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# Magic hypercube and index of welfare and sustainability[☆]

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## Abstract

This paper systematically surveys the use of different approaches for the magic square (MS) as an indicator of welfare, a formal system of necessary relations to deal with conflicts of socioeconomic objectives. The starting point is the article of Kaldor (1971) followed by contributions by the OECD from the 1970's resulting in a diagram which allowed a visual diagnosis of macroeconomic performance. Such representations were re-examined by Medrano-B and Teixeira (2013), who introduced a required normalization of the variables. Here, we show that this approach was marred by an oversight, namely the issue of the ordering of variables along the axes. In order to avoid this problem, we propose the use of a new mathematical approach involving a Hypercube Graph, which we call magic hypercube, which produces the same index, regardless the ordering of the variables. An application of the new concept is offered using economic data from Brazil and Chile.


*JEL classifications:* F34; G01; G18; E12

*Keywords:* Magical square; Index; Hypercube

## Resumo

Este artigo pesquisa de maneira sistemática autilização de diferentes abordagens para o Quadrado Mágico,um indicador de bem-estar que cria um sistema formal de relações necessárias para lidar com conflitos de objetivos socioeconômicos. O ponto de partida é o artigo de Kaldor (1971), seguido das contribuições da OCDE a partir da década de 1970, resultando num diagrama que permitiu um diagnóstico visual do desempenho macroeconómico. Tais representações foram reexaminadas por Medrano-B e Teixeira (2013), que introduziram uma normalização necessária das variáveis. Mostramos que essa abordagem foi prejudicada por um descuido, ou seja, a questão da ordenação de variáveis ao longo dos eixos. Para contornar este problema, propomos o uso de uma nova abordagem matemática envolvendo um Gráfico de Hipercubo, que denominamos Hipercubo Mágico, que produz o mesmo índice, independentemente da ordenação das variáveis. Uma aplicação do novo conceito é dada usando dados econômicos do Brasil e do Chile.


*Palavras-chave:* Quadrado mágico; índice; hipercubo

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## 1. Introduction

By the mid-1960s, Kaldor became increasingly interested in domestic and international economic policy, partly as a result of being a Special Adviser of the British Chancellor of Exchequer from 1964, and also as a resident of Great Britain, a country with the slowest postwar growth rate among major industrialized countries in Europe. He focused in particular on the search for empirical regularities related to inter-country and inter-regional growth rate comparisons. In this vein, he produced a number of articles (see, for example, Kaldor, 1970, 1971, 1976) primarily concerned with fundamental policy issues linked to socioeconomic management such as finance, monetary and fiscal requirements for sustainable growth, distribution and stability.

One of his most stimulating, albeit not too well known, essays on macroeconomic policy is "Conflicts in National Economic Objectives", originally delivered as Presidential Address to Section F of the British Association for the Advancement of Science (Durham, Scotland, September 1970) and printed by the Economic Journal in 1971. There, he deals with a comprehensive analysis of the performance of the British economy after the World War II. The main purpose of his article was to present a logical and empirical reconsideration of a basic macroeconomic framework of necessary relations to achieve some desirable targets, or economic policy objectives. It is our view that Kaldor's essay goes well beyond the scope of his country. Actually, he produced an explicit and successful attempt to extend the General Theory of Keynes (1936) to an open economy in which the government's economic policy constitutes a fairly unambiguous component of the power dynamics (decision making) within a mixed economy.

Following, to some extent, the policy announcements of successive governments in Britain, Kaldor's seminal paper considers four macroeconomic variables (GDP growth, employment, trade balance and inflation), all of them expressed as percentages. His formulation of the economic policy did not contain equations, tables or graphical illustrations. He assumed that a successful management of an open economy comprises at least the simultaneous attainment of explicit targets for the mentioned variables. The reader may wonder why these four variables are the relevant ones to be considered. Why are variables that measure fiscal policy or institutional performance, for example, not included? It is true that the two latter variables might be correlated with some of those that comprise the scope of his approach. Kaldor does not pay much attention to this problem following instead, with minor changes, the announcements of successive post-World War II Chancellors.

As mentioned, Kaldor's pioneering analysis did not benefit from quantitative nor graphical instruments. This absence was remedied by the introduction of the so-called "Magic Square" (MS), a graphical representation of Kaldor's approach. According to Dickhaus (2004, p. 354) and others, the credit for this corresponds to Karl Schiller, a German politician and leader of the Social Democratic Party (from 1966 to 1972) who was also Economics Minister of the Federal Republic of Germany. Since the 1970s, economists at OECD began using this instrument, with minor modifications, to deal with the performance of a single country or the comparative performance among a set of nations or regions.

Fig. 1 presents a diagram of the MS as it was conventionally used in the 1980s (Bernard et al., 1988). The annual variables considered in this Cartesian plane were: rate of GDP growth (%), trade balance (as percentage of GDP), rate of unemployment (%), and rate of inflation (%). Notice that, as measured from the origin, growth rate is supposed to take values from 0 to 10, trade balance values from −2 to 4, inflation from 10 to 0, and unemployment from 12 to 0 (the latter two variables on an inverted scale, given that higher values are less desirable than lower values). Alas, in such simple representation the authors did not bother with the different scales of the variables and they simply joined the four variables according to the axes. The ranges assigned to the macroeconomic variables are somewhat arbitrary but, for a magic square to be built, clearly some ranges had to be chosen. In addition, the correlation existing between some variables is recognized (e.g. Okun's Law – unemployment versus real GDP; Phillips Curve – inflation versus unemployment).

Medrano-B and Teixeira (2013) realized that such formulation contained a basic mistake since the original area of such figure has no useful meaning due to the non-uniform scales of the axes. To construct an adequate MS all four scales must be redefined to be uniform by normalizing the figure to a unit area. They also pointed out that the performance of any country, given by an area inside the unit square, is drawn not as a square but a diamond shape figure. Such geometric construction allows to quantify the inside figure as a proportion of the unitary MS. As a result, this work introduced a formal indicator, called Index of Economic Welfare. As an application the authors compared

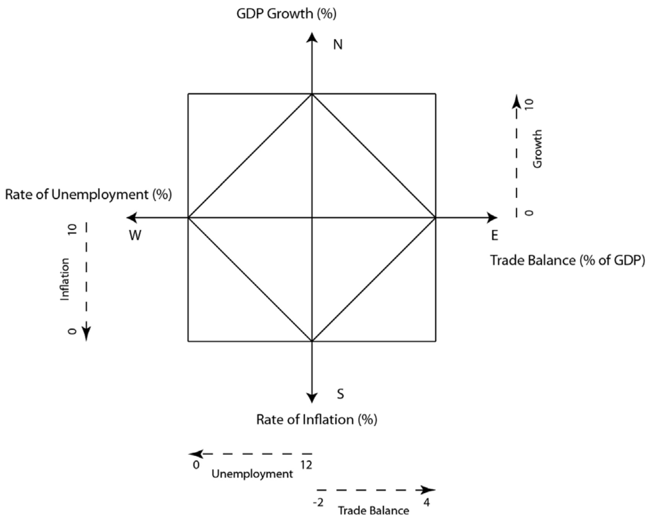


Fig. 1. Graphical representation of the macroeconomic variables.

Source: Bernard et al. (1988).

the macroeconomic impact of the recent global financial crisis in Brazil and Chile before (2004–2007) and during the crisis (2008–2011).

In the last few years a number of articles, presented in international seminars, colloquia and published in journals, took Medrano-B & Teixeira approach as theoretical framework. Thus, Firme and Teixeira (2014) published a macro-econometric analysis focusing on Brazil and another set of countries using this approach; Teixeira et al. (2015) employs the same analytical framework to provide a composite index to measure the overall performance involving both socioeconomic activities and environmental sustainability for China and the USA. Furthermore, Kucera (2012) presents an alternative mathematical formulation of the magic square.

During a recent series of seminars at the University of Brasilia, some of our colleagues noticed an important oversight related to the ordering of the variables along the vertices of the Medrano-B & Teixeira MS formulation. Numerical calculations show that alternative orderings of the four variables may produce different values for the index of welfare. We concluded that the previous result based on the original MS were valid only under special circumstances (symmetrical axes). This points out to the need for correction of the formulae upon which the index rests. This oversight is due to a not fully thought-out appreciation of ordering in the geometric transformation of a graph. Actually, as pointed out by Bergson (1959), the scientist tries to fit in all facts in a certain order, but forgets that other orderings are possible and may lead to different outcomes.

In this paper, after this introduction, we deal with the oversight just mentioned. An alternative treatment is given to the existence of a proper solution involving a Hypercube Graph, which we call, by analogy with the magic square, the magic hypercube. In order for this paper to be self-contained, Section 2 revisits the configuration of the magic square. Section 3 uncovers the oversight regarding the concept of revised MS and its consequences. Section 4 introduces the concept of Magic Hypercube Graph and Section 5 shows some numerical results of the new formulation using the same data in Medrano-B and Teixeira (2013) for Brazil and Chile. Section 6 offers some concluding thoughts.

## 2. The original magic square re-formalization

As we have mentioned in the introduction of this article, Medrano-B and Teixeira (2013) revised the original version of Magical Square correcting a mathematical problem this version presented. The diagram is conceived in its four cardinal directions (N, E, S and W) indicated by $\gamma$, $\tau$, $\varphi$ and $\zeta$. All four variables (axes) are originally drawn at different scales expressed in percentages and the adjacent vertices are joined by straight lines. The fact that the scales of these variables are not uniform implies that the original area of such figure has no useful meaning. To construct the revised MS all four scales will be normalized so that each of the new variables assumes values between 0 and a

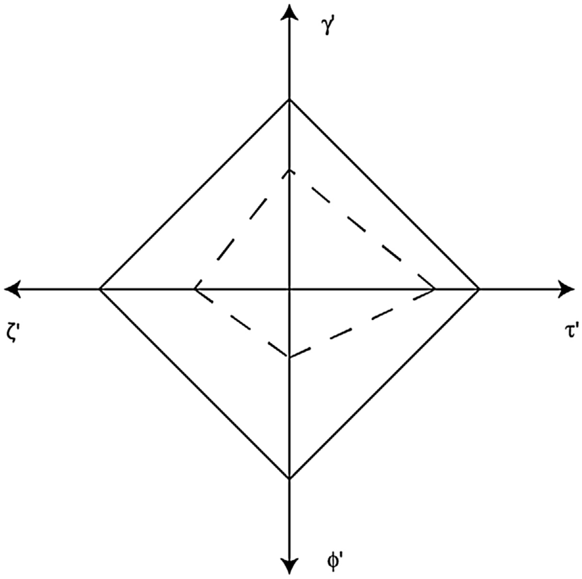


Fig. 2. The magic square.

Source: elaborated by the authors.

constant value $\beta$ and the maximum (the wonderland or ideal economy) area of the MS is 1. From these conditions it follows immediately that the maximum value $\beta$ of the normalized variables is $\sqrt{2}/2$.

Given the original boundaries on our four variables, as stated in the previous section, it follows that the new normalized variables (identified with a prime superscript) are

$$\gamma' = \frac{1}{\sqrt{2}\times 10}\gamma \tag{2.1.1}$$

$$\tau' = \frac{1}{\sqrt{2}\times 6}(\tau+2) \tag{2.1.2}$$

$$\varphi' = \frac{1}{\sqrt{2}\times 10}(10-\varphi) \tag{2.1.3}$$

$$\zeta' = \frac{1}{\sqrt{2}\times 12}(12-\zeta), \tag{2.1.4}$$

where the new "primed" variables will obey the following restrictions:

$$0 \le \gamma' \le \sqrt{2}/2; \quad 0 \le \tau' \le \sqrt{2}/2; \quad 0 \le \varphi' \le \sqrt{2}/2; \quad 0 \le \zeta' \le \sqrt{2}/2 \tag{2.2}$$

In summary the Medrano-B & Teixeira magic square index is given by the expression:

$$\frac{1}{2}\left(\gamma'\cdot\tau' + \tau'\cdot\varphi' + \varphi'\cdot\zeta' + \zeta'\cdot\gamma'\right) \tag{2.3}$$

In other words the area of the inner magic square enclosed by the full square, the latter corresponding to the wonderland economy where all variables take their maximum value, equal to $\sqrt{2}/2$ (see Fig. 2 below).

Some observations are worthwhile at this juncture. Firstly, let us note that the term "wonderland economy" is a rhetorical expression and does not imply that we consider, for instance, a zero inflation rate (or, for that matter, a zero rate of unemployment) as ideal. We are well aware that a zero inflation rate is not a healthy feature of a modern economy as widely accepted by most economists (see, for instance Mankiw, 2009). Secondly, the choice of extreme values for the four variables considered is somewhat arbitrary but, nevertheless, as Kaldor and Schiller did in their time, we need to select some values in order to construct the magic square index. Notice in particular that an inappropriate choice of extreme values may lead to a misleading comparison of the performance of different countries. For instance, any country with a trade balance at its minimum value would have an index equal to zero and thus a worse performance than any other country with a strictly positive value for its index, regardless of how the other macroeconomic variables compare. Moreover, its performance could not be distinguished from that of any other country with identical (minimum) trade balance. These objections can, of course, be easily lifted by an appropriate choice of extreme values.

## 3. A not inconsequential oversight

The main interest of the quantitative index introduced by Medrano-B and Teixeira (2013), the magic square, and its advantage over the simple geometrical illustration that preceded it, reside in its ability to proceed with inter-temporal explorations and cross country comparisons. The cited paper indeed compares Brazil and Chile over the period 2004–2011, claiming that, albeit this index showed Chile in a better light than Brazil, the dynamic examination revealed a better situation for Brazil than for Chile. On the other hand, one would expect any such comparison, based as it is on measurements of a collection of variables, to be independent of the particular ordering of those variables or, alternatively, the places they have on the planar axes. And here lies the oversight of the paper in question. As we shall see in a moment, the quantitative index introduced therein depends crucially on the order in which the variables are listed. More importantly, this order has also an effect on the results of comparison exercises, leading to undesirable inconsistencies.

In order to look systematically at this issue, let us first observe that the area of the magic square pictured in Fig. 2 can also be expressed by the formula:

$$\varepsilon_{\mathrm{I}} = ½\left(\gamma' + \varphi'\right)\left(\zeta' + \tau'\right) \tag{3.1}$$

A simple combinatorial argument shows that there are precisely 24 (4 times 3 times 2) ways of ordering these four variables. In addition, it is apparent from the expression above that these 24 orderings yield just three formulas that are not identical among themselves. We will adopt the following notation for these three expressions.

$$\varepsilon_{\mathrm{I}} = ½\left(\gamma' + \varphi'\right)\left(\zeta' + \tau'\right) \tag{3.1.1}$$

$$\varepsilon_{\mathrm{II}} = ½\left(\gamma' + \zeta'\right)\left(\varphi' + \tau'\right) \tag{3.1.2}$$

$$\varepsilon_{\mathrm{III}} = ½\left(\gamma' + \tau'\right)\left(\varphi' + \zeta'\right) \tag{3.1.3}$$

We will now examine how the values of these three expressions are related. More precisely, we purport to establish in this section the three propositions below, where to simplify notation we will substitute the variable names for the letters a, b, c, d, and we will do away with the normalizing factor ½. These three expressions of indexes can thus be written in this new notation:

$$\varepsilon_{\mathrm{I}} = (\mathrm{a} + \mathrm{b})(\mathrm{c} + \mathrm{d}) \tag{3.2.1}$$

$$\varepsilon_{\mathrm{II}} = (\mathrm{a} + \mathrm{c})(\mathrm{b} + \mathrm{d}) \tag{3.2.2}$$

$$\varepsilon_{\mathrm{III}} = (\mathrm{a} + \mathrm{d})(\mathrm{b} + \mathrm{c}) \tag{3.2.3}$$

It will be assumed throughout that a, b, c, d take non-negative values not exceeding $\sqrt{2}/2$.

**Proposition 1.** The three indexes $\varepsilon_{\mathrm{I}}$, $\varepsilon_{\mathrm{II}}$, $\varepsilon_{\mathrm{III}}$ are not identical and thus take, in general, different values as functions of their arguments.

**Proposition 2.** Albeit all three indexes are increasing functions of their arguments, the difference between two of them may either increase or decrease in response to a change in one of the arguments.

**Proposition 3.** A combined variation in the arguments may produce changes in different directions in the three indexes. More precisely, a particular change in the values of the arguments may lead one of the indexes to increase in value while another index will see its value decreased (or unchanged).

The proof of these propositions is rather simple and it is helped by the following formulas for the difference between the various expressions, themselves easily derived from their definition.

$$\varepsilon_{\mathrm{I}} - \varepsilon_{\mathrm{II}} = (\mathrm{a} - \mathrm{d})(\mathrm{c} - \mathrm{b}) \tag{3.3.1}$$

$$\varepsilon_{\mathrm{I}} - \varepsilon_{\mathrm{III}} = (\mathrm{a} - \mathrm{c})(\mathrm{d} - \mathrm{b}) \tag{3.3.2}$$

$$\varepsilon_{\mathrm{II}} - \varepsilon_{\mathrm{III}} = (\mathrm{a} - \mathrm{b})(\mathrm{d} - \mathrm{c}) \tag{3.3.3}$$

A look at (3.3.1)–(3.3.3) should be sufficient to prove Proposition 1, as these expressions have no reason to be identically zero. Anyway, it will be sufficient, for the sake of that proof, to exhibit a single combination of values for a, b, c, d that yields different values for the three indexes. For instance, if

$$\mathrm{a} = 0.5,\ \mathrm{b} = 0.2,\ \mathrm{c} = 0.6,\ \mathrm{d} = 0.7,$$

we obtain:

$$\varepsilon_{\mathrm{I}} = 0.91,\ \varepsilon_{\mathrm{II}} = 0.99,\ \varepsilon_{\mathrm{III}} = 0.96$$

As said before, this example suffices to prove Proposition 1.

The proof of Proposition 2 is even simpler. We just need to notice that expressions (3.3.1)–(3.3.3) depend positively on some of their variables and negatively on others. This means, as Proposition 2 asserts, that a change in these variables will make the difference in the indexes either rise or fall.

Let us deal finally with Proposition 3. In this case we need to look at the partial derivatives of each one of the three indexes in a given direction. Once again, it is sufficient to choose an appropriate direction and we will compute the partial derivatives following a direction that is a linear combination of a, b. The coefficients $\lambda$, $\mu$ of this combination will be selected conveniently so as to prove the proposition. A simple calculation yields:

$$\partial_{\lambda \mathrm{a}+\mu \mathrm{c}}\varepsilon_{\mathrm{I}} = (\mathrm{c} + \mathrm{d})\,\lambda + (\mathrm{a} + \mathrm{b})\,\mu \tag{3.4.1}$$

$$\partial_{\lambda \mathrm{a}+\mu \mathrm{c}}\varepsilon_{\mathrm{II}} = (\mathrm{b} + \mathrm{d})\,(\lambda + \mu) \tag{3.4.2}$$

$$\partial_{\lambda \mathrm{a}+\mu \mathrm{c}}\varepsilon_{\mathrm{III}} = (\mathrm{b} + \mathrm{c})\,\lambda + (\mathrm{a} + \mathrm{d})\,\mu \tag{3.4.3}$$

The expressions on the left of the preceding equations stand for directional partial derivatives. Thus, for instance, $\partial_{\lambda \mathrm{a}+\mu \mathrm{c}}\varepsilon_{\mathrm{I}}$ stands for the partial derivative of $\varepsilon_{\mathrm{I}}$ in the direction of the vector $\lambda + \mu \mathrm{c}$. If we now make $\lambda = 1$, $\mu = -1$ within these expressions, they become (with obvious notational simplification):

$$\partial\varepsilon_{\mathrm{I}} = \mathrm{c} + \mathrm{d}\text{-}\,\mathrm{a} - \mathrm{b} \tag{3.5.1}$$

$$\partial\varepsilon_{\mathrm{II}} = 0 \tag{3.5.2}$$

$$\partial\varepsilon_{\mathrm{III}} = \mathrm{b} + \mathrm{c}\ \text{-}\ \mathrm{a} - \mathrm{d} \tag{3.5.3}$$

If, finally, we select the values of the variables in such a way that $\mathrm{a} = \mathrm{c}$, $\mathrm{d} = 2\mathrm{b} \neq 0$, we have:

$$\partial\varepsilon_{\mathrm{I}} = \mathrm{b} \tag{3.6.1}$$

$$\partial\varepsilon_{\mathrm{II}} = 0 \tag{3.6.2}$$

$$\partial\varepsilon_{\mathrm{III}} = -\mathrm{b} \tag{3.6.3}$$

which completes the proof of Proposition 3.

Let us close this section by summarizing what was here accomplished: we made explicit the fact that, contrarily to what geometrical intuition might suggest, the quantitative index associated with the magic square depends crucially on the position of the variables in the axes. More precisely, we showed that there are three possible indexes associated with all possible orders and that these three indexes do not behave consistently; not only their values will generally be different but also the indexes may respond in different directions to changes in the values of the variables.

Table 1
Macroeconomic variables for Brazil and Chile, 2004–2007 and 2008–2011 (percentages).

| | Brazil | | Chile | |
|---|---|---|---|---|
| | 2004–2007 | 2008–2011 | 2004–2007 | 2008–2011 |
| Growth | 4.7 | 3.7 | 5.2 | 3.3 |
| Trade balance | 1.2 | −1.9 | 3.2 | 0.4 |
| Inflation | 5.3 | 5.6 | 3.0 | 3.8 |
| Unemployment | 8.7 | 5.5 | 7.9 | 6.4 |

Source: Medrano-B and Teixeira (2013).

## 4. An alternative concept: the magic hypercube

The rather negative results obtained in the previous section prompt us to search for an alternative index, one that will be independent of the ordering of the variables and that, consequently, will allow for comparison exercises. Fortunately, there is a surprisingly simple answer to that search that, in addition, can be generalized effortlessly to any number of variables. This is the "magic hypercube" (MH) whose associated quantitative index is the multi-dimensional volume of that geometric construct. If we are dealing, in general, with n variables, the hypercube[1] resides in the n-dimensional Cartesian space. Once the variables are normalized in such a way that their values lie in the interval [0,1], the *magic hypercube* consists of the n-dimensional parallelotope with edges along the axes going from the origin to the value of the corresponding variable. The n-dimensional volume is then simply the product of the values of all variables. By its construction this index is constrained to take values between 0 and 1. The value of 1 corresponds to some ideal situation whereby all arguments take their maximum (normalized) value of 1. On the other hand, for the value of 0 to be obtained it will be enough for one of the variables under consideration to reach their minimum value, a fact that needs to be kept in mind when making the selection of extreme values for the variables considered.

If we apply this general concept to the four variables under consideration, $\gamma$, $\tau$, $\varphi$ and $\zeta$, the new index is simply:

$$\mu = \tilde{\gamma}\cdot\tilde{\tau}\cdot\tilde{\varphi}\cdot\tilde{\zeta} = \frac{1}{7200}\gamma(\tau+2)(10-\varphi)(12-\zeta) \tag{4.1}$$

$\tilde{\gamma}$ , $\tilde{\tau}$, $\tilde{\varphi}$ and $\tilde{\zeta}$ being the "revised" normalization of the original variables $\gamma$, $\tau$, $\varphi$ and $\zeta$ (such that the maximum value assumed by this index is indeed 1).

More precisely,

$$\tilde{\gamma} = \frac{1}{10}\gamma \tag{4.2.1}$$

$$\tilde{\tau} = \frac{1}{6}(\tau+2) \tag{4.2.2}$$

$$\tilde{\varphi} = \frac{1}{10}(10-\varphi) \tag{4.2.3}$$

$$\tilde{\zeta} = \frac{1}{12}(12-\zeta) \tag{4.2.4}$$

A final word is necessary in relation to the use of this index (as well as the three MS indexes) when comparing different moments in time for a given economy. The proper formula to compute the variation in the value of the index

[1] In geometry, a hypercube is an n-dimensional analogue of a square (n = 2) and a cube (n = 3). It is a closed, compact, convex figure whose 1-skeleton consists of groups of opposite parallel lines segment aligned in each of the space's dimension, perpendicular to each other and of the same length. The interested reader may consult Harary et al. (1988) for further details about the hypercube and its associated graph.

Table 2
Average value of indexes associated with the magic square (MS) and the magic hypercube (MH) (percentages).

| | Brazil | | Chile | |
|---|---|---|---|---|
| | 2004–2007 | 2008–2011 | 2004–2007 | 2008–2011 |
| $\varepsilon_{\mathrm{I}}$ | 19.0 | 11.3 | 36.8 | 20.6 |
| $\varepsilon_{\mathrm{II}}$ | 18.7 | 10.4 | 33.7 | 20.3 |
| $\varepsilon_{\mathrm{III}}$ | 18.7 | 9.5 | 36.1 | 19.8 |
| $\mu$ | 3.24 | 0.15 | 10.78 | 3.82 |

Source: Medrano-B and Teixeira (2013) and computations by the authors.

Table 3
Geometric average variation of indexes associated with the MS and MH (percentages).

| | Brazil | Chile |
|---|---|---|
| $\varepsilon_{\mathrm{I}}$ | −12.2 | −13.5 |
| $\varepsilon_{\mathrm{II}}$ | −13.6 | −11.9 |
| $\varepsilon_{\mathrm{III}}$ | −15.6 | −13.9 |
| $\mu$ | −53.6 | −22.8 |

Source: computations by the authors.

from time t to time t + n is given by the expression below (where, to simplify the notation, we let "1" stand for t, "2" for t + n):

$$variation = 100 \cdot \left( \sqrt[n]{\frac{index_2}{index_1}} - 1 \right), \tag{4.3}$$

giving us the yearly variation that, when compounded n times, produces the total variation of the index. The geometric average presented (Formula (4.3)) seems to be much more adequate for the comparison than the simple average used by Firme and Teixeira (2014) and Medrano-B and Teixeira (2013). The analogy with the difference between simple interest and composed interest can be observed.

Let us stress once again that the procedure to construct the new index implies on multiplying all four variables. As noticed before, this could create problems if a country reaches extreme values for the variables that would produce a null value for the index (for instance, an unemployment rate of 12% or a trade deficit of 2% of GDP). Such a situation can be dealt with easily by a judicious choice of extreme values for the variables that will prevent these occurrences. Such a change in extreme values would alter the values of the index but not the comparisons made with it.

## 5. Some numerical illustrations

An example using figures from the real world will allow for a comparison of the index just introduced with that (those) associated with the magic square. For the sake of this comparison it will be best to use the same figures appearing in Medrano-B & Teixeira, which refer to Brazil and Chile and are annual averages for two periods in time: 2004–2007 and 2008–2011. Let us reproduce in Table 1 the values that our four variables take, in each period, for each of these countries.

We will now compute the four indexes discussed so far, namely the three indexes associated with the magic square given by expressions (3.1.1)–(3.1.3) and the index associated with the hypercube given by expression (4.1). The results are displayed in the Table 2, where the indexes are expressed as percentages.

The first row reproduces the index of magic squares of Medrano-B & Teixeira. We can see that the other two possible indexes associated with the magic square, and corresponding to alternative orderings of the variables, assume, generally, different values. In all cases, including as well the index associated with the magic hypercube (last row), the indexes go down from the period 2004–2007 to the period 2008–2011 for both countries and they have higher values for Chile than for Brazil.

Let us finally look at the variation in these indexes from one period to the next. As we are dealing with a four-year difference between both periods, we need to apply the formula presented in (4.3) for that variation: if "index" refers to any of the four indexes under consideration and the subscripts 1, 2 refer to the periods 2004–2007 and 2008–2011 respectively, we have:

$$variation = 100 \cdot \left( \sqrt[4]{\frac{index_2}{index_1}} - 1 \right)$$

For the sake of completeness, we need to mention that Medrano-B & Teixeira used a different formula for variation, taking the simple (rather than geometric) average of the total variation; as said before, we believe the geometric average (formula (4.2)) is more suitable for this sort of analysis and we remade the computations of that paper according to the revised formula. The Table 3 summarizes the results of these computations.

As indicated above all values are negative. However, only the first index shows a worse performance of Chile relatively to Brazil, as claimed by Medrano-B & Teixeira. The two other indexes associated with the magic square and corresponding to alternative orderings of the variables point towards the opposite conclusion as does, more strongly, the index associated with the magic hypercube. Given that the indexes associated with the magic square yield inconsistent results it seems safer to adopt the index associated with the magic hypercube and this indicates a much better performance of Chile, with a yearly decrease of 22.8%, compared to Brazil, with a yearly decrease of 53.6%.

## 6. Concluding thoughts

A few remarks may be made by way of conclusion. The importance of the contribution of Medrano-B & Teixeira's article, emphasizing the necessary normalization of the variables used in the construction of the Index of Welfare based upon the magic square (MS), cannot be denied. However, they overlooked the crucial point that, except in the case of particular symmetry, the ordering of the variables around the square is fundamental to avoid multiple values for the index. This is the issue we are dealing with in the present contribution. Surely, the ordering is irrelevant if only three or less variables are considered, but this is not true as soon as we depart towards a higher number of variables.

The approach developed to deal with this situation involves a new logically consistent analysis featuring the magic hypercube (MH). Such modelling clarifies the theoretical matter in a rigorous and concise way. The new formulation for the Index of Welfare is shown to present a unique value, independently of the ordering of the variables. In other words, this reformulation, in its formal simplicity, appears complete and synthetic. The described empirical illustration, comparing the performance of Brazil and Chile, based upon the new theoretical apparatus, is very effective.

This article has not attempted to identify the "best" set of variables for the construction of the welfare index and the choice of variables for such an end is of course an open question worthy of further research. Given that the main aim of the paper was to deal with the oversight remarked in the original paper by Medrano-B & Teixeira we did not want to innovate in relation to the choice of variables. The same can be said about the selection of ranges for the four variables considered. Further applications of the methodology herein proposed would involve an adequate selection of the set of variables (not necessarily four of course) and the ranges appropriate for the particular application.

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