if the multiples of 3 pattern continued to hold. Reflecting on the work she had done so far, Mary says:

> M: So that doesn't work….there's something that 3 and 6 have in common that may have in common with other numbers, but not with 5, 8, or 9…Pretty much, you're going to need the denominator, which is 2 – A, to be a multiple of the numerator. Or vice…I'm trying to think of if there's another way to rewrite the C value -2A over 2 – A… So I have 2 – A equals -2A times an unknown I'll call E.

Next, Mary completed the algebra in the right column of Figure 22. Once she got to the end, she realized that she had not gotten any new information:

M: Which is the exact same as this except…

N: Solved for *A* and calling *E C*, exactly.

M: And the positive are now negatives and the negatives are now positive…Which doesn't really…help much (laughs)

Mary and I then discussed the fact that this happens because one can switch the roles of *A* and *C* and it essentially rotates the rectangle by 90° but does not change the area and perimeter. This leads Mary to say: "So pretty much if I look and find one [a number that works], then the number I find would also work." This notion of paired numbers shows up later in the interview.

Feeling somewhat stuck, Mary goes back to trying examples. She tries *A* = 7, 10, 12 4, finding that only 4 works.

M: Ok. So -2 times 4 over 2 -4 is -8 over -2 which is 4 which would give you the square that it works for…Which would also prove that I have them all. There would only be the two, wouldn't it, because if 4 is switched and it's still four, wouldn't that be the middle part?

N: What do you mean the middle part?

M: If you think of it as a time line. Or a number line I guess…

N: Can you draw what you're talking about?

M: Yeah. So like I know when you're factoring out something, you do, like say you were factoring out 6, 1 2 3 6. (See Figure 23)

N: Ok

M: And then like if you were doing, like, 9 it would be 1 3 and 9.

N: Ok

M: So like this 3 is the middle part for 9 and every number on one side of 3 has a match on the other side.

N: Oh, ok.

M: But that doesn't work for this… For 6 the factors are 1 2 3 and 6. So the middle part is right here between 2 and 3…And so everything on this side has a match on this side. So like we found that 3 4 and 6 works. 4 works for the 4….So that would be the mirror part and 3 goes to 6 and because we know 2 doesn't work because it gives a 0 in the denominator, and the 1 doesn't work because it gives a negative…But I don't know because it might be possible to have a number, like where both the numbers are bigger than 6.

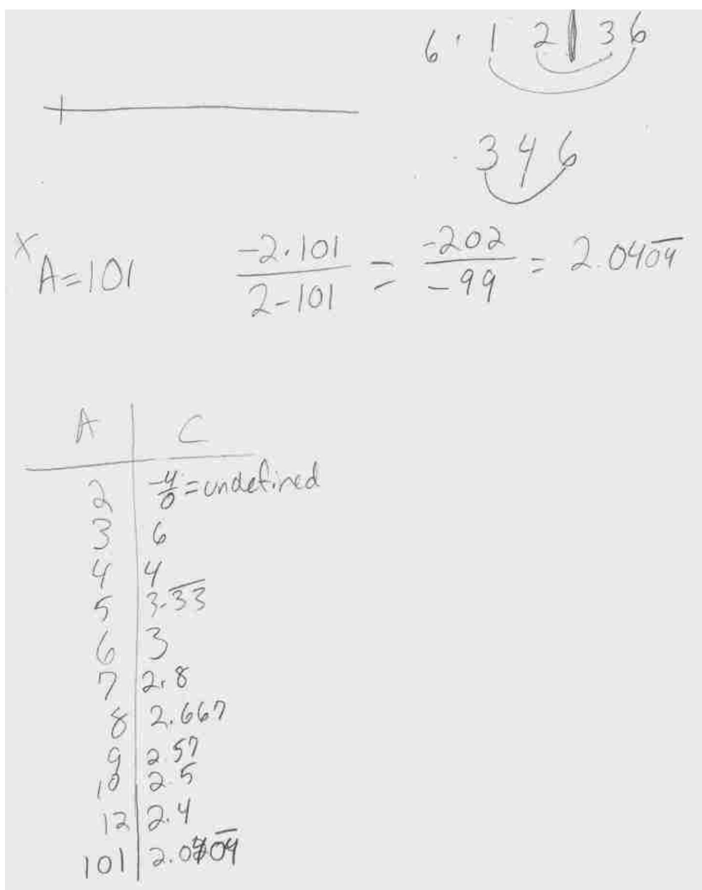


Figure 23: Mary's work on Question 1 (2 of 3)

After this discussion, Mary tries one more example, *A* = 101 which she found to not work. She used 101 (a prime number) because it came up in relating the method she was talking about to a way to check if a number is prime (one only has to check the integers up to its square root for divisibility). At this point, I suggested she try looking at the examples she had tried in decimal form. She did so, collected the numbers in the chart in Figure 23. This leads her to conclude that she had found a proof:

> M: So that would also help prove that I have them all… It keeps going down…And so, because at 12 we're at 2.4 and 2 can't show up on the a list…In order to get to get lower, like even at 101, it's above 2, but a very small amount so it's like approaching 2…And therefore we know that the 2 can't show up on the a list…because it would have an asymptote at 2.

I asked Mary if she could remember the proof, write it up and bring it in for the next interview and she said she could. The proof she brought, however, did not match this line of reasoning. The proof she brought can be found in the next figure.

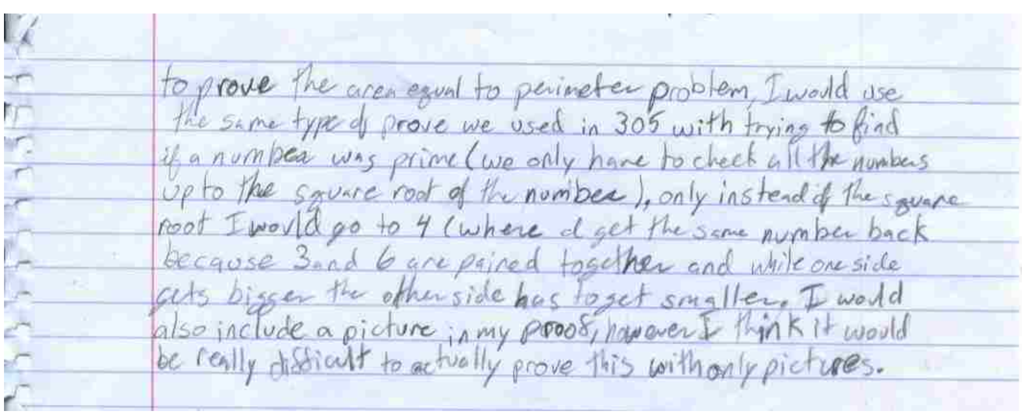

to prove the area equal to perimeter problem, I would use the same type of prove we used in 305 with trying to find if a number was prime (we only have to check all the numbers up to the square root of the number), only instead of the square root I would go to 4 (where I get the same number back because 3 and 6 are paired together and while one side gets bigger the other side has to get smaller, I would also include a picture in my proof, however I think it would be really difficult to actually prove this with only pictures.

Figure 24: Mary's work on Question 1 (3 of 3)

Mary either had forgotten the proof she came up with in the previous interview or she was no longer convinced of it. However, there was a hint of what she had done in the last interview. She mentioned when discussing her proof that "you have pairs and so one side's going down while the other side's going up. So they overlap." She is making use of the decreasing nature of the function $C = \frac{-2A}{2-A}$, if *C* were taken as a function of *A*. This decreasing nature is something she noticed last time. Because *C* decreases as *A* increases, increasing *A* above 4 means that *C* must be less than 4. Thus, there can be no pair where both numbers are greater than 4.

Mary's proof is not complete, but when combined with what she mentioned in the last interview one can see what she means. This is a semantic proof because used examples, and algebraic manipulations get gain an understanding of the problem. She then used this understanding to give a proof, albeit a partial one.

Mary provides evidence for a couple different proof schemes here. First, using the decreasing nature of the function indicates that Mary has a transformational proof scheme. This reasoning is based on anticipating the result of increasing one of the variables in the equation. That being said, Mary also demonstrates an empirical proof scheme. She becomes convinced of the decreasing nature of the function based on the examples she looks at. In the first interview she makes the point that checking examples alone does not give a proof: "Without just plugging, and checking and with infinity it doesn't work." However, she does not recognize that this sort of reasoning is used in the proof she provides. Even though she does not realize it, she did become convinced by inductive reasoning and thus reveals an empirical proof scheme.

<u>Question 2a</u>

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9. (For example, if a = 4 and b = 2, then 42 – 24 = 18, a multiple of 9.)

Mary began this problem by asking what I meant by "multiple of 9." "But, does 0 times 9 work?" I told her that for our purposes, that 0 would work as a multiple of 9 because 0 could be written as 9 times another integer. I also told her that would allow for negatives to be considered multiples of 9.

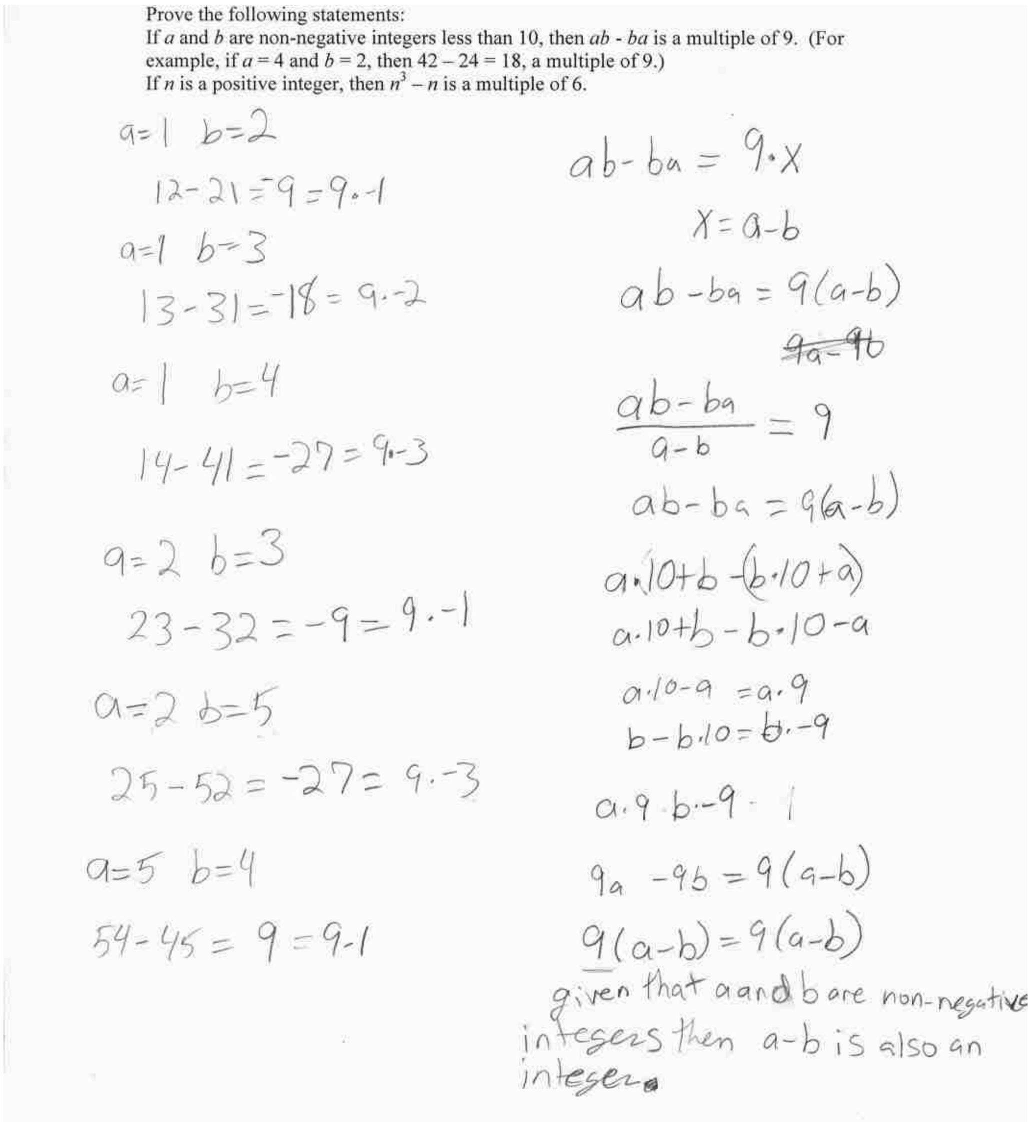
After getting the terminology straight, she began the problem by looking at examples: "So, I'm going to go through and plug in some of them, not all of them because that would take for ever." Initially, I had thought this meant that Mary had the idea that there were infinitely many pairs of *a* and *b* to check. However, later in the interview she said "I'm trying to think of a way I could do it without numbers... because I could do all of them and it still wouldn't be a real proof." This told me that she did realize that all of the combinations could be checked. To her, however, that would not count as a proof.

So, Mary began checking some examples. See Figure 25. After a few examples were tried, Mary noticed the pattern: "Is it always, or a coincidence that the multiplier is pretty much the difference between the numbers *a* and *b*?" I told her that was for her to decide and that is when she made the statement above about finding "a way to do it without numbers."

Mary then proceeded to set up the algebra in the right hand column in Figure 25.

> M: So if you have $ab - ba$, that would equal 9 times x where x is the multiple and x would also be $a - b$…And so by plugging $a - b$ in for x, you have $ab - ba = 9(a - b)$.

After rearranging this last equation for a bit, Mary decides to take a different approach: "So I'm taking the *a* out of the *ab* – *ba*… So I broke it up where it's *a* times 10 plus *b*…And I did the same for *ba*." From there it did not take Mary long to complete the problem:


Prove the following statements:
If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9. (For example, if a = 4 and b = 2, then 42 – 24 = 18, a multiple of 9.)
If n is a positive integer, then $n^3 - n$ is a multiple of 6.

a=1 b=2
12-21=-9=9·-1
a=1 b=3
13-31=-18=9·-2
a=1 b=4
14-41=-27=9·-3
a=2 b=3
23-32=-9=9·-1
a=2 b=5
25-52=-27=9·-3
a=5 b=4
54-45=9=9·1

ab-ba = 9·x
x=a-b
ab-ba=9(a-b)
~~9a-9b~~
(ab-ba)/(a-b) = 9
ab-ba=9(a-b)
a·10+b-(b·10+a)
a·10+b-b·10-a
a·10-a =a·9
b-b·10=b·-9
a·9 b·-9
9a -9b = 9(a-b)
9(a-b)=9(a-b)
given that a and b are non-negative integers then a-b is also an integer.

Figure 25: Mary's work on Question 2a

M: But then if you take the *a* times 10 minus *a*, that equals *a* times 9…And *b* minus *b* times 10 equals negative 9, *b* times -9. And you have *a* times 9 and *b* times – 9 equals 9(*a* – *b*), right? … Which equals 9(*a* – *b*) as a quantity. So you get 9(*a* – *b*) = 9(*a* – *b*)… But that also proves that that would be a multiple of nine…

N: So, is *a* – *b* going to be, so you've proven that *ab* – *ba* is going to equal 9(*a* – *b*), right, so have you proven that *ab* – *ba* is equal to nine times an integer?

M: If *a* and *b* are non-negative integers, then it works.

The proof Mary provides is a bit difficult to classify. On the one hand, she began working on the problem by looking at examples, typically something one does to get a handle on the problem. And this exploration did lead her to finding a pattern that she was eventually to prove holds. Usually, this change of events would constitute a semantic proof due to the effort made to understand the problem before starting the proof. However, just because Mary had a pattern she thought might hold does not mean that she had an understanding of why it held. The operations she performs to arrive at the proof could have easily been accomplished without seeing the pattern beforehand. In fact, the key to solving the problem was when she decided to separate the variable a from the rest of the expression in $ab - ba$. I feel this has little connection to the pattern she had found. Having the pattern in mind was important, however, in that she knew she was done once she derived it. Because she did not turn an intuitive understanding of the problem into a proof, Mary's proof is a syntactic one.

Mary displays a transformational proof scheme with this proof. She makes a point to avoid an empirical argument ("…I could do all of them and it still wouldn't be a real proof.") even though it would have been valid in this case. Instead, Mary relies on mathematical operations performed on the expressions involved to arrive at her proof.

<u>Question 2b</u>

If n is a positive integer, then $n^3 - n$ *is a multiple of 6.*

Mary began this problem like she did the previous one. She started with examples and made sure to note the "multiplier" (for example, if $n = 2$ the multiplier is 1 because $6 \cdot 1 = 2^3 - 2$). Her work can be seen the figure below.

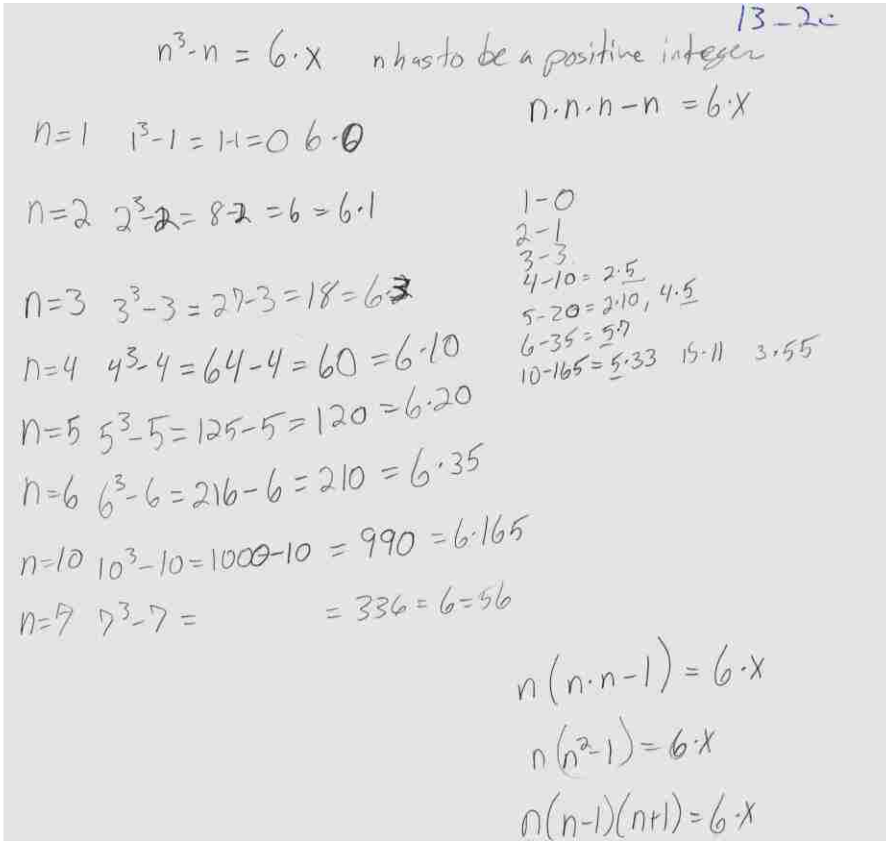


Figure 26: Mary's work on Question 2b

N: So if you just stepped back for a second, you'd say that you're overall goal is to find a pattern in the multipliers again?

M: Yeah

N: Because it worked for the last one or is that just sort of a general strategy you have?

M: A little bit of both…I might be missing something but I'm not really seeing the pattern.

N: Ok.

(thinking, starts writing the n = 7 case)

N: So you thought about trying another one and kind of decided that it probably wouldn't help?

M: Kind of…I'm also kind of looking at the difference between the result, the multiplier that I got for 6 and the multiplier that I got for 5 and 4 and stuff…But 3 being 3 and 4 being 10 kind of messes with it, because I was noticing that the 35 and 20 is a difference of 15, 20 and 10 is a difference of 10.

N: Sure. Oh, so you kind of expected a difference of 5 before that.

M: Yeah, but that's also kind of just a small one because it doesn't really fit.

Unable to see a consistent pattern, Mary goes back to the original expression: "If I just factor, if I went back to the $n \cdot n \cdot n - n$ equals 6 times x, I could factor out an n…That is the difference of squares, right? …So that is $n(n - 1)(n + 1) = 6x$." After not seeing anything particularly useful, she considers some algebraic manipulation but does not see anything that would help. "Pretty much, I just don't really know what to do from here because I was to bring the 6 over, it would equal the multiplier, but it still wouldn't really get me anywhere, right?" After some more thought, she goes back to finish the $n = 7$ example she began earlier:

M: I'll try 7 just to see if we get a 5 out of it, because it looks like after 4 we do.

N: Ok. So it looks like after 4, there's a factor of 5 in the multipliers.

M: Yeah.

N: So you want to try seven…336, if you divide that by 6, you get 56.

M: And that doesn't have, is not divisible by 5.

(thinking)

M: I'm drawing a blank.

At this point, the interview was over and I asked Mary to look at the problem some more between interviews. She said that she would.

She did not bring in any work the next time we met, but it was not because she neglected to work on the problem.

> N: So did you go back, did you write anything up for the second one, or did you not really look at it?
>
> M: I didn't because I couldn't figure out a pattern.
>
> N: But you did look at it a little bit, but there just wasn't really anything worth writing?
>
> M: Yeah, I just went around in circles, kind of like I did in here a little bit. Because I remember I was working at factoring and stuff like that and it was just circle after circle.

The rest of the interview was spent rehashing what she did with the problem the previous interview. At the end of the interview, I gave Mary a chance to continue working on the problem but she declined: "Yeah, I don't really know a path to take."

Although Mary did not complete a proof, it is possible to classify her work with a proof type because she was working towards creating a proof. Mary used her method from the previous problem to try to come up with a proof for this question as well. Although she tried the same techniques with the same motivations, Mary's proof attempt here is classified as semantic rather than syntactic. The reason this gets a different categorization is that she is trying to gain an understanding into the problem. While she made this same effort in the last question, understanding alone did not lead to the proof she presented. In this case, the fact that she does not come up with a proof prevents this switch in proof type.

Mary's proof scheme, however, is unchanged from the last question to this. Like with the last problem, her proof scheme is still characterized by operations performed and the anticipation of their results. This is the main feature of the transformational proof scheme.

Question 3

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

Mary started this problem like one would with any induction problem: with a base case. She began with a base case of n = 2, but she did not say why. From there, she wrote out her induction hypothesis and the moved to the inductive step (see Figure 27). "Ok, so we're assuming that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}$ all the way up to $\frac{1}{2^n}$ is greater than or equal to $1+\frac{n}{2}$. And we're to prove that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}$ all way up to $\frac{1}{2^{n+1}}$ is greater than or equal to $1+\frac{n+1}{2}$." Mary ran into a little trouble in that she simplified a little too much on the right hand side of the inequality, but she quickly realized that in order to use the induction hypothesis she would need to have things separated out:

M: So this part is the same as up here. And since we're assuming that, then if I break this side up again, instead of having it under the common denominator, it's $1+\frac{n}{2}+\frac{1}{2}\ldots$

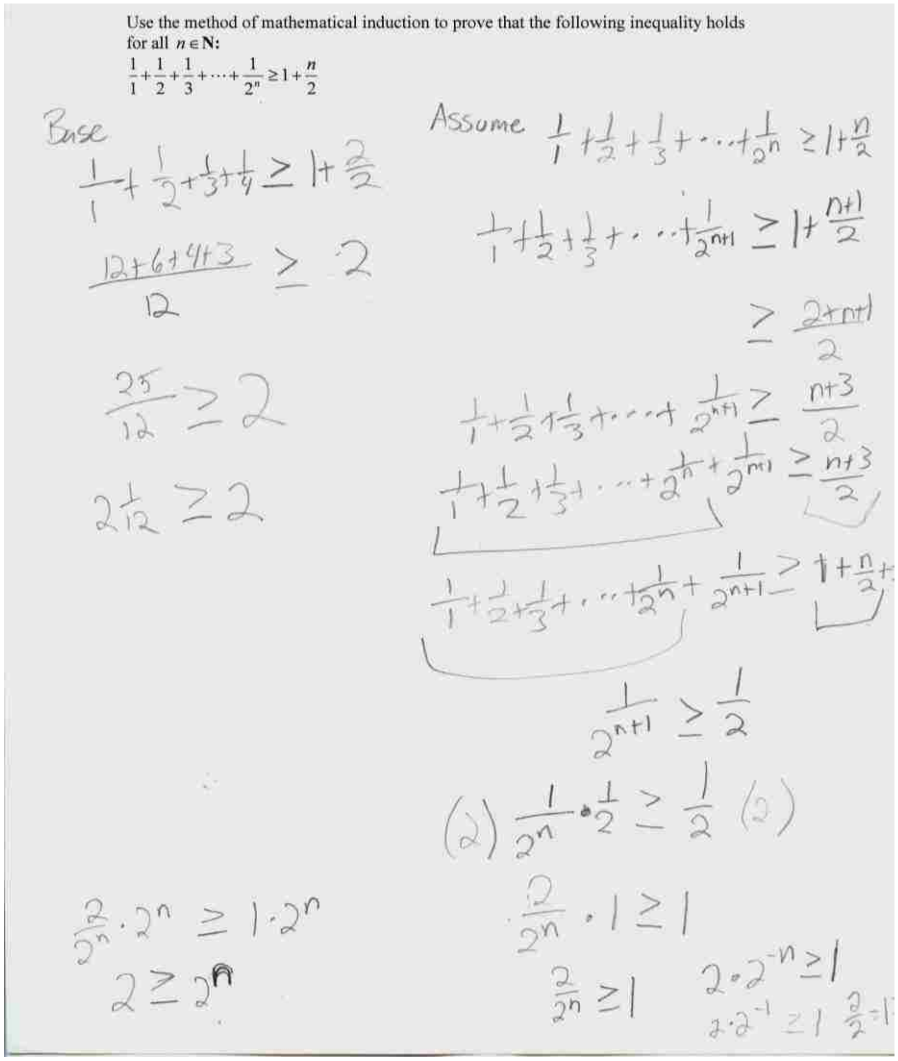


Figure 27: Mary's work on Question 3 (1 of 2)

M: (continued) So the $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}$ to $\frac{1}{2^n}$ is greater than or equal to 1 $+\frac{n}{2}$, so we can take it out and we're left with $\frac{1}{2^{n+1}}$ is greater than or equal to $\frac{1}{2}$. Which, this is $\frac{1}{2^n}$ times $\frac{1}{2}$, which is greater than or equal to $\frac{1}{2}$ because $\frac{1}{2^n}$, where *n* is a natural number is always, like is a multiplier to $\frac{1}{2}$, always has to be bigger than or equal to because if *n* is 1, then it's 1…Like, 2^1, is $\frac{1}{2}$, which is still a multiplier.

N: Ok

M: And, except that would be a quarter, which is less than a half.

Mary had momentarily thought she had completed the problem until she realized that multiplying by $\frac{1}{2^n}$ would actually make a number (in this case one half) smaller. She saw that she had reached a trouble spot, the fact that $\frac{1}{2^n}$ is not greater than or equal to $\frac{1}{2}$. However, she did not take this to mean she had done something wrong up to this point. Instead, she attempted to rearrange the inequality to get it to work out the way she would like. “I’m just looking to see if there’s another way I can write the $\frac{1}{2^{n+1}}$ is greater than or equal to the $\frac{1}{2}$. Because this way seems to just make it go backwards, not true.” Her attempts at making the inequality true via algebraic manipulations can be seen on the bottom of Figure 27. Eventually, Mary ends her attempt: “I don’t know any other way to re-arrange it to get it to where it’s not getting to a dead end at this point.”

At that point, it was time to begin the reflection portion of the interview and eventually I asked Mary what she thought might be the problem: “One part I thought it might come from would be where it, the $\frac{1}{2^n}$ to the $\frac{1}{2^{n+1}}$…there might be a term in between that I’m missing or something.” Mary reasoned that there was more going on than she had thought because “the three $\left(\frac{1}{3}\right)$ is kind of difficult to get with 2 as a base.” Mary realized that the sum on the left contained more than simply powers of 2 in the denominator. From there, Mary left the interview intending to look at the problem some more between interviews.

Mary worked on the problem between interviews and made some progress but did not complete the problem. The work Mary completed betweens interviews can be seen in the following figure:

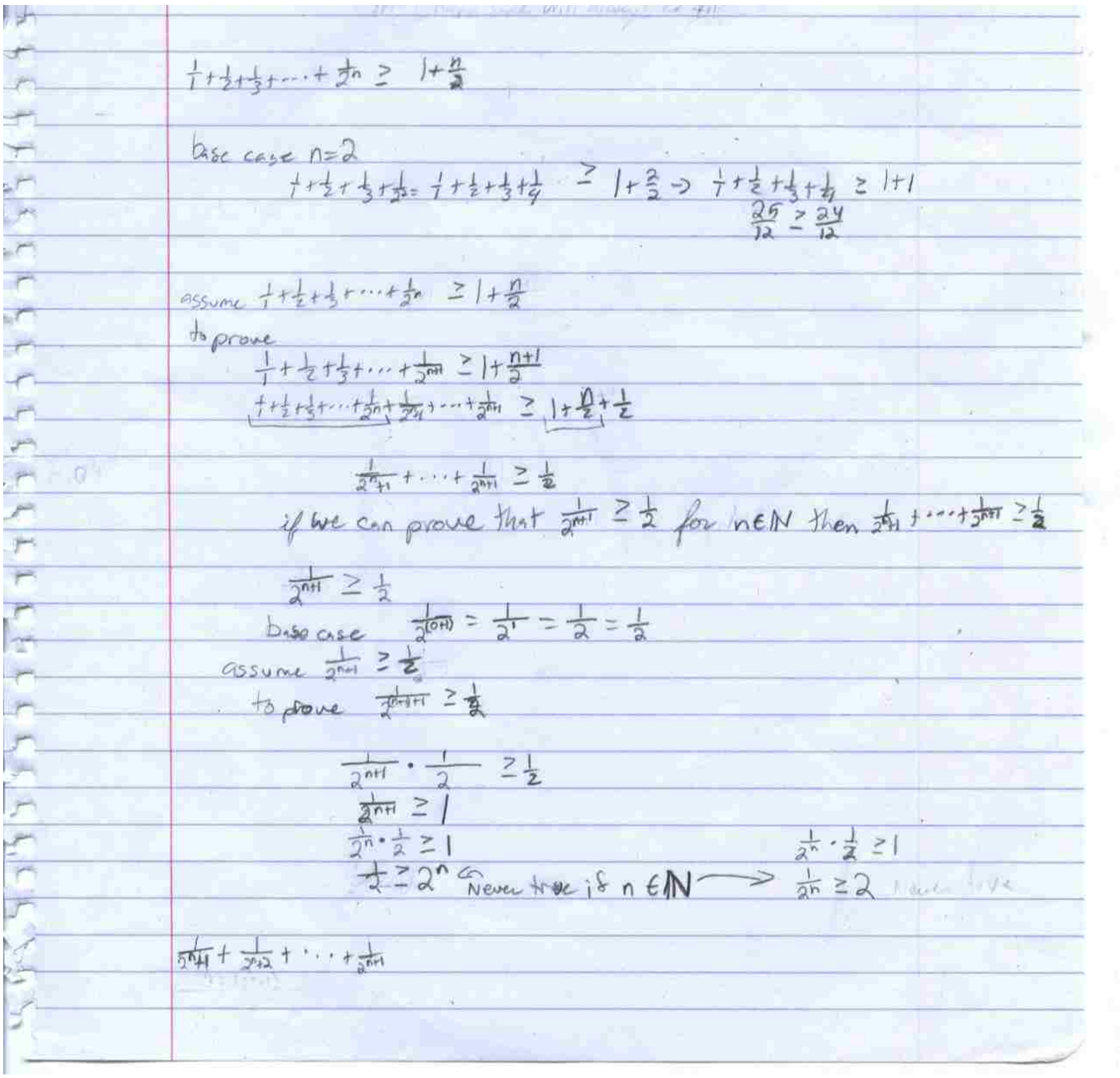


Figure 28: Mary's work on Question 3 (2 of 2)

Mary began the problem as she had before, but now she made a point to include the terms in the sum she had been missing before. The trouble was, she did not use them as she should have to finish the problem:

> M: And then I broke it down to proving that just one of the terms, and I chose the $\frac{1}{2^{n+1}}$ is greater than or equal to $\frac{1}{2}$, because I figured if we can prove that, because we're just adding on to that, then it would be true.

So, Mary got herself right back to where she had been before with the problem. She did not try only algebraic methods this time, however. As can be seen in the figure, she attempted to complete this problem with another induction argument: "And so I tried it for a base case, and it worked and so I assumed it and tried to do another proof by induction, but absolutely nowhere that was true." At that point Mary quit working on the problem. We then spent the rest of the interview talking through the problem together.

As with the last question, Mary did not complete the problem. However, like before, one can still classify the type of proof she is attempting. Because this is an induction argument, it is classified as a procedural proof. Of the sub-types of procedural proofs, this is considered a process proof because there are a few global steps to be completed. An algorithmic proof would have had every step laid out exactly. Here, the completing the inductive step required more than straight forward algebra operations.

This proof provides evidence that Mary has a transformational proof scheme. She uses operations on expressions to try to arrive at a proof. Also, while it is true that Mary does not completely understand the process of induction (she does not use the proper base case), she does seem to have an otherwise adequate knowledge of the procedure. Since she does not understand the process completely, her proof scheme can be labeled as an internalized proof scheme.

<u>Question 4</u>

Examine the following proof by contradiction:

<u>Prove</u> $\sqrt{-1} \le 0$

Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \le 0$.

<u>Prove</u> $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \le 0$, *then* $-\sqrt{-1} \ge 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \ge 0$, *so* $-1 \ge 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

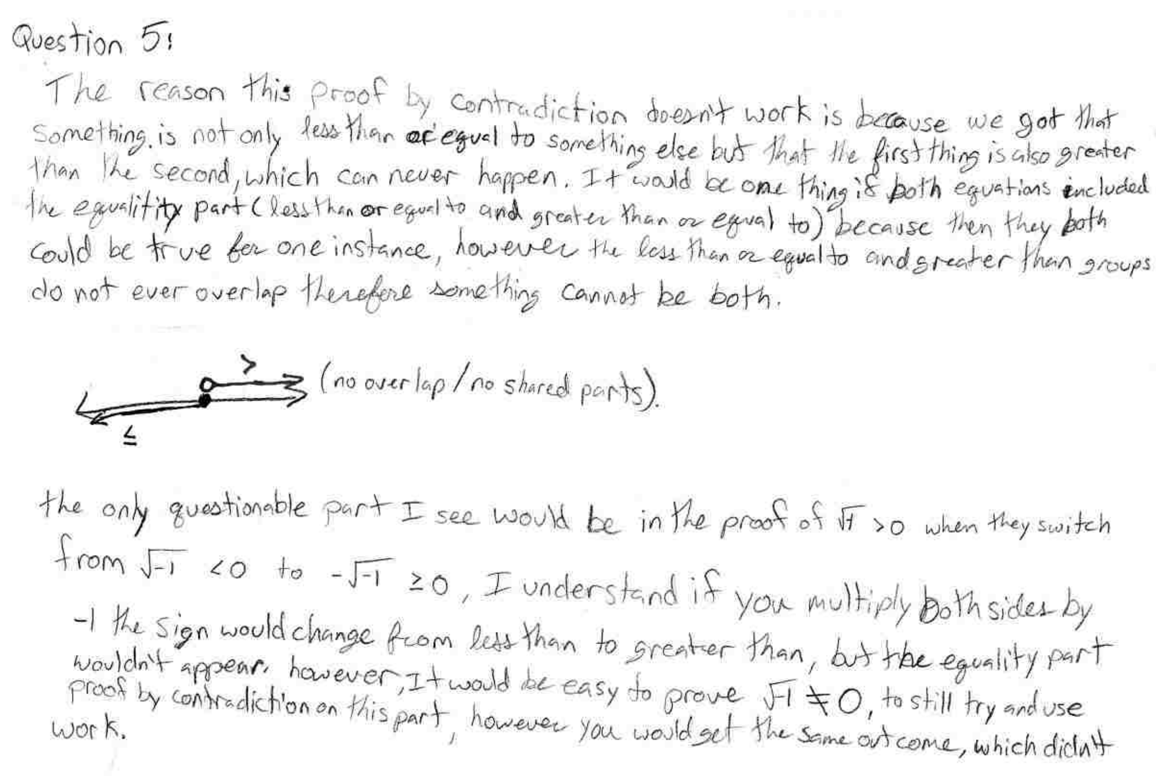
Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This problem was on Mary's midterm exam that she completed the semester before the study began, while she was taking MATH 305. The solution Mary provided on the midterm can be seen in Figure 29. Notice that Mary basically says that the proof provided is not working because it reaches contradictory results. She finds potential algebraic flaw in the proof but concludes that it would not change the outcome of the proof. (The copy of the proof on the midterm contained a mistake that had been corrected for the study.)

Initially during the interview, Mary mentioned the same flaw as before: that contradictory conclusions had been reached.

> M: The proof by contradiction isn't working because when you try a proof that it's less than or equal to 0, you get that it's true because the other thing's absurd…And then when

you're trying to prove that the square root of negative one is greater than 0, you get that as well in proof by contradiction and it can't be…


Question 5:

The reason this proof by contradiction doesn't work is because we got that something is not only less than or equal to something else but that the first thing is also greater than the second, which can never happen. It would be one thing if both equations included the equality part (less than or equal to and greater than or equal to) because then they both could be true for one instance, however the less than or equal to and greater than groups do not ever overlap therefore something cannot be both.

the only questionable part I see would be in the proof of $\sqrt{-1} > 0$ when they switch from $\sqrt{-1} < 0$ to $-\sqrt{-1} \geq 0$, I understand if you multiply both sides by -1 the sign would change from less than to greater than, but the equality part wouldn't appear. however, It would be easy to prove $\sqrt{-1} \neq 0$, to still try and use proof by contradiction on this part, however you would get the same outcome, which didn't work.

Figure 29: Mary's previous work on Question 4

I then pointed out to Mary that the question is really asking is what is allowing for those contradictory conclusions to be reached. From there she proceeded to go over both sub-proofs in detail, trying to find a flaw.

One of the potential flaws she considered was the fact that in the second sub-proof, something was getting squared:

M: It was, I know that it was squared properly, but I don't know if that was the right approach to take.

N: I see, ok. So why might've it been not the right approach to take, just because it's leading to something so strange? Or is there something else about it that you're not ok with?

M: It just seems like it would missing a lot of information, I don't know if information would be the right word.

N: Sure

M: It's just like, you could take -3 times -3 and you would get 9, which is greater than 0.

N: Right

M: But it doesn't really say anything about the original.

After some thought, however, she decided that perhaps trying to come up with a proof on her own might help her see the problem: "Now I'm just trying to figure out another operation or something that you can do to come up with a proof if you didn't square it…Just to see if that would help show it is a flaw or whatever." This leads Mary to consider the possibility that the proof is not actually contradicting itself:

M: It's kind of like saying, you can say the cat is black and look at the cat and say 'Yeah, it's black' but if it's black and white you can say that cat is black but somebody else could say the cat is white and they're both saying the same thing, but…But then it's a hard thing because it has the whole number line is where the square root of -1 could be. I mean it could fall anywhere in there.

N: Ok

M: And I know that the square root of -1 is i and we tend to use that not as a variable but as a constant.

Mary is back to facing the fact that the proof is indeed contradictory. She thinks that i should be at some fixed point on the real number line and that point is either to the left or right of 0. Mary spends the rest of the interview looking back over the proofs trying to find flaws. By the time we needed to start the reflection period, Mary still had not found a flaw although she thought there must be one: “I think that there is a flaw but to say exactly where it is…”

Due to the nature of this question, there is no proof attempt to classify. However, some of the things Mary and I spoke about did give some insight into her proof scheme. First of all, she spends a lot of time analyzing the individual steps of the two sub-proofs and is concerned with the implications of those steps. This is typical of the transformational proof scheme.

Question 5

The next interview was the last of the semester and served as a debriefing session for the first half of the study. Since Mary did not attempt a proof, there is nothing to classify. However, some of her comments did support some of the observations made earlier in the semester. For example, when I asked Mary what she thought was necessary for completing a proof, the first thing she mentioned was:

> M: First of all knowing all of the terms and what the question’s asking…That’s the biggest part, I think.
>
> N: Ok
>
> M: And knowing kind of what your solution should look like, or have sort of game plan of how to get there, or a path or something.

N: So, do you think that path would come from other proofs that have been completed and are similar, or just really understanding the hypothesis, the statement of whatever you're proving?

M: A little bit of both because you need the prior techniques and stuff like that as well as, like, knowing the individual problem.

This matched what happened in the interviews because in at least half of the interviews, they began with Mary and I talking about terminology and other concepts related to the problems. Also, at least with Questions 2a and 2b, Mary spent a good amount of time trying to figure out what patterns she was dealing with in order to work towards it. The business of having a plan and purposely moving toward a goal are characteristics of a transformational proof scheme, which showed up in every one of Mary's proof attempts.

When I asked what was helpful but not necessary for completing a proof, the first thing Mary said was "Not messing up on your algebra, stuff like that…" The fact that this was the first thing she mentioned tells me that Mary finds the particular steps performed in a proof to be important. Again, this meshes with a transformational proof scheme.

Later in the conversation, while we were still talking about things that help while proving but are not necessary, I mentioned the fact that she would often try examples when beginning a proof. To this she replied: "It helps me to start the problem and make sure I have an understanding of it before I start trying to prove it for everything." This is refers reminiscent of the semantic proof attempts Mary completed earlier in the study (for Questions 1 and 2b).

While this interview did not yield any new insights into Mary's ideas about proof, it did support some of the observations made earlier in the analysis.

Question 6

Prove that $\forall n \in \mathbb{N},\ n \equiv n^3 \pmod 6$.

Mary and I began this interview by discussing modular arithmetic, something she had not seen too often. To begin the problem on her own, Mary verified that the result holds for $n = 7$.

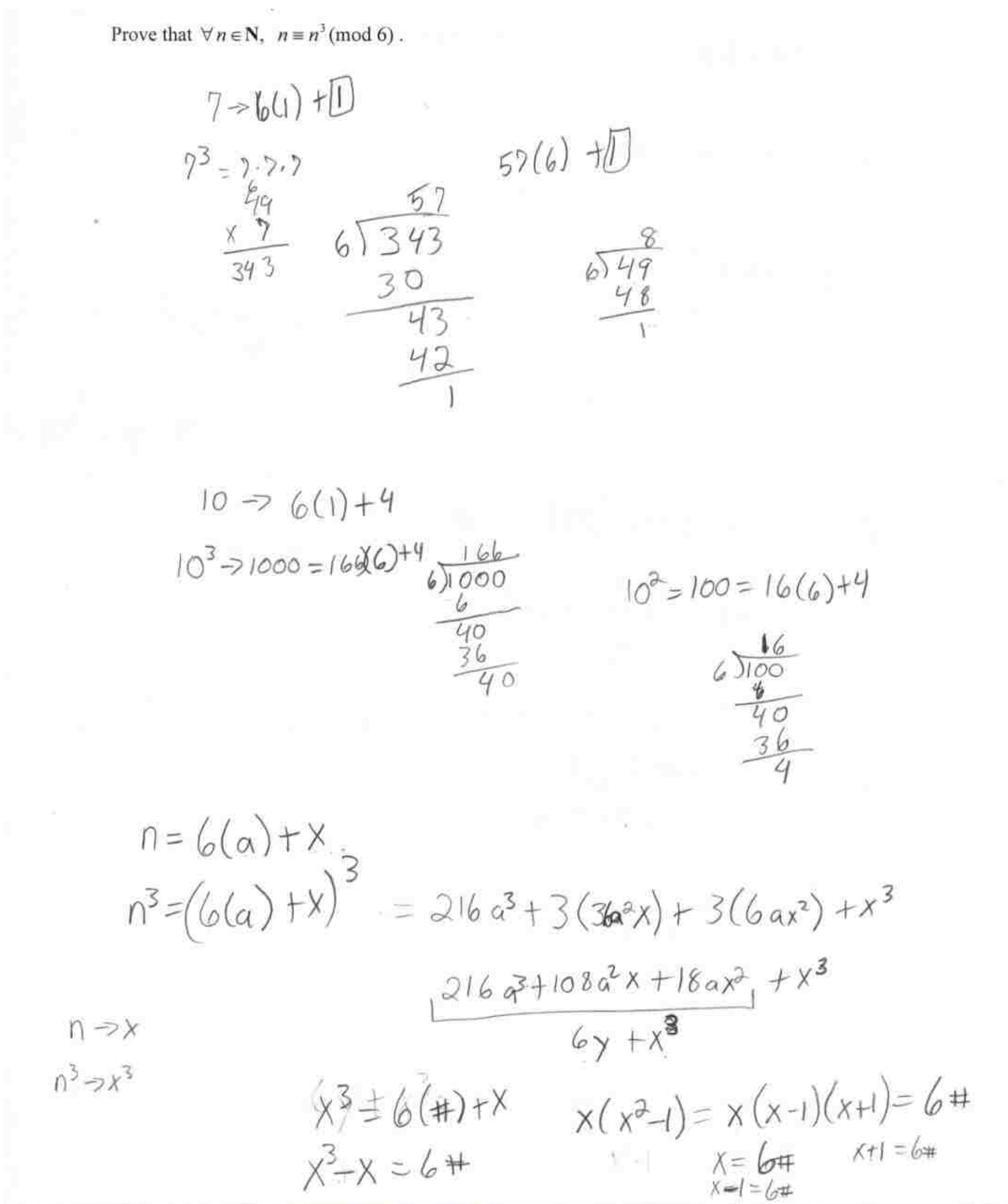


Figure 30: Mary's work on Question 6 (1 of 2)

Mary sees that it works for 7 and notices that it would also work to compare 7^2 and 7. She then decided to try it for a different number to see if it would still work:

M: So now I'm trying it out with 10, again, just to see, like…

N: Ok, see if it's going to keep going.

M: Yeah, first to make sure that n and n^3 are congruent and then I was going to check the n^2 just to see if it was with the 7 or what.

As can be seen in the figure, if 10^2 is also congruent to 10 and 10^3. Mary does not pursue this line of inquiry, however. Instead, she shifts her focus to the fact that she is working modulo 6:

M: I was just looking at 6 which is, like, I was trying to figure out because I know 2 can go into it and 3 can go into it…And so I was, like, factoring it out thinking whether it had something to do with the mod 6 or…Or whether if it was mod 5 if it would still work…Or if mod 6 made it so it where it's specifically for n^3 and n… But nothing was really popping out at me, like 'Oh, well, this makes sense' or…

From there, Mary goes back to the notation she had for the examples at the top of the page and uses it in the general case: "So I have $n = 6a + x$. Where a would just be any multiplier ...and x would be what's remaining…And then, n^3 is…that whole thing cubed." After cubing that out, she realizes that the problem boils down to comparing x and x^3.

M: So then you pretty much have to prove that x is similar to x^3 where it could have a number that's, like, divisible by 6 that would give you the remainder…So like x^3 should be in the form of a number times 6 plus the original x…in order to get it congruent.

Mary had basically returned herself to the original problem but with one difference: the values for x were now limited to 0 – 6, which could be checked. She did not realize this at the time, however, and continued:

> M: So then $x^3 - x$ would be 6 times a number. I was just rewriting it to see if something popped out, or…So then I factored it out to $x(x - 1)(x + 1)$ which is still equal to 6 times the number.

At this point, Mary gets a little confused and relates this last equation to find the zeros of a polynomial:

> M: So I just went through and found what the zeros where be of the $x(x - 1)(x + 1)$.
>
> N: Ok
>
> M: So it's like x could equal 6 times the number or $x - 1$ is 6 times the number, or x would be 6 times the number + 1.

Mary did not see much hope in this method either, because she moved on from it and went back to some previous work on the next page (Figure 31):

> M: I feel like it should be, like, popping out at me…So I just copied down real quick what I got for 7 and 7^3…So I just wrote out x times x times x for x^3…equals $x + 6$ times the number, which is what, kind of what I was going over on the back, on the other side.

She then also tried $n = 3$: "So then I'm using 3 just to see, for like a number less than 6, it's just 6 times 0 plus that number." At this point, she felt like she had reached the end of what she could do: "I kind of feel like I hit a wall at this point."

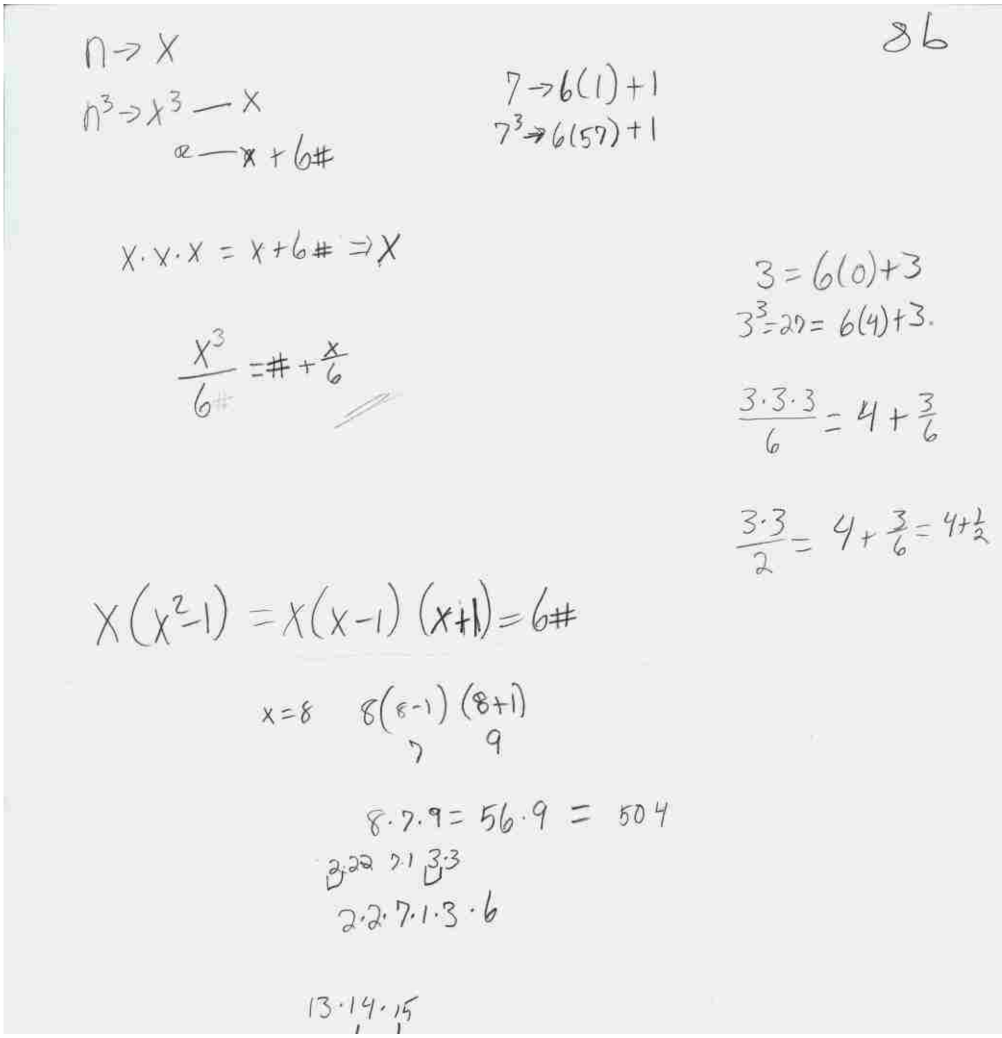


Figure 31: Mary's work on Question 6 (2 of 2)

At this point, I began talking Mary through to the end of the problem. I did this because she was one of the last participants to work on the problem and most of the other students completed it so I knew I would not be spending time talking about it during the next interview. Also, Mary did not get the chance to complete Question 2b, which was identical to the remaining portion of this problem.

The advice I gave Mary was to try individual numbers into the factored form of $x^3 - x$. She tried $x = 8$ first and I asked her if she could verify that the product was a multiple of 6 without actually multiplying it out. She was also able to apply the reasoning when $x = 14$.

“Then you’d have 13 times 14 times 15, and you’d get the 2 from the 14 and the 3 from the 15.” When asked to explain what she was doing, Mary said: “Because for any 3 consecutive integers, which is what you have, then you can always, one of them will always be even…and because it’s 3, one of them will always be divisible by 3.” Mary also realized that this finished the problem for her.

As with Questions 2b and 3, Mary did not complete the proof on her own and so I will be classifying her proof attempt only. The main thing to notice about Mary’s work is that she tried numerous methods to gain an understanding into the problem. When talking about the things she tried during the reflection, Mary said “…I figured doing it this way if it was something to do with the squared, it would come out” and “I was just trying to think, like with a graph or something like that, that would be where the intercept is, see if that had anything to do with it” and finally “That was just to see if there was any connection, like, between the multiplier…the connection between the 10 and the 7, I didn’t really see anything…between the 57 and the 166.” In all these instances, Mary is working with various representations of the problem to gain an understanding that can be turned into a proof. This is indicative of a semantic proof attempt.

As with Mary’s other work, she displays a transformational proof scheme here. She is definitely working towards a deductive proof, which shows she has an analytic proof scheme. However, her work shows no reliance on previous results, axioms or undefined terms.

<u>Question 7</u>

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

We spent the first part of the interview reviewing some basic concepts with sets. Like with most of the problems she worked on in the study, Mary started this problem by considering a specific example:

> M: Ok, so if there's *n* elements, then…there's going to be more than *n* possible subsets because each, like for the {*a*, *b*, *c*}, {*a*} would be a subset of the {*a*, *b*, *c*} set…And then you would also have, like all of them that would include 2 of them, so like {*a*, *b*} would be a subset of {*a*, *b*, *c*} and {*b*, *c*} and {*a*, *c*}…I'm trying to think, like, of how…that'd be *n* over 2…or, 2*n*…or $n^2 - n$ actually. Because then you could have the first number…it could be any of them and then the second number would be any of them except for the first.

Mary sees that there are n^2 ways to arrange 2 out of *n* things, but *n* of those arrangements involve the same thing twice. Since Mary needed a refresher on sets and subsets earlier in the interview, I felt it was necessary to mention something that did not come up at the beginning of the interview:

> M: So then you have *n* times $n - 1$ and get $n^2 - n$.
>
> N: Ok, so one thing I would say is that the…order doesn't matter in a set, so like {a, b, c} is the same set as {b, c, a}.

This triggered a memory for her and got her thinking about counting principles:

> M: Right now I'm trying to remember, like when the order doesn't matter, where it could be {*a*, *b*} or {*b*, *a*} is the same thing. I'm trying to remember how we did that, like to

find it with like the n and the $n - 1$…I remember we divided by something. And I'm trying to remember whether it was…like what is was that we divided by for that.

She asked for the formula for combinations, which I gave her, and she quickly applied it to the problem: "So for this, for just figuring it out where there's 2, n would be just n, the number of elements in the set, and then k would be 2 because we're choosing 2 of the elements…So…that would be, instead of the $n^2 - n$, it would be n! over 2! times $(n - 2)$!" This is the first work that can been seen in Figure 32.

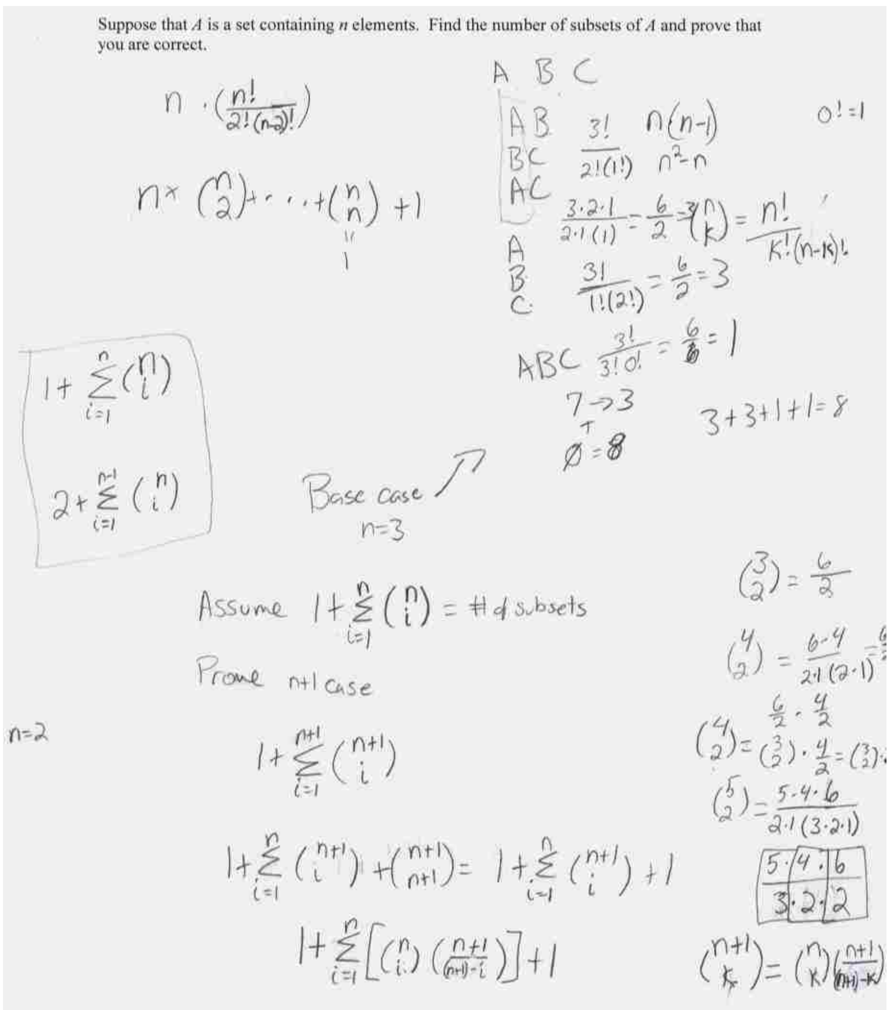


Figure 32: Mary's work on Question 7 (1 of 3)

Next, Mary applied this reasoning to subsets of different sizes, not just size 2: "And so then that would go up, like it would be *n*, which is pretty much *n*C1…And then *n*C2, *n*C3, all the way up to *n*C*n*." To test her new formula, she went back to the example she had been considering: "So I'm going through and list out the possible subsets of the {*a*, *b*, *c*}." She listed out the subsets and I mentioned the fact that the empty set is a subset, which is something she had not considered.

> M: So now, because I have the {*a*, *b*} and {*b*, *c*} and all that, I'm trying to go through and figure it out using the combinations, like with the *n*C*k*.
>
> (working)
>
> M: So then it would pretty much be 3 + 3 + 1 then + 1 for the empty set. Which is 8, so…I got the same number listing them out as I did mathematically.
>
> N: Ok
>
> M: Which I was just double checking to make sure, like…

She was now convinced that her summation method would work and shifted her attention to how to prove that she had the correct formula.

> M: And now I'm trying to think of the different proofs that I can do.
>
> N: Ok
>
> M: Like induction, or…I was thinking proof by contradiction, but that would be…I don't know exactly how I would go about that.
>
> N: Ok
>
> M: Because I would be proving that, or I would be trying to prove that that's not the number…

N: Right

M: …that just sounds like it would get messy.

N: Ok

(thinking)

M: I'm going to try induction just because that's the only thing I can think of that would help.

During the reflection, Mary added: "And usually when it deals with numbers, induction's the thing that pops into my head."

Mary begins the induction by claiming that the n = 3 case would serve as her base case and laying out the necessary steps her induction argument. She quickly saw that she needed to relate her summation back to the induction hypothesis somehow: "I broke it down to 1 + the summation as i goes from 1 to n of (n + 1)Ci, but I took out the (n + 1)C(n + 1) case…And I'm trying to think of how to get the (n + 1)Ci just into nCi."

To see how to compare (n + 1)Ci to nCi, Mary began looking at the examples in the bottom right of Figure 32. Through this work, she felt confident that she had found the correct relation: "So it appears that (n + 1) choose the number is, which the number would be k there, equals nCk multiplied by the quantity (n + 1) divided by the quantity (n – k + 1)." Armed with this new relationship, Mary went back to the summation formula she had come up with and rewrote it. She was unsure what to do from there: "So I plugged the thing I just found in for (n + 1)Ci. So when I'm multiplying within this summation, can I break it up, or no?" I told her that was for her to decide and she tried her formula a few different ways with n =2, in Figure 33, to decide.

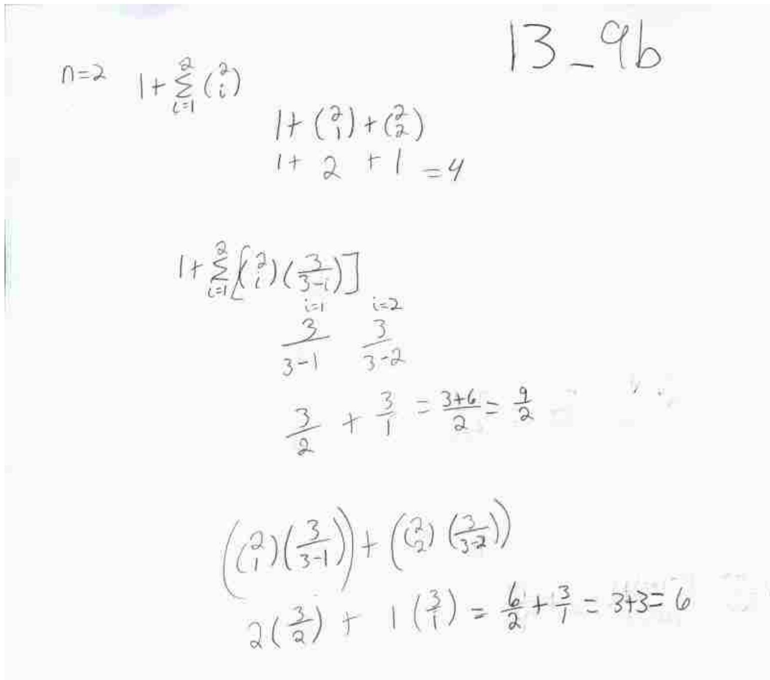


Figure 33: Mary's work on Question 7 (2 of 3)

Because she does not get the same results from both ways of computing her sum, Mary concludes that she can not simply separate it out as she hoped. At this point in the interview, it was time to stop working and begin the reflection. I asked Mary to continue working on the problem between interviews and she said she would.

Outside the interview, Mary did not make much progress:

M: So I knew that I had to get, like, the n case, which was the part that I was assuming and then another part but the other part couldn't, like, rely on k or i…so I would be able to take it out of the summation…and it's pretty much like as far as I got and I still hit like a wall and I couldn't figure out how to get the k out of there.

She did, however, verify more formally the identity she had been using during the previous interview, that $\binom{n+1}{k}=\binom{n}{k}\left(\frac{n+1}{n-k+1}\right)$. See Figure 34.

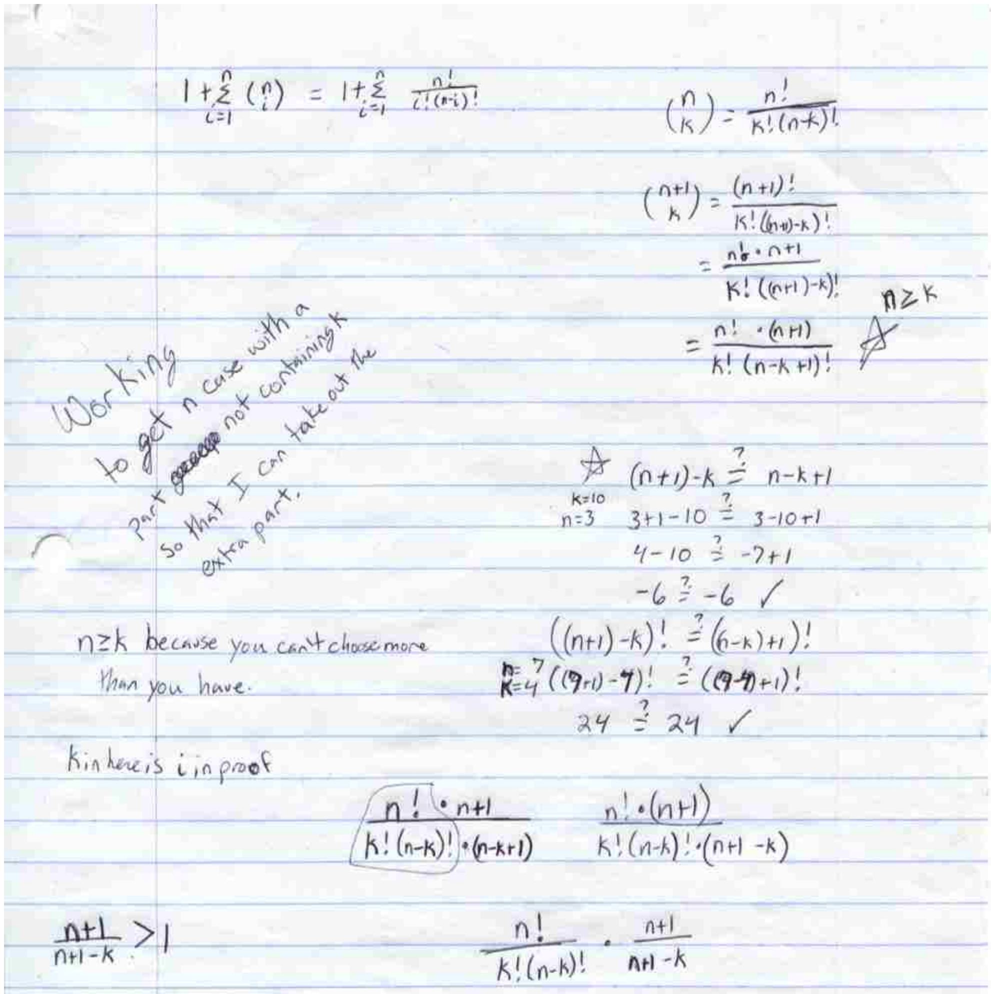


Figure 34: Mary's work on Question 7 (3 of 3)

While Mary used general algebra to verify what she has been using, she still tried some examples to convince herself that what she was doing was alright:

> M: So I could get the 1 out of there and so then I went through I went through a couple of trials to make sure that still worked, that what I was doing was legit, not…
>
> N: Right
>
> M: …and stuff. So that allowed me to take out the *n*! over *k*! times (*n* – *k*)! ...

Finding the expression was important to Mary because she felt it would allow her to use her induction hypothesis.

M: And I was still just trying to figure out a way to isolate the k so I could get that away from the extra little part...And then I just got it down to like where the extra part was n + 1 over $n + 1 - k$.

Because Mary was unable to get the k isolated from the rest of her induction hypothesis, she was stuck. I asked Mary if she had considered whether an explanation of the formula could serve as a proof. She said:

M: Well, I considered the fact that it would sort of have listed out here, how it would include all of that (the $\{a, b, c\}$ stuff in Figure 33), but I didn't know to prove that that was it, it was completed.

N: So, like, that's all that's included you mean, or…?

M: Yeah

N: Ok. So you don't know how to proof that this (the sum) gets everything, kind of?

M: Yeah.

N: Ok

M: Like how to prove that it's getting everything that you want, so yeah.

This was not the only potential issue Mary had with calling an explanation a proof. She went on to tell me about an experience she had in one of her classes:

M: I know like, I've known in the past I've been really good, especially after like one of the problems where it was the 3 consecutive integers. And then I had that thing in my number theory class…And I hand-wrote out everything that I, like, because you have (the product of) n, $n + 1$ and $n + 2$, that has to be divisible by 3 because one of them is

> divisible by 3 because it's 3 consecutive. And he was, like the teacher said "Well, you need to show it as a proof, you can't just explain it in paragraph form." So that's kind of …

Mary had been told by a professor that and explanation did not suffice for a proof in a different setting and she then assumed that it would apply to this setting as well. Mary did not know what she would do next, so we spent the rest of the interview going over a few different solutions to this problem.

Because Mary tried to give an induction argument, this qualifies as a procedural proof attempt. However, Mary did not follow specifically laid out steps, so this does not fall under the heading of algorithm but instead process. Mary knew of a few global steps that needed to be completed but did not follow explicit steps to complete them. Also, since she did not complete the proof on her own, this qualifies as an attempt only.

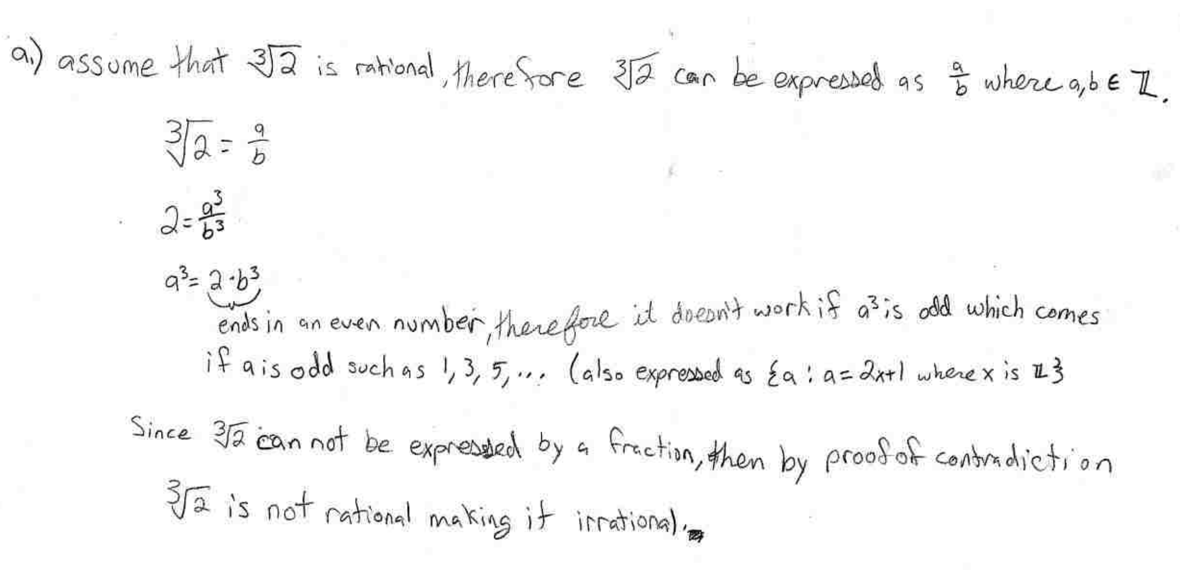
Mary shows some evidence of a couple proof schemes here. First, and probably most prominently, is the empirical proof scheme. A couple times during the course of her work, Mary uses empirical evidence to decide for herself what she can and can not do algebraically. She did so both to verify what she had done (in Figures 33 and 34) and also to decide what expressions should look like in the first place (Figure 32).

Mary also shows that she has an external proof scheme a couple ways. One, when she uses $n = 3$ as a base case for her induction argument, she is showing that she does not completely understand the method of induction. She uses it anyway because it has been established as an accepted way to proof something. This is evidence of a ritual proof scheme. Also, she is reluctant to use an explanation for her formula as a proof because a professor told her that an explanation would not work for a different problem. Here, she is relying on an outside authority to decide for her what can be considered a proof. This is indicative of an authoritarian proof scheme.

<u>Question 8</u>

Prove that the cube root of 2 is irrational using a proof by contradiction.

This problem was on Mary's midterm exam when she took MATH 305. On the test, Mary did not provide a correct proof. Her midterm response can be seen below.



a.) assume that $\sqrt[3]{2}$ is rational, therefore $\sqrt[3]{2}$ can be expressed as $\frac{a}{b}$ where $a,b \in \mathbb{Z}$.

$\sqrt[3]{2} = \frac{a}{b}$

$2 = \frac{a^3}{b^3}$

$a^3 = 2 \cdot b^3$

ends in an even number, therefore it doesn't work if a^3 is odd which comes if a is odd such as 1, 3, 5, ... (also expressed as $\{a : a = 2x+1$ where x is $\mathbb{Z}\}$

Since $\sqrt[3]{2}$ can not be expressed by a fraction, then by proof of contradiction $\sqrt[3]{2}$ is not rational making it irrational.

Figure 35: Mary's previous work on Question 8

Mary seems to make the case that assuming $\sqrt[3]{2}$ is rational leads to a contradiction because initially *a* can be either even or odd. Then one deduces that *a* must be even, contradicting the fact that it could be either or odd. Mary did not express this idea entirely, but it is an argument that others in the study provided and matches what she actually wrote. Therefore, I feel confident in saying that this is what she actually meant.

In the interview, I mentioned to Mary when she tried this problem on the midterm, she had access to the argument that $\sqrt{2}$ is irrational and that if she wanted, she could try that proof first. She declined however, reasoning that she did not know how to do either and they seemed to be equally difficult:

M: Like I tried to figure out in my head which way would be the easy, which way might give me more insight, or whatever, but I was like, "Well, if I go this way, I'll hit a block here and if go this way, I'll hit a block there, so…'

N: Ok, so you sort of figured they'd be about the same difficulty?

M: Yeah

Unlike most interviews, Mary did not need any help going over the concepts involved with this proof. She was able to get started straight away:

M: So the cube root of 2, in order to prove by contradiction that it's irrational, you're going to try to prove that the cube root of 2 is rational.

N: Ok

M: And so a rational number can be written as some *p* over *q*, where *p* and *q* are just integers, or…

N: Ok

(thinking)

M: So that's saying x^3 equals 2.

Introducing this new variable did not seem to help much, though, so Mary quickly restarted the problem: "So I went back to the cube root of 2 can be written as *p* over *q*, so it equals some number p over q. And then I multiplied both sides by q to get rid of the fraction." She does some manipulation, realizes that p^3 must be greater than q^3 for the expression to hold and tries to do something with that. See Figure 36.

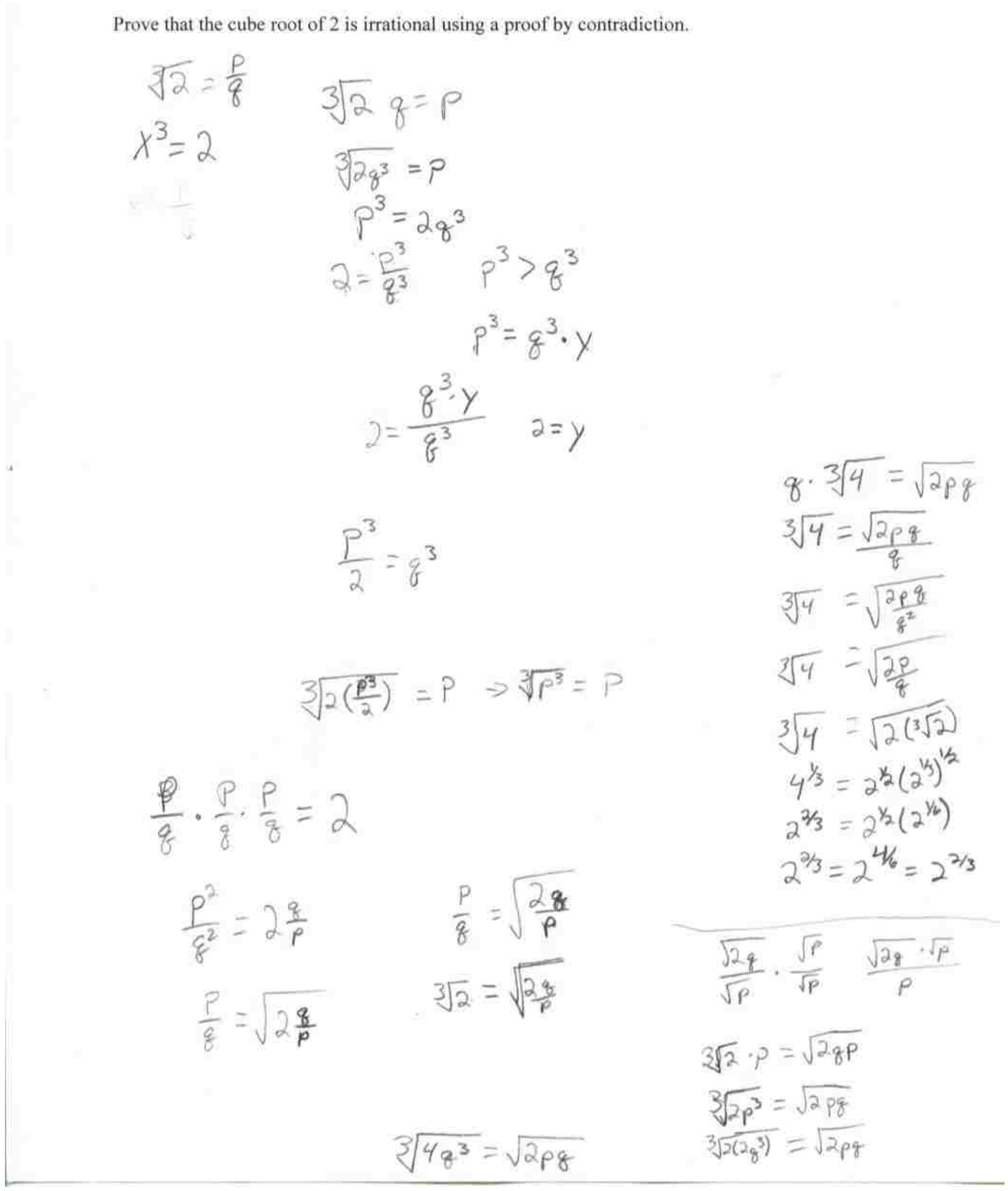


Figure 36: Mary's work on Question 8 (1 of 2)

This new variable (y) did not help much either, unfortunately:

M: In order to get a number that's bigger than one, since 2 is bigger than one.

N: Right

(thinking)

M: And in order to get it where it's a whole number, you'd have to have it where q^3 goes evenly into p^3.

N: Ok

M: So…

(working)

M: So then I just put that p^3 equals q^3 times some number y.

N: Ok

(finds that $y = 2$)

M: That didn't really work. So I plugged the q^3 times y in for p^3…

N: Ok

M: ..and so then I got 2 equals, or the quantity q^3 times y all over the quantity q^3. Which then would, the q^3's would cancel out and that gives you y equals 2.

Mary then goes back to the beginning of the problem: "So I'm going back and looking at different ways to write, like, the p^3 equals $2q^3$…Just to try and see, like, is there a way that way that I can get the contradiction to fall out." Mary then starts working on the algebra seen in Figure 36. She keeps running into the same problem, though, saying at various times "And that pretty much would bring me around again to going through and, like, all of the steps that I did up above were I…were we worked out pretty much that p^3 equals 2 times q^3." and "So now I'm just trying to look back through, like, all my work and see if something pops back out at me where I could re-write it where it wouldn't necessarily go back around in a circle."

Eventually Mary thinks she might have come to a contradiction:

M: And so I got the $\sqrt[3]{4}$ equals $\sqrt{2\left(\sqrt[3]{2}\right)}$…I think it's a contradiction, but I would have to do more to figure out, like make it more where you could see exactly.

N: Sure. So one way I suggest you might be able to do that is with fractional exponents.

She takes my suggestion and sees that this equation is in fact true.

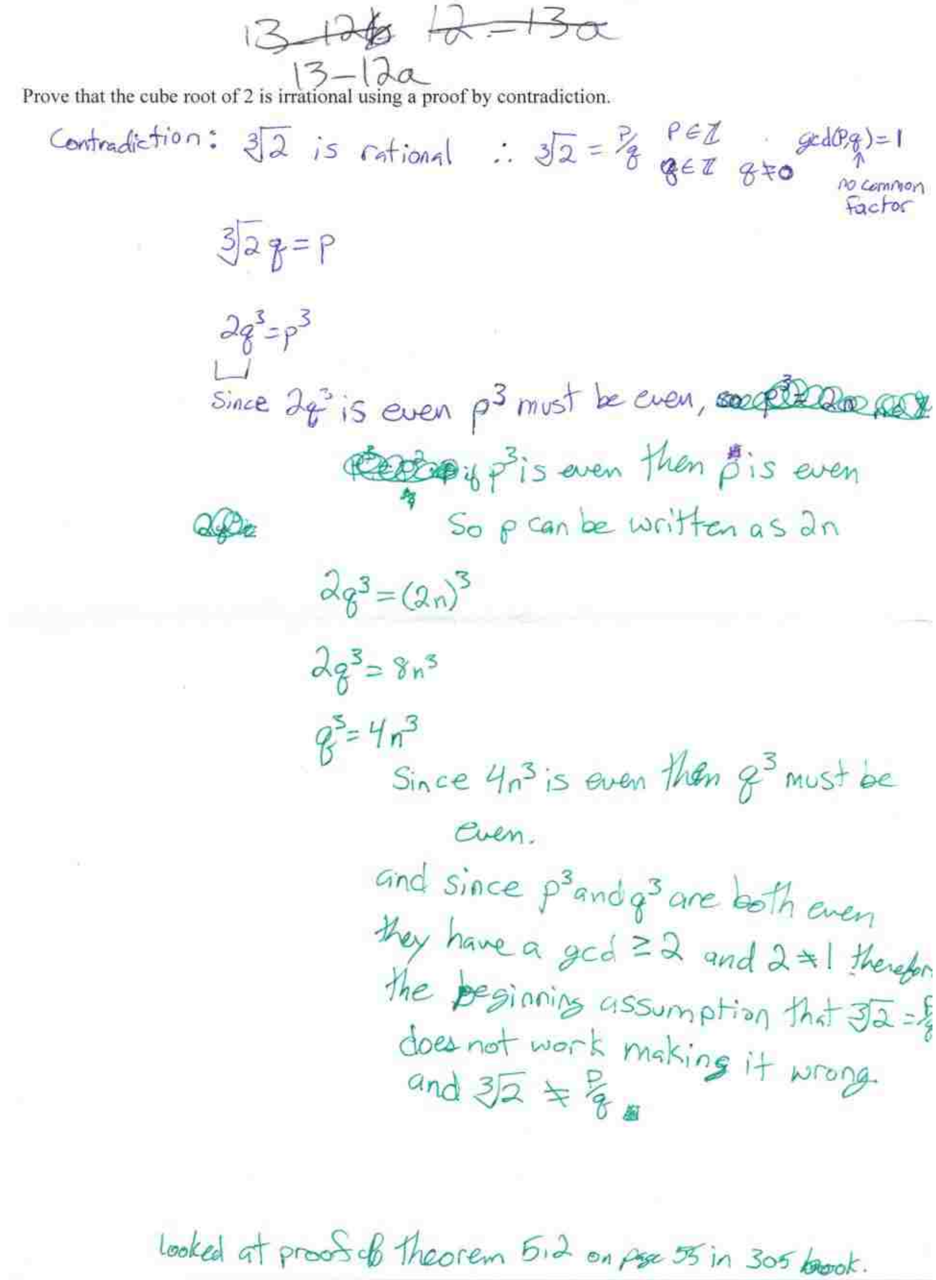
At that point it was time to reflect on what she had done. As she left the interview, I suggested that she go back to her notes from MATH 305 and look at the proof that $\sqrt{2}$ is irrational, because she had that proof available during the midterm, to see if that would help.

In between interviews, Mary did look up the proof I referred her to and she said it "helped out a whole lot." This is clear, because she brought me the proof that can be seen in Figure 37. As she notes in her proof, she did not look up the proof in her notes or on the midterm exam, but rather from her MATH 305 textbook.

At the time, I failed to notice that Mary found her contradiction in the fact that both p^3 and q^3 were even, not that p and q were both even. After having Mary explain the proof she provided, I asked her about one of the steps:

N: So the only thing I wanted to ask about then is the statement here that since p^3 is even, p has to be even. Did you give any thought to verifying that? Or is it just so obvious that you write it, or…?

M: Well, I just figured, I thought about it as an even times an even times an even…And that would always be an even, it's really hard to get an odd out of that.


13-12a

Prove that the cube root of 2 is irrational using a proof by contradiction.

Contradiction: $\sqrt[3]{2}$ is rational $\therefore \sqrt[3]{2} = p/q$ $p \in \mathbb{Z}$ $q \in \mathbb{Z}$ $q \neq 0$ $gcd(p,q) = 1$ no common factor

$\sqrt[3]{2}\, q = p$

$2q^3 = p^3$

Since $2q^3$ is even p^3 must be even, if p^3 is even then p is even

So p can be written as $2n$

$2q^3 = (2n)^3$

$2q^3 = 8n^3$

$q^3 = 4n^3$

Since $4n^3$ is even then q^3 must be even.

and since p^3 and q^3 are both even they have a gcd ≥ 2 and $2 \neq 1$ therefore the beginning assumption that $\sqrt[3]{2} = p/q$ does not work making it wrong. and $\sqrt[3]{2} \neq p/q$

looked at proof of theorem 5.2 on page 55 in 305 book.

Figure 37: Mary's work on Question 8 (2 of 2)

Although Mary does not provide a correct verification for the fact that p^3 is even implies p is also even, her response did tell me that she had considered making backing up that step. This is important. Because she based this proof so closely off the proof for $\sqrt{2}$, this proof qualifies as an algorithmic type procedural proof. There were precise steps that needed to be completed, and Mary completed them. Generally one would associate an algorithmic proof with an external proof scheme. However, this is not the case here because Mary makes sure that she is validating the steps herself. She does not consider the proof valid because she follows the steps provided or because it simply looks like a proof. Instead, she understands the operations taking place (or at the very least thinks she does) and uses the results of the operations to validate her proof. Thus, Mary is displaying a transformational proof scheme here.

Question 9

This question involved asked Mary to evaluate a version if Cantor's diagonalization argument. Thus, Mary did not complete or attempt a proof and therefore there will be no proof attempt to classify. During the course of the interview Mary did give some insight into her proof scheme, which will be discussed.

At the beginning of the interview, Mary and I spent a few minutes discussing what it meant for a set to be countable or uncountable. She had seen the topic briefly during MATH 305 the previous year, but not since then. After getting the definition straight, Mary read through the proof silently and when she was finished, I asked her to explain how it worked.

M: So it starts off by saying they're going to prove it by contradiction, so they're assuming that it is countable.

N: Ok

M: And then they broke it down where instead of looking at the whole thing, they're going to look at one small interval because if the whole thing is countable, then the small interval would also be countable.

N: Ok

M: And, so from there they figured out a way to like number, or list the numbers in the interval.

N: Sure

M: And then…from there they, like, picked out a number.

N: Ok

M: And that was the *B*.

N: Right

M: And they defined the digits of it, of that number…and then from there to, from there they went through and it's like 'Well, β_k is not equal to α_{kk}.'

N: Right

M: And that was, like that was the main point to proving the contradiction, like getting the contradiction.

I asked her to elaborate on this and it took her a bit to collect her thoughts, but eventually she was able to explain it:

M: And the other one, like, B is supposed to be in that list of functions.

N: Right, ok.

M: And so if you have the list, like the function list or whatever…

N: Yeah

M: …and it's, you find a number that's not included in that…

N: Right

M: …then it's hard to, then you can't prove that it's countable because it's not included.

N: Right, ok.

M; And then if you assume that it's included and then it shows that it's not, then your assumption is wrong.

Eventually, Mary was able to put the proof into her own words which led me to believe she understood the proof.

N: So you think it's true? Or do you think there's a hole in the logic or some [missing] detail or something like that?

M: I can't really think of anywhere where there could be a hole.

N: Ok. So you would say that this is a valid proof?

M: Yeah…I had a little bit of an expectation to find a hole, but…I tried to go into it with an open mind but kept an extra look out for…holes.

This exchange tells me that Mary is not showing an external proof scheme here. She is not convinced by the form of the proof or the fact it came from a mathematics authority figure (be it interviewer or the fact that it is a named proof). Instead, she works to understand the proof to determine its believability. It was simply a matter of Mary being able to sort through the notation and everything to understand the proof.

More evidence that Mary does not have an external proof scheme came from when I asked if she saw any hidden lemmas in the proof:

N: Is there anything like that in here that might need separate proof outside this proof?

M: Maybe a little bit with the subset, saying that's countable.

N: Ok

M: Like if you think about it, it kind of seems like it's commonsense, but you have to take a little bit of time to really think about it and be like 'Well, ok, I understand that.'

When she was confronted with a statement she was not completely comfortable with, Mary did not just take it for granted. Instead, she thought about it until she felt like she understood it and then moved on with the proof.

Because Mary wants to insure that she understands the proof before she makes a judgment about its validity, she is displaying an analytical proof scheme. However, her reasoning shows no signs of her understanding the axiomatic structure of mathematics. Instead, she is relying on the operations performed on and relationships between mathematical objects. Thus, this interview reveals that Mary has a transformational proof scheme.

<u>Question 10</u>

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, $6/9 < 5/7 < 7/9$.

Mary was able to dive right into the proof with this question, without needing to go through any terminology or anything. Since she was comfortable with the concepts involved, Mary went straight to the general case:

M: So I'm going to start off by naming the rational numbers for, that share common denominators, so like you could have *a* over *b* and *c* over *b*…And so then it's saying that $a + 1$ equals *c*…And so we're trying to prove or disprove that between *a* over *b* and *c* over *b* there's another rational number that has, like that could be *d* over *e*…where *e* is less than *b*.

Mary mentioned later in the interview, during the reflection, that she had no preconceived ideas about whether or not the statement would be true: "I just kind of figured, like if I started and went through, then it would either fall out one way or the other…I wasn't really going into it strictly to prove it or strictly to disprove it." Mary just figured that if she could manipulate the inequality in the right way, the need to verify or refute the statement would take care of itself.

The manipulations Mary performed can be seen in Figure 38. Mary made her way down the column on the left of the figure, then to the work in the top right of the page and then down to the bottom right of the figure. At various times, Mary saw that she was at a dead end, so she tried to move back within the problem to try a different path.

M: I was just trying to look up through my work to see if that will help me…So I'm looking at rewriting *e* as *b* – *n* where *n* can be any number between 1 and *b*.

N: Ok

M: Because if it was 0, then it would be *b* – 0 which is *b* which is not strictly less than *b*.

N: Yeah

M: And if it's *b*, then it's 0 which is strictly less than *b* but it's 0 and 0 in the denominator doesn't work.

Mary sums up her efforts by saying: "I'm just kind of like trying to manipulate it to see if something really, truly like spits out at me or…"

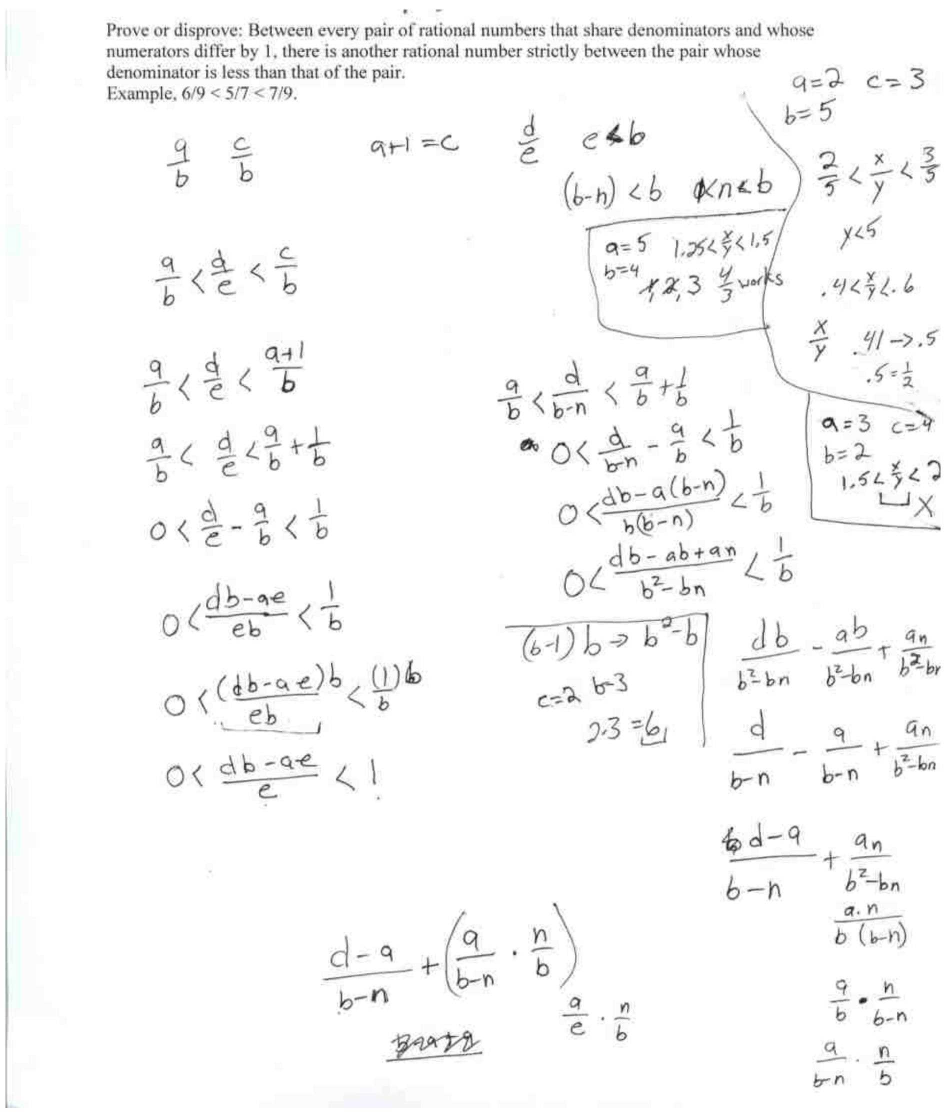


Figure 38: Mary's work on Question 10 (1 of 3)

Because nothing seems to be working, Mary goes back to the very beginning of the problem to start fresh.

M: Nothing's really sticking out to me, so I'm going to try, like you give an example 6/9 is less than 5/7 which is less than 7/9…So I'm just going to try like a couple more examples just to see, like …if anything pops out where, "Well, it doesn't work for this case", or stuff like that.

Mary begins by trying $a = 3$ and $b = 5$ and she sees that this example works. She then moves on to $a = 3$ and $b = 2$. This time, however, she is not so lucky.

M: So then I'm looking for a fraction between whose decimal value is between 1.5 and 2.

N: Yeah

M: Because 4 over 2 is 2.

N: Right

M: And I need y to be less than 2. And the only number that would be less than 2 that would work in the denominator would be 1.

N: Right

M: And there's no whole numbers strictly between 1.5 and 2.

Mary wonders if the fact that this example did not work had to do with the fact that 3/2 is greater than 1: "So now I'm going to try $a = 5$, $b = 4$ just to see if it was that case or see if another one over, where the fraction's greater than 1." After going through the possible denominators, Mary finds that 4/3 meets the criteria she is looking for. The leads her to conclude that: "So because I just found a fraction that worked there, then that shows me that it does work for most numbers over 1."

At this point, it was time to begin the reflection. I began by asking about whether she thought the result was true:

N: So, so you said, you found one example of rational numbers greater than 1 where it doesn't work and one where it does. But yet you said it works most of the time over 1, why did you say most of the time when you had one that works and one that doesn't?

M: Because I was just thinking, like, for the one that didn't work, I used a really low denominator.

N: Ok

M: And so there's a lot numbers higher than 2.

Mary had created some additional restrictions on the statement without mentioning it:

N: Ok, so then, so then you found this example where it didn't, doesn't work, so how did that fit in with the original statement, this example where a is 3, b is 2?

M: It kind of showed me, like…for a denominator, it has to be greater than 2, so…

N: Right, ok. So yeah, but I mean, I guess I'm just meaning, specifically in relation to the prove or disprove part.

M: That would be more towards, like, the disproving part, so…

N: Yeah, so would you say it's been disproved or not?

M: Yes, by 1 example. So unless you were to put the limitations of…the denominator having to be greater than 2 and…

Because she did not finish the problem, I asked that she try to work on it between interviews and she said she would. When Mary came back for the next interview, she brought the work that is shown in Figures 39 and 40. Besides a few changes in how she labeled the quantities in the inequality, her approached did not change much. One thing that was new was that instead of working with the double inequality, she broke it up into its right-hand and left-hand sides. Also, when she got stuck, she checked her work with an example she knew to work.

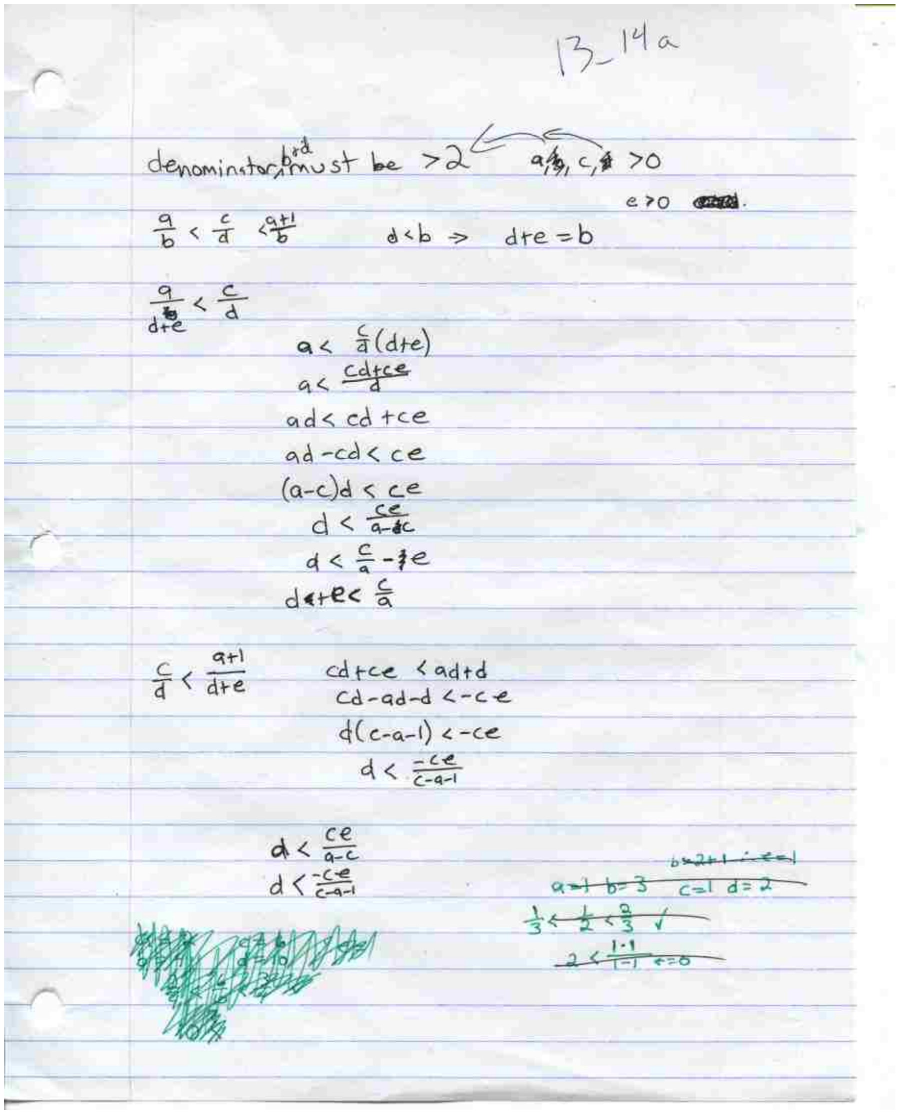


Figure 39: Mary's work on Question 10 (2 of 3)

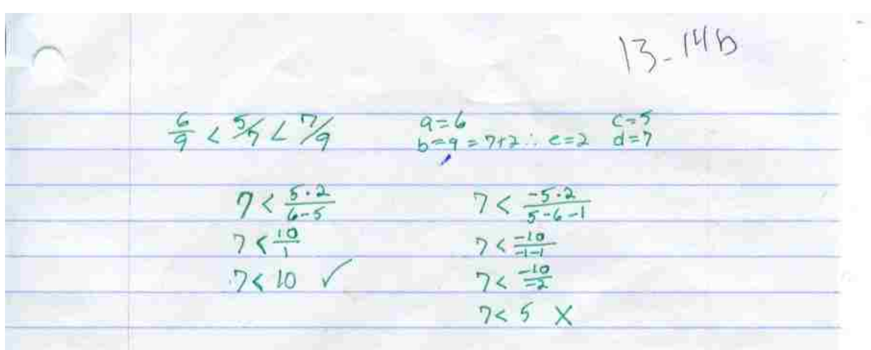


Figure 40: Mary's work on Question 10 (3 of 3)

Mary reached a stopping point with the problem when she checked the right hand side and found that it did not hold true with the example she expected to work.

> M: And *c* is 5 and *d* is 7. And then I went through and it worked for the left hand side I got 7 < 10, but on the right hand side I got 7 < 5, which is a contradiction. So I don't know if that was, I tried to go back over my algebra and stuff like that, and that took a lot of time and I just couldn't find it, so…

The reason Mary ran into trouble was because when she divided by $(c - a - 1)$ in Figure 39, she neglected to consider what would happen if this quantity was negative. It happens to be in the case Mary tried.

At this point, Mary did not know what to try next. In an effort to save time for the debriefing session that was also to take place that interview, I guided Mary through a proof of the problem given the proper restrictions.

Mary did not provide a complete proof, but it is possible to classify her proof attempt. While Mary spends some time looking at examples, the bulk of her time spent on performing algebraic manipulations. As such, she makes no overt effort to gain an understanding of the problem that might be turned into a proof. Thus, this is a syntactic proof attempt. As might be expected, the proof scheme Mary shows here is a transformational one. Her potential proof is based on performing operations that she hopes will lead to a proof. She is not relying on any external or empiric evidence, which means an analytic proof scheme is at play. Since her proof attempt relies on no previous results besides algebra properties, this proof shows that Mary had a transformational proof scheme.

Question 11

The last interview of the study was used to talk to Mary about the study overall as well as tie up the loose end of Question 10. During the interview, Mary mentioned a few things that had changed over the course of the study. One of them had to do with examples: “I noticed that I kind of quit doing examples right at the beginning (of problems).” When I asked if she knew why that was, she was not sure:

M: Not really, I just, like if I get stuck I’ll use them.

N: Yeah

M: But, a lot times when I was reading through the problem I would think of a way that would come to me to try out, like, to try out or something. Like while I’m reading it or whatever, I’ll be like ‘Oh, well I can do this’ or ‘I can do it that way’ and so…

N: So you sort of feel like you have less of a reliance on them?

M: Yeah

This was something that I had also noticed about Mary’s attempts over the course of the study. It is also something I felt like had hindered her while attempting some of the proofs, question 10 in particular.

Another thing that came out of the interview that matched something I noticed during the study was that Mary always made sure she understood a problem completely before she started it. This came up when I asked her what was necessary to complete a proof. The first thing she said was:

M: I think you definitely need an understanding of the problem for sure, like…

N: Ok

M: And if it's a problem where you're trying to, like, prove something with mod or something like that have a good understanding for that mathematically, like…

N: …the operation of modular arithmetic or whatever's being dealt with?

M: Yeah

For most of the interviews, Mary and I would spend the first few minutes going over the terminology and concepts involved in the problem.

The last thing Mary mentioned reinforced what was observed when she looked at completed proofs during the study. When I asked if she had implemented anything new into her proof techniques over the course of the study, she said:

M: Like, a lot of the problems where it was like prove or disprove or whatever I go into it with an open mind, I don't really try to like go at from one angle expecting to prove it or expecting to disprove it or anything, just…

N: Oh, ok. And that's kind of a change because before you were just expecting to prove it maybe?

M: Kind of, yeah

This was something she mentioned while going over Cantor's diagonalization argument. It also shows that Mary has an analytic proof scheme. She does not give credence to a proof simply based on what it looks like, who gave it to her or who supposedly wrote it. This is evidence for Mary's transformational analytic proof scheme because at no point in the study does she refer to the axiomatic nature of mathematics.

Mary's progression

Below is a chart containing the classifications for Mary's various proof attempts and the proof schemes she displayed throughout the study.

Question	Type of proof	Proof scheme
1	Semantic	Transformational, Empirical
2a	Syntactic	Transformational
2b	Semantic (Attempt)	Transformational
3	Process (Attempt)	External (Ritual)
4	N\A	Transformational
5	N\A	Transformational
6	Semantic (Attempt)	Transformational
7	Process (Attempt)	Empirical, External(Ritual & Authoritarian)
8	Algorithm	Transformational
9	N\A	Transformational
10	Syntactic	Transformational
11	N\A	Transformational

Table 2: Summary of Mary's work

When looking at the chart, it seems like Mary did not make steady progress in either the types of proof she produced or in her proof scheme. The proof scheme she displayed was fairly consistently transformational. This is good because it shows that she has a deductive view of proof for the most part. However this view is not ubiquitous. Even relatively late in the study, Mary demonstrates signs of both an empirical and external proof scheme. Also, Mary does not ever display evidence that she understands the role of axiomatics in mathematics. Of course, this does not mean that she lacks this understanding. However, I think such an understanding would be beneficial for a student in her position.

It is unclear how much of an impact her proof scheme had on her individual proof attempts. The type of proof attempts she provided varied and was often dependent on the type of problem at hand. It is also unclear whether her proof scheme had any influence on whether or not Mary happened to be able to provide a correct proof. In the cases where Mary was unable to complete a proof, it was usually a matter of her either not making use of the proper relevant facts (for example, Question 3) or not trying enough examples to give her a pattern to work with

(Question 10). In either case, it seems unlikely to me that a more axiomatic view of mathematics would have led to successful proofs in these circumstances.

4.3 Will

This case study looks at the progress that Will made throughout the course of the study. Will majored in Physics. During the first semester of the study, Will's mathematics classes were Euclidean and Non-Euclidean Geometry and during the second semester he took Partial Differential Equations.

<u>Will's Proof Attempts</u>

<u>Question 1</u>

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Will started this problem by going directly to the equation $2l + 2w = lw$. From there, before trying any examples, he started manipulating the equation:

> Will: Well, actually, I'm just kind of playing with it…is what I'm doing when I don't know what I'm doing. I like to play with the equations if I can't really think of where to go next. …Because if you're playing with it, at least you're doing something.

From there he starts considering whether the equation can ever be satisfied in a non-trivial way:

> Will: Well, I'm just sort of trying to think a case where it's true…so far all I have is zero (laughs)…Oh! Can I use a calculator to graph this?
>
> Nick: Yeah, absolutely.
>
> W: Awesome. Then, I'm going to solve for *l*.

Will had already set the equation equal to 0, a standard method of solving an equation in one variable. This reliance on established algebraic techniques was also evident as he moved on with the problem:

W: Ok…That's not really giving me anything…

N: So did you get it to graph?

W: I got it to graph...but…

N: Ok…do you notice anything about it?

W: Well it's only got one zero (x-axis) intercept which is at 0.

Again, Will is using a typical algebra method to solve an equation: he is looking at the graph of the function and trying to find its roots. He found the lack of roots discouraging and began looking mentally checking examples of squares to see if any of those worked:

W: I kind of expected the zeros to be there so I could just play with them, so…Right now, um …I don't know. It seems like …zero is the only one because anything larger than zero, I mean I've tried, one is even 1 1 1 1 is 4, 1 times 1 is 1. 2 2 2 2 is 8, 2 times 2 is 4. …3 3 3 3 is 12, 3 times 3 is 9 – I'm getting closer…Like 4 is 4 4 4 4 , uh is 16. 4 times 4 is 16. That's it! It works.

From there, I asked if the solution he had found had anything to do with the graph. "4? Um, no, not really… I mean it's just kind of on the downward slope…Yeah, the graph is not giving me anything on that."

After a little more work trying some examples with the equation, I directed Will's focus back to the graph:

W: The pair 4 4 works. I'm just having trouble finding another one that works.

N: Ok, so let me, um, just for the sake of time, so the pair 4, 4 makes this equation true, right?

W: Right.

N: So does that tell you anything about the point (4, 4), like whether or not it should be on the graph?

W: Well…it should…hit the graph, right? … Oh, I could just do an input…input output table, right?

Will then proceeds to create a table which displays only integer inputs.

W: So the table start at 1 and have it go up by 1 so that…should give me a list of all integers for *x*…and we'll have positive integers…so we'll have 1 and -2 but that…That makes the equation true, but -2 can't be a length…So, uh, 3 and 6, 4 and 4, 6 and 3, of course…

N: Ok so you skipped 5, didn't mention 5, why not?

W: Well because it doesn't have…it doesn't work…it's got a decimal, so that's not an integer length. …So, unless this approaches 2… Maybe I can prove there's an asymptote at 2…that'll show that I've shown all possible solutions.

From there, Will took the limit of the equation $\frac{-2w}{2-w}$ as *w* approaches infinity and saw that the limit was 2 (in his written work he wrote 2*w* as the numerator, but mentioned -2*w* when discussing what he was doing).

W: So, this goes to 2, so I know that it'll approach 2 but never cross it, so…Yeah, so, looks like that pretty much proves it….but I mean, I could write down more of a proof, but…I mean that pretty much shows that, um, I've got the only two answers, 4 and 4 and 2 and 6, I mean, 3 and 6.

He brought a completed proof to the next interview in which he provided pretty much the same argument. In it, he did not go through the work of showing that $l = 2$ is an asymptote but he did mention it. See Figure 41 below.

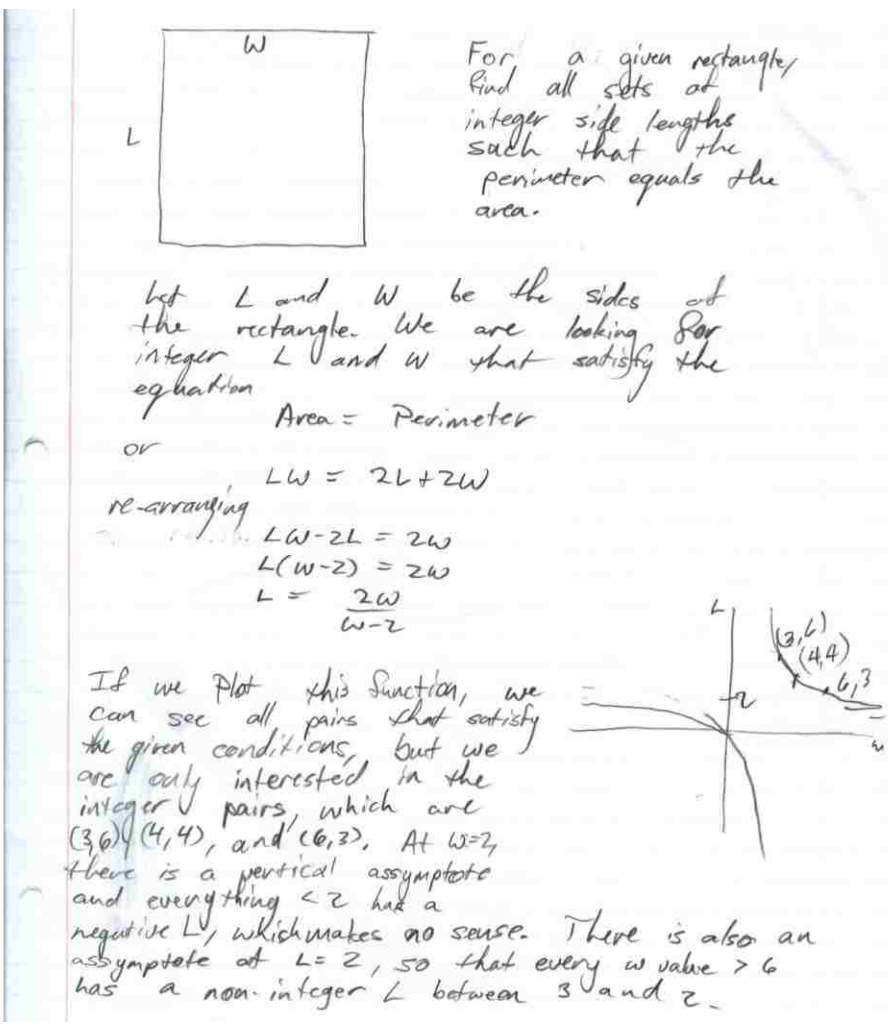


Figure 41: Will's work on Question 1

The proof Will provides is a semantic proof. Although the methods he employs are rather typical algebra arguments, he mentions early on that his proof will come from an understanding of the problem. When I asked if he had any ideas of how to start the problem, Will says: “Yeah, I’ve got an idea…proving it, uh I don’t know how much of a proof you want, but… I can tell you, just try to figure it out first. And then prove it from there.” This can be seen in the proof also. It was not until Will had an understanding of the problem that he provided a proof.

The proof scheme Will displays here was a transformational proof scheme. He operated on the equation by solving for one of the variables and treating the equation as a whole as a function of the other. He finished the problem by them considering how that function behaved as the input changed, specifically as it approached positive infinity.

Question 2a

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9.

Will spent the first few minutes working on the problem going through the details of what the problem was really asking. Since he was the first person to attempt the problem, the statement he saw did not make clear the fact that “*ab*” was meant to describe the two digit number with *a* in the 10’s place and *b* in the 1’s place rather than *a* times *b*. Also, he went through a justification of why one only needs to consider the case when *a* is greater than *b*.

> W: And *a* and *b* are not equal?...Well I guess it’s 0 times 9, so I guess that is fair. Um, gosh, this is kind of uh…Well, I suppose if you flipped the order, like, 27 – 72, is negative, make it 72 – 27...I could may be say the absolute value of the difference is equal to some positive integer *n* times 9.

After getting those ground rules straight, Will begins working on the problem. He starts by looking at the subtraction *ab* – *ba* in general terms.

> W: I'm not really used to working with numbers like that, so…I'm trying to kind of break up the subtraction so I can say, um, oh, wait, if *b* is smaller than *a*, then you're going to have to make a substitution, so this is going to be, um, *a* – 1 minus *b* is going to be you're first digit, we'll call it *c*… And then *b* minus *a* is going to be *d*…this is *b* plus 10 right here…minus *a*. And *a* – 1 – *b* so I want to prove that, um, ok I know that there's a rule that for every multiple of 9, the sum of *c* and *d* has to be 9.

Will is referencing a rule for checking a number for divisibility by 9 (a number is divisible by 9 so long as it's digits add up to a multiple of 9). He then takes the expressions he had for *c* and *d*, adds them together and sees that the result is always 9:

> W: Shoot, the *a*'s cancel out and the *b*'s cancel out. And I get 10 – 1, 9 equals 9. And that's an equation that works, but it says that *a* and *b* are irrelevant here…Which I guess is kind of what I want to say, right?... No matter what *a* and *b* are, you get this equation and it works.

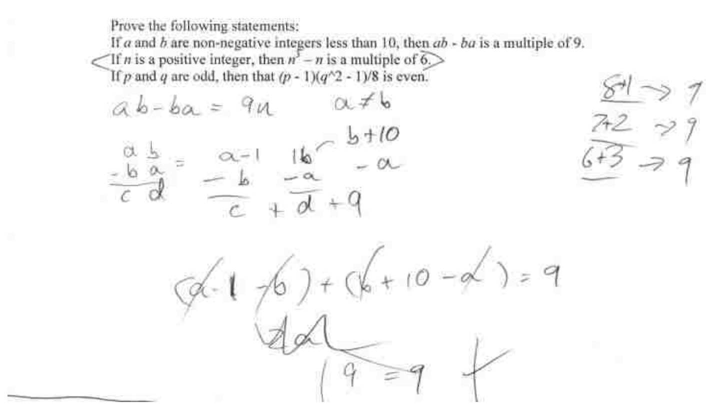


Figure 42: Will's work on Question 2a

Will was a little unsure that he had actually solved the problem at this point. However, it did not take him long to decide he had solved it, "…assuming that this rule works, which I've sort of explained does, so yeah. We're getting 9 = 9, that's a true statement." It should be noted that Will did not really explain why the divisibility rule worked, but he did display confidence that it does.

There were still a few issues for Will to consider. First, $c + d$ did not equal 9 in the case of $a = b$. However, this case was easy to check. Will had also earlier made the case that all 2 digit multiples 9 had digits that summed to 9, with the exception of 99. However, he pointed out that this could be dismissed because no combination of a and b had a difference of 99.

This proof is a syntactic proof. While it does rely on Will's understanding of place value, he does not use any sort of representation to gain understanding into the problem which is then turned into a proof. He does, however, manipulate expressions in a logically permissible way to construct his proof.

Will gives evidence for the presence of both types of analytic proof schemes here. The bulk of the proof points to a transformational proof scheme because the bulk of the proof is manipulating algebraic expressions. However, Will demonstrates that he understands the reliance of his proof on the divisibility rule when he says "assuming that this rule works…" This also might be construed as evidence for an external proof scheme because he never actually provides adequate verification of the rule. Although that is the case, Will does not find the divisibility rule to be valid because he has been told by an authority figure. He believes the rule holds because he believes he understands why the rule holds. It is possible that he has faith in the rule simply because he has seen it work ever since he learned it. If this is the case, that would constitute an empirical external proof scheme. There is no evidence that this is the case, however likely it might seem.

Question 2b

If n is a positive integer, then $n^3 - n$ is a multiple of 6.

Will started by discussing different ways to go about it. He mentioned graphing $n^3 - n$ and $\frac{n^3 - n}{6}$ together to see what happens (he did not try this because he said his calculator was not working right) and also trying to do a proof by contradiction (he said he could not make sense of saying something was not a multiple of 6). Not knowing how to go about solving the problem, Will mentioned he was not even sure if it were true: "Yeah, I'm not convinced that's a fact, I kind of want to see why it's like that before I try to say, well, this is why, so…Yeah, so I'll try to figure this out…"

To do this, he tried to think if there were any quick multiplication tricks that could be used in this problem. He also mentioned that the difference of multiples of 6 would be a multiple of 6 also. This led him to consider induction:

> W: So…let's say n^3 and…n and $n + 1$ are my two integers…Oh! I can do this by induction! Cause if I can prove that every step is a multiple of 6, by one step is a multiple of 6 and find some base case, where it's a multiple of 6, then I can prove that… that it's true for all….(n^3 – n) minus (n + 1)^3 – (n + 1)…

The choice Will made for the 2 integers lead him to think of induction, but the idea that led him there did not go away, even if it was not the way induction needed to be applied to the problem. He continued to look at the difference (n + 1)^3 – (n + 1) minus (n^3 – n) and was confused with what he was trying to show and what he was assuming (for example, he placed *n* with 6*q* in one expression even though he had assumed *n*^3 – *n* = 6*q*, see Figure 43).

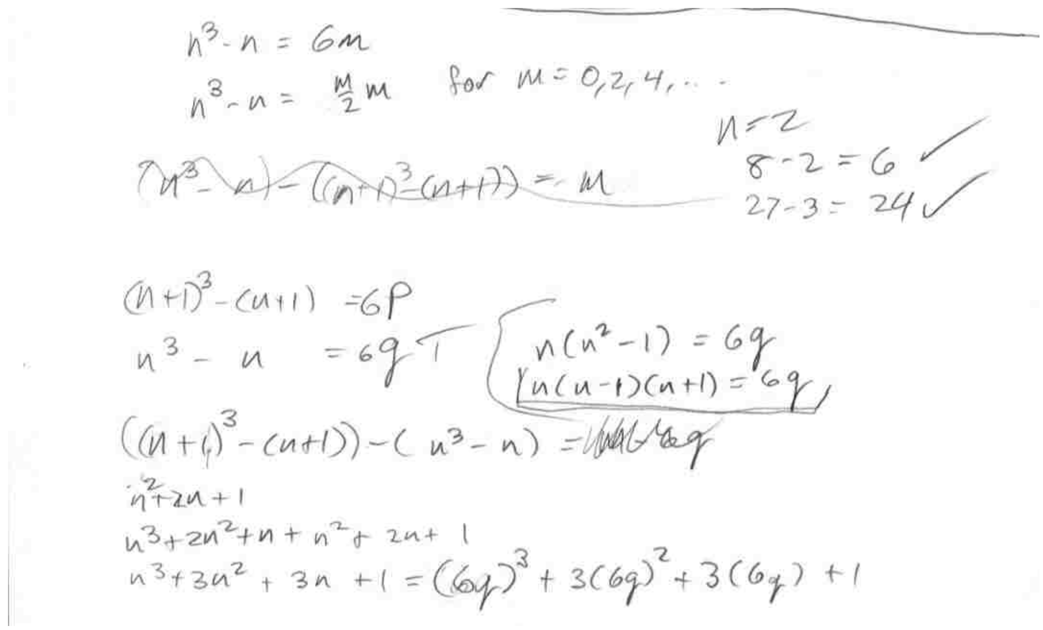


Figure 43: Will's work on Question 2b (1 of 2)

Eventually, through his algebraic manipulations, he factored *n*^3 – *n* in an effort to isolate *n*:

> W: I factored it to…$n^2 - 1$ is $(n + 1)(n - 1)$…I know the roots to make this zero, but…this hasn't really helped me to show it's a multiple of 6, does it?

At this point, Will asked for a hint and I suggested he look at this latest expression with a particular number plugged in. He then tried $n = 4$ and $n = 3$, realizing that this gave him 3 consecutive integers. After that, he replaced $n(n + 1)(n - 1)$ with $(n + 1)(n + 2)(n + 3)$ to make this more apparent. He thought that this new form might be more conducive for induction and left the interview with that idea in mind. When Will came to the next interview, he had the problem figured out.

W: So for this one, as soon as you realize that these are three consecutive integers, you know that you have … one of the integers has to be a multiple of 3…Because in any of these over here, in any interval of 3, there's always a multiple of three. You can see it here (see Figure 44) I didn't really go into much of a proof for that, but…

Will also provided a written up version of this argument. In his write up, he treated the numbers as if the multiple of 2 and multiple of 3 were always different numbers. He did not consider the case where one of the three is both.

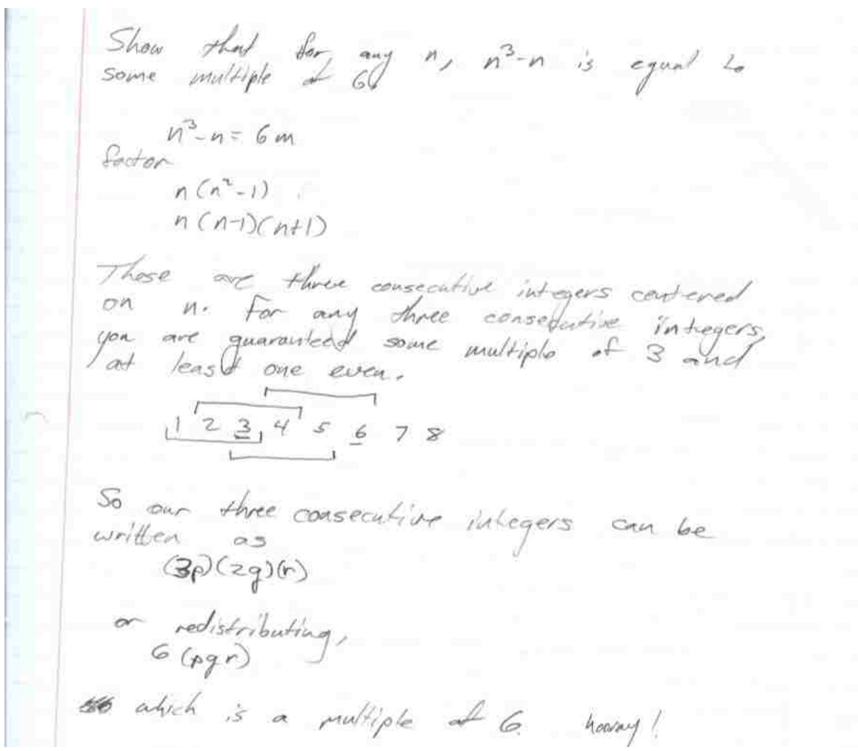


Figure 44: Will's work on Question 2b (2 of 2)

This is a semantic proof attempt. Although most of Will's time is spent working towards an induction proof, the proof he actually presents comes about only after he understands the form of the numbers in question. Once this was understood, it was turned into a (fairly) formal proof.

Through his work on this problem, Will displayed a transformational analytic proof scheme. He performed operations on the algebraic expressions at hand (via factoring) and used the results of those operations to prove the result.

<u>Question 3</u>

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$

Will had little trouble starting this problem. The one issue that did come up was that he used a base case of $n = 2$ instead of $n = 1$. This was a common issue with the students in the study. Other than that, Will performed the necessary steps of induction quite well. He started with the right–hand side of the inductive step, re-writing it as

$$1+\frac{n+1}{2}=1+\frac{n}{2}+\frac{1}{2} \quad \leq \quad \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}+\frac{1}{2}.$$

From there, however, he got a little stuck in comparing this new right-hand side to

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{n+1}}.$$

W: So…now I'm just trying to see what the other side is, the $\frac{1}{1}+\frac{1}{2}$ da da da + $\frac{1}{2^{n+1}}$. Basically, I want to show that the difference from $\frac{1}{2^n}$ to $\frac{1}{2^{n+1}}$ is less than $\frac{1}{2}$, I think…right? Or is greater than $\frac{1}{2}$.

Will had arrived that the difficult part of the problem. He first needed to address what was going on between the terms $\frac{1}{2^n}$ and $\frac{1}{2^{n+1}}$:

> W: So, the integer gap from 1 to $\frac{1}{2^n}$, is going to be the same as the integer gap from 2^n to 2^{n+1}. So…I still haven't showed that second gap is greater than $\frac{1}{2}$. How do I do that in general? (see Figure 45)

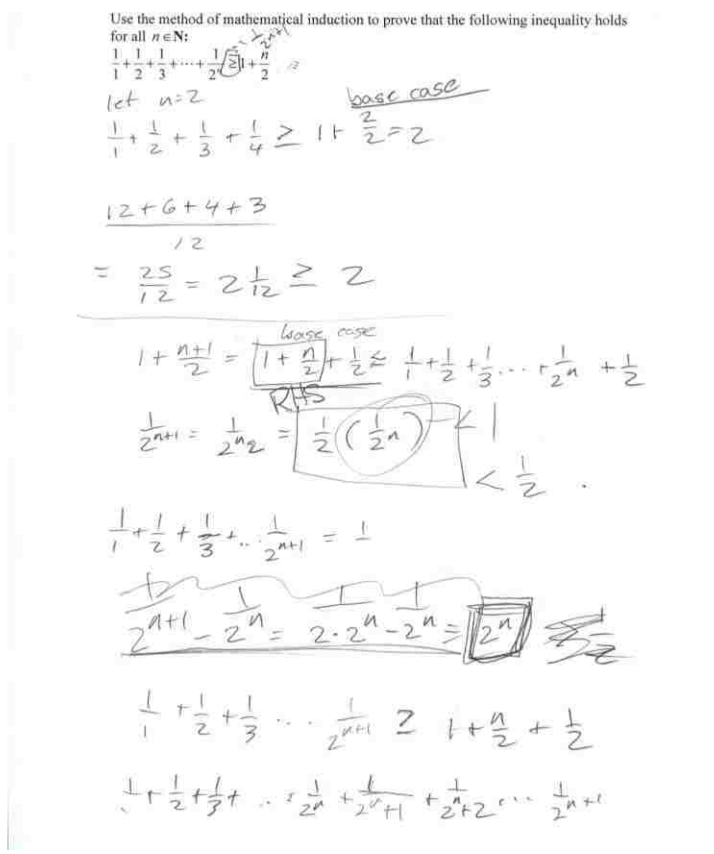


Figure 45: Will's work on Question 3 (1 of 2)

Will left that interview knowing what he needed to do to complete the problem. He just did not know how he was going to do it: "And then I just have no idea where to go, how to do that sum in general terms. I mean, without finding a least common denominator and multiplying everything together, which is not going to be pretty."

In between interviews, Will did not get a chance to look at the problem. He picked up where he left off the nest time we met. Because he did not work on it since the last interview, he began where he said he would. Specifically, he looked at taking the least common denominator of the left-hand side of

$$\frac{1}{2^n+1}+\frac{1}{2^n+2}+\frac{1}{2^n+3}+\cdots+\frac{1}{2^{n+1}}\leq\frac{1}{2}$$

for simplification purposes. During the reflection, Will said, “I was looking for something I could pull out or get rid of and throw away and just have something there that was neat. I was kind of looking for neatness, is what my motivation was.” He did not see anything that made the sum any simpler and asked for assistance completing the problem. We then talked through the problem together to finish it. See Figure 46.

Because this was an induction problem, the proof attempt is of the procedural type. Of the sub-types, this attempt was a process proof. Here, Will did not complete strictly laid out steps, but rather attempted a few global ones. He performed a base case, made his induction hypothesis and applied it to an inductive step. The characteristic that really separates this from an algorithmic procedural proof is that completing the inductive step did not have a set algorithm that would guarantee success.

Will shows some evidence of a couple different proof schemes here. First, since he is adhering to the method of mathematical induction, it may seem that he has a ritualistic external conviction scheme. However, this scheme is reserved for those that become convinced of a proof because of the steps being complete look as if it is a proof. However, here Will understands the steps being completed. This is a type of transformational proof scheme (what Harel and Sowder call an “interiorized proof scheme,” p 264).

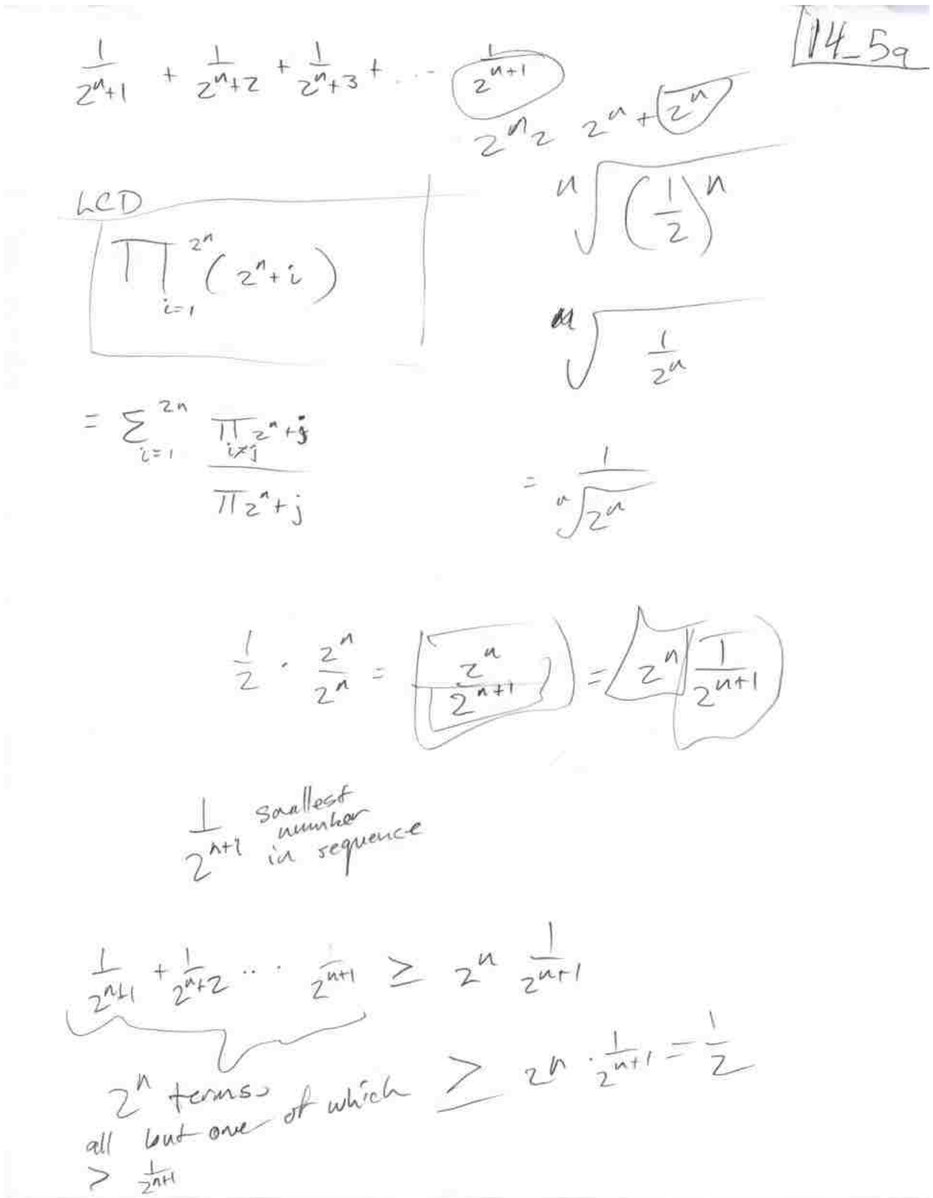


Figure 46: Will's work on Question 3 (2 of 2)

More evidence for a transformational scheme can be seen when Will tries to simplify $\frac{1}{2^n+1}+\frac{1}{2^n+2}+\frac{1}{2^n+3}+\cdots+\frac{1}{2^{n+1}}$ in order to get it more comparable to $\frac{1}{2}$. When explaining this, he said "Yeah and I mean, in physics, if you have 2 really complicated, ugly things, you can usually you know, work them down, simplify them out a little bit and you'll find terms that'll drop out." Will is speaking of performing operations and anticipating the results of the operations. This method did not work this time, but it does show what he thinks it takes to complete a proof.

Question 4

Examine the following proof by contradiction:

Prove $\sqrt{-1} \leq 0$

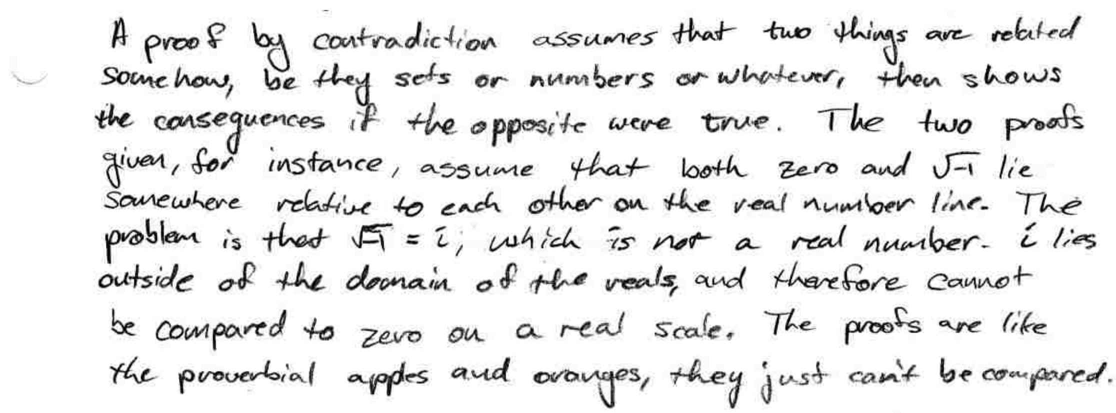
Suppose $\sqrt{-1} > 0$*, then* $\sqrt{-1} \times \sqrt{-1} > 0$*. This implies* $-1 > 0$*, which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

Prove $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$*, then* $-\sqrt{-1} \geq 0$*. This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$*, so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This problem was first given to Will on his midterm exam when he took MATH 305. On the midterm, he made the case that using a proof by contradiction makes the assumption that 2 things are related somehow. In the case of this problem, the number *i* is not on the real line and thus not relatable to 0 in the sense required: since "*i* is not in the domain of real numbers…it cannot be compared to 0 on a real scale."



A proof by contradiction assumes that two things are related somehow, be they sets or numbers or whatever, then shows the consequences if the opposite were true. The two proofs given, for instance, assume that both zero and √-1 lie somewhere relative to each other on the real number line. The problem is that √-1 = i, which is not a real number. i lies outside of the domain of the reals, and therefore cannot be compared to zero on a real scale. The proofs are like the proverbial apples and oranges, they just can't be compared.

Figure 47: Will's previous work on Question 4

Will remembered his answer and did not change it in the interview: “I think the argument I made for this one is that… it’s an imaginary number and it’s outside the set of the reals, so the comparison to the reals was not working.” The interview setting did allow me to go into more depth with the question than was seen on the midterm. I asked Will what his answer meant for the rest of the argumentation provided. His response:

> W: Well, I guess it just kind of says that…I mean, you can make this argument, $\sqrt{-1}$ is *i*, by definition, so what you’re saying is *i* is less that or equal to 0, *i* is greater than 0, I mean, you can say that all you want, but that doesn’t mean anything…if there is no interpretation for what $i > 0$ is, then it’s a completely irrelevant argument. So even though this looks fine written out here, it just, you’re not saying anything…

Because Will did not attempt a proof, there is no proof type to be dealt with here. However, he does reveal an analytic proof scheme. When talking about what to make of the proof provided, Will was aware that it was meaningless because it was based on something that was meaningless. Because Will recognizes this dependence on starting assumptions, he is displaying an axiomatic proof scheme.

<u>Question 5</u>

The following interview was the last interview of the first semester of the study and served as the first of two debriefing sessions. Like the last interview, Will did not attempt a proof so there is nothing to classify as a proof type. However, Will did mention something that has shown up while classifying previous proof attempts. When I asked if he felt like he had improved over the course of the study, Will said "I think it's just knowing what I'm actually doing and being able to make that plan rather than just jumping in and 'that didn't work,' do something else 'that didn't work.'." This attitude towards how to go about starting a proof is evident by the number of his previous proofs that were semantic in nature.

Will also provides evidence for the proof scheme that showed up most often during the first half of the study: the transformational analytic proof scheme. When discussing his improvement, Will said:

> W: I have a better idea of how to start on a proof…And what kind of, I mean, I'm still doing the same things I've always done but now I have more of an idea of what I have and I can think about it a little bit better and plan it a little bit better.

This is evidence for the transformational scheme because he feels like he has gotten better at anticipating the results of the things he does while completing a proof.

He also provides some weak evidence for an axiomatic proof scheme. While discussing the sorts of proofs he does in his physics classes, Will says that they start with basic principles, like Newton's laws, and work from there:

W: So you'll have to take stuff, fundamental blocks in physics and you'll need to use those to build up into the bigger equations. …Take what you have, put it together and see what, see if works, see if you can see something that shows why this happens.

N: Yeah. And I think that serves as kind of a nice analogy for math also, except on the math side of it, those fundamental blocks of physics knowledge are kind of replaced with axioms and you kind of deduce from there.

W: Yeah

I called this weak evidence because Will merely agreed with something I said; he did not articulate it himself. However, I do think that he really does see this parallel between mathematics and physics. For this reason, I feel Will is displaying the axiomatic proof scheme.

<u>Question 6</u>

Prove that $\forall n \in \mathbb{N},\ n \equiv n^3 \, (mod\ 6)$.

Will needed a review of modular arithmetic before getting started on this problem. He also worked a long division problem to help re-familiarize himself with the roles of the divisor, quotient and remainder.

W: So *n* over 6 is going to be some integer *m* + *a*, let's say, if that's remainder *a*…remainder is what doesn't get taken out…[tries 16 divided by 5 for reference] So I have one fifth. Ok, so I'm dividing by 6 so I'm going to have some *a* over 6, is going to be my…Ok, so *n* over 6 is going to equal to some other integer *m* plus *a* over 6. So now I want to prove that n^3 over 6 is going to be equal to some other integer *p* plus *a* over 6. So let's take this *m* plus *a* over 6 and cube it…

Will then proceeds to cube $m+\frac{a}{6}$, his expression for $\frac{n}{6}$, in the hopes that the expanded version of $\left(m+\frac{a}{6}\right)^3$ will be comparable to $m+\frac{a}{6}$. As he said, "That doesn't help as much as I hoped…" (See Figure 48).

From there, Will takes a step back and, after some thought, says:

W: I haven't shown that this works yet, so let's try 2. Or, let's try something greater than 6, let's try 7. It equals 1…So, it's 343 over 6, it's 30, it's 5…7 and 1. So *r* is 1. So yeah, I get the same remainder for that one. 57 plus one sixth. … So it works.

Working this example makes Will realize that a different characterization of the problem would work better:

W: Uh oh. I just cubed *n* over 6 instead of just *n*. I don't want to do that. So this…I need to try again. Ok, so *n* over 6 equal to *m* plus *a* over 6. So *n* is equal to 6*m* + *a*. Now that's what I want to cube.

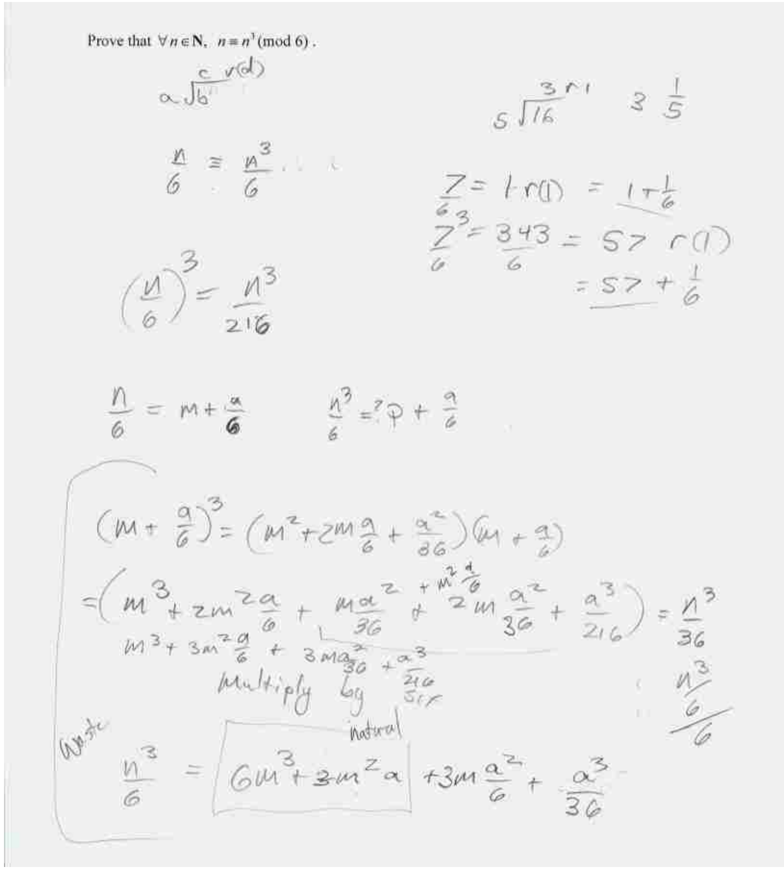


Figure 48: Will's work on Question 6 (1 of 2)

Will goes about doing this and gets to the middle of page 14_8b (Figure 49).

After performing the cubing, Will divides each side by 6. He realizes that many of the terms are divisible by 6, so he collects those together and labels them *p*. He sees that the a^3 over 6 term is not what he needs and he addresses that by adding and subtracting *a* over 6. “So a^3 over 6 minus *a* over 6 is an integer and I’ll have my *a* over 6 remainder.”

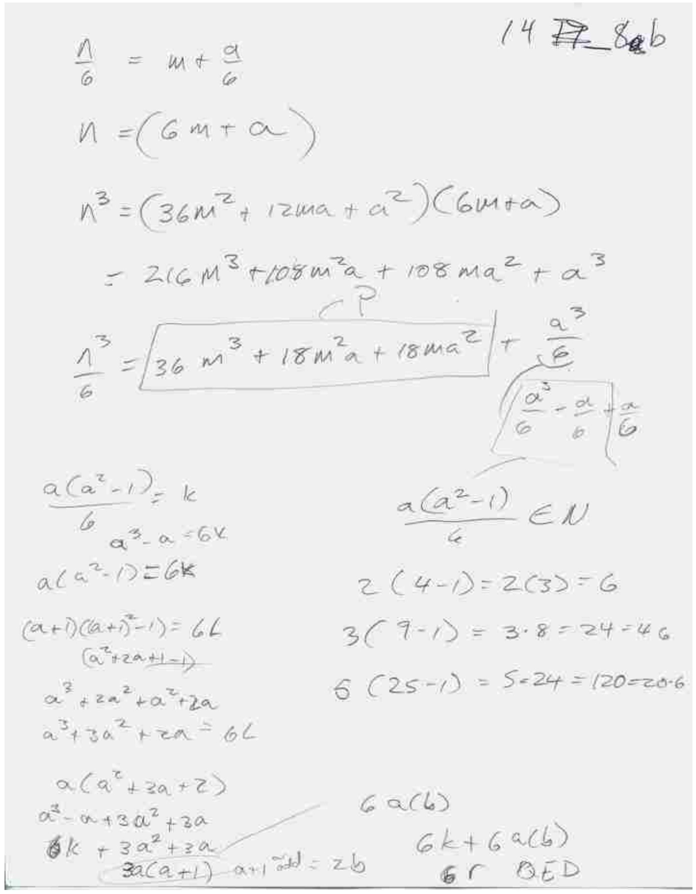


Figure 49: Will's work on Question 6 (2 of 2)

To show that $\frac{a^3}{6} - \frac{a}{6}$ is an integer, he does a little simplifying and says: "So I need to show that somehow *a* times $a^2 - 1$ is a multiple of 6." Next, he tries some examples, sees that it is working and proceeds to do the induction argument on the left side of Figure 49. He gets a bit stuck: "$a^3 - a$ is already 6*k* from up here…I need to show that this ($3a^2 + 3a$) is some integer multiple of 6…Now if *a* is even, that's easy. But *a*'s not necessarily even…" Eventually, Will factors the terms he is worried about and realizes that everything is fine in the case that *a* is odd as well:

> W: Oh, then I get an even number anyway (from the $a + 1$ factor). So, is equal to some…and I'll use *r* because I'm running out of numbers. And that is what I set out to

> prove in this little part. I really hope that there is a little easier way to go about this than the one I did. Ok, so I had to show that this right here is a multiple of 6, $a^3 - a$ is a multiple of 6.

This is a syntactic proof. Will began the interview by making sure he understood modular arithmetic, but this is separate from trying to gain an intuitive understanding of the problem itself (which would have classified the proof as semantic). Once he had straight the concepts necessary to attack the problem, it was a matter of manipulating the expressions at hand to verify that the identities held true.

As is typical with syntactic proofs, this proof reveals that Will has a transformational proof scheme. This is definitely a proof based on logical deductions, but the deductions do not show the reliance on previous that would be needed for this proof to be evidence for an axiomatic proof scheme.

<u>Question 7</u>

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

When Will first read this problem, he immediately thought it would be related to Pascal's triangle: "Ok. So you're going to have…so it's probably going to be related to Pascal's triangle at some point." During the reflection, I asked why he thought this and he said: "you know, it's going to keep getting bigger and bigger the more and more subsets you take." It turns out that Will was more correct about this assumption than he realized at the time. It did not take long, however, for him to abandon this. After drawing the triangle in Figure 50, Will said: “I don’t know, the triangle’s probably not the best place to start this.”

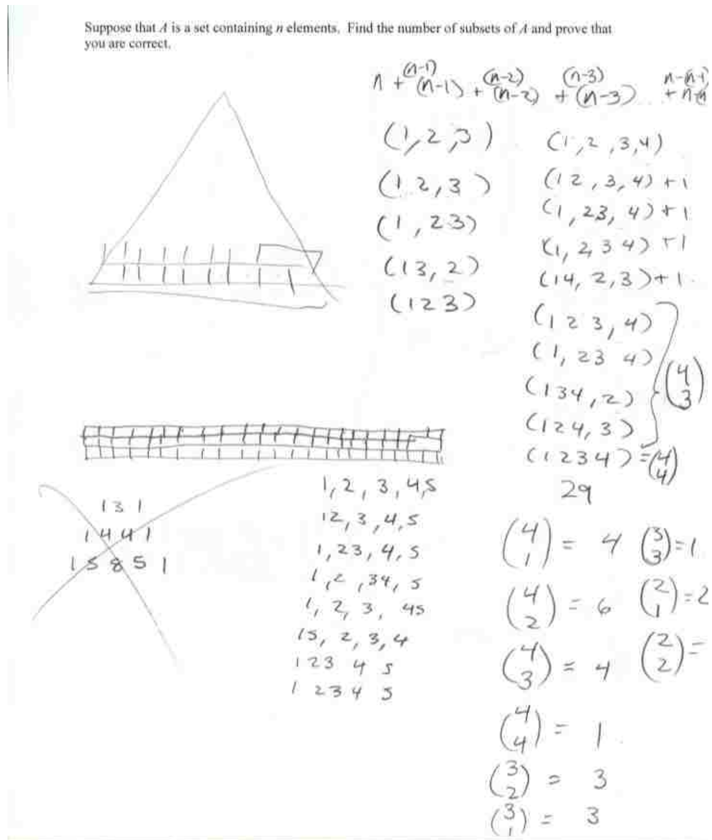


Figure 50: Will's work on Question 7 (1 of 4)

After this, Will gives the problem some thought and attempts to go straight to a formula:

W: So, if take A and divide it into your n elements, you’re going to get n subsets and then you can group, you know, 2 of them together. And you can do that, so then you’re going to have $n - 1$ elements that way, and…you’re going to have $n - 1$ of those…and then, similarly for $n - 2$…all the way down to $n - (n - 1)$, or 1. So…now I have to prove it…Is a picture proof ok?

This leads him to draw the rectangle of boxes in Figure 50. This attempt is also quickly discarded.

In an attempt to restart the problem, Will begins to write out the concrete example of a set with 3 elements: {1, 2, 3}. "Let's just start easy.…So you have *n*, 3, plus *n* – 1 times *n* – 1, so…2 times 2, so 4…oh, no, you don't. …Turns out this doesn't work after all." Will tried to extrapolate what he thought he was seeing in the example to a general formula. At this point, I asked him to explain the notation he was using:

> W: Oh, I'm just grouping 1, 2 and 3 into subsets. I've got 1 2 3 and then I've got 1 2 and 3.
>
> N: So is this supposed to be one subset of the set 1, 2, 3?
>
> W: No, this says, you know, 1, 2, 3; this [(1, 2, 3)] is a subset of 1, a subset of 2 and a subset of 3. This [(1 2, 3)] is a subset of 1 2 and a subset of 3.

Here, instead of simply listing each subset, Will is listing all the ways he can think of partitioning the set {1, 2, 3}. This will lead to trouble as he continues with the problem.

Will then moves on to a set of size 4 and again writes out what he thinks are the all the ways to partition the set {1, 2, 3, 4}. At this point, Will thinks he sees a pattern: "There should be 4 of these…So here I have 1 3 1, 1 4 4 1, this is starting to look vaguely like Pascal's triangle but not quite." Will sees that when he made his list for the 3 element set, there was <u>1</u> collection of subsets with 1 element each, <u>3</u> partitions of the set involving a subset of size 2 and 1 other element and <u>1</u> subset with 3 elements (1 2 3). Then in the column with the partitions of {1, 2, 3, 4}, there is <u>1</u> collection of 1-element subsets, <u>4</u> partitions each where the largest subset is size 2 and 3, and <u>1</u> subset containing all 4 elements. He then says: "So I'm going to posit that my next one, if I do 5, I'm going to get 1 5 8 5 1. That's just a guess, but…Let's find out."

Will begins to partition the 5 element set, but stops because he realizes that he forgot some partitions in the case of the 4 element set. “Oh crap. I forgot about additional groups. Because here I can do 1 2 3, and then 4 5…” What Will realized is that the subset {1, 2, 3} can be joined with the subset {4, 5} or the pair of subsets {4} and {5} to complete partitions of {1, 2, 3, 4, 5}. This led him to go back up to the 4 element case and add on some additional partitions in number only (as can be seen in Figure 50).

> W: Ok, alright…let’s go back to 4, make sure I can get all my set’s out of 4…+1, +1, +1, so for each of these I have one more because I can group the two remaining sets together. And for this one, it’s actually correct. This is the 1 2 3 4 and the 2…(counts the sets written out) …29.

Will then shifts his method again and starts using the choose function to count out subsets:

> W: What do I have here, 10, 15, 24, 25… one more, 26 but not 29. The reason I was thinking the choose would work is because initially you have *n*, well in this case we’ll say that we have 4 elements. So if you to pick, you could do 4 C 4 is going to be choosing all 4 in one set.

Will realizes that *n*C*k* will give you the number of ways to start with *n* things and choose *k* of them to place in a subset. However, he is still stuck with his notion that all the elements of the superset need to be accounted for when writing out the subsets. This accounts for the 3C2, 3C1, and so on that appears at the bottom of Figure 50. This is even more evident in the following figure:

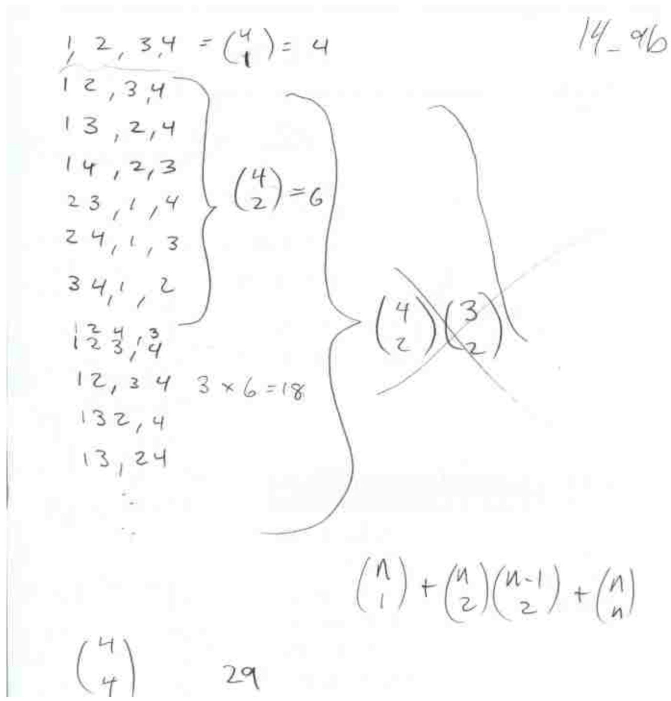


Figure 51: Will's work on Question 7 (2 of 4)

Will explains what he's doing here:

> W: So actually…this is 4 2…ok, and then for each of these I can additionally do, group the remaining 2 into a set, which is actually 3 choose 2…3 2 will cover the sets 1 2 3 …4 and 1 2 3 4, so both ways I could choose 2 from here…

Will is using the multiplication principle of counting. He sees that there are 4C2 ways to form a 2 element subset of a 4 element set and then adds the step of arranging the 2 additional elements into the partition in some way, which he believes to have 3C2 options. He concludes that the (4C2)(3C2) also accounts for the 3 element subsets: "Yeah, for every 1 2 and 3 4 there are 3C2 ways to group these into 2 additional sets, so 1 2 3 and 4 and 1 2 4 and 3 and then 1 2 and 3 4." All that remains, then, is to add 4C4. He mistakenly adds the 4C2 = 6 when he says: "10, 28, 29…So the plus side is I just got 29. The down side is I don't think that's the number I want anymore." According to his method, he should have added 4C1 + (4C2)(3C2) + 4C4 = 23. He then wrote out his formula for a general *n*.

At that point it was time to end the work portion of the interview and start the reflection. During the reflection, Will mentioned that he was skeptical about his method: "This doesn't really make sense to me…the whole 4 2, 3 2 [4C2 and 3C2]." We also discussed that the method he was using was flawed in that it allowed for repeats. I asked Will to work on the problem between interviews and he said he would.

Between interviews, it was clear that Will had made some progress:

> W: Well, after your hint from last time, where I was counting sets repetitively, I basically just came up with, right away, this new expression where it was the sum of *n*C*i* from *i* is 1 to *n*…Basically, this is the number of ways you can take a single element and call that a subset and then, you know, subsets of 2 elements would be *n*C2…And then you do all the way to *n*C*n* where you take the entire set as a subset…So, I basically just came up with that and I was like 'This is awesome, this should work perfectly.' And it does, if I plug it in for the set 1, 2, 3, it's got 3 elements, 7 subsets…and then when you actually do this calculation, you get 3C1, 3C2, 3C3 is 3, 3 3 and 1, 7 hooray.

It should be noted here that I intentionally did not mention the empty set as a subset until later in the interview. Will then proceeded to try to prove his new formula by induction. To do this, he tried a number of algebraic manipulations, as can be seen in the following figures.

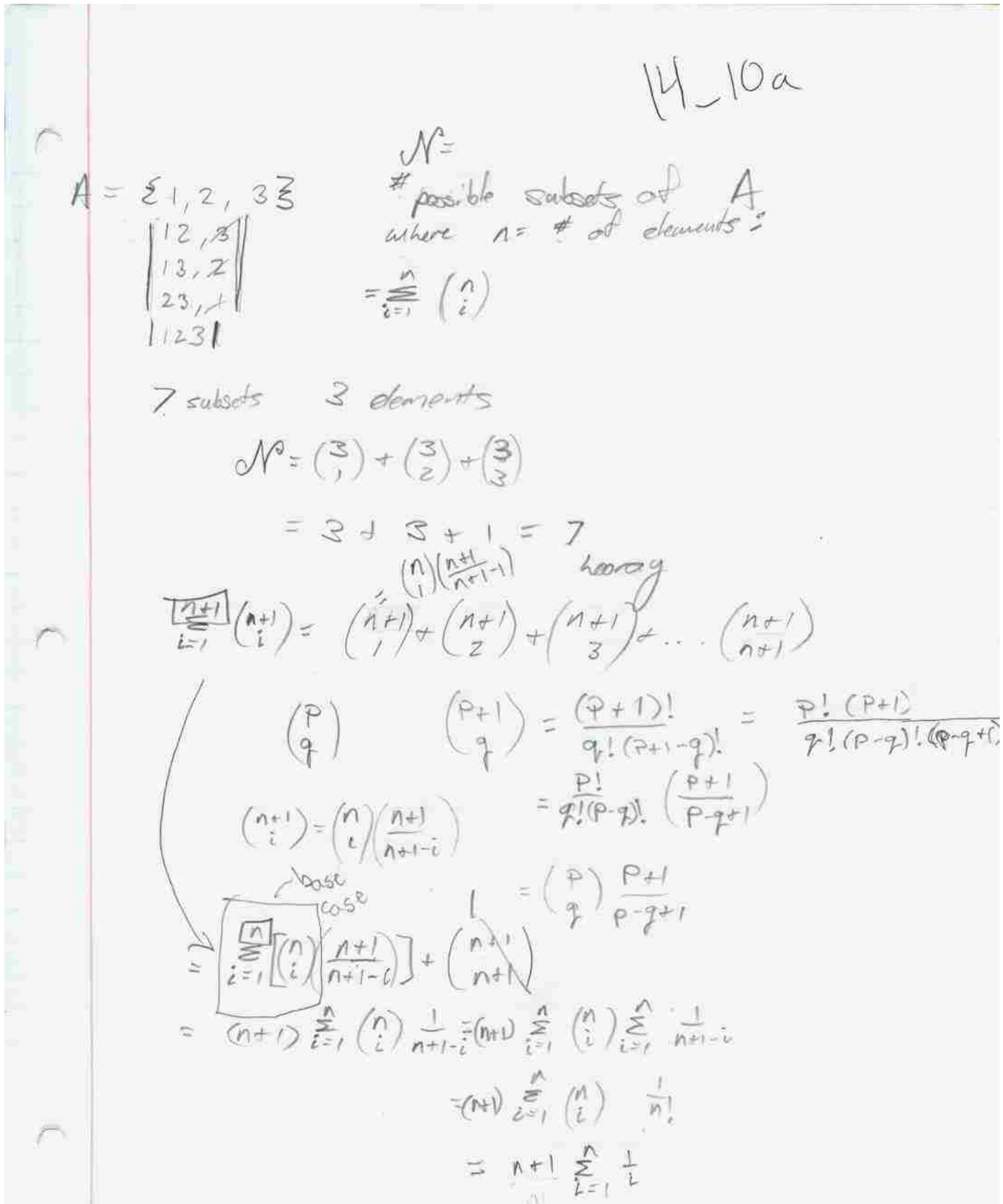


Figure 52: Will's work on Question 7 (3 of 4)

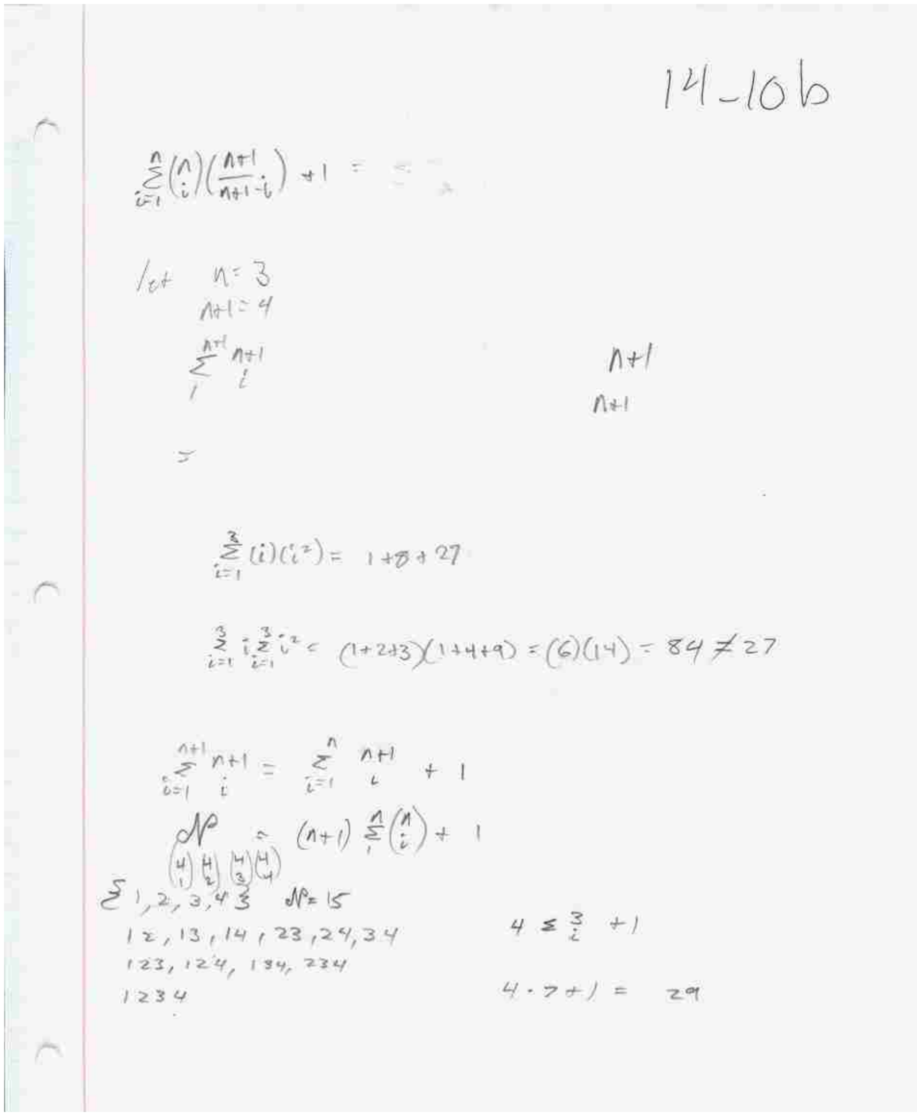


Figure 53: Will's work on Question 7 (4 of 4)

Will had come to a formula that he was comfortable with, he just was not sure how he was going to prove it:

W: So it works. Then I tried to do an induction by taking $n + 1$ and trying, you know, to get something that looks like n choose, or $(n + 1)$Ci trying to get to get something that looks like nCi. I kept making stupid mistakes.

Will knew that he had made a mistake along the way (by the time he reached the bottom of Figure 53) because his formula gave him an answer of 29 when he was expecting 15 (not counting the empty set).

I asked Will if he had considered the possibility that induction was unnecessary:

N: So did you ever give any thought to the fact that maybe this formula maybe just sort of inherently was a proof?

W: Well, logically, it works. It's built out of logic.

N: Right

W; I kind of built it that way so...I don't know. It works, it's pretty clear why it works. I understand very well why it works. I was trying to use one of these proof techniques to show it works. But, no I didn't really want to consider that a proof.

I then asked Will about the fact that he did not really know what he was supposed to end up with at the end of his inductive step:

W: Yeah…I don't know, that was something I was noticing, that I wasn't really going anywhere with this. And that's kind of what I was getting down here (bottom of Figure 51), from working on this, I realized that I didn't really know what I was looking for…I was just looking for, just hoping that something would fall out and I'd have one of those aha moments…

N: Sure, right

W: But I guess that never really happened and I ended up like just kind of flailing around a little but until finally, 'Screw it.'…Just kind of gave up, you know?

Next, I asked Will what he would he would turn in if this were a homework problem:

> N: So, just like a hypothetical, if this was a homework assignment, you know, you go through this flailing around a little...would you just eventually say "Ok, I have to turn something in, let me explain where this comes from"?
>
> W: Yeah, that's pretty much what I do on my homework assignments. I work it as much as I can and then I look at the clock, I've got 5 minutes to turn this in, so I just say 'This is my work. Partial credit?' Thank God for partial credit.

From this statement, I gathered that Will was still confident that he had the correct formula, but not that he'd be able to provide a proof. We spent the rest of the interview looking at alternative ways to solve the problem.

Although Will did not provide a proof, per se, the work he does on this problem does constitute a semantic proof attempt. The first interview working on this problem was spent trying to figure out the structure of the problem. Between interviews, he got to a point where he felt like he understood the structure of the problem and then tried to turn that understanding into a formal proof. This is why his proof (or at least proof attempt) is classified as semantic.

His proof scheme, however, is less straightforward. He tries to use induction, which is evidence for a transformational scheme given how well he understands the method of induction (as with Question 3). However, the fact that Will does not take his justification for the summation formula and use it to provide a proof gives support for a ritualistic external conviction scheme. Ritualistic proof schemes are generally schemes held by those who become convinced by something that simply looks like a proof. Applying the contrapositive of this definition, we see that it applies to Will because he thinks that his argument is not a proof because it does not look like one: "I mean I probably could have written something out that made logical sense and called that a proof, but…if you can't prove it numerically, it's kind of hard to say that it's proven."

Question 8

Prove that the cube root of 2 is irrational using a proof by contradiction.

This problem was on the midterm that Will completed when he took MATH 305. It was meant to help gauge any progress Will might have made since the middle of that class. On the midterm, Will made the mistake of proving that the square root of 2 was irrational, not the cube root. He initially made the same mistake during the interview, as we will see. Other than this mistake, Will had no issue with the problem on the midterm. See the figure below:

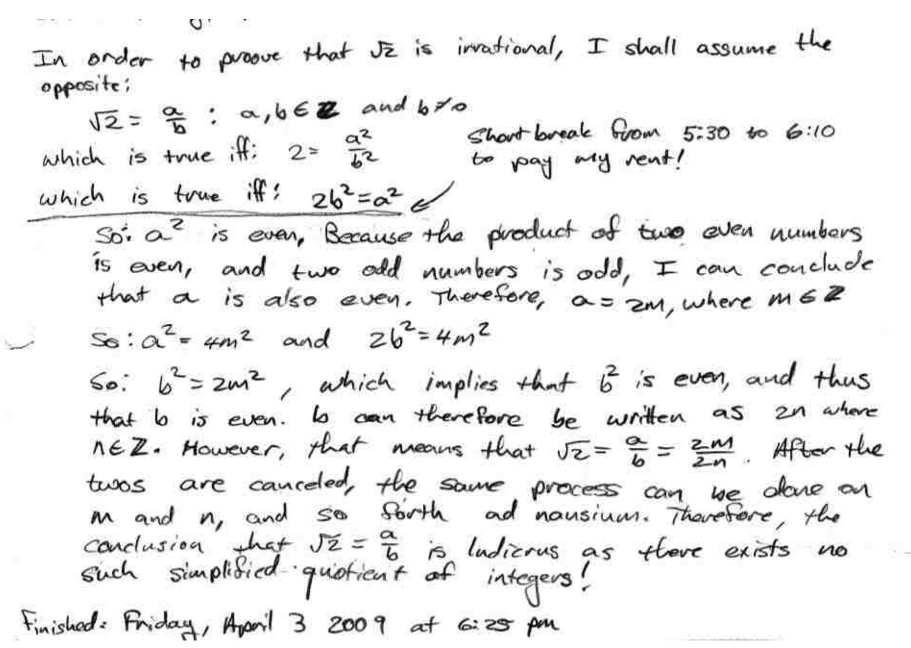

In order to proove that $\sqrt{2}$ is irrational, I shall assume the opposite:
$\sqrt{2} = \frac{a}{b}$: $a, b \in \mathbb{Z}$ and $b \neq 0$
which is true iff: $2 = \frac{a^2}{b^2}$
Short break from 5:30 to 6:10 to pay my rent!
which is true iff: $2b^2 = a^2$
So: a^2 is even, Because the product of two even numbers is even, and two odd numbers is odd, I can conclude that a is also even. Therefore, $a = 2m$, where $m \in \mathbb{Z}$
So: $a^2 = 4m^2$ and $2b^2 = 4m^2$
So: $b^2 = 2m^2$, which implies that b^2 is even, and thus that b is even. b can therefore be written as $2n$ where $n \in \mathbb{Z}$. However, that means that $\sqrt{2} = \frac{a}{b} = \frac{2m}{2n}$. After the twos are canceled, the same process can be done on m and n, and so forth ad nausium. Therefore, the conclusion that $\sqrt{2} = \frac{a}{b}$ is ludicrus as there exists no such simplified quotient of integers!
Finished: Friday, April 3 2009 at 6:25 pm

Figure 54: Will's previous work on Question 8

Will did have more trouble, at least initially, when working on the problem during the interview. As I said above, Will began the problem by working to show that the square root of 2, not the cube root, was irrational. See Will's work in Figure 55:

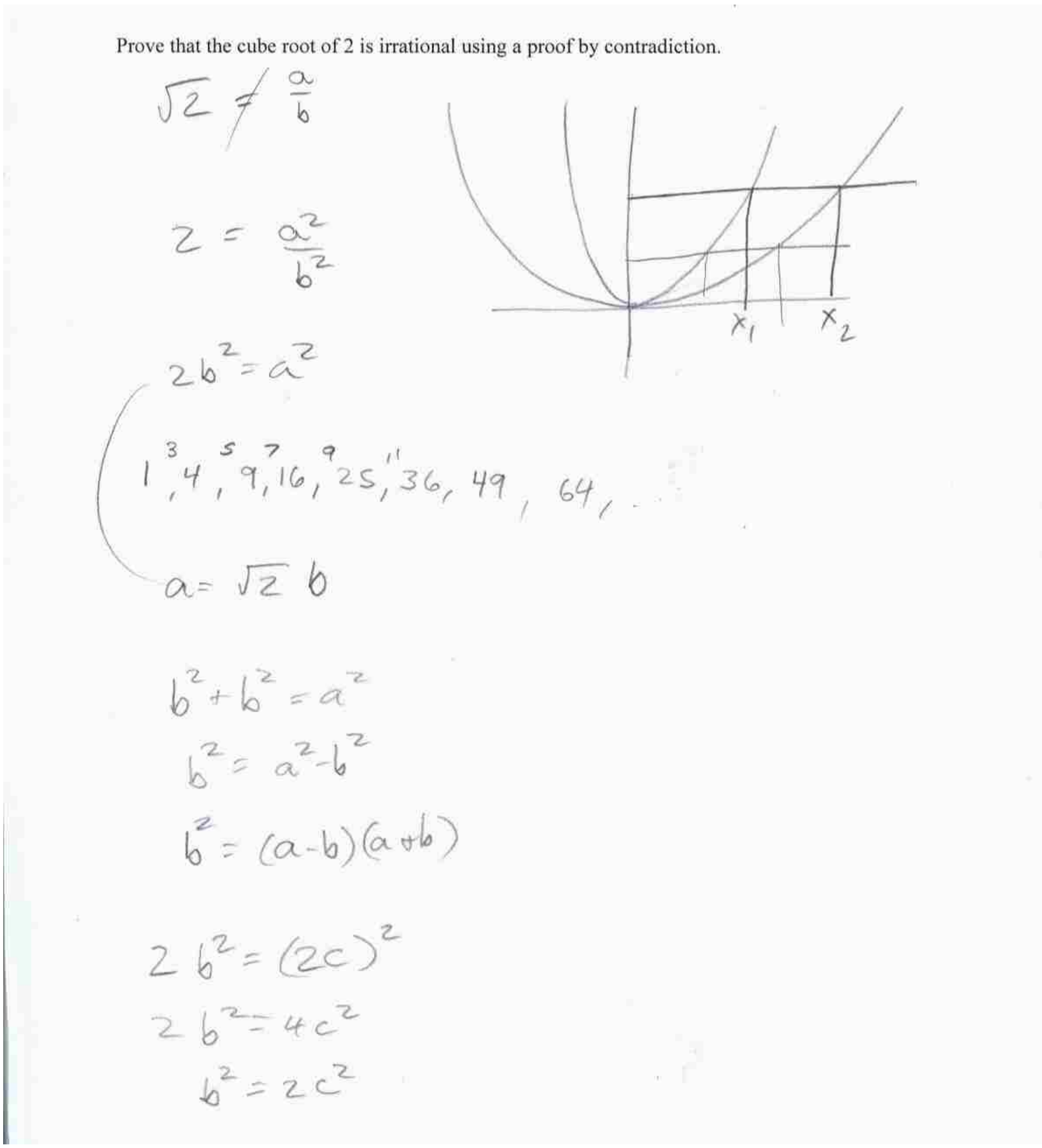


Figure 55: Will's work on Question 8 (1 of 2)

Will started the problem fine in that he assumed that $\sqrt{2}$ is a rational number and trying to derive a contradiction from that. The first thing he attempted was looking at a list of squares and trying to get his contradiction from that:

W: And the difference appears to be, between each consecutive square appears to be the series of odd integers, 3, 5, 7, 9, 11. … From that, I'd imagine that there's no way you can have a square that's double another square…Because any…I don't know…Right now what I'm trying to do is show that is no way that you can have a square and another square that's double that.

He abandoned that fairly quickly and moved on to considering what would happen if $2b^2$ and a^2 were treated as functions:

W: If I broke this ($2b^2 = a^2$) into a transcendental equation, and set each side equal to…Yeah, I think, I don't think there's any intersection here. Where $2b^2$ would equal to a^2 except for at 0. So 0 over 0 is 0 and square root of 2 is not equal to 0. So, therefore, square root of 2 cannot be equal to that ratio.

I pointed out to Will that just because the graphs he is referring to only intersect at the origin does not mean that $2b^2$ cannot equal a^2:

N: I agree with you that the equation $y = 2x^2$ and the equation $y = x^2$ only agree where x is 0…But you can find sort of separate a values and b values that can make that true…like if you had b is $\sqrt{2}$ and a is 2, then this would be fine… You know, you just sort of tried to make the claim that $2b^2$ could never be a^2.

After a little more discussion, Will moved on to another potential method:

W: I mean…I don't know, would it be out of line to actually calculate of few digits or say something along the lines of, you know, 1^2 is 1, or the square root of 1 is 1, the square root of 4 is 2, so…the square root of 2 must be somewhere in between there?

I: Um, yeah, well, I mean, I'm not sure it would get you anyway because, let's just pretend right now for the sake of argument right now that the square root of 2 is 1.5. So, 1.5 is definitely between 1 and 4, right?

W: Right

N: But *a* being 3 and *b* being 2 would satisfy that equation, then, right?

W: Oh yeah. Shoot… So, it would actually come back to this one ($2b^0 = a^2$). How could I show that?

From there Will had one more method he wanted to try: "So, what I just did was I split my $2b^2$ into $b^2 + b^2$ …and I subtracted one of the b^2's from the a^2. I got

$$b^2 = (a - b)(a + b).$$

And I'm not seeing how this is helping me either."

Normally I would not have given Will as much help as I did. However, this question was presented to see what progress he had made since taking MATH 305. Since he had been given the proof that $\sqrt{2}$ is irrational in that class and was then asked to adapt it to $\sqrt[3]{2}$, I gave him the argument for $\sqrt{2}$ at this point. This can be seen at the bottom of Figure 55. From that point on, Will had no problem at all providing a proof for the irrationality of $\sqrt[3]{2}$. See his proof in Figure 56:

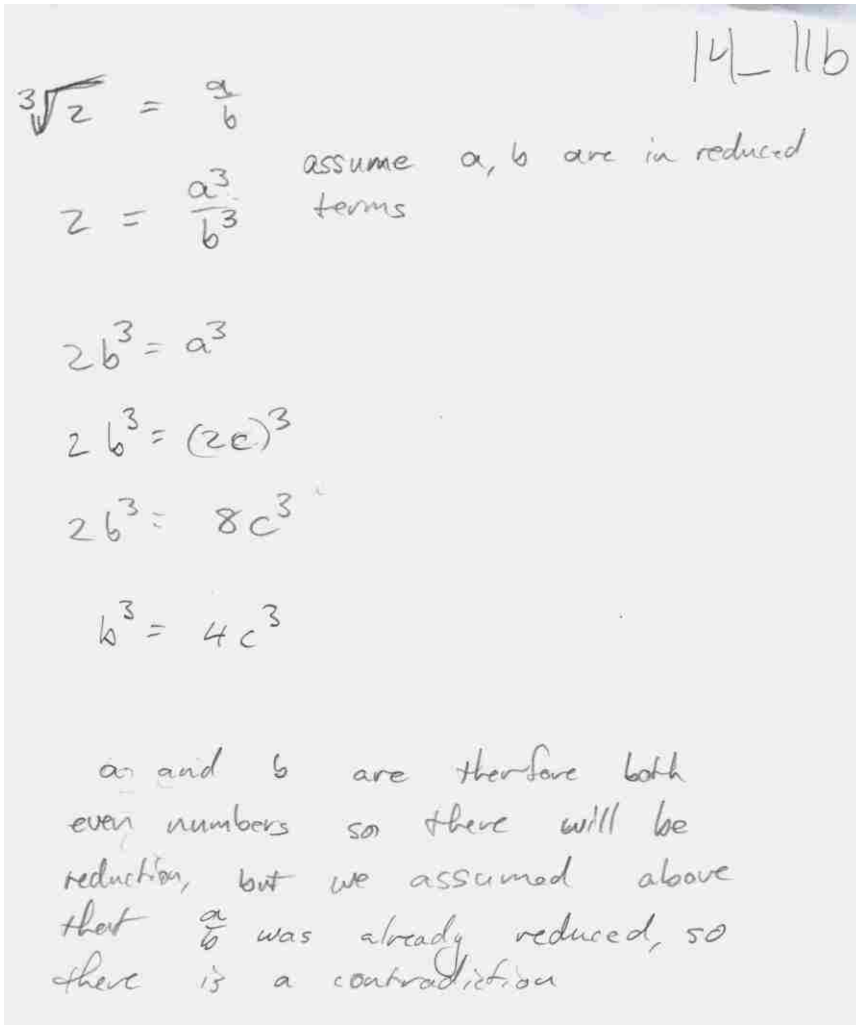


Figure 56: Will's work on Question 8 (2 of 2)

The proof Will provided for this problem (once he had seen the argument for the $\sqrt{2}$ case) is a algorithmic procedural proof. There where specific steps that needed to be completed and he accomplished those steps by modifying the steps from a similar proof. That being said, I do believe the steps were meaningful for Will; I do not think he was simply blindly following along. I believe this to be the case because he had no problem explaining the gist of the problem: "I said *a* and *b* are therefore both even numbers so there will be reduction, but we assumed above that a over b was already reduced, so there is a contradiction."

Because Will had a good idea of the steps he was completing, he reveals an analytic proof scheme here. Had he merely completed steps, seemingly not knowing what he was doing, then that would have constituted an authoritarian external proof scheme. As it is, Will performed operations he understood well and used their outcomes to complete his proof. The proof relied heavily on algebraic manipulation and the anticipatory action of assuming *a* and *b* are relatively prime. Thus, Will is displaying a transformational proof scheme here.

Question 9

Like Question 4, the next interview involved showing Will a proof and having him evaluate it. The proof was a version of Cantor's Diagonalization Argument and can be seen in the appendix. Since Will did not complete a proof for this problem, there is no proof to classify. As before, however, discussing this proof gave some evidence for Will's proof scheme.

As is to be expected, Will spent the first portion of the interview getting used to the notation in the proof:

> W: It's a number…ok. α_{ji} represents the *i*th digit in the decimal expansion of *f*(*j*)… β_j is 1 if α_{jj} equals 2…So…*B* is another number and…so that's where the diagonal comes in? The α_{jj} and the β_j? So *B* is just the numbers in the diagonal? The functions, or the numbers…if you have a function of β, it'd just be the α's that are in the diagonal's?
>
> N: Well, sort of, the individual numbers in *B* are sort of dictated by what is in that diagonal. So yeah, I mean, that diagonal you just circled is where the term diagonalization comes from.
>
> W: Ok. So, if α_{11} is equal to 2, then β_1 is equal to 1. So this is just a string of 1's and 2's. So since *f* is onto, there exists a *k* such that, or *k* a subset (element) of ℕ such that *f*(*k*) is

equal to *B*…However, if β_k is not equal to α_{kk}, by definition of β_k, because…it always has to be…yeah….Oh, ok. So *B* is the list of β's, so there's some function *k*…or *B*…

After getting this all straight, Will gave the proof overall some more thought and decided that he understood most of the proof: "Ok, so I'm fine up until this last paragraph." We then discussed what the proof was saying there and Will started to understand:

W: Yeah, ok, yeah. So…but it's still saying that *B* has to lie somewhere on this…one of these horizontals…Yeah, ok. So, but... β_k is not equal to α_{kk}, so…*f* of …So *b* of some digit, some number, some β, is not equal to α there, so…so that's bad. (thinking) …Ok, so *f*(*k*) is defined as *B*. That determines where *B* is in this list. This is kind of like a β_k, or whatever. *f*(*k*) is equal to *B*, but β_k is not equal to α_{kk}, so whatever function is already there…your β_k, one of the digits is guaranteed not to match, therefore they can't be the same. So…how does the countable assumption lead to the absurdity?

Will then goes back through the proof and makes some more progress towards understanding it:

W: Ok…so write the outputs as, ok, that now makes a lot more sense…So, it has to be, it is a problem with 1 – 1, isn't it? Because suddenly you have 2 functions, you know, for *k* and *n*, even though *k* is equal to *n*, you have 2 different decimals.

N: So, say that again.

W: So, basically, it's established that you're, if you let your *k* equal to *B*, and there's some *B* in this list that since *k* is a subset (element) of ℕ and we're taking all the *n* that are a subset (element) of ℕ, there has to be some n out there that is equal to *k*.

N: *Right*

W: So when you take that decimal expansion, you should get the same decimals because, I mean they're the same, they're 1 – 1.

N: Well, they're the same number, yeah.

W; Yeah. So, but you're going to get different one's because *B* is not equal to whatever *f*(*n*), because that α_{jj} is not going to be able to be equal to β_j.

N: Yeah

W: So, you've shown right there that they can't be 1 – 1, it has 2 different numbers and therefore the set is not countable.

Will is displaying some confusion about function terminology. The contradiction he is explaining does not violate the fact that *f* is supposed to be one-to-one, but the fact that *f* is a function in the first place. Besides this error, Will understands the proof.

This interview displays that Will has a transformational proof scheme. His understanding of the proof depended on his understanding of the relationships between objects developed over the course of the reading through the proof. He also recognizes that the validity of the proof depends on outcomes of the operating on objects at hand (in this case, taking the number *B* and relating it back to the list).

It is worth noting that Will is only convinced up to a point. When I asked him if saw any holes in the proof, he responded by saying:

W: You want me pointing out holes now? ...As far as my level of understanding goes, I can't really say that there's no holes, I mean… If I had 5 minutes to answer the question, I'd probably say "No it's fine."

Will did not go into the evaluation of the proof looking to debunk it. This might make one think that he expected it to be true. If this was the case, that would be evidence for some sort of external conviction scheme. However, the fact that he qualifies his conviction with “As far as my understanding goes” tells me that he is relying on his understanding alone (and not the structure of the proof or some other authority) to decide whether or not to believe the proof is valid.

<u>Question 10</u>

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

Will started this problem by going straight to a general formula:

W: So…you have *a* over *b* and *c* over *b* and between those, oh, there we go … less than *d* over *e* … less than *c* over *b* … whose denominator is less than that of the pair…I would, what happens is you take the average of those 2…(thinking)…Ok…Oh, numerators differ by 1, so this (*c*) is *a* + 1 over *b*.

Will seems to have a number of ideas regarding the problem as he begins. He first writes down an inequality involving general rational numbers, 2 of which share a denominator, and then realizes that he can say more that this: that $c = a + 1$ due to the restrictions given in the problem. In between, he has the thought that maybe using the average of the 2 outer numbers might given him what he needs. After a little more thought, Will discovers a counter-example:

> W: Oh, wait a second what about 1/2 + … I guess 2 halfs, doesn't really work does it? Because that's 1. … I don't know is that legitimately…I guess 2/2 works. I don't know what's between those, just 1/3…Ok, well 1/3 isn't less…that would say that there has to be an integer between there. There's no integer between 1/2 and 1….So there's a contradiction.

I asked Will what that meant for the problem and he said:

> W: It means that it must be disproved because 1/2 and 2/2 are both rational numbers that numerators differ by one and they have the same denominator. However, the only number less than 1, or less than 2 that can go in there is 1.

From there, I asked Will if he could place any restrictions on the problem that would make the statement true. "I suppose maybe if I said so long as the denominator is greater than 2, maybe, because 1/3 and 2/3 is 1/2. What about 0 and 1/3, there's nothing in there…"

Will did not let these counter-examples stop him from considering the problem. To investigate what was going on, he laid out some numbers to see if he could notice anything about what was going on.

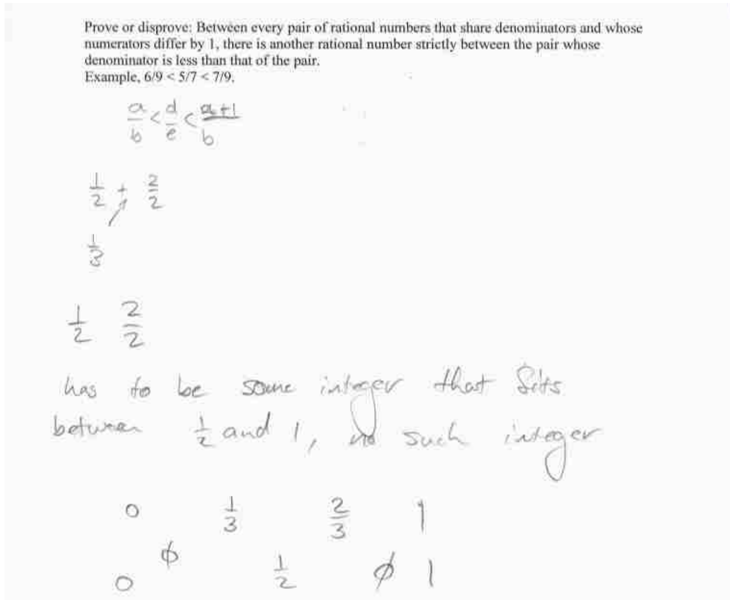


Figure 57: Will's work on Question 10 (1 of 4)

Will quickly moved on to an improved version of his diagram: "So, let's say I have 0 and then…1/3, 2/3, 1. In here there's nothing, 1/2, nothing. So…you have 0, 1/2, 1, you're going to get, well actually let's build this the other way around."

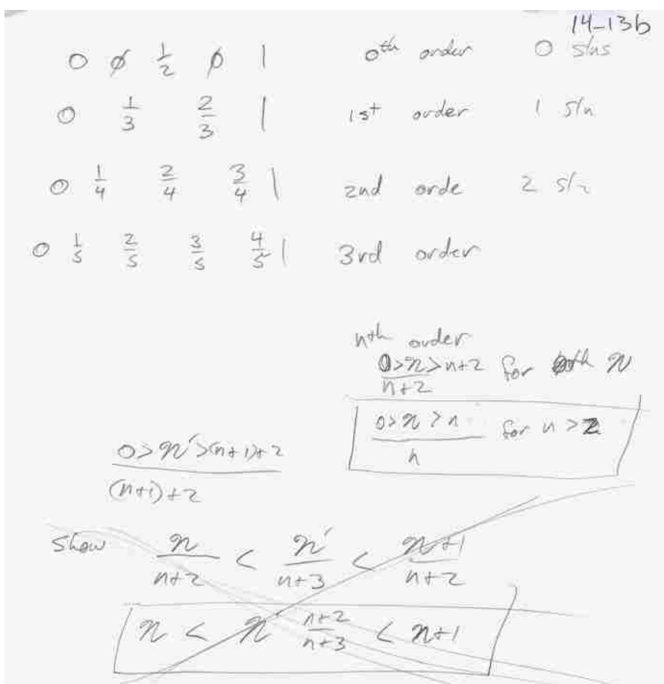


Figure 58: Will's work on Question 10 (2 of 4)

W: So we're going to start with 0, 1/2, 1, there's nothing in between there, though. I go down to 1/3, though, I have 0, 1/3, 2/3, 1. So 1/2 lies between then. I go to fourths, I have 0, 1/4, 1/3, I have 2/4, 3/4, 1. This is 1/2, so that doesn't really help much, but yeah, I have 1/3 and 2/3 in between there. So this is…I'll call this 0^{th} order, 1^{st} order, 2^{nd} order, so here we have 1, here we have 2, here we have 0, that's why I chose those for my orders (the number of solutions for each line is the "order"). I'm just going to keep going for a bit. 0 to 1/5, 2/5, 3/5, 4/5 and 1. So yeah, third order, you're going to have 3. So the pattern that seems to be emerging here is that for an n^{th} order…denominator, the denominator is going to be n+2…and so long as your numerator is… greater than 1 and less than…n +2, it looks like you're going to have something there…greater than 0, less than or equal to 1. Should I prove that?

Will is basically saying that the property holds so long as you neglect the parts of the unit interval from 0 to $\frac{1}{n}$ and $\frac{n-1}{n}$ to 1.

When I asked Will to prove this, he said "So…call that a base case…trying to think of how to do this. The *n*'s kind of suggest that it should be an induction proof, but…" He proceeds to then set up the inductive step, where he is going to induct on the value of the denominator (see Figure 58).

W: So…call this *n*' (the new numerator that's going to be between in the inductive step). This is, I want to show that there's some *n*' in between (script) *n* and (script) *n* + 1 over this (*n* + 2). So…show that from (script) n defined up here over … less than *n*' over, call it *n* + 3, (script) *n* + 1 over *n* +2. I don't think I'm going to be able to prove that by induction.

He then gets a new sheet of paper and draws a diagram similar to the one he had before, only this time he uses tick marks instead of numbers: "I'm going to look at this pictorially. So this is my 0th order case. I've just divided this in 2, so this is 1/2. Divided into thirds…and this into fourths…it's a little off…but hopefully there will be something really illuminating."

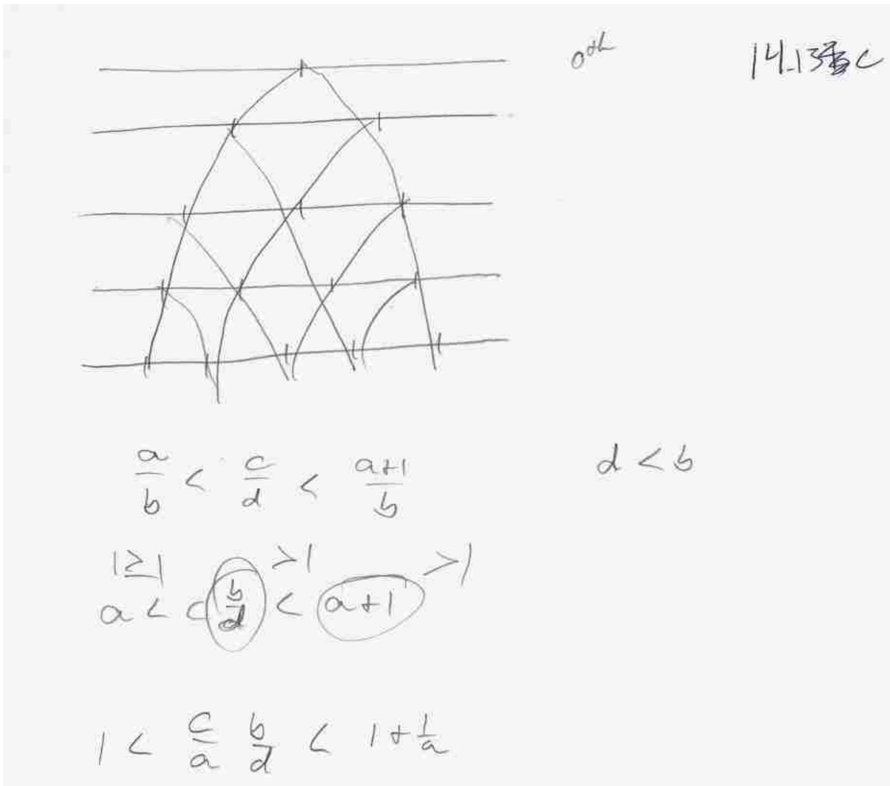


Figure 59: Will's work on Question 10 (3 of 4)

W: Gosh I just don't know how to prove this…statement here. I mean this picture proof almost does it because you're always going to have this…this boundary expanding and, I mean, you can clearly see that there's always something in between here. … But I'm not happy with that as a proof…It's kind of frustrating.

Will continues his restart of the problem by going back to a general inequality as can be seen in the figure.

W: Ok, this is kind of a throw back to this (crossed out work in Figure 58), but I'm just going to write a over b and then a+1 over b and then c over d where d is less than b…Just going to see if writing it in terms like this helps me see anything…this gets me nowhere.

At this point, I began the reflection portion of the interview. As we talked through the work he had complete, I referred to the pattern that had begun to emerge in his drawing in Figure 59. "Yeah, you kind of get a…shell of available values that's like … (draws outer-curves in the diagram) you know, 0 to 1, you're going to get something like this."

I mentioned to him that he seemed pretty convinced by the pattern he saw and I asked if it might be enough to serve as a proof for someone else:

W: I just don't think that that's rigorous enough to call a proof. And I suppose if I took the time to sit down here and write these functions (the ones that would define the outer-curves of the shell), this function, you know, this function (draws in inner-shells)…And, you know, I'm sure something might fall out of this…Yeah and as far as just an illustration, that…doesn't really…convincing, but not satisfying.

The other main question I had for Will was regarding the fact that he had only worked in the unit interval up to that point. He said that he had not thought about anything outside that, but quickly saw that once he had that figured out, the rest would be easy:

W: I wasn't really thinking about that when I did this, I don't know why I decided to do 0 to 1, but it would work because if you can find solutions in this interval, you can find solutions in any interval you…any interval higher than that…Yeah, all you're doing is adding some integer.

At the end of the interview, I asked if could find some time to try to prove the pattern he found, and he said he would try.

Will did not get a chance to look at the problem between interviews. Because it was the last interview of the study, I wanted to make sure that I had time to conduct the debriefing part of the interview. So, Will and I talked through the solution of the problem, starting with me making explicit the pattern he saw the last time we met:

N: (Referring to the diagram in Figure 58)…for example, between 2/5 and 3/5, what number, what rational number that's going to fit the criteria is going to be in there?

W: 1/2

N: Ok, what about between 1/5 and 2/5?

W: 1/4

N: Yeah, ok, what about 1/4 and 2/4?

W: 1/3

N: Right, so do you sort of see any pattern sort developing from there? Any relationships?

W: Well, it looks like your denominator is just going down by 1 every time you step over.

Once Will saw this he went about setting this pattern up in a general way. He set up 2 inequalities. To show the left side of $\frac{a}{n} < \frac{a}{n-1} < \frac{a+1}{n}$, Will set up the inequality on the top of

Figure 60: "Yeah, because if this distance minus this distance…" He is referring to showing that the distance between $\frac{a}{n}$ and $\frac{a}{n-1}$ is less than the distance between $\frac{a}{n}$ and $\frac{a+1}{n}$, which is exactly $\frac{1}{n}$. He then set up the right-hand side of the inequality in a more straightforward manner and worked with that one, as can be seen in the figure. Note that Will only works on the second of these sub-inequalities, something that neither of us noticed during the interview.

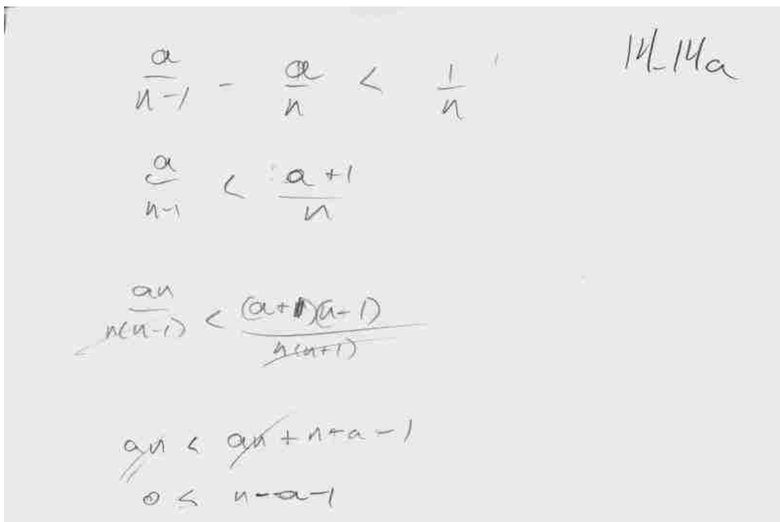


Figure 60: Will's work on Question 10 (4 of 4)

When working on the right-hand side, Will talks through the final bit of justification:

> W: So that's…yeah, an – a + n – 1. So…take an out. Can I just say that…n – a – 1. Ok, well…I know that a is less than n…and what were the restraints I put on what a can be? It had to be greater than 0 and less than n, right? Yeah…so I know that 0 is less than n – a – 1…Oh yeah, because a has to be less than n…minus 2…Yeah, so I was right, this is true. n – a – 1 is greater than 0, ok…Yeah, that does it…. That's what I set out to prove.

The proof Will has constructed constitutes a semantic proof attempt. He does not finish his proof on his own, and it's complete, so it is considered an attempt only. However, it has the tell-tale signs of a semantic attempt. Through a few starts and restarts, Will eventually uses a pair of diagrams that he hopes will give him insight into the problem. He tries to convert this insight into a proof, getting nearly all the way there once I make the pattern he saw on his own more explicit.

As with most of the questions over the course of the study, Question 10 provides evidence that Will as a transformational proofs scheme. The proof he provides is logically deduced and does not rely on empirical evidence or external verification. It also depends on operations carried out on mathematical objects, not previous results, for its validity. Producing this type of proof attempt places demonstrates that Will has a transformational proof scheme.

Question 11

The final interview of the semester concluded with a debriefing session in which Will and I discussed how he felt the semester went with regards to proof. He did not produce a proof, so there won't be a proof to classify.

While Will did not provide a proof, he did reinforce some of the things observed over the course of the study. For example Will often, especially early in the study, would try to gain an understanding of the proof instead of trying to go straight to it. When I asked him about the role of examples in proofs, Will responded: "I guess seeing that the proof (result) actually works somewhere is a good first step…Because if you can't find, you know, an example of where the proof (result) works, then it's kind of a crappy proof (result)." Besides just giving reassurance that a result holds, Will points out that examples can also help while constructing a proof:

> W: If I have nowhere else to go, it never hurts to look at the example and just see what's there, I mean…If it's not like you can just sit down and write the proof instantly…staring

> at the proof for a while and just trying to understand, staring at the example, excuse me, for a while and just trying to figure it out…Figure out what's going on in the example, what you can take away from that, what you can use from that.

This attitude towards understanding a problem before trying to provide a proof is why Will had so many proof attempts classified as semantic over the course of the study.

Will's tendency to hold off on a proof until he had an understanding did not show up in every question over the course of the study. For example, in Questions 7 and 10, Will made some attempt to go straight to the proof and these were the Questions that Will was least successful with. Will saw the classes he had been taking as having a role in his proving:

> W: I'd say if anything my proofs wanted to be more construction proofs…. Yeah, but like the classes I have now, it's not so much contradiction proofs and induction proofs, it's like "Show that this is true"….or "Carry out this integral to show that this is actually equal to this."… Yeah, I think it's kind of like what kind of math you're exposed to… If you looking at what kind of thinking you're going to do in a given time interval, you…it's probably going to be related to whatever you're expected to do in other classes and what not.

In the classes Will took during the second half of the stuffy, Will was presented problems that could be approached in a straight forward manner. This led him to try more direct proofs during the second part of the study. When he ran into trouble he was still able to take a step back and look for that intuitive understanding, but that served as a fallback option when his direct proof attempt failed.

Like with his preferred proof type, Will also highlighted the proof scheme that showed up most often over the course of the study. When I asked him what he thought it took to have a successful proof attempt, Will said, “Patience. A little creativity. Do something clever once in a while… A little outside the box thinking.” I think Will is referring to the need to manipulate the objects involved with the problem he is working on. As such, Will is displaying a transformational proof scheme.

Will’s progression

Below is a chart of the proof types Will provided along with the proof schemes he displayed over the course of the study:

Question	Type of proof	Proof scheme
1	Semantic	Transformational
2a	Syntactic	Transformational
2b	Semantic	Transformational
3	Process	Transformational
4	N\A	Axiomatic
5	N\A	Transformational, Axiomatic
6	Syntactic	Transformational
7	Semantic (Attempt)	Transformational; Ritualistic
8	Algorithm	Transformational
9	N\A	Transformational
10	Semantic (Attempt)	Transformational
11	N\A	Transformational

Table 3: Summary of Will's work

When looking at the chart it does not seem like Will displayed much change over the course of the study. This is true, as both the type of proof provided and proof scheme displayed did not change much. However, while most students felt like they had improved over the course of the study, Will did not.

Will had noticed the fact that he did better overall with the questions from the first semester (since there were 2 problems that he did not complete on his own the second semester). When I asked how he thought he did with proofs semester, Will said “I think I’ve kind of slipped a little bit. I'm out of practice since I don't have a class I'm taking that's just strictly, you know, writing proofs and reading proofs and stuff."

While Will felt like his proof ability had regressed, it did not show up in the classification of his proof types or proof schemes. This reinforces point that student's conceptions of proof can not always be seen by observing their work alone (Stylianides & Stylianides, 2003; Stylianides & Al-Murani, 2010).

4.4 Helen

This section focuses on the progress made by Helen from the beginning of the study to the end. First, I will briefly describe each of her proof attempts and their classifications. Then, I will discuss changes in Helen's understanding of proof and proof structures based on this and the reflection portions of our interviews. Helen majored in art and was also working on a minor in mathematics. During the first half of the study, Helen took Euclidean and non-Euclidean Geometry and Ordinary Differential Equations and during the second semester, Helen took Number Theory and History of Mathematics.

Helen's proof attempts

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Helen began the problem by writing out the formula $2l + 2w = lw$ and trying a some examples (see Figure 61). She tried a few examples that didn't work and then hit on the solution where $l = w = 4$. She then mentioned that she wanted to try induction because "it's what we spent a lot of time on last semester." As this was still early in the study (before I let the participants work independently before reflection), I pointed out the reasons for not using induction and suggested she solve her equation for a single variable. From there, she solved for w and plugged in various l values to see which gave integer solutions. This led her to finding the solution $l = 3$ and $w = 6$ as well as confirming the 4x4 square she found earlier.

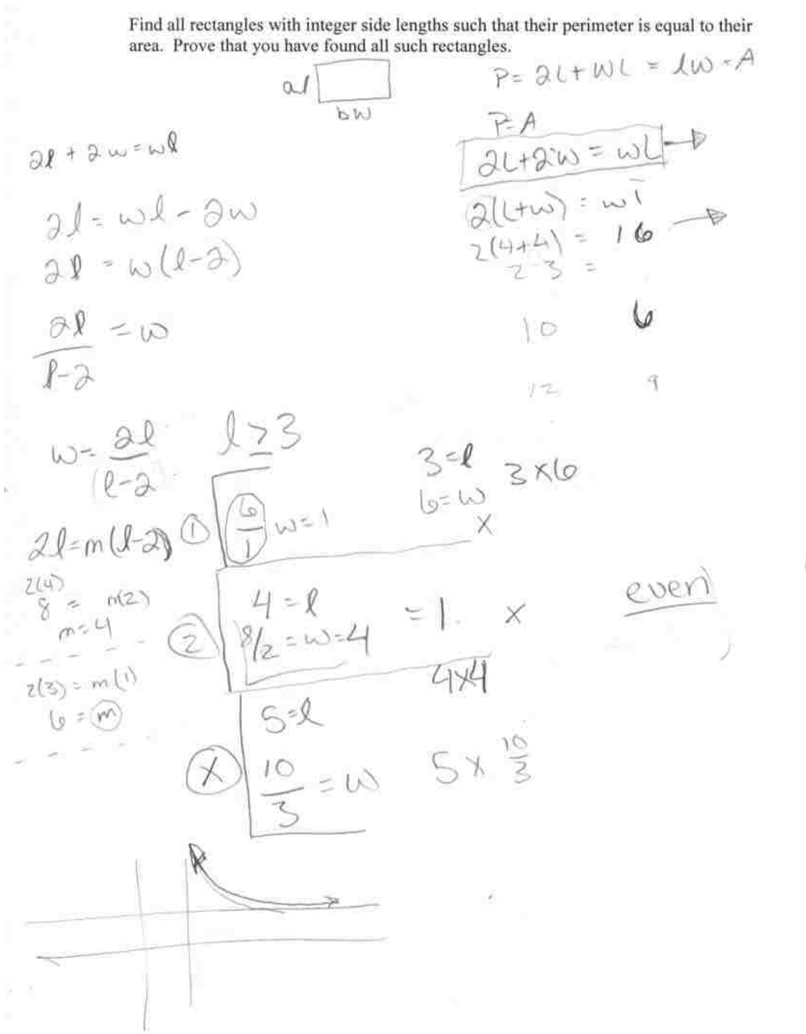


Figure 61: Helen's work on Question 1 (1 of 2)

After finding an l value that did not give an integer (l = 5), she went back to the equation $w = \frac{2l}{l-2}$ and discussed what it would mean for l – 2 to divide 2l. It was at this point that I suggested looking at the given equation as a function and she proceeded to look at the graph. We discussed what the asymptotes of the graph meant for the problem at hand. She was able to see that asymptotes at y = 2 and x = 2 implied that all solutions had indeed been found.

Helen: Yeah, it looks like it has …has a vertical asymptote at (two) and it looks like it's got another one…

Nick: A horizontal one at …

H: It never actually …it starts heading towards 2…

N: Yeah, you have like a minute and a half right now, so I'm going to sum things up, if that's ok. So it looks like it's heading towards 2, right?

H: Uh huh.

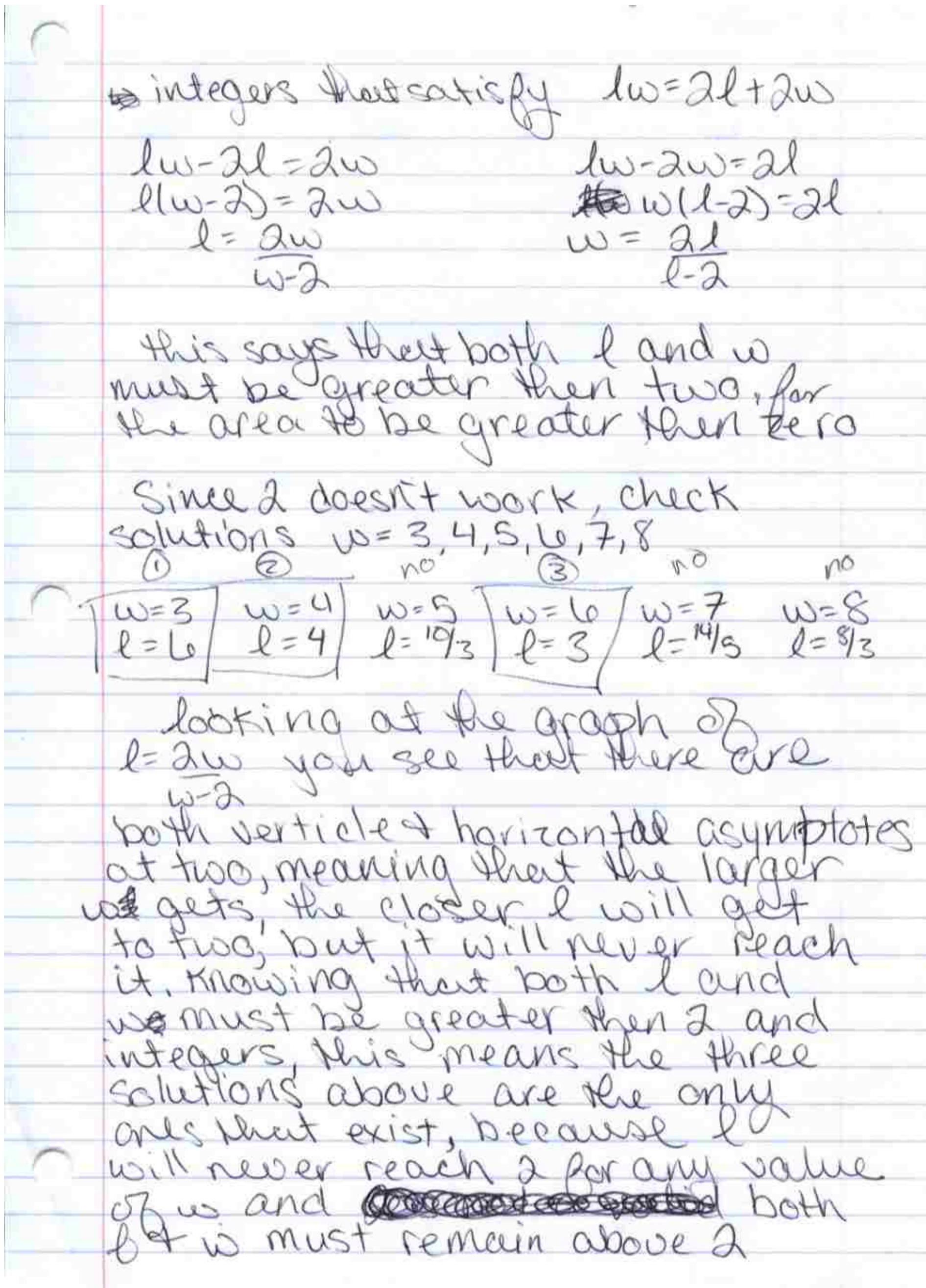
N: Is it ever going to get to 2?

H: No…

N: So what does that tell you?

H: That those are the only two.

I asked her to then write up a formal proof for the problem and bring it to the next interview. She did (Figure 62) and showed the examples that worked and provided a verbal description of the graph and why it meant there were no more solutions. This is a syntactic proof. However it was based on understanding from help, not intuition. This excludes the proof from classified as semantic. Also, the proof did not involve prescribed steps laid out elsewhere and therefore can not be considered procedural. This is also evidence that Helen has a transformational analytic proof scheme. The proof involves operations on objects (the function discussed) and the anticipation of the results of the transformations (how the outputs of the functions change as the inputs approach 2 from the left and infinity).



integers that satisfy $lw = 2l + 2w$

$lw - 2l = 2w$ | $lw - 2w = 2l$
$l(w-2) = 2w$ | $w(l-2) = 2l$
$l = \frac{2w}{w-2}$ | $w = \frac{2l}{l-2}$

this says that both l and w must be greater then two, for the area to be greater then zero

Since 2 doesn't work, check solutions w = 3, 4, 5, 6, 7, 8

①	②	no	③	no	no
$w=3$	$w=4$	$w=5$	$w=6$	$w=7$	$w=8$
$l=6$	$l=4$	$l=10/3$	$l=3$	$l=14/5$	$l=8/3$

looking at the graph of $l = \frac{2w}{w-2}$ you see that there are both verticle & horizontal asymptotes at two, meaning that the larger w gets, the closer l will get to two, but it will never reach it. Knowing that both l and w must be greater then 2 and integers, this means the three solutions above are the only ones that exist, because l will never reach 2 for any value of w and both l & w must remain above 2

Figure 62: Helen's work on Question 1 (2 of 2)

<u>Question 2a</u>

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9. (For example, if a = 4 and b = 2, then 42 – 24 = 18, a multiple of 9.)

As before, Helen began by looking at examples. Here, though, she took the examples a bit farther by organizing them based on the difference between the *a* and *b* values. For example, she made a chart in which one column had $a = 5$, $b = 3$ and $a = 6$, $b = 4$ because $a - b = 2$ in both cases. The chart, in Figure 63, was organized in this way because she had noticed from her examples that $ab - ba = 9(a - b)$.

H: It seems like there's lots of patterns and none of them help.

N: Ok, so what are some of the patterns that you've noticed besides the multiplier being *a* - *b*?

H: That you would have 7 and 3 and 6 and 3 and 8 and 3, they're all multiples 3, 4, 5 so that's 27, 36 and 45.

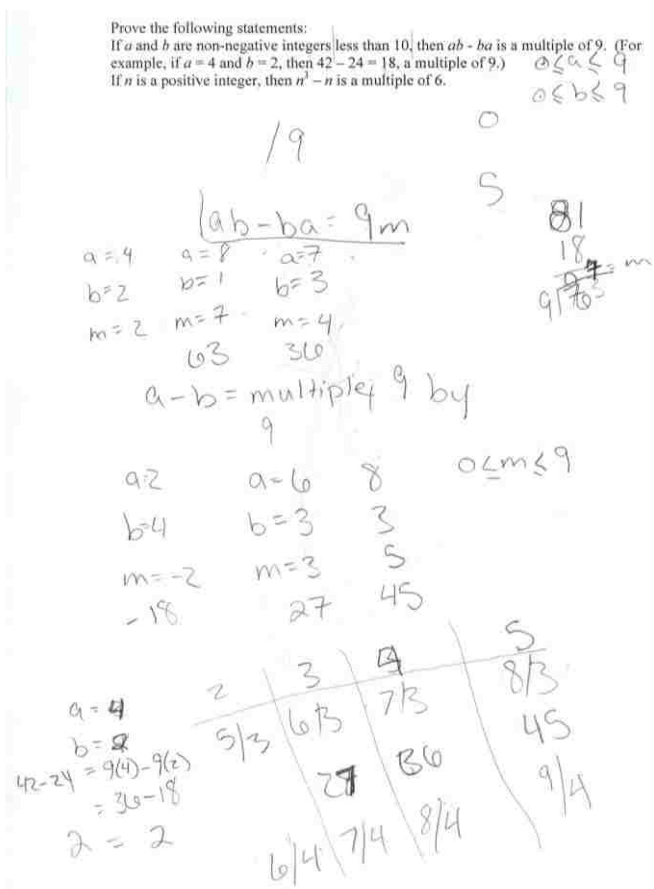


Figure 63: Helen's work on Question 2a (1 of 2)

Helen was actually referring to different manifestations of the same pattern.

From there, she began her attempt at proving this equation would always hold. She then rewrote the equation in those terms and set 9(*a* – *b*) = *cd*. She saw that this was new, but did not see how she was any closer to proving it. This brought her to realizing that in *ab* – *ba* having the numbers *a* and *b* locked in their respective places was a problem. I prodded her to realize how take 42 and write it as 4*10+2. She was quickly able to put this idea in general terms and completed the algebra with *ab* – *ba* = 10*a* + *b* – 10*b* – *a* = 9(*a* - *b*).

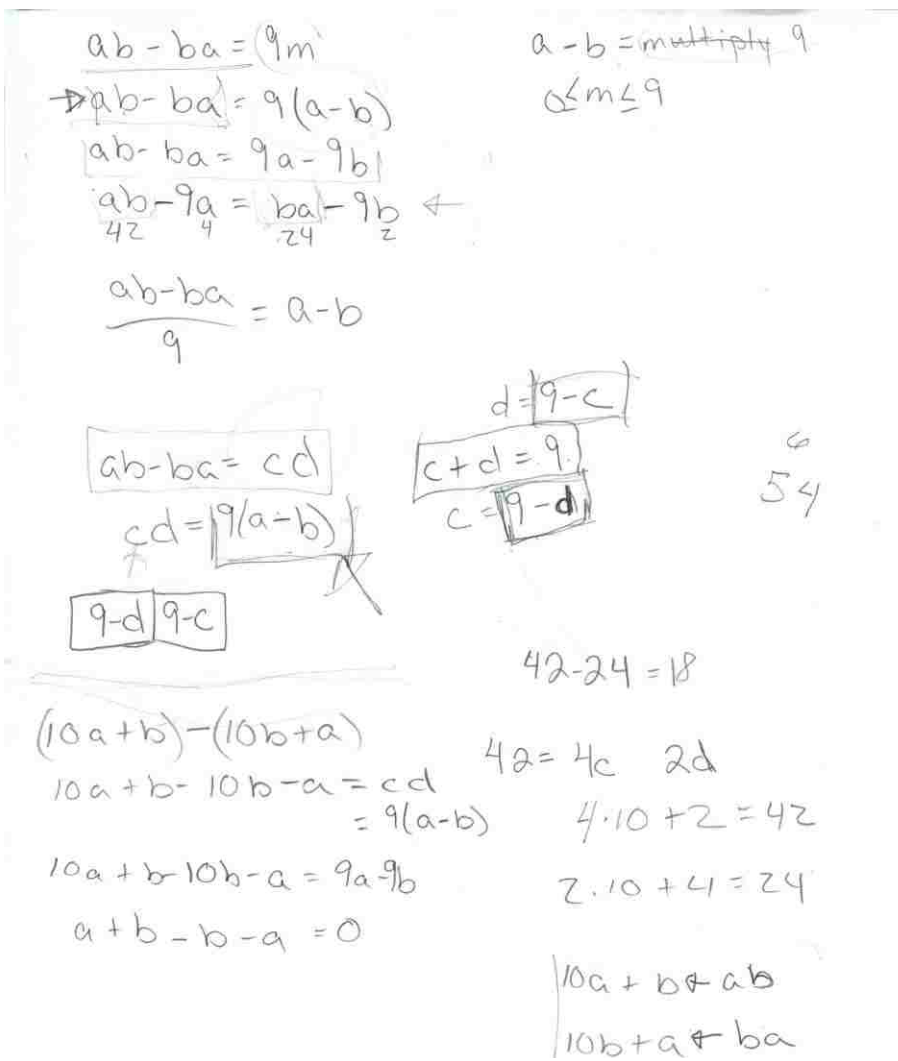


Figure 64: Helen's work on Question 2a (2 of 2)

Again, I would classify this attempt as a syntactic proof because there are no prescribed steps, and the argument is not based on her intuition. However, she is "using definitions and other facts in logically permissible ways" (Weber, 2004, p. 4). This is also more evidence that Helen possess a transformational analytic proof scheme because she is using operations on algebraic expressions to justify claims.

<u>Question 2b</u>

If n is a positive integer, then $n^3 - n$ is a multiple of 6.

This is the first problem in which Helen deviates from a syntactic proof attempt. She applied the method of induction to this problem, which is an algorithm proof. She completed this proof attempt on her own between interviews. Because we had spent so much time on the previous question, we merely discussed what she had done outside the interviews. We could have spent more time on the problem because she did not complete it correctly, but I wanted to make sure that she stayed caught up with the rest of the students in the study.

She began the problem as she had the previous two, by looking at examples. Prior examples had other influence as well, however. Like in the last problem, when looking at the examples, she focused on the multiplier (the *m* in *n*^3 – *n* = 6*m*). However, this didn't yield a pattern she felt she could work with as before. So from there she provided what she thought may have been an induction argument. Helen's work from the interview is in Figure 65. During the discussion, she mentioned that she wasn't sure if it served as a valid proof. She began the proof correctly, but then replaced $n + 1$ with x in the inductive step and said that since $(n + 1)^3 - (n + 1) = x^3 - x$. Since this final expression had the same form as $n^3 - n$, she reasoned it must be a multiple of 6 as well. She admitted when discussing this point that "I'm not entirely sure that's how induction works".

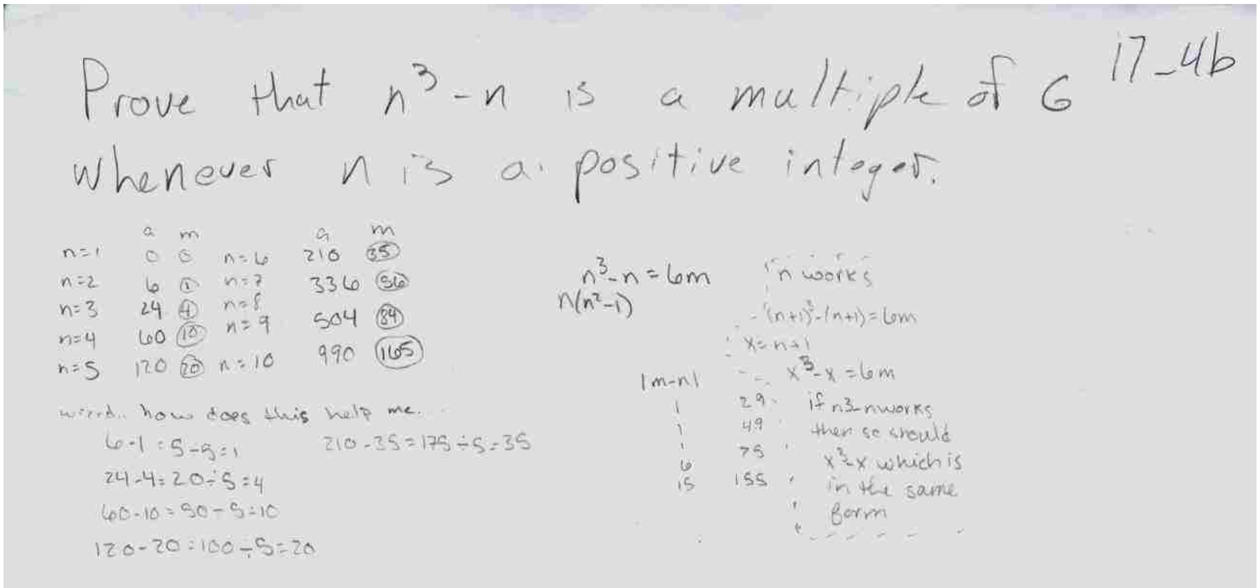


Figure 65: Helen's work on Question 2b

While this is a procedural (algorithm) proof attempt, there are some elements of what could have been a semantic attempt. She began by looking at examples hoping to gain insight into the problem (potential for semantic). She never came to any understanding that she tried turning into a proof, so this is not a semantic attempt. By instead relying on the process of induction, she also reveals that facets of her proof scheme are external. She places some trust in the induction process, even though she is not sure how it works:

> H: And then I decided that induction, I don't really understand it, because if you put the $n + 1$ case cubed, and if you set a number, $x = n + 1$, it's the same thing, so that should say that it works, but then I don't know if that's how that works.

Not only is she not sure how induction works, she is not sure if she has completed the steps properly. Because this is the closest thing she presents to a proof, she is displaying a ritualistic external conviction proof scheme.

Question 3

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

Here, since the problem explicitly asks for the use of induction, Helen's attempt used induction and is classified as a procedural proof. She showed one misconception with induction. She chose a base case of $n = 3$ instead of 1 (she also made an arithmetic mistake at this point, but more on that later). This was because she believed the higher number would lead to more insight into the problem without being too unwieldy. She did not realize that even with a correct induction proof from there she would not have verified the property for $n = 1$ and 2. This was the only difficulty she displayed with the process of induction itself.

The mathematics involved with completing the steps of induction, however, did provide difficulties. Initially when working with the inductive step, she simplified without regard to when or how the induction hypothesis would be used – showing a reliance on old habits.

> H: I went from 1 + (n + 1)/2 to (3 + n)/2, because that's what that simplifies into, but that's not what you're supposed to do with induction, you're supposed to get it so that you have the n case plus the n + 1 case, so I went back up here so that it separated out so that you have 1 + n/ 2 + 1/2, which is what this breaks done into. (See Figure 66)

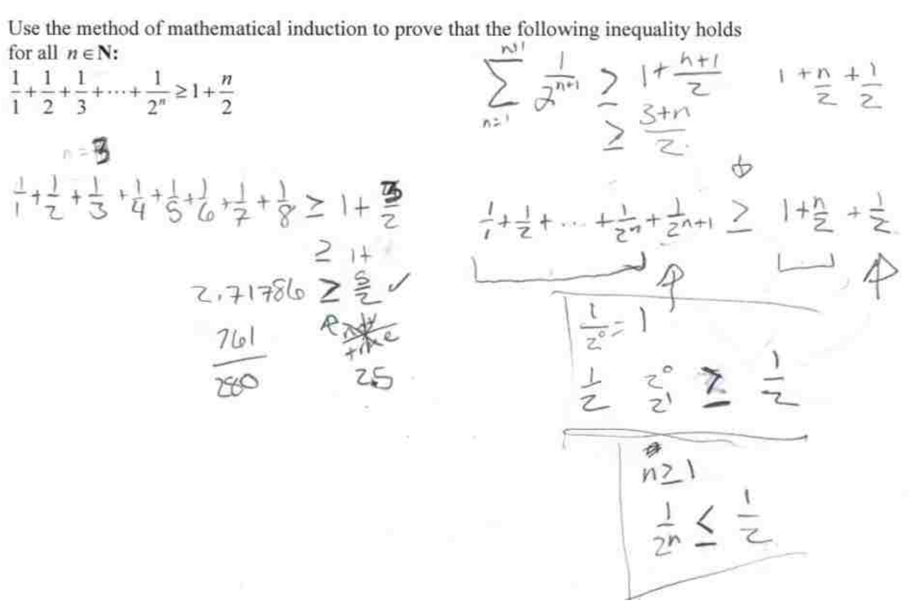


Figure 66: Helen's work on Question 3 (1 of 3)

She then broke up the problem to use the induction hypothesis, but this revealed another misconception she had: when she moved to the $n + 1$ case, she added only a single term to the sum on the left. Instead of adding all the numbers $\frac{1}{2^n+1}+\cdots+\frac{1}{2^{n+1}}$, she simply added $\frac{1}{2^{n+1}}$ to the existing sum from the induction hypothesis. This led to trouble in that $\frac{1}{2}$ was being added to the right hand side and she knew that for $n \geq 1$, $\frac{1}{2^n} \leq \frac{1}{2}$. This prompted her to re-evaluate the proof overall and she found an arithmetic mistake from earlier. This error led her to thinking that it wasn't true for the base case. She didn't pay attention to the numbers initially because she assumed it would work out. "Yeah, well, the problem is to prove that it is true, so I assumed that the base case was going to be true, so I just wrote down the numbers without actually looking at them."

When I asked about it, she realized that she would be done at that point because one counter-example is enough to disprove something. I pointed out the mistake she made (that 8 =

2^3, not 2^4) and that gave her confidence in the result again. She left the interview thinking that the left hand side in the induction hypothesis would be big enough to make up the deficit created due to $\frac{1}{2^{n+1}}$ being less than or equal to $\frac{1}{2}$.

In between interviews, she tried more examples, and came to the question of how the $n + 1$ case differed from the n case. In the examples, she used her mistaken notion of how things changed. However, noticing that the next number in the series with $n = 3$, $\frac{1}{9}$, was not in the form of $\frac{1}{2^n}$ lead her to question how the sequence was acting and that's where she quit working. See writing directly under "another question" in the figure below.

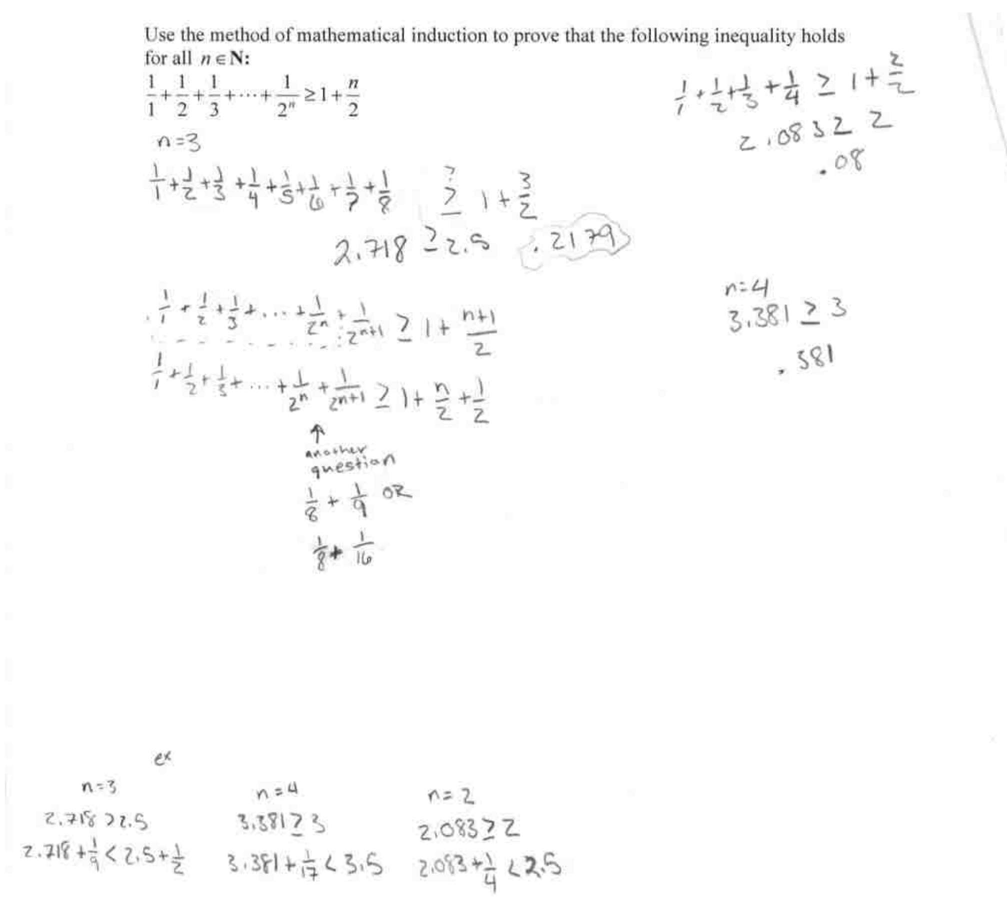


Figure 67: Helen's work on Question 3 (2 of 3)

Part of her decision making process was trying the different interpretations to see which one made the example true ($\frac{1}{9}$ and $\frac{1}{16}$ were too small to be bigger than $\frac{1}{2}$). She continued from there in the next interview. She ends up deciding through examples that correct interpretation is the right one (see figure below).

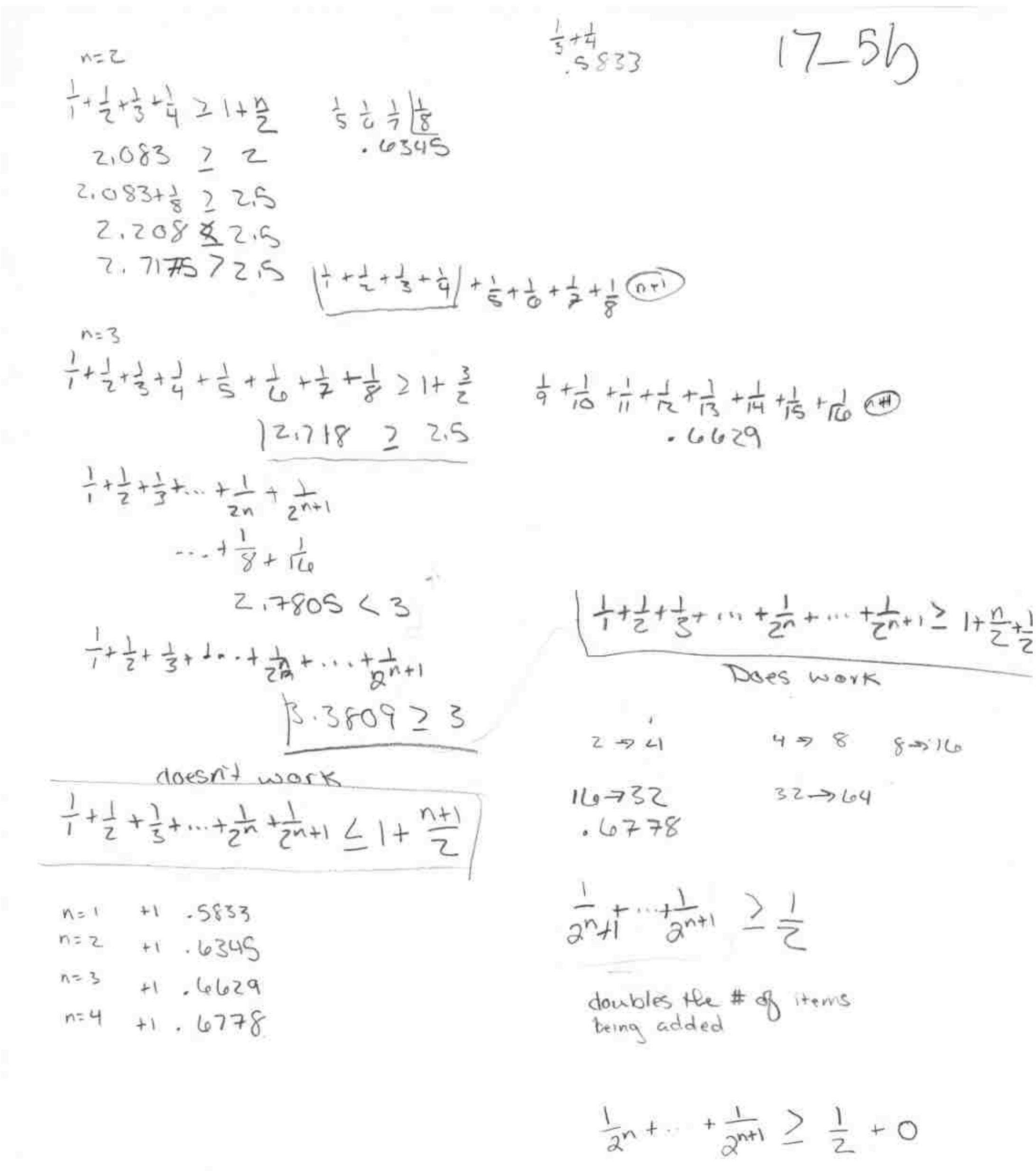


Figure 68: Helen's work on Question 3 (3 of 3)

She then considers the series $\frac{1}{2^n+1}+\cdots+\frac{1}{2^{n+1}}$ and asks about ways to decide if it's increasing, converges or goes to infinity. She then works a bit more and realizes that increasing n by 1 doubles the terms in the series and I talk her through the rest of the proof.

Although this is an algorithmic proof attempt, it still reveals some things about her proof scheme. For example, when discussing her attempt, she says "Well, as far as the induction goes, I'm pretty sure that I did this part right, that this is the actual way to do induction." However, not much later in the interview she asks "Did I do the induction right?" So, while she largely had the induction set up properly, she still wanted verification from an authority figure. Combining this with the fact that she did not use the correct base case reveals that she has not yet interiorized the method of mathematical induction. Thus, she is displaying an internalized transformational proof scheme.

This is not the only proof scheme Helen shows, however. The fact that she asked if she did the induction correctly goes with her disregard of her base case (she assumed it was true without verification) and points to an authoritarian proof scheme. She also displays an empirical proof scheme by checking her various interpretations of how to perform the inductive step by checking examples.

<u>Question 4</u>

The following interview involved the evaluation of the following proof:

Examine the following proof by contradiction:

<u>Prove</u> $\sqrt{-1} \leq 0$

Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

<u>Prove</u> $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$, *then* $-\sqrt{-1} \geq 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$, *so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This problem had been given to Helen the previous semester on her mid-term exam. As she was not asked to complete a proof on her own, this interview can not be classified using Weber's criteria. However, evidence of proof scheme does exist. The proof she gave on the midterm is below.

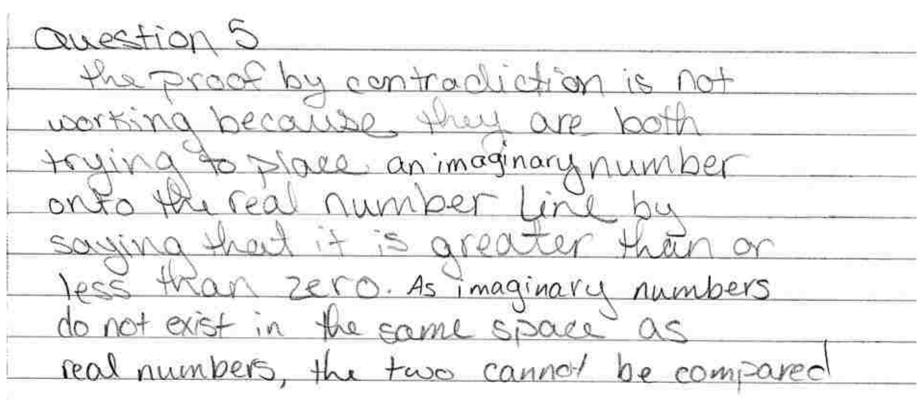

Question 5
the proof by contradiction is not working because they are both trying to place an imaginary number onto the real number line by saying that it is greater than or less than zero. As imaginary numbers do not exist in the same space as real numbers, the two cannot be compared

Figure 69: Helen's previous work on Question 4

During the interview, Helen finds a few different problems with the proof presented. First, she mentions the problem of applying real number properties to imaginary numbers. "So…can it be, since it goes against everything that that all the real numbers…are? By being the square root of a negative number, maybe it can't be held to the same rules." Here, she is saying that a problem rests in the fact that operations are being performed on objects to which they do not apply. This presents evidence of an analytic proof scheme because she is stating that the conclusion has not been deduced properly.

When I asked whether she would have believed the proof if only the top half had been given, she said:

> H: Yeah, I think that this original statement here (first inequality that says 'Prove') has a problem because that would be mapping it to a real number space and it doesn't belong in that same space.
>
> N: Ok, so you're saying that somehow making this comparison makes the square root of -1 a real number?
>
> H: It compares it to a real number.
>
> N: Right
>
> H: And I'm not sure that you can do that.
>
> N: Ok
>
> H: Can you do that?
>
> N: Well, that's kind of all part of this and I'm asking you.
>
> H: I think that even without the 2^{nd} one, that I would have a problem (with that).

This implies that Helen is not relying on the fact that contradictory results had been attained and still harbors the doubts she had when she did it on the midterm. She also seems prepared to accept my word on whether the comparison can be made. However, since Helen does not actually do this, I will stop short of calling this evidence of an authoritarian proof scheme.

Helen's display of an analytic proof scheme is hard to pin down into a sub-type (transformational or axiomatic). Her reasoning above has transformational elements to it (the involvement of operations and anticipation of results). However, there is not enough evidence to say for sure. Instead, there is clear evidence that she is thinking deductively. She says that when you do not have a real number you don't know…

H: …technically, where that is in comparison to 0.

N: Ok, so location somehow plays a role in this?

H: Yeah. Because saying a number is less than or greater than 0 is like drawing out a number line…

N: Ok, and seeing if it's on the right or left of zero.

H: Yeah

This also matches what she said on the midterm. It emphasizes that she sees the need for certain conditions to be satisfied before deductions can be made.

She also sees the combination of the 2 provided proofs as a proof of what she's saying, that *i* is not on the number line: "And I think that the two of them together maybe proves that's not on the line. Because you can't be both greater than and less than and equal to 0." This shows, possibly without her knowing, that she has a grasp of how proof by contradiction works.

Question 5

The next interview was a debriefing session that occurred at the end of the first semester of the study. In it, we discussed how Helen felt the semester had gone in regards to proof. Again, since she did not produce a proof, Weber's classification will not apply.

The first thing she mentioned was moving way from the view that all proofs are procedural proofs:

> H: Last semester it (proving) seemed very strict. Like, a proof was a certain way, and it had to be a certain way and it had to be very rigid. And that's maybe not the case. It seems a little more…so it started out, in here especially, where you would give me a problem and then I would only look at those certain ways of doing it, like the steps he made us go through to do things that certain way. And it never ever worked until that problem where you set it up to specifically use induction. … (In the proof writing class) they focus on, like, three things. And then they put you into these little boxes and then it gets really hard to get out of them.

This shift in view implies to me that Helen is beginning to take on more of an analytic proof scheme and away from an external ritual proof scheme. She goes on to say:

> H: I think, being able to think, to see the patterns, but not always being confined to what you already know. … Because I know that was my problem a lot. Just looking at it and trying to apply what I already know to it instead of reading what it says and then…
>
> N: Ok, so kind of like, if you have one equation that talks about, that says something about the problem maybe, I don't know if it's the case for you, but some people would keep manipulating that one equation, hoping that something would come out of doing that.
>
> H: Yeah

An example of what she's referring to occurred in Question 2a. There, she recast $ab - ba = 9(a - b)$ as $ab - ba = cd$, then $cd = 9(a - b)$. Here, she is not actually doing anything new with the problem. What she needed instead was a transformational analytic approach where she rewrote $ab - ba$ as $10a + b - 10b - a$. This sort of transformation would have needed the anticipation that isolating the individual digits was necessary for finishing the problem. This anticipation is a hallmark for the transformational analytic proof scheme.

The discussion also included whether she thought she improved over the course of the semester:

> H: I think that's a really, really hard question to answer. Because the more, I noticed this in this class and in here, that the more I work with them, the more problems I have with them…Yeah, I think I might have gotten better, but then I find new problems that I don't know how to answer…Like, I think that I'm doing better, but then, I guess more knowledge of what I'm doing points out more problems with it.

Another topic included was what led any improvement she did see: "Do it more. And more. … and then knowing more kinds of proofs… Like, understanding that there's more than one way to prove something, like you can't prove everything through induction."

<u>Question 6</u>

Prove that $\forall n \in \mathbb{N},\ n \equiv n^3$ *(mod 6).*

This was the first problem of the second semester and it was a recasting of sorts of the problem that was 2b. Helen did not recognize this, however, until I pointed it out in the reflection. Like earlier problems, she started by looking at a few examples and then began a proof by induction. When asked why she tried induction, she said "(b)ecause all the other ideas I had didn't seem to work. Like I didn't know how to prove it by contradiction with the congruent part…I didn't know how you would say that n is not congruent to n^3."

In this attempt, she set up the induction properly and performed the steps properly. She used the induction hypothesis after multiplying out $(n + 1)^3$. After subtracting 1 from each side, this left her to show that $3n^2 + 3n$ was congruent to 0 (mod 6). She made this deduction from the applying the division algorithm that she had recently learned in number theory (knowing that congruent to 0 (mod 6) meant $r = 0$). From there, she factored out the 3 and reasoned via cases (n even and n odd) that $n^2 + n$ was in fact an even number.

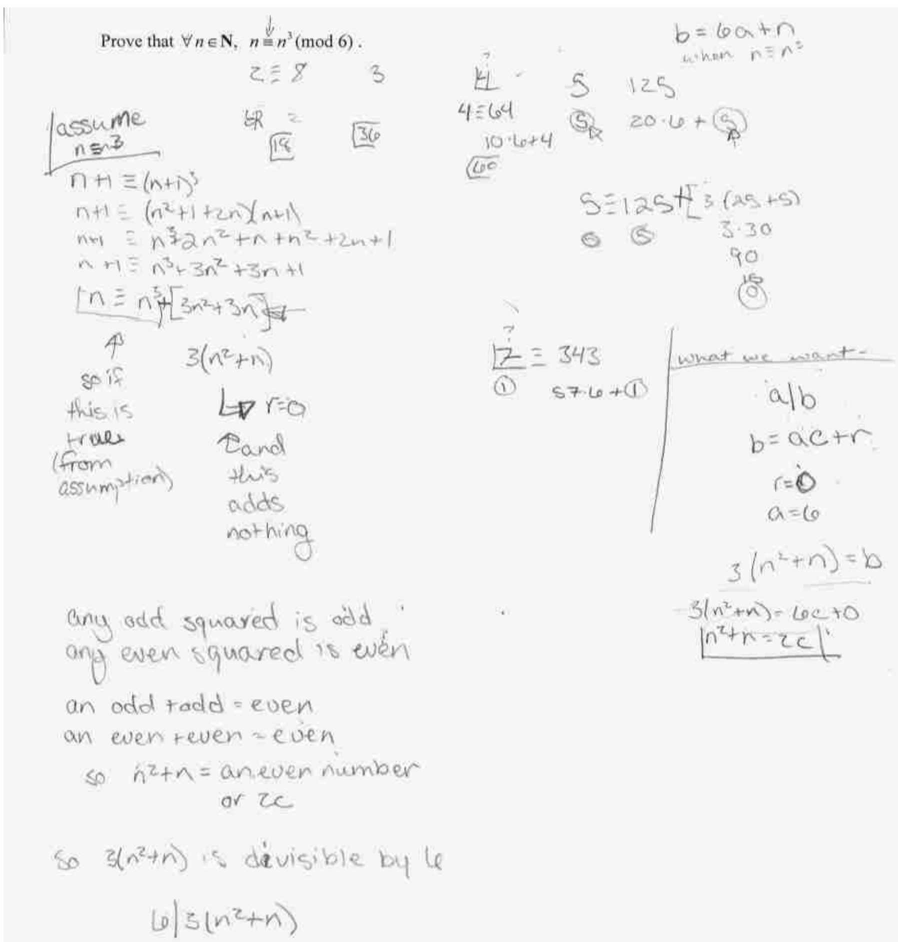


Figure 70: Helen's work on Question 6

This is a procedural proof, as it makes use of induction. It is not algorithmic, however, because she does not follow prescribed steps throughout. She had to adapt and come up with her own technique when completing the inductive step. Since this proof does not typical of the induction arguments she saw before (where the inductive step involves only algebraic manipulations to verify an identity), the proof she gives is a process procedural proof. This differs from the other induction proofs Helen has attempted and shows that some progress has been made.

Her proof scheme has not changed much here. She asked a few different times through the course of the proof attempt whether she was allowed to perform certain operations. She was satisfied when I told her which were allowed with modular arithmetic (for example, subtraction is but division is not).

H: How does the congruent, this part work? With like operations.

N: So, what do you mean?

H: Could you subtract across it, could you …

N: Um, yeah, so you can subtract across it, you can add across it, I think you can multiply as well, but you can't divide.

H: 'K

This points to the authoritarian external conviction proof scheme because she was comfortable taking my word without seeing proof.

Helen did complete a proof by induction, so there is evidence of the transformational scheme as well. Because she does not write up a formal argument (and because I did not think to ask at the time), it is not clear whether or not Helen views an example she tried as the base case for her induction argument or if she tried $n = 1$ mentally. Because of this ambiguity, I will refrain from labeling this an internalized transformational proof scheme.

It is worth noting that during the reflection, when I pointed out that this is really the same as a problem she's seen and she said: "This one's way easier. … I'm excited about the idea that maybe I'm just smarter."

Question 7

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

Helen started the problem by looking at examples of sets of size 4 and 5. She initially did not consider the empty set a subset, which became important as she moved along. She proceeded by counting up the numbers of subsets of each size, but not adding these numbers to come up with the number of subsets of a set of size, say, 4. She thought that the total number of subsets would not help her. Instead, she was looking for a pattern that would give the number of subsets of a given size of a superset of different given size, for example, how many 3 element subsets there are for a set of size 4. She would then check to see if that formula would give the number of subsets of size 3 for a superset of size 5. This was based on a guess for the number of subsets of size 2, which she thought was $(n - 1) + \ldots + 2 + 1$, due to not including the empty set. Her reasoning was there were $n - 1$ things to pair with the first element, $n - 2$ ways to create a new 2 element subset with the second element and so on. (See Figure 71.)

Once she began finding the total number of subsets, I told her about the empty set and this led her to guess the 2^n formula. From there she started noticing the numbers she was finding for subsets of particular sizes matched rows in Pascal's Triangle. She left the first interview with this in mind and I told her that she could use the fact that the rows of Pascal's triangle sum to 2^n because it was proven in 305.

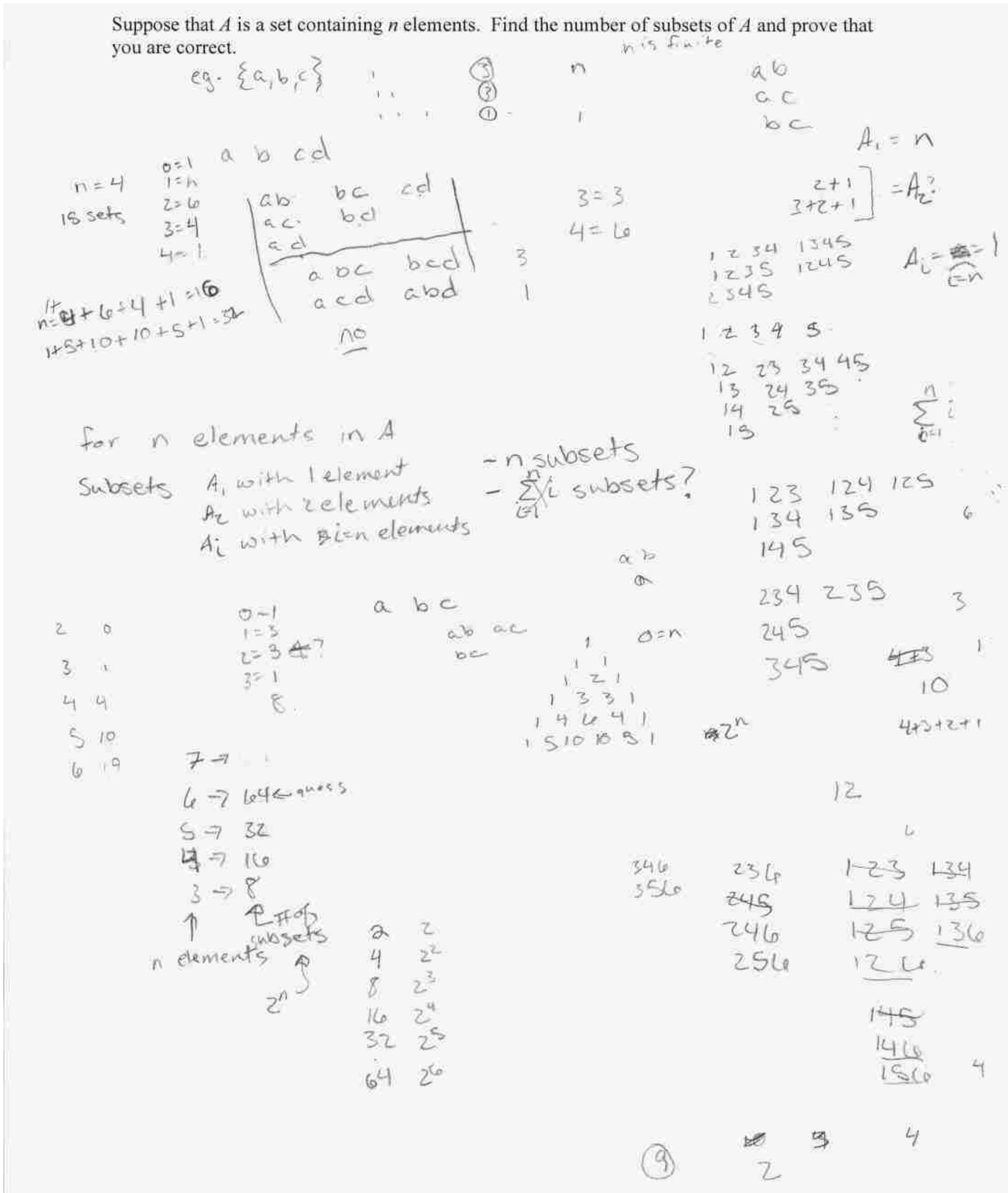


Figure 71: Helen's work on Question 7 (1 of 2)

In between interviews, she tried thinking back to her probability class. Eventually, she realized that the number of subsets of size k of a superset of size n was nCk. Also, she remembered the results from Pascal's triangle. Helen put these ideas together but thought she needed to somehow prove more and gave consideration to using induction.

> H: Well, it says to prove that you're correct, that that's the number of subsets of *A*. But then, like I guess the reason that I didn't just go through with all of the induction is that, like I don't…I don't see how going through all of the induction would really help prove that.
>
> N: Why is that?
>
> H: Well, *n*C*r*, is the way to find a number of subsets of a larger set with a certain number of things in it.
>
> N: Right
>
> H: And that's…I guess, like, that was kind of a stopping point for me because that's just how you do it.
>
> N: Ok
>
> H: So I didn't see how induction would really make it more true than it already is.

Figure 72 shows the work she did between interviews, including the work she did while going beyond her "stopping point". Helen realized that since the total number of subsets was the sum of the number of subsets of each particular size and because *n*C*k* gave the coefficients in Pascal's triangle, she could use the result from 305 that gave the number of subsets to be 2^n. Helen was also able to verbalize what her proof would look like if she were asked to turn one in for homework.

> H: I think if I wrote it out the way that I was thinking about it, I would then (be comfortable calling what I did a proof).

N: If you wrote out your reasoning or just wrote out what…

H: From beginning to end, like stating that this is an unordered pick of r numbers out of n things without replacement.

N: Right

H: Like, that's exactly how to find the set that it's asking for, the subsets that it's asking for.

N: The number of subsets of a particular size.

H: Yeah. And so that's all the subsets of that size in that set.

N: Right

H: And then that goes directly into this that says 2^n is the total number of subsets in that size of a set.

This is a semantic proof. Helen uses her understanding of the purpose of the choose function to come up with a formula that is self-explanatory and therefore is proof. This proof displays an axiomatic proof scheme due to her reliance on previously proven results (the work done in 305 in particular) and her knowledge that she would have to prove the result if it had not been proven already.

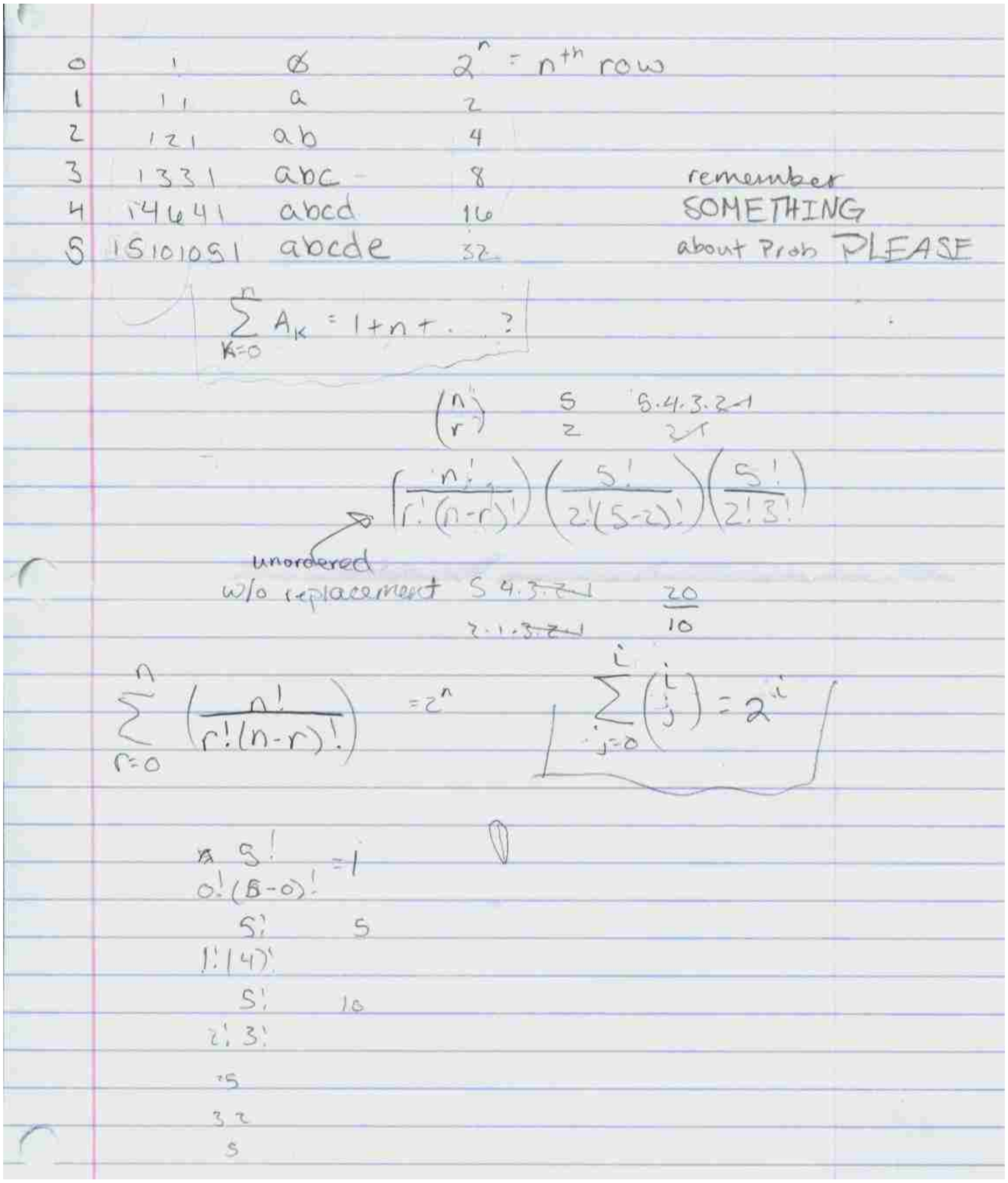


Figure 72: Helen's work on Question 7 (2 of 2)

Question 8

Prove that the cube root of 2 is irrational using a proof by contradiction.

Like Question 4, this question was a repeat from Helen's mid-term exam in 305. Her response on the midterm is in the figure below.

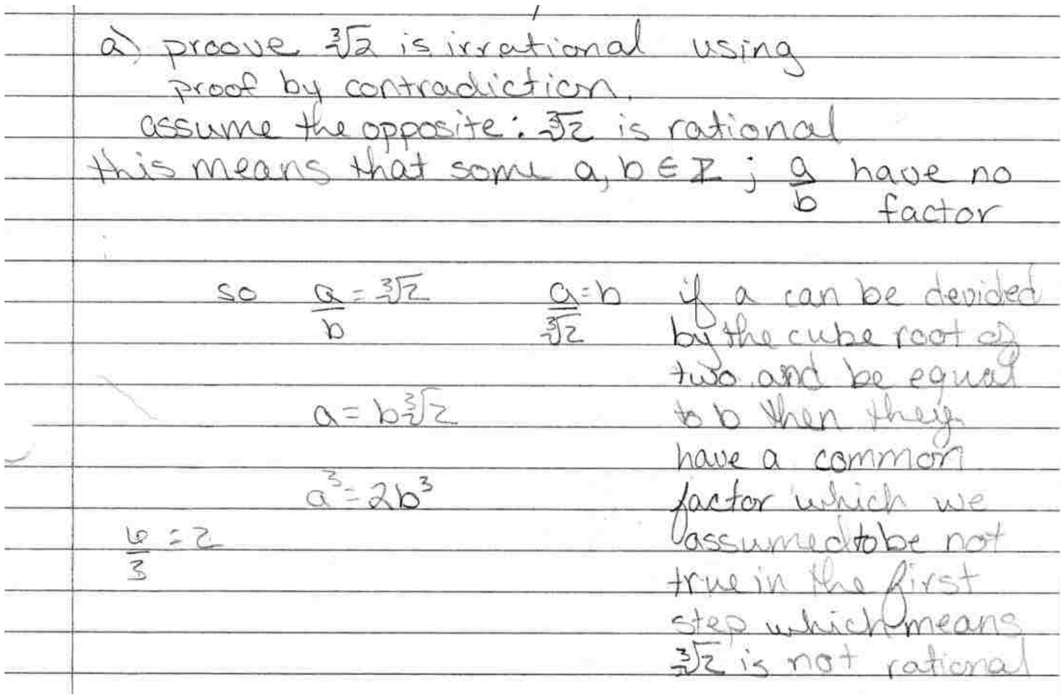


Figure 73: Helen's previous work on Question 8

Unlike Question 4, Helen made some progress here because she was not successful on the mid-term. She had seen the proof for the $\sqrt{2}$ in class shortly before the mid-term so at the beginning of the interview, I suggested that she attempt that proof first if she was unsure of how to complete this one. She spent the first interview in which this problem was discussed doing that other proof. See Figure 74 for this work.

She did not have trouble setting up the contradiction in either setting (interview or on the mid-term). She started the problem properly, assuming that $\sqrt{2} = a/b$ where *a* and *b* are integers and getting to the point of $a^2 = 2b^2$, where she assumed (from memory) that *a* and *b* had no common factors.

H: I think, are you supposed to say something about *a* and *b* being the smallest numbers possible and that's where the contradiction was?

N: It's possible. I mean, I'm being purposefully non-committal here. If that's what you think it might be, try to get something out of it. Does that make sense?

H: Yeah…I think my problem is I don't know where I would go with that.

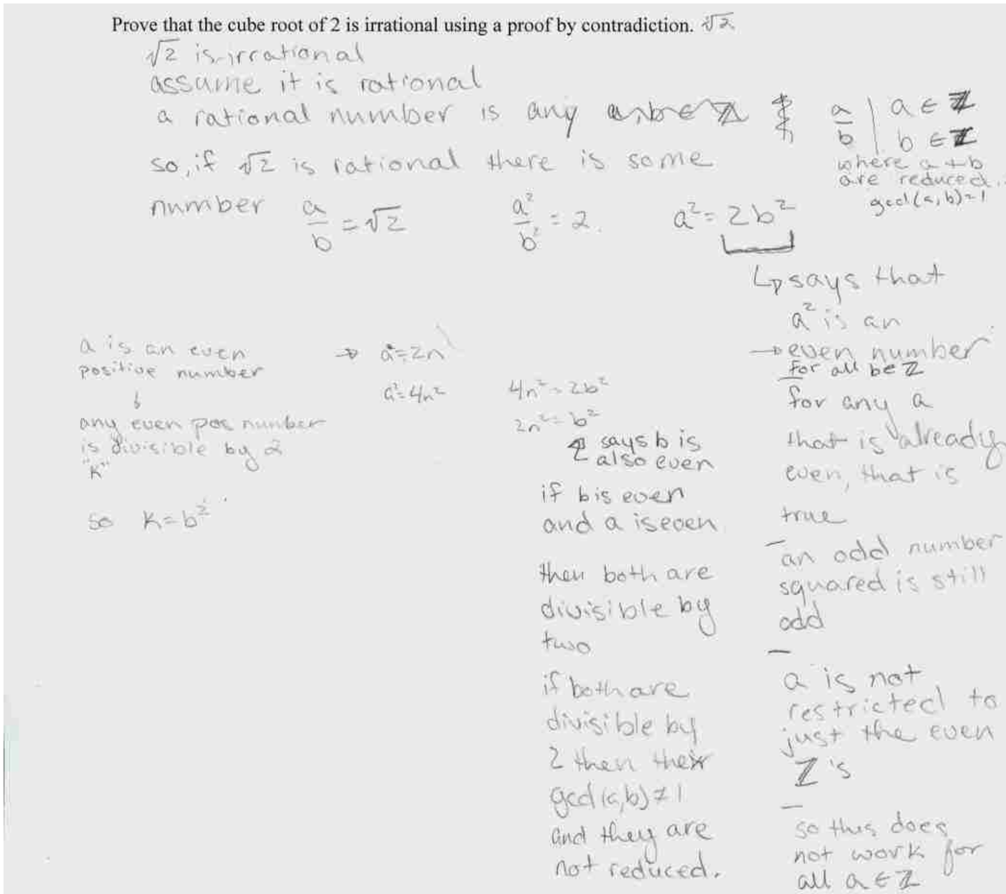


Figure 74: Helen's work on Question 8 (1 of 2)

She thought that she deduced a contradiction from there. Her argument (down the right side of the figure) was that this last equation implied that *a* was even, although it need not be from the starting assumptions. I explained that she didn't finish and she accepted the fact that she didn't reach an absurdity because *a* being an integer allowed for it to be even.

She then picked up work using the deduction that a^2 (and thus *a*) was even. She deduced that *a* was even via contrapositive. She finished the argument for $\sqrt{2}$ when I reminded her of the definition of an even and asked her what it would look like to use that. She then deduced that *b* must be even as well. I reminded her of the starting assumptions and she realized that it was a contradiction. She was happy with her argument because she knew it had to use the fact that *a* and *b* were assumed to have no common factors. She left the interview knowing how to finish the $\sqrt{2}$ case, and returned with the cube root case written out nicely.

Once she understood the $\sqrt{2}$ case, the $\sqrt[3]{2}$ case was straightforward (see Figure 75). In fact, she remarked that she had finished it on the bus ride home after the initial interview. Because the $\sqrt[3]{2}$ case is so similar to the $\sqrt{2}$ case, and that was where the real work was, this is the proof that I will address. This was a process proof attempt. While this distinction is generally reserved for induction proofs, this proof fits the criteria as well. The main reason for this is that she had a general idea in her head of the steps required and the work was in completing them. She knew that she was to assume the opposite and an additional assumption:

H: I was fairly sure that this needed to be said ("where *a* and *b* are reduced").

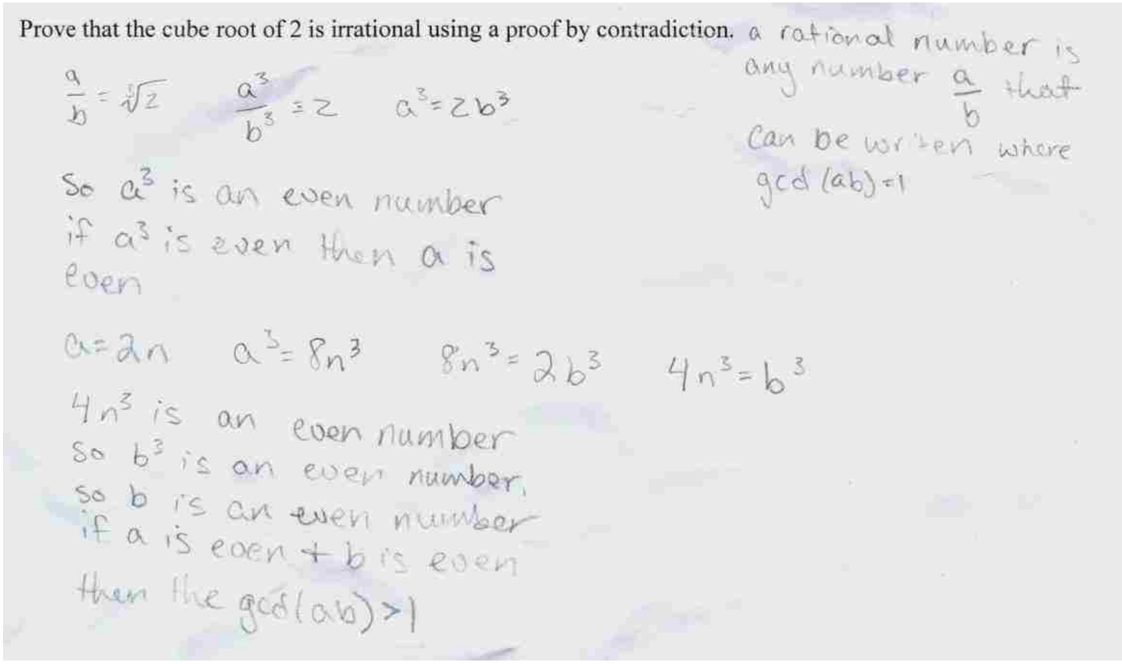
N: Ok

H: Like, I remember that being important in the proof.

N: Right

H: But I didn't know…like how this was going to…

She needed help completing the required steps, but she definitely had a global goal that needed to be accomplished in mind while attempting the proof.


Prove that the cube root of 2 is irrational using a proof by contradiction.

a rational number is any number $\frac{a}{b}$ that can be writen where gcd (ab) = 1

$\frac{a}{b} = \sqrt[3]{2}$ $\frac{a^3}{b^3} = 2$ $a^3 = 2b^3$

So a^3 is an even number

if a^3 is even then a is even

$a = 2n$ $a^3 = 8n^3$ $8n^3 = 2b^3$ $4n^3 = b^3$

$4n^3$ is an even number

So b^3 is an even number,

so b is an even number

if a is even + b is even

then the gcd(ab) > 1

Figure 75: Helen's work on Question 8 (2 of 2)

Although this was a process proof, Helen was using a transformational proof scheme. While she followed general steps in completing the proof, she did not rely on seeing those steps alone as validation for the proof. She understood the implications of the operations performed (even if she needed prodding to perform the correct ones). Once those operations where performed, she saw the starting assumption that was violated, completing the proof by contradiction. While her proof relies heavily on operations, Helen realizes that the proof hinges on the anticipatory act of assuming *a* and *b* are relatively prime. This is kind of action is a tell-tale sing of a transformational proof scheme.

<u>Question 9</u>

This question was another time in which I asked Helen to evaluate an argument. Therefore, I will not be using Weber's classification system here. For this problem, I gave her a version of Cantor's diagonalization argument. The reader is referred to the Chapter 3 to see the proof, as it is quite long.

Helen needed to be reminded about some terminology (including natural numbers, real numbers and countable sets) before reading the proof. She also had trouble with the concept of talking about a function when she did not know how it was defined. Once she was comfortable with the fact that the function was just saying where each number was in the list, she was comfortable with the argument as a whole. Initially, she related the notion of countable to the well-ordering principle, which she had seen in her number theory course. This led her to say that she had not seen a proof like this go "past the list part". She mentioned that it took reading through it a few times, but that she understood the proof.

When I asked if she saw any hidden lemmas, she mentioned the statement that every subset of a countable set is countable. She did not give much the statement thought while reading the proof because "(i)t seems silly to think that…if the bigger thing is countable then the smaller thing isn't." This brought out a different misconception, however, because she went on to say, "Well, it seems kind of absurd that if an infinite set is countable that some small finite part of it wouldn't be countable." This is shows that a couple things. First it pointed out her belief that every subset of an infinite set is finite. After a brief discussion, she seemed comfortable with the fact that this is not the case. Her statement also implies that she thought that (0,1) was a finite set as well. I believe she did not make that connection, however, because she did not object to the idea that every decimal had a corresponding natural number later in the proof. She seemed to be alright with the statement regarding subsets of countable sets because oftentimes in proof assertions are made that aren't backed up.

> H: Well, like for most of these proofs I notice that there's a lot of places that happens…Where you're like working through the proof and you get to a point where, at no point on that piece of paper, what you're saying has been proven…So if you're just, like if someone who didn't know anything about it were looking at it, you would just be making random claims…about how things work.

In this case, the assertion made was alright because "(w)ell, like, in ℝ you're already assuming that everything is countable. Like, everything is countable and then you're taking this teeny tiny section, comparatively, of ℝ and saying that it is countable. So that just makes sense." Of course, this development opened up the possibility that she didn't understand the proof as well as she thought. However, at that point I was confident that she did.

In this interview, Helen displayed an axiomatic proof scheme. She understood that the proof relied on definitions and previous results, even if her conceptions of the definitions and results involved were shaky at times. There are shades of a ritualistic proof scheme in that she related this proof to that the reals are not well-ordered. However, this was mentioned in passing and had little bearing on whether or not she found the proof to be valid.

<u>Question 10</u>

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

Helen didn't finish this problem in the first interview. In her proof attempt, she labeled the numerators and denominators in the inequality with variables, $\frac{b-1}{a} < \frac{n}{c} < \frac{b}{a}$, and then tried various re-arrangements (Figure 76). I was surprised that she did not try more examples as she had in the past. She mentioned a few as they came to her as she was doing the algebraic manipulations. However, these were lower numbers (denominators of 1 and 2) and led her to questioning whether the result was true. The low denominators also led her to considering negative denominators, but I told her to disregard those.

> H: And I made that [the inequality mentioned above] and I started messing with it, so I got rid of all the denominators and put them up on top…And that's this (second row of manipulations in the figure). And then…like you can move that around a bunch of ways. … But, like, it's still the original statement.

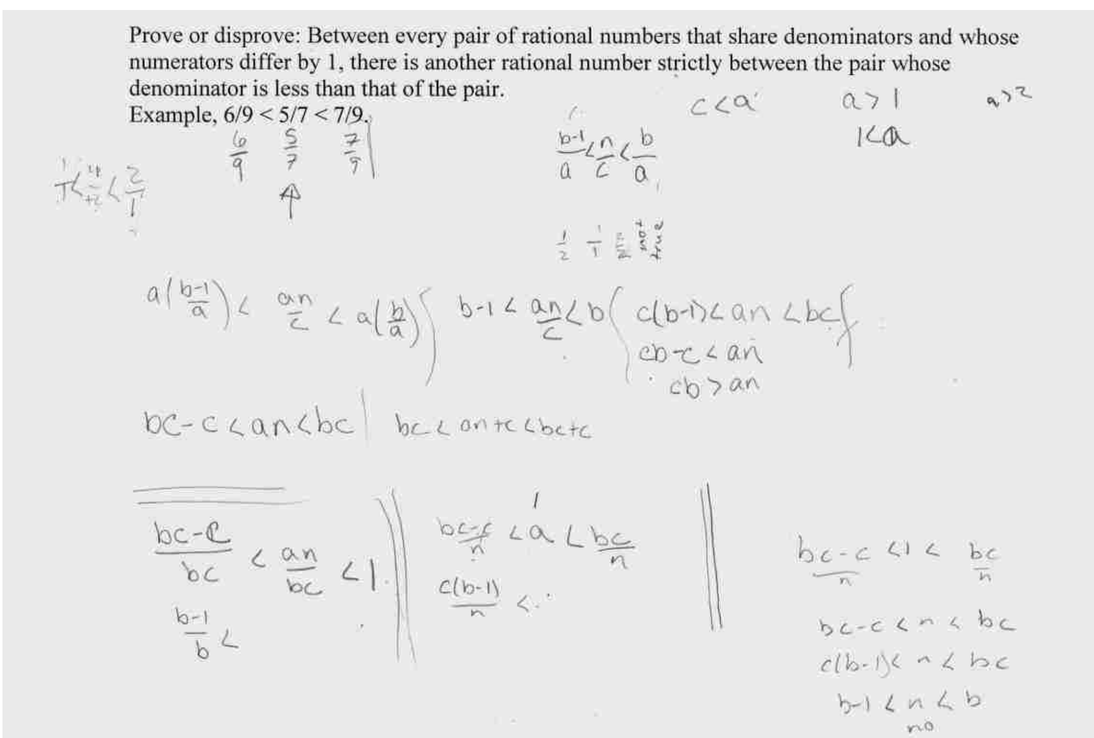


Figure 76: Helen's work on Question 10 (1 of 2)

Helen sees the relative futility in what she's doing, but she doesn't know what else to do. She considered other courses and they seemed less fruitful. Helen mentioned considering induction and also that part of her manipulations were made in an attempt at a proof by contradiction, but there was no evidence of either from her work. She left the first interview knowing it was not true in all cases and I asked if she could find some restrictions that would make it true.

In her out-of-interview work, she looked at some examples and came up with the necessary restrictions but made no attempt to prove it (see Figure 77). She got the restrictions by using circles to represent the unit interval.

> H: So, I went through one by one. 1 doesn't work and 2 doesn't work, I found that out when I was here last time.
>
> N: Yeah
>
> H: So I tried 3 (for the denominator) and…half of it works (the line for 1/2 in the circle falls between 1/3 and 2/3).
>
> N: Ok
>
> H: I didn't check any of them between 0 and 1 and then the last one and 1 because it makes sense that there wouldn't be any in between those.
>
> N: Right
>
> H: Because they're the smallest pieces, nothing is smaller than that because that's the smallest piece so far.

She didn't consider fractions outside the unit interval on paper. She also looked at many numerical examples, but did not focus on the pattern that would have led her to a proof. She did see a pattern in a few examples in which the numerators increased by moving left to right, but only after the numbers were put into reduced form. “Well, like for something to be in between, like this is 1/2, this is 2/3, this is 3/4.” From there, I pointed out and proved for her the pattern that works to save time.

In this proof attempt, Helen is showing signs of both a syntactic and semantic proofs. I use both classifications because neither part of her work can be considered her “main” attempt. During the first interview, she resorted to symbol pushing when she saw no other recourse (syntactic attempt). Between interviews, however, she used diagrams to gain an intuitive (at least partially) understanding of the problem but failed to convert this understanding into a formal argument (semantic attempt). The proof scheme being displayed here is the transformational scheme. In both the initial work and her work outside the interview, she is performing operations on mathematical objects. There is no real reliance on previous results (besides real number properties) which would imply an axiomatic proof scheme.

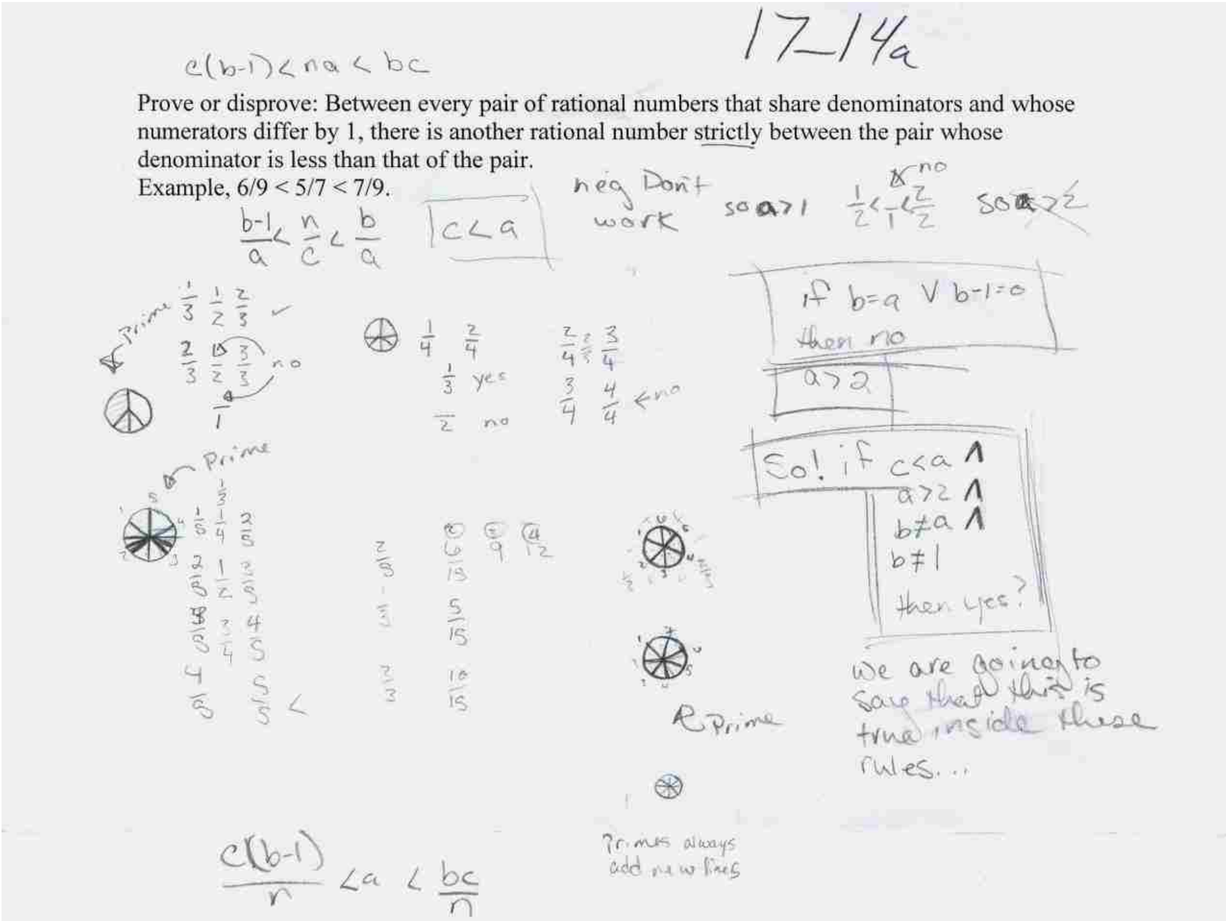


Figure 77: Helen's work on Question 10 (2 of 2)

<u>Question 11</u>

Like Question 5, this interview was a debriefing session that concluded the semester. This was the last interview I had with Helen.

Helen said that she definitely felt as though she had improved at proving over the course of the year, and this semester in particular:

> H: I feel much more comfortable with it. I think that last semester I was all like “Oh my goodness, induction”, like everything, I was like “how do I do induction with this?”
>
> N: …(W)hat do you think led to the improvement you saw?
>
> H: Practice

N: Practice?

H: Practice that leads to different kinds of proofs.

N: Ok

H: Where it's absolutely impossible to use whatever you used last time… I think it was a combination of the class that I'm in (number theory) and just how many how many we've done.

N: …Ok, so yeah, so you think you saw a lot of, you saw and were forced to use a lot of different methods in number theory, along with just sheer practice.

H: And I think that that was something that hadn't really been dealt with a lot, like proofs that didn't really follow like some strict set of rules, like this is how you do a proof step by step.

She also mentions that taking the time to understand a problem before attempting the proof has helped her comfort level with proofs.

H: (L)ooking (at a problem) without immediately applying some rigid form of…structure…Just looking at how it works first.

N: Instead of just plowing through and…

H: ...trying to force it into some little box.

To this end, she said of examples: "I think they're kind of huge. They show you where the patterns are." This matches the fact that both of the times she attempted semantic proofs came later on in the study.

When asked what was necessary to complete a proof and what was helpful but not necessary, Helen said that examples are necessary, as is an understanding of the problem and the language of math (notation) and how to manipulate it (rules). Also, in some circumstances, taking a break is necessary:

H: I think that the taking a break is helpful but I kind of think it's also necessary.

N: Ok

H: Because like with the last problem, if you just sit there and you go over it and over it and over it, I would never have seen the…

N: …pattern that helps?

H: Yeah. But if you walk away from something you can get a fresh perspective…

N: Right

H: ..and look at it new and refreshed.

Like in the other semester ending interview, Helen talks about moving away from a ritualistic proof scheme to an analytic one. This matches her latest attempt in which she went looking for understanding in how the problem was working in hopes that it would lead to a proof. This also occurred while she was working on Question 7.

Helen's progression

Below is a chart of each question and the type of proof Helen used and the proof scheme displayed:

Question	Type of proof	Proof scheme
1	Syntactic	Transformational
2a	Syntactic	Transformational
2b	Algorithm (Attempt)	Ritual
3	Algorithm (Attempt)	Trans. (Internalized); Authoritarian; Empirical
4	N\A	Analytic
5	N\A	Analytic
6	Process	Transformational; Authoritarian
7	Semantic	Axiomatic
8	Process	Transformational
9	N\A	Axiomatic
10	Syntactic; Semantic (Attempt)	Transformational
11	N\A	Analytic

Table 4: Summary of Helen's work

On the face of things, it may not look like Helen made much progress in proof at all. The type of proof employed varied with the problem at hand and there is no evidence she gravitated toward any particular proof type as she got more comfortable with proof. Syntactic proofs (or attempts) were used in 3 of the 8 opportunities that Helen had to complete a proof. Procedural proofs were also common (4 of 8) but, again, were mostly a by-product of the fact that induction was used often, especially early in the study.

Where Helen showed the most progress, however, was in the proof schemes she displayed while working on the problems. Although the various proof schemes do not follow a set progression (e.g., evidence of a transformational analytic scheme early in the study), she does move to the point where the second half of the semester she displays almost exclusively one of the 2 types of analytic proof schemes. This also comes up in both debriefing sessions, where she mentioned that her view of proof has changed over the course of the study. Originally, Helen saw the act of proving as being rigidly set with prescribed steps that need to be accomplished to a view where proofs can go many different directions and knowing which direction to take depends on an understanding of the problem at hand. This realization even influenced what she thought could have led to an even greater improvement on her part. Besides simply doing more proofs to improve, she said forcing the issue of figuring out what steps to take would help:

> H: Maybe a 305 class that made you actually think about what you needed to do instead of just giving you a problem and telling you how you needed to do it… Because, like everything we did with induction, we were told to do induction. Everything we did with contradiction, we were told to do contradiction…And we weren't, like there was no way to find out like what would work in what situation better.

On one hand, I believe she is referring to a necessary evil, so to speak, of the introduction to abstract mathematics class. Until students are comfortable with methods like induction and contradiction, I feel it is unreasonable to ask them to decide which method to use. On the other hand, Helen sees that eventually students are going to have to branch out on their own to make those decisions. She also sees that being forced to do so leads to progress in proof in general.

4.5 Barbara

This section documents Barbara's progress over the course of the study. Barbara was a Mathematics major who planned on becoming a secondary teacher. During the first semester of the study, Barbara took Euclidean and Non-Euclidean Geometry, Linear Algebra and Statistics and Probability. During the second semester, she took Number Theory and History of Mathematics.

Barbara's proof attempts

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Barbara started this problem by trying a square of side length 1 and saw that it did not work: "But that doesn't work if you have … if each side equals one that would be perimeter 4, which obviously isn't the area." I then asked about a few more squares and she saw that they did not work either, until we got to side length 4. After finding that a square worked, Barbara moved on to a non-square rectangle (1x2) and saw that it did not work.

After this, Barbara and I went back to discussing the case dealing with squares. We talked about the fact that for the squares with sides shorter than 4, the perimeter was greater than the area. We also noted that for a side length of 5, the area is larger. I asked if this observation would help with deciding whether or not any more square solutions exist, and Barbara said:

Barbara: Yeah, because the area would just keep getting bigger. Right?

Nick : Sure, it will, but the perimeter will get bigger too.

B: Well yeah, but it'll stay bigger, I think…Where the 4*x* is just limited to 4 times the number, but the number times the number, as it gets bigger would get even bigger.

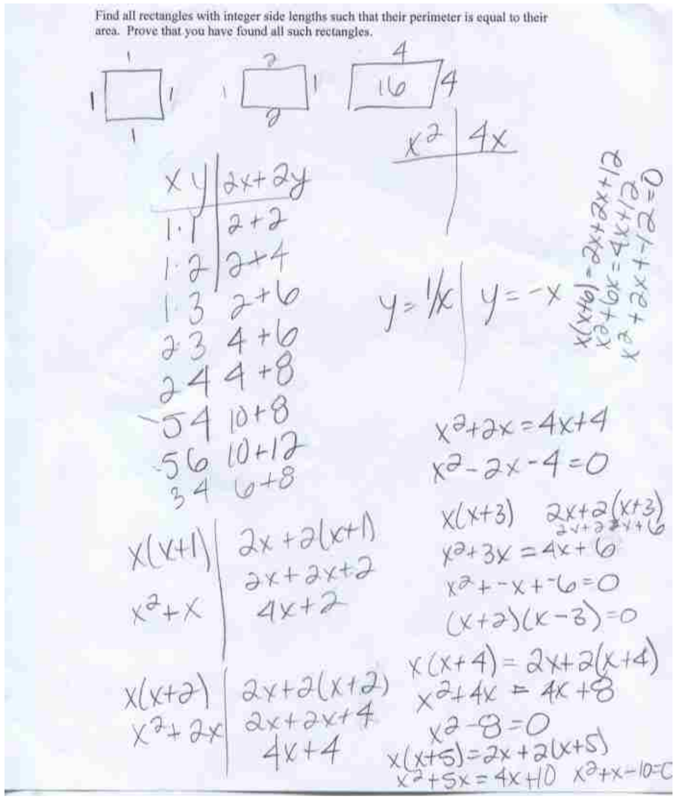


Figure 78: Barbara's work on Question 1 (1 of 2)

I then mentioned that we look at the derivates of 4*x* and x^2, the formulas for perimeter and area, to see that the area does indeed continue to grow faster than the perimeter. Barbara was confident that we had the square cases covered and then we moved on to rectangles. When I asked how she would proceed, Barbara said: "It's probably not the easiest way, I guess but…I don't know, I would just start with, I would just start plugging in numbers. I would, like, say, *x* is 1 and *y* is 1 and then start increasing *y*." The result of Barbara's trials can be seen in Figure 78. She does not stick strictly to her method too long, as she begins to increase the *x* values as well as the *y* values after just a few trials.

Eventually, she decides that she has checked enough: "Yeah, we were trying to look for it but it doesn't look like there's a rectangle." I asked Barbara if she could verify that and she began to look at ways to generalize what she had been doing.

B: Well, y would equal 1 over *x*…right? In one of them? And then…*y* equals…

N: Well, *y* is one plus *x*, right?

Putting this new plan into action, Barbara said: "So you'd end up with, if *y* equals *x* plus one, then it would be x^2 plus *x* versus 4*x* plus 2…Which obviously is bigger because that one's squared." Barbara proceeded with this method, trying the case where *y* is 2 more than *x*:

B: So this time I kind of added both sides together so I got $x^2 + 2x = 2x + 4$.

N: Ok.

B: Because when they would equal would be when it would be true.

N: Oh, that's a good idea.

B: I got $x^2 – 2x – 4 = 0$ … and the only thing I could come up with is not an integer answer so… if an answer at all.

Barbara needed encouragement to continue with this method because as she said: "Well, there wouldn't be a stopping point. You'd have to keep going." I suggested trying one more, because I knew she would find another rectangle soon:

B: Sure (laughs)

N: What's that?

B: Well, I'm pretty sure you can factor that one…*x* equals, well *x* equals -2. So, you get *x* equals 3…So, when one side's three, the other side's 6.

N: Ok, so does that work?

B: Yeah. 3 times 6 is 18 and 6 and 6 and 6 is 18.

Barbara realized that the negative answer did not yield anything meaningful. She tried a few more examples and eventually noticed a pattern developing. She was able to predict what polynomial she would end up with if $y = x + 7$: "$x^2 + 3x - 14$..Well, you get $7x$ minus $4x$…and then 2 times 7. On this side you'd have $7x$ and on this side you'd still have $-4x$."

We had reached the end of the interview and I asked Barbara to continue to work on the problem and she said she would. The work she brought back can be seen in the following figure.

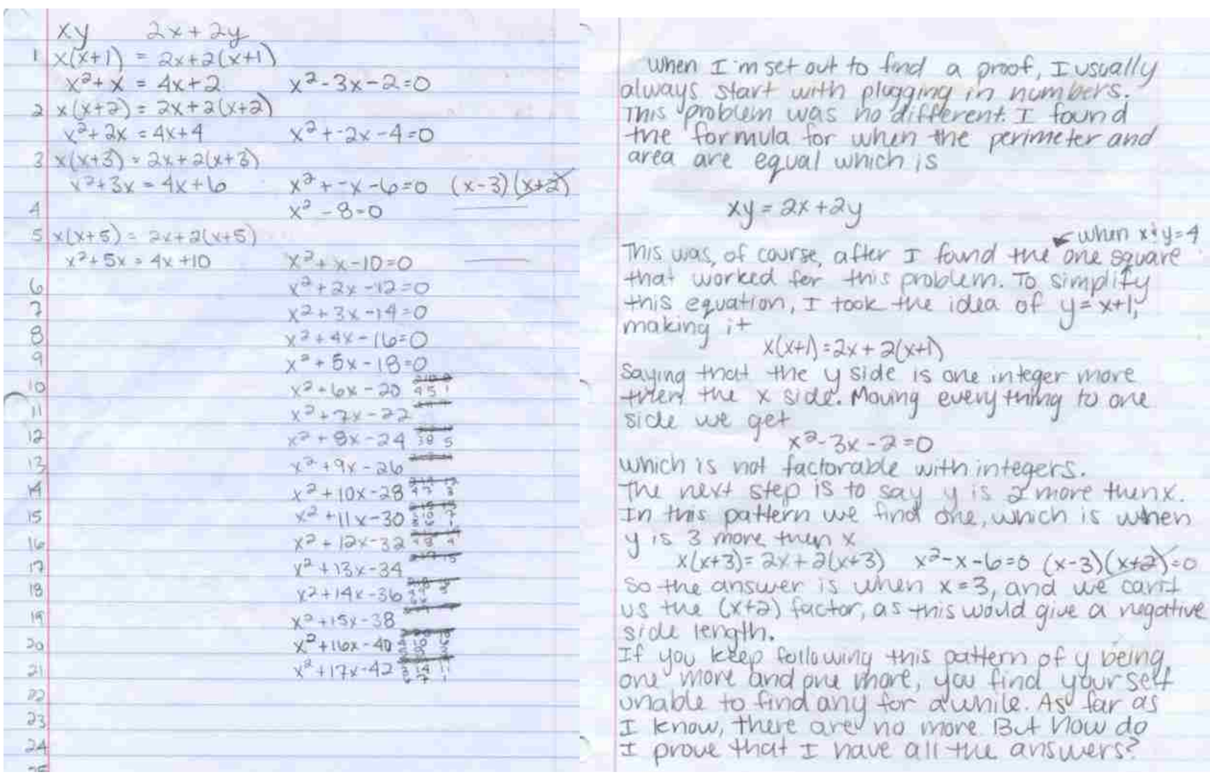


Figure 79: Barbara's work on Question 1 (2 of 2)

As she states, Barbara does not know how to verify that she has found all the rectangles that fit the criteria. She is, however, convinced that she there are no more to be found: "I feel like it's obvious that it's not going or factor anymore…But that doesn't mean that I'm not wrong."

Barbara does not provide a proof here and does not even attempt to, strictly speaking. When presenting her between interview work, Barbara says: "But I don't know, I wasn't really thinking about how I would prove it as much as just trying to find another one." However, this does not mean that Barbara is not trying to work towards a proof. When talking about her method during the first interview, I reminded her of what happened when looking at the case of squares:

> N: Remember when you tried the squares something sort of happened that made you realize that you didn't have to go any further.
>
> B: Yeah.
>
> N: So, I'm guessing if you keep messing with these polynomials something eventually will happen to let you know that you don't have to go any further.
>
> B: Ok.
>
> N: And if that happens, then you're done.
>
> B: Right. Ok, that makes me happy.

So, even though Barbara says that she is trying to find another solution (even when she's sure she will not), she is also hoping for something to happen that will let her know that she can stop. In other words, she is looking for some break through in understanding that will lead to a proof. Because she is looking for understanding could be turned into a proof, this attempt is classified as semantic.

Barbara's proof scheme is complicated. On one hand, she states that she is personally convinced by empirical evidence a couple different times. One example of this is when she says it is "obvious" that no more of her polynomials will factor. It also happened a few times in the first interview. After trying the case when one side length is 1 longer than the other ($y = x + 1$) and I asked about the case where $y = x + 2$, she said:

B: I still don't, I feel like it'd still be the same, like there wouldn't be one, but.

N: Ok, so what are you basing that on?

B: Because we didn't find one that was 1 bigger.

So, on a personal level, Barbara displays an empirical proof scheme. However, it is also clear that she realizes that this would not suffice for a full-fledged proof. This can be seen in her work from between interviews and in the fact that she is discouraged by her method's need to check all the possible difference in side in lengths: : Well, there wouldn't be a stopping point. You'd have to keep going." This implies that Barbara has an analytic proof scheme as well. From her work, we can see that none of reasoning is based on the axiomatic nature of mathematics. Instead, her work is based on operating on mathematical objects and the results of those operations. This is indicative of a transformational proof scheme.

<u>Question 2a</u>

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9. (For example, if a = 4 and b = 2, then 42 – 24 = 18, a multiple of 9.)

After Barbara initially read through the problem, she mentioned how she generally starts with examples: "Well, I don't know, like I wrote in my thing (write-up from previous interview)

I always start with plugging the numbers in." The first example she looked at, ($a = 1 = b$) led her to ask about the definition of multiple. I told her that we would take a multiple of 9 to be a number that could be written as 9 times an integer. She was then comfortable with the fact that 11 – 11 = 0 was a multiple of 9, as was her next example: $a = 1$, $b = 2$. It did not take Barbara long to find a pattern:

B: Well, it just looks like it's $b – 1$, right?

N: Well, technically, it's $1 – b$, right?

B: Right, because it's negative.

(does more examples)

N: … So now you did 2 and 5 and got – 27.

B: Oh, so that's, well, that would be $b – a$ times 9… But if I did 5 and 2 then it would be $a – b$ times nine.

After finding the pattern, Barbara and I talked about whether that was a proof:

B: So that's the answer?

N: What's the answer?

B; Well, ($a – b$) times 9.

N: So that proves that you always get a multiple of nine?

B: Well, no, it's a formula, I guess, for what we're trying to prove.

Barbara and I then talked about her options for proving her formula was right. We briefly talked about some methods and she mentioned that she felt like she was lacking in the content knowledge to use them:

B: See that's the thing, I forget, like, what the heck is an integer? Or what is a multiple, I have to have, like, a list that I have to look at to say, like, what is a multiple, like what's the definition of it. So when you say induction, like I don't even remember what that is.

Barbara also mentioned that she would not know how to go about using a proof by contradiction. Eventually, she says: "I don't know what I would do with this. There's nothing in my tool box."

Eventually, Barbara is able to pinpoint the main difficulty she is having with the problem: "I think I'm getting caught up in the fact that you're not multiplying the numbers." What she is saying is that she is not used to seeing a pair of variables written next to each other and having that mean anything but multiplication. I then told her that might be something for her to focus on, suggesting she start with her last example (52 – 25) and move on to something more general from there. When I asked what she thought she could do with that, she said:

B: Well, the 52 would have to be the 5 and you can't 5 out of 52. And you can't get 2 out of 25.

N: What do you mean by that?

B: Well, I'm thinking dividing. (See figure below)

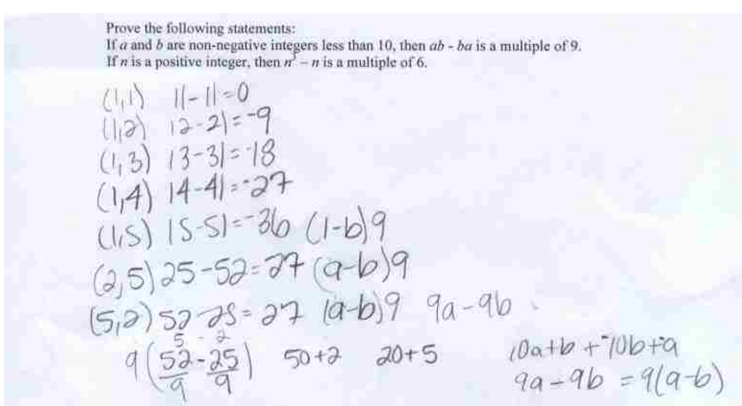


Figure 80: Barbara's work on Question 2a

At this point, Barbara was frustrated and ready for help completing the problem. I began to point out the place values of the digits and Barbara quickly was able to see where I was going:

N: Ok, can you write 52 as a sum that makes it clear what's in the 10's spot?

B: Well, it'd be 50 + 2, right?

From there, Barbara was able to adapt the argument to the general case and complete the proof.

Because Barbara did not complete the proof on her own, this episode constitutes a proof attempt only. This proof attempt is a syntactic proof attempt. While Barbara did not try too many different things to work towards her pattern, what she did try involved algebra, i.e., symbol pushing, which is a major characteristic of syntactic proofs. It should be pointed on that on the surface of things, this proof attempt does not look to be too different from Question 1 and yet it is classified differently. Both featured Barbara looking through examples and being unable to develop anything into a proof. There is a key difference between the attempts, however. The difference is that in Question 1, she was looking through examples hoping that something (some form of insight) would come to her that could be turned into a proof. Here, she used the examples to find the pattern and once she found it its verification became her focus. She did not purposefully use examples to give her understanding that could be turned into a formal proof, thus making this attempt syntactic.

Like with the last question, Barbara displays a couple proof schemes here. First, after a few examples, she becomes convinced that her pattern will hold. This supports an empirical proof scheme. The evidence is not a strong as with the previous question (when explicitly mentioned her certainty), but she does work with the pattern exclusively once she finds it. This implies that she believes it is true. Once she found the pattern, she tried using algebra techniques to verify it. This is typical of a transformational proof scheme.

<u>Question 2b</u>

If n is a positive integer, then $n^3 - n$ is a multiple of 6.

Barbara started this problem like she had the previous 2, by looking at examples. She tried *n* values of 1 through 6 before looking back to see if she could find a pattern (see Figure 81):

B: I guess I'm trying to go through in my head, any kind of links in the numbers. To try to come up with the formula …

N: Ok, so you're trying to come up with a formula to try to come up with, to go from 2 to 6, from 3 to 24, or a link to go from three to four?

B: Both.

N: So it's kind of like last time, you looked at examples, found a pattern. Now you're looking at examples…

B; And trying to find a pattern. That's my method.

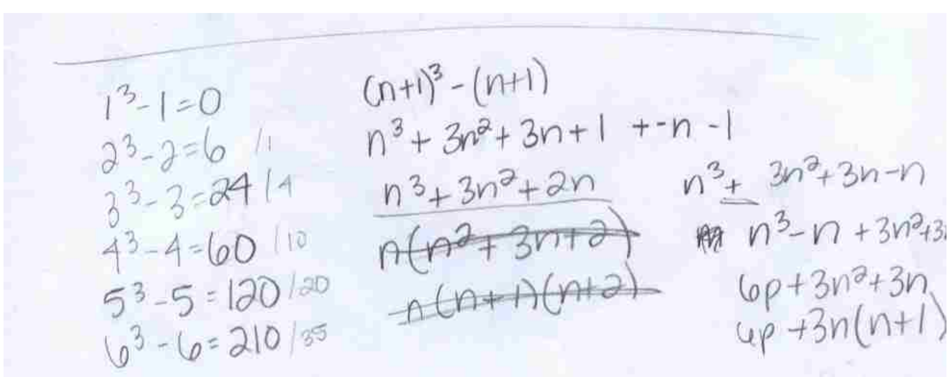


Figure 81: Barbara's work on Question 2b (1 of 2)

After thinking a bit and not seeing anything useful, I asked Barbara if there was anything she thought she might want to try. The first thing she mentioned was induction, so she decided to give that a try. Because she had tried examples that could serve as her base case, Barbara went directly to the inductive step, deciding whether or not $(n+1)^3-(n+1)$ is a multiple of 6 when assuming $n^3-n=6p$. After multiplying the expression out, Barbara simplified things:

B: n^3 + 3n^2 +2n. Now I want to factor it.

N: Just because that's sort of standard practice?

B: Yeah, so first I take the *n* out and I get $n^2 + 3n + 2$. That's $n(n + 1)(n + 2)$, right?

Barbara needed to be reminded that if she was going to construct an induction argument, she need to make use of the fact that $n^3-n=6p$ at some point. That is when she went back and got down to the last line on the bottom right in Figure 81. She then knew that it was left to show that $3n(n + 1)$ is a multiple of 6. This was at the end of the interview, but we discussed what it would take for that to happen:

B: Well, 3 times any number is going to be…

N: Ok, so 3 times 4 is 12, that gives you a multiple of 6. 3 times 5 is 15 that…

B: That's not a multiple of 6. So, even numbers.

She left the interview knowing that what was left to show is that $n(n + 1)$ is an even number.

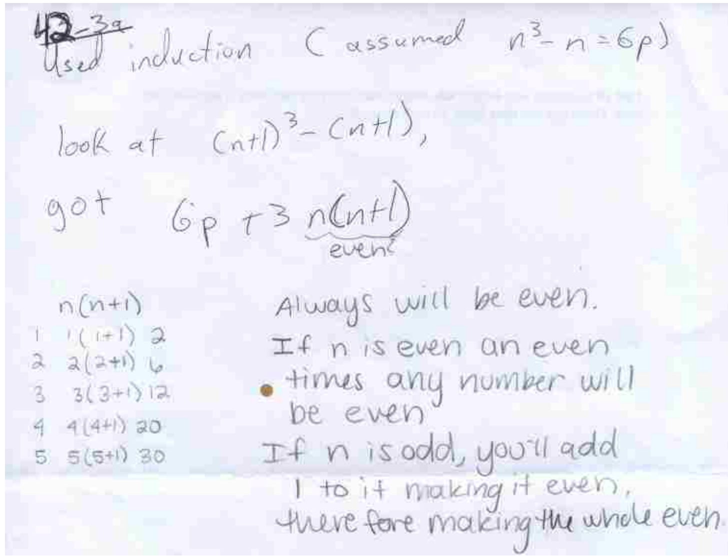


Figure 82: Barbara's work on Question 2b (2 of 2)

Barbara did not take long to finish the problem at home. I had written up a summary of the things she had done and what was left to do and she took it from there. At home, she tried a couple examples and saw that for all n values, either n or $n + 1$ would be an even. Her work can be seen in Figure 82.

The proof Barbara provides is an algorithmic procedural proof. It involves performing prescribed steps to complete her proof. The amount of guidance Barbara needs to complete the proof is what prevents her proof from being simply a process proof. Not only did Barbara need to be reminded the steps required in an induction argument, she needed to be reminded to make use of the induction hypothesis – an essential component of a proof by induction.

B: Just trying to form a pattern by plugging in the numbers to see how it was working out. And then it was, so I was like I'll try induction…

N: So you didn't really consider anything else, you just kind of went straight to it because it was something you were comfortable with?

B; Something, I guess yeah, something I was comfortable with, even though I didn't really know how to do it…I mean I knew it was tool … I guess I picked that because I knew it was a tool.

This is also evidence of Barbara's proof scheme: internalized transformational. She is not convinced of the proof's validity simply because of the way it looks (which would be evidence for a ritualistic scheme). At the same time she does not display the thorough understanding of the process of induction necessary of a typical transformational scheme. This sort of episode reinforces the idea that deductive proofs need not be meaningful for those completing them (Weber, 2004).

<u>Question 3</u>

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$*:*

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

Barbara worked began this problem by doing a base case of $n = 2$ because, as she said, "$n = 1$ is just simple." In Figure 83, it shows that Barbara made a mistake when filling out the inequality, which will be discussed more later. (She crossed out the extra fractions later.)

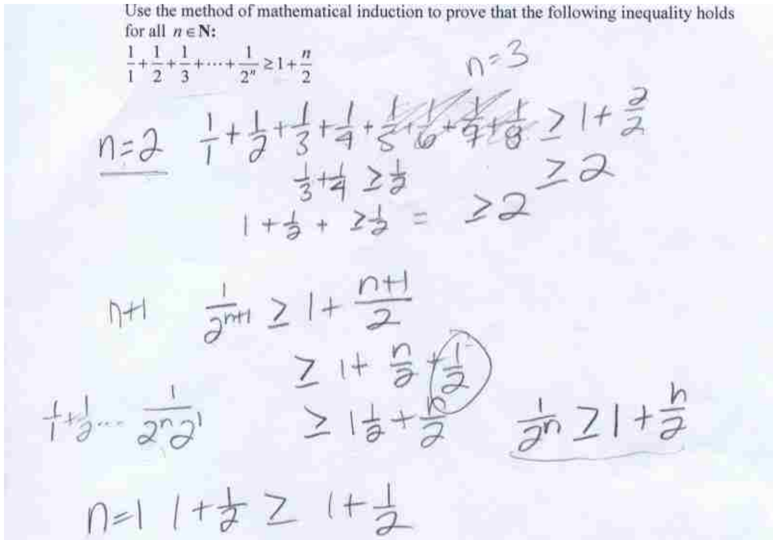


Figure 83: Barbara's work on Question 3 (1 of 3)

After writing out the inequality, she said:

> B: Well, I'm not going to add those up, but I'm not concerned about it…(laughs)
>
> N: So you're confident it works as is.
>
> B; Well, yeah, because 1/3 and 1/4 is greater than 1/2, which would mean that it's greater than 2.

She noticed that only the fractions that should have been there were needed to make the statement true. After that, she moved on with the induction argument:

> B: So I'm saying the base case is true, now it's the $n + 1$ case… I'll just start on the right side to simplify it…So the right side is saying that 1 and 1/2 plus n/2. So you're just adding a half. But it has to be greater than…

She then moves on to the left hand side of the inequality and rewrites $\frac{1}{2^{n+1}}$ as $\frac{1}{2^n \cdot 2^1}$.

At this point, Barbara got stuck and said:

B: I guess this is the part of induction that I don't quite understand, is the using what you're trying to prove… I guess I just don't know how to use it. I'm pretty sure this is where I would want to use it.

It was not clear to me at the time, but Barbara did not have a clear understanding of how the inequality changed when *n* was changed to *n* + 1. To her, the expression on the left stayed the same, but for the last term which changed.

B: So it's just this extra one half that it has to be greater than or equal to. And then on the left side, when you have the $\frac{1}{2^{n+1}}$, you get the $\frac{1}{2^n}\cdot\frac{1}{2}$…I guess the problem that I'm trying to figure out is that when you times something by one half, it's going to get smaller.

N: Uh huh

B; I just, now I'm going to work on another base case. Because if I had *n* = 1, then it'd be 1 + 1/2 greater than 1 + 1/2. And that's why you have to say that it's the equal to…because otherwise the left side would be greater.

At this point, Barbara was stuck so we began the reflection. It was going back through her work that we realized that she had made a mistake with the initial trail of *n* = 2. She was fine simply crossing out the extra terms because, as she said, the reasoning she used before still applied. It did not, however, help with the problem she mentioned earlier:

B: Well, I mean, if it was times 2, then it would be "Well, ok, then it's going to be bigger."

N: Sure

But because you're dividing it, how can you say that it's going to be bigger when you're making something smaller…Because basically, this is like saying, like if it was the $n + 3$ case it would be, or $n = 3$, it would be saying that you're dividing this by a 1/2, right?

N: To go from the $n = 2$ case to the $n = 3$ case? Because, kind of, that's what you did here, if you stop at 1/4, that's the $n = 2$ case but you've written out, so it's kind of a lucky accident because you can kind of see what happens here, right?

B: Yeah…

N: Yeah. And so this (the right hand side) goes up by that one half that you found here, and this, the last term does get smaller, the last term was 1/4, becomes 1/8 but you're adding this other stuff.

B: Oh, I see…I guess I was looking at this and not understanding, because the way that you just said it, was like you're changing this by one half to get to the third case, which is true. And then you'd add these ones.

N: Right. So it kind of sounds like you were looking at this left hand side as staying the same except for the last term.

B: Yeah.

I asked Barbara to look at this problem more between interviews, so she did. Unfortunately, between interviews, she basically repeated the things she had done in the previous interview. See Figure 84:

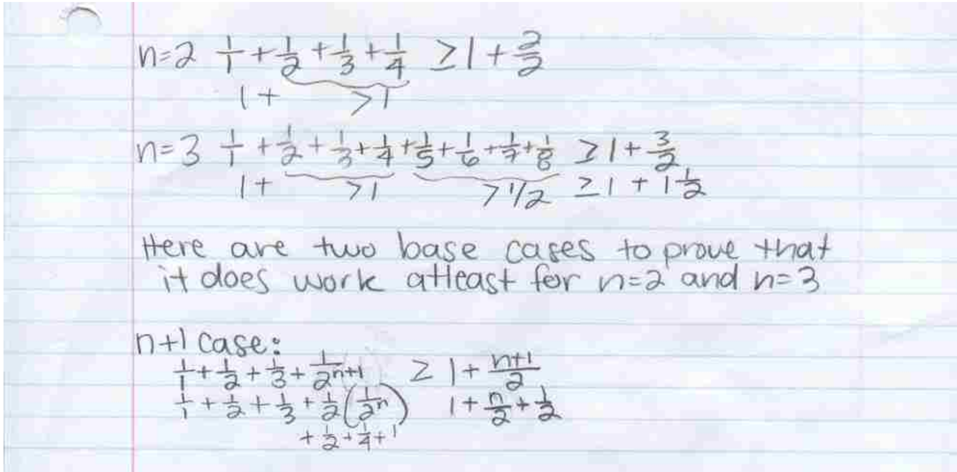


Figure 84: Barbara's work on Question 3 (2 of 3)

When I asked Barbara to explain what she had done, she said:

B: Basically, I did 2 examples of a base case, $n = 2$ and $n = 3$, and obviously it was true in those cases. So I went on to the induction part, which is $n + 1$, so I was that was plugging that in, $n + 1$ for n. I worked on the right side and tried to work on the left side and got stuck again.

N: Ok

B: Which is where I got stuck last time, and then you said something that made it click and then I left and that was it.

After talking about what she did between interviews, I let her review what she had done during the last interview and that reminded her of what we talked about at the end: “The bottom number’s…the last number of this one and the last number of this one…that’s half of that…Now I remember what I was thinking.” With the reminder in mind, started up the problem again, in Figure 85.

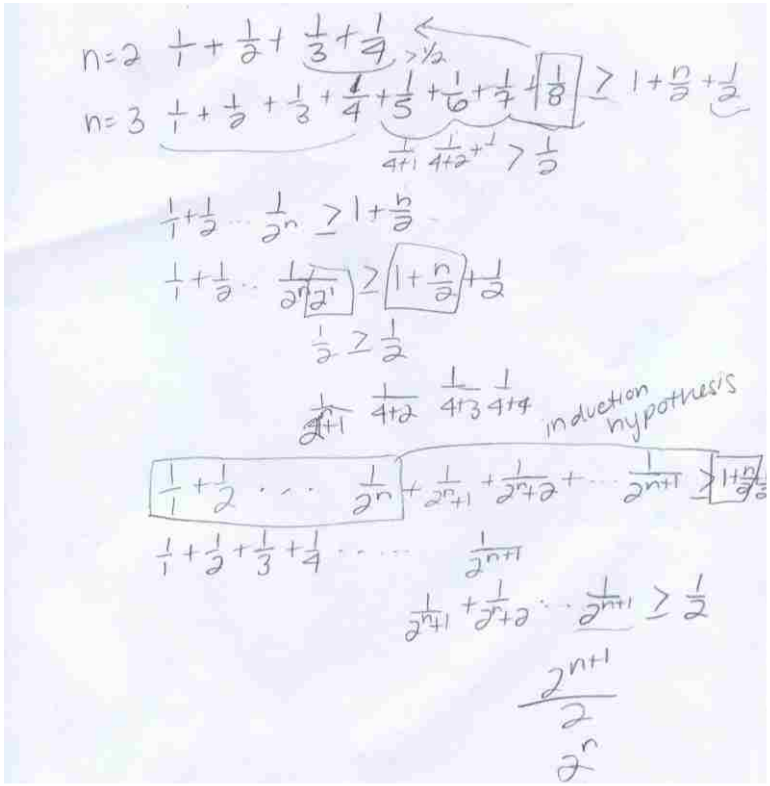


Figure 85: Barbara's work on Question 3 (3 of 3)

Barbara saw the mistake at the top of Figure 83 and it reminded her that she needed to add a number of new terms to make up for the fact that the last one was being reduced each time *n* was increased. She also was able to articulate the sort of reasoning that would eventually solve the problem, even though she was not able to apply it to the general case without help.

> B: Well, I know that this is greater than 1/2. These numbers (1/5 – 1/8, underlined) are greater than 1/2…and these numbers (1/3 and 1/4, above and underlined) are greater than 1/2, which is really what we needed to show. I guess it's using the hypothesis.

Even though she has an understanding of what she is supposed to do, she does not know how to do it in the general case: "Well, I, it's like, the point where I need to use the hypothesis, but I don't know how to, like, say it."

At that point, Barbara said she was stuck and asked that help her finish it. We then talked through generalizing her reasoning. This work was completed below the $\frac{1}{2} \geq \frac{1}{2}$ in the middle of Figure 85.

Barbara did not complete the proof on her own, so I will be classifying her attempt only. Although she did not complete this proof, Barbara did show some growth when this attempt is compared to her work on Question 2b. Specifically, she has a much better grasp on the process of induction. After she was done working on the last question, we talked through induction and it seemed like that brief refresher was all that was needed for her to understand induction. That shows up here as well. The difficulty she had with this problem was due to the question itself; the process of induction was never the issue. In fact, this was something she noticed: “Yeah, I know how it works in theory, just trying to apply it to a problem is where I get stuck.” Because Barbara did not try to follow a number of specific steps, but rather a few global ones, this is not an algorithmic proof attempt, but rather a process procedural proof attempt.

This new, better understanding of induction does not change her proof scheme, however. While she does know more about induction in comparison to when she worked on the last question, her understanding is not complete: in her attempts, she still uses improper base cases. She clearly still is under the impression that showing the property holds for any *n* will suffice. Because she lacks this thorough understanding, her proof scheme is the same as it was before: internalized transformational.

Question 4

Examine the following proof by contradiction:

Prove $\sqrt{-1} \leq 0$

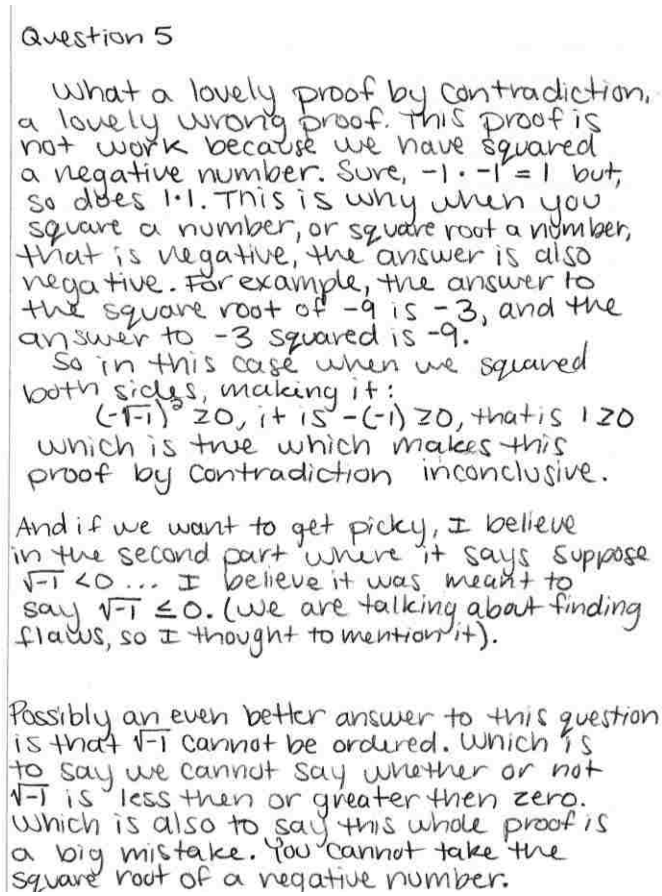
Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

<u>Prove</u> $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$, *then* $-\sqrt{-1} \geq 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$, *so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This question was first given to Barbara on the take home midterm exam she completed in MATH 305. Her midterm response can be seen in the figure below. The main flaw she sees is based on a misconception regarding real numbers: she claims that $-3^2 = -9$. I think she has made this mistake due to trying it on her calculator without realizing that her calculator sees -3^2 as $-(3^2)$. Because of this, she says that the second part of the proof should have read $1 \geq 0$ instead of $-1 \geq 0$ which would mean that a contradiction was not reached.



Question 5

What a lovely proof by contradiction, a lovely wrong proof. This proof is not work because we have squared a negative number. Sure, $-1 \cdot -1 = 1$ but, so does $1 \cdot 1$. This is why when you square a number, or square root a number, that is negative, the answer is also negative. For example, the answer to the square root of -9 is -3, and the answer to -3 squared is -9.

So in this case when we squared both sides, making it:

$(-\sqrt{-1})^2 \geq 0$, it is $-(-1) \geq 0$, that is $1 \geq 0$

which is true which makes this proof by contradiction inconclusive.

And if we want to get picky, I believe in the second part where it says suppose $\sqrt{-1} < 0$ ... I believe it was meant to say $\sqrt{-1} \leq 0$. (we are talking about finding flaws, so I thought to mention it).

Possibly an even better answer to this question is that $\sqrt{-1}$ cannot be ordered. Which is to say we cannot say whether or not $\sqrt{-1}$ is less then or greater then zero. Which is also to say this whole proof is a big mistake. You cannot take the square root of a negative number.

Figure 86: Barbara's previous work on Question 4

Her "possibly better answer" is indeed better, if incomplete. If one takes the statement "You cannot take the square root of a negative number" to mean "you cannot take the square root of a number and expect it to be on the real number line" then she has gotten it.

Barbara said as much in the interview. It did not take her long to remember her "better answer": "The fact that you're trying to say that, because the square root of -1 is imaginary, trying to put it in a number line is impossible." From there, I asked Barbara about what that meant for the individual steps:

> B: Well, you can do whatever you want to it, it's not going to make it have order…Well, following the steps there's no flaw, but what you're trying to prove is absurd, so it doesn't matter what the proof is or what you're answer is.

Barbara realizes that if you start with something that makes no sense the subsequent steps will be meaningless as well, even if they look fine.

Because Barbara did not attempt a proof, there is no proof to classify. However, this does give evidence that Barbara has an analytic proof scheme. Although we do not discuss the axiomatic nature of mathematics in general, she does show that she understands that the mathematics depends on deducing from starting assumptions, at least on a small scale. This is evidence for an axiomatic analytic proof scheme.

<u>Question 5</u>

The next interview was a debriefing session in which Barbara and I talked about the progress she had made over the first half of the study. She did not attempt a proof, so there will be nothing to classify with a proof type. However, the interview did provide support for some of the things observed in the proceeding interviews.

For example, when I asked Barbara how she thought the semester had went proof-wise, she singled out induction as something she improved on: "I don't know…I feel like I learned something from coming here…I have a general idea of how to do induction now." This matches the progress seen between Question 2b and Question 3. This also matches something she said later in the interview. I asked her to compare the progress she made while taking MATH 305 to the progress she made during the first half of the study. She said that she felt like she made progress during both semesters, but that the type of progress was different:

> B: But it's not, I really didn't know any proof techniques and stuff like that before. So obviously, I did make an improvement last semester, but it was more like gaining knowledge of what proofs are and techniques to do them.

N: I see.

B: And this semester was more like actually using those techniques and learning how to use them to prove something.

She felt like the difference was one semester was spent filling up her proverbial tool box and the next was learning to use them: “I have all the definitions; I just need to apply them. All the tools and using them.”

Another way in which this interview reflected what was seen earlier in the study was when I asked Barbara what she felt was necessary to complete a proof. Her first response was: “Just making logical steps with stuff that’s true.” She sees the formality in necessary in proof, but recognizes that that is not the whole story when it comes to seeing a proof to the end:

B: Well, yeah, it’s using the right steps to get where you need to go. If you need to go left and you’re going right…but in the same sense it’s, you know, I can make up a bunch of rules, but they have to be valid rules.

This statement reflects her ideas about Question 4: that one can make any statement they like, but if it is not valid, it is meaningless.

I also asked Barbara about what was helpful, but not necessary, in completing a proof. For this she mentioned a couple different things that boil down to coming to an understanding of the problem at hand:

B: It would just be trying the wrong ways.

N: Ok

B: I think is what my answer would be to that.

N: Ok

B: It's just, obviously you don't need the wrong ways, but it helps you, it probably helps you get in the right direction.

N: Yeah, I mean, for me anyway, I think you get a lot of insight and it's very useful.

B: Like I did a lot of examples when we were trying to work the other proofs. It's just you have to see kind of where it's going so you have a better educated guess.

Barbara mentions examples as helpful, which definitely matches her behavior while working on the problems this semester.

Because so much of this interview matched what she said and did during the first half of the study, it is not surprising that she also reinforced the proof schemes she displayed earlier on. Her focus on the necessity of logical steps highlights the axiomatic scheme she displayed while working on Question 4. Also, she mentions that being able to take the time to go through problems thoroughly led her to a better understanding of induction. This is the sort of process that Harel and Sowder (1998) describe when a student's internalized transformational proof scheme becomes an interiorized transformational scheme. This is also something observed with Barbara, although the process is not complete (at least in the case of induction).

Question 6

Prove that $\forall n \in \mathbb{N},\ \ n \equiv n^3$ *(mod 6).*

This interview started with Barbara and I reviewing what modular arithmetic was and how it work. Barbara then began working on the problem as she had many in the first semester: by trying examples. After seeing the property hold a couple times, she moved considering how she would prove the problem. She considered contradiction, but settled on induction because she knew how to start it. During the reflection, we talked about this decision:

N: So by this point (when she cubed out $n + 1$, see Figure 87) you had already considered contradiction and deemed it unworthy to try?

B: Yeah

N: And again, that, the reason for that was what? You just didn't know where to go with it?

B: Yeah

N: So you did have kind of a feeling of where to go with induction?

B: Yeah, well yeah, it's just $n + 1$.

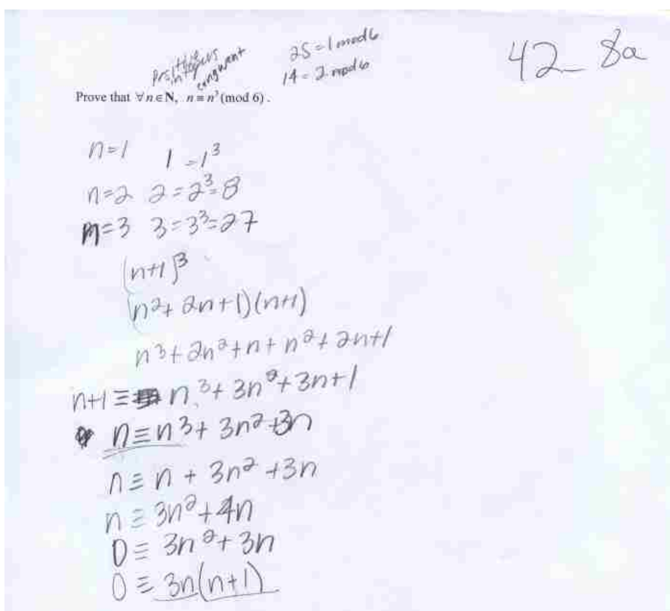


Figure 87: Barbara's work on Question 6

After Barbara had simplified things down, we talked about the operations she was allowed to perform. These questions on her part were related both to the use of modular arithmetic and with the process of induction:

B: But induction, you're not supposed to move things around, are you?

N: You can…

B: Like on, to either side?

N: You can as long as what you're doing is, like an equivalent, you're trying to show that some statement is true. And as long as you keep coming up with equivalent statements, everything's ok. You know, you can't just say, "Well, let me add 9 to this side but not that side."

B: Well yeah. But I mean, like these 1's would cancel out.

N: Ok

B: So I can do that?

N: Yeah, sure.

At this point Barbara was stuck and I, as I had done with previous induction arguments, asked her about using the induction hypothesis:

N: So what would it look like in this situation to use the induction hypothesis?

B: I have no idea.

N: Ok

B: This is the part I always get stuck at.

N: Ok. So since n and n^3 are the same thing…

B: Yeah, I can put n in for this.

Eventually, this led Barbara to conclude "So then it would be n is congruent to $n + 3n^2 + 3n$" and, after canceling, $0 \equiv 3n^2 + 3n$. I then asked her what it would mean for something to be congruent to 0 (mod 6) and she said: "Divisible by 3, or 6." She then factored the expression on the right and considered what would happen for *n* values of 1 and 2.

N: Ok, so if it's 1 or 2, it works.

B: And then it will keep going.

N: Ok, so…

B Because either this'll be even or this will be even.

N: Right, ok. So then…

B: I'm done.

Barbara had finished the problem using the same reasoning she had in Question 3. I pointed out during the reflection that this was in fact the same problem as Question, something she said she noticed but did not mention.

This is a process procedural proof, as were the most recent induction proof that Barbara provided (Question 3). The proof required a few global steps that Barbara knew were to be completed (even if she did not know how to complete them without help).

Barbara's proof scheme has not changed much either. Again, Barbara has no problem with the steps required to complete an induction argument. She does have some issues with how to complete them. For example, she needed help talking through how to use the induction hypothesis but she knows it needed to be used. Also, she had the misconception that one was not "supposed to move things around" when dealing proving the inductive step. While the first issue I mention is problem specific (how to use the induction hypothesis varies from problem to problem), the second issue reveals a limited view of induction. She is used to seeing induction problems in which two expressions are set equal to each other and one or both is simplified to the point that they obviously agree. This issue, combined with the fact that she still seems to think that any *n* value serves as a valid base case, shows that Barbara has not yet completely interiorized the method of mathematical induction. Thus, Barbara's proof scheme here is internalized transformational.

<u>Question 7</u>

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

Barbara and I began this interview going over some details, as usual. We briefly talked about what it meant for a set to be a subset and the fact that for the sake of this problem, we were only going to look at finite sets. Barbara mentioned that she did not know how to start this problem.

> N: Ok. So are there things that you generally do when you don't know how to get started with a proof?
>
> B: Well yeah, I use examples of how it works. But this isn't one of those cases were you can do that.
>
> N: Why not?
>
> B: Well, there's no number, there's no formula.

Eventually, Barbara decided that she could look at some small sets to serve as examples (see Figure 88):

> B: So, I've written out a set that has 3 in it. And that came with 6 possibilities, and…
>
> N: Ok

B: 4 had 10 possibilities.

N: Ok

B: I mean, what I’m thinking is that it’s a kind of factorial thing.

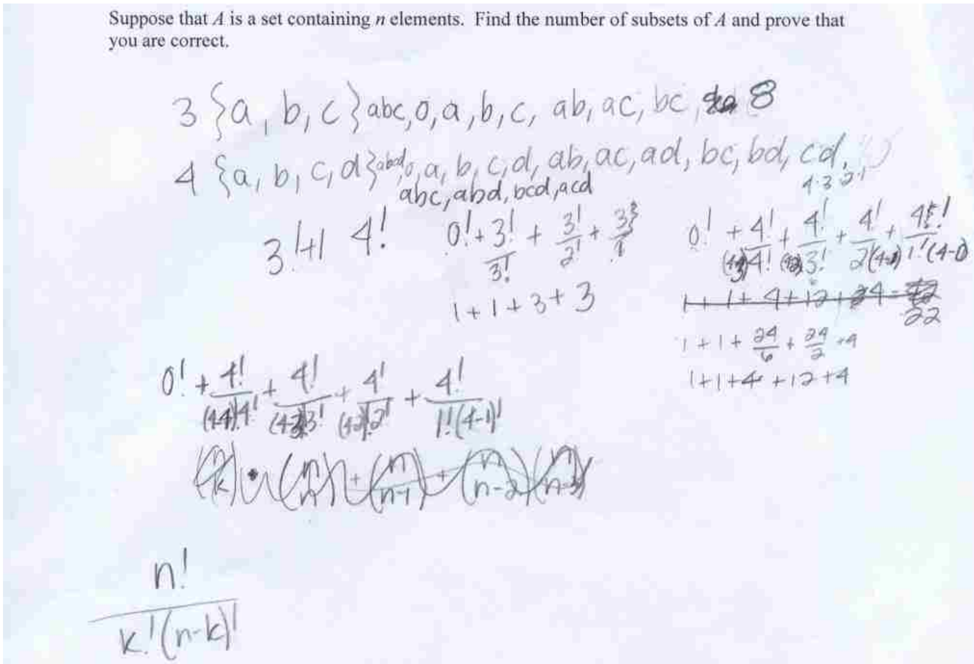


Figure 88: Barbara's work on Question 7 (1 of 2)

From there, Barbara spent most of the rest of the interview deriving an equation that would give her the number of subsets of a given size, given a particular *n* value. It did not take her long after she started this process to see that she had missed some subsets in her lists: the 3 element subsets in the case that $n = 4$ and the empty and the set itself in both cases.

Her progress in deriving the formula she was looking for (eventually she arrived at the formula for the *n*C*k*) was slowed down somewhat by that fact that an incomplete expression almost yields the proper results in the case of $n = 3$. Her method began with $\frac{n!}{k!}$ where *n* is the number of things in the superset and *k* is the number of elements in the subset. As one can see in the figure, this formula nearly works for $n = 3$, she had mistakenly wrote that $\frac{3!}{1!} = 3$ initially and discovered the mistake when looking back at that example for guidance while working with the n = 4 case.

Eventually, by looking at the numbers she needed to get out of her expression and trail and error, Barbara arrived at the proper formula (which can be seen directly above the crossed-out line in the figure). At that point, I pointed out that she had figured out the choose function.

B: I didn't really figure it out…

N: Well sure you did.

B: ...more like slowly remembered.

Whatever the case was, Barbara had a formula she was comfortable with and she then asked: "So now I have to prove something, right?" To Barbara, the next step was to state her formula in general (her attempt is the crossed-out line in the figure).

B: I don't know.

N: So what don't you know about?

B: I don't know this formula.

N: You don't how to get…

B: I don't know this *n*C*k* thing and like how, what the formula is for that.

It was near the end of the interview, so I gave Barbara the general formula for *n*C*k* and asked what she would try next.

B: I'm thinking induction.

N: Yeah? So why induction?

B: Because it has the next case…and I know how to do it now.

Barbara had done an induction proof successfully in number theory, so her confidence level was high with the method. Also, I think her next step would have been to try her formula in the case that $n = 5$ (an approach she used in the past). This also reminded her of induction: "Because you're going to the next number, so why not plus 1 it." With that, Barbara left the interview agreeing to work on it again between interviews.

She did not work on it between interviews, though, and so we spent the first few minutes of the next interview going over what she had done in the previous one. After we talk about her work, I ask her how she thinks she would go on from there:

B: I guess it's just like, I want to pull out a tool or something but…

N: Oh, ok. So what sort of tools are you considering?

B: Induction.

Earlier in the interview, she had mentioned that one of the problems with trying to prove the general formula is that there are two variables. I think this was a deterrent to her trying induction, as was the perceived complexity of the *n*C*k* formula.

B: So now I'm thinking to pull out contradiction…I don't know how that would apply, though…Maybe a direct proof?

When I asked why she did not want to try contradiction, she said:

B: I wouldn't, I mean, because it's some number, it's not, like…it's not like *n* or *k* or something, it's … some number irrelevant so I can't say this is not the number of sets.

N: Oh, ok. So you would say you'd try to come up with some sort of contradiction and then…ok. So would be fair to say that you wouldn't have something to compare it against, at some other, some point down the line because you don't know what the subsets would be without using this.

B: Yes

Barbara also discussed what it would look like to do what she called a direct proof: "Like, I would write n!, 0!, (n-0)!." At the time, I misunderstood what she was talking about and said that she could use the notation for *n*C*k* instead of writing out the formula every time. She responded: "I don't know how I would do a direct proof without…" She was referring to simplifying the formula in some why that might help. This is the work she did in Figure 89.

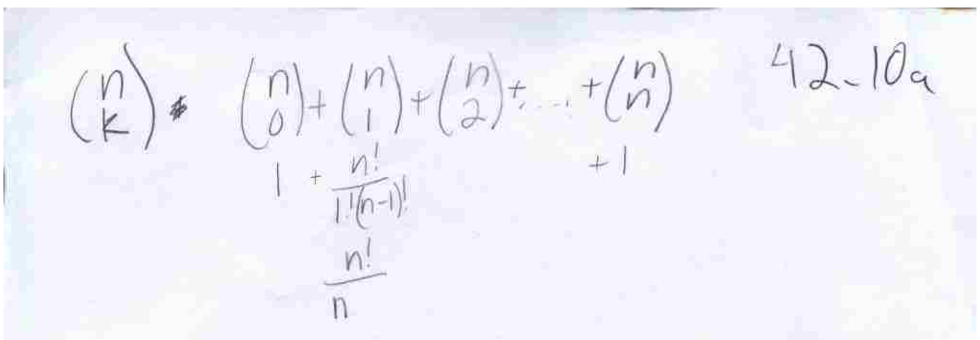


Figure 89: Barbara's work on Question 7 (2 of 2)

About this work, Barbara said: "I guess I was hoping it would help because, but then it's (n - 1) (in the denominator), I was thinking it was relating to the n + 1, which would be … like n! and …but it's minus 1, not plus 1." This made me think that she had gone back to trying an induction proof: "No, no… I was hoping to like, simplify this, I guess…But you can't pull n! out of (n – 1)!." At this point, Barbara was out of ideas and said "I don't know how I would prove this."

At this point, I asked her if maybe the formula she had could serve as a proof: "I guess I feel it's kind of like saying 'Duh, this works. So it's true'." When I asked what she would do if this had been a homework assignment, Barbara said: "Well, I would probably say how I got to it for some kind of partial credit, but definitely not as, like, an answer to a proof." I then told Barbara that I thought what she did could serve as a proof as long as the explanation was airtight. She seemed alright with it at that point, saying: "I guess in a sense, it's kind of a proof by definition. Because you're defining your formula."

Because Barbara did not provide what to her seemed like a proof, this work is classified as an attempt. The bulk of her work was spent trying to use her examples to formulate an expression for *n*C*k*. However, she I did not get the impression that she was using her examples for insight. Rather, she was using the examples to confirm or rule out her guesses about what the formula should look like. Once she had the formula, she mentioned a few things to try to proof that she had the correct one. The only thing she tried, however, was algebraic manipulation before giving up. Her proof attempt is syntactic because, although she does not see it through to what she thinks is a proof, her focus is on the expression and manipulating it and not an understanding of the problem itself.

Because Barbara's focus is so skewed to getting a pliable expression she can deal with, the main proof scheme she displays here is transformational. In the little work she did do in an attempt to verify the formula, she also referred to algebraic simplification as the goal she was working towards. She also gives evidence for an empirical proof scheme, however. When she was working on finding the expression for *n*C*k*, she became convinced after seeing it work for both $n = 3$ and $n = 4$. In fact, this is evident later on as well. When discussing whether or not an explanation of her summation formula worked as a proof, Barbara said:

B: I understand what you're saying, it's just choosing the right words to explain it because, like, I did these 2 examples and I was like "Ok, it works" or like I plug numbers

> to make sure the formula works and then it's proved to me. I don't feel like I have to do induction to prove it for all cases. It's like "Oh, it works for these few, so it probably works for all of them."

Barbara is referring to the difference between what convinces her and what would constitute a proof to others. While she knows there are higher levels of rigor, to her seeing it work for a few examples is good enough, verifying that she does have an empirical proof scheme.

<u>Question 8</u>

Prove that the cube root of 2 is irrational using a proof by contradiction.

This is a proof that Barbara saw on her midterm take-home exam that she took while in MATH 305. Barbara's work from the midterm can be seen in Figure 90. As one can see, Barbara had no problem completing the problem on the test. At the time, she had recently seen the proof that $\sqrt{2}$ is irrational and her proof on the midterm is an adaptation of that.

In the interview, Barbara was not as successful. She began the problem just fine, knowing how to set up a proof by contradiction. From there, however, she relied on algebraic manipulations of the equation you get out of assuming that the cube root of 2 is rational. This work can be seen in the right column of her work in Figure 91.

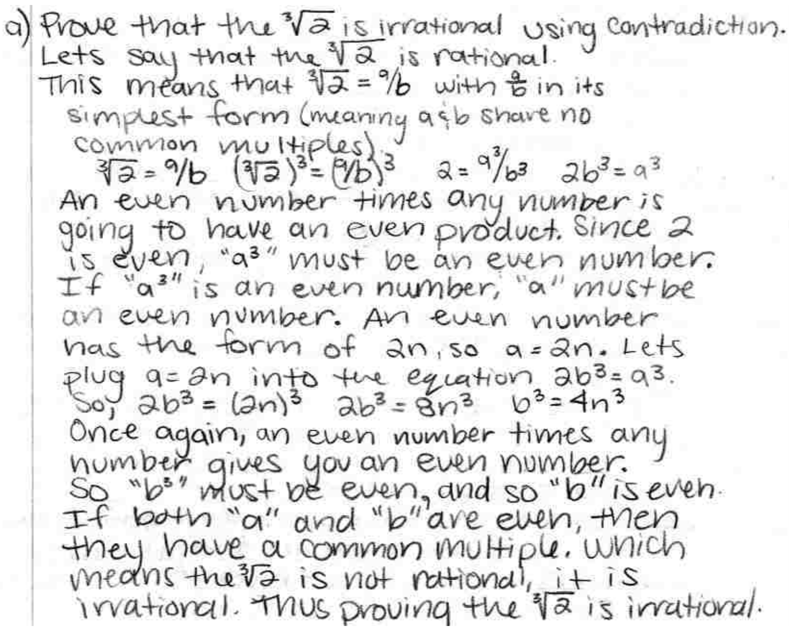


Figure 90: Barbara's previous work on Question 8

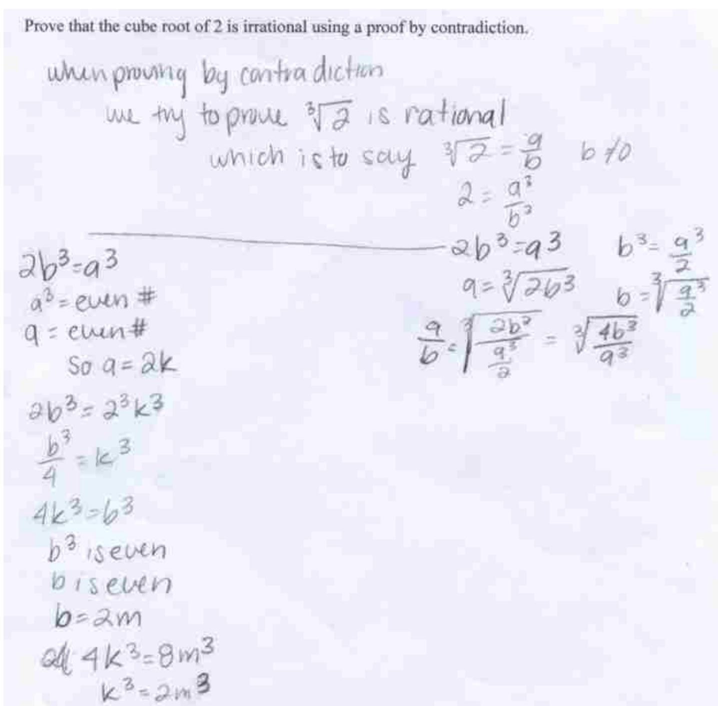


Figure 91: Barbara's work on Question 8 (1 of 2)

When she got to the bottom of that column, she was stuck and did not know where to go from there. She told me that if she was to work on the problem at home at all, she would "straight to my notes." I thought that it would be more fruitful for the study if I gave her a hint so that I could see what she did from there.

N: This equation right here, $2b^3 = a^3$?

B: Yeah

N: That tells you that, what kind of number is a^3? …

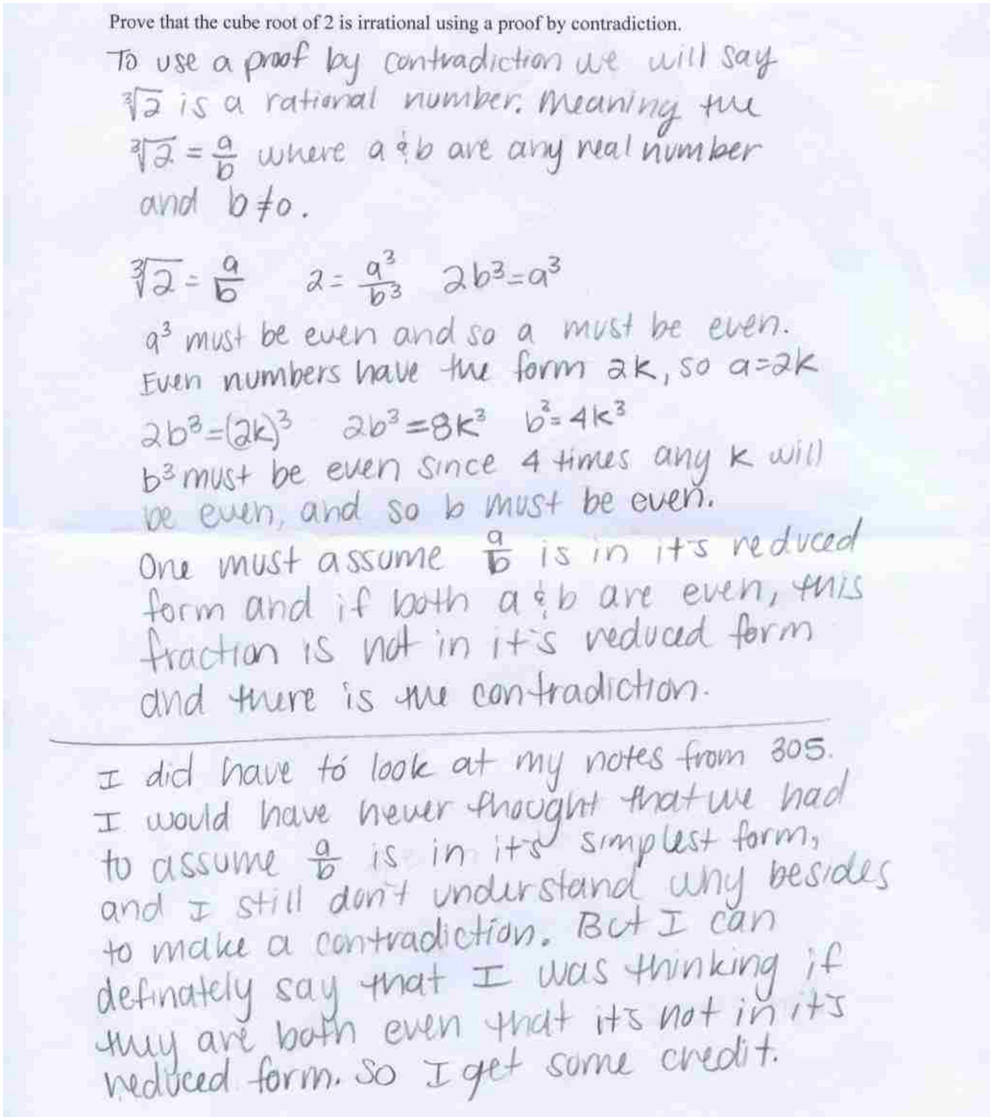
B: I have no idea.

N: Since a^3 is equal to 2 times b^3 …

B: Oh, it's even.

This insight led her to complete all the work in the left hand column of Figure 91. She got to the bottom of that column and knew she was onto something: "I remember using that, but I don't remember where that went." She thought about her conclusion for a while and eventually said: "Well, it's just like it's this idea, I'm just supposed to, like, I end up saying *a* is odd or something, right? I'm just not to the point of seeing it to the end." I felt that telling her to start with the assumption that *a* and *b* were relatively prime would be too much of a hint, so we ended the interview. I asked her to think about the problem some more before she looked at her notes and she said she would.

At the beginning of the next interview, Barbara brought the work that is in Figure 92. She did look at her notes to finish the problem, but she was close. Barbara asked me about the issue she mentions at the bottom of the figure. She does not understand why a/b would have to be reduced based on the assumption that $\sqrt[3]{2}$ is rational. I explained that she was right, it would not have to be but it could be and that additional assumption was made in order to reach the contradiction. She seemed alright with the idea at that point.



Prove that the cube root of 2 is irrational using a proof by contradiction.

To use a proof by contradiction we will say $\sqrt[3]{2}$ is a rational number. Meaning the $\sqrt[3]{2} = \frac{a}{b}$ where a & b are any real number and $b \neq 0$.

$\sqrt[3]{2} = \frac{a}{b}$ $2 = \frac{a^3}{b^3}$ $2b^3 = a^3$

a^3 must be even and so a must be even. Even numbers have the form 2k, so $a = 2k$

$2b^3 = (2k)^3$ $2b^3 = 8k^3$ $b^3 = 4k^3$

b^3 must be even since 4 times any k will be even, and so b must be even. One must assume $\frac{a}{b}$ is in it's reduced form and if both a & b are even, this fraction is not in it's reduced form and there is the contradiction.

I did have to look at my notes from 305. I would have never thought that we had to assume $\frac{a}{b}$ is in it's simplest form, and I still don't understand why besides to make a contradiction. But I can definately say that I was thinking if they are both even that it's not in it's reduced form. So I get some credit.

Figure 92: Barbara's work on Question 8 (2 of 2)

The proof Barbara provides here is a procedural proof. She is given specific steps to follow (either by me in the interview or from the analogous argument regarding $\sqrt{2}$) and she follows the proofs to complete her proof. Of the 2 types of procedural proofs, this is an algorithmic proof because for the most part, she is following very particular steps. The classification of process proof is reserved for proofs that follow "a shorter list of global qualitative steps" (Weber, 2004, p. 2). This is not the case here.

Barbara's proof technique is transformational. While she does not come up with the steps she completes entirely on her own, she does understand then as she goes. This rules out an authoritarian or ritualistic proof scheme. Also, these steps involve algebraic manipulations and the results of them. These are characteristics of a transformational proof scheme.

<u>Question 9</u>

During the next interview, I asked Barbara to read and evaluate a version of Cantor's diagonalization argument. Barbara did not work on a proof so there is no proof or proof attempt to classify, but the interview did reinforce some of the things seen in previous interviews and gave insight into Barbara's proof scheme.

The interview began with making sure Barbara knew all the terminology involved, which she did. From there she read through the problem and had the same issue as she did with the last question. She said, "I guess I just don't understand why you would define *B* like this." Barbara is referring to the fact that she does not understand why one would take the action the proof writer did. This is similar to her issue with the proof for Question 8 where she did not understand why someone would make the decision to assume that *a* and *b* were relatively prime. Again, I explained that this was not necessary but that it helped with the proof.

This led Barbara and I into a conversation of the last paragraph of the proof. Like with all participants her understanding of this part required some discussion, mostly to get used to the notation and to get everything straight. Once she felt comfortable with what was going on, she was able to sum up what was happening with the proof:

> B: Umm…because the…because there has to be a function of some *k* that equals *B*, but in our definition of the *B*, it's never equal to *k*.

N: Right

B: So it, like, jumps around rather than going straight there in a sense.

N: So, what do you mean by that?

B: I don't know, I guess it's just like the onto, like, every number has to go to the same number, or, like, it can't go to 2 different numbers.

She is actually referring to the definition of a function, not onto, but she gets the idea.

At that point, I began to ask Barbara if she believed the argument because she seemed on the fence about it. This led her to become suspicious:

B: It seems to flow to me.

N: Ok. But you're unsure of it, right?

B: Well, you're making me unsure. I guess it's like a teacher tells you something and you just assume that it's true.

N: Ok

B: I mean, going through it, seems like it would be true. There's nothing missing.

This is evidence that Barbara has an authoritarian external proof scheme. She mentions her tendency to believe statements that come from a teacher. She may be referring to believing the proof because of the way it was presented. It is also possible that she is reacting to my line of questioning regarding the proof's validity. This would be similar to episodes in classrooms where a student rethinks his or her response when a teacher questions it rather than simply moving on. In either case, Barbara is using the actions of an authority figure influence her belief in the proof.

The interview also provided some evidence that Barbara has an axiomatic proof scheme. During the reflection portion of the interview, I asked her if she thought the proof needed any of its statements verified or not. She said:

> B: I think it's all pretty self-contained… I mean, I just like, it could. Sometimes when you see proofs, like to try and point out, like, "this is a lemma" when they don't say it because…you're like "Oh, I know that's true, so…"

I pointed out that the statement "since every countable subset of a countable set is countable" might be something that required proof. When I asked why she did not think that needed a lemma, she said "I guess I just…I guess it's just because I....I knew it was true, I assumed it was true. Kind of give and take between those two."

This exchange is weak evidence of an axiomatic proof scheme because she is acknowledging the reliance of proof on other results. In this particular case, she decides that the verification of this other result does not need to be provided. In a broader sense, though, she points out that this reliance occurs in other proofs, even if not explicitly stated.

<u>Question 10</u>

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

Barbara began this problem by looking at examples. She tried all the examples in Figure 93 before moving to the general case in the left hand column in the figure.

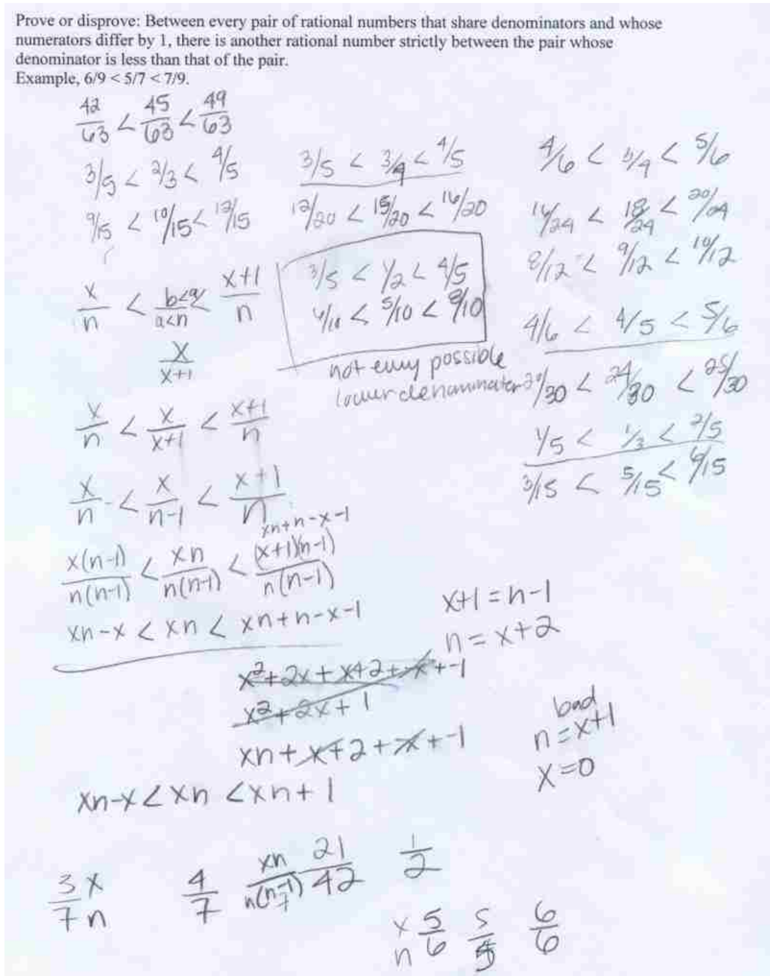


Figure 93: Barbara's work on Question 10

After the examples and writing the inequality with the variables, Barbara did not know where to go.

B: Now I'm stuck.

N: Now you're stuck?

B: Yeah, I tried to come up with a line that…

N: Ok…Let's see so you have *x*/*n* less than *b*/*a* where *b* is less than *n,* and less than *x* + 1 over *n*. So it looks like you're after…you're try to prove it.

B: Right

I asked Barbara about what she had done at that point:

N: So did you look at those examples to decide for yourself whether or not it was true or did you do it to try to get some insight into the problem? Or both?

B: Mostly insight.

N: Ok

B: I was mostly assuming that it was true…because, I mean, the basic idea that there's a rational number in between them is definitely true.

N: Yeah

B: But to say that there's one that's just with a denominator less than that…the example was true and so I was like 'Ok'.

At this point, Barbara was fairly sure the result held, but she did not know how to go about proving it. I then suggested that there was a pattern in the examples that she could use, but that not all of the examples fit it. Eventually, I pointed out which examples I was referring to and she was able to see the pattern:

B: It's (the middle number) always one less in the denominator…when they're (the middle and left numerators in the inequality) equal. Then the denominator is the numerator in the next one.

N: So let's just pretend for a second that that always worked. Ok?

B: Ok

N: Would you be done?

B; Well, you have to show that it always works.

So, she believes the result holds but she knows that she has not provided a proof even though she has an algorithm developed to find the number she is looking for. She goes onto apply this method to the general case (moving down the left hand column in the figure).

> B: I just multiplied it all out so I had the same denominator…and if the center one is *xn*, the one below it is less than that because you're minusing *x* and this one's going to be bigger because…you're adding stuff…but you're subtracting stuff too.

To finish, she uses a property that holds in the examples she based the pattern on but does not hold in general. During the interview, I did not notice this either. The fact she used was that the numerator on the right equaled the denominator of the middle number ($x + 1 = n - 1$). Applying this supposed equality allows Barbara to arrive at $xn - n < xn < xn + 1$, which she deems to be true.

I then pointed out some potential problems with this by having Barbara examine counter-examples. I suggested that she look at the pairs 5/6, 6/6 and 0/2, 1/2. This led her to conclude: "It can't be 1 or 0." She was referring to the fact that neither of the bounds could be either 0 or 1 for the property to hold (if $x = 0$ or $n = x + 1$). We then discussed how these cases affected the general inequality she had and the fact that the property holds when these restrictions are satisfied. We also talked about how one could also look at rational number outside the unit interval so long as one treated them as mixed numbers and dealt with their fractional parts only.

Barbara's proof is not semantic despite the fact she used examples to gain insight into the problem. The insight she gained eventually came in the form of a pattern that could be turned into a proof. It is an important distinction to realize that her examples did not lead her to an understanding of how the problem was working. Instead, they led Barbara to an expression she could verify through algebraic manipulations. Although she needed help turning that pattern into a proof, she did eventually get there (in her mind). Remember, some of the details of the proof were based on the faulty identity $x + 1 = n - 1$. Despite this, she could have used relationships that come from the restrictions to correctly complete her proof. Because her proof is largely made up of operating on her expression, it is a syntactic proof.

While she used examples for "mostly insight" she also used them to convince herself that the result was true. However, this was accomplished fairly early in the process. The following exchange occurred when I was asking Barbara about how she decided if she would try to prove or disprove the property:

> N: So, do you sort of, was it just sort of an intuition that told you that it was true, that made you not look for a counter-example?
>
> B: I guess it was just seeing that the first example was true and so I was like 'It's true for something.'
>
> N: And why would I ask…
>
> B: Well, it's not so much that 'Why would he…'
>
> N: Oh, ok
>
> B: …and give me an example that worked?' But, it was just more like 'Well, it's true, so…it's at least true for something.'

So, seeing it work once along with her reasoning that it would certainly work without the restriction on the denominator led her to believe it was true. Because she was convinced via examples, Barbara displays an empirical proof scheme.

Barbara also displays a transformational scheme. She knows that while she has become convinced by examples, they do not serve as proof. Because the proof she provides relies on logically valid mathematical operations and the outcomes of those manipulations.

<u>Question 11</u>

The next interview was the last of the study. Barbara spent it discussing the progress she felt she had made over the course of the study. Due to the nature of the interview, there was no proof or proof attempt to classify.

The interview helped to highlight some of the observations made earlier in the study. For example, when I asked Barbara if she thought she had improved with proofs over the course of the study, she singled out induction as something she improved on: "Definitely because I actually knew how to do an induction proof on a test." This was something that she had mentioned earlier, although before she referred to a quiz. She even mentioned later that she had tried to use induction more because she had gotten better at it. "Well, I mostly learned how to do induction. And so I was able to use that."

Barbara's response when I asked about the role of examples in proof also matched what she said and did earlier in the year:

> B: Oh, ok, then yeah definitely, they play a very big role because, I mean even if there isn't a pattern to notice, it still gets you to like a place of where it's going, and idea of where it's going…Like how it works.

When I asked Barbara to differentiate between what is necessary to complete a proof and what is helpful, she again brought up examples, indirectly: "Yeah, I think it helps just to write it out and, instead of just trying to see patterns in your head, like write them out on paper or…at least for me." Here, she is referring to writing out examples to see a pattern that might be helpful in solving the problem.

As far is what is necessary, she said: "Well, what's necessary is logical steps to…gauge an answer." I feel this matches what she said during the last interview of the first semester: "Just making logical steps with stuff that's true." While she often used examples to convince herself of the validity of a statement, she realized that formal deduction was required for something to be considered a proof.

While much of our discussion revisited some of the things that had shown up over the course of the study, Barbara does not give evidence for all the proof schemes she displayed throughout the year. For example, she does not mention becoming convinced by examples. Instead, she pointed out their ability to provide insight. Also, she does not discuss proof's reliance on previous results like she had in a few interviews. Everything we talked about, though, showed that Barbara has an analytical proof scheme. Her focus on logical steps reinforces that. Since her analytic proof scheme is not axiomatic, it is transformational.

Barbara's progression

Below is a chart displaying the classifications of Barbara's proofs and proof attempts along with the proof schemes she had along the way.

Question	Type of proof	Proof scheme
1	Semantic (Attempt)	Transformational, Empirical
2a	Syntactic (Attempt)	Transformational, Empirical
2b	Algorithm	Transformational (Internalized)
3	Process (Attempt)	Transformational (Internalized)
4	N\A	Axiomatic
5	N\A	Transformational, Axiomatic
6	Process (Attempt)	Transformational (Internalized)
7	Syntactic (Attempt)	Transformational, Empirical
8	Algorithm	Transformational
9	N\A	Authoritarian, Axiomatic
10	Syntactic	Transformational, Empirical
11	N\A	Transformational

Table 5: Summary of Barbara's work

As is typical, the type of proof Barbara gives or attempts varies with the kind of problem she is asked to solve. When one looks at Barbara's proof schemes throughout the study, one does not see much change. The biggest difference Barbara shows is in her improved understanding of induction, although it is not evident from looking at the chart. I presume she eventually got over her misconception regarding base cases, since she referred to completing two different problems correctly on graded work.

Other than with induction, though, she does not show much progress. In both Question 1 and Question 10, she mentions becoming personally convinced by empirical evidence. Barbara does not let this affect her view of formal proof, though. Part of the reason she seems to not make much progress is that she started the study with a fairly formal view of proof. This can be seen both in the number of times her proof scheme is deemed transformational and also in the fact that her description of what is necessary for a proof does not change between the end of the first semester and the end of the study. She also shows a fairly consistent reliance on examples throughout the whole study. In fact, in the last interview Barbara summed up pretty well how she sees proof: "Everything just…just start with examples and make sure your tool box is full of the right tools."

4.6 James

This case study looks at the work James did over the course of the study. James was a mathematics major. He took Statistical Methods, Probability Theory and Ordinary Differential Equations during the first half of the study and History of Mathematics, Statistical Methods II and Mathematical Statistics during the second semester.

James's proof attempts

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

James started the problem by looking at a diagram of a rectangle "just to get an idea of what we'd be looking at." From there he went to the equation $2a + 2b = ab$.

James: And then from there…what would I do…just trail and error, what it could be.

N: OK

J: I could…right off the bat the first one that came to mind would be let a = 4 and b = 4 and you're going to get 16 for the perimeter and 16 for the area.

James then started the smaller chart shown in Figure 94. He had found one solution but recognized the difficulty in using a chart to prove that he had found them all.

J: It's easy, I guess, if you look at it like it's squares and just do like 1 1, then you get 4 but an area of 1. That doesn't, but then you'd have like one 8, but then there's just too many options to look at in a table format.

James then went to algebra as an alternative to using a chart. First he isolated *b* and then he used

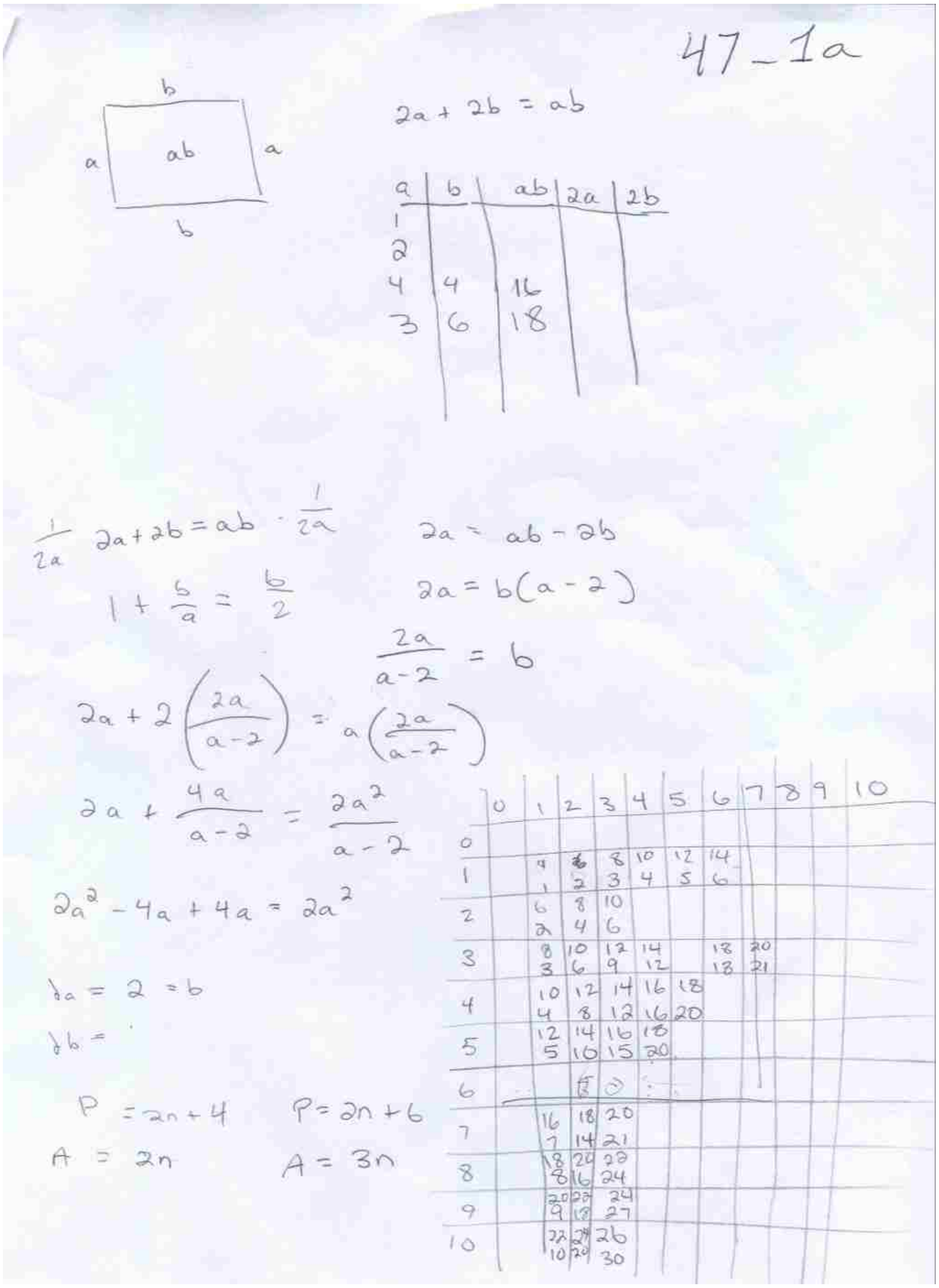


Figure 94: James' work on Question 1 (1 of 2)

his new express to replace *b* in his original equation.

N: And so, what'd you come up with?

J: 0 (=) 0

We then talked about the fact that he is basically plugging an equation back into itself and his method typically works when you have 2 equations and 2 unknowns. James then resorted to looking at the partial derivatives of the equation, but realized that would not help either. When I asked why he tried this, he said "Trying to do stuff I've done recently because that's about the only math I can come up with right now."

From there, he seemed to take a step back. After thinking quietly for a bit, he said:

J: Yeah…um, I just thought of another one, though.

N: Thought of another what?

J: Um, one that would work.

N: Which is?

K: Going back, to 3 and 6.

James put his new solution in the small chart above and decided that maybe a bigger table would not be such a bad idea:

J: Yeah. I should do a table, like 0, through 10. Just do kind of like a grid pattern.

N: Ok. So, along the x and y axis, I guess?

J: Yeah. And just seeing if it shoots off.

N: Ok

J: Or a pattern goes anywhere there.

He then went about setting up the chart in the bottom right of Figure 94. He began filling it in by working across the 1 row, and then stopped:

N: So, you've stopped filling out the 1.

J: Yeah, you can …there's a, a pattern emerges that, the perimeter is going up by 2 but the area is only going up by 1.

N: Ok

J: But there's a possibility that it could also deviate from that pattern, but it's followed that pattern for the last 6, so I'm just going to assume that it's going to continue also. There isn't really there's not much reason to continue there.

N: Ok. So, now you're going across the 2 row.

J: Yep. The 2 row. (filling it in) …and that one's holding to a pattern as well. Each one's going up by 2 but they're not, area's not gaining on perimeter and perimeter is not…

James then moved on to row 3, noting that he should find an answer there. "But then, continuing on after that, the one right after that, it, area surpasses it by 1, so…Yeah, so there isn't going to be any more in that row." The same sort of thing happened when James moved on to row 4: "Another pattern, 16 and 16 and that's where the 4 4 came in…and then continue on, area jumps over it again." Moving on to row 5, he says: "And then the area on 5 is just going up too fast, it just hops over…So there isn't one…"

Upon moving to the 6th row, James makes an observation that serves him well later:

J: And I know at 6 and 3 there's another one because 3 and 6…

N: Right

J: …is there, so…I know that one will be there, and can assume that property that they're not going to be before that…

N: Ok, so why is that?

J: Um, you can look back, we were in row 2, 2 and 6 we know that one wasn't, so we can assume 6 and 2 won't be one.

James noticed that the chart is symmetric about the main diagonal and eventually that observation would allow him to eliminate a large portion of the table.

He then went back to filling in the rows.

J: So 7 surpasses within 3 spots.

N: Ok

J: I guess it's just going to just keep getting quicker and quicker until it just shows that you'll be down to one and it won't happen any more.

James then did the remaining rows out to the third column, seeing that in each the area surpasses the perimeter in the third column but not before. At that point, he thought he was done: "So you get 3 and 6 and 4 and 4. I don't think there was any others. Yeah, so there's just the two. I'd say as far as saying that's an absolute…that'd be a little more difficult."

Generalizing what he had found up to that point, James said: "That happens after the last…so after 6 and 3, every column after that, in the third row, you're going to have area larger than perimeter, in the 2nd row, it's going to be perimeter larger than area." James then talked about how that could be turned into a proof, saying:

J: I would like to see, I think, if you gave it to someone and were able to come up with, you know, an equation for what the 2nd row is going to be and what the 3rd row is going to be. And they could throw in any number that they want, they can choose…

N: Sure

J: …and it always came out like that, then…

James then came up with the 2 sets of 2 equations to the left of the chart in the figure, the $P = 2n + 4$ and $A = 2n$ pertaining to the second column and the $P = 2n + 6$ and $A = 3n$ referring to the third. James the talked about how the proof could be completed:

J: Yeah, and so, yeah, starting, say 7, you can definitely see just from there that area's always going to be larger. Just because…7 is larger than 6, 8 is larger than 6, 9 is larger than 6, so just from looking at that formula that area's always going to be larger than perimeter…and then…yeah perimeter is always going to be larger when you're looking at column 2.

N: Ok

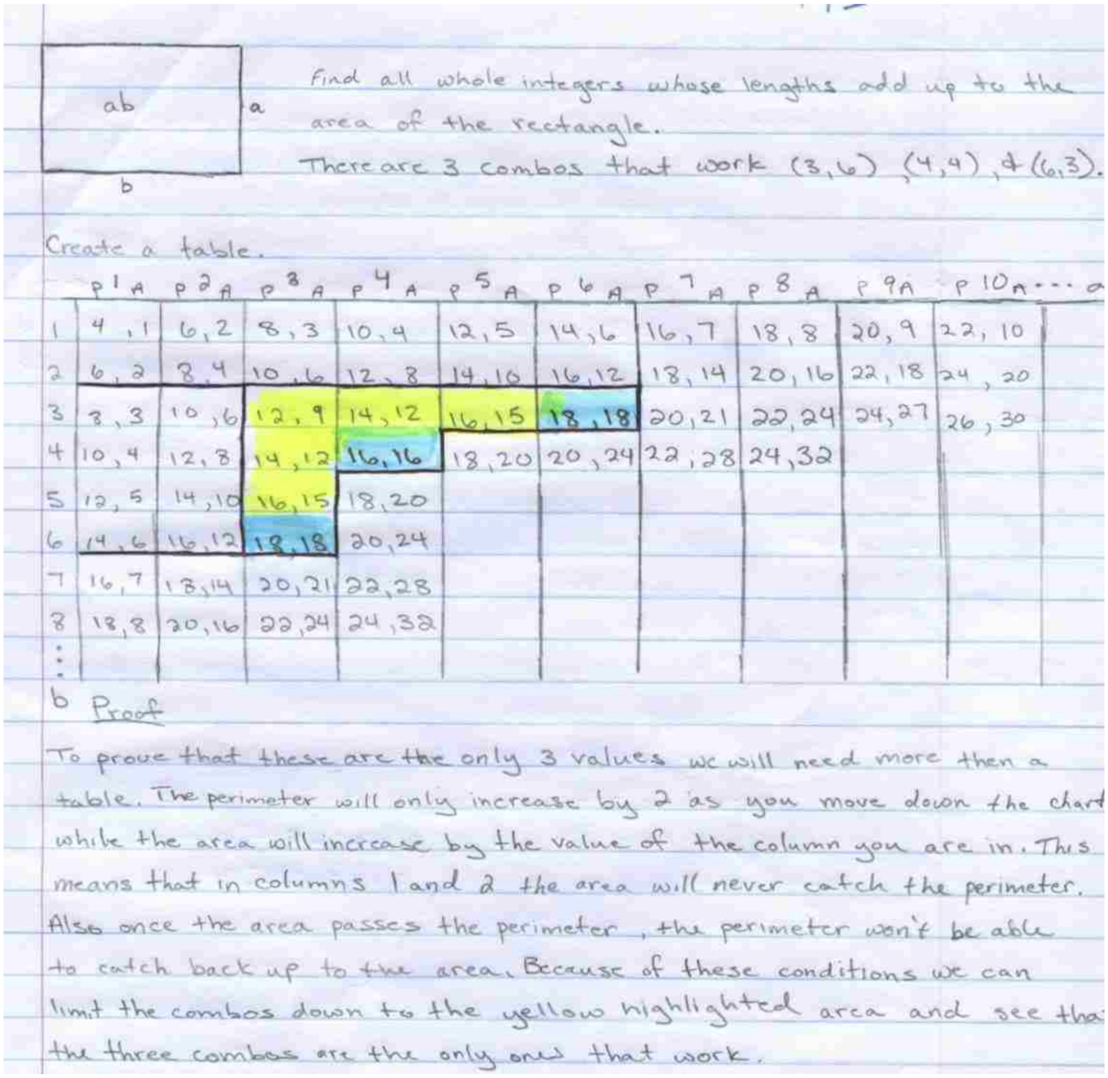
J: For 7 on. And then it was shown, I think that…

N: You think that'd do it? Do the trick? Basically, I'm asking, you know, does this sort of thing cover all your bases?

J: I think, yeah. I could show that it would. The fact from looking at from 7 on, you could actually take the table and actually cut it down so that you only you're only looking at the 6 by 6…section, which is quite a bit easier to show and to fill up that table with values and…

James' plan had not yet tackled the columns past 6 in rows 1 through 6, but he addressed that by saying: "Well, 'cause you can show that from seven on the columns it won't work and then seven down in the rows…Just because, when you're looking at the rectangle, they're going to be interchangeable."

I then asked James to write up his proof and bring it back to the next interview. The work he brought back is shown in Figure 95:


Find all whole integers whose lengths add up to the area of the rectangle.

There are 3 combos that work (3,6) (4,4), & (6,3).

Create a table.

	P1 A	P2 A	P3 A	P4 A	P5 A	P6 A	P7 A	P8 A	P9 A	P10 A ... a
1	4,1	6,2	8,3	10,4	12,5	14,6	16,7	18,8	20,9	22,10
2	6,2	8,4	10,6	12,8	14,10	16,12	18,14	20,16	22,18	24,20
3	8,3	10,6	12,9	14,12	16,15	18,18	20,21	22,24	24,27	26,30
4	10,4	12,8	14,12	16,16	18,20	20,24	22,28	24,32		
5	12,5	14,10	16,15	18,20						
6	14,6	16,12	18,18	20,24						
7	16,7	18,14	20,21	22,28						
8	18,8	20,16	22,24	24,32						
⋮										
b										

Proof

To prove that these are the only 3 values we will need more then a table. The perimeter will only increase by 2 as you move down the chart while the area will increase by the value of the column you are in. This means that in columns 1 and 2 the area will never catch the perimeter. Also once the area passes the perimeter, the perimeter won't be able to catch back up to the area. Because of these conditions we can limit the combos down to the yellow highlighted area and see that the three combos are the only ones that work.

Figure 95: James' work on Question 1 (2 of 2)

The work James brings back is incomplete compared to what he did in the interview. He does not mention the symmetry in chart that allows him to conclude that his search is over. A thorough explanation of this would have sufficed in making this a complete proof.

The proof James provided in the interview is a semantic proof. He uses the chart to understand the relationship between the area and perimeter and he turns that understanding into a proof. James's proof scheme is not so cut and dry, however. On one hand, he seems convinced by empirical evidence a few different occasions when looking at the patterns in the table. While he acted as if he was sure based on empirical evidence, he actually acknowledged that the pattern might deviate. So, his conviction was really more of a hunch and he eventually verified his conjecture. The proof scheme he actually displays here is transformational. He operates on the objects at hand (in this case the relationships between the area and perimeter in each of the rows) by observing their behaviors as he moves down the rows. He then makes use of the relationships to provide a proof.

<u>Question 2a</u>

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9.

James started this problem by looking at examples. Like with Question 1, when he as dealing with the patterns in the columns, James seemed sure the pattern would hold but knew that was not a proof: “Yeah, 13 and 31, got -18. 45 and 54, -9. So it definitely works, but as far as proving it…” After a little more writing, James moved on to an attempt at a formal proof:

> N: So you wrote $(a + 1)(b+ 1) – (b + 1)(a + 1)$. So there you’re using the parenthesis to separate one spot from the next…
>
> J: Yeah, just separate them out.
>
> N: So why $a + 1$ and $b + 1$?
>
> J: Just hoping that induction would go somewhere. So you could look at that and say, yes it works for those values, ab and ba and if it has to be under 10, then it should work for $(a + 1)(b + 1) – (b + 1)(a + 1)$…I don’t see it actually taking me anyway unless we separate it all out.

James then left that method alone for a while, going back to examining his examples for insight.

> J: I'm just trying to see why by switching *a* and *b* you're always getting a multiple of nine…by taking 23 and switching it to 32…it's always going to be a multiple of nine. It has to do with just switching those numbers.
>
> N: So you're thinking of how that change…
>
> J: …is guaranteeing the multiple of nine.

After some more thought, James restarted the problem again, this time looking at the operation of subtraction in more depth:

> N: So you're looking at a0 – b0?
>
> J: Yeah, just looking at the, just a quick way of…
>
> N: …sort of examining the process of subtraction?
>
> J: Yeah, I'm just getting rid of the last *a* – *b* and…so you're always going to end up with 10, 20, or the negative as well of those (multiples of 10). And with *a* and *b*… *b* – *a*, that will just take down the (multiple of) 10 to the actual number.

This work can be seen in the middle of Figure 96. James had come up with a way to examine the individual place values in the subtraction *ab* – *ba*. This consideration of place value would play a big role in James' eventual proof.

He did not see how his new way of writing the subtraction would help, so he went back to looking at examples:

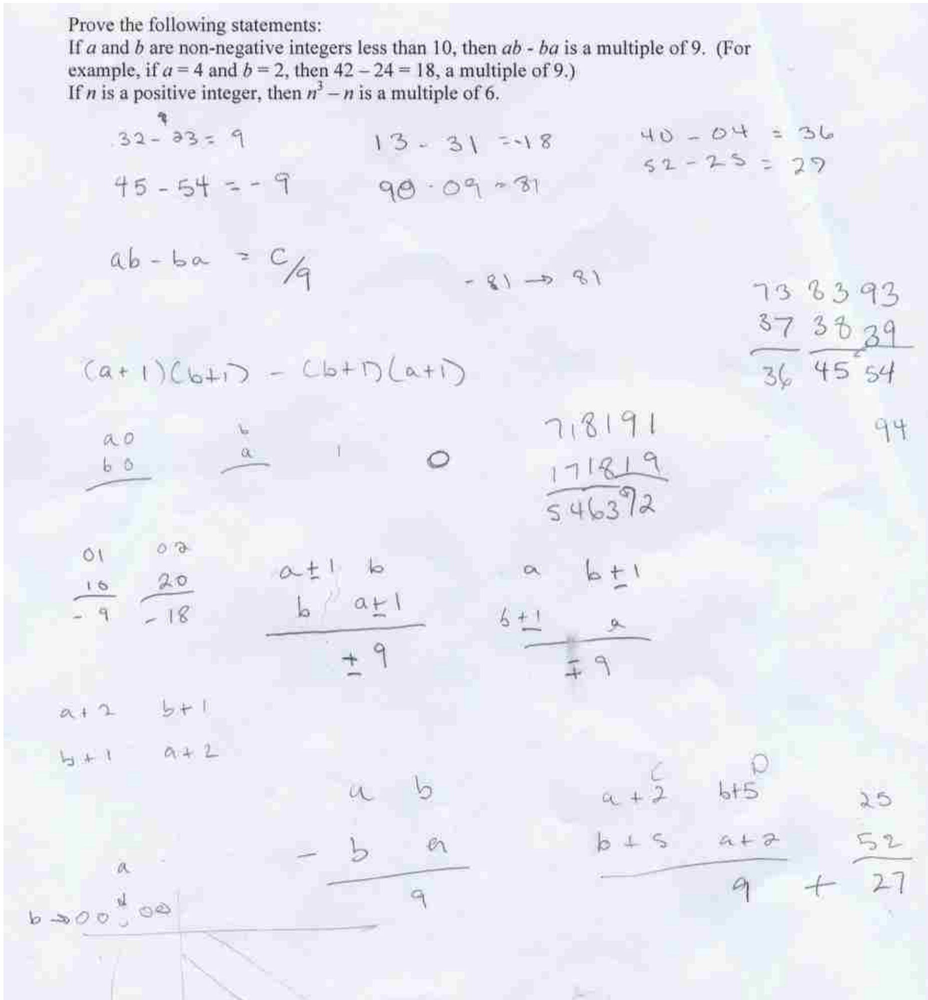


Figure 96: James' work on Question 2a (1 of 2)

J: I just started looking at 91, 81, 71, just decreasing them by ten. And then getting 72, 63, 54 so decreasing powers of nine.

N: Ok. So you're decreasing the *a* value, right?

J: Yeah, decreasing the *a* value, *b* value stays constant. Drops that *a* value by 10…

(working, does same thing with b = 3 instead of 1)

J: Yeah, so that pattern always holds.

N: Ok, so do you see a reason why, or do you just sort of believe based on these examples?

J: I'm just going by the examples, I can't think of anything right now…But that does make sense, because as the *ab* decreases by 10, the *ba* is only decreasing by one so the difference will always be 9.

By looking at place value in conjunction with a change in *a*, James was starting to see way the property he was proving holds. After trying some examples in which $a = 0$, James quickly applied the same reasoning to changes in *b* value: "I'm increasing *b*, but it's the same pattern as …It's just doing the opposite, you're still going to get a difference of nine, but it's just going to be a negative nine because the *b* will be larger."

After writing out the generalized subtractions with 1 added and subtracted to *a* and *b* separately, James went on to consider how to turn his understanding into a proof: "Ok. So there's our pattern…Would a proof, I feel like using proof by induction by…" James is able to explain how the proof would work if only one variable were allowed to change, but the presence of 2 had him stuck.

J: I guess the only problem I really saw with showing it this way is that, say you start with *a* and *b*, you're only going to be changing one variable at a time, you're not changing both.

N: Yeah, so can you reconcile that, then? If you had to change both?

J: Yeah, because if you looked at *a* plus or minus 2, *b* plus or minus 3, you can go through the same process.

To see how to handle the case when both variables change, James looks at the last example in the bottom right of Figure 96.

J: Any of the numbers that you're adding on, like say let this 2 be a *C* and that 5 be a *D*, then you can separate them because of addition, you can look at the *ab* – *ba* plus the *cd* – *dc*, we've already proved that it works …

N: I see, you've proved that, well you need to be careful because you've proved that it works if you change one of the variables. And it doesn't matter which one you've changed. But you haven't, I don't think…

J: Yeah. I see that now. Because we haven't actually proved it that *ab* – *ba*…

James was hoping to handle the case when both variables needed to change by pulling out the changed part, leaving him with *ab* – *ba* + *cd* – *dc*. He realizes, though, that this would not work.

James leaves this last idea and comes up with a new one: "Wouldn't these two, just these two statements, won't they cover, just 1 through 9 for all the foundations?" I asked him if he could use his reasoning to change cases from 00 – 00 to 25 – 52. "Well, yeah because if you're starting out with 00 we know that's leaving *b* the same, so let's start out with 05 and 50 and start going down and get 15 and 51…" James then draws the beginning of a chart which is the in lower left corner of the figure. He reasons that it works for 00 – 00 and from there one could verify any possible difference by changing *a* values (moving up or down columns) and *b* values (moving between columns) as necessary.

J: Yeah, I don't see why, I think the hardest part that I'm running into now is that it seems verified just from me looking at it, I don't see what else really needs to be stated by saying that, use *b* to get to any column, you know you get to that *b*, it doesn't matter. Then with that *a*, the *b* has no effect, so you can use that to go down that column and hit for every *a*.

James was confident he could turn this argument into a proof and he say he would bring it back for the next interview. The proof James brought is shown in Figure 97:

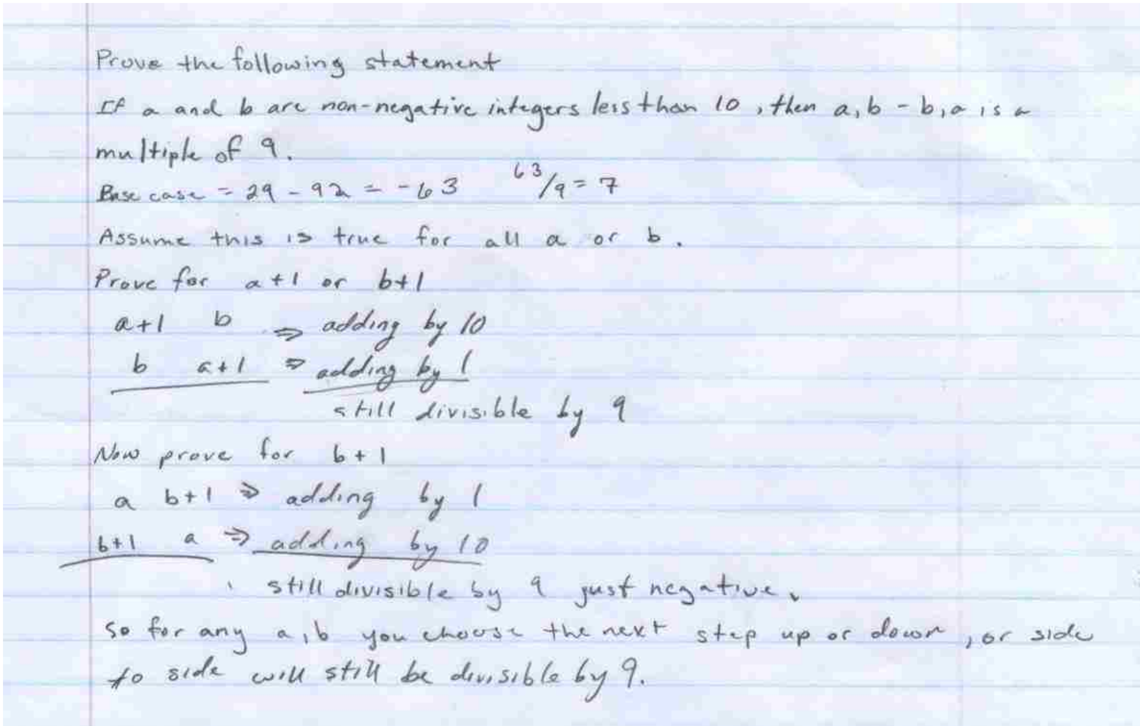

Prove the following statement
If a and b are non-negative integers less than 10, then a,b - b,a is a multiple of 9.
Base case = 29 - 92 = -63 63/9 = 7
Assume this is true for all a or b.
Prove for a+1 or b+1
a+1 b ⇒ adding by 10
b a+1 ⇒ adding by 1
still divisible by 9
Now prove for b+1
a b+1 ⇒ adding by 1
b+1 a ⇒ adding by 10
still divisible by 9 just negative.
So for any a,b you choose the next step up or down, or side to side will still be divisible by 9.

Figure 97: James' work on Question 2a (2 of 2)

As was the case with Question 1, the proof James brings back is an incomplete version of what he had in mind during the previous interview. Strictly speaking, his proof only works for *a* values of 2 through 0 and *b* = 9 (since he did not discuss subtracting 1 from *a* or *b*). However, one can see that he had his broader argument in mind because he mentions moving up, down or side to side as was the case with his grid idea.

The proof James provides is semantic. He looks at examples to find insight into the problem. Eventually that insight is found by combining observations about place value and the change seen in the difference *ab* – *ba* when one of the individual values changes. Once he has this understanding, he turns it into a proof. This is exactly the sort of process that constitutes a semantic proof.

James' proof scheme is similarly straight forward. He operates on the difference $ab - ba$ and uses the results of this operation to complete his proof. The fact that James uses a base case of 29 – 92 may lead one to conclude that he does not thoroughly understand the proof he provided. However, I believe this was an oversight on his part because during the discussion that took place in the interview, James realized that if he was merely adding to either a or b, then one would start at 00 – 00. Also, it is possible wanted to use the fact that one could add or subtract 1 from either variable and still have it work. In any event, James displayed understanding during the interview and thus he is displaying an interiorized, rather than internalized, proof scheme.

<u>Question 2b</u>

If n is a positive integer, then $n^3 - n$ is a multiple of 6.

Unlike the previous question, James did not begin this one by looking at examples. Instead, he tried a few different methods first. The first thing he did was factor the expression.

J: Kind of broke it apart, just to see if anything comes out, so $n(n^2 - 1)$ and then $n(n + 1)(n - 1)$ just to see if anything would come from there…

N: And it didn't?

J: No, didn't get anything from there. Thought maybe I'd graph it, just for fun.

N: *And that didn't do much for you?*

J: Yeah, it shows that all the spots are going to be multiples of six, but that's not going…

N: Yeah, that's not going to show…

J: …to be a huge help. So then I just started listing out, this (left side of Figure 98) is the initial part, so it's not nearly as clear, so…

N: So, what is this initial part?

J: This is just the same as this (middle of figure), this is just written up a little nicer so I could actually see…

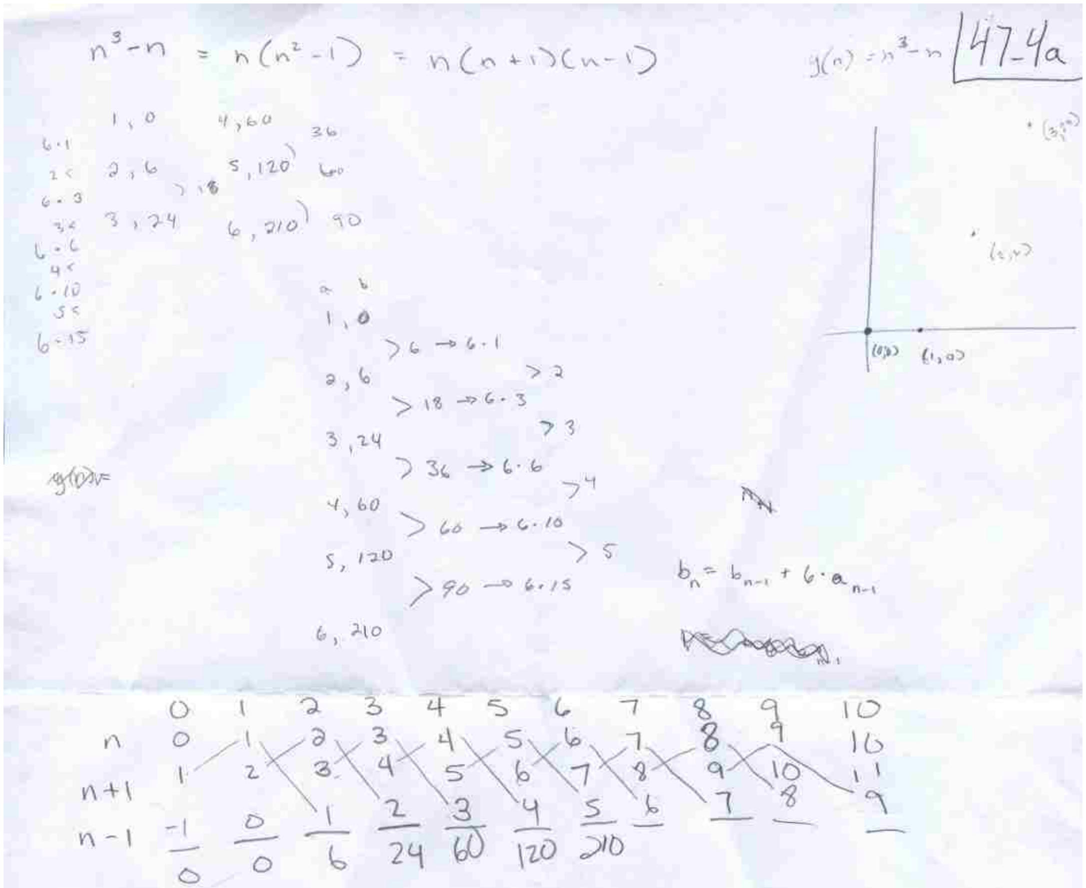


Figure 98: James' work on Question 2b

The work that James completed on the problem during the interview stopped at the middle of the page, just above the grid of values at the bottom. He had begun to look at the successive differences between $(n + 1)^3 - (n + 1)$ and $n^3 - n$, saw that they were multiples of 6 as well and then looked at the successive differences between their multipliers.

N: So you saw these successive differences are multiples of 6, so then when you divide those by 6 to get the multipliers of the differences.

J: Yeah.

N: And you saw that there was another successive difference thing going on there, because the difference between those multiples was 2, 3, 4, 5.

J: Yep. And then I tried looking at a way I could look at these patterns and turn them into some equation, where…

N: Like a recursive relationship.

J: Yeah, so I just called the *n*'s I changed them to *a*'s and the *b*'s for the multiples of 6 that were showing up. So then, your b_n, for them is going to be the previous $b_n + 6$ times…Yeah, and that's pretty much where we stopped with those.

During the reflection, I asked why he had tried a few different things before looking at examples. James said that "(f)actoring was easy, and it wasn't, it was more with a function, *ab* wasn't a function, well it was a function, but it was so simple…" He mentioned that the expression he started with looked like a function, which also explains why he wanted to look at the graph.

As we were talking about what he had done, he had a new idea:

J: I was just kind of looking at…and it's maybe just because…I was looking at with 5 you get, you know in this (factored) form, for when, you know, 5, 6, 4, so…I was just trying to think with 4, that'd be 4, 5, 3…

N: Ok…

J: Which, when you broke it had a 6 in it, but yeah, I don't know how you, in the general form…

N: So it sounds like you noticed that for all your n values, you can sort of tell what these, what each of these factors is going to be, that's what this factored form gives to you…

J: Yeah

James had noticed that the factored form of the expression allowed him to see the factors that make it up. It also allowed him to see that, at least in the cases he considered, there was a 2 and a 3 to be found once you broke it up.

At that point, the interview was over and I sent the paper home with him so he could continue to work on the problem. When he came back, he began to look at the individual factors *n*, *n* + 1 and *n* – 1 for different *n* values. This is the grid that it at the bottom of Figure 98.

J: Yeah, I went back to just looking, from when I split up the function to *n* times *n* + 1 times *n* – 1.

N: Ok

J: And started looking at those individually, and just wrote down the three defined relationship…and, you know, a pattern emerges.

N: So what pattern emerges?

J: Just, you can start with *n* + 1, it's kind of just a re-circulating, it moves up to *n*, then down to *n* – 1.

This, however, was as far as he got:

J: Yeah, and that's kind of where I got stuck. I was trying to figure out a way where, you know, as you're moving up the number line, you're getting , you know, you're either getting a 3 times a 2, or…a three times a 4 and the 5, well, you can just disregard the prime and you'll have a 3 and 2 still.

James was trying to articulate that for any n, between $n - 1$, n and $n + 1$ at least one of these numbers would have to be a multiple of 2 and one would have to be a multiple of 3. He just did not know how to prove it. As far as he got was identifying how numbers get shifted around as the n value is increased. In order to save time for the next problem, I talked James through the rest of the problem.

James does not complete a proof here, but he does provide a semantic attempt. He looks at a couple different ways to represent the expression $n^3 - n$ in hopes that he will find something that leads him to a proof. The closest he comes to a proof is the pattern he finds in the grid at the bottom of the figure. It is easy to see why James was looking for such a pattern; it was finding a pattern that led him to his proof for Question 2a.

Like with the previous question, James' work here suggests a transformational proof scheme. This is most evident in the way he observes the change in factors of $n^3 - n$ from one n value to the next. While this method of investigation may have been borrowed from the previous question, it does show that James is looking at how change affects a mathematical object (in this case, the expression $n^3 - n$). Thus, James shows a transformational scheme.

<u>Question 3</u>

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

James began this problem by moving directly into the induction argument. He worked silently for a few minutes and thought he had completed the problem:

J: So I'm pretty sure I'm done.

N: You’re pretty sure you’re done?

J: But, there’s just one value that’s kind of iffy.

N: So, what do you mean by one value? One statement that you’re not sure about?

J: Yeah, if, when breaking up this 1/1 + 1/2 + 1/3 + and so on to $1/2^{k+1}$, that you can break that up into $1/2^k + 1/2$.

James then looked at $1/2^{k+1}$ and $1/2^k + 1/2$ for some values of k because he was unsure of his statement. The values let him know that “1/2^$^{k+1}$ is actually $1/2^k$ times another 1/2, it’s not another plus 1/2.” He erased this work (see Figure 99) but left a note of what happened.

At that point, he went back to work. As he mentioned later during the reflection, James was working with the misconception that the only change that occurs on the right hand side of the inequality when moving to the $k + 1$ case is that the last term in the sum changes from $1/2^k$ to $1/2^{k+1}$. This, combined with accidentally changing $\geq$ to $\leq$ in the last line in the figure (they have since been switched back) led James to again think he was done. The thinking was that removing the factor of 1/2 from the last term would make the sum larger and get it back to the induction hypothesis. Then, he used his mistaken induction hypothesis to say the sum was less than $1 + k/2$, which is clearly less than $1 + k/2 + 1/2$.

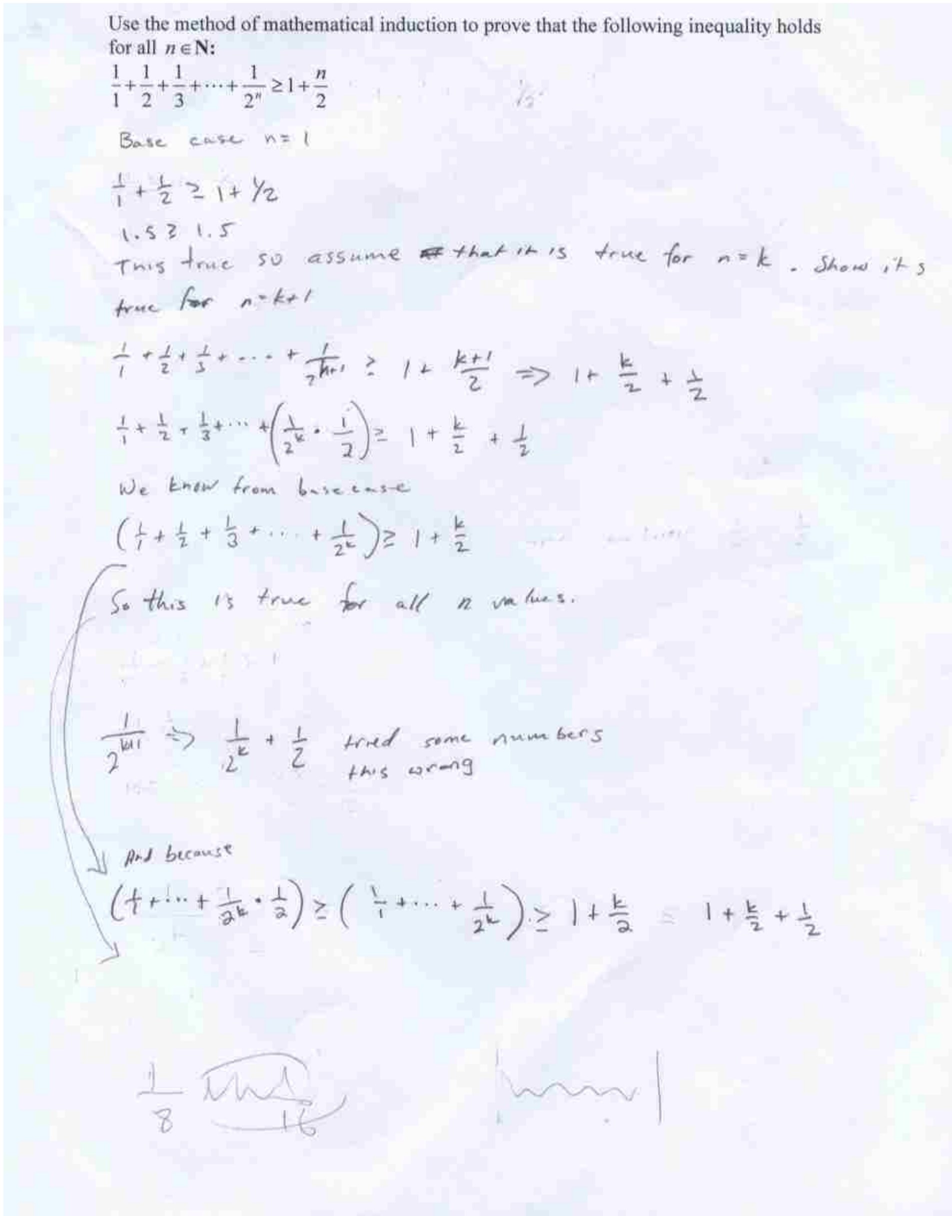


Figure 99: James' work on Question 3 (1 of 4)

James then moved to the work in Figure 100 in order to try to justify the first step in the string of inequalities at the bottom of Figure 99. He tried to do it by integrating a couple different functions: $1/x$ and $1/2^x$. He settled on working with $1/x$ when the other function got to unwieldy. His plan was to integrate it from 1 to 2^k and from 1 to 2^{k+1}. The idea was that if he got a greater number out of the second integral that would show that the sum was greater when the final term

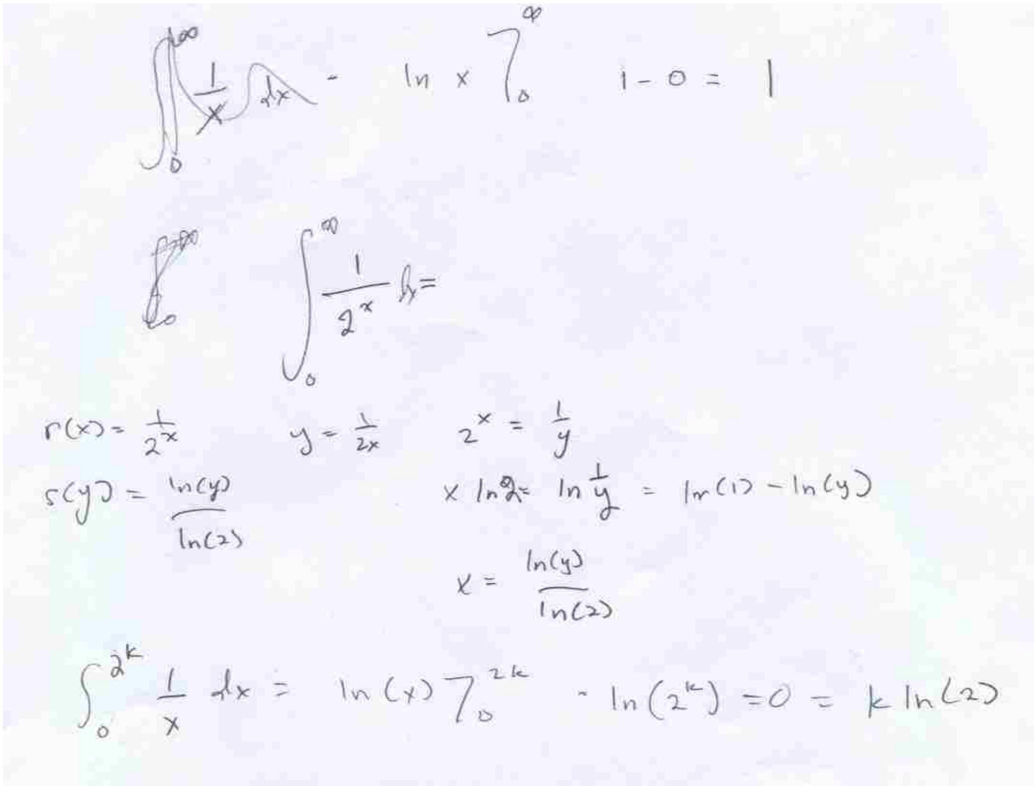


Figure 100: James' work on Question 3 (2 of 4)

was $1/2^{k+1}$. When he was explaining his idea, he realized a couple things. First, he realized that using integrals was unnecessary.

> N: But you're still kind of going at it with this method, with the stringing inequalities together method and you're just trying to work on justifying this first step.
>
> J: Yeah
>
> N: Ok, and that's what this stuff was about, this integral stuff.
>
> J: Yeah, and I can definitely, I'm actually positive now that that's true, it's just…when you have this, $1/2^k$ would have, you know, the sum of all up to that, and $1/2^{k+1}$ is going to be the sum of that, the same plus whatever else is in between, so if it's up to 1/8, then,

that's 2, then the other one's going to be 1/16, all of this added is going to 2^{k+1}. So, it's going to be a larger sum.

James was also able to correct himself regarding his misconception:

N: Yeah, so I think that here, until you made that realization, it seemed to me like you were treating this left side as having the same number of terms in both the k case and the $k + 1$ case.

J: Yeah, because I was looking at it as timesing it by that 1/2.

N: And having that be the only difference between the two?

J: Yeah.

With a fresh perspective on the problem, James left the interview planning to work on the problem before coming back for the next one. The work he brought back can be seen in Figure 101.

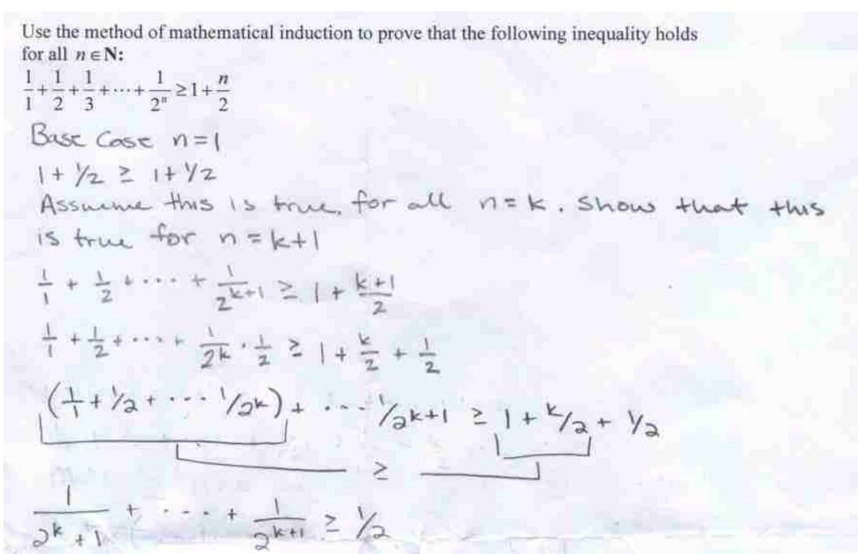


Figure 101: James' work on Question 3 (3 of 4)

James was able to successfully restart the problem and apply the induction hypothesis. At that point, he said he tried a couple different algebraic manipulations that did not lead anywhere and he ended up erasing.

Despite hitting this road block, James did make a breakthrough of sorts while waiting for the interview to start. This is why this new work did not show up on the page he brought in. He did, however, continue working during the interview and this new work can be seen in Figure 102.

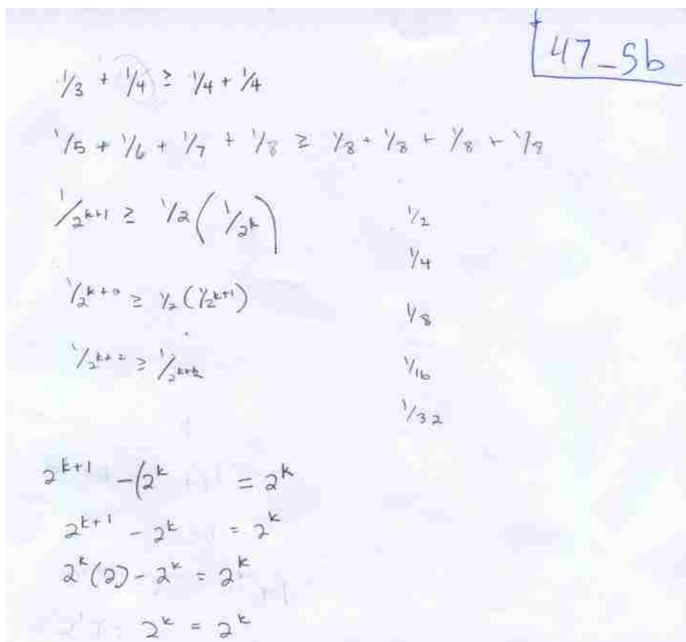


Figure 102: James' work on Question 3 (4 of 4)

J: So, then when I was kind of wandering around outside this morning because I showed up early, I tried looking at, I forget what it was…So, I was just trying to think of one pattern…just kind of dealing with this and separating that 1/2 into the 1/4 + 1/4. And so then, the 1/3, yeah, is greater.

James had considered the last thing that needed to be done with the inductive step for a particular *k* value. He also felt that his method could be extended to prove that it works for all *k* values.

J: And I think that it's going to correspond out that you're going to have the same amount of terms on each side.

N: Ok

J: And…if that pattern would continue, if the last term up here in this string would always equal what this repeating term is on this side, then everything before this…

N: ...would definitely make it bigger, right, because those denominators are smaller.

J: And that the, just looking at every time *k* goes up, is there a, I don't know what the terms is, so that both sides have the same amount of fractions.

James knew that for any *k* value, he could always break 1/2 up into a sum of identical powers of 2 that matched the last term on the left. The trouble was making sure the number of terms on each side of the inequality would match up if he did.

To this end, he did some work, even starting beginning another induction argument. Eventually, though, he realized this was unnecessary: "Well, I was trying to find how many terms there are going to be between $2^k + 1$ and 2^{k+1}." It was left, then, to show that the number of terms between would be equal to 2^k (which is what you get when you divide 1/2 by $1/2^{k+1}$). It did not take James too long, then, to verify that this is case. I asked James if this meant that he had completed the problem, and he said:

J: But, yeah, I think that I've shown that the left side is going to have the same number of values as the right hand side, plus I've shown that the last value, the smallest value on the left hand side is going to equal all the repeated values on the right hand side. Everything before the last, the smallest value on the left hand side is going to be larger, so…

N: Yeah

J: Seeing how there's the same number of values on both sides, the smallest value on the left hand side equals the constant value, so the left hand side has to be larger than the right hand side so that statement's true, combined with the initial statement (the induction hypothesis, completes the proof).

The proof James provides here is a process procedural proof. He clearly understands the process of induction and sees it as a few global steps rather than a complete list of instructions to follow. The bulk of the work he does involves completing the inductive step, something for which he had little guidance.

His proof scheme is transformational here. Because this is an induction proof, the proof relies heavy on what happens as one moves from one case to another (from $n = k$ to $n = k + 1$) and what happens as a result of that transition. Also, James seems to thoroughly understand the process of completing a proof by induction, so this is not an internalized transformational proof scheme.

Question 4

Examine the following proof by contradiction:

Prove $\sqrt{-1} \le 0$

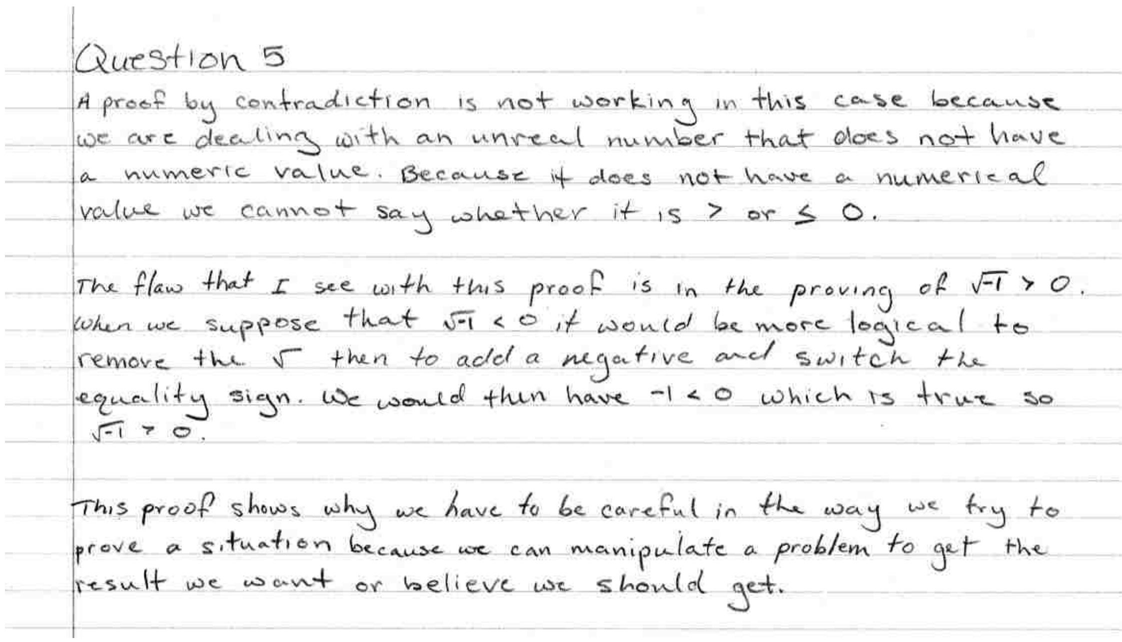
Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \le 0$.

<u>Prove</u> $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$, *then* $-\sqrt{-1} \geq 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$, *so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

This question was on the midterm exam James completed while taking MATH 305. The solution he turned in on the test can be seen in the figure below:


Question 5

A proof by contradiction is not working in this case because we are dealing with an unreal number that does not have a numeric value. Because it does not have a numerical value we cannot say whether it is > or ≤ 0.

The flaw that I see with this proof is in the proving of $\sqrt{-1} > 0$. When we suppose that $\sqrt{-1} < 0$ it would be more logical to remove the $\sqrt{\ }$ then to add a negative and switch the equality sign. We would then have $-1 < 0$ which is true so $\sqrt{-1} > 0$.

This proof shows why we have to be careful in the way we try to prove a situation because we can manipulate a problem to get the result we want or believe we should get.

Figure 103: James' previous work on Question 4

James points out two different aspects of the proof here. One is that because *i* is imaginary it has no “numerical value” and, presumably, no position on the real number line making a magnitude comparison impossible. He also mentions that a different step could be taken that would be “more logical.” He does not actually claim what it done is incorrect but he does saw that it is a flaw in the proof, as if the proof would have been improved had his step been taken.

During the interview, James initially thought he had found an algebraic mistake: "When you have 0 on the 1 side, it allows you to change the left hand side in any way you want it." I pointed it that this was not quite true; one also had to flip the inequality when multiplying by a negative. He quickly abandoned this and went back to this first response on the midterm:

J: It's not a real number…Yeah, that's what I'm getting at with the *i* part, that we don't have a numerical value. It's kind of hard to compare it to something that does.

N: Ok. So, what does that mean in terms of the rest of the proof, then? I mean, you're being asked to find the flaw in the proof.

J: The whole proof's flawed, right from the beginning.

James did not complete or attempt a proof during this interview, so there will be nothing to classify. However, this interview did yield some insight into his proof scheme. James realizes that the proof's flaw is that it starts with a faulty assumption: that *i* can be compared to 0 in the first place. Because this is not true, the rest of the proof is bogus. Thus, James is displaying an axiomatic proof scheme because he sees the dependence of mathematics on starting assumptions. In this limited setting, James realizes that building up from something false does not yield true results. In this way, this question can be thought of as a microcosm of the axiomatic nature of mathematics and James understands this structure at this smaller scale at least.

This is not to say, however, that James is completely satisfied with his answer. He stands by the flaw he finds but finds himself looking for something more:

J: Yeah it seems like there should be something that stands out as, you know, this is wrong, but there isn't…there's some iffy things, especially on that second one, but…

N: Yeah, but nothing overtly incorrect?

J: No.

(thinking)

J: *i* isn't a real number, that's the flaw.

James was hoping to find some algebraic mistake that would allow him to conclude the proof was incorrect. His uneasiness with the flaw he found in the proof tells me he was out of his comfort zone when it comes to proof. This implies that he thinks of proof mostly (but obviously not completely) in terms of logical deductions from one step to the next. So, while James does show that he has at least a weak axiomatic proof scheme, his desire to focus on individual steps demonstrates that he has a transformational scheme as well.

<u>Question 5</u>

The next interview was the last of the semester, and it James spent it discussing the first half of the study. There was no proof attempt, so there is nothing to classify but the interview reinforce some of the observations made earlier in the study.

First of all, it helped to highlight James' behavior in previous interviews. When I asked James what was necessary to complete a proof, he said: "Understanding what's actually going on." This matches what he did in many of the interviews. He often started with examples, charts or other forms of given expressions to get an understanding of the problem that could be turned into a proof. This attitude should not be surprising given the number of times James produced a semantic proof.

J: I've definitely learned that with proofs you have to sit down, read over it, you know, maybe even go back and look at a more simple version of what's going on.

N: Ok

J: Just to get an idea of where I really need to go and how to get there.

This focus of knowing where he needs to go and how to get there is also typical of a transformational proof scheme, something that showed up in all of James' interviews.

James also displays an axiomatic proof scheme, something that he gave hints of in the previous interview. James says that this view of mathematics is something he's noticed as he has moved into more advanced classes.

J: Now that I've noticed, especially in my other class, I've gotten better at recognizing…and using other proofs.

N: Besides in induction?

J: Well, and then stuff that's already been proved to build off of.

N: Ok, sure.

J: And that's been the biggest thing with my other class that I've had to do is that, you know, taking a basic proof for a problem and you have to go back and use 2 or 3 other proofs or theorems to manipulate because "We proved this in this theorem and we know that that's true, we can apply that to this so that we can…"

N: Right, so you're starting to see how, not just how an individual proof comes together but how sort of the structure of proofs and math…how it all works together a little bit.

J: Yeah, I've noticed that the more basic proofs, it's easy to just say "Oh, ok, here's induction" and boom, spit it out real quick.

N: Yeah

J: And now, going on, the higher you get up in math, the more proofs are relying on other proofs.

N: Yeah

J: That everything just kind of builds off each other.

N: Right

J: That's the biggest thing that I've been noticing this year.

So, while the evidence for an axiomatic proof scheme was weak during the last interview, it is strong here.

Question 6

Prove that $\forall n \in \mathbb{N},\ n \equiv n^3$ *(mod 6).*

James and I began this interview going over modular arithmetic because he had not seen it since he was in MATH 305. After that, James worked silently for a while, getting down to the rows of numbers and their equivalents mod 6 (see Figure 104). At that point, he felt stuck and we discussed what he had done. He tried the problem for an example (n = 2) and then looked a couple ways to represent division: both with numbers, 9 and 183, and variables, $n/6$ and $n \cdot n \cdot n/6$. James decided to use $n/6$ as shorthand for the operation of division by 6. When working with this notation, he made a mistake in simplifying n^3 over 6, something he repeated when looking at his n = 2 example.

At that point, he went back to trying some examples, n = 10 and n = 3. It was here that James noticed the fact that remainders cycle and so you get a repeated pattern when looking at numbers mod 6.

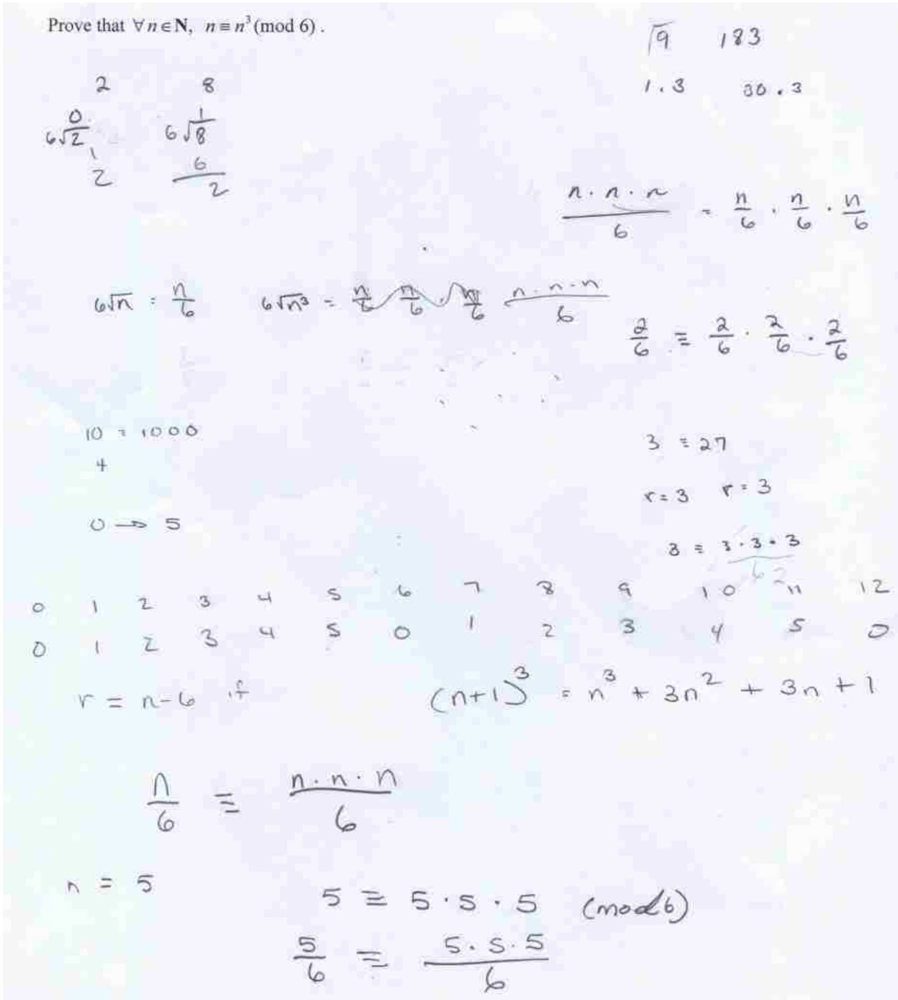


Figure 104: James' work on Question 6

N: And then…did some rehashing with that sort of stuff…looked at another example, so another number and its cube. So then you looked at a whole bunch of numbers and their cubes, well, a whole bunch of numbers mod 6, not necessarily their cubes.

J: Yeah, just looking at, yeah, it just runs 0 through 5.

N: Ok

Yeah, that’s what the remainder’s going to be and it’s just repeating…

I then asked James if he had given any thought to which proof technique he might be able to apply to this problem and he said he had not gotten that far yet:

J: No, I was more looking more just trying to understand it.

N: Ok and the proof would come out of the understanding?

J: Yeah hopefully. Yeah, I don't even know where I would end up going with it to start proving it…Yeah, that's kind of what I always hope for.

N: Ok

J: And then, I guess, with the understanding that it would lead to one of the proof methods.

This discussion of proof methods led him to expand out $(n + 1)^3$ to see what an induction proof might look like, but he did not see it going anywhere. Seeing the expansion seemed to complicate matters, but he was still having trouble with how to deal with the remainder when dividing a variable by 6.

N: Right, so you're sort of worried about having that whole expansion…

J: Well, and trying to find what that remainder is going to be of just the *n*. Just because you not, that's kind of the part that I keep getting stuck at is that you're not showing that *n* divided by 6 is going to equal…

N: Right, it's just the left over part.

J: Yeah, you can't show a left over part of something that's a variable…just *n*. *n* doesn't have anything left over when you divide by 6.

N: Well, you have no idea what it would look like, or what it would be, I guess.

J: Yeah and so trying to say that, through induction, you'd hit that spot and then say "Ok, I can't show what this remainder should look like."

N: Ok

J: As far as I know.

This difficulty of what to do with the remainder, and the notation he developed earlier led to confusion and abuse of notation when looking at the equation $\frac{n}{6} \equiv \frac{n \cdot n \cdot n}{6}$.

J: So…can I look at something where if I'm multiplying both sides by 6/*n*, that you know that on this side some remainder, whatever it is…And then you just end up with *n* times *n* on the other side, which is going to be a whole number as well.

In an attempt to represent the division, James got caught up in notation. We the talked how what he was proposing really had no meaning because *n*/6 was not representing what it normally does.

We had reached the end of the interview and James said that he would continue to work on the problem between interviews. He said that he did work on the problem but he did not bring anything in. He said that he had just kept running into something he had noticed during the interview: "Really just how the remainders are working out, how it's a repeating cycle of 0 – 5 but that makes sense because it's 6." He said he was trying to make use of that since modular arithmetic deals with remainders but he didn't know how or where to go next. I then talked through a solution with James, making use of the division algorithm. He had seen it a bit in number theory and took to the solution of the problem well.

Although James did not complete a proof, the work he does constitutes as a semantic proof attempt. Instead of going straight into a proof, he first explores the problem hoping to find some understanding that would lead to a proof. He mentions explicitly that is what he is doing, but his actions back it up. The fact that he gravitates to the fact that the remainders repeats also shows this is a semantic attempt. That is the main insight he finds and he spent his time between interviews trying to turn it into a proof.

His proof scheme here is mainly transformational. In various attempts to understand what is going on with the problem, he tries to get a handle on the remainder in such a way that will allow him to manipulate it. His attempted manipulation goes a bit awry at one point when he tries to multiply by 6/*n* when he gets to the bottom of Figure 104. The way he chose to represent division by 6 led him to resort back to the conventional meaning of *n*/6 and methods to manipulating that. This abuse of notation represents a symbolic external conviction proof scheme. It is not a strongly held scheme, however, and I feel he resorted to that sort of operation out of desperation only.

Question 7

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

James did not take long to get into the problem, recalling counting techniques he had seen before:

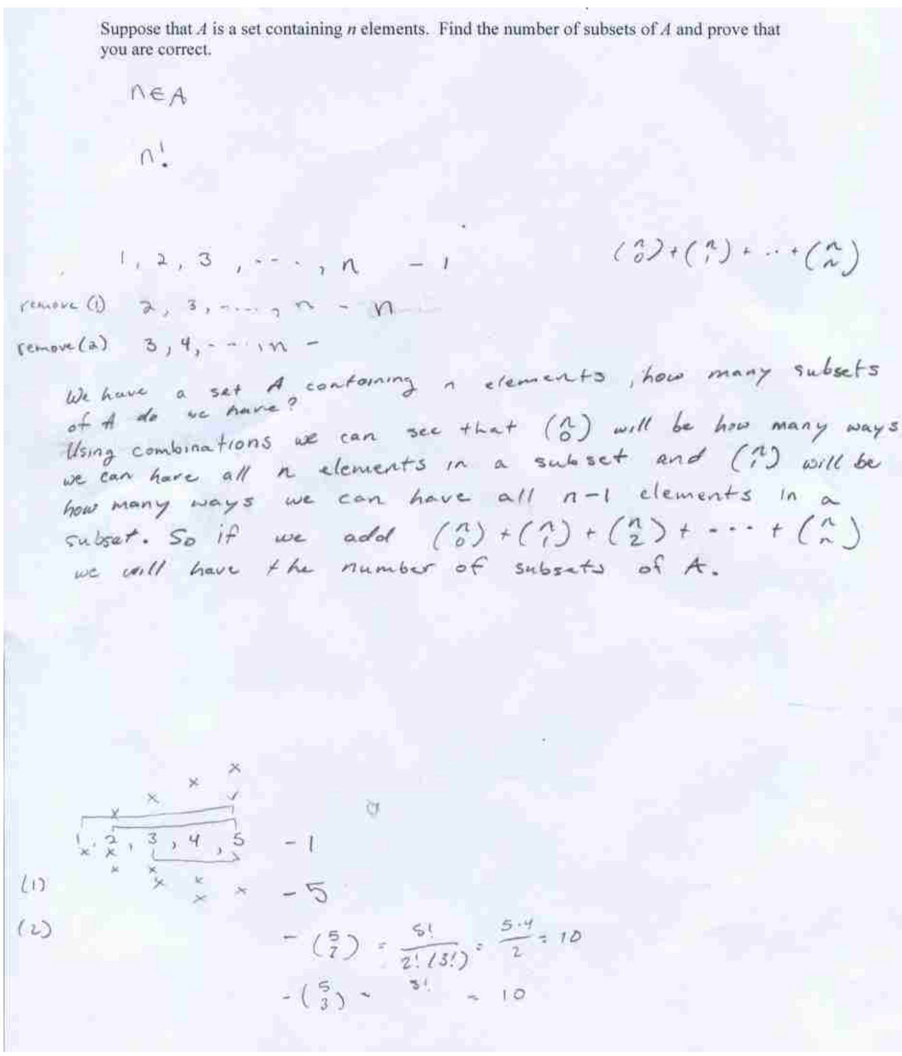
> J: So, I'm trying to figure out to, how many ways we can pretty much arrange 9, or *n* different things, right?
>
> N: Ok
>
> J: Would be kind of looking at it?
>
> N: Sure…yeah, that's one way of thinking of it.

J: Because I'm just thinking of it as *n*!...right, since if you had *n* things you can arrange *n* things *n*! different ways.

N: Right, one thing to think about is the fact that the set 1, 2, 3 is the same as the set 3, 2, 1.

J: True

James then went straight to the summation formula in the right middle of the figure below:



Suppose that *A* is a set containing *n* elements. Find the number of subsets of *A* and prove that you are correct.

$n \in A$

$n!$

1, 2, 3, ..., n − 1 $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}$

remove (1) 2, 3, ..., n − n

remove (2) 3, 4, ..., n −

We have a set A containing n elements, how many subsets of A do we have? Using combinations we can see that $\binom{n}{0}$ will be how many ways we can have all n elements in a subset and $\binom{n}{1}$ will be how many ways we can have all n−1 elements in a subset. So if we add $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$ we will have the number of subsets of A.

1, 2, 3, 4, 5 − 1

(1) − 5

(2) $-\binom{5}{2} = \frac{5!}{2!(3!)} = \frac{5\cdot 4}{2} = 10$

$-\binom{5}{3} \sim 3! = 10$

Figure 105: James' work on Question 7

J: Prove that you're right…It has to be *n*C0 + *n*C1 + all the way up to *n*C*n*. Now I'm just trying to think of a way that I can prove that I'm correct.

N: Ok…so just saying that that's what it is…

J: Doesn't really work, just saying that…saying that's the way combinations work.

At this point, James does the work that can be seen on the left side above the paragraph.

J: So I started looking at if you have 1, 2, 3 to *n* different elements in there…

N: Ok

J: …there's 1 way you can have those.

N: What do you mean have those?

J: Well, just have, one way that those can be in a subset all together…

N: *Right. So there's 1 subset that has all the elements in the original set.*

J: Yeah, and then if you want to remove 1 of those elements and find out how many combinations are from there, then you do *n*C1, for how many different ways you can remove 1 element from your *n*.

N: Ok

J: And then you can just continue down, so then you want to remove 2 of the elements.

N: Ok

J: And I looked at more of a simple, just 1 through 5.

N: Ok

Just to see, and started out by picking, you know going through and matching 'Ok, so I have 1 through 5, well, there's one way. So now let's exclude 1.' [a single element, not the specific element 1]

N: Yeah

J: Ok, so there's 5 ways you can do that and then I went through and started counting how many different ways you can remove 2 and saw that was just going to be the 5 C 2. So I just generalized that for *n*.

I asked James if what he had done was a proof and he said:

J: I think it could be refined into one, and molded.

N: Ok

You know, but…and maybe it would depend on who was going to be looking at the proof, you know, how in depth you'd go in refining it. But I think if I would stick the definition of the combinations…

N: Sure, and you wouldn't really have, like, pretty much anybody reading would know what that means, so that would be ok I think.

James then wrote out the paragraph in the figure, ending by saying: "Yeah, sounds pretty good." I asked him about what he thought the proof would have been like if he was working on his own (and I did not tell him he did not need to justify the choose formula):

J: Yeah, I think I would have ended up putting a lot more into it. I felt like I needed…yeah, I would have felt I needed to go through and state, you know, what combinations…

N: Ok

J: …how they work, and not really thinking about that I'd be writing, you know, the proof for people who would understand…

N: Ok

J: …all of that.

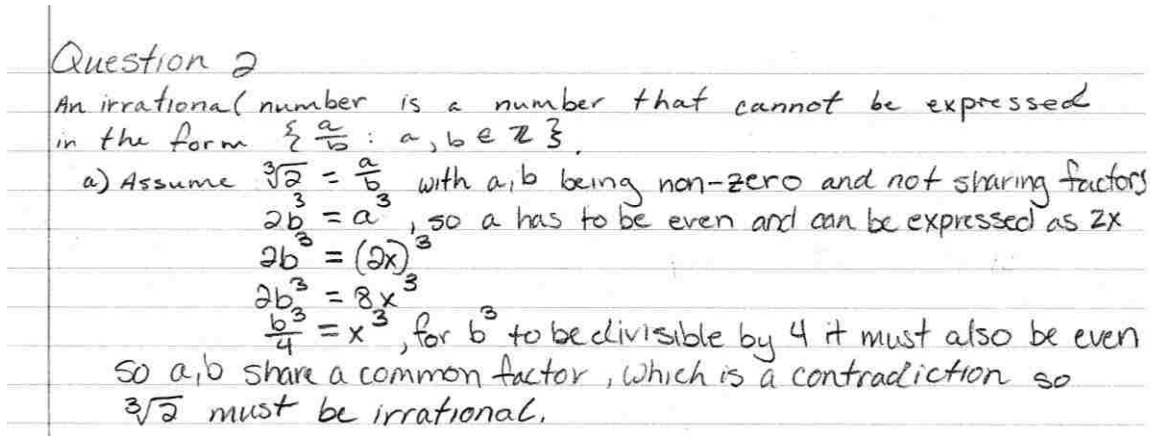
James's proof is a semantic one. Even though he goes right into the general case without much exploration, he arrives at his proof by examining how to calculate the number of subsets when no elements are removed, 1 is removed and so forth. The result of this examination led him to an understanding of the problem that was eventually turned into a proof.

His proof scheme is almost as straightforward. James arrives at his proof via operating on the set by seeing what happens when elements are removed. This is a transformational proof scheme. However, there is also a possibility of an external conviction proof scheme here. James seems unsatisfied with leaving his proof as an explanation. He talks about the fact that he might be inclined to include more details into his proof. It is unclear whether this inclination is result of not knowing if the reader would understand the choose function or if it is because he feels if more mathematics needs to be included for his proof to count. If the later is the case, it would be evidence for a ritualistic external proof scheme. While I am not certain, I believe the former is true. I base this on the fact that James seemed more comfortable with his proof once I told him that he could assume that the reader was familiar with choose. For this reason, I conclude that James is demonstrating a transformational scheme only.

<u>Question 8</u>

Prove that the cube root of 2 is irrational using a proof by contradiction.

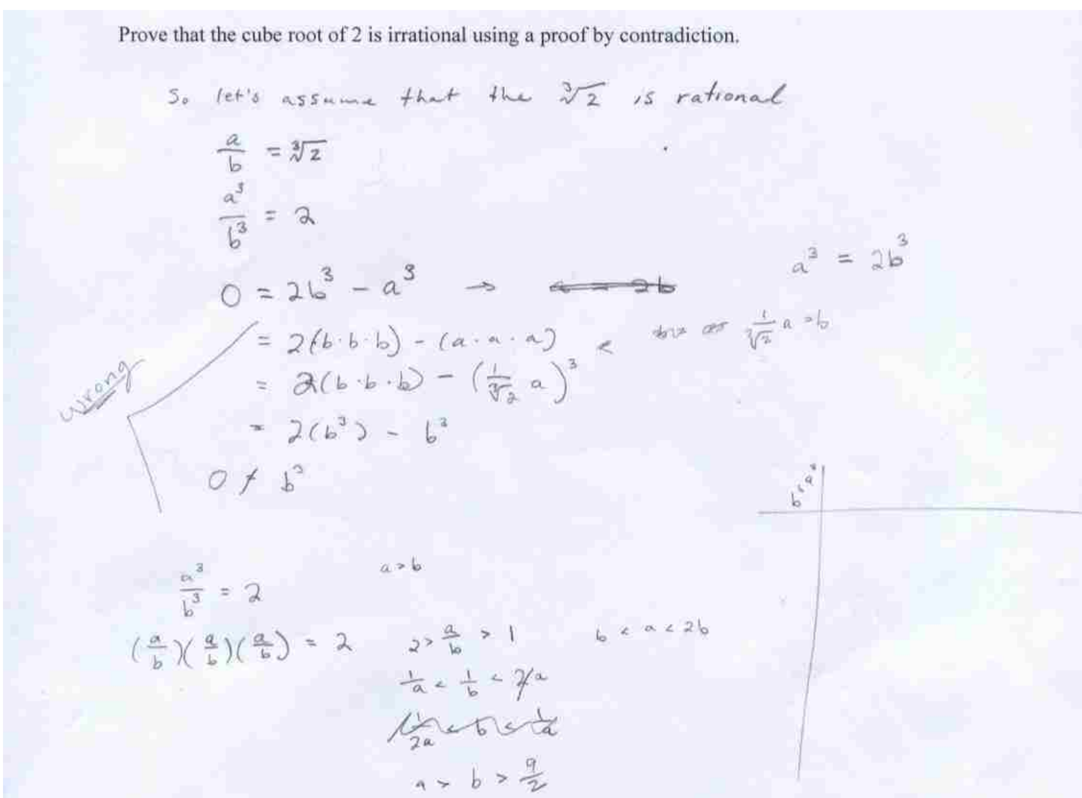
This question appeared on the midterm example James took while in MATH 305. He had no issues with the problem on the midterm, as can be seen in the following figure.



Question 2
An irrational number is a number that cannot be expressed in the form $\{\frac{a}{b} : a, b \in \mathbb{Z}\}$.
a) Assume $\sqrt[3]{2} = \frac{a}{b}$ with a, b being non-zero and not sharing factors
$2b^3 = a^3$, so a has to be even and can be expressed as 2x
$2b^3 = (2x)^3$
$2b^3 = 8x^3$
$\frac{b^3}{4} = x^3$, for b^3 to be divisible by 4 it must also be even
so a, b share a common factor, which is a contradiction so $\sqrt[3]{2}$ must be irrational.

Figure 106: James' previous work on Question 8

James started his work on the problem as he should have, by assuming $\sqrt[3]{2}$ is rational. He then set it equal to *a*/*b* and began manipulating the equation. After a few minutes, James thought that he had arrived at a contradiction. In Figure 107, one can see that James took two different versions of the equation $a^3 = 2b^3$ and substituted on into the other.



Prove that the cube root of 2 is irrational using a proof by contradiction.

So let's assume that the $\sqrt[3]{2}$ is rational
$\frac{a}{b} = \sqrt[3]{2}$
$\frac{a^3}{b^3} = 2$
$0 = 2b^3 - a^3$
$= 2(b \cdot b \cdot b) - (a \cdot a \cdot a)$
$= 2(b \cdot b \cdot b) - (\frac{1}{\sqrt[3]{2}} a)^3$
$= 2(b^3) - b^3$
$0 \neq b^3$
wrong
$a^3 = 2b^3$
$\frac{a^3}{b^3} = 2$
$(\frac{a}{b})(\frac{a}{b})(\frac{a}{b}) = 2$
$a > b$
$2 > \frac{a}{b} > 1$
$b < a < 2b$
$a > b > \frac{a}{2}$

Figure 107: James' work on Question 8 (1 of 2)

Unfortunately, James made a mistake in his substitution (he inadvertently substituted $\frac{1}{\sqrt[3]{2}}a$ in for *a* instead of *b*. When I pointed this out, James labeled his attempt "wrong" and when back to work.

After that James worked silently, trying the rest of the work in Figure 107. First, he tried to come up with an inequality that he hoped would lead to a contradiction:

J: Yeah, and then tried just…bounding it…

N: So you were, what were you hoping to do with that, then? Anything, or just looking at it?

J: Kind of just looking at it, hoping that something would make sense.

When James saw nothing that he could use from that, he looked at the chart on the right side of the figure.

J: Yeah, I was going to look at, because *a* and *b* would have to be integers…And so, you're just looking at, if you looked at…like 1, 2, 3 , 1, 2, 3 just like for the a^3 and the b^3…And then just looking the, divide them out and you're going to be, they're not going to be at 2…Even though they would have to be.

N: So, were you hoping sort of like their ratios would go off and go in the same direction and never work again?

J: Yeah, that would show that they're exponentially getting…

N: further away?

J: Yeah, which would happen really fast with cubes.

I then pointed out to James that no matter how big *a* became, there was a *b* whose cube would be roughly the half of a^3.

By the time we were done talking about the things he had tried, it was time to end the interview. I asked James to look at the problem some more between interviews and he said he would. During that time, James recalled that he had seen a proof that $\sqrt{2}$ is irrational in his History of Mathematics class: "Yeah, I remembered that, the square root of 2, and so I flipped back through my notes and…Yeah I saw the 'You can't have common, or they have to be in the lowest terms'." Seeing this jogged him memory regarding the current question. Although he had lost the paper that had the work he had done between interviews, he was able to quickly provide the work in the figure below.

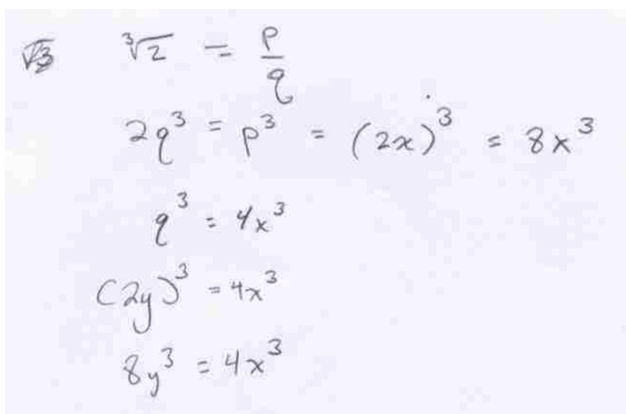


Figure 108: James' work on Question 8 (2 of 2)

As he went, James explained his steps and completed the proof, although I did have to help him articulate the conclusion. Earlier in the interview, James mentioned that he had not finished the problem but that he was close. I think this was because he was having trouble explaining the conclusion.

The proof James provides is a procedural proof. Because he used the proof that $\sqrt{2}$ is irrational as a guide, James had been given steps he used to complete this proof. However, in the interview James mentioned that he merely used the other proof to get started; he did not use it for step-by-step instructions. Because of this, James' proof for this question is a process and not algorithm proof.

Although James needed help at the beginning and end of his proof, he still exhibits a transformational proof scheme. The reason for this is because he understands the steps he is completing as he goes. If he had completed his steps because he had been told to or because he was simply copying a different proof, that would have been evidence for an external conviction proof scheme. As it was, James explained what he was doing and that understanding is what convinced him of the proof's validity. Thus, James is not relying on an external source to tell him the proof is complete and so he displays a transformational scheme.

Question 9

For the next question, I had James evaluate a version of Cantor's Diagonalization argument. Because James did not attempt a proof there will be nothing to classify. I was, however, able to use the interview to find some evidence of James' proof scheme.

Unlike most participants, James remembered what it meant for a set to be countable from MATH 305. Because he had a good understanding of the idea of countability, he went right into reading the proof. He found the proof confusing the first time he read it. "Yeah, I got a little bit confused, you know, just with the, why they were doing the β_j equaling 1 or 2 if α_{jj} equals 2 or α_{jj} doesn't equal 2."

After reading through the proof a second time, though, he found it to be much clearer.

J: I understand what they did.

N: Alright, so what do you think of it?

J: After I read through it and as long as I think I understand…it's a neat way of doing it. I thought it was going to be a, when I first I read through it the first time I thought it was a really complicated…

N: Sure

J: …way of doing it, but…after I, you know, read through the first part, made sure I definitely understood what they were doing there, and then just read through this last part a couple times…

N: Right

J: Yeah, it's a kind of neat way of doing it.

James was even able to point out that the choice of comparing each α_{jj} to 2 was arbitrary:

J: Because I was just kind of confused at why they picked…

N: The jj?

J: Well, and also then why they said if it equals 2 or doesn't equal 2.

N: Oh, ok

J: And then, I mean, as long as I'm right, it wouldn't really…it wouldn't matter if they chose something else…

N: Exactly

J: They could let it be 2 or 3 or 4.

This was conformation for me that James really understood the proof as he said he did.

I then asked James what he thought was the biggest hang up in reading through the proof. He said:

J: Just going through it and going over it quickly, it was just, yeah, a lot of different notation in there and…reading it just all the way through that first time and not really stopping to analyze what they're actually doing with this first part.

N: Ok

J: And then getting to the second part and, you know, having to know what was going on in this…

N: …the first part?

J: Yeah, to even make sense of how they were being able to say that …

N: Ok

J: …that relationship.

James is referring to the fact that he did not know why the proof writer did what he did and so he did not take the time to focus on what was happening earlier on in the proof. He made a similar point later in the interview, saying that not knowing why things were done added to the confusion the first time through.

J: And the first time I read through that, it seems kind of like a, you're just throwing that out there.

N: Oh, sure

J: But then when I see what they're doing with it I understand that that's perfectly reasonable…

James recognized the importance of performing steps within a proof that will pay off later, even if they seem unmotivated at the time. This is a key aspect of a transformational proof scheme. One of the main characteristics of the transformational proof scheme Harel and Sowder mentions is the “application of mental operations that are goal oriented and anticipatory” (p. 261). James sees that this is an important aspect of proof and reinforces that he has this proof scheme.

Question 10

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

James started this problem by going straight to a general inequality. He began the problem by separating the 2 parts of the inequality and working with them individually. It did not take him long to realize that he had too many variables floating around, and so he took a cue from the provided example, letting $x = a - 2$ and seeing the implications of that (see Figure 109).

J: Yeah, I was starting to looking adding my own little restrictions to it to see if I could get there.

N: Ok, yeah, I noticed one thing you did, you said x had to be…

J: $a - 2$

N: Where did that come from?

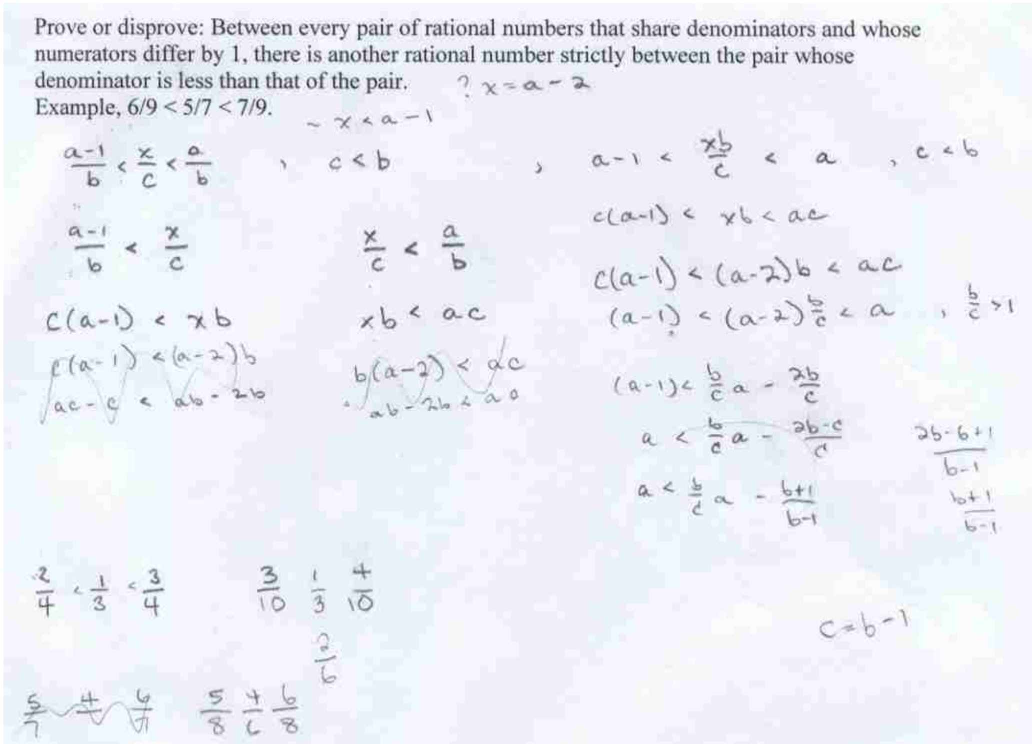


Figure 109: James' work on Question 10

J: That came from looking at, just the…the little example that you had put up there, and then stuff that I had jotted down, I was just trying to figure out if I could…

N: …generalize a pattern?

J: Yeah, if I could get a pattern that, you know, if I tried like 3 or 4 of these, could I find an x that was $a - 2$ over something.

N: Right

J: That would work for it. You know; if I could get a pattern going that way, then maybe I could…yeah, get it to work.

After looking at the algebra on the left side of the figure, James then tried a few examples to make sure that the restriction he had applied made sense. It did in two of the four cases he looked at, made a couple arithmetic mistakes in another and did not attempt the last, apparently because it would was not easy to do mentally.

After that, he went back to working algebraically, eventually adding a new restriction: $c = b - 1$.

J: So I was looking at, with $c < b$…

N: Yeah

J: …that, if that's true, then for c to actually be a whole number, then that, the largest that it would be able to be, then, is $b - 1$.

James followed through with that for a bit but that did not seem to be leading anywhere either.

This was at the end of the interview and I again asked James to work on the problem for the next interview. He did not have time, however, so he did not work on it. Because the next interview was the last of the semester with a different agenda, and because he did not want to work on it alone any more, we agreed to talk through the solution quickly.

Because James did not complete this proof, I will be classifying his attempt only. This is a syntactic attempt. James concentrated on algebraic manipulations, making few efforts to do much else besides imposing restrictions. While I have no doubt that the operations he was performing were meaningful to him, he did not engage in much beyond "symbol pushing." Proofs (and attempts) with that as a main characteristic are syntactic.

James' proof scheme here is transformational. Like with syntactic proofs, transformational proof schemes are often characterized by the manipulation of mathematical objects. Although the particular operations James performed did not lead to a proof in this case, they were carried out in hopes that they would.

Question 11

The rest of the last interview was spent discussing the study with James and the progress he felt he made over the course of the year. As such, there is no proof to classify. However, the discussion was used to find insight into his proof scheme and reinforce some of the observations made earlier in the study.

The biggest thing that came out of the interview was James' focus on understanding problems before trying to prove it. This matches the proportion of semantic proofs that James provided. When I asked James if he felt like he had gotten better at proofs over the course of the semester, he said:

> J: Yeah instead of having to spend time making up big long lists of stuff, I still have to make small lists, but not the big long lists. You know doing little basic examples, being able to sit down more and actually just work through it.
>
> N: Ok, so would you say you sort of relied a little more on…like understanding the problem before you sort of just start going into it?
>
> J: Yeah
>
> N: Ok
>
> J: Yeah, you know, reading it over a bit more carefully like with the…oh, I forget which one it was, the neat one.
>
> N: The proof (Cantor's Diagonalization argument)?
>
> J: Yeah
>
> N: Ok

J: Yeah, something like that where, just importance of reading over it and fully understanding, you know, the whole thing instead of jumping in and just trying to make lists or…

N: Right

J: Because yeah, it can save a lot of time because once you understand that one, it's "Oh, perfect sense."

This focus on understanding a proof also showed up when I asked James what he thought was necessary to complete a proof:

J: Having the, at least the basic knowledge of what the proof is trying to get to.

N: Ok

J: And then, yeah, just so you can understand it.

N: So the understanding is necessary also you think?

J: Yeah, I think, if you don't understand what the proof is trying to do, then there's no way you're going to have a successful attempt at…

N: Sure, right

J: …trying to prove it or understanding what has been proved.

This also highlights James' transformational proof scheme in that he recognizes the importance of knowing where a proof is going if one is to either complete it themselves or understand one completed by someone else.

James' progression

Below is a chart of the proof classifications and proof schemes observed in James' work during the study.

Question	Type of proof	Proof scheme
1	Semantic	Transformational
2a	Semantic	Transformational
2b	Semantic (Attempt)	Transformational
3	Process	Transformational
4	N\A	Axiomatic, Transformational
5	N\A	Axiomatic, Transformational
6	Semantic	Transformational, Symbolic
7	Semantic	Transformational
8	Process	Transformational
9	N\A	Transformational
10	Syntactic (Attempt)	Transformational
11	N\A	Transformational

Table 6: Summary of James' work

As can be seen in the table, both the types of proofs James provides and the proof schemes he displays stay fairly constant over the course of the study. That is to say, he does not make much progress. One could easily argue, however, that he did not have much progress to make. Of course, there is always more content that can be learned.

At the same time, I believe that James has a fairly mature approach to and view of proof. Except for one incident, which I believe occurred out of desperation, James always displays an analytic proof scheme. Also, he generally works purposefully on his proofs. Again, this has one exception, Question 10, where he seems to be performing the algebraic manipulations without knowing where it will lead.

So, with few exceptions, I would argue that James' approach to proof is exactly the one we would like all undergraduate mathematics majors to have. He is thoughtful in his approach and knowledgeable about the axiomatic and logical nature of mathematics. As James proves, lack of progress is not always a bad thing.

4.7 Robert

This section describes the work that Robert did over the course of the study. Robert was a mathematics major. During the first semester, he took Introduction to Real Analysis and Ordinary Differential Equations and during the second semester he took Introduction to Complex Analysis and Deterministic Models.

<u>Robert's proof attempts</u>

<u>Question 1</u>

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Robert started this problem by drawing a picture to "interpret the problem." After he had a picture, he decided to look at squares first:

> Robert: (W)ell, we could try that first just to see if a square works…So, we'll have $4x$, so $4x$ is going to be the perimeter equal to the area, so it'll be x^2… That's x. It's equal to 4. I solved for x to kind of get an idea of what it was…So, I mean, proof wise that doesn't really prove anything, but it does help me understand that squares are going to work… Cause 4 is the only…the left hand side is $4x$ and the right hand side is x^2, and the only time it's going to be 4 times 4 equal to 4 times 4 is when x is 4.

From there, Robert moved his attention to the more general case of rectangles, writing out the equation $2x + 2y = xy$.

> R: Right.. I suppose I can…the only thing now is I'm going to have to solve for y because it's sort of …

N: Sort of logical thing to do from there?

R: Well, I could …it just seems…I gotta try to do something, even if that's the wrong way of going, at least I'll know that…

Robert then solved his equation for *y*, as he said he would (see figure below).

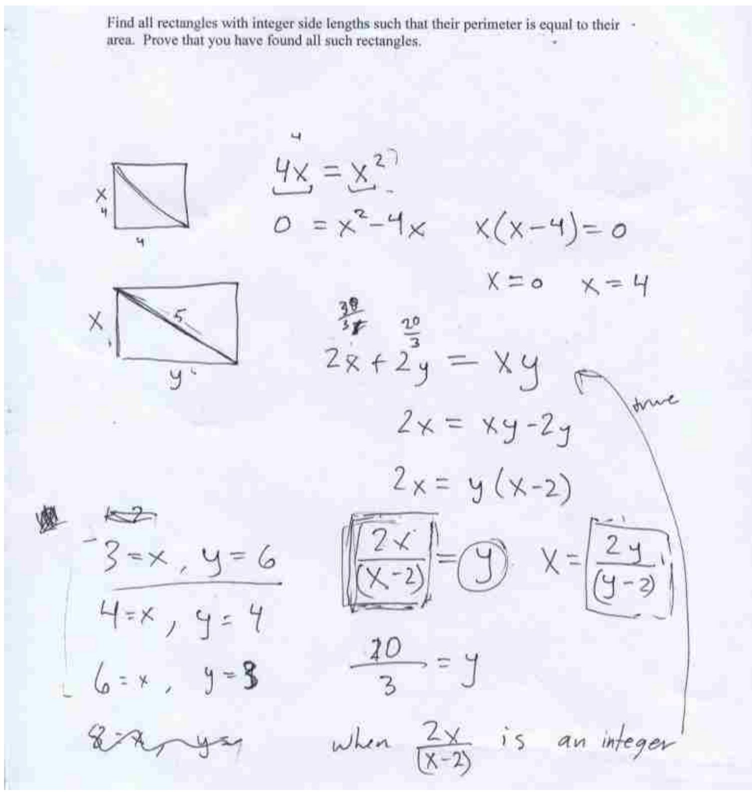


Figure 110: Robert's work on Question 1 (1 of 3)

After giving his new formula some consideration, I asked Robert what would happen if $x = 5$. He solved to see that when $x = 5$, $y = 10/3$ and then checked that the pair solved the original equation as well. It took some questioning on my part, but eventually he realized that even though the pair made the equation true, it did not satisfy the criteria given in the problem:

R: Oh, it's not an integer, though…10 thirds is not an integer.

N: Right, so there is not rectangle with one side length 5 that has, that you know, that fits the criteria.

R: Right. Ok…So, *y* would have to be…so *y* has to be an integer…so now I'll kind of work my way…so this part here I mean isn't this…this is sort of a clue it shouldn't…that's it's not going to be an integer…

N: Well, I mean

R: I guess there's going to be some … so when $2x$ divided by $x – 2$ is an integer, then it will work.

At this point, Robert said: "Ok, so, I mean, maybe I could start actually just plugging in some numbers and see if I can get it to work" and he found the remaining solutions.

After trying some more values and not finding any more solutions, Robert said: "I guess the trouble I'm having is I know this is supposed to be a proof." With that, he moved to a new sheet of paper and rewrote his formula, this time setting it equal to *n* instead of *y*. See Figure 111:

N: Ok, so you've switched the *y* to an *n* instead, just because *n* is typically reserved for an integer?

R: Yeah, and I mean, I keep thinking that…everything is too much…So now that I have a new, I'm thinking…when I had the *y* there I was thinking too much that it depends on *x* and now that I have an *n* there, it's just an integer now, so …But it does depend on *x*, *y* does depend on *x*.

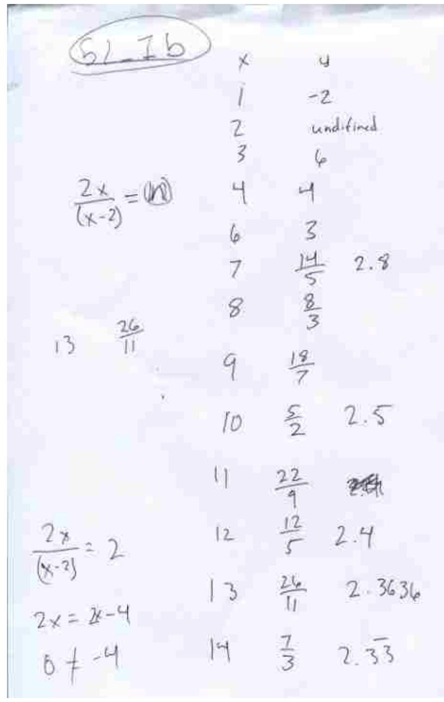


Figure 111: Robert's work on Question 1 (2 of 3)

This insight would eventually become important to his proof. Although he had realized that he could view *y* as a function of *x*, he did not put it to use right away. He still felt like he had only 2 options.

> R: Well, there are sort of the two tools that I have right now…I can rearrange the equations or plug in some values…And it seems like if I do any more manipulations of the equations, I'm just going to get back to the original.

Robert knew that his first option was not a very good one so he tried more examples. This time, though, he recorded the values he was finding. This still was not too helpful until I suggested he record the *y* values expressed as decimals as well as fractions.

> R: So it looks like they're going down, but I guess that doesn't really…So the next one is going to be 2…

N: Ok, right, so eventually… so the hope I guess, to find another one is that, since they're all 2 point something…eventually, if you go far enough it'll get down to 2, the *y* value will be 2.

R: Right, then eventually get down to 1…so you could just put that in for the *y*.

Robert then tried his idea, seeing that no *x* value would give $y = 2$. He also saw that *x* would need to be negative for $y = 1$.

R: Well, after 1, as the *x* values increase, the *y*'s going to go down to 0, which…

N: Which doesn't work

R: After that the *y* values are going to be negative…

Next, Robert turned his attention to making proving the function would in fact continue to decrease. He realized that he could use the derivative to do this and we talked through applying the quotient rule to the function at hand. After working through it, Robert seemed comfortable with his conclusion:

R: So, that's … so to prove… we have the 3 6 the 4 4 and the 6 3 … so there's really only two?

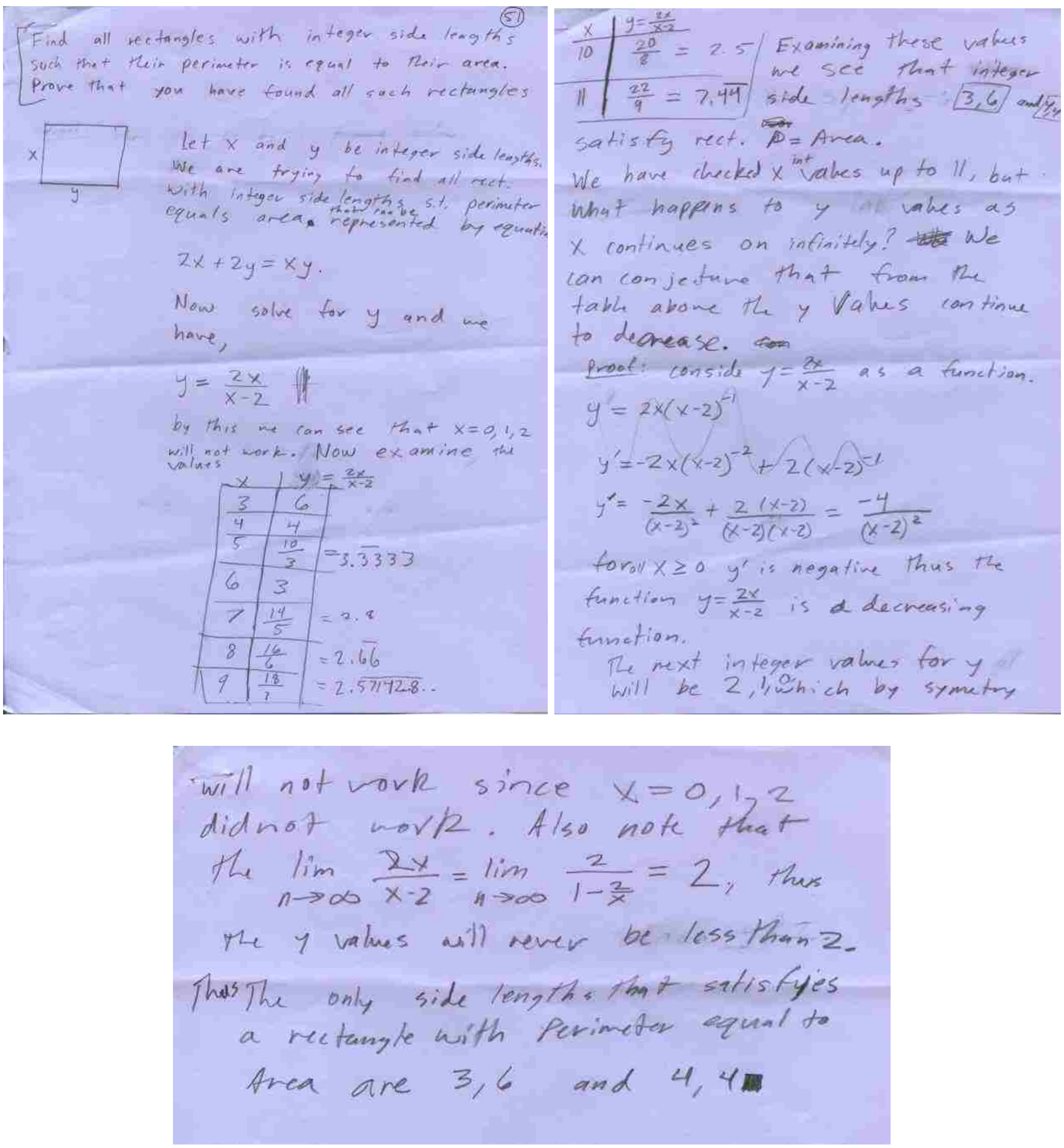
N: Right.

R: Because….2 wouldn't work and 1 wouldn't work…so then…ok…so we don't have to worry about formally proving it?

N: Right

R: Because it makes sense now…I understand that now…

When Robert asked about whether or not we had to "worry about formally proving it" he was referring to the fact that neither $y = 1$ or $y = 2$ work for this problem. I asked Robert to write up a solution to the problem and he brought it to the next interview. His write up can be seen in the following figure.



Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles

let x and y be integer side lengths. We are trying to find all rect. with integer side lengths s.t. perimeter equals area, that can be represented by equation

$2x + 2y = xy$.

Now solve for y and we have,

$y = \frac{2x}{x-2}$

by this we can see that $x = 0, 1, 2$ will not work. Now examine the values

x	$y = \frac{2x}{x-2}$	
3	6	
4	4	
5	$\frac{10}{3}$	$= 3.\overline{3333}$
6	3	
7	$\frac{14}{5}$	$= 2.8$
8	$\frac{16}{6}$	$= 2.\overline{6}6$
9	$\frac{18}{7}$	$= 2.\overline{571428}$..
10	$\frac{20}{8}$	$= 2.5$
11	$\frac{22}{9}$	$= 7.\overline{44}$

Examining these values we see that integer side lengths $\boxed{3,6}$ and 4,4 satisfy rect. P = Area.
We have checked x int values up to 11, but what happens to y values as x continues on infinitely? We can conjecture that from the table above the y values continue to decrease.

Proof: consider $y = \frac{2x}{x-2}$ as a function.

$y' = 2x(x-2)^{-1}$

$y' = -2x(x-2)^{-2} + 2(x-2)^{-1}$

$y' = \frac{-2x}{(x-2)^2} + \frac{2(x-2)}{(x-2)(x-2)} = \frac{-4}{(x-2)^2}$

for all $x \geq 0$ y' is negative thus the function $y = \frac{2x}{x-2}$ is a decreasing function.

The next integer values for y will be 2, 1, 0 which by symmetry will not work since $x = 0, 1, 2$ did not work. Also note that the $\lim_{n \to \infty} \frac{2x}{x-2} = \lim_{n \to \infty} \frac{2}{1-\frac{2}{x}} = 2$, thus the y values will never be less than 2.

Thus The only side lengths that satisfyes a rectangle with Perimeter equal to Area are 3, 6 and 4, 4 ■

Figure 112: Robert's work on Question 1 (3 of 3)

The proof Robert provides here is semantic. He uses the tools he has, examples and the equation $y = \frac{2x}{x-2}$, until he has an understanding of the problem that can be turned into a proof. The understanding he eventually found was that the *y* values depend on the *x* values and that this function was decreasing. He combined with that the fact that the side lengths in the rectangle were interchangeable to verify that he had found all the solutions.

Robert's proof is a formal one, so he is displaying an analytic proof scheme. However, his proof is based on manipulating the function and anticipating what will happen as the input increases and not any previous mathematical results. Thus, he is displaying a transformational analytic proof scheme rather than an axiomatic one.

Question 2a

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9.

As was typical with this problem, Robert and I began by talking about what "*ab – ba"* means in this particular problem. Once he was comfortable with the notation, Robert began trying some examples.

R: I'm getting kind of caught up, you said that it isn't multiplication, but it's…there's not really an operation going on there, it's just like …where you would put the non-negative integers.

N: Yeah, I mean that's just sort of atypical, so that's sort of tough.

R: Yeah, but it seems like it's going to work, though.

N: Yeah, because you've tried three or four cases and it's worked so far?

R: Yeah….So I don't think you'd be able to find a counter-example…

From there, Robert tried a few ways to re-characterize the problem (see the right side of Figure 113), but he kept coming back to the crux of the problem:

R: So I'm getting caught up on the digit places. I think that seems like it's the…

N: That's sort of the main sticking point, or…

R: I mean…I just know how to define that, you know? How I could use that to my advantage.

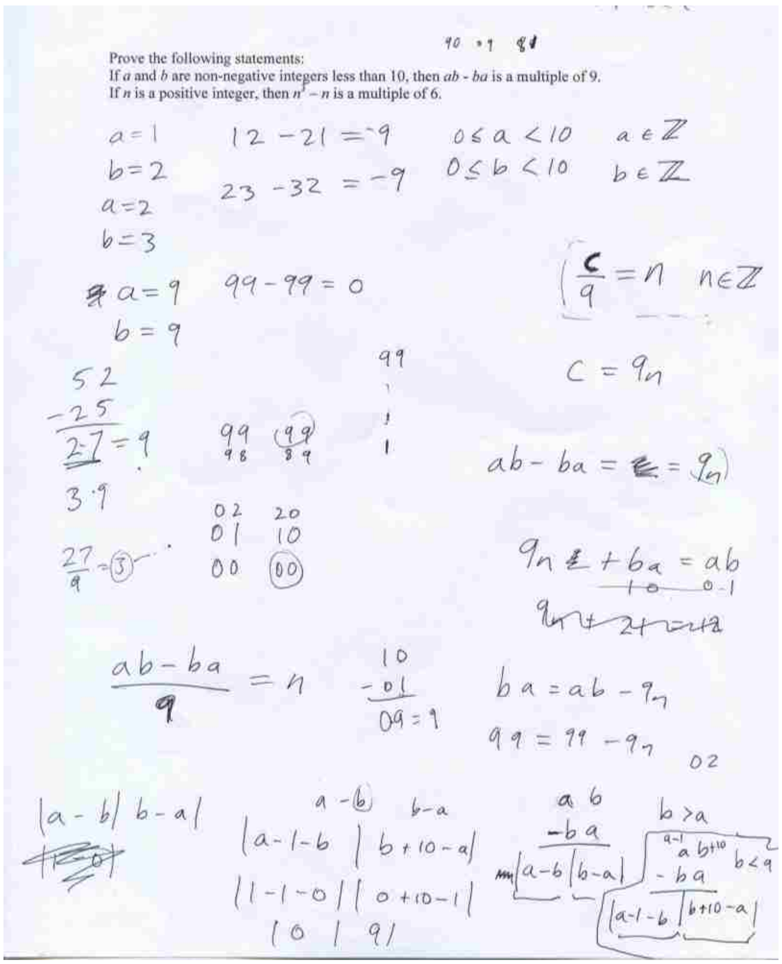


Figure 113: Robert's work on Question 2a (1 of 2)

At this point, Robert began using a generalized form of the subtraction algorithm he had learned as a child. Eventually, he would conclude that $ab - ba$ would result in $a - 1 - b$ in the 10s position and $b + 10 - a$ in the 1s place. This can be seen in the bottom right of the figure above. Later, while reflecting on the work he had done on the problem, Robert said the idea "just came from…if I wouldn't had done a large…I think if I wouldn't have done 52 – 25, I might not have written it like that." Initially, he had a hard time coming to grips with what that subtraction would look like in general, writing *a* – *b* and *b* – *a* in the 10s and 1s place respectively. "I don't know, I'll have to…like because if the *a* is bigger, that seems kind of tricky." He then thought about it and realized that when one "borrows" 1 from the 10s place 10 is added to the 1s position. He then checked his work with the example 10 – 01.

Next, he revisited his earlier guess, which had come to represent the case when *a* > *b*. In this case, he tried 02 – 20 and saw that it did not yield what he thought it would, -18.

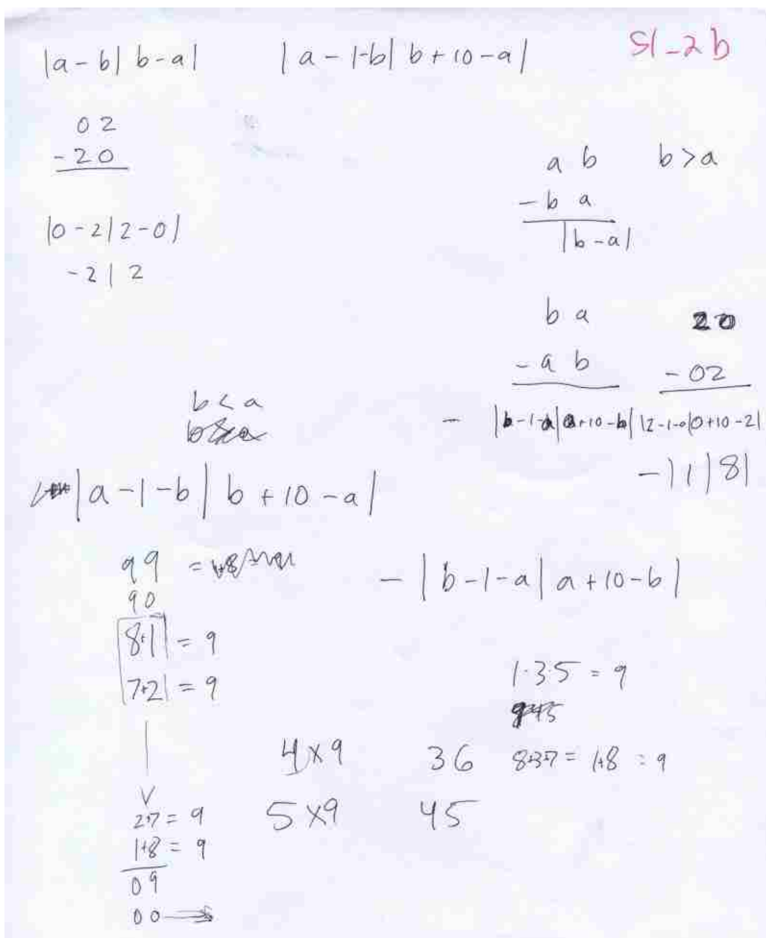


Figure 114: Robert's work on Question 2a (2 of 2)

Robert revised that guess to $-|b - 1 - a|a + 10 - b|$, which he found to work.

R: Ok, so that, I think that helps me understand that…at least I kind of broke down what that was.

N: Ok, what the subtraction looks like in general terms?

R: I've never actually done that, that's kind of weird. You know, you always just put a 1 there and move it over. It's kind of weird to …

N: Realize that taking 1 from the 10s spot is really adding 10 to the 1s spot?

R: Right, yeah. Ok, so I have 9, so…do multiples always add up to a 9? They have to add up, right?

Robert and I talked about the division rule for 9, which says that if the digits of a number add up to a multiple of 9, then the number itself is a multiple of 9. He mentioned that he thought all the 2 digit multiples of 9 had digits that added up to 9. This was based on a multiplication trick he learned growing up:

R: When I was in elementary school, when I learned how to multiply nines, it was always when you had, you subtract one from this [the multiplier] and then this would have to add up to nine.

N: Ok, so that was a multiplication rule.

R: Yeah, so 5 times 9, 5 – 1 is 4 and you have to add 5 to get nine.

Robert began to check the two digit multiples of 9, on the left side of Figure 114. He became confident that his idea was correct, with the exceptions of 0 and 99, which were easy to check.

Next, Robert turned back to his generalized subtraction expressions to see how everything fit together. “Well, it’s always…let’s see how that…When you add those guys you always get 9 (because the *a* and *b* cancel out in both cases), which is a multiple of 9, I mean I don’t know if I could use that.” I told Robert that he could take that rule for granted and he recapped what he had done:

R: So this statement, take two digit number, flip the digits, take the difference. The resulting two digits will add to nine.

N: Yeah. That’s proven.

R: And so by that rule, it will be a multiple of 9.

Here, Robert provided a semantic proof. He used examples to get a feel for the problem and then turned that insight in to a general proof. During the reflection, Robert mentioned this deliberate move from example to proof: “Well, I mean, I felt like you had to use this *a* and the *b*, you had to get it somehow involved, in general. I think that’s the tricky part, going from your examples to the general.”

Robert’s proof scheme here is axiomatic. He understands his proof depends on the divisibility rule that I mentioned. He even cites the rule when summing his proof up. Before I told him that he was free to assume the rule, he said: “But I don’t know if you can use that…it seems like you’d have to prove that too.” Here, Robert is displaying an acknowledgement of the fact that his proof is only as valid as its parts.

One could make the case that Robert is also displaying an authoritarian proof scheme because he was willing to take my word regarding the rule’s validity. I am not certain this is all there was to it. He checked a number of the 2 digit multiples on the page and more in his calculator. I am not sure whether or not he checked them all. Granted, even if he checked them all, he actually checked only the converse of the divisibility rule. It is also possible that he had some doubt regarding the rule and that is why he added “by that rule” when summarizing his proof. By adding that provision, he may have been trying to transfer responsibility to me for that portion of his argument. If this is the case, then he did not simply take my word for it and thus does not have the authoritarian proof scheme. In any event, I am not certain of such a scheme here and will label as axiomatic only.

Question 2b

If n is a positive integer, then $n^3 - n$ *is a multiple of 6.*

Because this question was on the sheet that Robert saw during the second interview, he had worked on it before the third interview but did not bring any of his previous work. When he began working on the problem in the interview, he started with examples. It did not take him long, though, to remember the line of reasoning he had been working on before. Between interviews, he noticed that the when *n*3 – *n* is factored, it is the product of three consecutive integers. It also reminded him of something he was going to ask when he came back:

> R: I was going to ask if there was some rule…that’s what I was going to ask about…because if *n* is 3, then you’re going to have 4 and 2. If it was a prime number like 5 it’s going to be 6 and 4. So it has a 6 right there…If you choose an even, an even number is a multiple of two and it’s, an even number is going to be either…
>
> (Robert writes 1 2 3 4 5 6 7 8 9, circles 3, 5, 7, see Figure 115)

R: …the number in front of it is either going to be prime, if the number in front of it is prime, then the number after it is going to be divisible by three. I think, that's what it was…if a number is odd and not prime, then it should be divisible by three, right?

N: No, well, 35, right? It's composite.

R: Ok, never mind. So that was the thing I was going to ask about. That makes sense; I should have gone out further.

(writes *n* = 8 case, 8(7)(9) and (2*4)(7)(3*3), circles the 2 and a 3)

R: At first I thought there would be different cases, because this is like you're looking at these three consecutive numbers being multiplied.

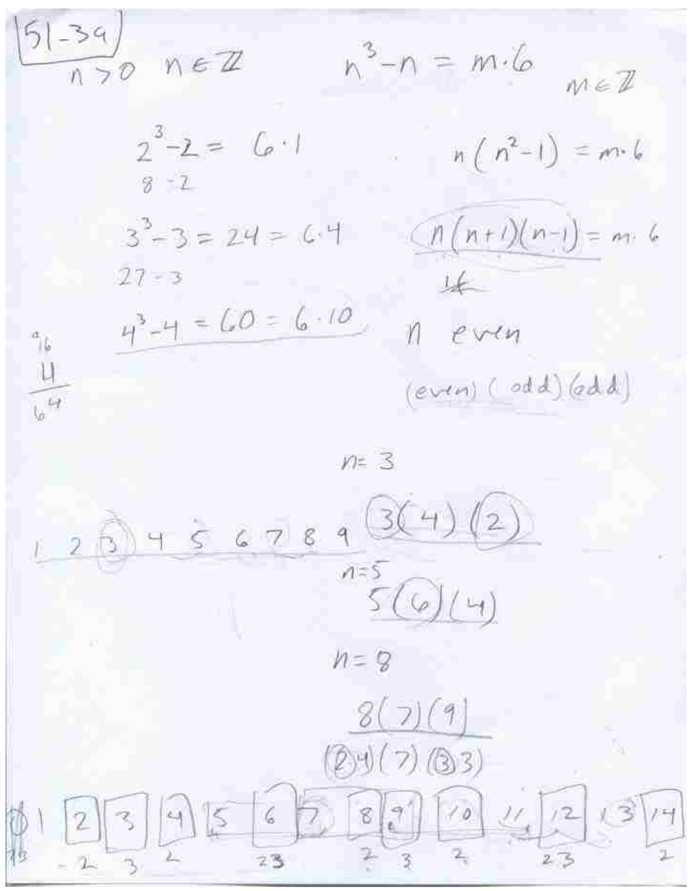


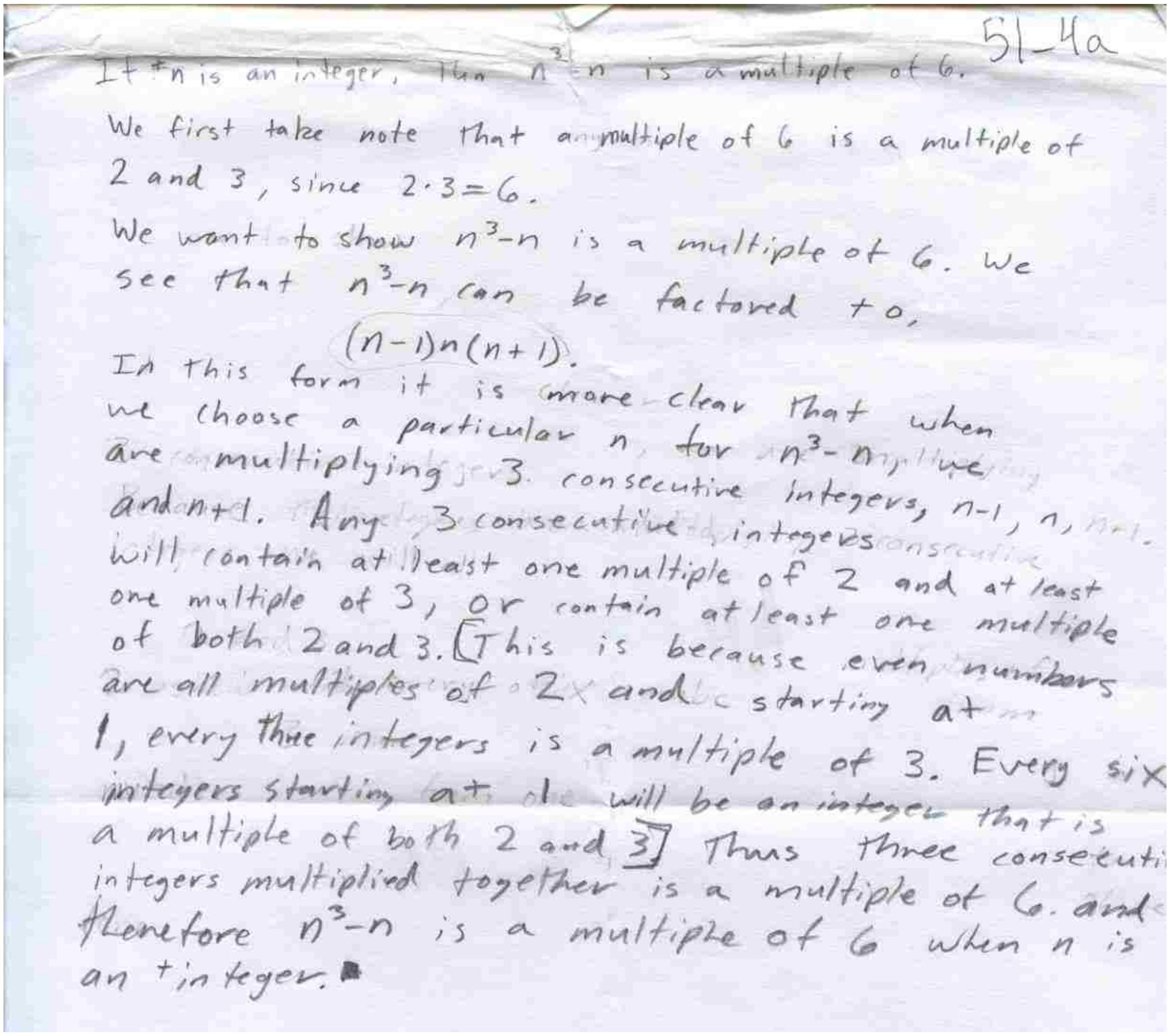
Figure 115: Robert's work on Question 2b (1 of 2)

Robert had been working with the assumption that an odd number was either odd or a multiple of 3.

His idea was not without merit, though, and he wrote out the numbers at the bottom of the figure, looking at three numbers at a time and putting boxes around the numbers that provided the necessary factors of 2 and 3. “Well, this…there’s a pattern because the 2s go every 2 and the 3s go every 3.” Robert was eventually able to verbalize what this pattern meant to the problem:

> R: So that’s how you get your 2 and 3. But I was just trying to break it down in here. Because I did want to see what happens when you have different cases. I drew this line to look at if there was a weird spot where…
>
> N: Where you could find three consecutive integers that wouldn’t work.
>
> R: Yeah.
>
> N: Try to fit a group of 3 between a multiple of 3 and a multiple of 2.
>
> R: And then I just saw the 3s because I wanted to, I was curious about the 3s…it’s weird because when you think about numbers you think about them individually, when you’re thinking about ‘are they divisible by 3’ but then to write them out and actually see – 3 is divisible by 3 then you go 3 spots, 6, 9, they’re going every 3 spots which makes sense because you’re adding. It’s like you add 3 things, then you add 3 more things.

Robert had verified that a multiple of 3 would indeed show up in every group of 3 consecutive integers. Applying the same reasoning to 2 would not be difficult. Turning this into a proof was something that Robert did not think would be so easy: “Because it feels like the proof would be…it seems like this would be hard to write this out…" I asked him to try and bring it to the next interview. The proof he brought back is in Figure 116.


51-4a

If n is an integer, then n^3-n is a multiple of 6.

We first take note that a multiple of 6 is a multiple of 2 and 3, since $2 \cdot 3 = 6$.
We want to show n^3-n is a multiple of 6. We see that n^3-n can be factored to,
$(n-1)n(n+1)$.
In this form it is more clear that when we choose a particular n for n^3-n, we are multiplying 3 consecutive integers, n-1, n, and n+1. Any 3 consecutive integers will contain at least one multiple of 2 and at least one multiple of 3, or contain at least one multiple of both 2 and 3. [This is because even numbers are all multiples of 2 and starting at 1, every three integers is a multiple of 3. Every six integers starting at 1 will be an integer that is a multiple of both 2 and 3] Thus three consecutive integers multiplied together is a multiple of 6. and therefore n^3-n is a multiple of 6 when n is an integer. ■

Figure 116: Robert's work on Question 2b (2 of 2)

Robert's proof attempt is semantic. By looking at examples, he comes to understand the factored form of numbers that can be written as $n^3 - n$. He then applies his understanding of the integers to conclude that of the numbers n, $n - 1$ and $n + 1$ one had to be a multiple of 2 and one had to be a multiple of 3 (or one was both). This understanding was eventually turned into a proof.

Robert's proof scheme here is transformational. He factors $n^3 - n$ and considers what happens to the expression $(n)(n - 1)(n + 1)$ for different n values. He sees that even though n may change, the fact that this product has a factor of 2 and a factor of 3 does not. Because he is purposefully manipulating a mathematical object and it results in a proof, Robert is displaying a transformational proof scheme.

<u>Question 3</u>

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

Robert started the problem by looking at the inequality for a couple of different n values. He looked at $n = 2$ "to make sure the inequality held." He wrote out the case of $n = 3$ for a different reason, though: "And then I just wrote out $n = 3$, just to see what this side would look like." (See Figure 117.) This was important because, unlike many participants in the study, Robert did not spend much time trying to decide what happens in the inductive step. While most students either thought that the left hand gained a single term only, $\frac{1}{2^{n+1}}$, or had its last term change to $\frac{1}{2^{n+1}}$, Robert knew that all the terms between $\frac{1}{2}$ and $\frac{1}{2^{n+1}}$ were added as well.

After this preliminary work, Robert began to set up the induction argument. I asked about his choice of base case and he said "Well, 1, 1 is equal…Well it holds, so I wanted to use that." After talking about his choice of base case, Robert said:

R: And so after, that, I had set this up - didn't really know where to with the induction, so I didn't really keep going with the base case, assume true, and prove that, because I hadn't really had an idea of how I would do it yet, so I thought I should try I could, like, break it apart and put it back together and stuff.

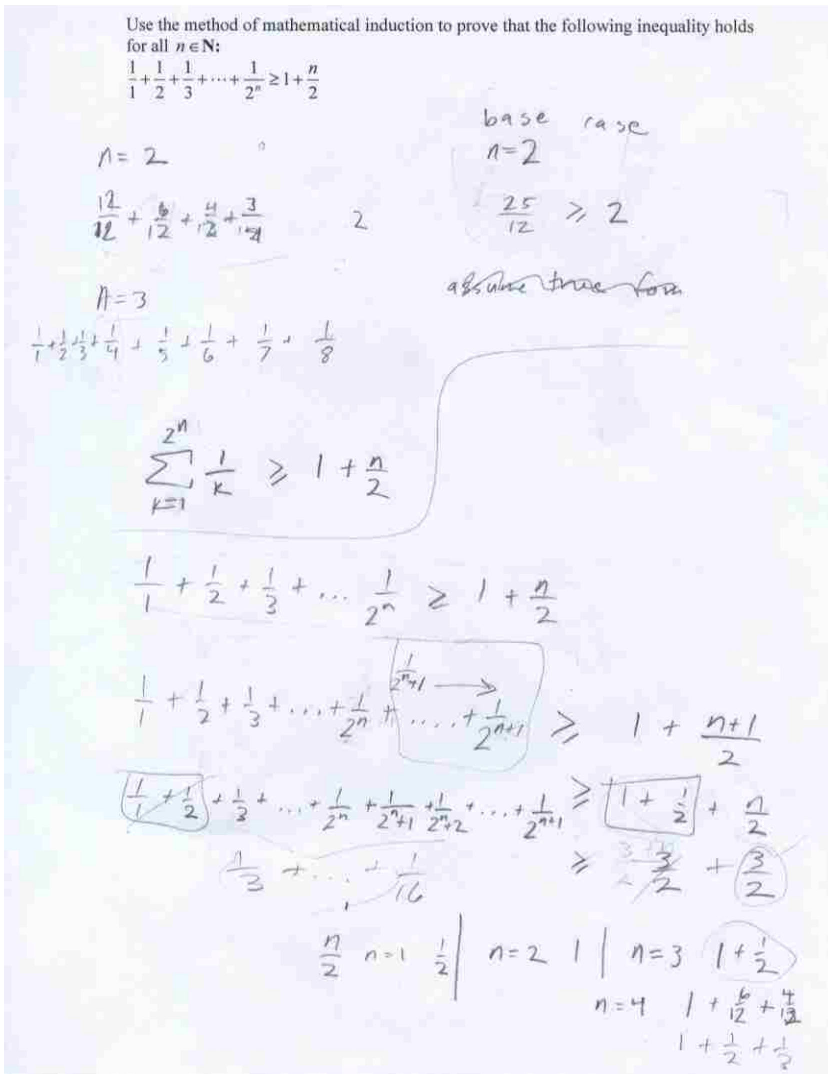


Figure 117: Robert's work on Question 3 (1 of 5)

Although Robert started to put the argument together formally, he wanted to make sure he knew how he was going to do it.

For the remainder of the time he spent working, Robert worked with the same idea: “And then I wrote the other side out…the n + 1 case too, and then I sort of started to see that there’s this … 1 and the 1/2, both happen on both sides…” His focus became comparing $\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2^{n+1}}$ to $\frac{n}{2}$. He noticed that when n = 3, the right hand side was $\frac{3}{2}+\frac{3}{2}$ and this made him consider writing the right hand side as the sum of fractions. This did not lead to much insight, though, and Robert started fresh on a new sheet, Figure 118.

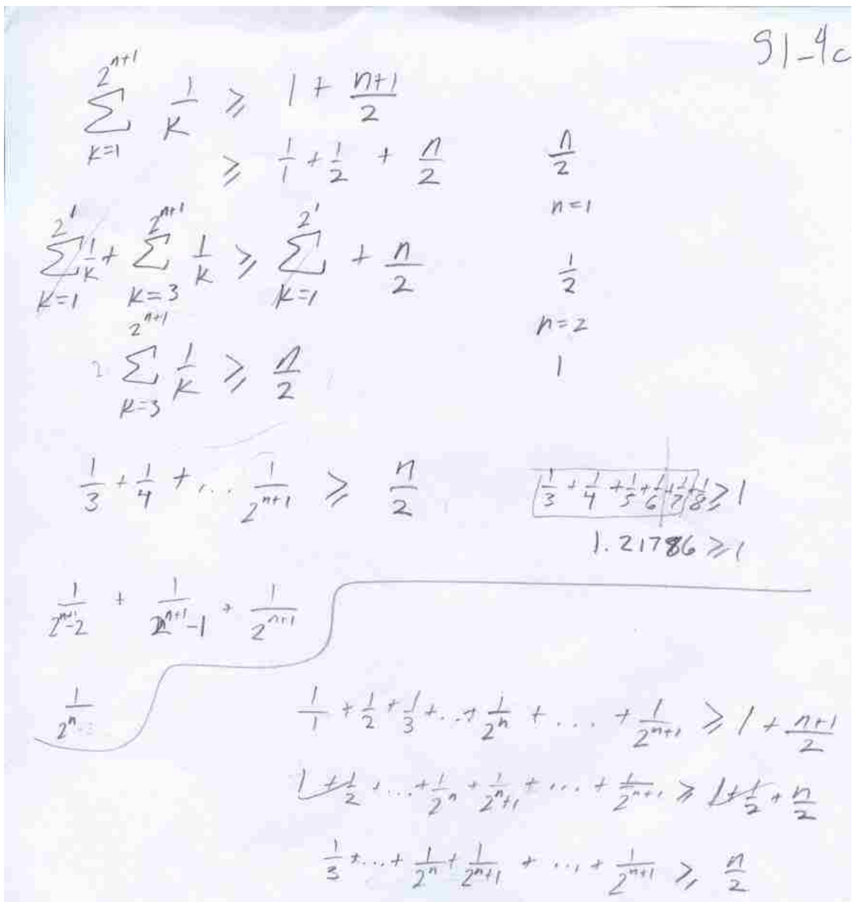


Figure 118: Robert's work on Question 3 (2 of 5)

The work on this sheet was similar to his previous work. He tried using summation notation, but it did not help much either: :Yeah, and then I thought maybe it would work out nicer with this sort of thing, but it just got me back to …basically, it’s summation notation, but…same thing.”

Next, he thought that maybe some insight could be gained by seeing how many terms were necessary for the inequality to hold, at least in the case of $n = 2$: "Yeah, and I started subtracting, so this is like, from up to here it doesn't work, but it works if you go up to 1/8, but it also works up to 1/7." With that, Robert looked at his inequality one last time and then it was time to end the interview. I asked him to look at the problem more between interviews and he said he would.

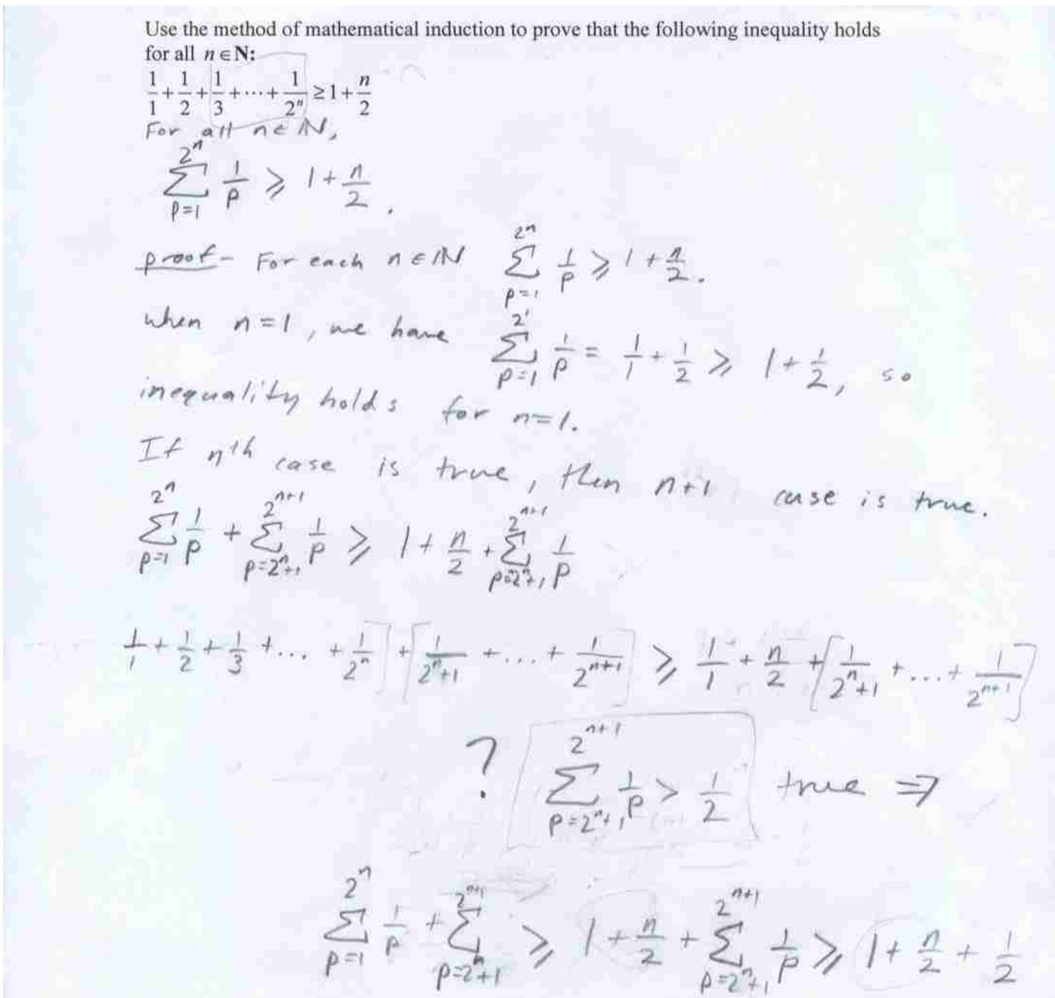


Figure 119: Robert's work on Question 3 (3 of 5)

The work Robert came back with is in Figure 119. He started over with the induction argument, setting it up formally. At first, he just added the terms he knew would come from moving to the inductive step. He quickly abandoned that, however, and thought about how he could explicitly use the induction hypothesis. This made him realize that the inequality in the box was what he had to do to finish the proof. Robert said that the idea to make use of the induction hypothesis came from looking in a different book:

> R: Yeah. I actually was trying to, like, actually, because I was in the library and just, like, grabbed a math proof book and I was like "Oh, I should look up induction just real quick," just to make sure I was, like just for, whatever.

This insight had gotten him to the heart of the problem.

The rest of the interview, Robert worked on finishing this part. The remainder of his work is in the following figure.

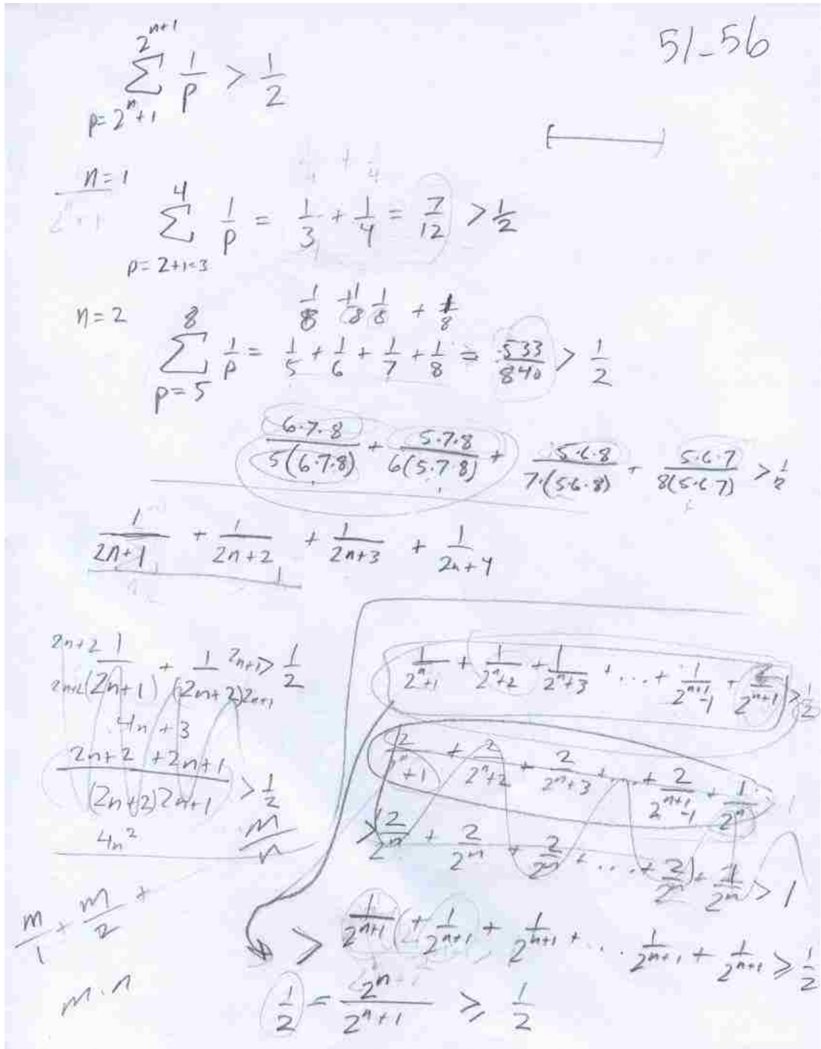


Figure 120: Robert's work on Question 3 (4 of 5)

Robert first went back to looking at his new inequality for a few examples, n = 1 and n = 2. He did not gain the insight needed to finish the problem right away, but it can be seen in the n = 2 case where he replaces each term in the sum with 1/8 to compare the left hand side to 1/2. Part of what he did in the previous interview showed up here as well. In addition to getting a common denominator in that case, he also thought that perhaps only the first 2 terms were needed on the left to make the inequality true. This led to the algebra seen in the left hand side of Figure 120. When he realized this would not work, he moved to the bottom right corner where he would eventually finish the problem.

Robert returned to the expanded out form of the inequality he wanted. He began manipulating the inequality by multiplying each side by 2, clearing the right hand denominator. For whatever reason, it was not until this point that Robert noticed that every numerator on the left was the same.

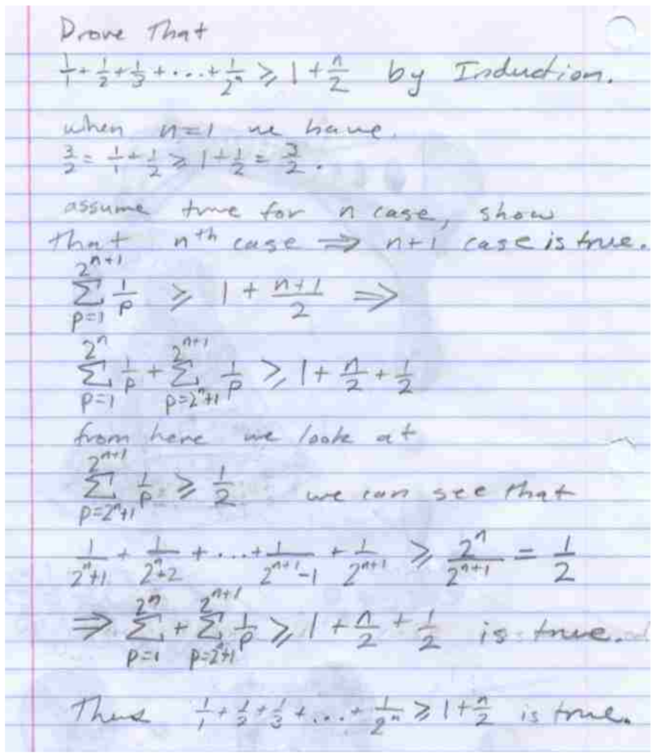
> R: Yeah, and I don't, it must have been because I was, like, missing it up here and I was thinking about, these are different and by making them the same, then I thought "Oh well…"
>
> N: Right, and for some reason the fact that they all were the same escaped you when they happened to be 1, but when they were 2…
>
> R: Yeah, and I don't know why.

At that point, Robert thought that it would be nice if the denominators were all the same as well. He used ease of reduction to decide which denominator to use: "I think it was because I thought that, well, the 2 (2^{n+1}), this is the easiest one that would cancel or something." After making this change and performing the cancellation, Robert realized that he had made the original left hand side smaller by what he had done. At this point, I gave him a hint because he was very close and we where running out of time:

> N: So you have, so…I kind of want to get to the reflection so I know we have enough time, and you're so close. So you have 2^n of these things, right?
>
> R; Yeah, over 2^{n+1}, so then greater than or equal to 1/2?

I did not feel too much guilt in pointing this out because it was something that Robert understood: "Oh yeah, well, so like the number of 1's…is 2^n because you have 2^1 is 2 is 2 terms. 2^2 is 4 terms, so that's how…"

From there, Robert was able to see how the pieces would go back together and complete the proof. I asked him to bring a written up version back next time and it can be seen in the figure.



Prove That

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^n} \geq 1+\frac{n}{2}$ by Induction.

when $n=1$ we have.

$\frac{3}{2} = \frac{1}{1}+\frac{1}{2} \geq 1+\frac{1}{2} = \frac{3}{2}$.

assume true for n case, show that n^{th} case $\Rightarrow$ $n+1$ case is true.

$\sum_{p=1}^{2^{n+1}} \frac{1}{p} \geq 1 + \frac{n+1}{2} \Rightarrow$

$\sum_{p=1}^{2^n} \frac{1}{p} + \sum_{p=2^n+1}^{2^{n+1}} \frac{1}{p} \geq 1+\frac{n}{2}+\frac{1}{2}$

from here we look at

$\sum_{p=2^n+1}^{2^{n+1}} \frac{1}{p} \geq \frac{1}{2}$ we can see that

$\frac{1}{2^n+1} + \frac{1}{2^n+2} + \dots + \frac{1}{2^{n+1}-1} + \frac{1}{2^{n+1}} \geq \frac{2^n}{2^{n+1}} = \frac{1}{2}$

$\Rightarrow \sum_{p=1}^{2^n} + \sum_{p=2^n+1}^{2^{n+1}} \frac{1}{p} \geq 1+\frac{n}{2}+\frac{1}{2}$ is true.

Thus $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^n} \geq 1+\frac{n}{2}$ is true.

Figure 121: Robert's work on Question 3 (5 of 5)

Robert's proof is process procedural proof. He follows a few broad goals in completing the proof. If he had had all the steps laid out explicitly for him, this would have constituted an algorithmic proof. However, this was not the case; completing the inductive step was especially difficult and required a method he had come up with on his own.

Robert's proof scheme here is transformational. Students who understand induction display the interiorized transformational proof scheme (Harel & Sowder, 1998). The fact that

Robert had used a base case other than 1 in the first interview made me question whether or not he understood the importance of the base case. However, during that reflection, he had mentioned considering it but not writing it down. Also, the 2 subsequent times Robert began his induction argument, he did use n = 1 as the base case. For these reasons, I do not see enough evidence to call his proof scheme internalized here. Instead, I think the mistake made in the first interview was an oversight.

Question 4

Examine the following proof by contradiction:

Prove $\sqrt{-1} \leq 0$

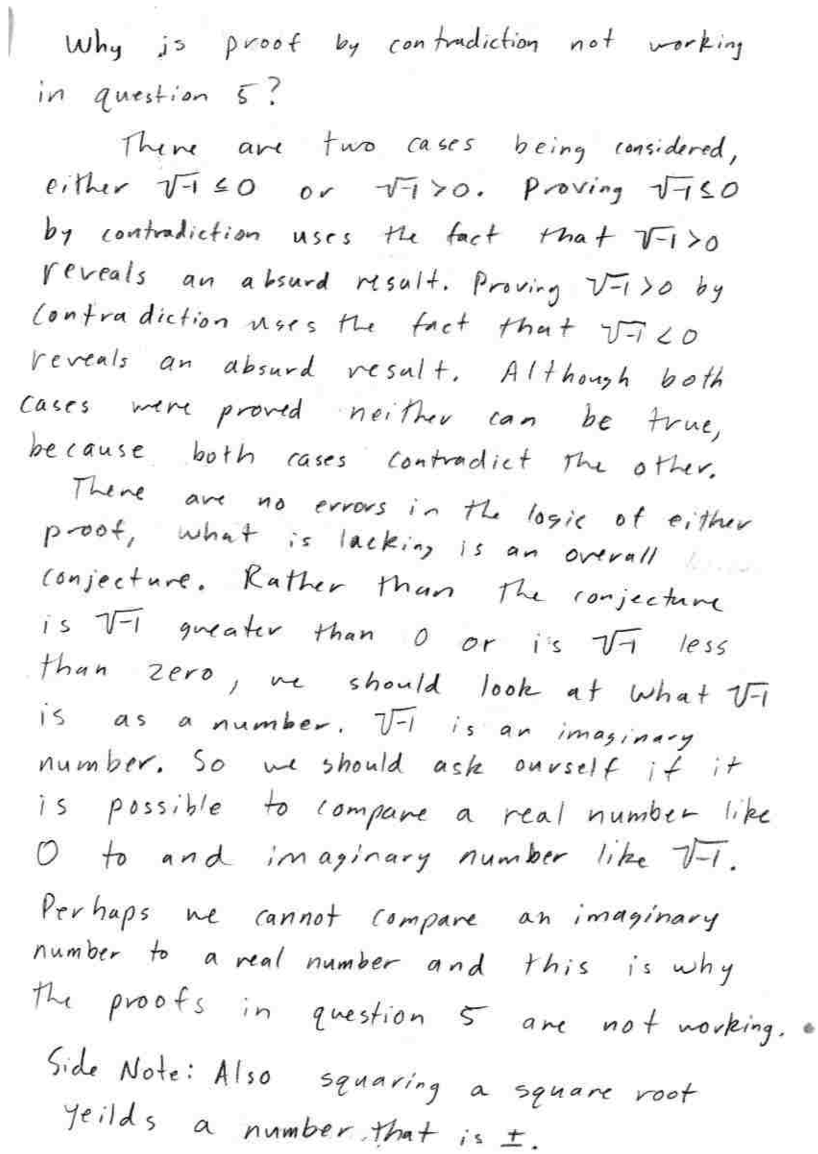
Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

Prove $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$, *then* $-\sqrt{-1} \geq 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$, *so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

Robert originally saw this question on his MATH 305 midterm the semester before the study began. Robert's response to the question on the exam is in Figure 4.51.13.



Why is proof by contradiction not working in question 5?

There are two cases being considered, either $\sqrt{-1} \leq 0$ or $\sqrt{-1} > 0$. Proving $\sqrt{-1} \leq 0$ by contradiction uses the fact that $\sqrt{-1} > 0$ reveals an absurd result. Proving $\sqrt{-1} > 0$ by contradiction uses the fact that $\sqrt{-1} < 0$ reveals an absurd result. Although both cases were proved neither can be true, because both cases contradict the other.

There are no errors in the logic of either proof, what is lacking is an overall conjecture. Rather than the conjecture is $\sqrt{-1}$ greater than 0 or is $\sqrt{-1}$ less than zero, we should look at what $\sqrt{-1}$ is as a number. $\sqrt{-1}$ is an imaginary number. So we should ask ourself if it is possible to compare a real number like 0 to and imaginary number like $\sqrt{-1}$. Perhaps we cannot compare an imaginary number to a real number and this is why the proofs in question 5 are not working.

Side Note: Also squaring a square root yeilds a number that is $\pm$.

Figure 122: Robert's previous work on Question 4

Initially Robert says, basically, that the proof is not working because it is not working (it gives contradictory conclusions). He goes on to say that "perhaps" the comparison can not be made to begin with but never concludes making this comparison is the reason for the problem. It is unclear how he intended his "Side Note" to apply to the problem, but he must not view the mistake as too detrimental because earlier he says that there "are no errors in the logic of either proof."

After reading through the question in the interview, Robert tried to think back to what he had done on the midterm: "So, I'm trying to remember what I did, but I think I did something with, just, imaginary numbers… Yeah, and then I think I might have said something about comparing an imaginary and a real, 0." He remembered the conclusion he eventually came to on the midterm, but vaguely.

He was not too convinced by it, though, because he looked back at the question and found what he thought was an algebra mistake. "But what I just saw was, this part, it's like you're multiplying both sides by the square root of -1, right? So the sign should have flipped, I don't know if that's…" He was referring to the second sub-proof. I reminded him that we were assuming that $\sqrt{-1}$ was negative, so $-\sqrt{-1}$ would then be positive, making the step in question correct.

After that, Robert went back to his previous argument: "Well, I think it was something like it's an imaginary number, like, it's not on the same, it's not on the real line, it's sort of off." He even took things a step farther. He began to see the question as a proof for what he was saying: "If you're not able to compare them, then it seems like this would happen, that you'd have, sort of, that's why you can't compare a real number to an imaginary because this sort of thing happens." He expounded on this idea later in the interview:

> R: (I)t's sort of like assuming that, well, you're not really assuming anything because you're trying to prove it, but…it seems like these are sub-cases of "Prove that you can't compare an imaginary to zero" or something.
>
> N: So, overall, there's really 2 proofs by contradiction here.
>
> R: Yeah

N: So the combination of those two kind of form a contradiction and that is contradicting the assumption that you can compare i to 0 in the first place?

R: Yeah, that you can compare an imaginary with 0.

Robert did not provide a proof in this interview, but we can find evidence of his proof scheme here. Robert definitely shows an analytic proof scheme here. There is no hint of either an external conviction or an empirical proof scheme. Instead, he focused on the logical deductions in the 2 sub-proofs. There is also no evidence for an axiomatic proof scheme here. At no point does Robert mention any other previous results or a dependence on starting assumptions during the discussion. In fact, he mentions at one point the lack of starting assumptions: "you're not really assuming anything because you're trying to prove it." Instead, he views the ability to compare an imaginary number to 0 as up in the air to begin with. Then, after a careful consideration of the steps in the sub-proofs, he concludes making the assumption that the comparison can be made is wrong.

R: But the contradictions come from the other one, you know, so it's…so that's, once you check everything you go back and you don't (find an error) and it has to be the comparison, you can't actually, you can't do the comparison.

Robert's proof scheme is transformational here because he examined the steps that follow from assuming i and 0 can be compared. He then looked over the steps carefully (because he saw these manipulations as important) and when he found them to be sound, he knew the comparison was the trouble from the beginning.

Question 5

Like Question 4, Question 5 did not ask Robert to construct a proof. Instead, the next interview was used as a debriefing session because it was the last interview of the first semester and marked the half way point of the study. Due to the nature of the interview, there will be no proof attempt to classify. Instead, the interview was used to look for clues about Robert's proof scheme and to reinforce observations made previously in the study.

One of the recurring things observed in Robert's work is that he did a number of semantic proofs. Robert's approach to proof involves first understanding the problem and then moving on to the proof. This is approach also showed up in the interview. When I asked what he sees as necessary to complete a proof, Robert said:

> R: It's almost, you have to, like, understand it…I guess, too, like, sometimes in the inductive ones…you get down to this formula, and you can kind of plug them in and it sort of works. Then your only job is to, like, it's sort of like works, you have the feeling that it works, but sort of need that actual, proper work laid out so it's sort of formal.

He also mentioned that he feels more comfortable with proof. Over the course of the semester, he got more secure in exploring with a problem: "Because I think I felt like I needed to know what was actually going to happen before but now, it's just, you're not always going to know you just have to sort of…you know, get going…" I feel also reflects his preference for semantic proofs. He seems confident that even if he does not know how to approach a problem he can play around with it until he does.

Another thing Robert mentioned that mirrors his behavior in the interviews came out when I asked him what helps with completing a proof but was not necessary: "If you get stuck, just to like, walk away, come back…" This was especially evident with Question 3, where Robert restarted the problem a number of times.

Robert also provides some evidence for the proof schemes observed during the study. While discussing a proof he saw in his Real Analysis class that took two class periods to finish, he said:

> R: You go through this big, long proof and sometimes it's like you have to prove this thing on the side and then you can go back. Those proofs are…
>
> N: Yeah, they're tough.
>
> R: But once you have that, you get to use the definition (result) for another proof.

Robert was referring to the axiomatic nature of mathematics where previous results can be used to prove subsequent ones.

Robert also provided evidence for the transformational proof scheme that showed up in earlier interviews. When discussing the proofs he sees in Real Analysis, Robert said:

> R: Yeah, like that, kind of came out of a proof, over all it sort of, it kind of…he starts proving something and he gets to the end, and "Ok, it makes sense now," but how he got there is a little bit…

He is referring to completing steps within a proof that do not have apparent motivation. Robert sees that sometimes in order to prove something, sometimes one needs to be purposeful and anticipatory while working towards an end. This view of proof is a characteristic of a transformational proof scheme.

<u>Question 6</u>

Prove that $\forall n \in \mathbb{N},\ \ n \equiv n^3$ *(mod 6).*

Robert and I began this problem by discussing modular arithmetic. It had been covered in MATH 305 briefly as an example of a way to partition the integers, but most students had not seen it since and were unfamiliar with it. I had mentioned that 14 was congruent to 32 (mod 6) as an example and accidentally gave Robert more than I had intended when discussing how to reduce numbers:

R: 2 mod 6 would be 2 because that's…

N: Right

R: 6 would be 0, no…yeah…right?

N: Yeah, 6 equals 6 times 1 + 0.

R: You may have given me a clue there.

N: Did I?

R: Maybe

I inadvertently mentioned applying the division algorithm, something a few students were able to use successfully for this problem. Robert did not use it right away though. Instead, he thought for a while, before saying: "I can't figure out this mod thing, that's the problem. Because, like, you can't really do anything algebraic until you figure out how this is working, like pulling it apart." Eventually, he did try to adapt what I had said to the problem, leading to the equation $6(r) + n \ = \ n^3$ at the top of the figure.

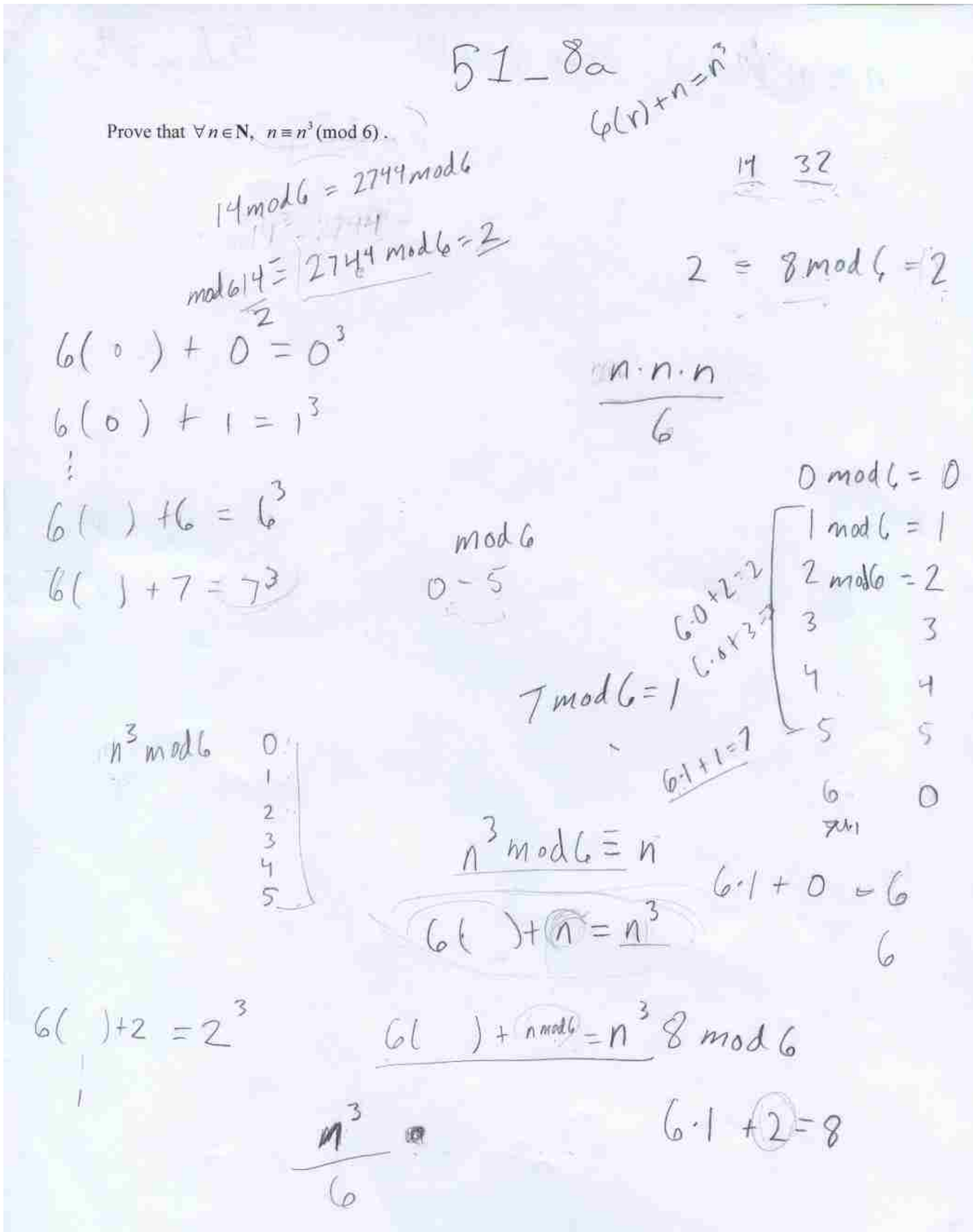


Figure 123: Robert's work on Question 6 (1 of 3)

After he has completed most of the scratch work in the figure (except the 0 – 5 list on the left towards the bottom), Robert seemed stuck.

R: I'm not really seeing much.

N: Ok. Let me see what you've got so far.

R: Well, so I started looking at, so like…7 mod 6 would be 1, 6 goes in once, you add one and you have 7. So then I was trying to use…sort of the same thing to…yeah. Because 6 times something plus (referring to $6(r) + n = n^3$)…if this is true, right? Then this remainder would be *n*, so I'm just trying to, there should be something that I should get to, right?

N: Ok, yeah, one problem you might have is that *n* is too big to be a remainder. But you could sort of handle that elsewhere. You know, *n* might be 10.

R: Oh yeah

After some more discussion about modular arithmetic, Robert realized that the remainders were limited to 0 – 5. Still stuck, Robert started over with the problem on a new sheet (Figure 124) but the interview ended shortly thereafter. He did make some progress, though, finally successfully applying the division algorithm to the problem.

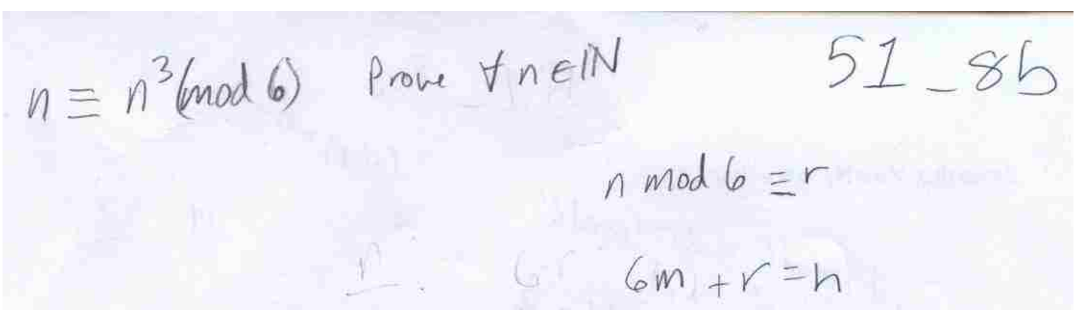


Figure 124: Robert's work on Question 6 (2 of 3)

I asked him to continue to work on the problem between then and the next interview. He said that he was not sure if he would continue with the equation he had just came up with ($6m + r = n$ when $n \pmod 6 \equiv r$. He also that mentioned the structure of the problem reminded him of induction problems so he might try that as well.

Between interviews, Robert did not get too far with the problem: “I felt like I kind of got stuck on this one I just…I don’t know. I just, yeah. I got stuck and I just kind of…quit working on it.” Since he did not really know where to go, we talked through a couple solutions together. I described roughly what it would look like to cube $6m + r = n$ and how that could be reduced to finish the problem. I also mentioned induction and that was the solution we ended up talking through. The work Robert did as we talked through the proof is in Figure 125.

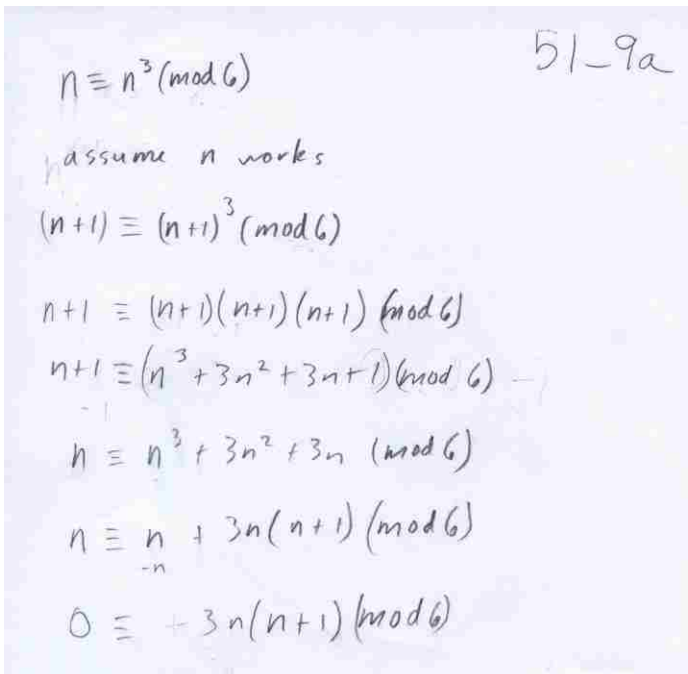


Figure 125: Robert's work on Question 6 (3 of 3)

Part of the trouble Robert was having had to do with being unfamiliar with the rules of modular arithmetic: “So that’s what I…I was thinking…can you just start replacing things? I’m just, I don’t know…this whole thing is kind of like playing a game with what you don’t really know the rules to, you know?”

Despite needing help to get there, Robert was able to finish the problem off himself:

N: Well, can I ask you think a second: What does it take for a number to be congruent to 0 mod 6?

R: It needs to be…it has to be a multiple of 6, right?

N: Right, ok.

R; So you're trying to say that $3n(n + 1)$ is a multiple of 6….So this looks like it should be.

N: Ok, why is that?

R: Well…because one of these is either even or odd. It doesn't matter, this could be even and this one's odd, so you could have an even number times 3 which is going to be…so an even times an odd times 3 is always a multiple of 6.

Robert did not complete a proof on his own, so I will be classifying his attempt only. As with his other proof attempts that did not involve induction, this is a semantic proof attempt. The scratch work in Figure 123 is done with the intention of gaining an understanding of the problem. When I asked Robert what he saw as the major obstacle in the proof, he said:

R: Just trying to understand, like, how things are working in this mod 6 and everything.

N: So, I mean, it doesn't seem like you explicitly made a proof attempt here. You're kind of trying to figure out how this mod stuff works while in parallel sort of trying to get a feel for what the problem is really saying, is that true?

R: Yeah

N: And then once you have that understanding of how the problem's working, what it's saying, then you'll probably go into the verification of this, the proof of it?

R: Yeah. Yeah, but I can't really do anything until I…because I feel like I'm still trying to understand…

The idea of getting an intuitive understanding into the problem first and then turning it into a proof is typical of a semantic proof attempt.

Robert's proof scheme here is transformational for a couple reasons. One, he becomes convinced of the statement via a proof by induction. This alone is enough to classify Robert's as a transformational proof scheme. Also, during the first interview in which he worked on the problem, Robert worked toward getting an expression he could "pull apart." In other words, he was looking for a way to manipulate the objects at hand in a more familiar way. This is also typical of a transformational proof scheme.

Question 7

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

Robert began this problem by writing out a general set with *n* elements. After a brief discussion as to why the empty set is a subset, Robert looked at the elements of his set and started mentally constructing different subsets.

R: I think it's sort of like the null set, so each individual one in a subset, this one (a_1) each, you know, paired with each one.

N: Ok

R: And paired with, sort of like 2, a combination of 2 of everything.

N: Ok

R; Just, so you have a subset of this set again, right? Isn't that a subset?

N: Yep, every subset is a subset of itself.

R: Ah man, I should have paid attention in that probability and statistics class. Would have helped a lot. Maybe not, I don't know. Because I'm thinking there's going to be the combinations of stuff, taking a count.

The idea of using combinations came from his consideration of all the different ways to construct 2 element subsets. Once he had that, he was able to go to the formula for the number of subsets.

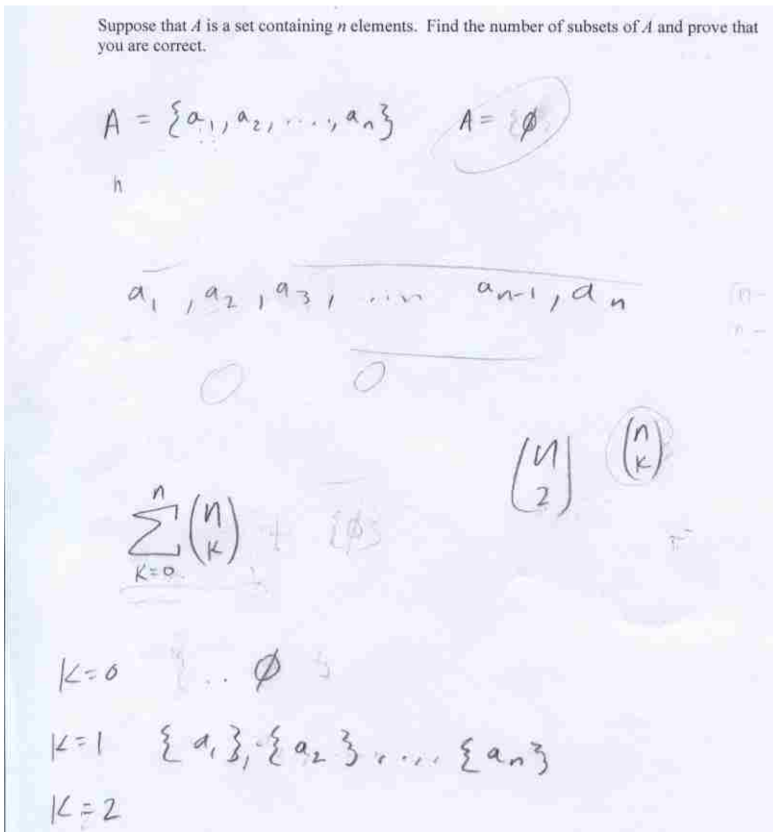


Figure 126: Robert's work on Question 7 (1 of 2)

R: Ok…Prove that you are correct.

N: So you've gotten to the point where you're happy with the formula you've got there?

R: Well, ok, let's talk about that. So you've got your *n* total things, the first little way of choosing 0 is just to not…not choose anything. If you have 1, you choose 1 so that's going to go through all of them and just pick 1 and put all those in a subset. And *n*C2 is going to take all the different combinations of…is that going to cover it, though, it's going to take the first one, the second one, third one, fourth one?

N: Yeah

R: Does that take it into account?

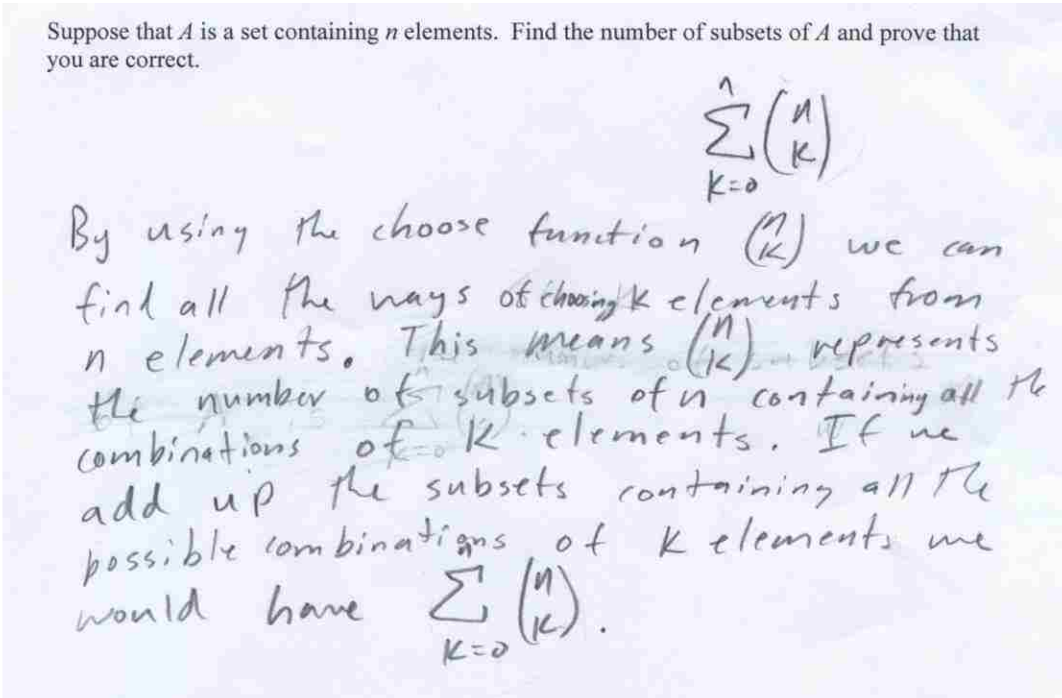
N: Yeah, you can put some faith in the choose function, that it's going to work, yeah.

R: But I'm just trying to see…and it'll just go through and we'll say *n*C*n*, take all of them and put it into a subset.

Robert seemed fairly sure he was correct, but he did not feel comfortable calling what he had done a proof. To gauge his confidence level in the proof, I asked him what he would do in if this had been a homework assignment.

R: I mean, I guess if I was going to turn this into homework, I'd sort of write some comments that like…you know, this thing sort of adds up all the different ways that we can choose from choosing nothing all the way up to choosing all the *n* things.

I told Robert that this uneasiness was something I wanted him to struggle with and then come back with a proof that he would turn in for homework. The work he brought back is in Figure 127. The proof is basically as he said it would be, the summation along with some commentary as to where it came from. Even though he wrote it up, he still said he was uncomfortable with it:



Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

$$\sum_{k=0}^{n} \binom{n}{k}$$

By using the choose function $\binom{n}{k}$ we can find all the ways of choosing k elements from n elements. This means $\binom{n}{k}$ represents the number of subsets of n containing all the combinations of k elements. If we add up the subsets containing all the possible combinations of k elements we would have $\sum_{k=0} \binom{n}{k}$.

Figure 127: Robert's work on Question 7 (2 of 2)

Robert then talked a little bit about the consideration he had given to using induction or contradiction on the problem. He had a hard time describing the issue he had with the other methods so I tried to help.

N: So let me sort of…so the problem you have with induction I want to re-visit that just a little, so the problem that I would have with induction and you can tell me if that's what you're trying to get at or not, is that you don't really have anything to shoot for exactly.

R: Yeah, because you don't know how many subsets there are.

He expressed a similar discomfort with contradiction.

The proof Robert provided here is semantic. As with most of his other work, Robert began by trying to understand what was going on with the problem.

N: Ok, so yeah, you read through this and the first thing you did was list out a set, a_1 to a_n.

R: Yeah, just sort of to get a feel for what it was going to be.

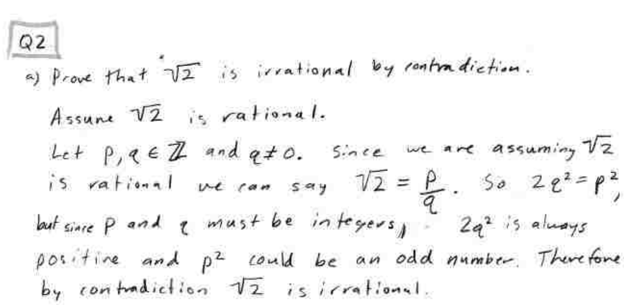
It did not take long and he did not have to look at examples, but Robert was able to turn his understanding of the way different subsets are formed (by choosing which elements are in them) in his eventual proof.

Robert potentially displays a couple proof schemes here. One is transformational. He considers what happens has you try to create different subsets and lets that guide him to a proof. The other proof scheme Robert seems to give evidence is a ritualistic external conviction scheme. Normally, one holds this scheme when they hold the opinion “if it looks like a proof, then it is a proof.” Here, Robert is uncomfortable with has proof possibly because it does not look enough like one. I am not sure that is the case, though, because at least some of his consternation has to do with the fear that he has not accounted for some possibility. For this reason, I will label his proof scheme as transformational only.

Question 8

Prove that the cube root of 2 is irrational using a proof by contradiction.

Like Question 4, this problem appeared on Robert’s MATH 305 midterm. His midterm response is in the figure.



Q2

a) Prove that $\sqrt{2}$ is irrational by contradiction.

Assume $\sqrt{2}$ is rational.

Let $p, q \in \mathbb{Z}$ and $q \neq 0$. Since we are assuming $\sqrt{2}$ is rational we can say $\sqrt{2} = \frac{p}{q}$. So $2q^2 = p^2$, but since p and q must be integers, $2q^2$ is always positive and p^2 could be an odd number. Therefore by contradiction $\sqrt{2}$ is irrational.

Figure 128: Robert's previous work on Question 8

Robert makes a few mistakes here. One, he attempts to prove that $\sqrt{2}$ is irrational, not $\sqrt[3]{2}$. Second, although he claims that "$2q^2$ is always positive" I think he means to say "even." This is a mistake a number of students made in the study. The thinking goes that *p* and *q* can be any integers, odd or even. Then once one deduces that *p* must even, it contradicts the assumption that it could have been odd.

During the interview, Robert began the problem properly:

R: So, assume that it's rational…Assume that it's rational and you can write it as *p*/*q*…

(working)

R: …lowest…reduced terms or however you say it. It's not…like if you had 4/2, that's…like this is the same as 2/1…

Even though Robert did not have his notes to guide him as he would have during the midterm, he did remember to include the assumption that *p*/*q* was reduced completely. Later, during the reflection, I asked him why he made that assumption right away: "Well, I sort of, I thought there was something about that that I needed because..." He was not able to give a reason, but he agreed that it was an idea that "nagged" at him.

After making his starting assumptions, Robert did the algebra work on the left side of Figure 129.

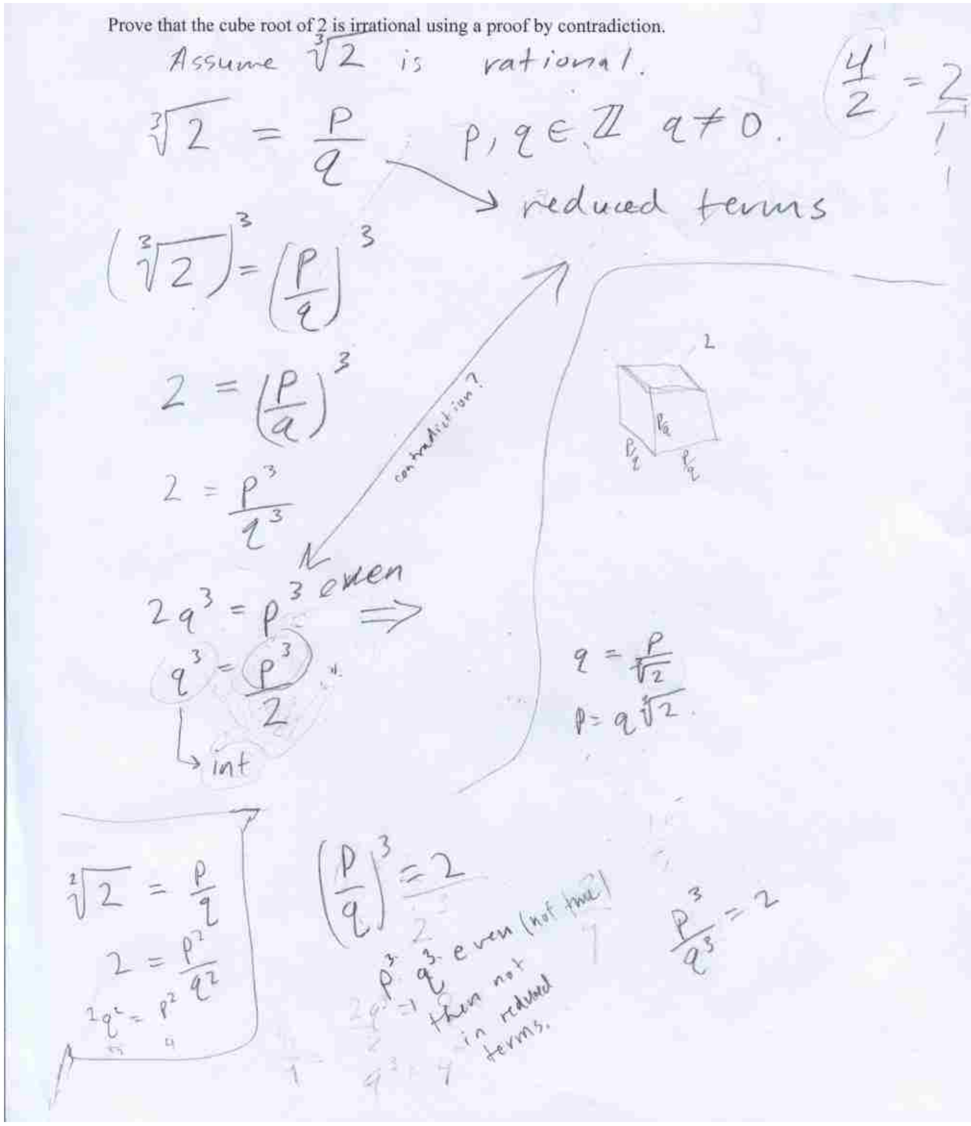


Figure 129: Robert's work on Question 8 (1 of 2)

After getting down to the end of his first portion of work (where q^3 is labeled “int”), Robert thought that he had arrived at a contradiction, but he did not know why:

> R: So doesn’t this mean that this (arrow labeled contradiction – the word ‘even’ isn’t added yet) is where you’re going to get you contradiction?...Because if they were in reduced terms, they wouldn’t be a multiple of …So I think that it’s that this is violating the reduced terms thing.
>
> N: Ok

R: When you can write it like this ($q^3 = p^2/2$)…assume it's rational…

N: So how is that violating the reduced terms thing?

R: I'll have to figure that out.

After giving it some more thought, he tried the remaining rearrangements of the expressions he had and the cube that is labeled with side lengths p/q. During the reflection, he said of the cube he drew:

R: I don't know, I thought it might help, because I was thinking maybe it was something geometrical, that I was…but I don't know. But yeah, that should be the volume of it, right? 2 should be the volume of the cube?

N: Right

R: And this, the length should be those, but that…I was just trying something else, just to…

N: Ok

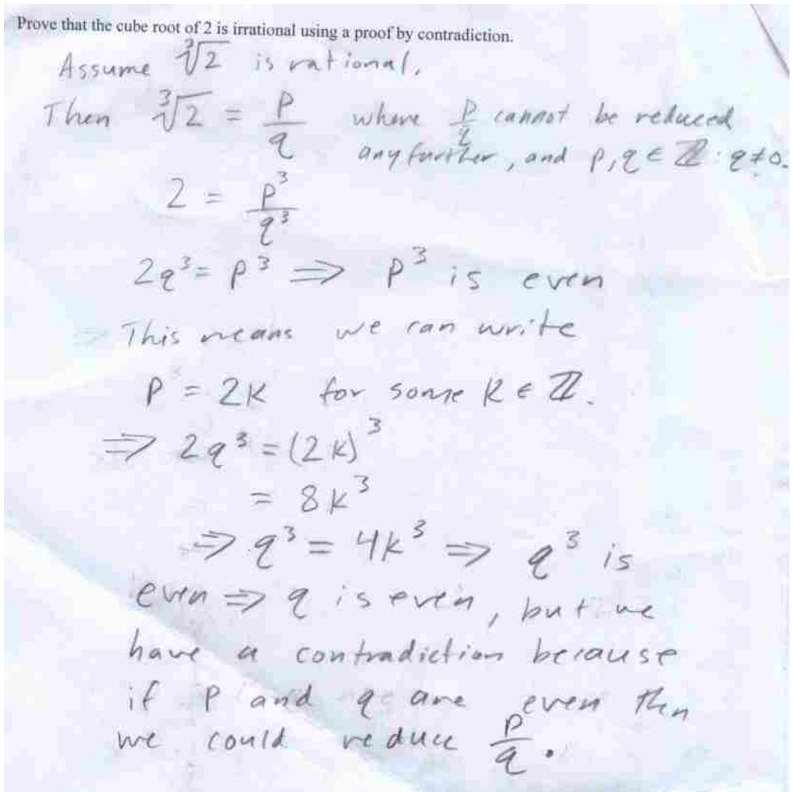
R: …maybe to try to break my head away from this business.

N: Yeah

Because for some reason I'm stuck on it. I feel like there's something about that.

He was referring to the assumption that p and q were relatively prime and its relationship to the fact that p^3 was even.

Since we had come to the end of the interview and Robert had not finished the proof, I asked that he go back to his notes and look up the proof that $\sqrt{2}$ is irrational because that would recreate the situation he was in while completing the midterm. The work he brought back is in the figure below.


Prove that the cube root of 2 is irrational using a proof by contradiction.

Assume $\sqrt[3]{2}$ is rational.

Then $\sqrt[3]{2} = \frac{p}{q}$ where $\frac{p}{q}$ cannot be reduced any further, and $p, q \in \mathbb{Z} : q \neq 0$.

$2 = \frac{p^3}{q^3}$

$2q^3 = p^3 \Rightarrow p^3$ is even

$\Rightarrow$ This means we can write

$p = 2k$ for some $k \in \mathbb{Z}$.

$\Rightarrow 2q^3 = (2k)^3$

$= 8k^3$

$\Rightarrow q^3 = 4k^3 \Rightarrow q^3$ is even $\Rightarrow q$ is even, but we have a contradiction because if p and q are even then we could reduce $\frac{p}{q}$.

Figure 130: Robert's work on Question 8 (2 of 2)

Robert did look the other proof up in the book and got some help from it.

> R: You get here, you know it's (*p*) even, but this is where I …like, somehow the q needed to be even for it to violate the lowest terms…So this was, I mean, this was the little trick that…That *p* could be written as a different thing.

The proof Robert provides here is an algorithmic proof. He used the proof that $\sqrt{2}$ is irrational as a guide for how to complete this one. He said "It sort of follows the same way as the square root." Although he was fairly close on his own and the steps were meaningful for him, he followed the other proof very closely – even to the point that he let it decide for him how much detail to include. When I asked about whether he considered the deduction that *p* was even based on p^3 being even, Robert said:

R: But I did, I actually did think about that a little bit but I just thought that you didn't, from the $\sqrt{2}$ proof it didn't seem like it said much…

N: Yeah

R: …about that.

N: Ok… So it's the fact that you're just sort of mimicking the other one, by and large, that and they didn't do much to verify it so you didn't feel like to verify it?

R: Yeah, but at the same time, it did make, it did make sense, though.

N: Right, yeah, no, it does. It definitely does.

R: I guess I didn't think of verifying it because in the other one it didn't really.

Robert's proof scheme here is transformational. The fact that he follows the steps of another proof does not imply that he is displaying a ritualistic proof scheme. This would be the case if he believes in his proof solely because it looks a different one. In Robert's case, he believes it because he understands it. That he knows to assume *p* and *q* are relatively prime even though he does not know what he is going to do with it is the sort of anticipatory step that is typical of transformational proof schemes.

<u>Question 9</u>

During the next interview, I gave Robert a version of Cantor's Diagonalization argument and asked him to evaluate it. Because he did not attempt a proof, there will not be one to classify. However, I did use our discussion to look for evidence of Robert's proof scheme.

I gave Robert the proof and he spent the first few minutes reading through it. Eventually, he said "I've read it quite a few times, and I know kind of what's going on." Although he had a general feel for what was going on, Robert went back and looked through the proof another time.

R: "every subset of a countable set is countable."

N: Ok. You don't like that?

R: No, I do.

N: Ok

R: I think.

N: Oh, you're just sort of going through the pieces of the argument again?

R: Yeah

N: Ok

R: So that seems…that seems, I'm convinced of that. It makes sense that's it's countable, but…

N: … So I've been asking people if they can identify these hidden lemmas and you just did. Like, it seems believable but…

R: It probably does need proof too.

Robert had found a potential hole in the proof. After this exchange I made the mistake of telling him that the result was a theorem in the textbook from MATH 305, leading him to say: "Well, if it's a…if it's a theorem then it has a proof somewhere…I feel like that's something that you could prove but…not necessarily…It depends on how…if you want to prove it or not, I guess."

After that, Robert went back to looking at the rest of the proof. It still took a little bit for him to become comfortable with the argument.

R: Well, like, I mean, this is just sort of, this area (definition of β_j on down) …This part is "Now we have our list, now da da da," this part gets a little…this is a trick where it's either like…if this wrong, then it's something in here, or if it's right, because I feel like all this stuff (from the beginning to end of the list) was fine… Well, I'm just, this defining *B*, defined by this part. It's a little bit…

N: You're sort of not seeing how to define the β's?

R: Well…

N: Is that…?

R: No, it seems like it changes…and it's changing it from…if α_{jj} equals to 2, you change it to 1, if it's not 2, you change it to a 2. And I was trying to figure out if that was like a legal…can you just do that?

N: Oh, I see. Well, we're just trying to decide what those digits are in *B,* right?... So what, I mean, what's making you think it might be illegal?

R: Well, that seems like a little key part to what's going on here…To get the contradiction, you have to…

Robert realized that he had to make sure he understood this construction because it was where the contradiction would come from to make the proof work. After a little more discussion of how to define *B*, he said:

R: Yeah…So it's saying there should be…β of *k* up here…Ok, so β_k is up here somewhere, check it with α_{kk} and this makes, ok, this makes sure that this isn't the same as that. Ok, alright, I'm cool with it now.

N: Alright

R: So that part looks good. … And so if α_{kk} is 2, you change this to a 1. So that makes, it's, if it's not 2, you change it to 2. Ok.

Now that Robert had a handle on how *B* was defined, he turned his attention to understanding the conclusion of the proof.

R: So you say this and then you get this result ($\beta_k \neq \alpha_{kk}$) from saying that because of the way you defined it.

N: Right

R: So, what I'm trying to say, then, is this legal in the first place, you know?

N: I see.

R: But, I see how you get the contradiction.

Robert was concerned about the ability to define *B* in the first place. I then talked through a summary of the proof one more time and at that point, Robert seemed comfortable with it.

R: Yeah, I mean, but I'm hesitant because there might be some little…

N: …some little thing you're not thinking of?

R: …trick.

N: Yeah

R: So, yeah…So in this one it looks like probably…I could probably stare at it for longer, but…

Robert does mention a little apprehension, but only in his ability to think of all possible "tricks" that might not be visible to him yet. I say he seemed comfortable with the proof because from that point in the interview on, Robert was able to explain the justification for the steps of the proof to me rather than the other way around.

Robert gives evidence for both types of analytic proof schemes here. First, when discussing the statement that every subset of a countable set is countable, he seems confident of its validity because “it has a proof somewhere.” He is not relying on the book’s word but rather the existence of a proof. Once he is comfortable with that statement, he freely applies it to the proof. This is evidence of an axiomatic proof scheme because he is acknowledge this proof’s reliance on previously justified facts.

Robert also gives evidence of a transformational proof scheme because he is able to identify actions take specifically for the purposes of the proof.

> R: Ok, so this is sort of like where the actual…so, what I’m thinking right now is that you say this (pointing to the last paragraph)… it’s like where the contradiction’s going to happen, but you say it and then the contradiction, like you get, I see where this is happening right now, like, you get this ($\beta_k \neq \alpha_{kk}$) from saying that (definition of *B*)…

Robert’s recognition of performing anticipatory operations for the purposes of a proof is typical of a transformational proof scheme.

<u>Question 10</u>

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

Robert began this problem by looking at examples, both the one provided and ones he tried on his own. When he looked at finding a rational number that fit the criteria between 0/4 and 1/4, he realized that he could not. This revealed, however, that he was not sure what that meant in relation to the problem overall.

R: So disprove…you pretty much just have to find a counter-example. But you could also…do sort of like a proof that disproves it, right?...

N: Well, yeah, so like, yeah, my question was to either prove it or disprove it, so like, have you disproved it?

R; Well, I'm not really sure if a counter-example means it's disproved…So would disproving…disproving would be that there's no...like, it would be proving that you can't do this, right?

N: Well, not necessarily.

R: Disproving means it either works for all of them…it works for all of them, it doesn't work for any of them, or it works for some and it doesn't work some…

N: Well, right, yeah, but like this statement, the statement that's there, you know, between every pair blah blah blah, that's a statement, right?

R: Yeah

N: It's either true or false. Right?

R: So the counter-example says that it's false, then.

Since Robert had decided that the statement as written was false, I asked him if he could place some restrictions on it so that it could be proven true. He then started working on the general inequality $\frac{n}{m} < \frac{n_0}{m-1} < \frac{n+1}{m}$ in Figure 131. During the reflection, I asked why he set the middle denominator to $m - 1$ when all that was required was that it be less than m:

R: Because that, I guess I was looking at these (the examples above).

N: Ok

R: Well they…it just has to be less than, right?

N: Yeah. And $m - 1$ certainly works as less than.

R: But in this case (the given example)…it's less by 2…Well, the reason I did it was because of this (1/4, 1/3, 2/4).

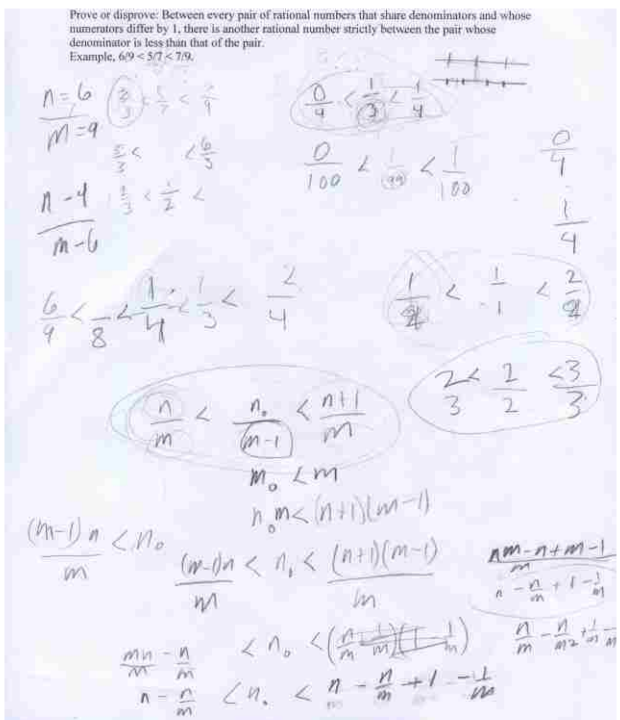


Figure 131: Robert's work on Question 10 (1 of 3)

He continued working on the general algebra on another page, see Figure 132. There is some erasing (on both pages) because Robert had made an algebra mistake along the way that I pointed out to him. I thought the interview time would be better spent having taken care of that quickly.

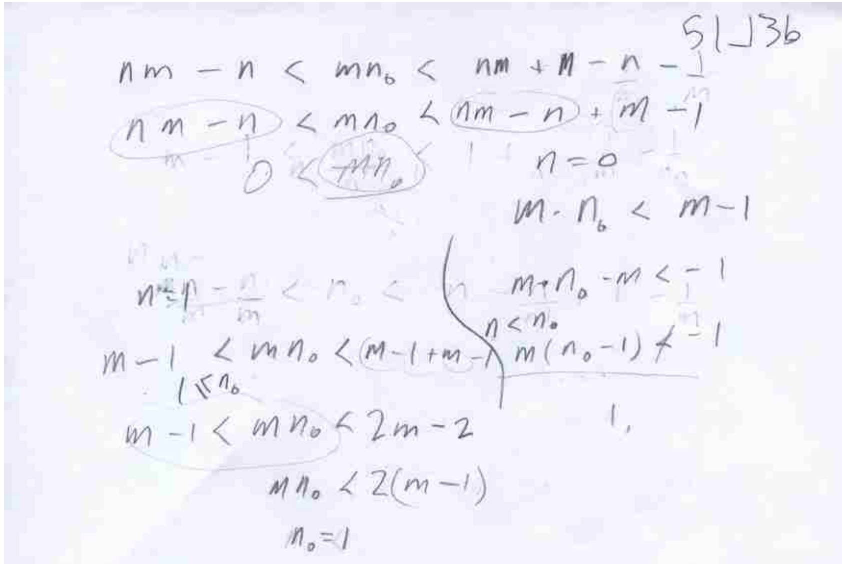


Figure 132: Robert's work on Question 10 (2 of 3)

After bringing the inequality from the bottom of the previous page, Robert broke it into its right and left parts and worked on them individually, seeing what value n_0 would need to take on if n = 1. He deduced that the inequality would hold if n_0 also equaled 1. He thought he had found a restriction that would allow him to say that it works:

R: I'm thinking that it's going to be when it's…when *n* is greater than 1.

N: When *n* is greater than 1?

R; Or equal to 1, I suppose.

He was discouraged, though, because he thought that he was now going to have to check all *n* values.

At that point, it was time to start the reflection. We talked about the examples he went through and he noticed something new about them:

R: You know, and there's something about, that I just now thought of when we went back you talked about this, that thing (6/9 in the given example) reduces quite a bit.

N: Right

R: I mean, not…it doesn't reduce, but…But these (the counter-examples) don't reduce…

N: Right

R: So…

N: Do you think it maybe has to reduce in order to work? Or not?

R: Yeah, I mean, I'll have to look at it.

After the reflection, I asked Robert to work on the problem between this interview and the next and he said he would. The work that Robert brought back was not a proof. Instead, he came up with a method that, for a given pair of rational numbers, would find another in between that fit the criteria if one exists. Robert's method can be seen in Figure 133.

Robert said that a given pair of rational numbers would define the pair *n* and *m*. Then the number between them, if it existed, could be written as $\frac{m+x}{n-y}$ where *x* could be any integer and *y* had to be strictly between 0 and *n*. In his original work, Robert used *j* and *k* instead of *x* and *y* but made the change when he decided to graph the inequalities. Then he worked the two parts of the inequality $\frac{m-1}{n} < \frac{m+x}{n-y} < \frac{m}{n}$ to get lines that he could turn into shaded regions.

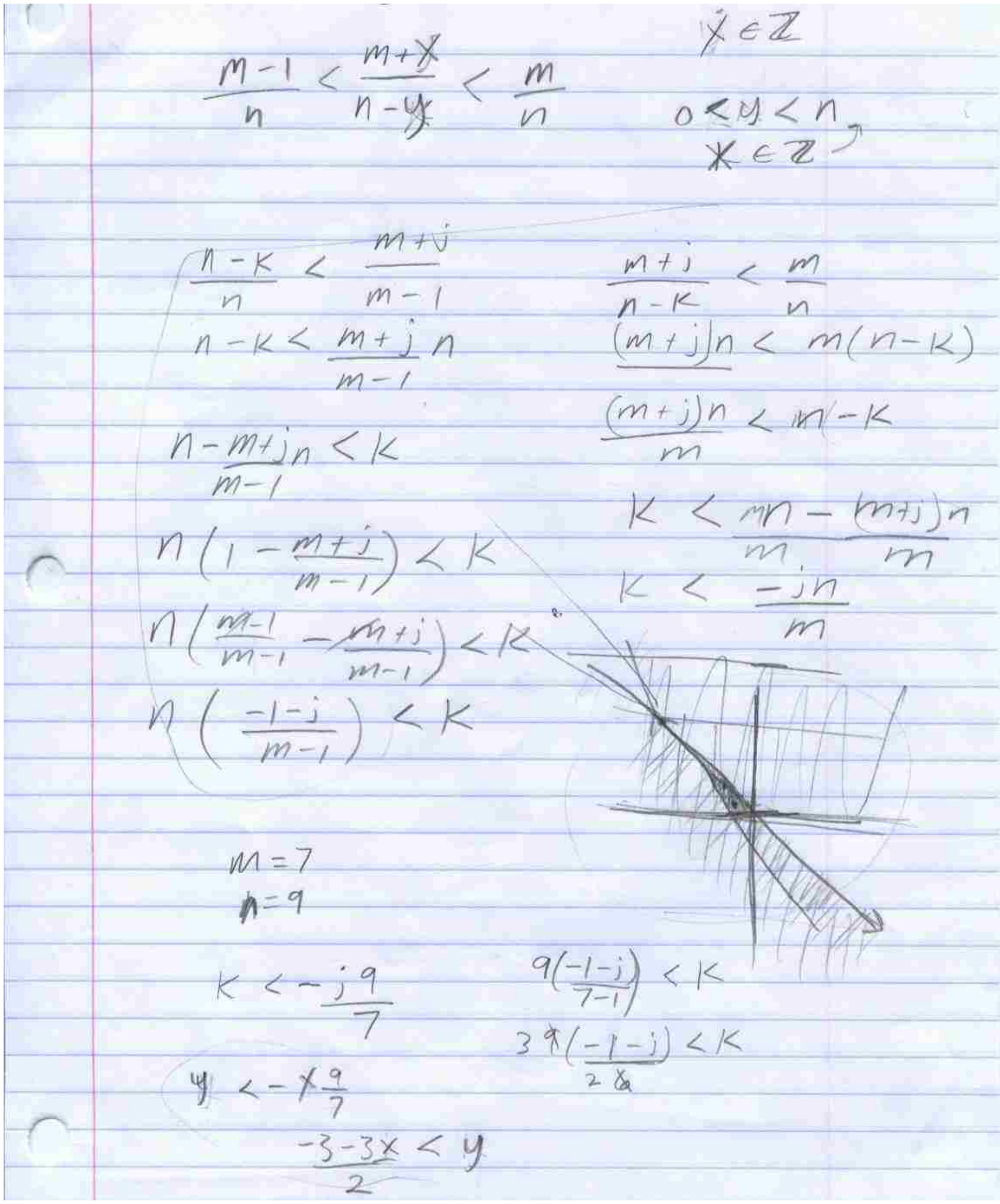


Figure 133: Robert's work on Question 10 (3 of 3)

R: And then kind of graphed it, this is kind of a rough graph of it. But, like with this restriction, the *y* has to be greater, it has to be less than *n*, so like so up here at 9, right? (he is trying his method with the given example, $m = 7$ and $n = 9$) So somewhere in this region. And then when you graph these 2, you get, well, these 2 lines and then what I was thinking is that there has to be like the integer, coordinates with just integers in there…Yeah, and it works because the negative 2 and positive 2 is right in there.

N: Right

R: And also the other one is like -1 and 1. So…so like what was the *n*, so like 6…6/8 works as well… That's how far I got, so…

Notice the heavy dots drawn in the graph he tried.

I mentioned to Robert that I had not seen anything like this before and I thought it was clever. To this, he responded "I mean, I'm not sure. What I did was I found this, but I haven't proved that it actually…this is more of like a method just to like find…" After going through his method, we ended the discussion of the problem by me walking him through a proof of the problem, given the restrictions that neither $\frac{m-1}{n}$ or $\frac{m}{n}$ was an integer.

Because Robert did not complete a proof, I will classify his proof attempt only. This is a syntactic proof attempt because most of Robert's work involves the manipulation of algebraic expressions. By the time he came up with his graphical method at least, these manipulations where certainly meaningful to him. However, his work is not at all based on an attempt to use an intuitive understanding of the problem into a proof. Rather, he is simply "pushing symbols" hoping that something useful comes from it.

Although he does not necessarily know where he is going with his manipulations, Robert does deduce logically as he works. Thus, his proof scheme is analytic. However, his attempt did not rely on previous results at all and therefore can not be considered axiomatic. Thus, Robert is displaying a transformational proof scheme here. This is not surprising given his focus on algebraic manipulations.

Question 11

Question 11 was in the form of a debriefing session with Robert. The interview in which we concluded Question 10 also concluded the study. Like was the case with Question 5, there was no proof attempt here so there will be nothing to classify but the interview did highlight some of the observations made during the study.

The first question I asked Robert was how he thought the semester has gone proof-wise. He said:

R: Well, it's just proof in general, it's like…it's becoming a little more comfortable, not that it's, like there's still little like gray areas here and there, but it's…I don't know. I feel like, I think that I've gotten this year is I feel a little more, like there's a little more room to move at times…Like at first you kind of come into proofs and you're sort of like "Well I could just do like, just like algebra, the regular little steps I was used to." But now you feel, you know what I mean?

N: Yeah and…

R: That was sort of not the best way to say it, but…

N: No, I think I get it, like you, like there's sort of freedom in understanding it to know what you're allowed to do, you're not sort of, you know, tied down by these rules that you've been told work and so you know they work and you don't want to deviate from that because you know they work and you don't what else might. But now you understand the proofs at maybe a deeper level and so that grants you the freedom to be just a little more…flexible I guess.

R: Yep

He felt like he had grown better able to adjust proofs and proof techniques to suit his needs and that made him more comfortable proving.

When I asked him what he thought led to this improvement, he thought it was a combination of repetition and variety:

> R: Probably just more exposure to proofs. You know, just seeing more…A well balanced diet of proofs.
>
> N: Do you think that's more important than just sheer number? Of course if you just have a sheer number, eventually you'll have a variety, but…
>
> R: I think, like, I think in the beginning, like learning induction you need like…
>
> N: Set practice?
>
> R: Yeah, you need like a few of them to get that to, those other things…but you know, at the same time it's sort of, yeah you need the repetition for those, just the methods…

Robert is referring to the process by which an internalized transformational proof scheme becomes an interiorized one. This also matches Weber's assertion that students can learn common proof techniques by viewing them as "mechanical procedures" (2003, p. 395). This also matches the number of times Robert displayed a transformational proof scheme through out the course of the study.

Robert also briefly alluded to the other proof scheme he showed during the study: axiomatic. When describing the nature of the proofs he saw in one of his classes, he said they were "straight forward type proofs…Or like, sort of like definition heavy proofs, there's not really like 'by the…whatever'." He is contrasting direct proofs to those that rely on previous results. By doing so, he acknowledges the axiomatic nature of mathematics.

Robert also highlighted his approach to proving: to gain an understanding before trying to complete the proof. I asked him if he had changed anything in the way he goes about proving and he said:

R: No, I guess, the only thing is, that's sort of the thing, you have to try to, like, yeah, I don't come into every problem saying 'Gotta do induction.'

N: Right

R: Like, I try to like, try to kind of respond to the problem in a way…I feel like whenever I get a problem I kind of have to sort of check it, like just with the example or whatever.

N: Yeah

R; Just do an example first then start kind of tinkering with how you can move things around…if that doesn't work then just sort of…add something in or, yeah.

N: Ok

R: Just start playing around.

N: Ok, so that, that kind of leads in pretty decently with number 5 here, "what role do you see examples playing in proof?" So it sounds like you try an example, or it's one of the first things you look to do is an example at least.

R: Just like a warm up lap or something, you know. It's just good to, just kind of get a feel for what's actually going on in the problem.

N: Right

R: Because sometimes when you just stare at the algebraic expression with numbers, with letters in there, it's like it kind of doesn't sink in exactly.

This approach was also evident when I asked Robert what was helpful and/or necessary when completing a proof:

R: What's helpful is like anything to sort of, well, understand what the problem's asking… Well, just having the understanding of the methods to solve it…But I think like doing like the…like rough work and you have to do, just feeling comfortable with just doing this rough scratch work and then sort of refining that.

N: Ok

R: Because if you're trying to do, if you're trying to start and do like this formal proof right off the bat...

N: Yeah

R: Well for me anyway, I feel like I would just sit there and stare at it…Well, to complete it you kind of have to do that formal and that, I don't know.

This matches nicely with the fact that of Robert's 8 proof attempts, 5 of them were semantic.

<u>Robert's progression</u>

Below is a chart of the types of proofs Robert produced over the course of the study and the proof schemes he displayed.

Question	Type of proof	Proof scheme
1	Semantic	Transformational
2a	Semantic	Axiomatic
2b	Semantic	Transformational
3	Process	Transformational
4	N\A	Transformational
5	N\A	Transformational, Axiomatic
6	Semantic (Attempt)	Transformational
7	Semantic	Transformational
8	Algorithm	Transformational
9	N\A	Transformational, Axiomatic
10	Syntactic (Attempt)	Transformational
11	N\A	Transformational, Axiomatic

Table 7: Summary of Robert's work

As can be seen in the chart, Robert was fairly successful in his attempts. His proof schemes were mostly transformational but always analytic; Robert maintained a deductive view of proof throughout the study. Also, from the beginning of the study Robert displayed a good approach to proof. He would explore the problems he was given hoping to find some understanding that he could then turn into a proof. Also, he did not give up when he felt like he was stuck on a proof, instead he frequently started problems over without being discouraged.

As was the case with James, Robert did not show much progress. Also like James, I believe that Robert has the formal view of proof that all students of mathematics would ideally have. Also, his tendency to develop semantic proofs and his persistence with problems he finds difficult is something most would hope for from their students.

4.8 Michael

This section looks at Michael's proof attempts and the progress he made over the course of the study. Michael major in computer science and so only took two mathematics classes during the study. During the first semester, he took Euclidean and Non-Euclidean Geometry and Probability and Statistics.

Michael's proof attempts

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Michael began this problem by working on the white board, but eventually decided that he liked working on paper better and copied everything he did from the board to the paper in Figure 134. Michael started the problem by drawing a rectangle and labeling the sides as m and n.

> Michael: Well, we want to find integers for m and n such that when the perimeter is found and the area is computed they're both the same. Is that…?
>
> Nick: Yeah
>
> M: It simpler to write it as an equation so, $2m + 2n$ = the area, which is equal to m times n.

After that, Michael begins to rearrange the equation into a form more useful to him (this work was done on the white board). Initially, he did not like the manipulations he did and so I helped him isolate the variable m, following the steps in the figure.

Once Michael had *m* isolated, I asked him what he planned to do with the equation.

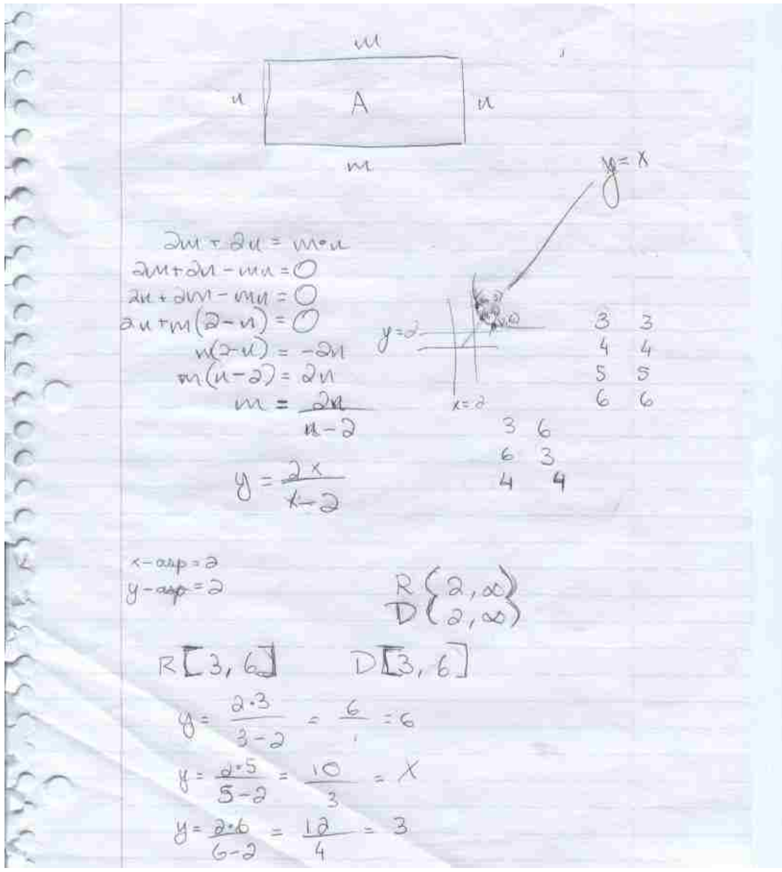


Figure 134: Michael's work on Question 1

M: If I want to, I could just graph it, probably.

N: Ok, and what are you graphing?

M: I'm going to graph…I'm going to convert *m* to *y* and *n* to *x*…I'll make life easy, I'll just look right here.

Michael was referring to using the table feature on his graphing calculator. After discussing the fact that non-integers, zero and negatives would not work for this problem, Michael began looking at his table again and said: "Yeah, I'm just going to pull up what I got here and … so the values I got here so far are 6 and 3, 4 and 4 and…where was it here…3 and 6."

I then reminded Michael that the second part of the problem asks him to prove that he had found all such rectangles.

M: Well, they're related to…can you give me a hint?

N: Sure, I would sort of examine the graph. Because, um, …

M: Because the, what do you call them, the limits

N: Yeah, so it kind of relates to

M: What do you call those things?

N: Asymptotes?

M: Yeah the asymptotes.

Michael identified $y = 2$ and $x = 2$ as the vertical and horizontal asymptotes, but he did not know how they were useful to the problem initially. Michael seemed stuck again, and so I reminded him of the criteria of the points we were looking for:

N: And essentially we're looking for what kind of coordinate pairs? They have to show up on that line, A, and what else? What other criteria do they need to have?

M: And they're integers

N: Yeah. So, um, like say past x value of 6. (6,3) is on the curve, right? So what do you know about the y values after x value 6 – for the …

M: Oh, yeah, they're all between 2 and 3 because of the asymptote at 2.

N: Right. So…

M: So therefore you cannot have a whole number between those right there.

Michael then mentioned that the same reasoning could apply when looking at the vertical asymptote. Then, he shifted his focus back to the inputs for his function.

M: I'm just messing around with the domains and seeing if…so…now I would just…I would just have to show that all values that all values between 3 and 6 are the values that you can use to find the … that show the perimeter and the area are the same. All the integers between 3 and 6…

N: So we've eliminated from 6 on. So why have you been able to eliminate everything from before 3?

M: Before 3? Well the asymptote is 2 and then once you get past the asymptote everything goes into negatives, which you can't really include.

N: Right

M: And you know everything (that has x value) between 3 and 2 is not an integer, it's a rational or irrational…

Michael's last hurdle to clear was what it meant to look only at his restricted domain:

M: Well, I'd have to show 16…or would that be it? There'd be 16 different ways I'd have to show it's …. maybe….

N: Well you've found …ok, so why 16?

M: Well for the *n*, you've got 3, 4, 5, and 6 that you can choose from and then for *m*, you also have 3, 4, 5 and 6 and since it's a rectangle it doesn't matter.

Michael had confused the fact that the sides of a rectangle were interchangeable with the idea of using one of the side lengths as an input for a function to find the other. I reminded him that he had such a function.

> N: Ok, so I kind of gave you this hint of considering this curve a function of x so what are you going to do with that?
>
> M: Just plug in all the x values.
>
> N: Sure. And see what?
>
> M: And see if the y output is an integer or not.
>
> N: Right, ok. And you've already done that for 4.
>
> M: For 4? Yeah, so I'll just do for 3, 5 and 6.
>
> N: Ok.
>
> (working)
>
> M: That doesn't work.
>
> N: So, what doesn't work? What did you just try?
>
> M: 5 doesn't work, that gives you an integer – you get 10 thirds.
>
> N: Ok.
>
> For, uh, 6…

Michael used the graph of the function and its asymptotic nature to eliminate all inputs besides 3 – 6 and then checked those cases by hand. By doing so, he knew that he had found all rectangles that fit the criteria given in the problem.

This was the end of the interview and I asked Michael to bring me a formal write up of the proof he had done. When he came back for the next interview, he said he had forgotten to do it, but we talked briefly about what the proof would look like if he had.

> M: Well, first of all, I would rewrite it again, make it more consistent, you know, linear. Get one idea through, the use that idea to get the next idea. And for this one, to find the, what do you call it?
>
> N: The asymptote.
>
> M: Yeah, the asymptote. I would actually show it by solving for y, you know $x = \frac{2y}{y-2}$ to show that it really has an asymptote there. Then I would just be a little more detailed.

Michael basically was saying that he would be more careful to be formal in his proof and justify all the steps he is making.

The proof Michael provides here is semantic. While I help him along, he does explore the problem to see what is going on with it. He observed the table and graph of the function he finds and then turns his understanding of its behavior into a proof.

When describing the proof he would write, Michael makes a point to mention that he would ensure that each step is used in justifying the next. This deductive view of proof is typical of an analytic proof scheme. He does not bring up, however, the use of any previous results to use in the construction of his proof. Instead, he uses algebraic manipulations of the equation he needs to satisfy to create a function he can use to complete the proof. This is indicative of a transformational proof scheme.

Question 2a

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9.

Michael started this problem by using modular notation to write down what it meant to be a multiple of 9. Although he would erase it later, he first tried the standard subtraction algorithm in general:

> M: Yeah, so we could rearrange this thing to where it's in the kindergarten form where the *ab* is over the *ba*.
>
> N: Ok
>
> R: Which seems like it would give you something, so…
>
> N: Ok
>
> R: Ok, so, but then we get a problem here because we don't know if *a* is bigger than *b* or *b* is bigger than *a*, so…

Michael abandoned this idea and turned to examples and it did not take him too long to notice something:

> M: So I think I got a pattern here. So 9 minus 1 equals 8, which 72 is divisible by 8. But this one is equal to right here, 7. And 63 is also divisible by 7 because 7 times 9 is also 63.
>
> N: Ok
>
> M: So, show lets see for a different case that without 9 this time…So this is (45-54) 9 and 5 – 4 is 1…

After giving this pattern some thought, he moved on to the work in the bottom of Figure 135.

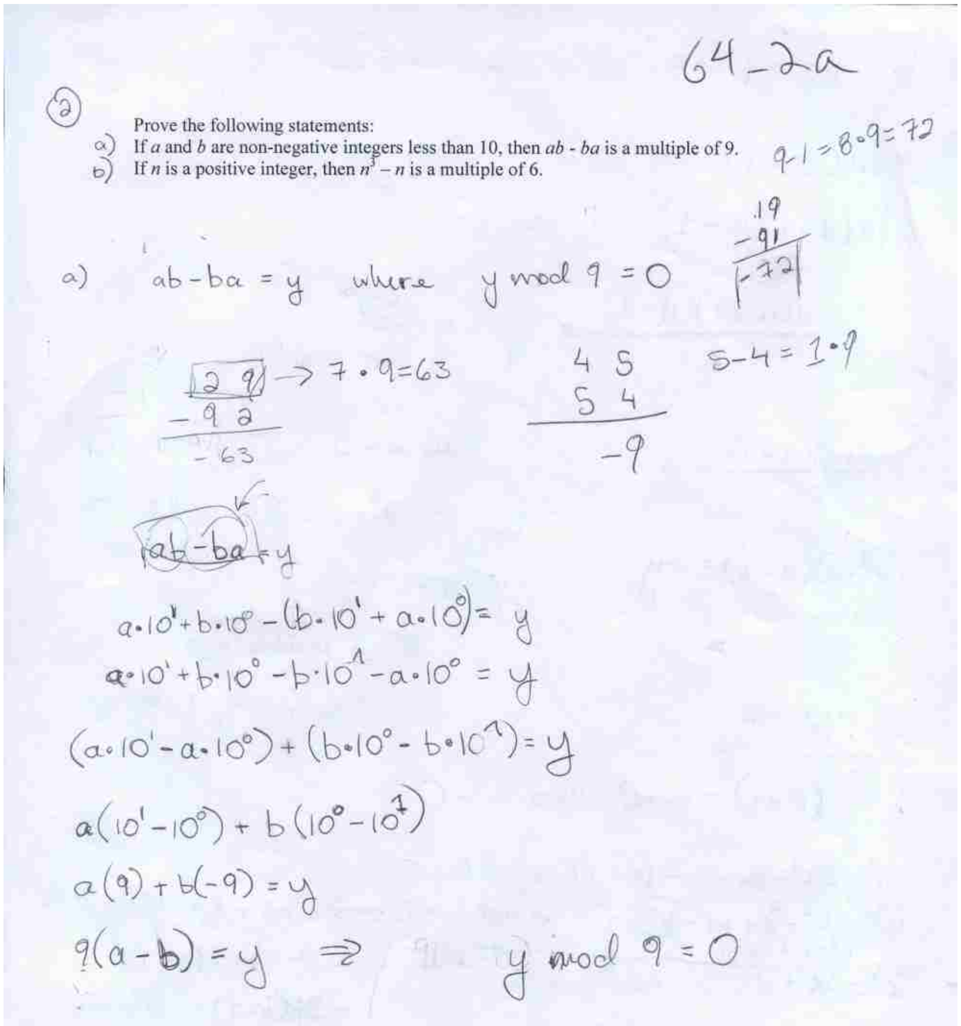


Figure 135: Michael's work on Question 2a

M: So *a* times 10 plus…So you can re-change the form of *ab* and *ba* to *a* times 10^1 times *b* times 10^0, plus *b* times 10^0, minus *b* times 10^1 plus *a* times 10^0…Alright, I'll just factor out a little bit more, try to clean it up.

(writing)

M: Put them together $(a*10^1 – a*10^0) + (b*10^0 – b*10^1)$…So we know that 10^0 is 1, so $a(9)$ and so we know that $10^0 – 10^1$ is –9, so $b(-9)$. Which is 9 times *a* plus *b* times 9, minus 9, sorry. So, we have $9(a – b) = y$.

N: Ok

M: So it's actually behaving similar to what I noticed up here, so say 29 – 92, and if…you subtract 2 from 9, you get 7, then times 9 is 63. Which is similar to the equation where $9(a – b) = y$. And since, it has 9 in it, it is divisible by 9, therefore $9(a – b) \bmod 9 = 0$. Which proves it, but I feel like there's a part missing to it, is that right?

N: Well, do you have a gap that you see or is it just an overall feeling that there's a gap.

M: Yeah, I see that I proved it, but I feel there's a gap somewhere, because I just showed right here that there's a, actually, no I think I just proved it. That's all there is. I proved it.

Michael used the nature of base 10 numbers to re-characterize *ab* – *ba* and show that it equals $9(a – b)$ which is clearly a multiple of 9.

It might seem like the proof Michael provides is semantic due to the pattern he found by looking at examples. This is not the case, however. Although he did find a pattern, he did not use it to come up with a proof. In fact, he only noticed that he arrived back at his pattern after he completed his work: "So it's actually behaving similar to what I noticed up here." While reflecting on the problem later, Michael said that his proof came about due to work he had done in other classes:

M: Actually, this part I remember this because of computer science, actually because we were converting using binary – it's actually really easy just to put two (to the) zero, one times ten to the…if you're converting and everything, so ….that really helps.

Since Michael gave a deductive proof that did not result from applying an understanding of the problem, this is a syntactic proof.

Michael's proof scheme here is transformational. Like with Question 1, he provides a logically sound deductive proof that in no way depends on previous results. Instead, it relies on performing operations on the mathematical objects at hand. Thus, Michael displays evidence of a transformational proof scheme only.

Question 2b

If n is a positive integer, then $n^3 - n$ *is a multiple of 6.*

Michael started this problem by factoring $n^3 - n$. He then looked at applying what it means for a number to be a multiple of 6:

M: And so since it's also a factor, well it's a multiple of 6…that would give you, if you divide by 6, that should also give you an integer.

N: Right.

M: Ok, so…

(writes $n(n + 1)(n - 1)$ over $6 = y$, thinking)

M: So $6y$ is equal to $n(n^2 + n)$…

N: If you need, yeah…So you're now…

Yeah, I'm just getting stuck right now, it's just one of those days…

At this point, Michael gave up manipulating the expression $n^3 - n$ and looked at some examples.

M: So let $n = 1$…would be 0.

(tries $n = 0, 1, 2, 3$ on his calculator and writes down the last case, see Figure 136)

M: Ok, let's suppose, let's try a proof by induction.

N: Ok

M: So $k^3 - k$ mod 6 = 0. So…

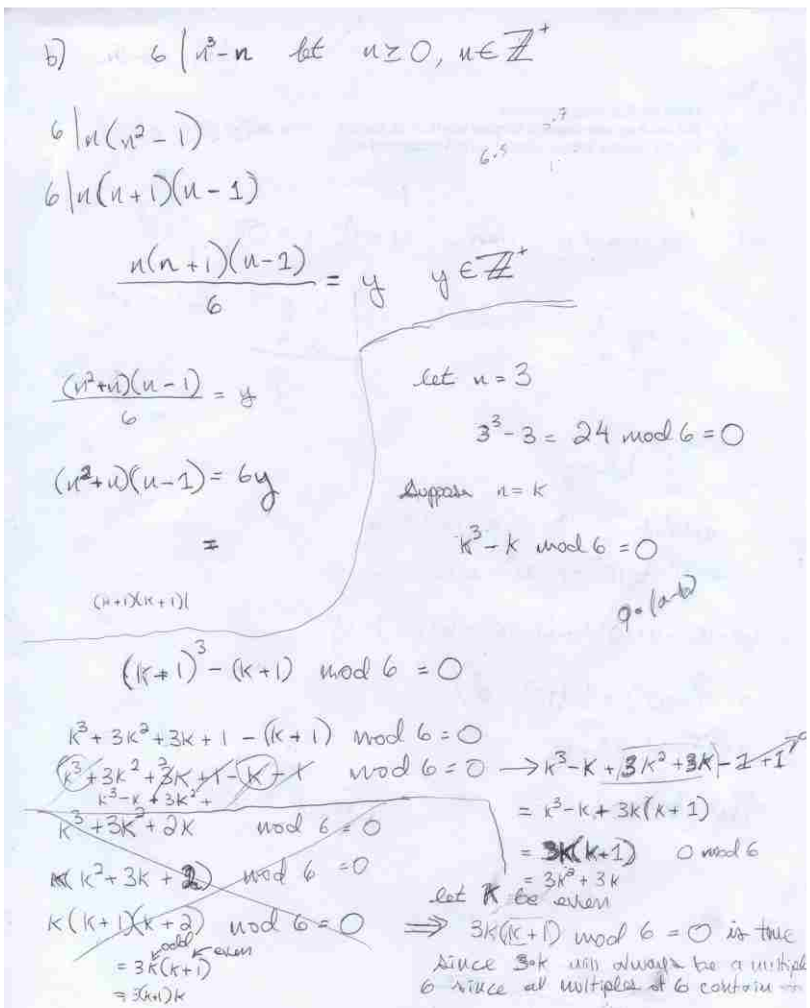


Figure 136: Michael's work on Question 2b

Michael moved to his inductive step on the bottom left of the figure. Initially, he simplified too much and lost his ability to use the induction hypothesis.

M: Well, maybe I should have stayed at the very being and see…and separate out another…because I have a k^3 right here. And then minus k, so I have the form I want. So $k^3 - k$…

(working)

M: We know that $k^3 - k$ is true, I just showed it right here, we got that form which is right here (in the induction hypothesis). So now we got the $3k^2 + 3k$, and we just ignored the 1s because they canceled out and now all we need to do is show that that ($3k^2 + 3k$) is a factor of 6. And if they're both multiples of 6, when you add them together, you'll still be able to divide by 6.

(thinking)

M: Well, we know that 3 divides anything that's a multiple of 6, 3's a factor of 6 so that would work also.

N: So, yeah.

M: Well divides those things…I'm kind of stuck here, can you give me a clue?

I suggested Michael think back to how he knew $9(a - b)$ was a multiple of 9.

M: Because it's multiplied by 9.

N: I guess that's not that helpful I guess.

M: Well, I have a three here, but the k would have to even for it to work, see? … ? But we can't tell if k is an even number or not, so…that only shows it for half of the cases. If k was even, then it would be a multiple of 6.

N: So, another hint is, you've covered half the cases, can you do something else in the case that *k* is odd?

M: In the case that *k* is odd…oh ok, that makes sense. So we go 3 times k …so we have 3 times *k*, which is 6 times the odd number which is still a multiple of 6. But if it's odd, then 3 times *k* is not a multiple of 6, but it makes *k* + 1 become even, which then, since it's a multiplied by 3, it's going to give you a multiple of 6.

Although Michael did not write it out completely, he had finished the proof.

The proof Michael provides is a process procedural proof. He uses induction, but does not follow explicit instructions every step of the way. Instead, he knows that there are a few broad steps that need to be accomplished (assume it works for *k* and show that it works for *k* + 1). He does try to explore the problem before getting into the induction proof but, as was the case with the previous question, his proof is not an intuitive understanding made into a formal proof.

Michael's proof scheme is internalized transformational. He did show at some points that he was proficient with induction. In particular, he realized that he needed to use the induction hypothesis while completing the inductive step so he started over and refrained from simplifying the expression too much. While seems to have a good understanding of proofs by induction for the most part, his use of 3 as a base case reveals that the method has not been completely interiorized yet. While reflecting on the problem, Michael said: "So I let *n* = 3, as the base case, I probably could have used anything else." Because he does not realize that the base case needs to be the smallest number for which the property is supposed to hold, Michael's proof scheme is internalized here.

<u>Question 3</u>

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$$

Michael drove right in with this problem. He tried a base case of $n = 1$ and assumed that the inequality held for $n = k$. It should be noted Michael's choice of base case. When I asked him about it later, he said the reason he chose $n = 1$ was because "(i)t's the beginning." Then, he wrote out what the $n + 1$ case looks like, noting that $2^{k+1} = 2 \cdot 2^k$.

Michael then moved into using summation notation and made sure to write it in such a way as to make use of his induction hypothesis.

M: Yeah, and since I know that part's true, I can kind of throw out that equation and not think about it for now.

N: Sure.

M: And then just get to this part where you have this summation is greater than or equal to 1/2.

Michael then explained the rest of the work he did in Figure 137:

M: and then, so I noticed that in this case, we know that from here...from $2^k + 1$ and all the way to 2^{k+1}, it's basically just 2^k doubled, because 2^{k+1}, you know, the power $k + 1$, is just the double of the original. So, you can also think of it the other way around, where you just start subtracting until you get to the original 2^k.

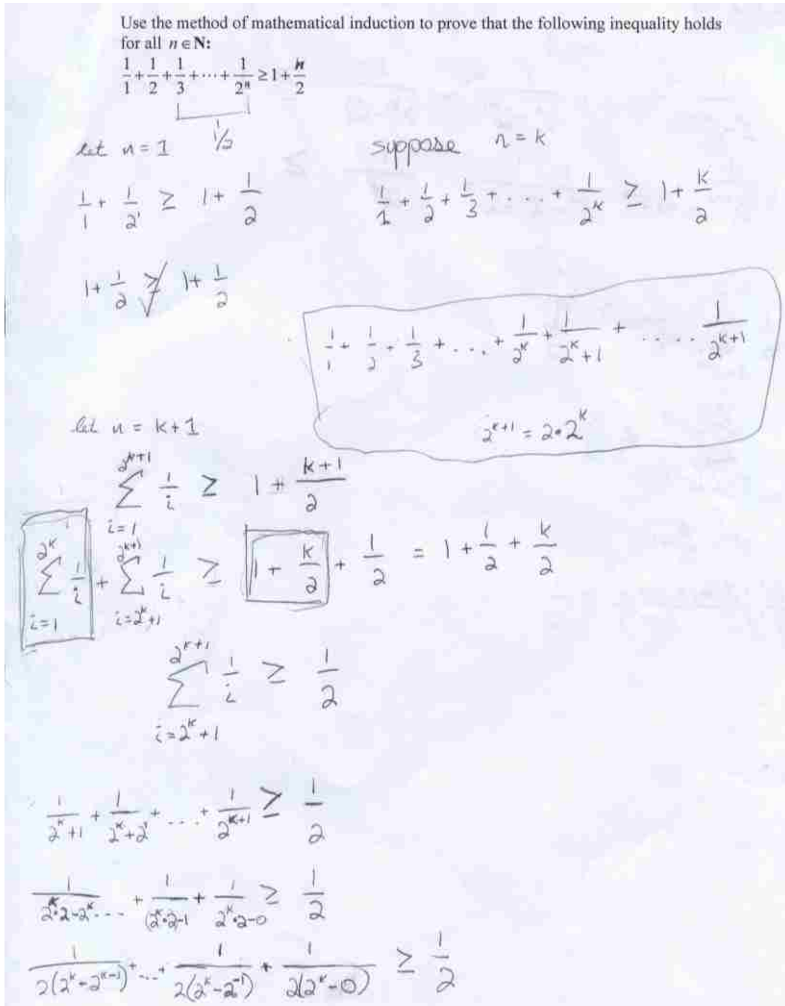


Figure 137: Michael's work on Question 3 (1 of 4)

N: Ok.

M: Instead of adding to get to the 2^{k+1}.

N: Sure, ok

M: Which is pretty nice, because if you get to that form, you can pull out a 2.

In the remaining summation he had after applying the induction hypothesis, Michael only wrote out a few terms. This was a problem because in those terms, Michael saw the opportunity to factor out a 2 and to have only the difference of powers of 2 remaining. He also viewed this as a sum working backwards, starting at $2^{k+1} - 0$ and moving "up" to $2^{k+1} - 2^k$. Moving to a new sheet (Figure 138), Michael used his factorized form to cancel the 1/2 on the right hand side.

> M: And so you can separate out the two, I guess it's kind of nicer not to deal with fractions and all that, so you get the one on this side, and you get kind of a nicer, cleaner looking form…Yeah, basically, and I wasn't sure I was going to get this out in that form, but right here, you have this 2^k, you know, and it increments. So, I guess the next part is to show…I guess I'll have to think about this thing here.
>
> N: So before we get too far into that, but keeping going, so what made you think you were done here? What did you notice, or what did you do to think you were done?
>
> M: Well, it was, you know, it was getting the same form as the original, I don't know. I was thinking, you know how the original was 1/1 + 1/2 + 1/3.
>
> N: Right, ok
>
> M: Well, you know, it's incrementing in the exact same way as that, but…I was thinking if there was a possible way to show the summation of all those, all the way is greater than or equal to 1.

Michael had gotten to end of the third line of Figure 138 and said he was finished, which prompted the exchange above. He then completed the work seen in the figure below. It is evident that Michael still has his mistaken conception of the way his remaining sum is behaving when 1/2 is factored out of each term. This idea, though, is important to the way he was viewing the problem at the time: "Just kind of…probably so it would be easier to get an original value of k out again." His goal was to get the sum of terms left over from applying the induction hypothesis into a form where it again, or as many times as necessary.

M: I was trying to make it go recursive. If it was recursive, then it'd be pretty easy to show that it just repeats over and over again. And then if it's true for all the repetitions, then you can just assume that's it's going to be all the way true until it …

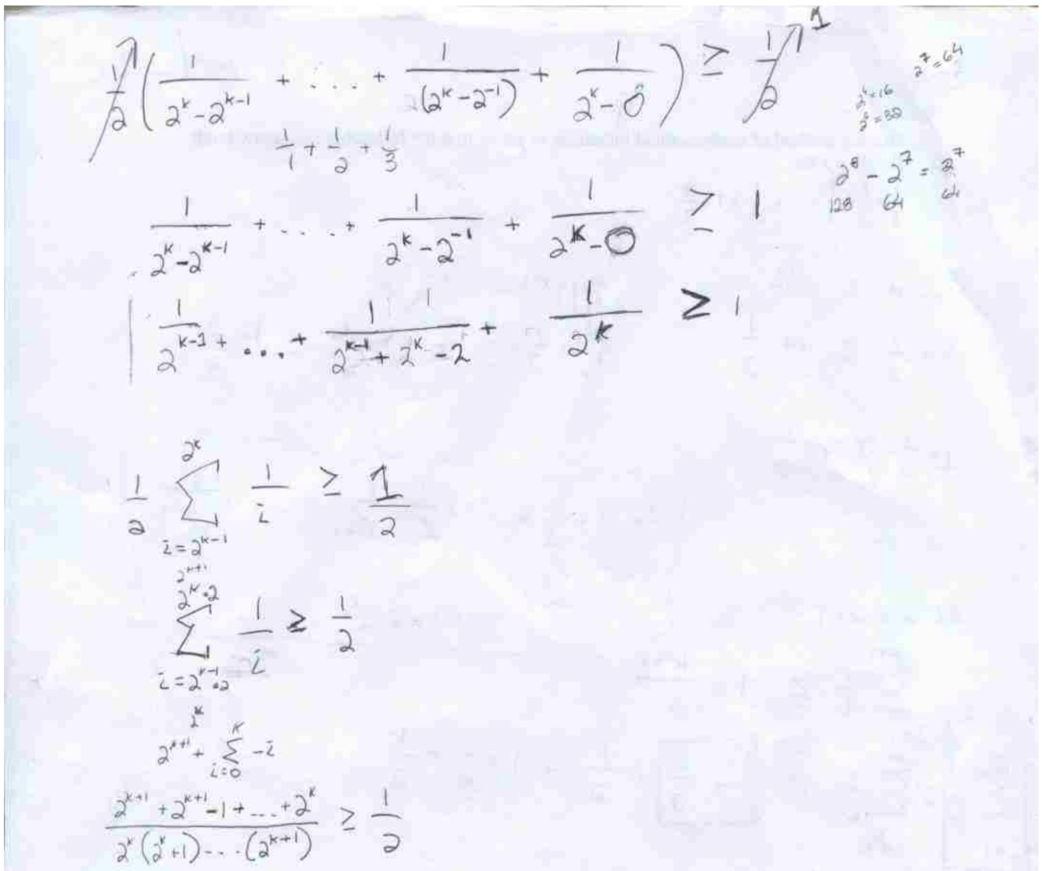


Figure 138: Michael's work on Question 3 (2 of 4)

Michael's last attempt in that interview involved the algebraic manipulation at the bottom of the figure. At that point, it was the end of the interview and I asked Michael to work on the problem before coming back for the next interview.

Michael did not have time to look at the problem between interviews, so he started fresh again the next time we met. The new work he did is in Figures 139 and 140. He started out similarly to the last time, writing the sum on the left in two pieces to make use of the induction hypothesis. As we were talking about the fact that moving to the $k + 1$ case doubles the number of terms, he mistakenly applied the idea of doubling to the right hand side. He quickly fixed this, though, lining out what this assumption would mean.

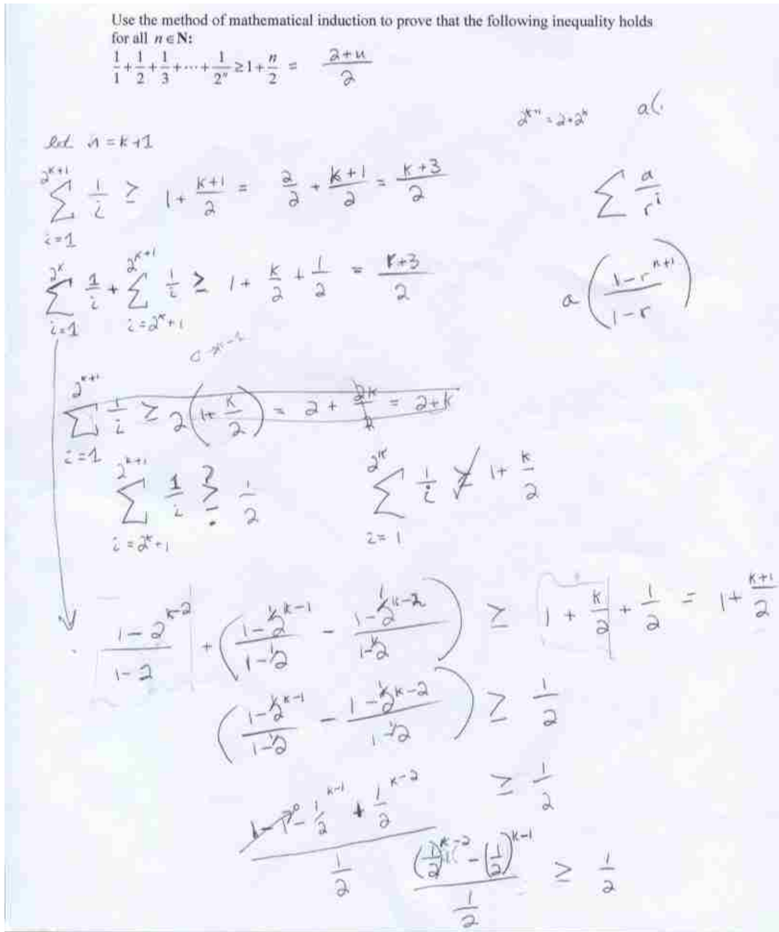


Figure 139: Michael's work on Question 3 (3 of 4)

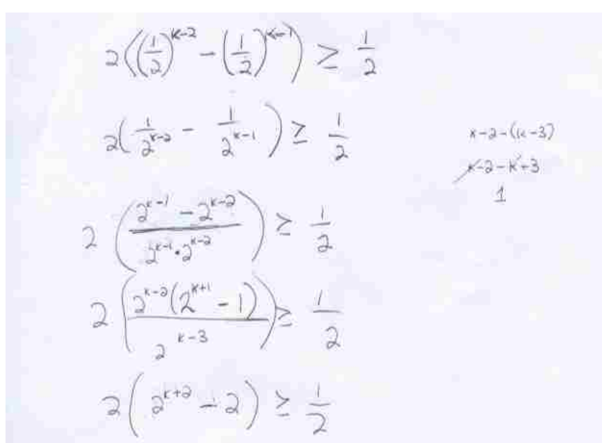


Figure 140: Michael's work on Question 3 (4 of 4)

After fixing this mistake, Michael came up with a new idea: "I could just show, using a geometric sum, where you would just find the geometric sum of 2^{k+1}, subtract 2^k and if that is less than 1/2…let me see this…do you know what the equation is for a geometric sum?" His idea was to apply the formula for a geometric series to both sums from $i = 1 \ldots 2^{k+1}$ and $i = 1 \ldots 2^k$ and subtract the later from the former. Then, he would be able to compare that difference to 1/2. Michael worked his way to the bottom of Figure 139 and said

M: Ok…I feel like I messed up on my algebra here back somewhere, but…it should…the concept…Ok, well we know this part right here will be positive because *k*, $(1/2)^{k-1}$ is smaller than $(1/2)^\wedge{}^{k-2}$, so…could probably write that nicer. So that part is definitely greater than 0, therefore it's still a positive number times…oh, wait, I guess I could factor out *k* right here, couldn't I?

(continues from the first line down n Figure 140)

M: I guess I would just quit right here because I see it, but I messed up somewhere on my algebra…

N: So let's see here…ok, and you're pretty confident (you would not get any further alone)?…so what if I told you, then, that this is not a geometric series?

M: I guess that would be completely…oh yeah, that's 1 to 2 to 3, I completely forgot about that.

N: Yeah, there needs to be a common ratio for a geometric series, so sorry about that, but I just felt like I had to let you know. So you still have 10 minutes of your time, that I said I'd let you work, I guess, so do you want to go back and look at again, or would you just rather talk about it?

M: I'd just rather talk about it.

Michael had become frustrated with the problem, so I talked him through the solution with the time that remained.

Michael did not complete a proof here, but I will classify his proof attempt. The work Michael provides constitutes a process proof. He used the method of mathematical induction so his attempt is clearly of the procedural type. Of the sub-types, this attempt falls under the category of process because he sees induction as a few steps to be accomplished, not as explicit directions to be followed. This can be seen in the variety of ways he tried to complete the induction hypothesis.

While Michael does not complete the proof, he does show some progress with induction when this attempt is compared to his previous proof. The most tangible evidence that shows Michael knows more about induction this time is in his use of $n = 1$ as the base case. Not only did he use the correct value, when I asked him about it he said he did so because it was "the beginning" not because it was the easiest to calculate or some other reason. This shows more understanding of induction than Michael displayed last time. This is not to say that Michael definitely learned more about induction than last time, just that he is now showing more of an understanding.

Also, when working on the last question, Michael did not look to apply the induction hypothesis right way. He did do so quickly after simplifying too much last time, but there was no hesitation to make use of it with this question. Because Michael shows no issues with his understanding of induction, he is displaying a transformational (interiorized, not internalized) proof scheme.

<u>Question 4</u>

Examine the following proof by contradiction:

<u>*Prove*</u> $\sqrt{-1} \leq 0$

Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \leq 0$.

<u>*Prove*</u> $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \leq 0$, *then* $-\sqrt{-1} \geq 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \geq 0$, *so* $-1 \geq 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

Michael saw this question for the first time when he took his midterm exam in MATH 305. The response he gave then is in the figure. Basically, Michael says that the reason the proof is not working is because it is not working and that there is no other flaw in the proof.

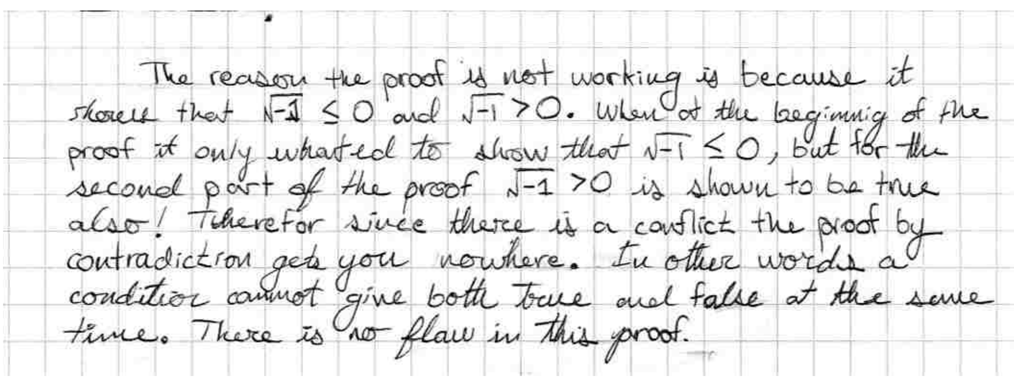

The reason the proof is not working is because it shows that $\sqrt{-1} \leq 0$ and $\sqrt{-1} > 0$. When at the beginning of the proof it only wanted to show that $\sqrt{-1} \leq 0$, but for the second part of the proof $\sqrt{-1} > 0$ is shown to be true also! Therefor since there is a conflict the proof by contradiction gets you nowhere. In other words a condition cannot give both true and false at the same time. There is no flaw in this proof.

Figure 141: Michael's previous work on Question 4

Michael read through the proof a few times and said: "Why it's not working? Let me see…When I first did it on the test, I just said because you're trying to prove something that was…like you can't…if you were trying to prove something in the imaginaries by using normal numbers…" Next, he recalls some things he knows about complex numbers, specifically the complex plane and what can happen to complex numbers (and their signs in particular) when you take them to various exponents.

After this discussion, Michael said: “Maybe it’s because they’re thinking, see, if you come back to the real numbers, maybe they’re thinking in terms of real numbers, maybe we have to apply new properties of imaginary numbers and not really…” He is referring to the proof’s use of real number properties on complex numbers.

M: (Y)ou’re trying to prove something with the incorrect tools which won’t work.

N: Ok, and in this case the incorrect tools being …

M: The real numbers…

N: …their operations?

M: Yeah, they’re trying to but all the properties of the reals onto the imaginaries, which not all of them are true for the imaginaries.

N: Ok, but some might and some don’t?

M: Like the ones for addition and subtraction will always hold true.

N: Ok

M: But for multiplication it won’t hold true, the i will always alternate between -1…

N: Depending on the power of the i.

M: Yeah.

N: Ok

M: So that can cause problems, so if you’re going to prove that way, you probably won’t even, it’d probably be best to just stay away from multiplication in these proofs, actually. It’d probably be better trying another way to do it, that way I’m not really sure…seems like it’s be better to do it that way.

Michael was able to put his idea more concisely when talking about this question in the following interview: “I think I compared it to tools, using your tools. You know how, for example, you can’t draw hyperbolic geometry using normal Euclidean tools.”

Michael and spent the rest of the interview discussing ideas on how one could set up a system to make comparisons between real and imaginary numbers. He settled on projecting them onto the real line. He also said that in any proofs involving comparisons, he would only use addition and subtraction.

Because Michael did not attempt a proof for this problem, there is nothing to classify. However, the interview did highlight Michael’s transformational proof scheme. His focus while talking about the proof provided was on the operations executed within it. He is not swayed by the fact that the proof looks like a proper proof. Instead, he considers the logical deductions performed and whether or not they should have been applied. This focus on algebraic manipulations is typical of a transformational proof scheme.

Question 5

The next interview was the last of the semester, so I used it to discuss the study with Michael. He did not attempt a proof, so there will be nothing to classify. The interview did give some insight into Michael’s proof scheme and underscored some of the things observed during the first half of the study.

At one point in the interview, I asked Michael what it took to successfully complete a proof. At first, he was not sure what I meant:

N: So if you sit down with a proof, and you complete it, you see it all the way through to the end, even if you don't get it all done in one sitting, say you eventually successfully complete the proof. What do you think it takes to get that done?

M: Well, even if I get a proof all the way through to the very end, I don't consider that successful, even if it is correct.

N: Ok

M: I like to write the thing again nicely so it's readable.

N: Sure

M: I don't know, well, that's basically what a proof is, it's just an essay in math…You have your conclusion, your body, I mean your intro, body and conclusion.

There are couple things of note here. First, Michael's view of proof is like an essay where an argument is presented by first saying what you are going to discuss (introduction), provide your evidence (body) and then put your evidence together (conclusion). This matches up nicely with his formal view of proof. For each of the questions in the first half of the study, Michael displayed an analytic proof scheme as he does here.

Secondly, Michael does not consider a proof successful even if he sees it as correct. Additionally, a successful proof must be one that he understands:

M: Actually, one extra thing for that part is when it's successful, even if I write it up nicely on paper, if I can think about it and completely visualize everything, and understand it, I'd consider it successful.

N: Ok, so success is not defined necessarily by a grade, but by how well you understand it?

M: Yeah, because there've been proofs I've done and it didn't, even though I couldn't visualize it, the logic just pointed to that answer and I was like "Ok, I'll call it a good proof."

N: Ok. So, but in your mind, that's not a successful proof because you don't, you didn't get some understanding out of it?

M: Yeah, it didn't feel that satisfying.

N: Ok

M: That's basically, yeah, if it's successful, it should feel really, the satisfaction at the very end.

This also matches what was seen during the first semester of the study. So far, Michael has only produced one semantic proof. Thus, he is not trying to turn intuitive understandings of problems into proofs. Instead, he waits for the understanding after. If the proof leads him to understanding the problem, then it is a successful proof. Then, that new understanding yields flexibility: "If you understand, you can usually easily put it in other proof forms without too much effort." Later, I asked Michael what was helpful in successfully completing a proof, but not necessary. He said that "what helps is probably writing a bunch on paper, doodling around, throwing ideas around, getting the ideas out." I think the fact that he sees exploring a problem as helpful rather than necessary also fits with the work he did earlier in the study (both in what he thinks it takes to make a proof successful and in his willingness to start over when stuck).

Question 6

Prove that $\forall n \in \mathbb{N},\ \ n \equiv n^3$ *(mod 6).*

After reading through the problem, Michael realized that he had seen this problem before: "That's weird; it looks like I've already done this one before…mod 6." He was right, of course, this is Question 2b in another form. Michael still went to work on the problem, first trying it for n = 1, 2, 3 and then moving into an induction argument.

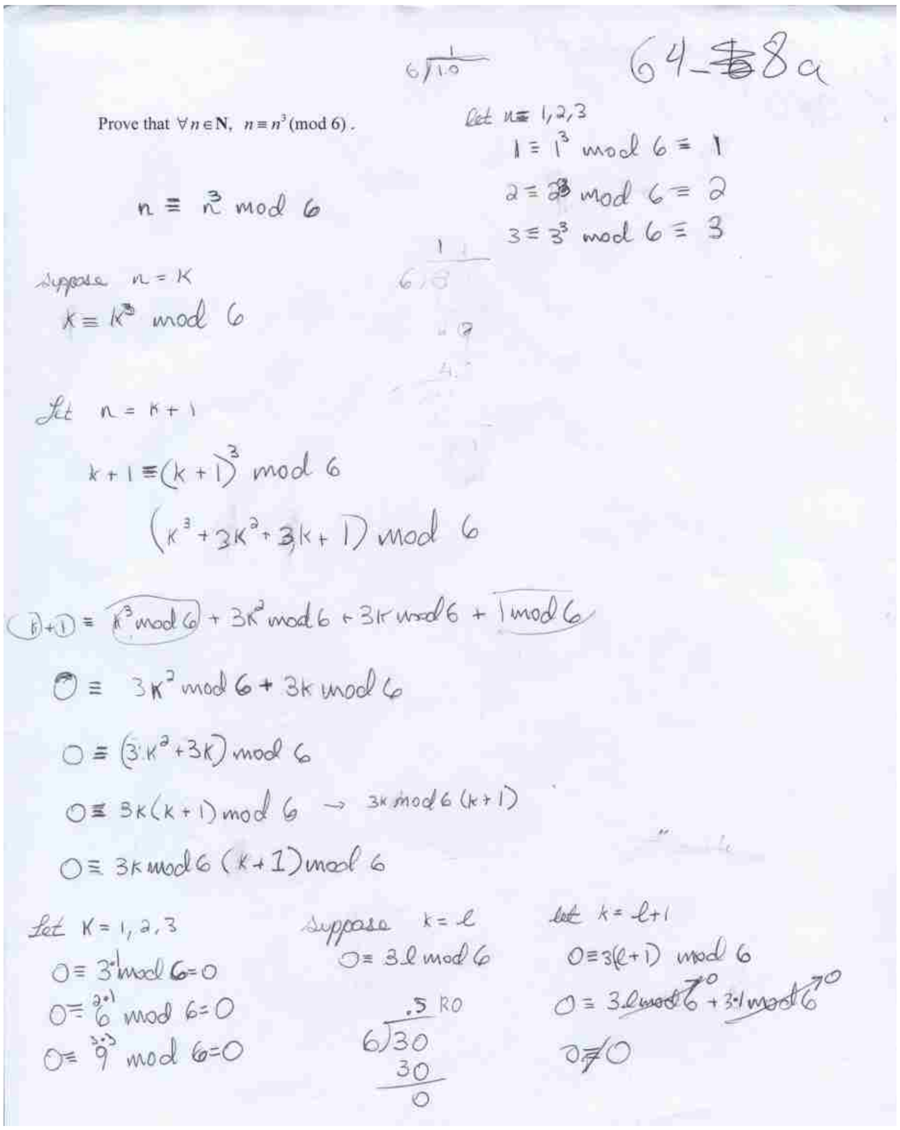


Figure 142: Michael's work on Question 6 (1 of 2)

Michael got down to the bottom of his induction argument and was somewhat stuck.

M: So now all we have to do is prove that 3 timesed by any number or just k alone, would also be 0 when you moded it by 6. Then you'd also have to show that 3 timesed by any number plus 1 also, the same thing, would also be 0 moded by 6… And then I get to here. [3*k*(*k* + 1)] Well, I thought to myself "Ok, yeah, I can just separate everything and do all that." And then since it's 0, one of them has to be 0 if we were multiplying. So, it was just left down to kind of guess which one would be most likely to be 0. So 3 is usually a factor of 6, it was the most likely one if you were to mod it by 6, 3*k* mod 6, it's most likely to be 0.

At this point, Michael showed that he had a misconception about modular arithmetic. When trying to decide what you get when reducing 3 (mod 6), he allowed decimals into the modular system. Because you get 0.5 (with nothing left over) when dividing 6 into 3, Michael concluded that $3 \equiv 0 \pmod 6$. In turn, this led him to conclude that both 3*k* and 3(*k* + 1) reduced to 0, finishing the problem.

I pointed out the mistake he made and he began working on a new sheet of paper. At first, he tried a few examples to see it working and then he remembered the other problem.

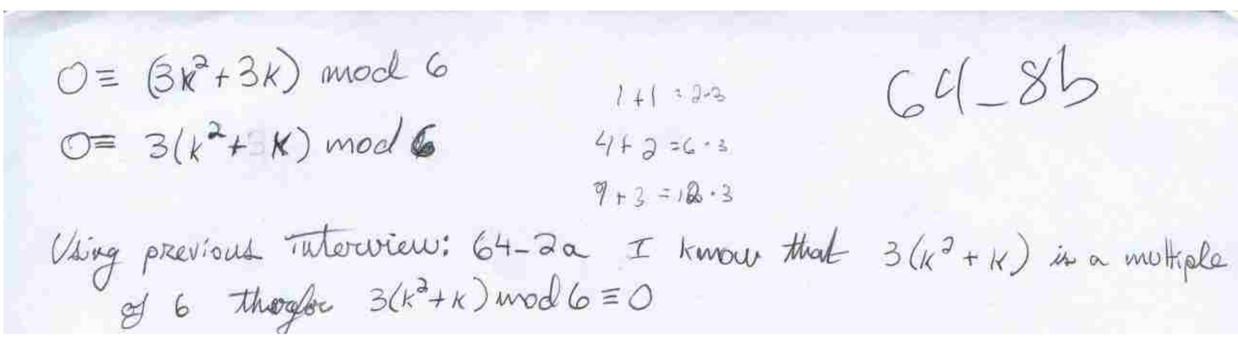


Figure 143: Michael's work on Question 6 (2 of 2)

M: I'll probably, might, have to show that this whole number [3*k*(*k* + 1)] here would always be 0 if you mod it by 6.

N: Ok

M: That's right, I'd probably have to show that it's something of 6, the value has a factor of 6…Can I look back to the earlier ones? Because I…isn't there one were we did where you actually…

N: Yeah, if you want to go back to what you did before, that's fine.

M: It doesn't, like, because I really don't feel like re-thinking that one over again.

Michael looked back through his old work and found what he had done to finish Question 2b (the citation he gives refers to the way I labeled that page of his work).

Seeing his old work gave Michael even more insight into the current problem:

M: They are almost, actually let me see…actually you could say that, see, look it, if you just did that, this way around. If you just made this 0 and moved it over here, it'd be the exact same thing.

N: Right. Yeah, that's exactly right.

M: So that's, I could have just basically did that.

During the reflection, we talked about the way he finished the problem before and it did not take him long to remember:

M: Yeah, I think I just did another little thing by induction for it or maybe not.

N: I don't remember, so…

M: Yeah, and then I was showing for even and odd which I'm assuming, yeah, because then if it (k) was odd, these (k and $k + 1$) would alternate, you know, k+1 would be even.

The proof Michael provides here is a process procedural proof. Like the other times he did an induction proof, it is clear that Michael understands the method well enough to see it as a few global steps rather than step-by-step instructions. This is most evident by the way he handles the inductive step. While other students would at times lament the fact that the problem they were working on did not lend itself to standard algebraic simplification, Michael is flexible in how he approaches the inductive step and does not simply rely on algebra.

Michael shows a couple different first schemes here. First of all, he successfully performs a proof by mathematic induction and shows no evidence that he has anything but a full understanding of the method. With this, Michael is displaying a transformational proof scheme. Michael also shows evidence of an axiomatic proof scheme in that he relies on a previous result to complete his proof. The acknowledgement and use of previous theorems or results is a hallmark of the axiomatic proof scheme.

<u>Question 7</u>

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

After reading over the problem and going over some terminology, it did not take long for Michael to know the route he was going to take:

> M: So if I'm thinking about this correctly, you're saying *n* elements, that's the *n* subsets, right?
>
> N: No, that's *n* elements. So if you had the numbers 1 2 and 3, the set that has just 1 2, 1 and 2 in it is a subset of the set that has 1, 2 and 3.
>
> M: Oh, ok. I see.

N: Yeah, so in that case, the set has 3 elements, so in that case *n* is 3. So how many subsets can you come up with?

M: I see what you mean. Yeah, so it's going to be combinations of elements that are contained within the set.

N: Right, yeah.

M: Well, then this becomes a combinatorial problem. Because if you don't consider order as important within sets…

Michael began started to work and we went over some notation conventions and then he asked about the empty set.

M: Alright, in this one, do we consider the null subset a part…

N: That's sort of up to you.

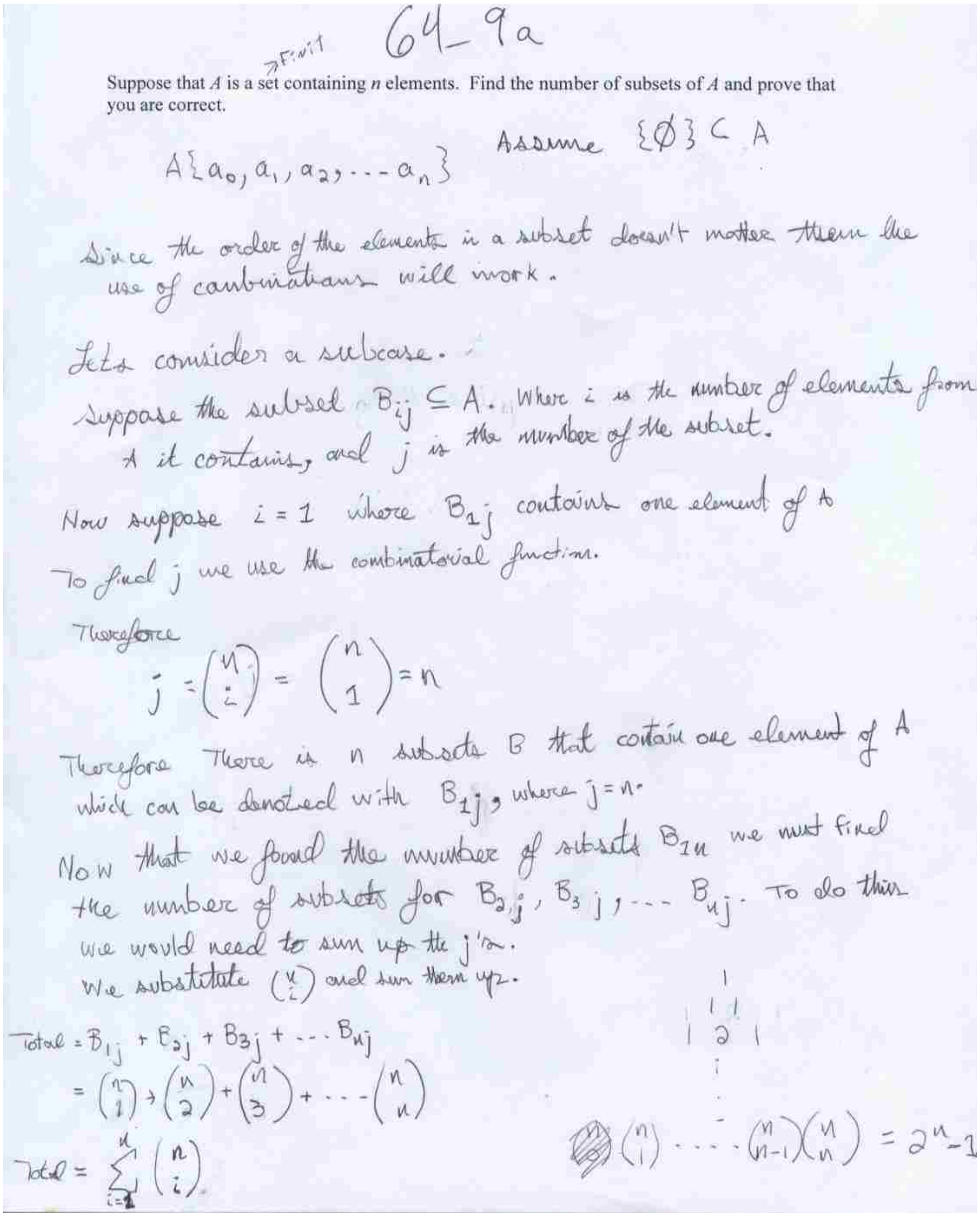
M: Ok, so I'll just state that myself later.

N: Ok. Yeah, that's one of the things I wanted you to wrestle with: whether or not the empty set is a subset.

Michael took what I said to mean that it was up to him to decide whether or not to count the empty set a subset of a different set. He did not initially, as one can see in the figure, but I did address it during the reflection.

Michael then went to work, completing most of what can be seen in the figure. When he had finished, I asked him to explain what he had done.

M: Maybe I should, I'm going to assume that the null set is not…Alright, well then, based on that assumption, then I continue. It kind of gets to a little bit of statistics, or



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Finit

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

$A\{a_0, a_1, a_2, \ldots a_n\}$ Assume $\{\emptyset\} \subset A$

Since the order of the elements in a subset doesn't matter then the use of combinations will work.

Lets consider a subcase.

Suppose the subset $B_{ij} \subseteq A$. Where i is the number of elements from A it contains, and j is the number of the subset.

Now suppose $i = 1$ where B_{1j} contains one element of A

To find j we use the combinatorial function.

Therefore

$$j = \binom{n}{i} = \binom{n}{1} = n$$

Therefore There is n subsets B that contain one element of A which can be denoted with B_{1j}, where $j = n$.

Now that we found the number of subsets B_{1n} we must find the number of subsets for $B_{2j}, B_{3j}, \ldots B_{nj}$. To do this we would need to sum up the j's.

We substitute $\binom{n}{i}$ and sum them up.

$$\text{Total} = B_{1j} + B_{2j} + B_{3j} + \ldots B_{nj}$$

$$= \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots \binom{n}{n}$$

$$\text{Total} = \sum_{i=1}^{n} \binom{n}{i}$$

$$\binom{n}{1} \ldots \binom{n}{n-1}\binom{n}{n} = 2^n - 1$$

Figure 144: Michael's work on Question 7

what do you call it, probabilities, combinations, where it doesn't really matter the order the numbers are in. The set, if the set contains in a different order the same numbers that are in a different set, technically they are the same set.

N: Right

M: So I can use that right there to know that one equation I can use to find the numbers is the combination right there.

N: Ok

And then I have to take another thing in mind, is that there are sets of 1 to *n* size within *A*.

N: Ok

M: Therefore I can safely assume if I sum up all those combinations from 1 to *n* I should get the number of subsets in *A*.

N: …So do you feel like you've proved it, then, or do you feel like you've just sort…you know, do you think it requires more proof or do you think it's as proved as it needs to be?

M: I think it's as proved as it needs to be. I don't see much more to it. It didn't have any other questions in it, just the number of subsets.

In his work, he took the sum to go from $i = 1$ to $i = n$, again because he chose not to consider the empty set a subset. Also, after he finished his explanation, I mentioned a different formula that could have been used which explains the rest of the work seen in the figure.

N: So what if I said that the number of subsets was 2^n.

M: 2^n? Oh yeah, because that's what do you call it? Pascal's Triangle.

Unlike many of the participants who came up with the summation formula for this problem, Michael was confident that explaining where it comes from is a proof. The proof he provides comes from his understanding of how subsets of a given set can be formed. Because he turns an intuitive understanding of the problem into a proof, this proof is semantic.

Michael's proof attempt is transformational here. His proof is logically deduced and does not rely on previous results or theorems. Another aspect of what Michael did that is typical of transformational proofs is the fact that early in the proof, he set himself up for what was to come. When I asked him about why he chose to handle the sub-case the way he did, Michael said that he had an eye towards what was to come.

> N: I'm talking about the, when you're coming up with the number *n*. You could have just said *n*, right? You could have said that there's *n* things and so if you take them one at a time it's *n*.
>
> M: Yeah I did that, I wanted to show that there was a function being used to find n.
>
> N: Right, so you did sort of have an inkling of what was going to come?
>
> M: Yeah
>
> N: It wasn't, I was just sort of wondering if you had an intuition that said 'Include this' but you didn't know why yet, but it sounds like you did know why and had an idea of what was going to come.
>
> M: Yeah, I thought of at the beginning, did I put the combinations? Yeah, I put the combinations right there, and I had to make use of the combination function in order to….early, so I could show them being summed up together.
>
> N: So you had sort of a rough outline of the rest of this even at this point already?
>
> M: Yeah, basically

This type of anticipatory action is a characteristic of a transformational proof scheme.

<u>Question 8</u>

Prove that the cube root of 2 is irrational using a proof by contradiction.

Like Question 4, the problem was on the midterm Michael took in MATH 305. At the time of the midterm, Michael had already seen the proof that $\sqrt{2}$ was irrational. The proof Michael did on the take-home exam is in Figure 145.

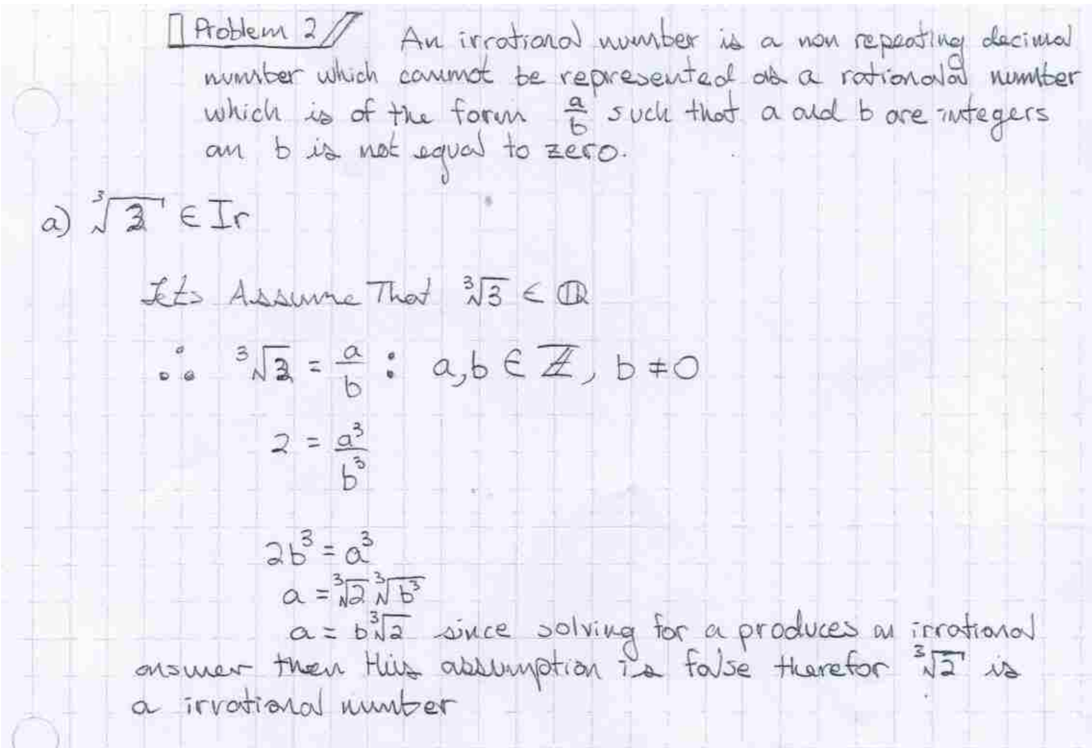


Figure 145: Michael's previous work on Question 8

In his test response, Michael apparently uses what he is trying to prove. He states that $b\sqrt[3]{2}$ is irrational using the fact that an irrational times an integer is irrational.

In the interview, Michael was not much more successful. As he did on the midterm, he started the problem correctly by making the proper assumption. He did not make the same mistake, though. Instead, he tried numerous different algebraic manipulations to try to arrive at a contradiction. Due to algebra mistakes, Michael did think he had arrived at a contradiction a couple different times. The first time was at the end of the blocked out work in Figure 146.

M: Ok, so assume cube root of 2 is rational. Therefore, in that case, if it's a rational, the cube root would be *a*/*b* where *b* is not equal to 0. Ok, so let's cube both sides. 2 is an element of a^3/b^3…let's make this equal, that'd probably be a better thing to do…Well a^3 in this case would also have to be a rational. b^3 is going to have to be a rational that's also not 0. Let's say we solve for like, let's say, a^3. So…So therefore a is equal to the cube root square…b…Which in this case, we get a little dilemma because it kind of ends up being recursive in a way, if I'm correct…

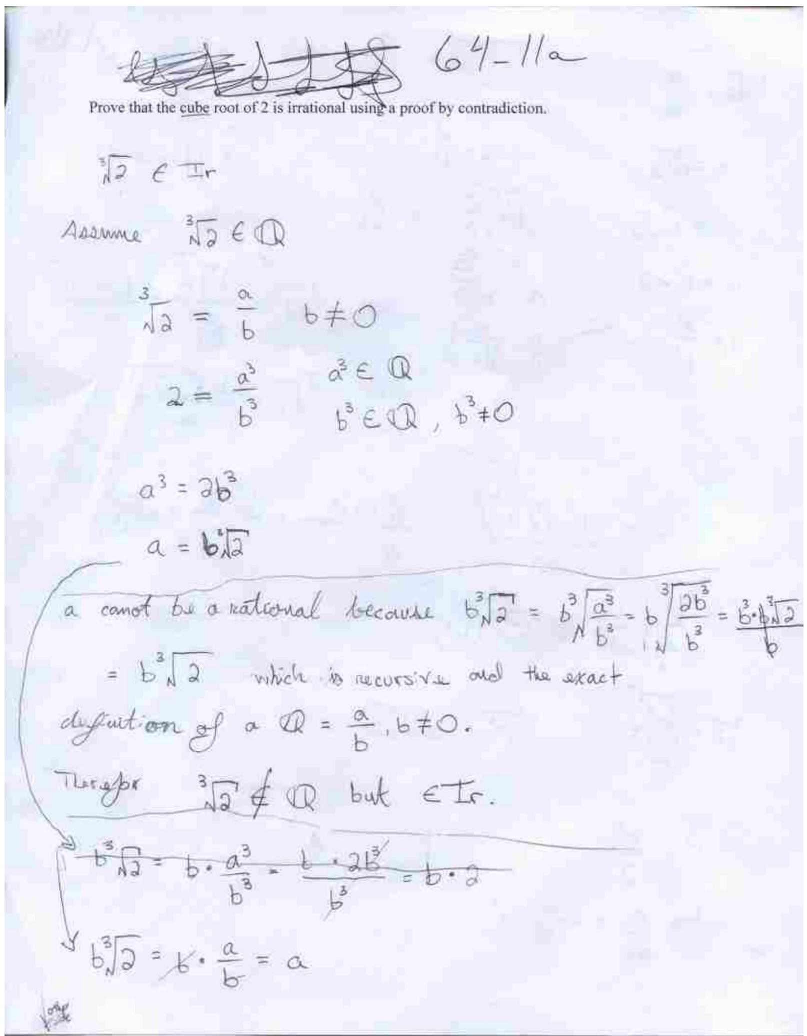


Figure 146: Michael's work on Question 8 (1 of 3)

Michael was not able to fully explain what he meant by "recursive" but he apparently saw getting back to where he started as a problem.

N: So, I guess I don't see why this (blocked out work) is a problem, right? Because if you're going to do a contradiction, you have to find a problem. And to say that *b* times $\sqrt[3]{2}$ is equal to b times $\sqrt[3]{2}$, I don't think that's a contradiction, is it?

M: Yeah, I guess that isn't contradictory, but…I was trying to see if there was a contradiction forming in here (top line, on the right, in blocked out work) you know.

There is also the issue of the superscript "3" in $\sqrt[3]{2}$ being misinterpreted as an exponent on *b*. When Michael thought about what I had said, he noticed this mistake. "Ok, I see where I went wrong on this. First of all, that should not be the cube root of *a*/*b*. I'm just going to put a line over this because I messed up on my algebra." He then finished the work in Figure 146 (at one point remaking the mistake he had just mentioned) and moved on to Figure 147.

On the new sheet, Michael continued to rearrange and substitute and simplify to come to an equation that was obviously untrue. He thought he had a few times, but realized that he had made an algebra mistake at some point in each instance. By the time he had completed the work in Figure 147, he realized that he was going in circles: "Ok…Well, I'm making this recursive also, just making it go like this continuously."

I asked that he continue to work on the problem before the next interview and he said he would. He did not look at it by the time the next interview came around, but I showed him where the proof that $\sqrt{2}$ is irrational was in the MATH 305 textbook and he said he would read through that and come with work for the following interview. This time, he was able to work on it. He did not provide a completed proof but instead what is in Figure 148.

M: I didn't write any logic down, I just kind of wrote it down to compare it to the book.

N: Sure

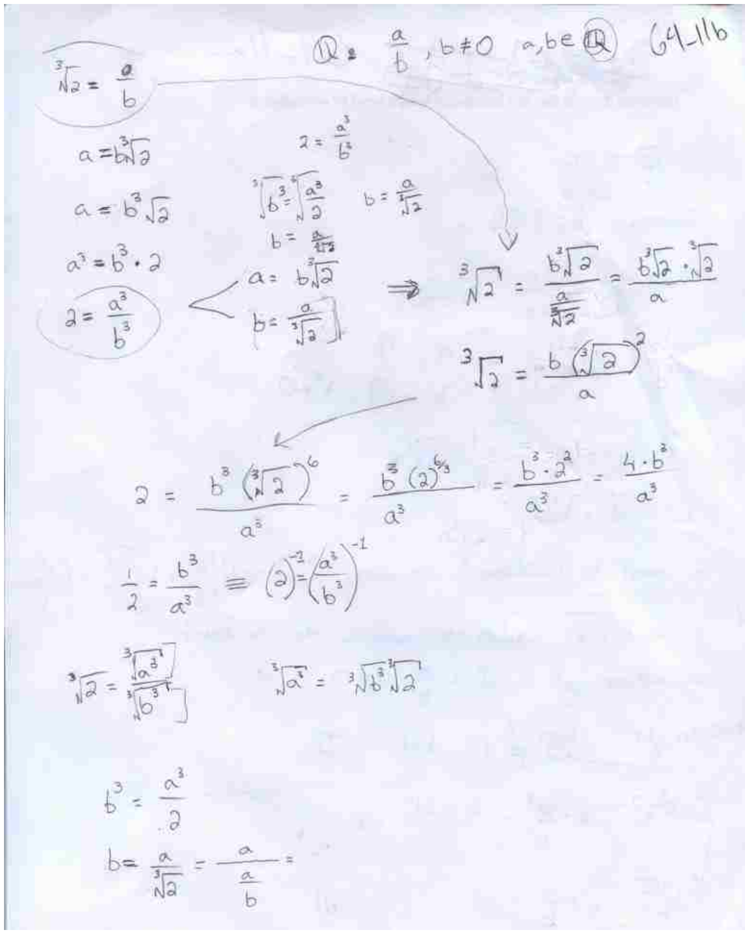


Figure 147: Michael's work on Question 8 (2 of 3)

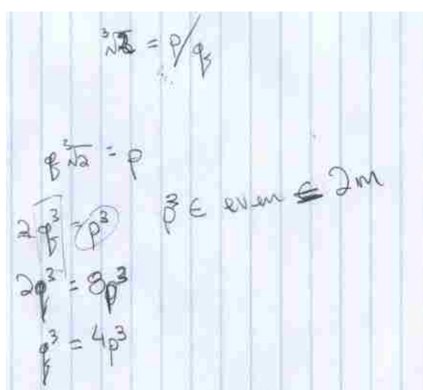


Figure 148: Michael's work on Question 8 (3 of 3)

M: And I guess it works the exact same way the book does it, if I'm correct, there's not much difference at all.

N: Yeah

M: Except the fact that everything's cubed instead of becoming, you know, squared. The 2, 4, becomes to 8 and…

The work Michael brought in was only the notes he used when following along with the proof in the book. Even so, I consider this a completed proof because he understands how the proof works.

Because he followed the exact steps in the book, Michael produces an algorithm procedural proof. Clearly, I do not think this means the steps were not meaningful to him. He was able to recognize the differences between the two proofs but that they were similar enough to allow for him to transfer most ideas over.

Although Michael is strongly influenced by the form of a different proof, he is not showing evidence for an external conviction proof scheme here. Because he understands the logical deductions that form the steps of the proof and how they apply to the proof he is doing, Michael is displaying an analytic proof scheme. The analytic scheme is transformational he because it relies on mathematical operations and not previous mathematical results.

<u>Question 9</u>

For this question, I had Michael evaluate a version of Cantor's Diagonalization argument. He did not produce a proof attempt to classify, but the interview was used to look for clues regarding his proof scheme.

Michael read through the proof a couple times and we discussed what was going on in the proof.

M: Ok, so…ok, so let me understand this, ok, so B is constructed technically from this kind of diagonal set right here, with all those α's."

N: Right

(goes back to reading the proof)

M: I see, I was doing this wrong, somehow I was assuming…I was confused….

(thinking)

M: It kind of makes sense, but at the same time I'm kind of confused about it, so…

N: Ok, oh sorry, go ahead.

M: Alright, so you've got *f*(*k*), from you contradiction, is, from what it sounds like so far, it sounds like the proof is, you can't see any problems with it but yet…Anyway, you have this set (the diagonal) right here and this…

N: Yeah…

M: …and this one right here gets kind of derived from this right here, whether it's 1 or 2, if I'm correct. So that kind of should, this whole diagonal set here should map also to like…β_1, 2…so should also map to β_4, but…supposedly when you get to β_k, it's not equal to it.

(thinking)

M: So I'm still kind of confused, I don't understand why β_k can not be equal to α_{kk}. Kind of confused about that, so…

N: Well, it's because β_k, right, to find out what β_k is, you have to look at α_{kk}.

M: But you don't know what α_{kk} is, right, you have…

N: Well, this list, after you have this list, then you construct this β_k.

M: Ok

After this discussion and some more thought, Michael began to see how the proof comes together.

M: Ok, it just made sense, I see how it works.

N: Ok, can you explain it a second?

M: Basically what you're saying is…ok, from this part right here, β_k can never be equal to α_{kk} because if it's 2, then β_k will have to be 1.

N: Yeah

M: But if α_{kk} is not equal to 2, then it'll have to be 2. In other words it must avoid being equal to, it cannot be equal to a…β_k and α_{kk} can not be equal no matter what, so this leads to the contradiction and that *B* and *f*(*k*) will always be off by 1 digit.

N: At least 1.

M: Somewhere done the line for infinity or beyond.

N: Yeah

M: And…basically you say they're equal in the first place right there, so they don't map to the same function. But then that leads to another thing that I could say makes this not work because I could say…well, actually, this is the…actually it would still never work for no matter what *f*(*k*) is, *B* wouldn't, you compare it to whatever, it still wouldn't be part of the set. And that would still be a real number…Yeah, it works I'd say.

Michael's proof scheme he is transformational. Most of our discussion focused on the notation in the problem and how objects and functions related to other parts of the proof. Also, it is clear that Michael is showing an analytic proof scheme here because he waits until he understands the proof to decide whether or not he believes it. He does not make this judgment based on how it looks or the fact that an authority figure gives it to him. At the same time, there was nothing in the interview that suggested that Michael had an axiomatic proof scheme. Thus, Michael's proof scheme here is transformational.

Question 10

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

Michael began this problem by looking at a picture in which he divided what he later said was the unit interval to see what was going on with the problem. He then wrote out the inequality for a general case. While doing so, Michael thought of the counter-example of 1/2, 2/2.

> M: Well, I don't know if…if this would actually, I don't know if you have set rules for this, but in this problem, would you assume 1/1, do you want to assume 1/1 is a considered a fraction because there's a fraction, because if I use 1/2 as…
>
> N: Right, I'm talking about rational numbers in general, so if you know, it looks like you're considering 1/1 a rational number, right?
>
> M: Yeah
>
> N: Yeah, so that's fair game.
>
> M: Which, in that case would break everything. I guess it would be an incorrect assumption unless you place the fallen assumption that it can't be 1/1 or the fraction had to be greater than, the denominator had to be …

I then asked Michael if there were reasonable restrictions that he could place on the problem that would make it true. He decided that he would make the restriction that the denominator you start with (the m value) would have to be greater than 2. From there, Michael worked silently on all the work in Figures 149 and 150.

At this point, it would be helpful to describe the order Michael completed his work. From the place he wrote "New Theorem," he worked on his general inequality straight down, then to the right where he wrote "$am = am$" and the line below it. Next, he followed the arrow back to the left and then wrote the last three lines in the bottom right of Figure 149. When he got to the bottom of the figure, he went back up to the top to look at an example ($1/4 < 1/3 < 2/4$). He used cross multiplication to check his work and that gave him the idea to look at his general inequalities on a number line in integer form.

At some point during this work, Michael got the idea that is shown in Figure 150. During the reflection, he explained what he was thinking.

M: Yeah, I got to a lead on something, I'm not really sure how to prove it, but you know, when you consider the fact that the denominator can only be 1 less…

N: Ok

M: It means that either the denominator, the number in between, this denominator is either going to be even or odd.

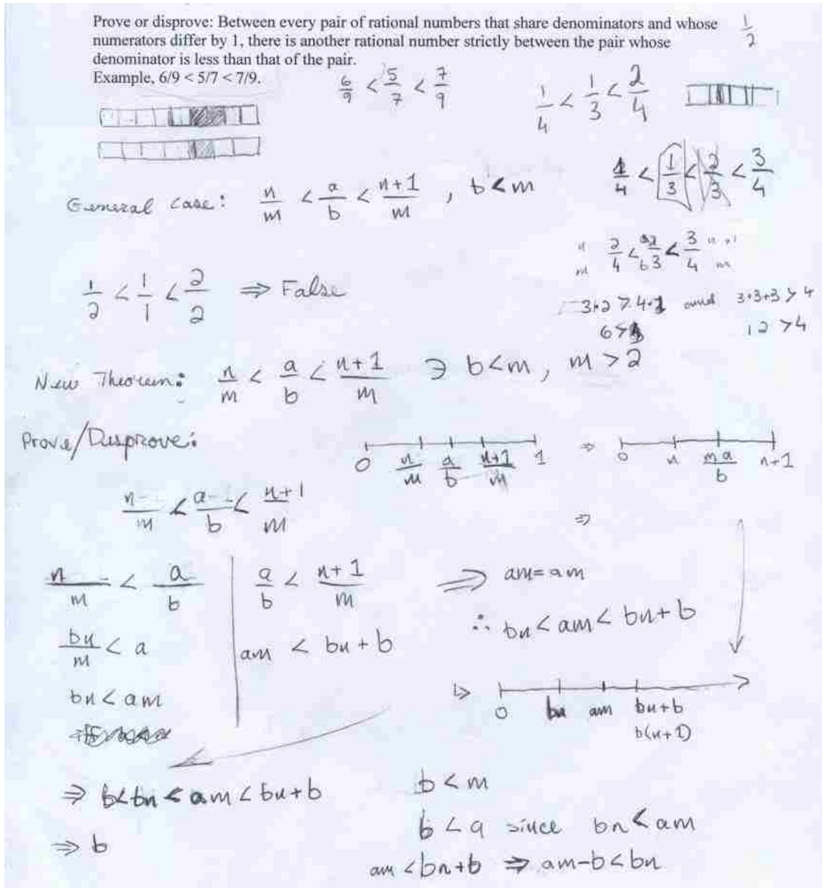


Figure 149: Michael's work on Question 10 (1 of 2)

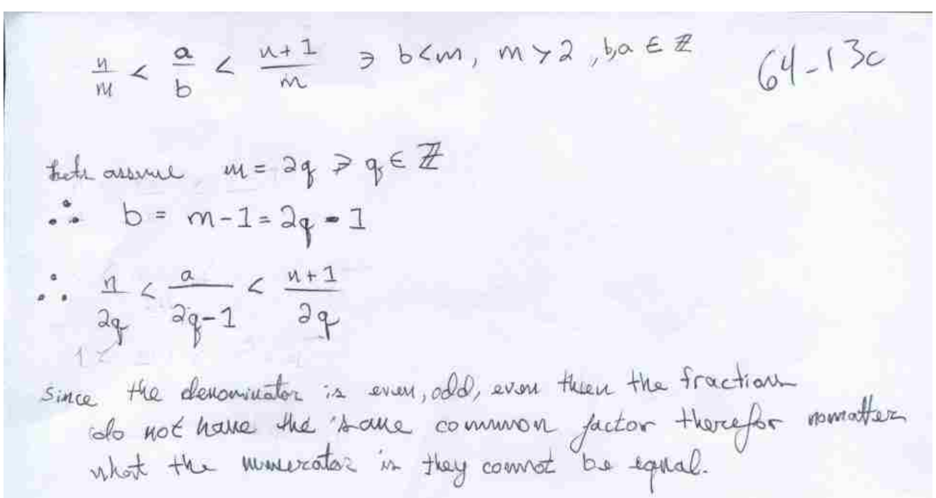


Figure 150: Michael's work on Question 10 (2 of 2)

N: Ok

M: It can't be the same….

N: …the same parity?

M: Yeah, it can't be the same parity.

N: Right

M: And basically when you're dealing with fractions, it's, there either even or odd, they'll never overlap, not matter what their numerator is.

I mentioned that the problem statement never said that smaller numerator had to be one less than the given one, but Michael realized that was ok: "Oh, you're right it is true, isn't it? Well, it still in a way works. It doesn't say, there isn't any rules on how I can apply that." I also asked him why he chose to let *m* be even. He said "It's usually easier to deal with evens than odds…in most cases" and that "if I can solve it for the evens, I usually just flip it around and make this one odd and make this one even and then…it kind of mirrors I suppose." During the reflection, Michael gave his idea more thought:

M: Well I guess, ok, I guess in this scenario too the numerator would have an effect too, if you wanted to make sure it was between the 2.

N: Right

M: So then numerator would also have to be an odd number, probably…Well actually, I could probably use both of these in here. I could assume that, yeah, I could actually somehow take this idea right here (in Figure 150) using the same formula for the denominator…

N: Ok

M: …in this and it'd probably make my life a lot easier because I'd only have one less variable to worry about (in the inequality at the bottom of Figure 149) instead of *b*, I'd only have, like, *q*.

N: Ok

M: And that would be easier, definitely.

When we had reached the end of the interview, I asked that Michael work on the problem more for the next interview. As with most of the problems I asked Michael to work on, he did not have time to get to this one. During the next interview (the last of the study), I used examples to guide him to conjecture that $\frac{n}{m-1}$ would fit strictly between $\frac{n}{m}$ and $\frac{n+1}{m}$ so long as neither was an integer. I then also helped him through the algebraic verification of this.

The work that Michael does here constitutes a semantic proof attempt. He does spend a fair amount of time simply manipulating the general inequality he had written down. However, he also spent time trying to gain an understanding into the problem (looking at examples and the number lines and boxes he drew). It was through looking at the example of $1/4 < 1/3 < 2/4$ that he realized that lowering the denominator by 1 would guarantee that the fractions would never be equal. That is, except in the case where the middle number and one other both equal 0. It is seems as though Michael did not get far enough into the problem to consider this case. In any event, because he tries to turn this intuition into a proof, his work on this problem constitutes a semantic proof attempt.

Like many other of his proof attempts, Michael's work here shows that he has a formal, deductive view of proof. While working on this proof, Michael focuses on the operations he is performing and the consequences of those operations. Thus, Michael's proof scheme is transformational here.

Question 11

The interview in which Michael and I finished discussing Question 10 was also used as a debriefing session in which we discussed the study as it wrapped up. Michael did not attempt a proof, but the discussion was used to look for clues about his proof scheme and also reinforce observations from earlier in the study.

The first thing I asked Michael about was whether or not he felt like he had improved over the course of the study. He did not think he had:

M: I'm not sure, I think I feel the same as usual…

N: So do you think you've gotten better at proofs this semester, or…it sounds like you, not really but you haven't gotten any worse either.

M: Yeah, it's kept me at the same level.

This matches what was observed during the study in that Michael was fairly consistent. In each semester, there was a single problem that Michael was not able to finish on his own and, for the most part, his proof scheme was always transformational.

I then asked Michael what he thought could have lead to more improvement and I thought that varying his methods might have helped.

M: Like I said, maybe I should get my mind off that one proof type structure that I like to do, by contradiction, not contradiction, induction.

N: Oh, ok

M: Maybe I could have worked more on doing contradictions and…well, there was that one square root of 2 thing, but that was kind of guided through.

Michael used induction three times over the course of the study. Nobody used induction more than he did, but most used induction twice so his use of the method was not inordinate.

I also asked Michael what role he saw examples playing in proofs. He mentioned a couple things. For one thing, he said that examples can help him understand a proof that someone else did.

M: Yeah, because there's sometimes some parts of the examples, it'll be talking about, "Well what do they mean by that?" and they just kind of breeze through it. Everything makes sense and then there's like that one liner that's like "Ok, I don't know how you get to that conclusion there." That's usually what happens with examples.

N: You use examples to make that leap down to that one line?

M: Yeah

He also said that they can be useful in creating a proof:

M: Well, like you were doing right now…because it kind of helps think of the structure of the proof quicker and see how you're going to prove it.

N: Ok

M: It definitely makes it a lot easier to understand.

That Michael sees gaining an understanding of a proof as important can also be seen in the number of semantic proofs he completes. Of the eight proofs Michael attempted, half were not procedural in nature. Of those four, all but one was semantic.

Asking Michael what he thought was necessary to complete a proof also highlighted some of the things noticed during the study. The first thing he mentioned in responding to the problem was the understanding he talked about earlier.

M: Well, to have a successful proof attempt is to be able to find…

(thinking)

M: Ok, so first of all, you have to be able to understand what you're doing.

N: Ok

M: That's probably one of the main things. Second of all you probably have to at least some type of a plan.

The second thing he mentioned is a plan. This is indicative of the number of times he displayed a transformational proof scheme, which has anticipatory actions as one of it characteristics. The next thing Michael mentioned should also not be surprising, given that he always displayed an analytic proof scheme.

M: Probably the third thing that helps is to …successful proof attempt is peer-evaluation.

N: Ok

M: Kind of, because even if I do prove something, I never really 100% sure if that proof's correct and it's always nice to have somebody take a look at it and say if there's a flaw in it…make sure there's no cases that were left out, or…

N: Oh, ok

M: Yeah, or…what helps but isn't necessary, probably…I'm not really sure what isn't necessary.

N: Ok

M: I personally find that everything when combined together, the doodling and all that, you know, it's all necessary.

His focus on insuring there were no flaws in his proof reflects his analytic transformational proof scheme.

Michael's progression

Below is a table of the Michael's proof types and his proof schemes.

Question	Type of proof	Proof scheme
1	Semantic	Transformational
2a	Syntactic	Transformational
2b	Process	Internalized Transformational
3	Process (Attempt)	Transformational
4	N\A	Transformational
5	N\A	Analytic
6	Process	Transformational, Axiomatic
7	Semantic	Transformational
8	Algorithm	Transformational
9	N\A	Transformational
10	Semantic (Attempt)	Transformational
11	N\A	Transformational

Table 8: Summary of Michael's work

Like the last two participants discussed (James and Robert), Michael does not show much progress over the course of the study. The reason for all three is the same: there was not too much progress to make. Michael did show some progress in using mathematical induction, but not too much as he began the study with a good understanding of the method. He also began the study with a formal view of proof, evident in the number of times he showed a transformational proof scheme.

Perhaps one would hope that a student in Michael's position would show more of an understanding of the axiomatic nature of mathematics. However, because of the nature of the study (as discussed in Chapter 3), it is possible that Michael has such an understanding without showing it. Given that Michael never gave evidence of anything other than an analytic proof scheme, I believe this to be the case.

Although Michael did not complete every problem he saw during the study (I consider the two he did not finish easily the two most difficult), I think Michael did quite well from beginning to end. This is not surprising considering his proof schemes shown during the interviews and he serves as another example of how making little progress is not necessarily a bad thing.

4.9 Chris

In this section, I will describe the proof attempts made by Chris over the course of the semester, the types of proof they represent and the proof schemes Chris displayed with his work. Chris was a graduate student at the time, with the goal of becoming a secondary school science and mathematics teacher. During the first semester of the study, the mathematics classes Chris took were Euclidean and Non-Euclidean Geometry, Introduction to Probability and Statistics and Teaching Mathematics with Technology. During the second semester of the study, Chris took one mathematics class: Number Theory.

Chris' proof attempts

Question 1

Find all rectangles with integer side lengths such that their perimeter is equal to their area. Prove that you have found all such rectangles.

Chris started the problem by writing out an equation, setting perimeter equal to area in a general triangle ($2l + 2w = lw$). He did not use it directly, though, instead he began mentally checking different rectangles to see if they made the equation true. After looking at a number of examples, Chris found a rectangle that worked.

Chris: (L)et's see what 6 and 4 is.

Nick: Ok

C: Which is going to be 24 and 10, right? So that's 20. Alright. So how about 6 and 3. Ahhh! I got one.

N: You got one?

C: Yeah, 6 times 3, equal to 18 and that's 9, times two is 18. (See Figure 151)

N: Ok, so just sort of for clarification, did you have any reason to the ones you tried?

C: It's just, the way I always do it is with manual perturbation, is my first way to try it. Um, figure out a case by starting at the beginning.

N: Ok, so what did you think of as the beginning?

C: Well, I started out with 1 and 2. And then 1 and 3, so that didn't work because I knew that 1 times 7 would never, well, 1 could never be one of the sides.

N: Ok

M: I couldn't happen. So then 2, was working in the same fashion, where if 2 as one of the sides, you always have 2 times 6, well, that's two of the sides. So 2 couldn't work for one of the sides.

N: Say that again.

C: Well, if you have a rectangle and 2 is one of the sides, it doesn't matter what the other side is, you're perimeter is always going to be 4 greater than the area. If you had a side of 2 and a side of 6, right, your perimeter, then is going to be 16 and 6 times 2, which is going to be your area, is 12.

Through examining examples and his equation, he was able to eliminate all rectangles that have either 1 or 2 as one of its side lengths. As he said, if 2 is one of the sides, the perimeter is always going to be 4 greater than the area. Although he does not mention it here, the perimeter will also be greater than the area in rectangles that have 1 as a side length.

Chris had worked his way up to rectangles that had side length 3 and mentioned that the rectangle he found would be the only one that works for this case. He did not say why he knew no other side could pair with 3, but I think he was using reasoning that was began evident when he discussed rectangles with a side length of 4.

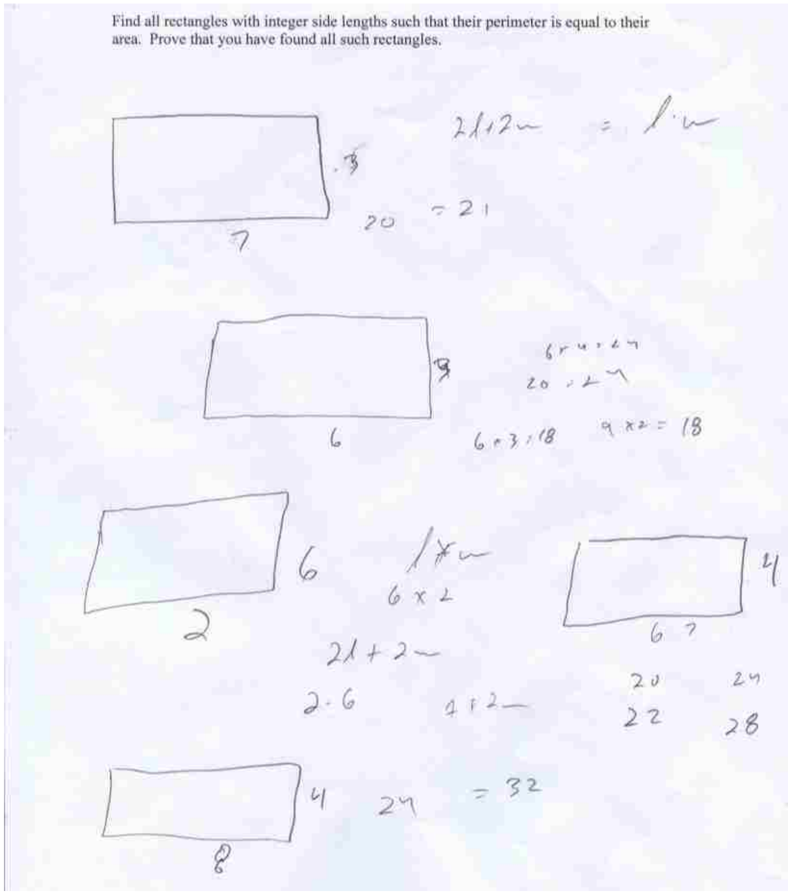


Figure 151: Chris' work on Question 1 (1 of 5)

C: So, I'm going to try and work this again. So I know the 3's not going to work, so I'll start with a base of 4 and see if any other side length will go with that.

N: Ok

C: Um…6, 6 times 4 is 24…so that doesn't work. 7 times 4 is 28, that doesn't work. And I think I'm actually farther away, 'cause this 6 times 4 is..

N: So, by further away you mean…

C: The area and perimeter are getting farther in distance, I believe, so let's see with 7 times 4…it was 4 and now it's 6. So, I don't believe that there's an integer with side 4 that's going to work. Because 5 doesn't work, 6 doesn't work, 7 doesn't work and they keep getting farther away.

N: *Ok*

C: Because of the way the relationship with perimeter and area works. The multiplying versus the adding. So I'm going to go…maybe there's one with a base 5.

There are a couple things of note here. First of all, when Chris says "3's not going to work" he's referring to pairing 4 and 3. This is important because Chris realizes that he has already eliminated that pair when considering the case where one side was 3. Second, it is the first time he explicitly mentions that he is comparing the ways in which the area and perimeters grow when one side of a rectangle is held constant and the other is increased. I mentioned the fact that he had forgotten to check the square of side length 4. Once he checked it, he saw that it worked as well.

Chris went on to check the cases of 5 and 6, again seeing the area grow faster than the perimeter. Only this time, the area also started out greater.

C: And I don't need to go to check the 3s and 4s and we did those.

N: Right

C: Which is why I'm starting at the square 5 5.

N: Right, ok.

C; So we'll try the square 6 6. So the area is 36, the perimeter is 24. So 6 and 7 is 42, 13, 26. Alright, are we really getting the same type of pattern again? Am I, 6 times 8 is 48, 14, times 2 is 28...So these are going up in increments of 2 and these are going up in increments of 6. Right, because, that's 54, 15 is 30. So it seems the farther we get away, the larger the numbers – they're not getting close any more...So they're increasing at different rates, so since they're starting off area is above perimeter already, perimeter is never going to catch up with area. The thing that was happening in the beginning is that area was below the perimeter.

N: Ok.

C: And so because, so for that reason, the area caught up with the perimeter. And that's when we found a point of intersection.

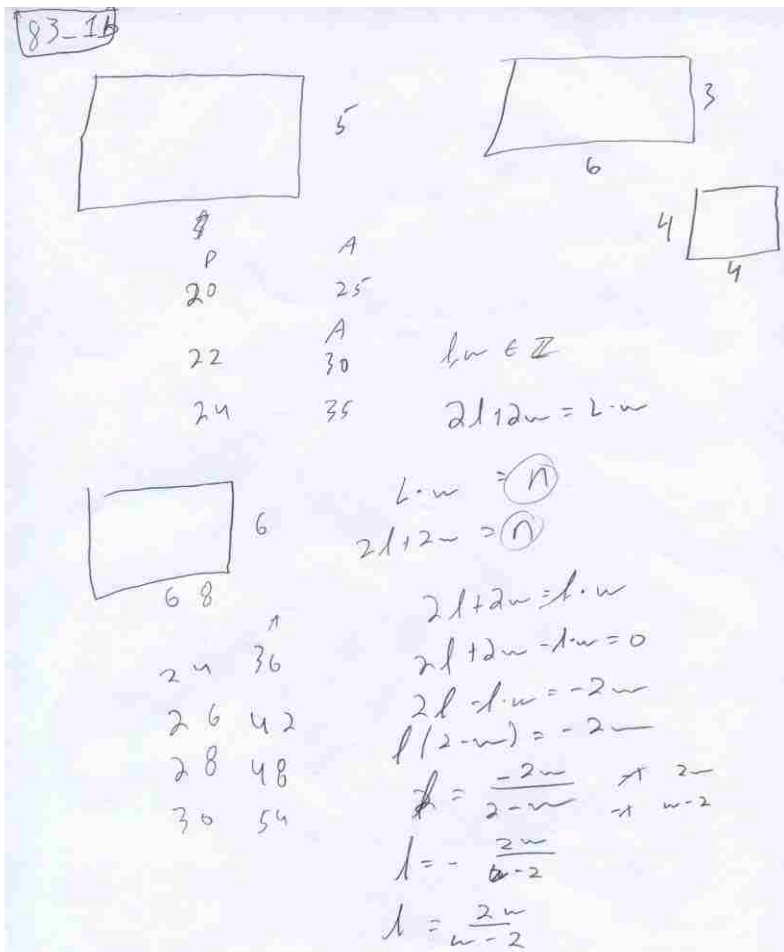


Figure 152: Chris' work on Question 1 (2 of 5)

Now that Chris had found a pattern he wanted to exploit, he turn to the general formula he had found earlier, as can be seen in the figure.

C: See, I almost want to set it to something and then use the system of linear equations, because I see I have two equations here. So, length times width has to equal something, we'll just set it equal to *n*.

N: Ok

C: And $2l + 2w$ also has to equal to *n*.

Chris had gotten to a point where he felt like he understood the problem well enough that he tried to come up with a proof. He also knew thought that continuing on his path would lead to an impossible proof by exhaustion. He explained what he was trying to accomplish with the formula:

C: Yeah, I was just kind of taking that step back, seeing if I could find that ratio and then maybe if that ratio would only work for a few integers, like the ones we have…

N: Ok

C: Then maybe I could come up with a proof.

After that, I gave Chris a hint that I thought would help him develop the ideas he already had:

N: But let me ask you this, when you were doing your exhaustion, it was sort of based on the fact that ok, every time I checked, the area is more than the perimeter and the area grows faster, so …

C: Well, this almost seems like you could graph it, you know, because you have 4 times l, which would be linear, and then you have l^2 which is going to be a parabola and they should, since this is always going to be a positive parabola, …I believe they should only intersect at one point, other than (0,0). Cause they're both going through (0,0).

This gave Chris the idea to compare the area perimeter graphically, which he discussed in more detail later (see Figure 153 below):

N: So let me ask you this again, with 5, explain to me way there's no rectangles that work beyond past what you've written. So like, you started with 5 and 5, 5 and 6, 5 and 7, why will 5 and 10 not work.

C: Well, just because here, I mean, the answer I said before is that the area is growing faster than the perimeter.

N: Ok

C: Is why that won't work, it just has greater slope and since the area starts higher, you know, you can imagine on the graph if you start at 10 and you start at 5 and the one at 10 grows faster than the one at 5, then they're never going to intersect on the right side.

N: Right.

C: In the positive. Cause rectangles, kind of stick in the positive.

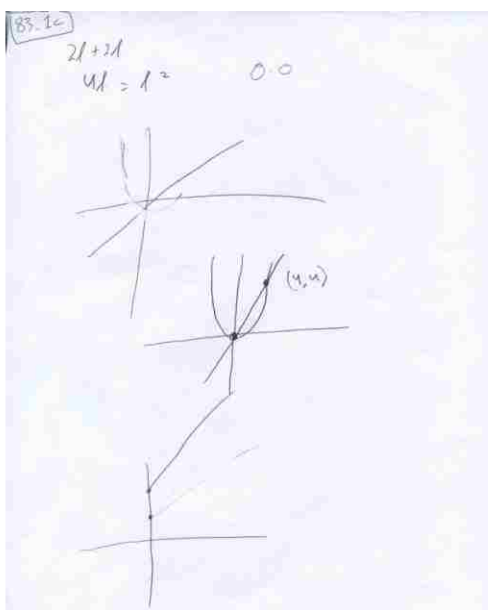


Figure 153: Chris' work on Question 1 (3 of 5)

Chris was making the case that if you fixed a side length at $l > 4$ and considered rectangles where $l \leq w$, the area would stay greater than the perimeter because if you looked at the lines representing area and perimeter (the bottom graph in 153) the area line would have a greater y intercept and greater slope. I asked that Chris put his reasoning into a formal argument to be brought back for the next interview and he said he would. The proof he came back with is in Figure 154 and Figure 155.

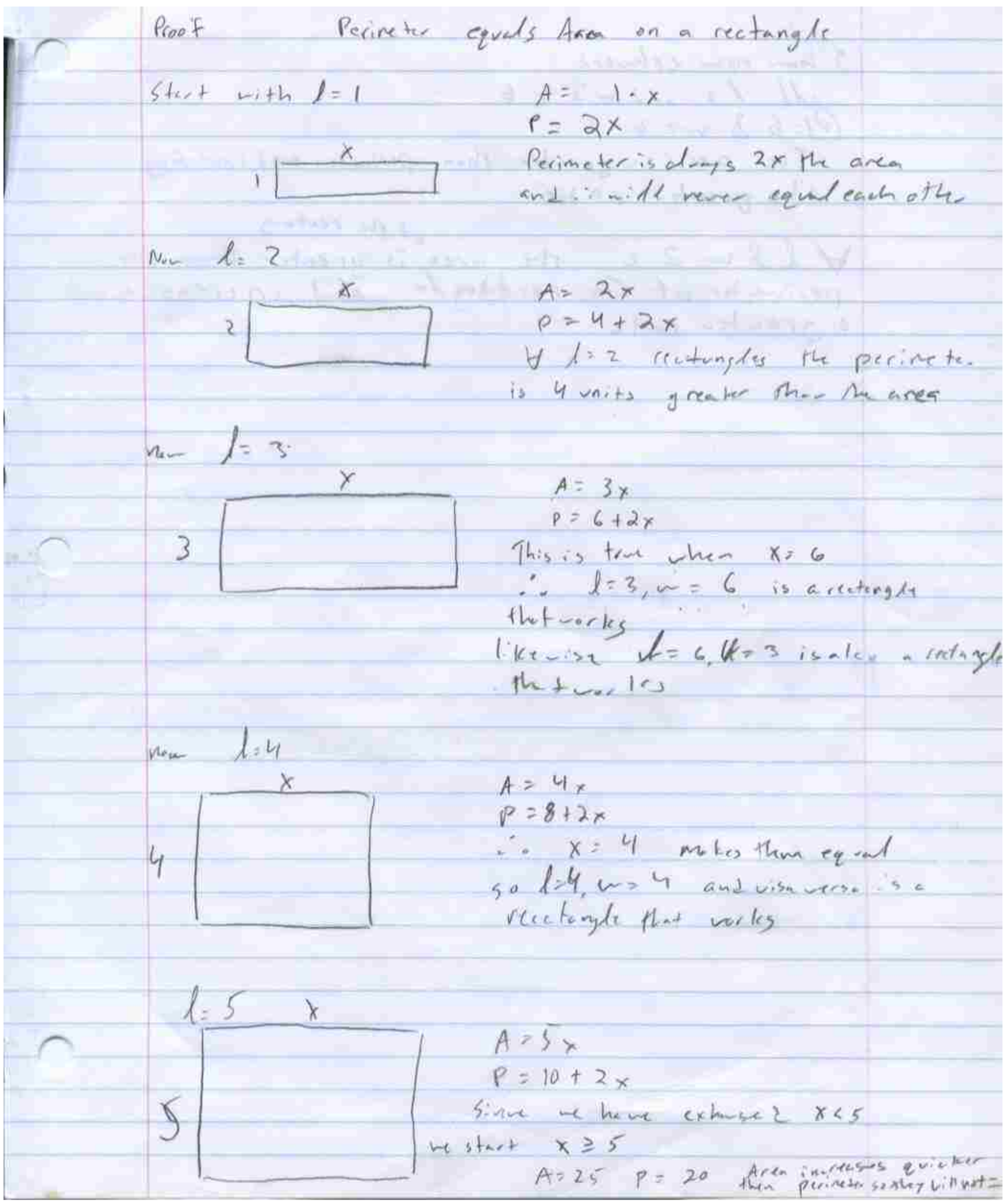


Figure 154: Chris' work on Question 1 (4 of 5)

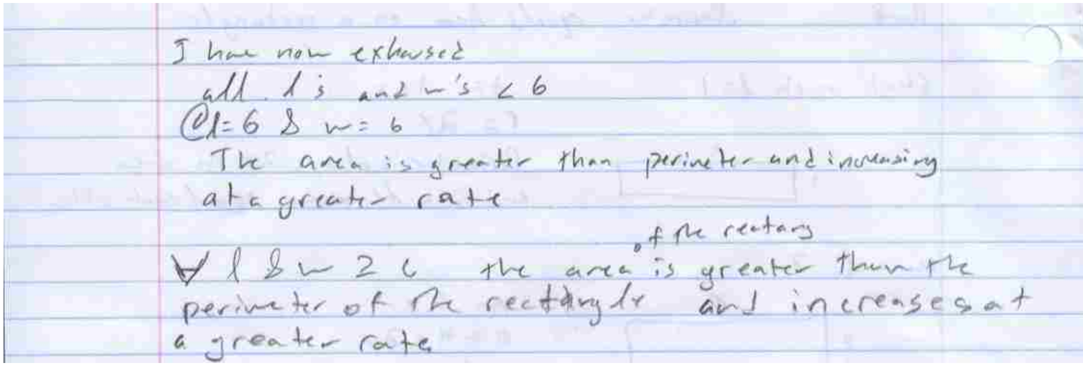


Figure 155: Chris' work on Question 1 (5 of 5)

Chris does not complete his proof. Based on the interview, he had all the pieces he needed to do so. He used his graph comparing l^2 and $4l$ to show that for squares of side length greater than 4, the area is greater than perimeter. Combining this fact with his reasoning about the way area grows in relation to perimeter and the idea that one needs only start by checking squares as the rectangles grow (to avoid redundancy) would have completed the proof. Because he did not put these pieces together, however, leaves his work as attempt only.

Even so, it is possible to classify Chris' proof attempt. Because he began by first exploring the problem to gain an understanding that he tried to turn into a proof, this constitutes a semantic proof attempt. Early in the interview, Chris said "…I know through our other stuff that at some point I have to get things in sort of general terms. But I figure I should figure it out in normal terms, then I'll figure it out in general terms." This is exactly the sort of effort that results in a semantic proof attempt.

Chris' proof scheme here is transformational. His work does not rely on previous results, but he does use formal reasoning. At one point, when describing what he was trying to do, Chris said "That's my reasoning, I know from my last class and from my classes now that, that this path is proof by exhaustion, and to prove something by exhaustion, that means you have to go through each and every case." So, even though it would appear that Chris is convinced by examples if one looked only the work he brings back, he actually has an analytic proof scheme. It is a transformational analytic proof scheme because his proof comes from a consideration of how changing one side length affects the relationship between area and perimeter.

Question 2a

If a and b are non-negative integers less than 10, then ab - ba is a multiple of 9.

Chris started this problem by looking at a couple examples. Unsure of where to go, he looked at the equation, even though he did not see it being very fruitful.

> C: I don't feel like with *ab* – *ba* that there's any equation, because it's not like it's *a* times *b* minus *b* times *a* is going to equal 9 times *x* where I could solve and do a proof by induction where it works for this case, does it work for the next case.
>
> N: Ok
>
> C: And obviously it doesn't work for anything past 10, because 108 and 810. Which would be cool if that was also a multiple of nine. Does that not also…?

Chris then tried a few examples of 3 digit numbers and saw that it works in that case as well. When talking about the written-up proof he eventually provides (during the following interview, Figure 157), Chris mentions that his proof allows for *a* or *b* to be other than single digit numbers and that is why it still works.

While inspecting the equation he had written, Chris notices something that could have been turned into a proof:

> C: Does that maybe have anything to do with as I increase that (*a*) by 1, that increases this (*ab*) by 10 and decreases this (*ba*) by 1? And that's 9.

N: Possibly.

C: And likewise if I increase this by…Does that make sense?

N: Oh, definitely it does. So if increase this by 1, what you're saying is you increase *a* by 1 *ab* becomes 10 greater and *ba* becomes 1 fewer. So you've added 9 over all. And if you increase *b* by 1, so what happens then?

C: If I increase *b* by 1, that goes 1 – 10, right?

N: And that's a negative 9. So the difference then decreases by 9.

C: Right. So it just has to do with the relationship between *ab* – *ba*, then? Not even necessarily having to look at 9*x*? I mean really, it's the fact that our numbers are base 10. Again, increasing by 3 over here, that's 30, and that's minus 3 over here, so that's 27. So that's cool, at least it feel like I'm getting somewhere…Everything's making it a multiple of 9. But even that, how would I prove that? Exhaustion. I can do 1 through 10. Right? Couldn't I do that by showing that if I increase *a* by 1 through 10, this is what it does, all those got multiples of (9). And then do the *b*'s . So if I just set out some type of chart…

Chris then began the chart in the bottom right of Figure 156.

C: And there's really not that many combinations, 81.

N: Well, 100, but…

C: So for homework, if this was an assignment, that's what I would do.

N: That's what you would do?

C: Absolutely. Just because I know I can get the right answer that way, if I do all possibilities. And it really wouldn't be too hard, I don't think, I mean I already have 10 of the possibilities. I just have to make a chart that's wide enough to go from 0 – 9 across the top and 0 – 9 for my *b*'s. You know what I mean?

N: Yeah, if that's what you would do, I'm not going to make you sit here and make you do 100 subtraction problems.

C: And that's one of the ways of doing it and I would like to know if there's another way. Try and work through that…I still feel like there's some relationship with our powers of 10s that goes on here.

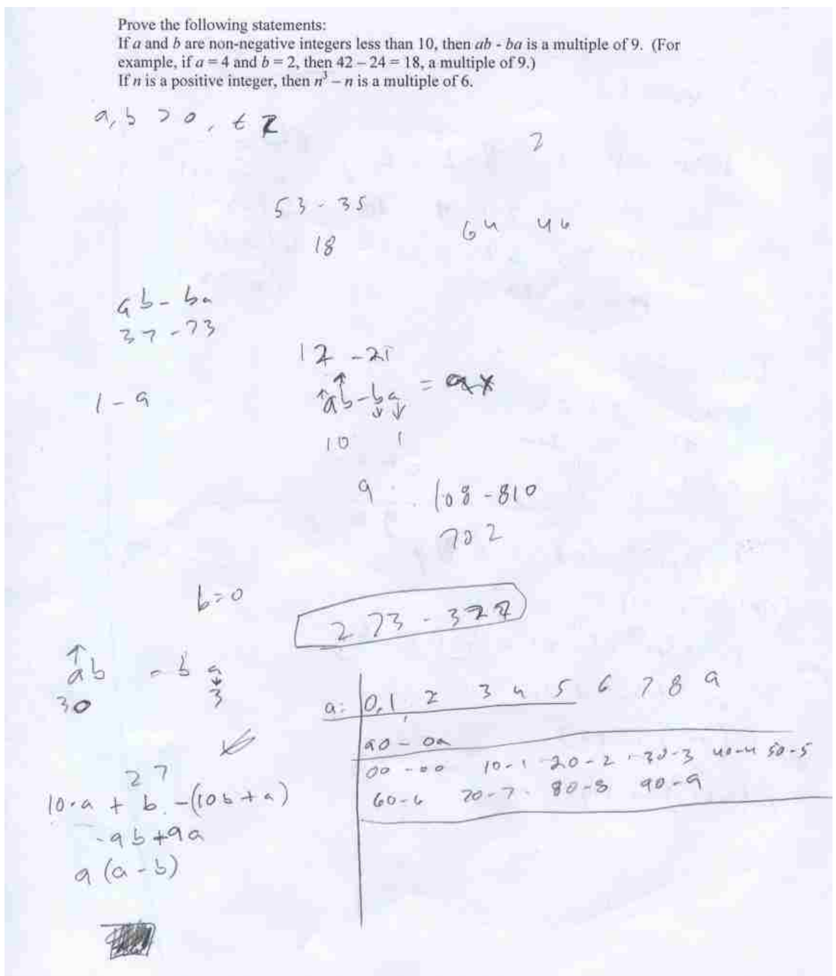


Figure 156: Chris' work on Question 2a (1 of 2)

Chris gave that some thought and an idea came to him.

C: Cause maybe, even being able to write this as 10 times *a* plus *b* minus 10 times *b* plus *a*. Right? So that's –9*b* and that's –9*a*, so no matter what you plug in, that's what happens.

N: …Ok, so that gives you a multiple of 9 how?

C: Well, once you factor out a nine from this and get *a* – *b*. And so…

N: And that's always a multiple of nine?

C: Yeah. So this is a true statement. In your equation, 10*a* + *b* – (10*b* + *a*), and so I just did that math and got 9 times (*a* – *b*) so no matter what *a* and *b* you plug in, it's going to equal that times 9. And therefore, it has to be a multiple of 9. And then I like put a little box and fill it in.

This work was done on the bottom of Figure 156. I asked Chris to write up a cleaner version of this argument and he did. During the next interview, he brought back the proof in Figure 157.

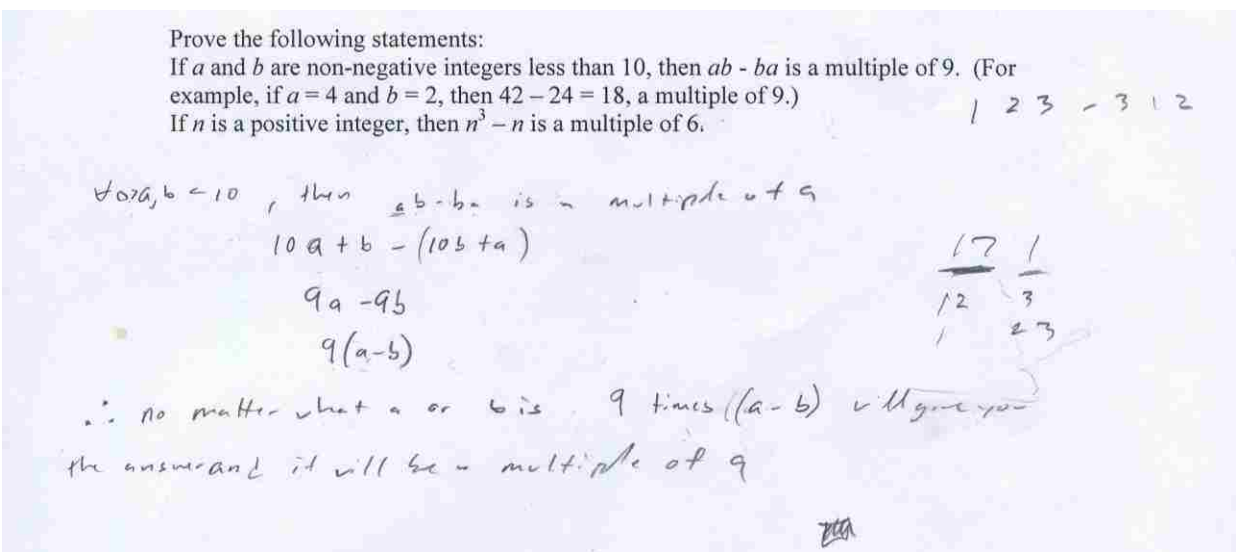


Figure 157: Chris' work on Question 2a (2 of 2)

The proof Chris provides here is semantic. The proof he ultimately gives depends on the fact that our number system is base 10. He gets this idea from considering what happens to the difference $ab - ba$ when a is increased by 1. Once he understands the way the expression can be rewritten [as $(10a+b)-(10b+a)$], he is able to turn his understanding into a proof. This is the definition of a semantic proof.

Like with Question 1, Chris is displaying a transformational proof scheme here. Strictly speaking, nothing in his final proof is transformed for the sake of the proof. Last time, his proof relied on considering what happens as one of the objects involved was changed. Here, this sort of reasoning helped Chris come to his proof. Even so, his proof depends on anticipatory algebraic manipulations. This sort of formal proof that does not rely on previous mathematical results is evidence for a transformational proof scheme.

Question 2b

If n is a positive integer, then $n^3 - n$ is a multiple of 6.

Chris began this question by looking at examples, and it did not take him long to decide what method he was going to employ.

> C: So, as I always do, I like to start with a base case (example). And if possible, man, do I love doing proof by induction. I really took to that one. I'll do $n = 2$, $8 - 2 = 6$. I just wonder if I can do a proof by exhaustion (induction). (Chris tries $n = 3$, mistakenly gets 18.) The other thing that I'm noticing is that our result is 6 times n. No, 6 times $n - 1$, oh wait, no that doesn't work, that's because I messed that up. So that doesn't work.

Chris tried a few more examples to see if there was a pattern in the number that 6 is multiplied by based on the original *n*. "So, that doesn't seem to be leading anyway productive. So I really would like try a proof by induction for this." His work on this problem begins in the figure below.

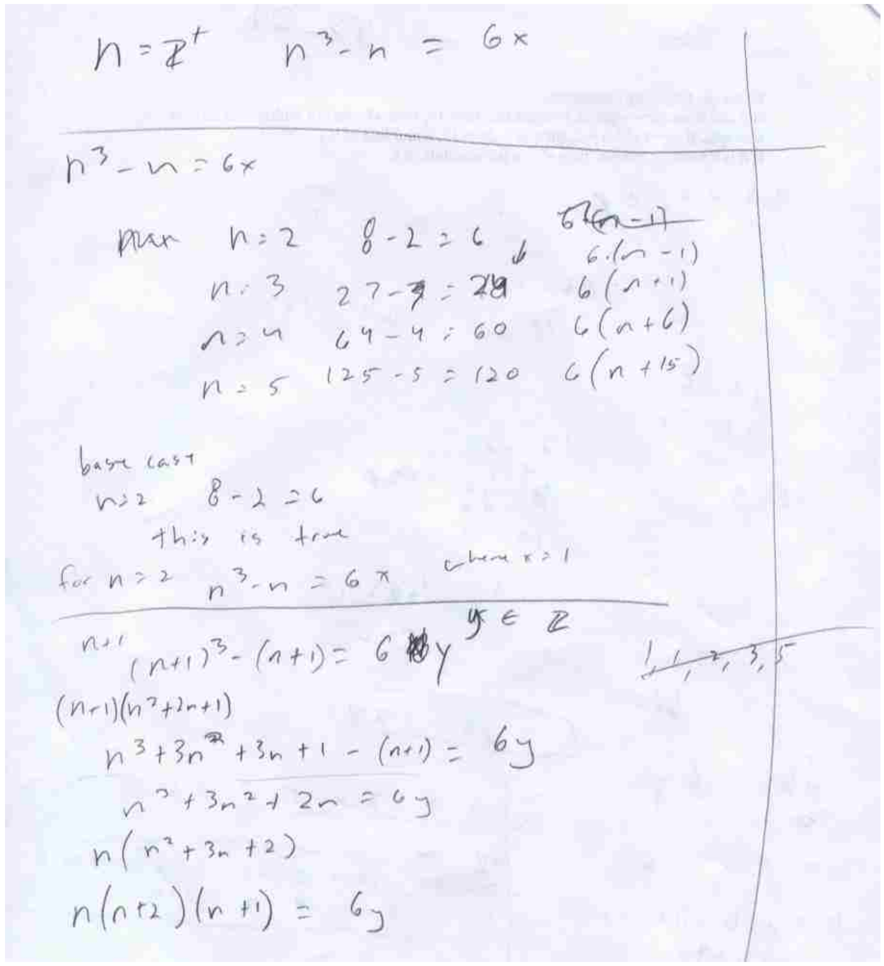


Figure 158: Chris' work on Question 2b (1 of 3)

Chris then proceeded to complete the induction argument. Unfortunately, he simplified too much and was unable to use his induction hypothesis.

C: And I know that in a proof by induction I am supposed to go back and use my original…typically…

N: So is there anyway you can see to do that here?

C: I mean if it did happen somewhere, I feel like it would be here at this line. And we want $n^3 - n$.

N: Ok

C: Well, if I go…back here maybe…because I'm not getting it here. I can see that much. This was just a bust.

Next, Chris got a new sheet of paper and restarted (in Figure 159 the inductive step at the point he thought he would be able to use the induction hypothesis.

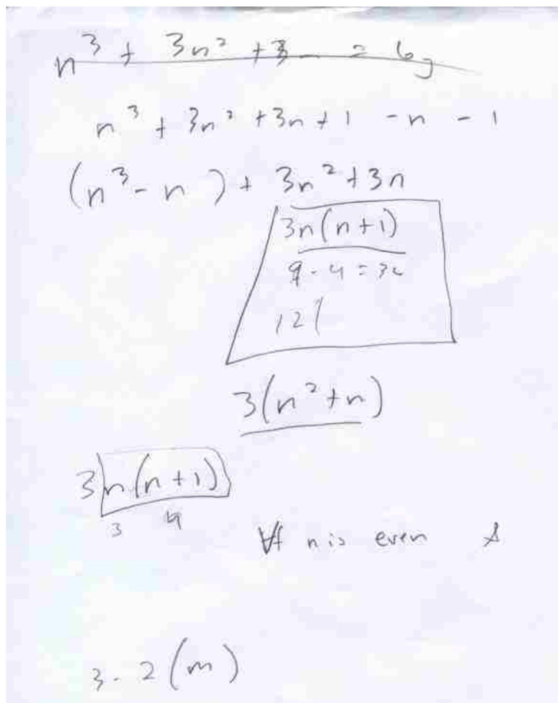


Figure 159: Chris' work on Question 2b (2 of 3)

C: Nice, well this is going to be $3n^2 + 3n$. And so I know that this $(n^3 – n)$ is going to be a multiple of 6, I showed that by the base case. So now I need to show that this is going to be. So if I pulled out a $3n$, I got $(n + 1)$. But, well when n is 1, that's 2 and when n is 2, that's 3… So 6 times anything is obviously, 2 isn't the best one pick, but….this is 9, times 4 and that's 36 which is still a multiple of 6.

N: So it still works of n is 3.

C: Right. If *n* is 4, 12…I just don't know why it works. Maybe if I just back up to 3…so this is where I am…[$3n(n + 1)$] I'm trying to think of any properties, anything I could use. $3(n^2 + n)$ is the same thing as $3n(n + 1)$, so obviously everything's divisible by 3, but I have to show that it's also divisible by 6. And really like to, if I could pull a 2 magically out of here…that would be fantastic.

Chris had hit upon what it would take to finish the problem. Knowing what he was going for made getting to the end fairly quick:

C: Well, if started…can it have to do with odds and evens, maybe?

N: Maybe

C: Because if this is, let's see, if *n* is 3, then that's odd and that's going to be an even. So that's always…

N: So if *n* is odd, $n + 1$ is going to be even and you can pull a 2 out of that even.

C: Right, and if *n* is even, n + 1 is always odd, then you're always going to get an even, right?

N: Yeah, if *n* is even, $n + 1$ is odd.

C: But no matter what, this quantity, $n(n + 1)$ is going to be even.

I asked Chris that he write up the proof formally and he did so. His proof is in Figure 160.

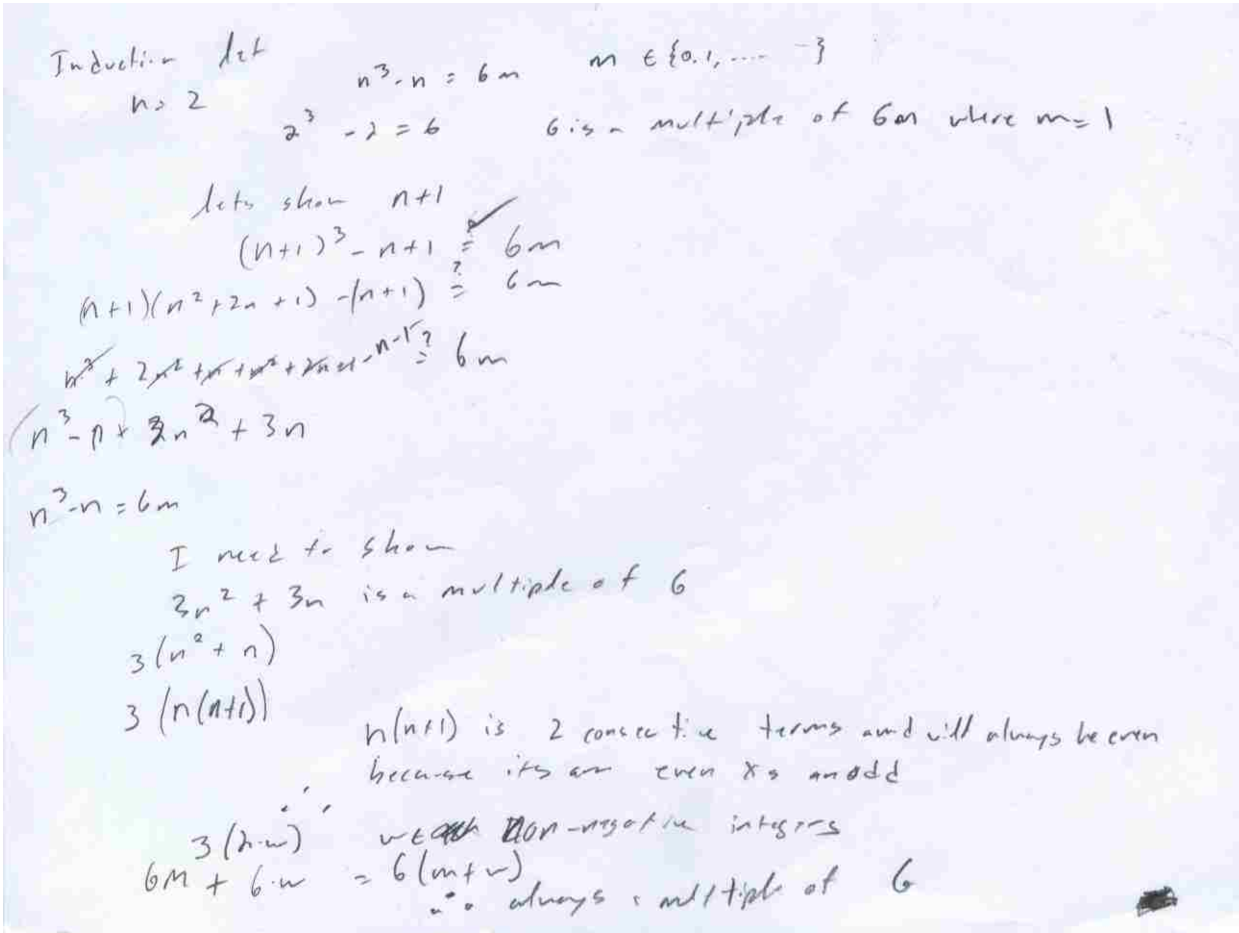


Figure 160: Chris' work on Question 2b (3 of 3)

The proof Chris presents is a process procedural proof. He completes a proof by mathematical induction, so Chris is following prescribed steps. The steps he completes are not all laid out for him. Rather, Chris views the method as a few broad goals:

> C: It makes sense. You're using integers, so if you can show that it works for the original integer, and then show for the $n + 1$ term, then it's proven. And so that, to me, seems like an easy way to do it. If you can get that base case substituted in, is where I've struggled.

He is able to articulate those broad goals and recognizes the difficulty that sometimes is involved with performing the inductive step. If every step had been laid out for him, this difficulty would not exist and Chris' proof would have been considered an algorithmic procedural proof.

While Chris was able to describe the steps of induction and he had a good enough understanding to complete the proof, there is evidence that he had not completely interiorized the method at this point. The first example of this is in his choice of base case. Even though he mentions that you have to start with the "original integer," Chris does not do so. In fact, he makes this mistake first in the interview and then later in the formal write up.

Another place a potential misconception appeared to arise during the reflection. When discussing his mistake of simplifying too far originally in the inductive step, Chris said

> C: So instead of simplifying the $3n$ and the $-n$, I just put the $n^3 - n$ together and so the next thing I did, I knew that that, from our base case, was $6y$, or $6x$ I guess is what I said, which was a multiple of 6.

He attributes the ability to use the induction hypothesis to the "base case." I think it is possible that this was merely a slip of the tongue and he meant to say induction hypothesis. However, I am not sure of this and when this is taken together with the fact that he uses $n = 2$ as his actual base case it suggests that Chris had not yet gained a complete understanding of induction. Thus, the evidence implies an internalized transformational proof scheme.

Question 3

Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbb{N}$**:**

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}\geq 1+\frac{n}{2}$$

Chris started this problem by subtracting 1 from each side of the inequality and then proceeding right into the induction argument.

C: Right, so it's…ok, so my new statement, then is 1/2 +1/3 up to 1 over 2^n is greater than or equal to *n*/2. So, base case, set, just set *n* = 2, make life easy. So you get 1/2 + 1/3 + 1/4, better be equal, greater than or equal to 2/2, which is one? Right? 1/2 + 1/3 up to 2^n, *n* = 2, 2^2, so 4. Ok, so put them all over 12, 6/12 + 4/12 + 3/12, is absolutely greater than 1, 13/12.

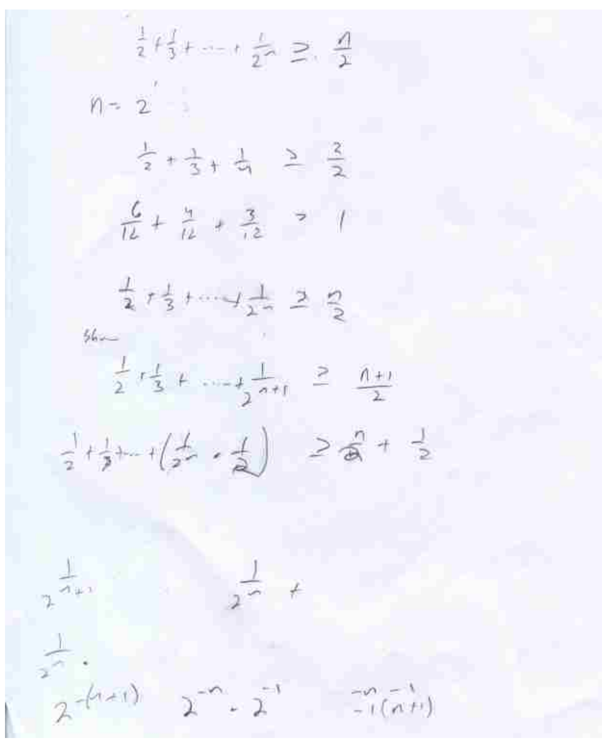


Figure 161: Chris' work on Question 3 (1 of 5)

After stating his induction hypothesis, Chris began to focus on how he could use it.

> C: . So that, here I'm going to get my $n/2$, and here I'm going to get up to, because I think I can rewrite this, $1/2^{n+1}$, that's what I have to figure out how to rewrite, because this is greater than or equal to $n/2$ plus 1/2, and there's my original $n/2$. So if I can get this to say …ok, so I have to mess with $1/2^{n+1}$, figure out how to write that as a $1/2^n$ plus something. Because that's my original…ok…

Chris saw how where the right hand side of his induction hypothesis could be found on the right hand side of the inductive step, but the left hand side was not working as well. Part of the problem he had due to the fact that Chris was not sure how the left hand side changed as he moved to the $n + 1$ case. He was able to fix this, though, when he looked at the difference between $1/2^5$ and $1/2^6$, in Figure 162.

> C: Oh, yeah, that's right. $1/2^n$ times 1/2. So, what I'm thinking is, so the term…except for it's 2^n, so I'm trying to think of what's the term before $1/2^{n+1}$, and it's not $1/2^n$. Because, it's squared, so it's actually an odd integer in between those two. Because if this was 5, this, well, there's a bunch, actually, right? Because this is, if n is 5, $1/2^6$…and so $1/2^5$ is quite a bit back, because this is 32 and this is 64, so there's a whole mess of integers in between there. 1/33, 1/34 all the way up to 1/64.

This new understanding led to the revised inductive step, also in Figure 162, and a clearer picture of what he had to do to finish the problem:

> C: So I have that this is greater than $n/2$. So I have to show that $1/(2^n+1)$, not in the exponent, up to $1/2^{n+1}$, in the exponent, is greater than or equal to 1/2. Because I know

that this term is greater than or equal to $n/2$ by my base case. And that term is 1/2 up to $1/2^n$.

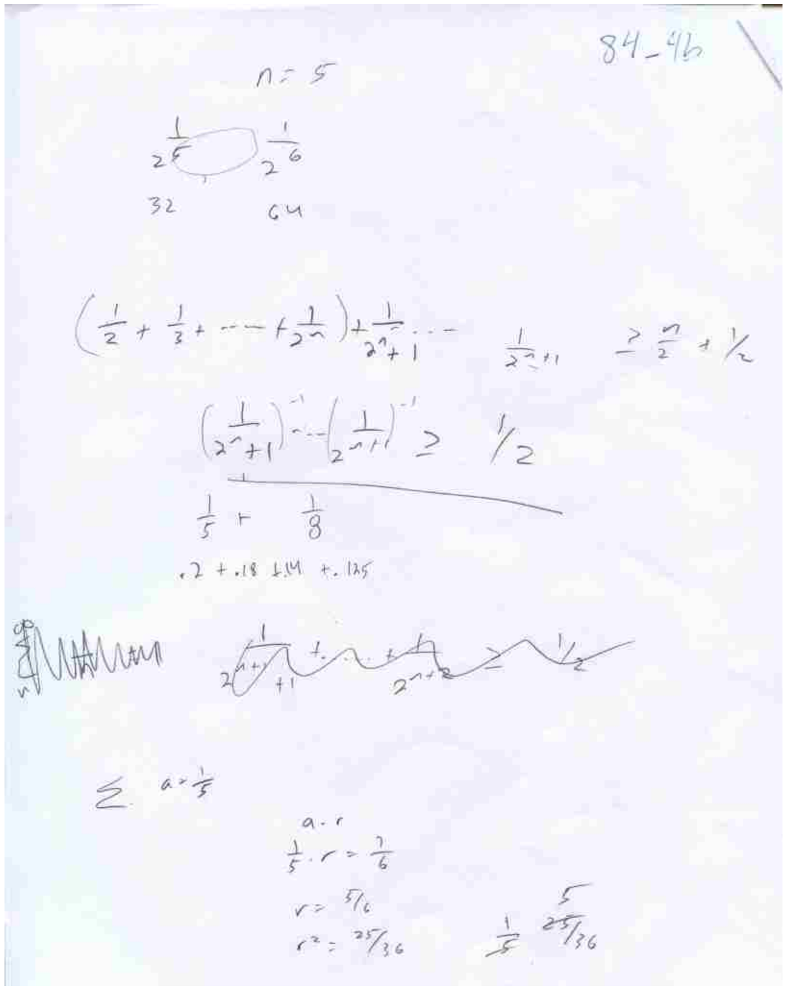


Figure 162: Chris' work on Question 3 (2 of 5)

Chris then thought back to geometric series, which I gave him the formula for once he asked, but he dismissed this idea because successive terms did not have a common ratio:

C: Ok, not what I have… Yeah, that part doesn't stay fixed, at least I'm not…1/5 times r to get to 1/6, so r here is 5/6 and r^2 is 25/36, multiply that by 1/5, just see if this works…and it doesn't. 5/36 is not the next term.

Chris finished the interview by doing a little more playing around with the sum (he considered reciprocating both sides of the inequality and realized he could not). I asked him to continue to work on the problem and bring back his work and he said he would. He left confident in at least some of what he had done.

C: I feel comfortable up to this step. I feel confident that this is my new thing, that if could just show that $1/(2^n+1)$ up to $1/2^{n+1}$ is greater than or equal to 1/2, that would show it. But I just…don't quite see it.

The work Chris brought back is in Figure 163. He started the problem out basically where he left off. This time, once he applied the induction hypothesis, he tried to use integration even if he could not articulate how integration related to the problem initially.

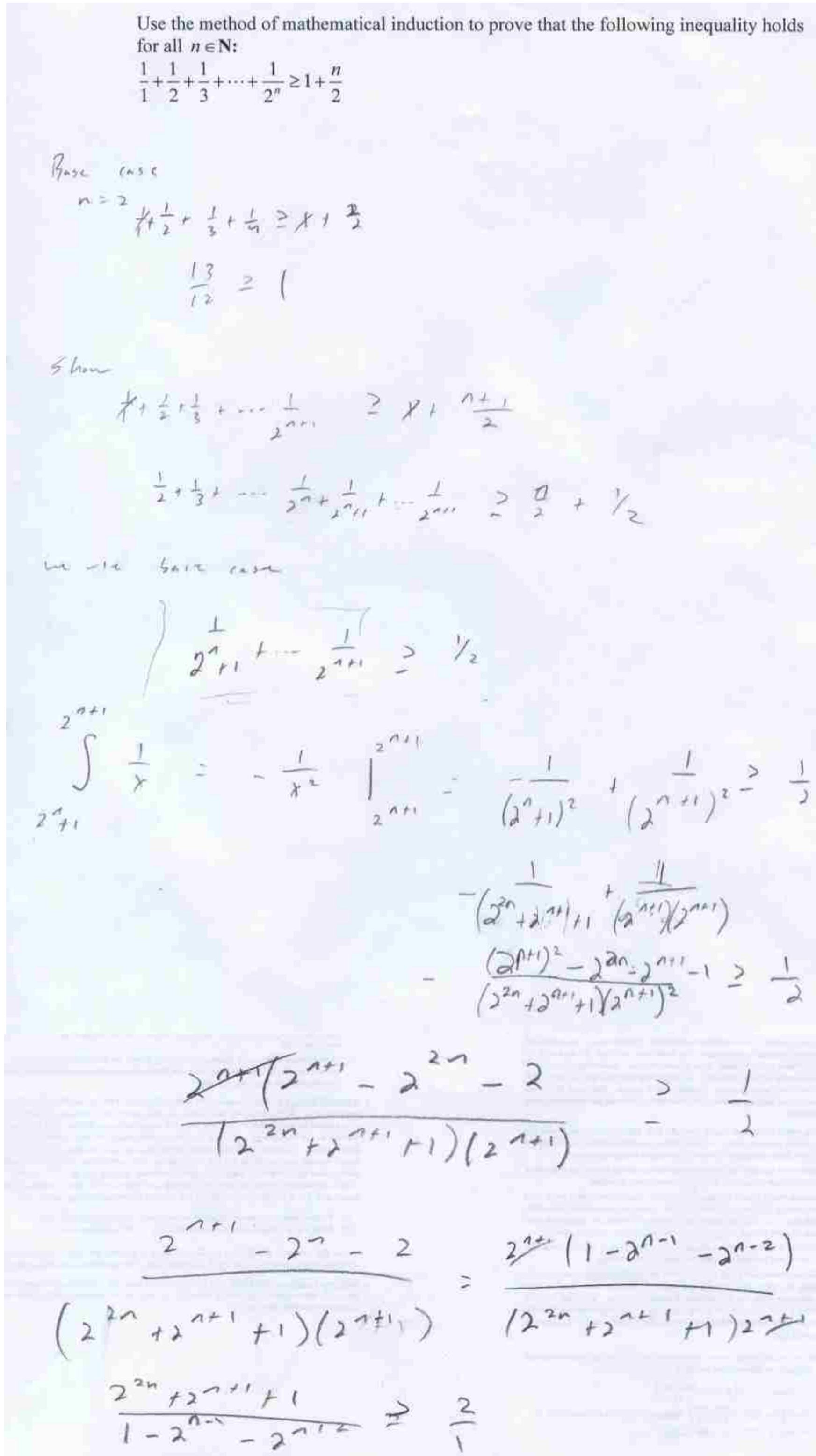
C: Well, then I knew this was the function $1/x$ because it goes, I mean at least I thought it was, like, 1/2, 1/3, 1/4, 1/5, because you're using those integers, that led me to believe you could use the function $1/x$…

N: Right, ok

C: …to explain it and then it's just a matter of evaluating because, you know, the area under the curve will gives us this value of what is $1/x$ from 2^n+1 to 2^{n+1}…Yeah, I mean, I guess my reasoning was…so the area under the curve, you know, I don't know. I guess I had thought about it that there were just these little boxes with fixed width, you know, the definition of the integral, they're getting smaller.

N: Yeah

C: And I just thought that…I have no good answer.



Use the method of mathematical induction to prove that the following inequality holds for all $n \in \mathbf{N}$:

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} \geq 1+\frac{n}{2}$

Base case

$n=2$ $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \geq 1+\frac{2}{2}$

$\frac{13}{12} \geq 1$

Show

$1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{2^{n+1}} \geq 1+\frac{n+1}{2}$

$\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{2^n}+\frac{1}{2^n+1}+\cdots\frac{1}{2^{n+1}} \geq \frac{n}{2}+\frac{1}{2}$

we use base case

$\frac{1}{2^n+1}+\cdots\frac{1}{2^{n+1}} \geq \frac{1}{2}$

$\int_{2^n+1}^{2^{n+1}} \frac{1}{x} = -\frac{1}{x^2}\Big|_{2^n+1}^{2^{n+1}} = \frac{-1}{(2^n+1)^2}+\frac{1}{(2^{n+1})^2} \geq \frac{1}{2}$

$-\frac{1}{(2^{2n}+2^{n+1}+1)}+\frac{1}{(2^{n+1})(2^{n+1})}$

$\frac{(2^{n+1})^2-2^{2n}-2^{n+1}-1}{(2^{2n}+2^{n+1}+1)(2^{n+1})^2} \geq \frac{1}{2}$

$\frac{2^{n+1}(2^{n+1}-2^{2n}-2)}{(2^{2n}+2^{n+1}+1)(2^{n+1})} \geq \frac{1}{2}$

$\frac{2^{n+1}-2^{n}-2}{(2^{2n}+2^{n+1}+1)(2^{n+1})} = \frac{2^{n+1}(1-2^{n-1}-2^{n-2})}{(2^{2n}+2^{n+1}+1)2^{n+1}}$

$\frac{2^{2n}+2^{n+1}+1}{1-2^{n-1}-2^{n+2}} \geq \frac{2}{1}$

Figure 163: Chris' work on Question 3 (3 of 5)

Regardless of why Chris chose to use integration, when he got to the point of integrating, he mistakenly took the derivative of 1/*x*. The rest of his time out of the interview was spent manipulating his evaluated integral to try to show that it was greater than or equal to 1/2. I pointed out to Chris that the anti-derivative of 1/*x* was ln *x* and he did the work in the next figure.

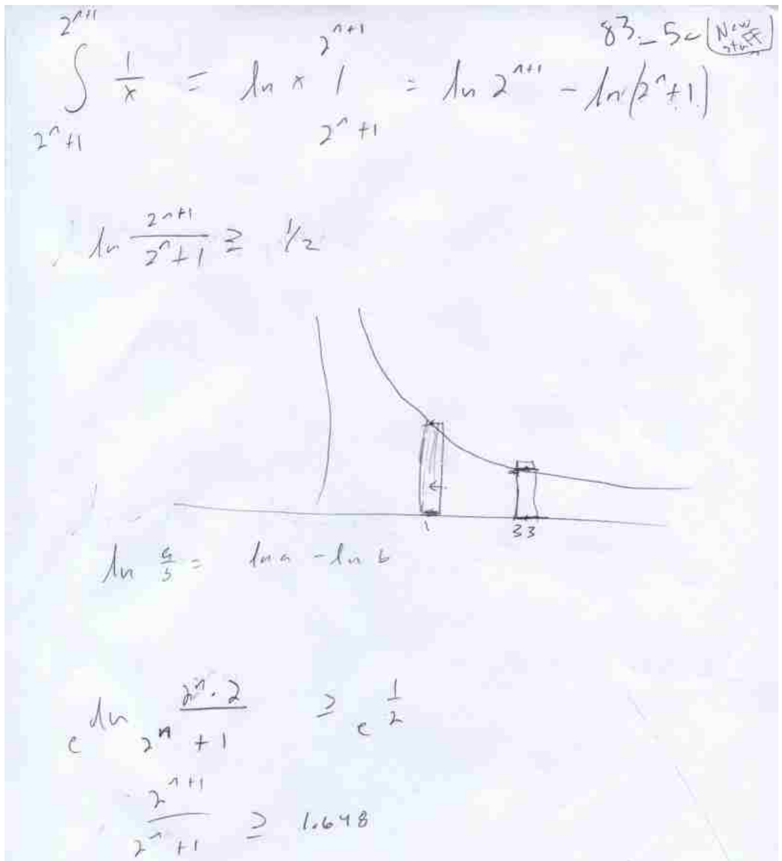


Figure 164: Chris' work on Question 3 (4 of 5)

Chris got to the end of his work here and was not sure where to go next. I told him that he could have more time to work on the problem if he liked, but he decided he would rather talk through the solution to the problem with me. The work we did together is in Figure 165.

We set the limits of integration so that the integral would be an underestimate of the sum $\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}}$ and hoped that it would still be greater than 1/2. After taking the

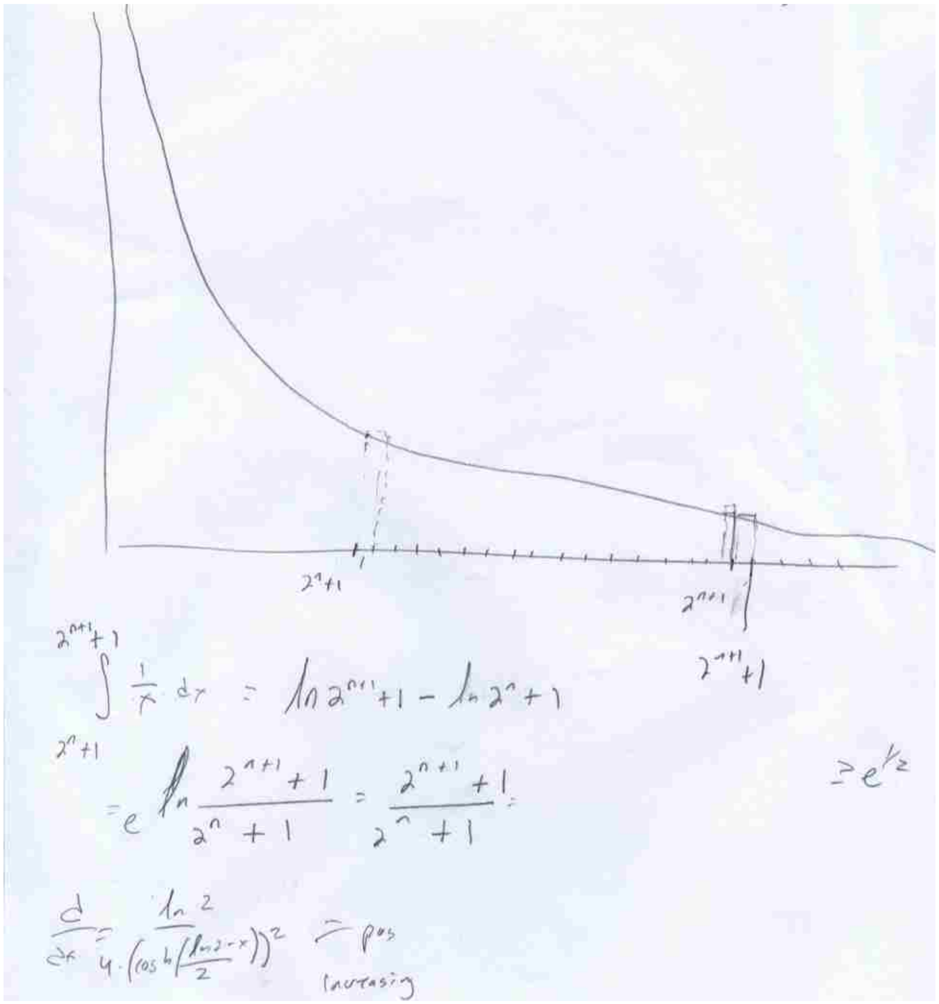


Figure 165: Chris' work on Question 3 (5 of 5)

integral and exponentiating each side, we showed that this was the case because the resulting function of n was strictly increasing (checked using the derivative) and already large enough when $n = 1$. Thus the proof was complete.

Because Chris did not complete the proof on his own, I will classify the work he did alone. The classification does not change, though, and because he was working on an induction proof, this is a procedural proof. Like with Question 2b, Chris does not follow a strict set of procedures. Instead, he sees the few general steps involved in induction and therefore his proof attempt is classified as a process procedural proof.

Also similar to the last problem, Chris gives evidence that he has not fully interiorized the method of mathematical induction. The most obvious support of this claim has to do with, again, his choice of base case. Like last time, Chris decides to use *n* = 2 as his base case.

> C: So I did my base case, *n* = 2, and that was 1/2+1/3+1/4, which ended up being 13/12 which is greater than 1, because 2/2.
>
> N: Right, ok
>
> C: So now…
>
> N: Is there any reason why you chose 2 instead of 1?
>
> C: No, I assumed 1 was going to make it equal because it's just 1/2 = 1/2.
>
> N: Sure. So you thought it would be just a little more interesting if you did *n* = 2? Because equality still satisfies this right?
>
> C: Right, yeah, it was more for my own…

Even though Chris says he considers the *n* = 1 case, he still does not choose to use it as his base case. He uses 2 because he thinks it may aid his understanding. However, even after he has the understanding of how the sum is acting (between interviews) he still uses *n* = 2. The fact that he sees the base case as flexible reveals that he still has an internalized transformational proof scheme when it comes to induction.

The other issue that arose last time dealt with his wording when referring to the induction hypothesis. In this case, it is not a misconception but rather an example of misused terminology. During the reflection period of the first interview, Chris was describing his efforts to separate $1/2^n$ from $1/2^{n+1}$: "Because I want this $1/2^n$, I want, I have to use my base case." While it appears as though Chris has one misconception where he was thought to have two, he still does not regard the base case properly and displays an internalized transformational proof scheme.

Question 4

Examine the following proof by contradiction:

Prove $\sqrt{-1} \le 0$

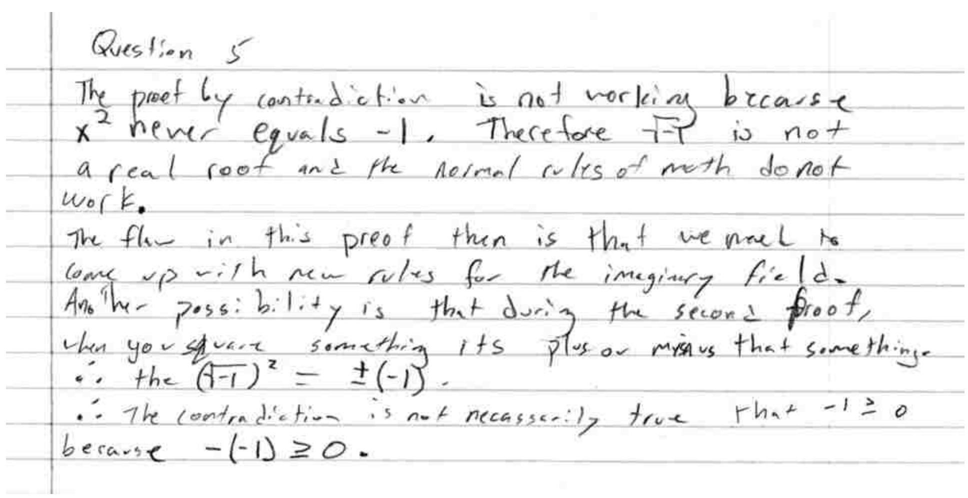
Suppose $\sqrt{-1} > 0$, *then* $\sqrt{-1} \times \sqrt{-1} > 0$. *This implies* $-1 > 0$, *which is absurd. Therefore,* $\sqrt{-1} \le 0$.

Prove $\sqrt{-1} > 0$

Suppose $\sqrt{-1} \le 0$, *then* $-\sqrt{-1} \ge 0$. *This implies that* $\left(-\sqrt{-1}\right)^2 \ge 0$, *so* $-1 \ge 0$ *which is, again, absurd. Therefore,* $\sqrt{-1} > 0$.

Why is a proof by contradiction not working here? Explain the flaw (if any) in the proof.

Chris first saw this problem on the take-home exam he completed while taking MATH 305. The response he gave on that test is in the figure.



Question 5

The proof by contradiction is not working because x^2 never equals -1. Therefore $\sqrt{-1}$ is not a real root and the normal rules of math do not work.

The flaw in this proof then is that we need to come up with new rules for the imaginary field. Another possibility is that during the second proof, when you square something its plus or minus that something.

∴ the $(\sqrt{-1})^2 = \pm(-1)$.

∴ The contradiction is not necessarily true that $-1 \ge 0$ because $-(-1) \ge 0$.

Figure 166: Chris' previous work on Question 4

Chris' answer is that because *i* is not a real number, it can not be handled with the same rules as real numbers. He also confuses squaring a number with taking the square root of a number, saying squaring something makes it "plus or minus that something."

During the interview, Chris found what he eventually came to consider the cause of the error he found on the midterm.

> C: This part, this greater or equal to 0 is using the, in my mind, the real number line and you have to, for the imaginary number line, to integrate the imaginary numbers into the number line I think is where I would find the fault.

He mentioned this before even before he read the problem. After reading the proof, he did not change his tune:

> C: Why is a proof by contradiction not working here? Find the flaw in this proof…I don't know if I see a flaw in the proof, but again, I would say…
>
> N: Sort of mechanically you mean?
>
> C: Right, the algebra, I don't see "Oh, this is wrong so this is why this is not…"
>
> N: Right, right
>
> C: So I don't see that.

After talking through a few of the steps in the proof to make sure he understood them, Chris went back to the mistake he saw originally:

> C: So to say that something's greater than or equal to 0, if it's not on the number line, it couldn't be greater than or equal to 0. Or less than zero because in my mind the number line is deciding the greater than or equal to 0 part.

N: Ok

C: And so because I can't place, you know, if you told me to put square root of -1 on the x axis, I would easily be able to say it's greater than or less than.

N: Sure

C: But it doesn't (exist) on that axis, it doesn't exist on that line, that number line.

N: Right

C: So if you can't place it there, then why would it have to follow the rules of being greater than or less than 0?

Because i is not on the real number line, it is not a real number and therefore need not follow the rules of real numbers. Chris then traced the main flaw back to the assumption that put i on the real line in the first place.

C: Now my answer would be, you know, if I had to write down "explain the flaw" or why the proof by contradiction isn't working, it's that our very first step of having to suppose something is on this number line that's greater than 0, you know, that's how it's defined, right, that a greater value has more, whatever, units I guess, to it, I don't know…

N: Yeah, more magnitude or to the right of 0.

C: Yeah, I like that the most, to the right of 0, but you can't even suppose that because it doesn't, it's not placed on, even adding it to the field and using it, it still isn't placed on our graph.

Because Chris does not create a proof here, there is no proof attempt to classify. However, this interview does make clear that Chris has a deductive view of proof; because the beginning of the proof is flaw and the rest of the proof is deduced from that, the rest of the proof has no reason to be valid. Chris does not allude to any previous results here and does not mention the necessity of relying on undefined terms or axioms so his scheme is not axiomatic here. Also, he does not do any manipulations that would be typical of a transformational proof scheme. This question did not yield evidence for the transformational or axiomatic proof scheme but it does support the idea that Chris has a formal view of proof. Thus, he is displaying the more general analytic proof scheme here.

Question 5

Like with the last problem, Chris did not attempt to create a proof on his own during the next interview. Instead, the interview was used as a debriefing session to look back over the first half of the study. Even though Chris did not complete a proof, it did serve to reinforce some of the observations made thus far in the study.

When discussing the progress he had made during the semester, Chris mentioned being better able to judge the level of rigor in the proofs he was attempting.

> C: I have definitely gotten to the point this semester, where I wasn't at before, where when before I say something in mathematics, I think about it in every case. You know, before I say that this is what it is.
>
> N: Sure, yeah.
>
> C: So…
>
> N: I think that's one thing you learn when you're doing proofs, just how technical and rigorous you need to be.
>
> C: And take that step back.

N: And really decide what you want to say?

C: Yeah, is this a true statement? Can I do that? Stuff like that.

This meshes with the fact that in every interview so far, Chris displayed an analytic proof scheme.

Most often, the analytic proof scheme Chris showed was transformational. One of the aspects of this proof scheme is the use of anticipatory actions in the completion of a proof. It is clear Chris recognizes this aspect of constructing a proof in this following exchange, which occurred after I asked him what he thinks it takes to successfully complete a proof:

C: Having the skills. Having that tool kit, I know (a professor) was always talking about that, wasn't he always talking about a toolkit, the skills you need?

N: Yeah, I think he's talked about that. I mean, it's a pretty common analogy.

C: Right, and I think that's essential for a lot of them but also having this idea of what's the final goal. Having different ways of solving it. You know, like when you and I were sitting down and you'd be like "Ok, what's the last thing we want?" Ok, so that's our goal, so let's work backwards, how can we build up to this. Something like that.

Chris sees knowing where you are going as necessary to completing the proof because it allows you to perform the steps needed to get there. In Chris' mind, this is not enough, however; understanding the problem is necessary as well.

C: Yeah, to get that complete, correct proof you have to have that background knowledge on the subject alone, to go along with those tools, but to even, you have to know what the proof is even asking.

N: Sure.

C; And then the skills come in and those tools come in on how to get to our final goal.

N: Ok, so I guess I'd kind of characterize that as tools and the thorough understanding of the problem, both what it's starting out as saying and where you have to get.

C: Yeah, definitely, without that stuff, and a lot of times (a professor) would just be like, he'd seem so frustrated because we wouldn't know his next step and it'd seem so straight forward to him. And maybe now looking back it was straight forward, but until you know where, he knows where he's going. He knows his plan. Of course he knows his next step. But when he asks us for his next step, we're always like "I don't know, how about this" or "you could try this." But you gotta have that plan, you gotta have that "Ok, here's where we are, this is what I want you to show" and then it goes into "Ok, I can show that because I have the toolkit to show.'

That Chris views gaining an understanding as necessary first also matches well with what he did during the first half of the study. At this point, half of Chris' proofs have been semantic and his other attempts have featured a lot of exploration aimed at figuring out how to finish the problem. When I asked Chris what helped in completing a proof but was not necessary, he mentioned the background to understand the problem. The purpose for this background was to attack the problem.

C: (E)xploration is very much so why you need the background. You can not explore a problem without knowing what the problem means and the different parts.

N: Ok

C: Way before you ever go towards proving it, you have to, you know, "Does this make sense in my mind?"

This waiting for an understanding before turning to a proof is a hallmark of a semantic proof attempt. "Well, you've seen, I always draw a picture, I always just try to figure stuff out for myself, (that) is my essential first step in everything."

As I mentioned above, this interview reflects some of the observations seen earlier in the study. Chris sees proofs as coming from the understanding of problems (which is why he provided semantic proofs) and realizes that proofs depend on taking purposeful actions towards a known end (echoing his transformational proof scheme).

<u>Question 6</u>

Prove that $\forall n \in \mathbb{N}, \; n \equiv n^3$ *(mod 6).*

Chris and I began the next interview discussing modular arithmetic but he was fairly familiar with it from seeing it in MATH 305. To remind himself, what it means for *n* and *n*3 to be congruent mod 6, he writes $\frac{n}{6} = r$ and $\frac{n^3}{6} = r$ (see Figure 167)

After that, he noticed that it was similar to Question 2b. "Ok…I don't know, I was just thinking, maybe, this *n* and n^3, you know, that's come up before with the, if you put them on the same side, that $n^3 - n$, we've seen that before."

Despite this recognition, Chris starts over with this problem:

C: So I guess, the way I normally, this has like, induction, for me, written all over it.

N: Ok

C: But I don’t know necessary whether a direct proof or whether induction would be the way to go.

N: Ok

C: So if $n = 2$, then this would be 2 on this side…is that congruent to 2^3, so mod 6…so that would, because that’s 2…that’s 8, so if this is 2 and 2, then it’s congruent mod 6. So I’m thinking, so I guess I’ll just do induction and see where that puts me.

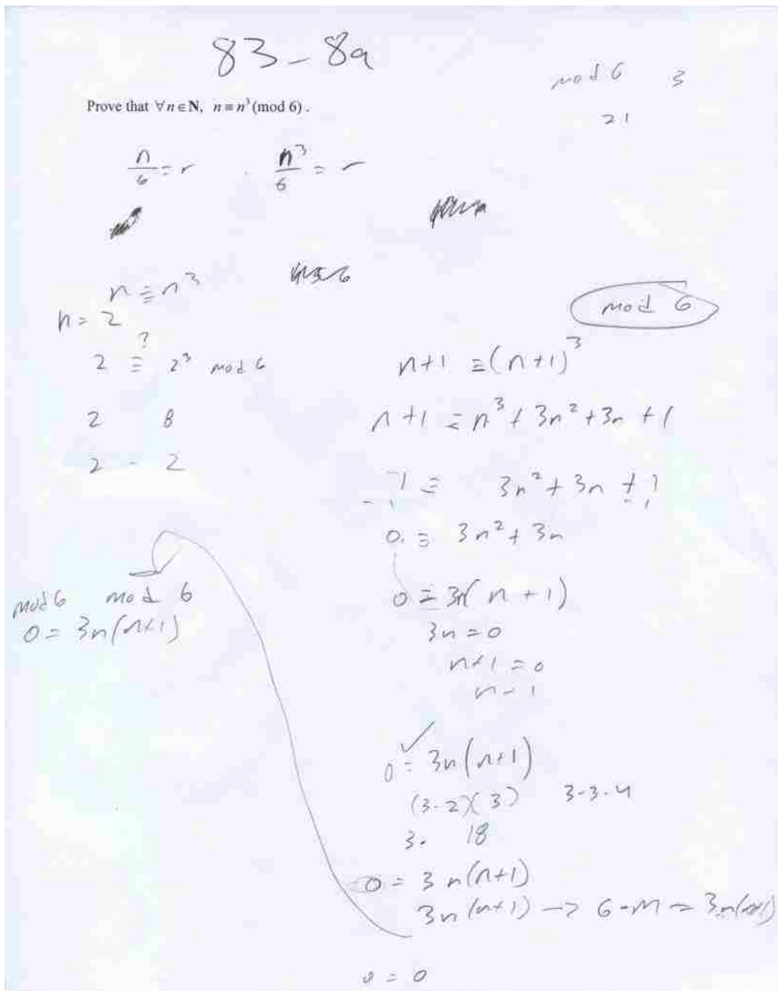


Figure 167: Chris' work on Question 6

Chris then proceeded to expand out $(n + 1)^3$ and use the induction hypothesis to arrive at the equation $1 = 3n^2 + 3n + 1$.

C: So I don't know if this is going to get me anywhere. I'm trying, I can't factor this… So I don't know if this is going to get me anywhere. I'm trying, I can't factor this…No matter what, we're multiplying 3 by an even, right, because we have 2 consecutive terms.

N: Right

C: So no matter what, this is a multiple of 6.

N: Right, ok

C: But whether that means it's not going to have a remainder…well, if you're taking 6 times an integer, yeah, I don't know, let's see. So for example, if we did 2, that'd be 6, so that's 6 times 3, so that's 18, which is 0 mod 6. If we did 3, that'd be 3 times 3 times 4. Because everything's a multiple of 6, it's not going to have a remainder, so this is a true statement.

Chris was still a little unsure that he had completed the proof, but soon he had talked himself into it.

C: See, I've never really done mod things before, I've never actually done in practice, or in proofs before, I only know the definition. So I wouldn't know, maybe, how, necessarily, to write this up…But I think I've proved it. Because this is a multiple of 6, so it's going to be 6 times some integer, it's going to equal, you know, $3n$ times $n + 1$.

N: Right

Because this is, one of these 2 has to be a multiple of 2 because it's 2 consecutive integers.

N: Right

C: And therefore it's not going to have a remainder, and so that's what this is saying on the other side.

The proof Chris did for this problem is a process procedural proof. He used mathematical induction and, as the other times he did, does not see the process as the following of a number of set instructions. Instead Chris sees the method as a few general guidelines.

Identifying Chris' proof scheme here is not as straightforward. One on hand, he still uses $n = 2$ as his base case. However, the reflection period of the interview shed some light into this choice.

N: Ok. Then you tried it for an example. So when you tried it, was that meant to be a base case for your induction argument?

C: Yeah

N: Ok, it wasn't a "Ok, let me see if it actually does work"?

C: It went for both.

N: Ok

C: It worked for both. I knew 1 worked too.

His choice of $n = 2$ was influenced by more than just how it would contribute to a proof by mathematical induction. It also helped him get a feel for the problem. This might have been part of why he chose $n = 2$ as a base case previously. However, in the past he continued using that value even after he knew how to complete the problem. Also, in this case Chris mentions the fact that it works for $n = 1$ without prompting. It is possible that my asking about it before prompted him to address that. This seems unlikely, though, because that interview occurred over two months prior to this one. It seems more likely to me that at some point in between the two interviews Chris gave consideration to the base case's role in proofs by induction and saw that it was necessary to start at $n = 1$.

At this point, I regret not asking Chris to write up a formal version of this proof. That way, I could see if he used $n = 1$ as a base case once exploring the problem was no longer an issue. As this is not the case, I speculate that Chris would have and therefore identify Chris' proof scheme as interiorized (rather than internalized) transformational.

<u>Question 7</u>

Suppose that A is a set containing n elements. Find the number of subsets of A and prove that you are correct.

Chris started this problem by making sure that he understood some of the basic ideas in set theory, like the fact that the empty set and the original set *A* are both considered subsets of *A*. I also told him that I only wanted to consider finite sets. From there, it did not take Chris long to come to a formula (see Figure 168).

> C: Ok…so, I mean, the systematic way to do it is to, is that each element could be a subset. So, you have A, you have the set A, you have the null set, you also have *n*…and then I assume you just have *n*C2 for all the subsets that contain 2 elements from *n*. Likewise, *n*C3, *n*C4, dot dot dot. *n*C*n* is A, so I guess the last thing you would do is *n*C(*n*–1)…Ok…It seems like a lot of work, I'll be honest. So I think what you'd have to do is, you would add all of these things together.

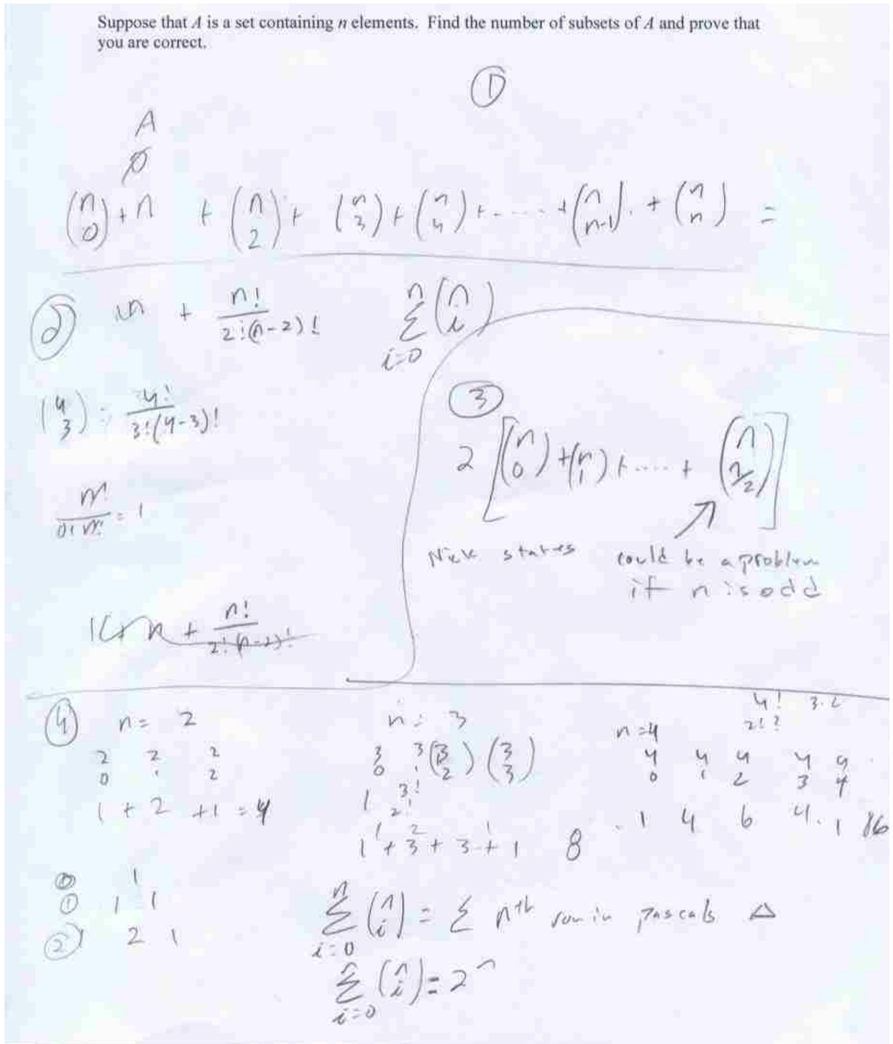


Figure 168: Chris' work on Question 7

Chris thought that the answer he was looking for would be more concise than this, so he looked at the choose function to remind himself how it works. He also wrote his formula in summation notation and turned his focus to a proof.

C: So now, I mean, I'm confident that this is all…the subsets.

N: Ok

C: Now the fun part and why we're here, the prove part. Right? So typically, when you give me an *n*, I want to do a proof by induction.

N: Ok

(thinking)

C: However…I don't know what the hell I'd do a proof by induction about. Because I don't have this equaling something, so I don't think that's the way.

Chris also considered listing all terms in the sum explicitly, but did not think it was feasible. He tried to use the symmetric nature of the choose function (*n*C0 = *n*C*n*, and so forth) to help with that, but I mentioned the possibility that *n* was odd.

Since he did not know where else to go, Chris decided to try it for a few values:

C: Ok, so this is all the subsets of A…I'm just going to plug in a number and see what that does. Kind of like a base case, you know. So if *n* = 4, maybe I'll see if there's a pattern, who the hell knows. I'll do *n* = 2, 3 and 4. So if *n* = 2, you have 2C0, 2C1, 2C2, so that's 1, 2C1, that's 2, plus 1, so that's (4). If *n* = 3, 3C0…3C2, 3C3, so this goes 1, this 3!/2!, so that's 1 + 3 + 3 + 1. Well, so far, I mean, you're looking at Fibonacci sequence, or Pascal's Triangle I mean.

N: Ok

C: So if that keeps up…then I'm going to go with something along those lines.

After adding up his sum for both 3 and 4, he noticed that he was getting successive powers of 2. I reminded Chris that during MATH 305, it was proved that the rows of Pascal's Triangle do indeed sum to powers of 2.

C: So if I, so this isn't what I'm trying to prove? Because this is known.

N: Yeah

C: So I have to show that this defines all of the subsets, more so than what I'm doing here?

N: Well, I don't know about more so, I guess. I mean, I guess what I'm probably asking you is…Once you've gotten to this point, you've made the claim that the number of subsets is this sum.

C: Right

N: So, at this point you have to decide if you've proven it or if it requires more proof. Right?

C: Right

Chris then described the reasons why he had convinced himself that his sum gave the number of subsets. I then reiterated the question I asked him above:

N: Right. I guess the question's become does the argument you've just provided, does that constitute a proof? And if not, what more is required to make it a proof?

C: A contradiction [laughs]. Ok, so my 2 arguments were subsets must contain *m* elements such that *m* is an element of the positive integers, *m* must be, it's bounded by 0, positive integers, up to our *n*. And it has to be an integer, ok? And my other argument is, then, that the, I don't know what that was called, the choose function…gives all possible, well not gives all possible…

N: …gives the number of all possible.

C: ..function gives the number of possible combinations within each subset. So as long as I list all possible subsets, and this function gives me all combination within each subset, then I should have given, therefore there's the total number of subsets.

N: Ok

C: And I don't see, and you're going to be the one that tells me, where there would then me a hole in my argument.

N: Ok, so like, if this was a homework assignment, you would just write up this argument nicely and hand it in?

C: I'd probably throw this guy (the formula on the top of Figure 168) in there somewhere…And therefore this equation gives all possible *m* subsets as well the number of combinations within each subset, is how I would end it.

The proof Chris provides here is semantic. Although he did not see it as a proof right away, his understanding of the problem is exactly (not merely the basis for) his proof. Determining the proof scheme Chris displays here is nearly as straightforward. His proof is certainly formal, based on his two main starting points and deducing from there without using any previous results (although he does acknowledge that the properties of the Pascal's triangle have been proven). This suggests a transformational proof scheme. The idea that Chris displays a transformational proof scheme comes from the way he eliminated potential proof methods.

C: I'm definitely looking at the outcome of the proof before I ever dive into it, you know what I mean?

N: Sure

C: Especially with contradiction, like where is your contradiction going to come from?

N: Or, where is the induction hypothesis going to be used.

C: Right, and that's the thing that I knew there was no way I could have done that because 1, it didn't equal anything and 2, there's no way of breaking up (n+1)C0 into nC0.

This sort of anticipatory thinking is typical of a transformational proof scheme.

Question 8

Prove that the cube root of 2 is irrational using a proof by contradiction.

Like with Question 4, Chris first saw this problem on the midterm he took in MATH 305. The response he gave there is in the figure.

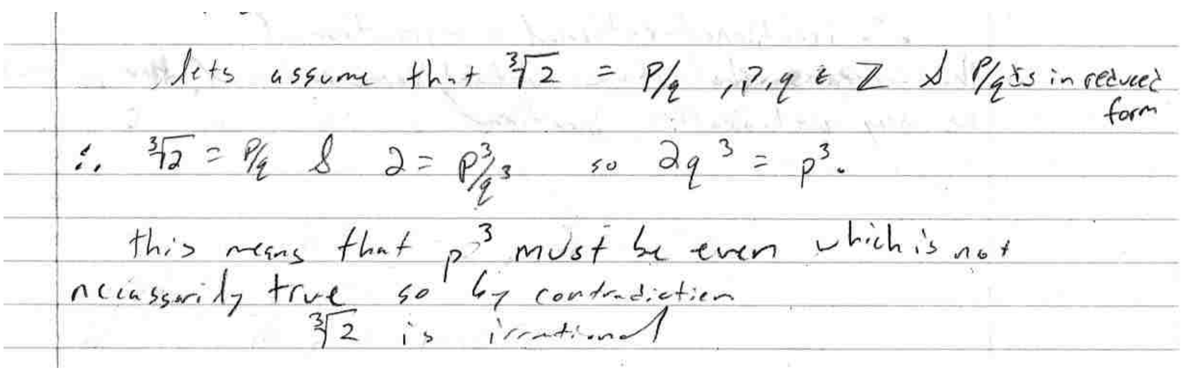


Figure 169: Chris' previous work on Question 8

On the midterm, Chris did not provide a correct proof. After assuming $\sqrt[3]{2}$ was rational, he deduced that the denominator, q, must be even. This was an apparent contradiction because initially q could be even or odd.

When Chris sat down to try this problem during the interview, I mentioned to him that when he saw the problem originally he had access to the proof that $\sqrt{2}$ is irrational. I also mentioned that if he wanted, he could flip the page over and do that proof first as a refresher, which he did. This proof is in Figure 170.

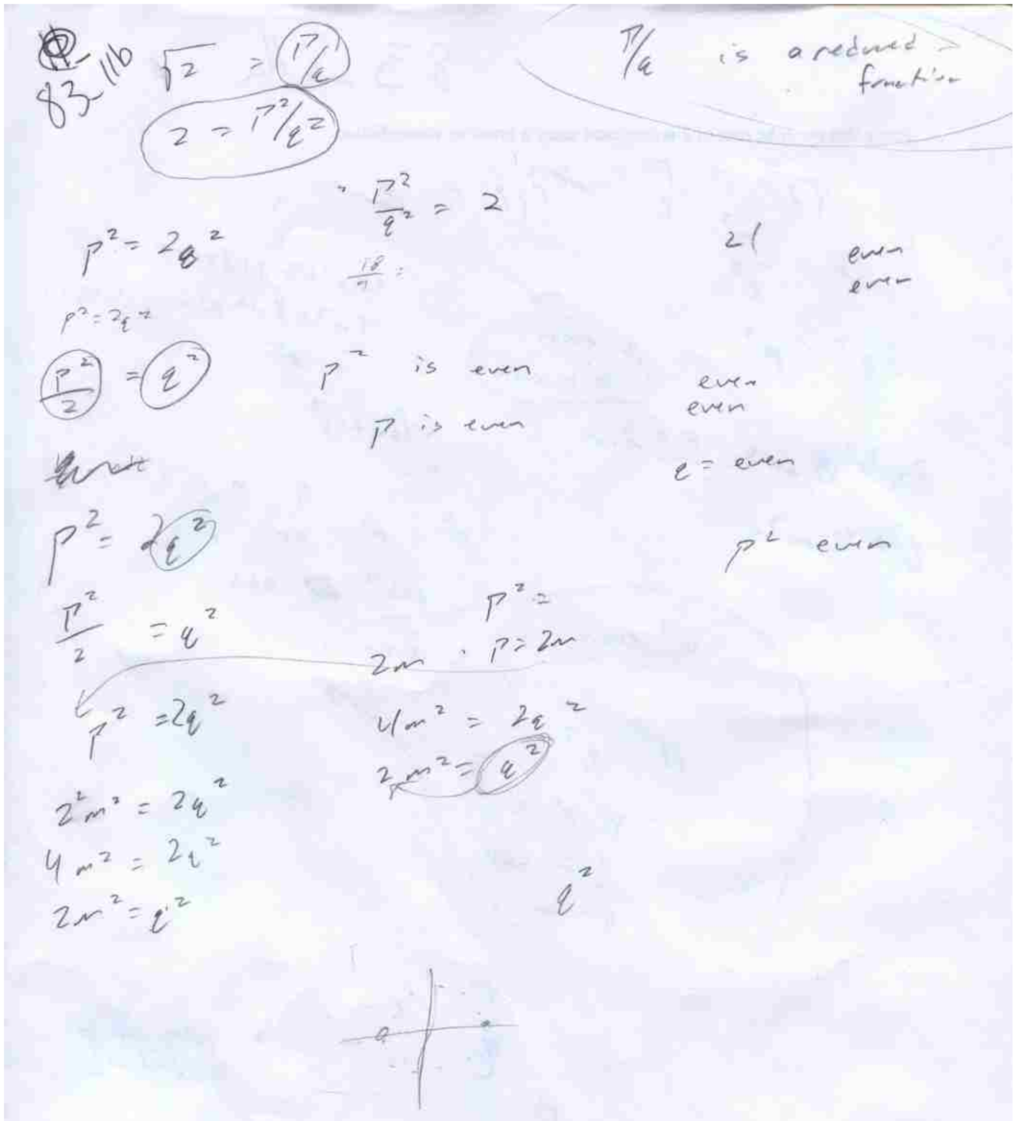


Figure 170: Chris' work on Question 8 (1 of 2)

C: I remember the big thing is like that this is a reduced fraction…I know the contradiction's here…p^2, let me think about this, so p^2 over q^2 equals 2…which means that p^2 is even, right? I know that I'm supposed to get that this is some how even over even…which says that it can be reduced and that was like where…because if p^2 is even, p's even. So I need to get it to say that q is even…Well, I have that p^2 is even and since p^2 is even, then so since p^2 was even I wrote, you know, p could be equal to 2m. Since p^2 is even, then p is even.

N: Right

C: So you could write p as $2m$, and if I plug $2m$ into this equation, I get $4m^2$ equals $2q^2$, which says that $2m^2$ equals q^2. Which, again, means q^2's even because it's a multiple of 2… Under the same argument that if q^2 is even, then q is even.

N: Ok…so yeah, you've reached the contradiction you wanted to reach.

C: Right, which says, then, that both p and q are even so you can factor out a 2, so there's your contradiction.

Chris had finished the $\sqrt{2}$ proof and went back to considering the proof for $\sqrt[3]{2}$. "So…do the same thing, cube both sides, 2 is equal to p^3 over q^3…And then the same argument, it's easy to see that p^3 is even. Now this seems so straight forward." He worked his way through most of the problem (down to $q^3 = 4m^3$ in Figure 171) and looked back over his work and realized there was something that needed justification:

C: (T)hat's the place where I'm stuck at, is just whether, is p^3 even implies p even. Although I know it is, is that what I then have to prove. So if p is even, p could be written as $2m$. So then that same argument, $8m^3$…q^3 is $4m^3$, so q^3 is obviously even as well. So, again, going through this circular argument, if q^3 even gets to imply q even…

Chris did not feel comfortable with the proof because he was using a fact that he did not know how to proof. This was not a concern with the $\sqrt{2}$ proof because "I knew this was true and I knew this wasn't my main problem. So I didn't want to spend a bunch of time working on it."

Chris did, however, spend most of the rest of the time working on this issue. He began by showing that an odd number cubed was still odd, unknowingly proving what he wanted to via contrapositive. The rest of his scratch work trying to complete this step is in Figures 170 and 171.

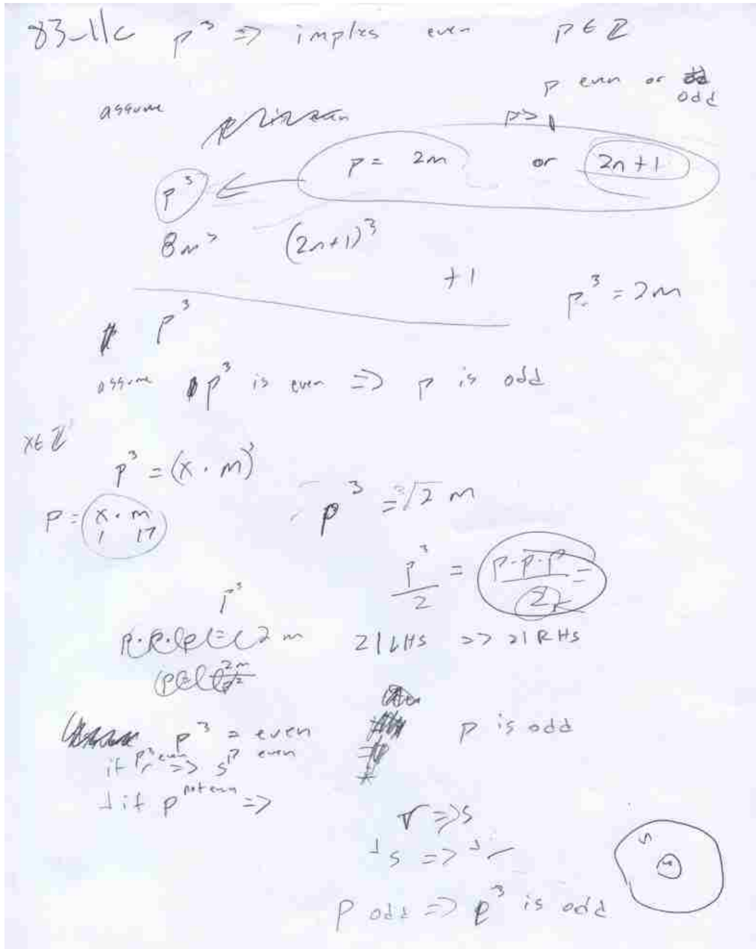


Figure 171: Chris' work on Question 8 (2 of 2)

Chris tried to complete his lemma with contradiction and other algebraic manipulations and eventually I explained that he had already finished the proof and reviewed proof by contrapositive with him. Once he saw that, he was happy with the proof.

> C: But…showing that something works like this (*n* odd implies n^3 odd) without the reasoning here (bottom of Figure 171)…
>
> N: Right
>
> C: …was what I needed to feel comfortable with this proof.

The proof Chris provides is an algorithm procedural proof. He recalls the proof that $\sqrt{2}$ is irrational and applies the same steps with little adjustment. Clearly, the steps Chris follows are meaningful for him as he spends most of his time in the interview verifying something he intuitively believes anyway.

Chris' proof scheme here is transformational. His proof relies on deducing that p is even, replacing it with $2m$ and deducing his contradiction from there. Like with many of Chris' proofs, he keeps his goals in mind while working on this problem. He remembers the contradiction he is to eventually find and works towards that goal. He even mentions that if he were to write up the proof formally, he would pay even more attention to anticipatory actions:

C: (T)hen you can write down the proof of all these little sub, lemmas or whatever the hell. Like this would be, this is the perfect lemma to do before you do this proof.

N: Right

C: Then you can just say 'By lemma 1A…'

Question 9

Like Question 4, I had Chris evaluate a proof. This time, I gave Chris a version of Cantor's Diagonalization argument. Since he does not attempt a proof, there will be none to classify. As usual, the interview will be used to look for insight into Chris' proof scheme.

Chris needed to read through the proof a few times and talk through it with me to really understand it. His main hang up involved making he connection between B had to be in the list, and therefore equal to $f(k)$, and why it could not equal $f(k)$.

C: So B is just another decimal expansion?

N: Yeah

C: That's it. And so…obviously there's an *f*(*k*) that equals *B*. There has to be an *f*(*k*) that equals *B*.

N: Given our assumption that it was countable to begin with.

C: …and *f*(*k*), *f*(*k*) is going to be α_{k1}, α_{k2} dot dot dot … equals β_1, β_2, but α …but β, ok so I should really get those in here (write out work in Figure 172), there's α_{kk}, there's β_k. β_k…absolutely does not equal α_{kk}. … Which means these don't equal, so these don't equal (crossing out equals signs from the bottom up) … So the contradiction, then, is that *f*(*k*) does not equal *B*.

(thinking)

C: It's gotta be in the list somewhere, right?

N: That's what you'd think. Yeah, so yeah, so if you think *B*'s in list somewhere, then there does in fact exist a *k* such that *f*(*k*) equals that *B*. Right?

C: Right, absolutely. But that doesn't happen.

N: But that can't happen so therefore *B* can not be on the list.

C: And if *B*'s not on the list, and *B*'s in between 0 and 1, then you didn't count *B*, you didn't include *B* in the set, so obviously…

N: …you didn't include B in the list.

C: Right. Yeah, that makes sense.

Once Chris had the argument down, he was able to articulate his trouble and sum up the proof nicely:

C: That's, yeah, by far the most confusing part is that there is this *B* that exists in this set, obviously it exists in the set, it's made up of 1's and 2's, it's point 1's and 2's, and so there has to be, if we've defined this set of everything, then there has to be an $f(k)$ that defines B.

N: Yeah

C: But $f(k)$ can't equal B because β_k doesn't α_{kk}.

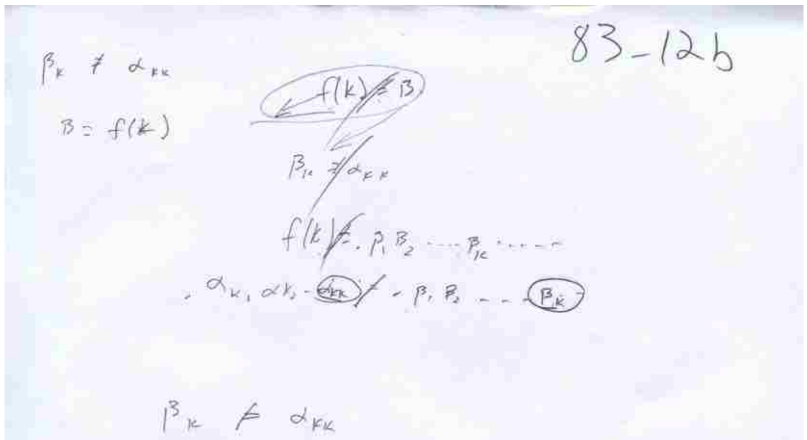


Figure 172: Chris' work on Question 9

Once I was convinced he understood the proof, I asked him if he saw anything in the proof that was not justified but should be.

C: I don't know, just the….let's see…Yeah, every subset of a countable set is countable.

N: Ok

C: That's a, that would be a claim that I would say I believe it, you know, like…

Chris believed the proof and since he was not the one who wrote the proof, he was willing to believe it because it made sense to him. He would have been more meticulous, though, if he would have been the one writing the proof:

C: If I've learned anything from proofs it's that anything that you write down, you need to be able to back up, and I've written statements down in proofs that I've gone home and thought, you know, 'I didn't back that up' and mentally I backed it up at home.

Chris was also able to see the implication for the rest of the proof had that statement failed: “If there was a hole in that statement, that’s the key statement, you know, one of the key statements that, of what we’re doing here. Without a doubt.”

Chris is displaying an analytical proof scheme here. He makes sure he understands the proof before he is comfortable saying he believes it. Also, he is able to see that a flaw early in the proof would mean that rest of it was no longer true. The form of the proof and the fact that I told him it was famous might have been enough to convince him. If it were, that would be evidence of an external conviction proof scheme. This is not the case, however, and Chris remains consistent with an analytic proof scheme.

<u>Question 10</u>

Prove or disprove: Between every pair of rational numbers that share denominators and whose numerators differ by 1, there is another rational number strictly between the pair whose denominator is less than that of the pair.

Example, 6/9 < 5/7 < 7/9.

Chris started this problem by trying it for a few examples, just to see if it was going to work. “I mean, if it works for 1/3 and 2/3, that’s the minimum, and I found 1/2.” Once he was satisfied that it would, he turned to how he could go about proving it. “Alright, well, I definitely don’t think this is induction. I don’t have too many tools, but I don’t think it’s induction.” During the reflection, I asked Chris why he eliminated induction so quickly and he said that it was because it “just seems like a lot of variables.”

Since one of his main tools had been taken out of consideration, Chris went to the other proof method he was comfortable with: contradiction. For this proof, he assumed that one could find a rational number between the two given ones, but that this new rational had a denominator greater than that of the pair, $\frac{a}{b} < \frac{c}{d} < \frac{a+1}{b}$ with $d > b$.

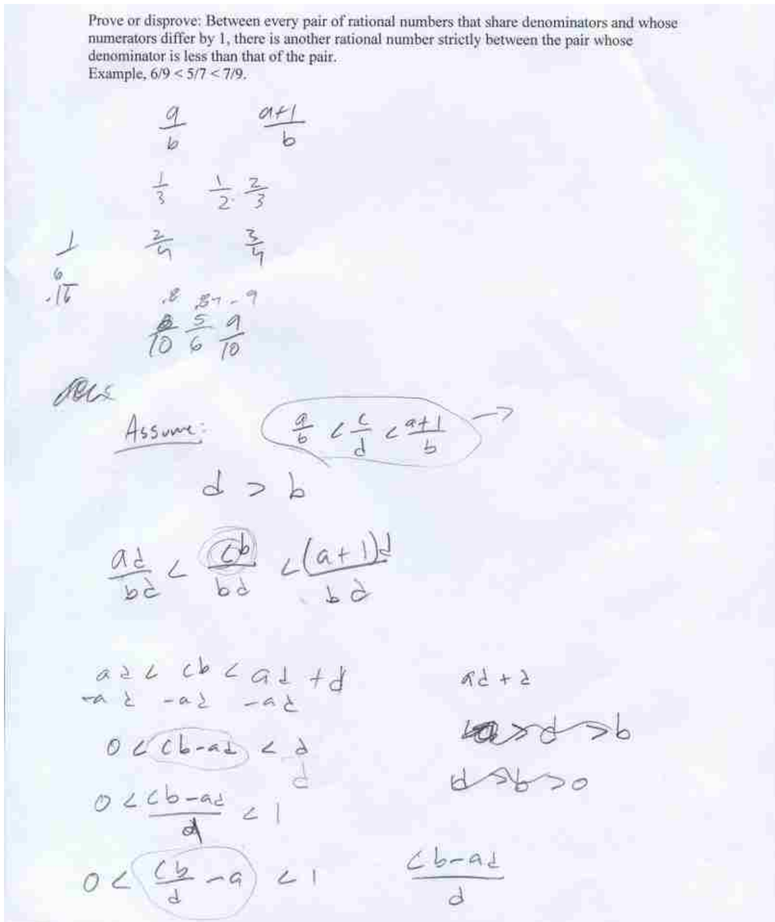


Figure 173: Chris' work on Question 10 (1 of 3)

Chris did some algebraic manipulation but could not come to a contradiction as he hoped: “I wish that I could say that this (circled expression at bottom of 4.83.25) is negative or something like that.” At that point, Chris restarted his work on a new sheet (Figure 174).

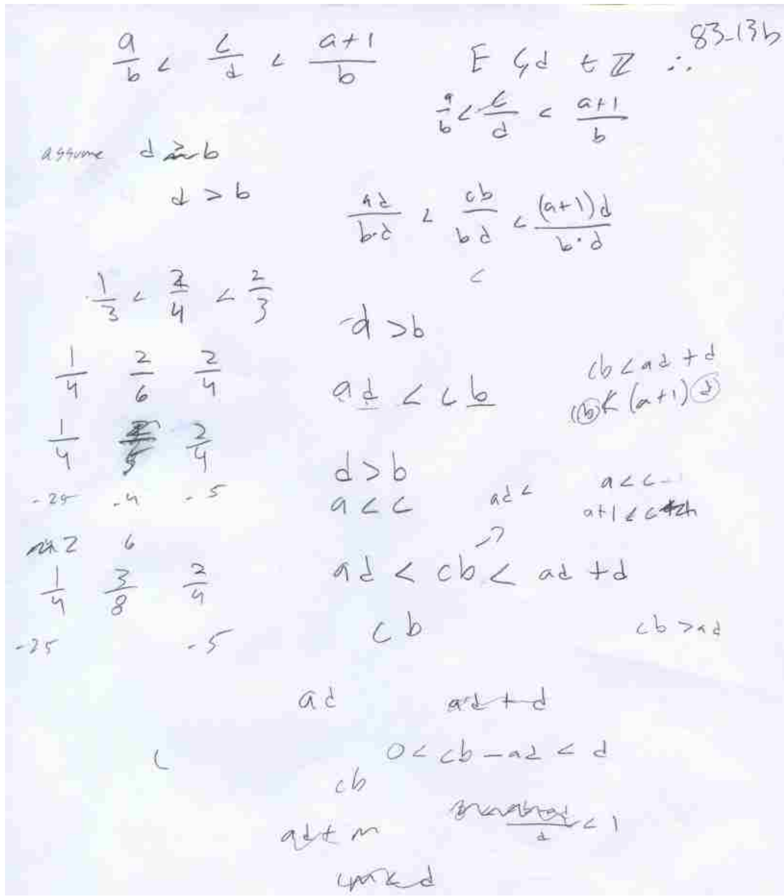


Figure 174: Chris' work on Question 10 (2 of 3)

Chris found some more examples but this time they matched the scenario he assumed to start his proof by contradiction. After he found them, he tried more rearranging in hopes of finding something obviously false. Because the interview was getting near its end, I pointed out to him that since he found such examples, there would be no way he could deduce something necessarily false from his assumption. He then scrapped that assumption but spent the rest of the interview continuing with his algebraic manipulations.

C: Yeah, I really don't know. This is, I'm just doing the same s&*! in a different form.

(thinking)

C: Yeah, I don't know. I don't know how to prove this by a direct proof.

By this time, we had run out of time and began to reflect. I asked him why, at the beginning of the interview, he said that 1/3 was the "minimum."

C: Well, you can't do 1/2 to 2/2, I mean, I guess you could, but…every pair of rational numbers that share, so yeah. I mean, I guess I could have started out with 1/2.

N: Ok

C: But then…it's interesting, because it's 1/2 to 1, but a denominator less than that is 1, and 1's never in between 1/2 and 1.

N: Yeah

C: So maybe I should have started out with that.

N: Could have. I would have just asked you to add some restrictions so that it would work.

C: Fair enough.

So, Chris realized that the statement was false as written but also agreed to work on the problem more for the next interview. He said he would and I asked him what thoughts he had on where he would go from there: "Yeah, it seemed like here (Figure 173) d was constantly $b - 1$ but we'll see. You know, that was the d I chose. I could always choose a d that was $b - 1$. So I'll see if that continues to hold."

When he came to the next interview, Chris had worked on the problem but did not bring his work with him. He was able to recreate his proof with a little time. The work he did is on the left side of Figure 175. He remembered the idea to let $d = b - 1$, worked with that for a bit and then remembered that he could also use the substitution $a = c$.

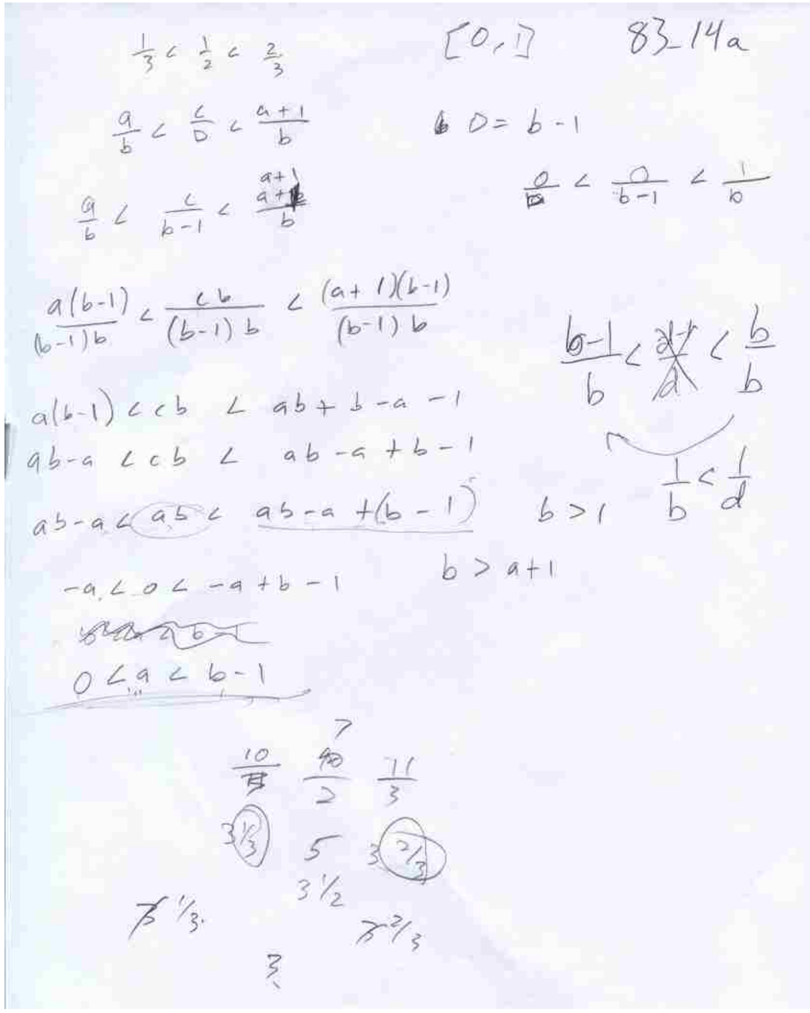


Figure 175: Chris' work on Question 10 (3 of 3)

C: Right, I'm pretty sure I just said "Ok, then there's an integer that lies, that there's room for an integer to lie in here. (see circled '*ab*') Oh, wait, or did I just say that this is a true statement? … Right, so *a*, *b* has to be greater…than *a* (+) 1, which makes sense because *b* is greater than *a* because I mean…if you're looking in between, if you're looking in the range 0 to 1, then it's very clear.

Chris thought that he had finished the proof given his restriction that the rationals be strictly inside the unit interval. Then, I talked through with him how to extend the result outside the unit interval, provided neither of the endpoints of the inequality were an integer.

The proof Chris provides here is a syntactic one. It is logically deduced and does not follow any prescribed steps. However, it is also not based on any intuition into the problem. Instead, the idea to restrict the numerator and denominator was based on the observation of examples. He did not know why such a restriction would work, only that it seemed to. Then, once he had that guess, Chris was able to come up with a proof.

The way Chris turns his guess into a proof displays a transformational proof scheme. His proof is composed of algebraic manipulations. While he likely did not know exactly where the operations he performed would lead, he did do them hoping to arrive at an obviously true statement which he eventually did. This is not the anticipatory action that would constitute a transformational proof scheme alone, but this combined with his algebraic manipulations does.

Question 11

The interview in which Chris finished Question 10 was the last interview of the study. After discussing the previous question, Chris and I talked about the study overall. Because Chris did not attempt a proof, there will be no proof attempt to classify. As is usual, though, the debriefing session will be used to look for clues into Chris' proof scheme and to reinforce observations made throughout the study.

I started by asking Chris if he felt like he had gotten better at proofs over the course of the semester. He said "I think so. I like to think so. I sure hope so." I asked him what led to the improvement he saw, and he said that practice was the biggest thing.

N: Ok, so you think just, you saw more, you had to do more proofs and that's what led to the improvement?

C: Yeah, yeah absolutely. And it's really just…same old same old proofs I haven't, I've stayed at the same level of proofs. I've gotten better with this, I would say, the same type of proofs, these number theory, these proof by induction, proof by contradiction.

When Chris says that he has "stayed at the same level of proofs" he is referring to the level of proofs he has seen and the skills necessary to complete them, not his ability. His focus on proof methods also came up later when I asked Chris what could have led to more improvement:

C: Well, just the skills, you got to have, the more techniques you have in math, the better off you'll be. Whether that means writing series, doing a better job with series, which I'm not good at, or…

N: Take analysis.

Complex or something along those lines. I think that's always, it's like you gotta know the language before you can do the work and to me going to class is the language and proofs are the work. It's the outcome, it's what we strive for.

N: Ok

C: So just the, I don't know how much farther my proofs could go with the background in math I have. I mean, obviously there's a lot of cool proofs out there, that I haven't done.

N: Right

C: But I believe I can do them, if they, you know what I mean?

N: Yeah, well, some of them, depending on whether or not the classes you've taken lend themselves to the methods necessary.

> C: Right, if I've had the material, if I have the background of "Ok, well, this is our goal, this is what we want, this is what you learned in class, now prove it in the general sense."

I believe this last statement refers to Chris' transformational proof scheme. Often over the course of the study, Chris would look to the conclusion of the proof to see how to get there rather than at the beginning to see how to start. Combining seeing where he needs to go with the knowledge of how to get there is the approach Chris takes to proof.

I also asked Chris if there was anything new that he implemented in his proof attempts:

> C: Contradiction. I feel like I've gotten better with the contradiction, writing them, being able to write the contradiction, that I've improved on…I mean, we've done contradiction before, like square root of 2, but I've gotten better at it…Yeah, if I can't do induction, I'm going to do contradiction.

This matches what Chris did with the last problem, where first he considered induction as an option and then worked for most of the interview on trying making a proof by contradiction work.

I then asked Chris what role he saw examples playing in proof. "I think examples essential, I could never imagine doing a difficult proof I've never seen before without examples, without writing down numbers, and just go try it, seeing if it's true." He mentioned the last problem as time when examples led him to see how to do a proof:

> C: Just like with this one, $b - 1$. I had to do all the examples and it still took me a full interview to think "Oh, $b - 1$, oh my god, every time $b - 1$." Likewise, every time, c, in my example, c over $b - 1$, c equaled a.

N: Yeah

C: Would never would have seen that. I don't know how you come up with that (stuff) without examples.

While his proof for the last problem was syntactic, I do not think it is a stretch to apply this line of thinking to his view of proof overall. In much of his work, proofs came from understanding he gained through the use of examples which is why semantic proofs showed up so often.

The next question I asked Chris had to do with what he saw as necessary to complete a proof and what he regarded as helpful but not necessary.

C: So obviously the background knowledge. If you're doing a complex (analysis) proof, you have to know complex.

N: Ok. That falls under the necessary…

C; Necessary, required. Unnecessary but really helpful is similar proofs you've seen before. Necessary and helpful is examples.

N: Ok

C: Absolutely, and then having these tools, having contradiction, induction, direct proof tools are necessary.

This last exchange sums up the things he said earlier in the interview which, in turn, dovetails nicely with the work he did during the study.

Chris' progression

Below is a chart of the types of proofs Chris attempted and the proof schemes he displayed over the course of the study.

Question	Type of proof	Proof scheme
1	Semantic	Transformational
2a	Semantic	Transformational
2b	Process	Transformational (Internalized)
3	Process (Attempt)	Transformational (Internalized)
4	N\A	Analytic
5	N\A	Transformational
6	Process	Transformational
7	Semantic	Transformational
8	Algorithm	Transformational
9	N\A	Analytic
10	Syntactic	Transformational
11	N\A	Transformational

Table 9: Summary of Chris' work

Judging by the table, it does not seem like Chris made too much progress over the year in which the study took place. The only place where there seemed to be improvement was in Chris' understanding of induction. Even there, the progress made was minimal. As was the case with others in the study, the lack of substantial progress is not a cause for concern. Instead, Chris did not make much progress because he had little progress to make.

As he mentioned in the final interview, there is certainly more mathematics he can learn. However, as far as proof goes, he came into the study with a mostly analytic proof scheme. Also, his tendency towards semantic proofs often led him to naturally caring out the first step of Polya's guidelines to problem solving: "understanding the problem." Once he had the understanding, Chris would look to the conclusion of the proof to see how to get there (Polya's "devising a plan") and then carry out that plan. Although it would generally occur after the interviews, Chris and I would often discuss alternative proofs for the problems he looked at. In addition to sheer practice, Chris mentioned in the last interview that this helped him learn.

N: Yeah, but I mean, the improvement you think comes from the practice?

C: Absolutely. And diversity…And doing one proof three different ways, stuff like that.

N: Ok, yeah, we did that in here once.

C: I like that stuff.

N: Ok, so…

C: But even, maybe we didn't, you and I always did it, though. Whenever we got done, you and I always discussed, "Oh, well, we could have done it this way" or "We could have done it this way" or "Oh, here's an interesting way" and that's what's cool, that's what…

This looking back at the problem completes Polya's fifth step in problem solving. From the beginning of the study, Chris followed problem solving steps that I believe most undergraduate mathematics professors would recommend. Pairing his approach to problem solving with his formal view of proof makes it easy to see why Chris' apparent lack of progress is not discouraging.

Chapter 5: Discussion and Conclusions

This chapter looks at the inferences that can be drawn from the data collected for this study. First, the observations made from this study will be compared to those found in a recent dissertation completed by VanSpronsen (2008), also at The University of Montana. Next, charts summarizing the proof types and proof schemes of all students will be examined for the purpose of getting a better overall picture of the data collected for this study. Based on this picture, some similarities between participants become evident and the following section looks at three groups of students that were formed based on these similarities. The third section discusses the conclusions that can be drawn from the data analysis and the fourth looks at the implications these conclusions have for teaching mathematical proof. In the final section, some ideas for further research are explored.

5.1 Comparison to VanSpronsen's Results

One of the benefits of choosing a research methodology based on task-based interviews involving proof is that it is very similar to the methodology used by VanSpronsen, who completed her dissertation at The University of Montana in 2008. Using the same methodology to study how students complete proofs put me in a unique position to verify the results of her study.

For her study, VanSpronsen (2008) interviewed 18 novice proof writers and studied the methods they used when completing proofs. Generally speaking, she found that students each had a particular strategy they employed while attempting to complete a proof and that strategy was consistent for the participants in her study across different questions. The strategies VanSpronsen found were the use of examples, the use of equations, the use of other visualizations and self-regulation. These strategies were also used by the participants in my study, but were not consistent across questions in this study as was reported by VanSpronsen. As will be seen, my study corroborated some of these findings and contradicted others.

5.1.1 Strategies

5.1.1.1 Use of examples

For all the questions in my study that required students to construct their own proof, at least some participants used examples in their proof attempt, with the exception of Question 8. That question (which asked them to prove that the cube root of 2 is irrational) was the exception, in my opinion, for a couple reasons. First of all, they were being asked to recreate a proof that they had already worked on, although this proof attempt came months later. Also, those who successfully completed the proof the first time were able to modify the steps from a well-known proof (that the square root of 2 is irrational). It is possible that even the students who did not complete the proof previously had encountered the more famous proof between attempts. In any event, all students had an idea of how the proof was supposed to go (at least how it was supposed to start) either from this previous experience with the problem or at least from the fact that the problem stated that they were to use a proof by contradiction.

That question aside, the use of examples was prevalent in this study as it was in that of VanSpronsen (2008).

> In general, participants were able to recognize the need to move past examples in their work, but not all could actually do so. They used examples as a tool to understand the definitions and new ideas posed and, in some cases, to form portions of the proof. (p. 324)

Participants in this study repeated mentioned the fact that examples do not serve as proof. Examples of VanSpronsen's other claims were also seen.

In Question 1, Barbara worked many examples but could not get past them in order to reach a proof. In that case, she was working on the Isis problem and she used as her examples various differences between the sides of the rectangle. She was able to see that she had found all the solutions for rectangles whose sides differed by as little as 0 up to rectangles whose sides differed by 21. However, she was not able to move beyond her method to show that no solutions other than the ones she had found exist.

The use of examples to help in the ways VanSpronsen (2008) mentioned also occurred many times in this study. For example, Chris' work on Question 2a was an instance where working with examples led to the proof that was eventually completed. The question asked him to prove that *ab* – *ba* is a multiple of 9 when *ab* is a two-digit number and *ba* is the number with the same digits reversed. Chris recognized that it would be possible for him to simply go through and check all such instances. However, in beginning to do so, Chris saw the pattern that he eventually used complete his proof. As was mentioned, this is just one illustration of examples aiding a participant in the completion of a proof.

5.1.1.2 Use of equations

Another method that was prevalent in VanSpronsen's study was "searching for equations and trying to manipulate those found towards a proof" (2008, p. 324). This was the case in my study as well, as were the results of such efforts.

> In some cases…, focusing on equations became distracting and was a factor in (students') inability to prove a statement. Overall, however, the use of equations was not a hindrance for most who attempted it, but seldom led to a full proof. (p. 324)

The problem mentioned was an issue for many students. In particular, this showed up for the students who did not readily remember the proof for Question 8 (in particular Mary and Michael) and for practically all students (except Barbara) when working on Question 13 (where students dealt with the inequality relating to rational numbers). It often seemed like working extensively with equations was something the participants did when they could not think of what else to do. The time spent on rearranging equations often took away from efforts that could have led to an understanding that could have in turn led to a proof.

Working with equations was not always a hindrance, however, and led to a proof in a few cases. For example, Mary used equations efficiently when completing her proof for Question 2a. In fact, although her proof ended up being very similar to Chris', they arrived at their proofs quite differently: Chris via examples and Mary by manipulating the equation only. At times, working extensively with equations was both a hindrance and a help, as was the case with John's work on Question 3 (the inequality relating to the harmonic series). During his initial work, John got distracted from the main idea of the proof, going so far as to recreate his equations (in this case, inequalities) in other forms. After leaving the interview setting and starting the problem over fresh, he was able to see a key insight that made his proof work out nicely.

The emphasis the participants in my study placed on working with equations is also evidenced by the number of proof types classified syntactic. Most (if not all) of the syntactic proofs discussed in Chapter 4 were labeled as such because the main focus of the participant was on working with equations via logically permissible means.

5.1.1.3 Use of visualizations

When VanSpronsen (2008) refers to visualizations, she is mostly referring to ways to organize information (p. 325 – 326). This was the case for some of the students in my study as well; in particular, see Chris' and James' work on Question 1 (the Isis problem). Helen also used visual aids effectively in her out of interview work on Question 13. Although it did not lead to a completed proof, it did lead to an insight that I feel is necessary to completing the proof.

VanSpronsen (2008) also mentions that these visualizations can be a hindrance if the visualization was "disorganized and random" (p.326). This was an issue at times as well. For instance, John's work on Question 7 (finding the cardinality of the power set of a finite set) quickly became complicated as he looked at different examples of sets and tried to make connections and keep track of all the subsets. It was not until he quit looking at these examples that he was able to come up with a proof. It should be noted, however, that he did eventually use a different visualization that may have been related to previous ones to come up with the idea that led to his proof.

Will also allowed visualizations to complicate his attempt at Question 7, both with the unrelated triangle he drew at the beginning of his work and the muddled way in which he began to keep track of the subsets of a four element set. Like with John, Will did eventually come up with a proof and, in the case of Will, the idea he used for his proof was more closely related to his previous work. So it is likely that even when the use of visualizations seems to be a hindrance due to disorganization, ideas that can lead to complete proofs may be found in work that might not seem fruitful at first glance.

5.1.1.4 Self-regulation

The examples of self-regulation VanSpronsen (2008) mentions mostly deal with using systematic versus random examples. Instances of this include some already mentioned. Chris' work on Question 2a and John's on Question 7 serve as examples where work began as disorganized but was cleaned up with positive results as work continued. Other examples include James' work on Question 1 and Mary's work on Questions 1 and 2b (proving that $n^3 - n$ is a multiple of 6).

Organized examples were not the only ways VanSpronsen (2008) decided the students in her study used self-regulation. "Overall, strategies used to monitor work, make goals, redirect work, and keep goals in mind were always beneficial and never a detriment for participants" (2008, p. 327). John and Chris were both conscious of the progress they were making on Question 8 and it affected the way in which they worked, allowing them to get their respective proofs more efficiently. Monitoring work and keeping goals in mind were also important for many students, especially in Question 2b where students who used induction (e.g., Michael) needed sub-proofs to complete the overall proof.

As VanSpronsen (2008) mentions, my study revealed examples where self-regulation was used by students to aid in their proof attempts and there was no evidence that self-regulation was ever a hindrance.

5.1.2 VanSpronsen's general discussion

While there were many results that VanSpronsen's study had in common with this one, one of her main findings differed from what was observed in this study. "The strategy use and difficulties experienced by each participant were fairly consistent from question to question, and the individual questions had little effect on changing the way the participant approached a proof" (2008, p. 330). As will be seen in later discussion, this was not the case for many of the students in my study. Most students had a fairly even split between the types of proofs they attempted (procedural, syntactic or semantic).

Those students who were fairly consistent in proof type, for example John and James, these students were consistent in their use of semantic proofs. However, the nature of semantic proofs is that the proof is dependant on gaining an understanding of the individual proof being completed. As such, when completing a semantic proof one is focused on the problem at hand and not on a preferred proof strategy. Also, at least in the cases of John and James, the times they deviated from using a semantic proof were mostly the times where they were asked to use a procedural proof (Questions 3 and 8). There were only two other times these two participants did not use a semantic proof. First was when John saw the presence of an '*n*' in the statement of Question 2b (a cue for him to use induction). Thus, in this instance, the work he completed was question-dependent. Second was James' work on Question 13, which was syntactic because he made no progress beyond manipulating equations.

I believe that this difference in findings between VanSpronsen's study (2008) and my own is due to a difference in methodology. While she used twice as many participants, she interviewed them only four times. On the other hand, I had far fewer students but was able to observe them over the course of a full academic year. Thus, it is my conjecture that the participant tendencies observed in VanSpronsen's study were short-term in nature and my longer study allowed for me to observe the students' changes in preference.

5.2 Charts summarizing categories

In this section, I will give a summary of the categorizations of each student's work. Included will be the classifications of both proof type and proof scheme for each participant. It is my hope that this will give the reader a general sense of the data compiled. Recall that Questions 5 and 11 served as debriefing sessions and the participants were asked to evaluate completed proofs in Questions 4 and 9, which is why there is no proof type designations for those questions.

A word about the tables is necessary here. Recall that many of the categories had sub-categories. So, for example, a procedural proof type could be either algorithm or process. Thus, there are two columns under the heading "Procedural". An "A" in a column under the Procedural heading means that the proof was of the algorithmic sub-type. If there are no sub-categories, an "X" is used designate the presence of that particular proof type or proof scheme. Also, "(a)" designates an attempted, but incomplete, proof. (See key on page 535.)

Participant: John

Question	Procedural*		Syntactic	Semantic
11			N/A	
10				(a)
9			N/A	
8	A			
7				X
6			X	
5			N/A	
4			N/A	
3		P		
2b		P		
2a				X
1				X

Question	Empirical	External**		Analytic ***	
11				T	A
10				T	
9				T	
8				T	
7				T	
6				T	
5				T	
4					A
3					N
2b				T	
2a				T	A
1				T	

Table 10: John's proof types and proof schemes

Participant: Mary

Question	Procedural*		Syntactic	Semantic
11			N/A	
10			X	
9			N/A	
8	A			
7		P(a)		
6				(a)
5			N/A	
4			N/A	
3		P(a)		
2b				(a)
2a			(a)	
1				X

Question	Empirical	External**		Analytic ***	
11				T	
10				T	
9				T	
8				T	
7	X	A	R		
6				T	
5				T	
4				T	
3				T(i)	
2b				T	
2a				T	
1				T	

Table 11: Mary's proof types and proof schemes

Participant: Will

Question	Procedural*		Syntactic	Semantic
11			N/A	
10			X	
9			N/A	
8	A			
7		P(a)		
6				(a)
5			N/A	
4			N/A	
3		P(a)		
2b				X
2a			X	
1				X

Question	Empirical	External*		Analytic ***	
11				T	
10				T	
9				T	
8				T	
7			R	T	
6				T	
5				T	A
4				T	A
3				T	
2b				T	
2a				T	
1				T	

Table 12: Will's proof types and proof schemes

Participant: Helen

Question	Procedural*		Syntactic	Semantic
11			N/A	
10			X(a)	(a)
9			N/A	
8		P		
7				X
6		P		
5			N/A	
4			N/A	
3	A(a)			
2b	A(a)			
2a			X	
1			X	

Question	Empirical	External**		Analytic ***	
11					N
10				T	
9					A
8				T	
7					A
6		A		T	
5					N
4					N
3	X	A		T(i)	
2b			R		
2a				T	
1				T	

Table 13: Helen's proof types and proof schemes

Participant: Barbara

Question	Procedural*		Syntactic	Semantic
11			N/A	
10			X	
9			N/A	
8	A			
7			(a)	
6		P(a)		
5			N/A	
4			N/A	
3		P(a)		
2b	A			
2a			(a)	
1				(a)

Question	Empirical	External**		Analytic ***	
11				T	
10	X			T	
9		A			A
8				T	
7	X			T	
6				T(i)	
5				T	A
4					A
3				T(i)	
2b				T(i)	
2a	X			T	
1	X			T	

Table 14: Barbara's proof types and schemes

Participant: James

Question	Algorithm*		Syntactic	Semantic
11			N/A	
10			(a)	
9			N/A	
8		P		
7				X
6				(a)
5			N/A	
4			N/A	
3		P		
2b				(a)
2a				X
1				X

Question	Empirical	External**		Analytic ***	
11				T	
10				T	
9				T	
8				T	
7				T	
6			S	T	
5				T	A
4				T	A
3				T	
2b				T	
2a				T	
1				T	

Table 15: James' proof types and proof schemes

Participant: Robert

Question	Algorithm*		Syntactic	Semantic
11			N/A	
10			(a)	
9			N/A	
8	A			
7				X
6				(a)
5			N/A	
4			N/A	
3		P		
2b				X
2a				X
1				X

Question	Empirical	External**		Analytic ***	
11				T	A
10				T	
9				T	A
8				T	
7				T	
6				T	
5				T	A
4				T	
3				T	
2b				T	
2a					A
1				T	

Table 16: Robert's proof types and proof schemes

Participant: Michael

Question	Algorithm*		Syntactic	Semantic
11			N/A	
10				(a)
9			N/A	
8	A			
7				X
6		P		
5			N/A	
4			N/A	
3		P(a)		
2b		P		
2a			X	
1				X

Question	Empirical	External**		Analytic ***	
11				T	
10				T	
9				T	
8				T	
7				T	
6				T	A
5					N
4				T	
3				T	
2b				T(i)	
2a				T	
1				T	

Table 17: Michael's proof types and proof schemes

Participant: Chris

Question	Algorithm*		Syntactic	Semantic
11			N/A	
10			X	
9			N/A	
8	A			
7				X
6		P		
5			N/A	
4			N/A	
3		P(a)		
2b		P		
2a				X
1				X

Question	Empirical	External**		Analytic ***	
11				T	
10				T	
9					N
8				T	
7				T	
6				T	
5				T	
4					N
3				T(i)	
2b				T(i)	
2a				T	
1				T	

Table 18: Chris' proof types and proof schemes

(a) = Attempt
* A = Algorithm
P = Process

** A = Authoritative
R = Ritual
S = Symbolic

*** T = Transformational
i= Internalized
A = Axiomatic
N = Neither, but still analytic

Before going any further, these results can be compared with those found by Weber (2004) in the studies he used to develop his proof type framework. There, Weber states that the categories were developed by observing proofs completed by students in two abstract algebra courses and an analysis course. He found that of the 139 proofs classified, 48 (34.5%) were procedural, 74 (53.2%) were syntactic and 17 (12.2%) were semantic. In this study, 71 proofs were classified with 29 (40.8%) being procedural, 14 (19.7%) syntactic and 28 (39.4%) semantic.

There are a few possible reasons for the discrepancy between the results of these two studies. One possible reason is the different populations in the studies. In Weber's, students were a little further along in their coursework. Although all the students in the current study had completed the transition to proof course, only one was taking analysis at the time of the study and none were taking abstract algebra.

Another more likely reason is that Weber did not classify proof attempts in which students made no progress. In particular, this (along with the instances of me giving the students help) may explain the greater percentage of semantic proofs. When students did not make any progress on the problem on their own, they employed various methods of gaining some understanding of the problem. If no progress toward a proof followed from that, I would have labeled that a semantic attempt while Weber would have refrained from labeling it at all.

However, this does not account for the fact that syntactic proofs were the most common by a wide margin in Weber's study and were by far the least common in mine. A few possibilities arise for this, although I am not sure which is more likely. One is that the problems used in Weber's study lent themselves to syntactic proofs much more readily than those in my study. Support for this comes from the fact that all the proofs that Weber classified from the algebra class were syntactic. It is also possible that all the differences seen can be attributed to the fact that the proofs were coded by different researchers. However, the consistency seen between the codes given by myself and the peer reviewers in this study suggests that the differences are due to a combination of the reasons listed, perhaps along with other possibilities not considered.

5.2.1 Notes on proof scheme

One of the first things one notices when looking at the charts is the relative lack of any proof scheme other than analytic. Outside of Helen and Barbara, no participants display an empirical or external proof scheme in more than one interview. In fact, every other student but two (Mary and James) showed an analytic proof scheme, and possibly others, while working on every question. In both of these cases, (Mary's Question 7 and James' Question 6) the participant did not complete a proof and, in my opinion, if they would have gotten to a point where they understood the problem well enough to do so, they would have given a rigorous proof thus constituting an analytic proof scheme.

I believe the analytic proof scheme was so ubiquitous because the participants had already completed MATH 305 and had been explicitly taught what does and does not constitute a proof. Even when displaying evidence for another proof scheme, students acknowledged the need for rigorous proof. For example, in Question 10 Barbara mentioned that based on the examples she had seen she believed that a rational could always be found that would make the desired inequality hold. Despite this belief, she still worked on trying to construct a proof. This highlights a difference between what serves as proof to her personally (she is convinced by examples) and what she decides will be considered a proof by others. Barbara clearly knows what does and does not "count as a proof" although she does not use the same criteria of verification to convince herself. It is unclear how many other students held similar beliefs over the course of study. It seems likely that other participants did as well but did not verbalize them.

Another factor leading to the preponderance of the analytic proof scheme is the fact that the use of equations mentioned in the previous section was so prevalent. The use of equations is present in almost all cases of students who displayed a transformational analytic proof scheme.

Exceptions to this include when participants used a proof with steps laid out for them. In most cases, this means the use of mathematical induction but also includes instances like Question 8 (where the participants were able to create a proof either based on previous work or on the similar proof showing that $\sqrt{2}$ is irrational).

Although Helen and Barbara were unique in that they did not display the analytic scheme almost exclusively as the other participants did, this does not mean that they were strangers to that proof scheme. On the contrary, looking at their respective charts shows that they too exhibited this type of proof scheme while working on every one of the problems in the study. So, again, it is possible that Helen and Barbara were not unique in their views of mathematical proof, only that they were more vocal while working and more willing to discuss ideas they held even when they knew it would not "count" as a proof.

One last note on the proof scheme categorizations is that in the vast majority of cases, the analytic proof scheme was either transformational or could not be classified into either transformational or axiomatic. As mentioned in previous chapters, the nature of the questioning in the interviews did not allow for material discussed to build upon itself. Among reasons for this was the infrequency of the interviews relative to other venues for discussing such material (i.e., classes) and the necessity for the questions to be accessible to students taking a variety of classes. Because of these limitations inherent to the study, it is unclear whether or not the axiomatic proof scheme was more widespread than it appears to be judging by the charts presented above.

5.2.2 Notes on proof type

There is a glaring difference that one notices when looking at the charts for proof type versus proof scheme. That is, proof scheme is far more consistent that proof type. For all students, the proof types employed vary a great deal. While it may seem like there is not much order to the breakdown of proof type, there are some things that can be gleaned from the charts. To make things clearer, below are more succinct charts breaking down the proof types seen by question (Table 19) and again by student (Table 20).

Question	Algorithm	Syntactic	Semantic
1		1	8
2a		5	4
2b	5		4
3	9		
6	4	1	4
7	2	1	6
8	9		
10		7	2

Table 19: Proof type by question

Student	Algorithm	Syntactic	Semantic
John	3	1	4
Mary	3	2	3
Will	3	2	3
Helen	4	3	1
Barbara	4	3	1
James	2	1	5
Robert	2	1	5
Michael	4	1	3
Chris	4	1	3

Table 20: Proof type by participant

The first and most straight-forward insight that can be made is that when students are asked for a particular type of proof, they are generally able to do so. Questions 3 and 8 asked the participants for particular types of proofs (induction and contradiction, respectively) and in each case, the students provided such a proof. This is reflected in the charts by the fact that all nine participants preformed a procedural proof for these problems: Question 3 because induction requires specific steps and Question 8 because the students had a previous proof that they were to mimic.

Beyond the observation mentioned above, not much is clear. While data for the study was being collected, the impression given by the data was that the type of proof attempted by each participant depended on the question asked more than on the students themselves. Looking at Table 19, however, evidence for this seems weak. Taking Questions 3 and 8 out of consideration, we find three questions (1, 7 and 10) that had yield proof attempts predominately of a particular type. This appears to support my earlier suspicion. When looking at the three remaining questions (2a, 2b and 6), however, we see that the proof types provided by the participants are fairly evenly split between two of the three possible. Interestingly, there were no questions that had proof types split evenly among all three possible types.

Since proof type is not clearly dependent on the question asked alone, one might posit that choice of proof type is entirely dependent on the student. Again, this does not seem to be the case. James and Robert are the students that come closest to displaying a preference for a particular type of proof. However, it can not be said that they display a strong preference as they each gave a semantic proof five out of eight times. Not only did their most frequently used proof type show up just over half the time, that type was semantic. As has been noted above, this type of proof is the most question-dependent of all.

Besides not having any participants display a strong preference for any one type of proof, two students (Mary and Will) came as close as is possible to evenly distributing the proof types among the three possibilities. It is interesting to note that when comparing the charts for Mary and Will, we see that they both used the same types of proofs for each particular question. I do not know of anything that can explain this besides mere coincidence.

All of the students who have yet to be discussed had a proof type they used 4 times, one they used three times and one used once. It is tempting to state that these 5 students each had a proof type they had an aversion to using, having only shown up one time. I do not think it is prudent to draw this conclusion, however. For one thing, the small number of classifications makes such an inference dubious. In each of the five cases, if a single question's proof type was changed to the type occurring only once, it would drastically change the balance of all types. Secondly, no such aversion to particular types of proofs was ever evident to me during the course of the study. It seems likely to me that such a preference would have become evident over the course of the year spent studying the students.

Given the ambiguity of what was discussed above, I believe that the question of whether proof type is more dependent on the question or the student answering the question is an open one. This, of course, excludes questions such as Questions 3 and 8 where a particular type of proof is asked for explicitly.

5.3 Three participant groups

This section divides the participants into three categories in an effort to draw comparisons between the different types of progressions the students made. The participants were divided up based on two different criteria: number of times they failed to complete a proof and change in proof scheme over time. These criteria were chosen for two reasons. First, because they were based on the classifications laid out before the study began (proof schemes) that turned out to be the most telling over the course of the study. Second, using these criteria leads to participant groups that match closely to the informal groups that I felt the students fell into as the study progressed.

The first, and largest, category includes participants who did not make much, if any perceptible progress yet were successful in most proof attempts. The second group includes students who also did not seem to progress much but were less successful with the problems. The last category includes only a single member, Helen, who did show signs of progress as judged by the criteria set out in this study.

5.3.1 Group 1

As is mentioned above, this class of participants is the largest by far. It also consists of the students most successful on the problems given during the study. This group includes a full two thirds of all participants: John, Will, James, Robert, Michael and Chris. The first criterion used to delineate the students was number of questions each student left a question unproved. All members of this group left three or fewer questions incomplete: John and Chris, one each; Robert and Michael, two each; James and Will, three each. Helen also successfully completed all but three problems but the reason for her exclusion from this group will become clear later.

While the number of complete, correct proofs is a rather rough measure of the work completed by these students over the course of the study, it is useful in grouping together these students. If one were to go back and read each of these students' summaries from Chapter 4, a recurring theme would become evident: all of these students displayed very little progression throughout the study. This may sound like a negative thing, but that is not actually the case. Instead, these students did not make much discernable progress because they all began the study with a fairly mature (and unchanging) view of proof.

This unchanging view of proof is evident in the near universal presence of the analytic proof scheme amongst members of this group. Between the eleven questions posed to these six students, evidence of anything other than an analytic proof scheme was only seen twice: James' Question 6 and Will's Question 7. In James' case, he displayed a symbolic proof scheme due to an abuse notation that occurred out of desperation (when he failed to see what else to do with the problem). In the case of Will's Question 7 work, the ritualistic proof scheme was evident because he failed to identify something that could have served as a proof because he did not recognize it as such (the contrapositive to believing something is a proof because it looks like one). Even in these cases, however, James and Will still displayed evidence of the analytic proof scheme.

Another thing common to the members of this group is their propensity to give or attempt semantic proofs. By definition, these are proofs that are meaningful to the proof writer and require an understanding of the proof beyond following prescribed steps or (potentially thoughtlessly) pushing symbols, even if the symbolic manipulations are correct and logically permissible. All members at least attempted semantic proofs at three times: Will, Michael and Chris, three times; John, four times; James and Robert, five times. Like the number of incomplete proofs, this criterion alone could not be used to group the students in the study as I have (Mary also used or attempted three semantic proofs). However, I do not think it is mere coincidence that students who were for the most part successful also used semantic proofs often. Typically, when one is struggling with a proof, a breakthrough occurs that allows him or her to complete the proof fairly quickly relative to the overall amount of time spent on the proof. These breakthroughs generally do not occur because the proof writer has finally applied the required steps correctly or happened upon the correct way to rearrange a given formula. Instead, these sorts of breakthroughs are typically due to some insight into the problem at hand that had eluded the proof writer until that point. It follows, then, that students who make an effort to gain a conceptual understanding of the problem they are working on would be more likely to solve it.

One last item of note with this group is that the two participants in this study who were not mathematics majors, Will and Michael, both expressed a feeling that they either regressed in their proof abilities (Will) or failed to improve (Michael). In both cases, they cited a lack of practice with proof outside the interviews as the cause. Personally, I think it is more likely that what they were experiencing had more to do with the increased difficulty of the problems given to them than with an actual regression of abilities. While all students encountered the increasingly difficulty problems, Will and Michael were unique in their estimation that they failed to get better at proof. I do not think this is a coincidence as they (Will and Michael) did not have the chance to offset their struggles in the interviews by becoming obviously more comfortable with proofs in proof-intensive classes. Regardless, both students were successful overall and were likely being a bit hard on themselves.

5.3.2 Group 2

The second group of students is considerably smaller, consisting of two students: Mary and Barbara. As is mentioned above, these two students did not display much progress based on the criteria used in this study, which was similar to Group 1. Unlike Group 1, however, these two participants were less successful in completing the problems presented to them. Both Mary and Barbara left five of the eight problems unfinished.

Despite finishing the same number of proofs, there were significant differences between Mary and Barbara, as judged by the results of this study. On one hand, Mary had much in common with the students in the first group. Like James and Will, Mary used a semantic proof three times. Also like the members of the Group 1, Mary displayed an analytic proof scheme on during most interviews. However, unlike the members of the first group, she did not show an analytic proof scheme while working on all questions. Relatively late in the study, Question 7, Mary gave evidence for the empirical, authoritative and ritualistic external proof schemes. This does not mean that she suddenly forgot what it takes for a proof to be considered valid. It does show, however, that she was still harboring some "bad habits" that became evident when she was dealing with a proof that made her struggle. This is different from members of the first group in that she does not present any evidence for an analytic proof scheme along with the empirical and external ones.

Another way Mary differed from members of Group 1 is that she does not ever show any evidence for the axiomatic analytic proof scheme. As has been mentioned a few times, the nature of the questions in this study made it difficult to exhibit this proof scheme. Many students did demonstrate evidence for this scheme, however. The amount of proofs labeled as transformational analytic is mainly due to Mary's reliance on manipulating equations while working on proofs. This tendency stayed with her throughout the study which, along with the number of proofs left unfinished, is why Mary was placed into the second group.

Barbara was also placed into Group 2, but differed from Mary in some important ways. For one thing, she relied more on examples than equations for many of the problems in the study. This is rather counter-intuitive when one considers that she attempted the fewest (one) semantic proofs. Barbara also differed from Mary in that her proof schemes were far more inconsistent. This is not necessary a bad thing, for example Barbara exhibited the axiomatic analytic proof scheme three times over the course of the study. She also demonstrated a non-analytic proof scheme more often: fives times compared to once. Even in doing so, Barbara gave evidence for an analytic proof scheme during every question.

This last trait is something Barbara shares with all the members of Group 1. Another thing Barbara has in common with the first group is that she did not show any progress based on the classifications from this study. The fact that she only finished three problems is not the only reason she was placed in this group, though. She also displayed non-analytic proof schemes over the course of the study. As late as the last question in which she was asked to complete a proof (Question 10), Barbara shows an empirical proof scheme. This was not an isolated instance, either. She also demonstrated an external proof scheme in Question 9 and an empirical scheme in Question 7. Unlike the students in Group 1, Barbara had real progress that she could have made and that would have been evident. Unfortunately, based on the criteria laid out for this study, Barbara did not make as much progress as she could have.

5.3.3 Group 3

The last group to discuss consists of only a single student: Helen. One thing that makes Helen unique is that one could make the case that she could be in either of the previous groups. For example she was fairly successful in completing the questions presented to her, leaving only three unfinished. This may imply that she belongs in the first group. Alternatively, one might note that she only attempted a semantic proof twice. This might lead one to conclude that she belongs in the second group. While both of these observations are correct, they both fail to point out an important aspect of the work Helen did. Over the course of the study, the proof schemes Helen displayed changed in a meaningful way.

Like the members of the first group and Barbara, Helen displayed some sort of analytic proof scheme while working on every problem in the study. This is countered by the fact that she also displayed a non-analytic proof scheme during while working on three of the questions, which was second most to Barbara (five questions). What is important to note, though, is when these three questions occurred in the study. Two were during the first semester of the study and the third was the first question of the second semester. Thus, Helen is the only participant in the study that showed discernable progress as judged by the criteria used in this study. To be sure, other students made progress as well (e.g., Barbara became more proficient in her use of mathematical induction). However, using the classifications decided on prior to data collection, Helen is the only student whose proof tendencies changed noticeably.

Helen's progress is also evident in ways other than her move away from non-analytic proof schemes. For example, although she used a semantic proof only twice, both instances came after the last time she demonstrated anything other than an analytic proof scheme. Also, two of the three problems she did not finish were Questions 2a and 2b, fairly early in the study. The last problem that she did not finish (Question 10) was difficult for all the students and she did actually come up with a proof, but not without help from me. It should be noted, however, that through exploring that problem she did come up with the proper restrictions that make the inequality hold and came very close to a key insight that would have, I believe, allowed her to complete the proof. Lastly, the two times that she displayed an axiomatic analytic proof scheme both occurred after the last time she exhibited a non-analytic proof scheme.

Given the signs of progress Helen displays, it is clear that she is unique and deserves her own category. While it likely occurred before the study for students in Group 1 and after the study for those in the second group, Helen made a discernable move away from non-analytic proof schemes during the study. Aside from improving particular skills, like becoming more proficient with induction, Helen is the only study to make clear progress. Not only that, the progress Helen makes is, in my opinion, the most meaningful of any made by any participant in the study.

5.4 Conclusions

At this point, we are ready to return to the question this study was designed to address: What, if any, identifiable stages exist through which students progress as they learn mathematical proof? One of the most basic criteria that could be used to identify student progress is whether or not the students complete a correct mathematical proof a higher percent of the time as they learn. This is not a very telling measure, however, because it depends on many things (experience with similar problems, proficiency with needed techniques, etc.). So, this study was designed to use two separate classification systems to track the work nine participants completed over the course of an academic year.

One thing becomes clear when looking at the summary of the classifications given in Section 5.2: there is no evident proof trajectory that is common to all the students who participated in the study. There are a few possibilities as why to this may be the case. First of all, it may be that classification systems used in this study may not be the ones needed to discern the progress that was made by the students. Of the frameworks available at the time the study was designed, I felt the two used were the most relevant and nuanced. The frameworks used allowed both what the participants actually did (proof type) and how they thought about proof (proof scheme) to be addressed. In addition, especially in the case of proof scheme, the frameworks also featured sub-divisions of the categories which allowed for a measure of refinement within the classifications themselves. Despite these advantages, there was a lack of discernible progress amongst the vast majority of the participants. At this point, it is unclear to me how these frameworks could be changed to allow for a better account of student progress.

Another potential reason that student progress could not be documented for all students is that much of the meaningful progress that might have been detected had already been accomplished. To me, this seems like a likelier explanation. As has been discussed, the students had completed the transition to proof course the semester before the study began. While I had expected that this class would primarily impart the basics of proof to the students, it appears as though the majority of the students in the study gained a fairly mature view of proof from that class or previous classes. Given the relative success of most of the students in the study (all of Group 1), I think it is safe to make this claim.

There are a few caveats that are worth mentioning. First, simply because two thirds of the students were in Group 1, and therefore considered successful, does not mean that they are reflective of all students in that class. It was well known to the students before the study began that they would be asked to complete proofs. It is silly to think that this fact was not taken into account when the students decided whether or not they would volunteer for the study. It is entirely possible that students not yet comfortable with proof would shy away from volunteering, thus leaving mostly those who felt confident in their abilities to take part. This is a conjecture, however, as I do not have any measure of student confidence from before the study to compare with comments made in either of the two debriefing interviews.

Secondly, just because the students in Group 2 did not make progress as judged by the criteria used in this study does not mean that made no improvements when it came to proof or that they have not made any progress since data collection from the study has ceased. As I have already mentioned, Barbara made noticeable progress with mathematical induction and Mary increased her use of self-regulation strategies and became more open to changing how she attempted to solve a problem toward the end of the study. I am sure that the improvements they made during this study are not the end of the line for Mary and Barbara and they have continued to become more comfortable with proof as they have taken more mathematics courses.

So, since the participants in Group 1 had relatively little progress to make and the students in Group 2 had some to make but did not, at least with regard to the classifications used in this study, we are left to what can be made of Helen's progress. Because she did make identifiable progress, it is possible to identify some rough stages that she went through as she became more comfortable with mathematical proof.

When it comes to Helen's choice of proof type, she moved from a tendency to use procedural and syntactic proofs early in the study to a propensity to attempt semantic proofs in later attempts. The exception to this was Question 8, which led all students into a procedural proof, as has been discussed. This shift in inclination is notable because it matches the general trend shared by the members of Group 1. As far as Helen's evident proof schemes, they changed somewhat as well, albeit less markedly. After the first question of the second semester, Helen exhibited exclusively analytic proof schemes. Helen's change in proof scheme is less pronounced than that of her proof types because she also displayed the analytic proof scheme earlier in the study and other proof schemes during only two questions in the first semester.

To sum up, then, the closest we can come to identifying stages that Helen went through is as follows. In the beginning of the study (Questions 1 – 3, 6) Helen attempted to complete proofs via the manipulation of equations (syntactic proofs) and/or following prescribed steps (procedural proofs). Later in the study, Helen began to use a more flexible approach in which she made an attempt to understand the problems, looking for insights that could be turned into a proof (semantic proofs) towards the end of the study. The students in Group 1 came into the study with a predilection toward semantic proofs and those in Group 2 did not attempt semantic proofs often and, when they did, those attempts came early in the study. To sum up, then, all of the evidence from this study suggests that as students become more successful provers, they tend to prefer and use semantic proofs more often when compared to procedural or syntactic proofs. There are exceptions to this preference, however. All the students in the group used procedural or syntactic proofs in at least one of the following scenarios: when they are explicitly asked for a certain type of proof, when the problem they are working on readily lends itself to a certain type of proof and when they are stuck with a problem and do not know what else to do.

5.5 Implications

Given the work done by the participants of this study, one can glean some potential implications relating to how undergraduates can become effective provers. The first implication is that when students are not sure about what to do with a proof, they are better off looking for ways to understand the problem at hand rather than manipulating equations or trying to apply a learned procedure. There were many instances in this study where students were able to successfully apply a process in the pursuit of a proof. Included were cases when mathematical induction could be used (such as Questions 2a, 2b, 3, 6 and 7) or when the students could rely on a proof for a similar problem with which they were already familiar (Question 8). However, by and large, the students who were successful were the ones that did not often get stuck retracing their steps within a set procedure and also could identify when they were rearranging an equation without making any progress. Instead, the students who were successful often used other methods, such as examples, tables or graphs, to get an understanding of the problem they were working on that could be turned into a proof. In other words, the students who were most successful on the questions used in this study were the ones that used semantic proofs and the students who were least successful were the ones who did not. Helen corroborates this in that she had more success during the second semester, after she had began to change her approach to proof to include more semantic proof attempts.

A certain amount of self-regulation and flexibility is necessary when attempting semantic proofs, and this leads to the second implication one can draw from this study. Students need to both be able to recognize when the approach they have chosen will not work on a given problem and be able to make a good guess as to what will work. As was noted in the summary of Helen's work in Chapter 4, students need to be taught and become comfortable with the different types of proof techniques they learn in a class such as MATH 305 before they can be expected to apply the appropriate technique on their own. This does not mean, however, that simply teaching students the different proof techniques is enough for them to be able to make such decisions. Helen said as much when I asked her at the end of the study what she thought could have led to more improvement for her:

> Helen: Maybe a 305 class that made you actually think about what you needed to do instead of just giving you a problem and telling you how you needed to do it… Because, like everything we did with induction, we were told to do induction. Everything we did with contradiction, we were told to do contradiction…And we weren't, like there was no way to find out like what would work in what situation better.

Although Helen may come across as critical of MATH 305, I do not think that she would disagree that first one needs to learn what the techniques are and how to accomplish them before one can decide for him or herself where it would be appropriate to use each technique. However, this does not eliminate the importance of one's learning which potential technique is most useful in which types of situations and why. Thus, part of learning how to self-regulate and complete semantic proofs is learning how to decide how one is going to proceed in his or her proof attempt. It seems to be Helen's stance, and it is mine as well, that the best way to learn these skills is to struggle on your own with such issues.

Therefore, part of any class similar to MATH 305 should be the inclusion of problems in which the students are given little guidance regarding which approach to take. For one thing, this would give students practice in recognizing what it is like to go down fruitless paths, increasing their self-regulation skills. It would also give students experience in identifying what proof techniques are most appropriate for different types of problems. Lastly, it would allow students the opportunity to discover which methods of exploring a problem (examples, graphs, etc.) are most useful to them, thus making future attempts at semantic proofs more fruitful and efficient.

I think it is worth noting here that this is not exactly new advice. In particular, it echoes some of Polya's suggestions first mentioned in Chapter 2. Not only is it reminiscent of the first two steps of Polya's five problem solving steps ("Understand the problem" and "devise a plan", Polya, 1945), it also recalls his address to all mathematics teachers: "*Let us teach guessing!*" (Polya, 1954, p.158)

5.6 Ideas for future research

This final section looks at suggestions for further research. First will be addressed what could have been done differently with this study. Secondly, some ideas will be discussed that could expand on the results from this study.

5.6.1 Ideas for improving current research

For all the planning that goes into research, one can not account for everything that may occur over the course of a study. Of course, this study is no exception and there are a few things I would have done differently if I were to conduct the study over again. First of all, I would have been more consistent with the level of interaction I had with the participants during each interview. As I mention in Chapter 3, I interacted with the students much more during the first few interviews than the subsequent ones. Beginning with Question 3, I was much more hands-off and allowed the students to work as they wished, often silently, until we discussed the work they had done during a reflection period at the end of the interview. As I mentioned before, this increased interaction on my part was due to both my inexperience as an interviewer and my desire to make the interviews as relatively stress-free for the participants as possible. Thus the interaction, especially for the very first interview, included more hints on the problem than in later interviews.

In retrospect, I think a better course of action would have been to set up informal interviews with the participants before the interviews included in the study were conducted. These informal interviews could have included the participants and me working on problems of their choosing, e.g., their own homework problems. This would give the students some time to become at ease in the interview setting without the pressure of trying to complete a new proof on their own. It would have also given me some experience in teasing out what the participants were thinking without revealing too much information about the problem at hand and it would have also likely given me a better idea of what happens when each individual student reaches the point at which they can no longer make progress on a problem on their own. If I would have conducted these informal interviews, once the interviews for the study began I could have let them go at it alone and gained a better idea of their work on those early problems. As it was, for those early problems, I was left to separate the work they did on their own from the work that was influenced by me and thus needed to be interpreted differently.

Once I did begin to allow students to work as they wished and go back to discuss their work during the reflection periods, I still did my best to not steer the conversation toward their beliefs about proofs but rather tried to directly discuss the work they did on that particular problem. My thinking was that their thoughts about proofs would come out via this conversation and that my questions could easily become too leading to obtain reliable results. Looking back, I think that I should have tried a bit harder to get at the students' beliefs. For example, on Question 10, Barbara mentioned becoming convinced about the problem based on the examples she tried, thus providing evidence for the empirical proof scheme. While this is definitely legitimate evidence for that proof scheme, what I am unsure about is how many other students had similar proof schemes. Although they may not have volunteered the information, I suspect that other participants held similar ideas. It is quite possible that they did not mention such thoughts because they knew that examples do not serve as proof, but it is also possible that they did not mention these beliefs simply because I did not ask. My guess is that the later is the case for a least a few of the participants in this study.

The last thing I would recommend changing if this study were to be repeated would be to change its timing relative to when the participants completed MATH 305. As has been seen, the majority of the participants did not have an extensive amount of progress to make as ascertained by the criteria set out in this study. It seems like to me that the students in Group 1 did not all begin MATH 305 with the tendency to employ the semantic proof scheme. Instead, I think that they all came to do so at different rates while working going through their transition-to-proof course. If this study were conducted while the participants were in that class, this shift in proof-writing approach would be more evident.

It is possible that this change in the study's timing relative to coursework would necessitate a corresponding change in the questions used in the study. However, at this time it is not clear whether or not this would be the case. It is also not apparent what new problems would be better if the questions were to change. It is possible that questions could be taken directly from the class, provided the participants had not seen them in advance.

5.6.2 Ideas for furthering current research

The first and most obvious thing that could be done to further this research is to repeat it. In many other areas of study, research is repeated by different researchers to verify results. To my knowledge, this does not seem to happen often in mathematics education. If these results were to be found again by other researcher it would this lend more validity to the results found here and would also expand the number of students and settings where the results were found. If a similar study was conducted and similar results were not found, then perhaps the differences in students and/or settings could shed some light on the different variables that influence how students learn to prove.

The second way this research could be furthered is by performing a categorical statistical analysis to see which has a greater bearing on the type of proof a student uses: the problem being asked or the participant who is attempting the proof. Because the types of proofs each student produced changed as they worked on different problems, it seemed to me that proof type depended on the question while I was going through each student's work individually. Taking in the data overall, however, this conclusion seems less likely. It is also difficult to look at the data accumulated and become convinced that the proof type attempted depends on the student alone. Because I did not anticipate encountering this sort of question, I am unprepared to answer it. I am confident, however, that someone well-versed in categorical statistics could answer such a question. Of course it is possible that a participant population greater than nine is necessary to complete such an analysis, in which case having this study repeated with a different population would be beneficial.

I am certain that others would be able to come up with other ways to further this and related research. I am hopeful that this research as contributed meaningfully to the body of knowledge regarding how students learn to mathematical proofs. This study has hopefully given a better understanding as to how, at least in the case of a single student, how one can improve one's abilities with mathematical proof and has found some characteristics common to students who are judged to be relatively successful in their proof attempts. I am also hopeful that this new knowledge can be used to improve the instruction received by those undergraduates who are learning mathematical proof for the first time.

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