Arbitrage opportunities across sponsored search markets☆

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ABSTRACT

We model and study arbitrage across sponsored search markets, created by search engines. We identify and focus on *traffic arbitrage* and *click arbitrage* by auctioneers. We derive and characterize equilibria of such arbitrage behaviors across multiple markets.

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1. Introduction

The search market has become a principal source of revenue for search engine companies. The business model relies on advertisement sales that provide advertisers with opportunities to introduce their products directly to potential customers. The primary model, the sponsored keyword auction, identifies each user of a search engine, who submits a keyword or several keywords to the search engine, as a potential customer to related consumer products. Web links displaying those products are listed along with the search results of the queried keywords. To win the slots for displaying positions of web links, advertisers compete through an auction process.

Some advertising positions draw more attention from users and generate more clicks than others. Therefore, different advertising positions have different *click-through-rates*, the ratio of the number of clicks on the advertising to the number of appearances of the advertising web links. For this reason, it is named the *position auction* by Varian [13]. The *generalized second price auction* (GSP for short), named by Edelman, Ostrovsky and Schwarz [8], is used as the primary prototype for sponsored link auctions to sell the advertising positions.

1.1. Arbitrage across markets

Though Google has emerged as a major player in the search engine business, several search engines are used worldwide and provide sponsored search opportunities to advertisers. As each search engine establishes a single market, prices are

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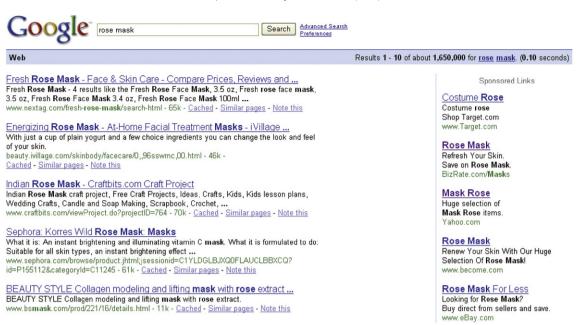
Fig. 1. The sponsor results when the keyword "rose mask" is typed into the search engine of Google. Yahoo.com and bitraze.com are sponsored links displayed along the search results in Google.

often different and arbitrage has become a possibility. We identify and model two existing arbitrage behaviors, and study the equilibrium properties.

1. Traffic Arbitrage: The concept is motivated by the behavior of some participating websites of the AdSense market model of Google. They are assigned advertisements by Google on their websites. They are paid by Google on the number of times those advertisements are displayed or clicked on their websites. Increasing the traffic that goes into their websites is an obvious way to increase their income. One way to do it is to purchase, at a lower rate, advertisement slots of (the same or closely related) keywords sold at search engines. In the same spirit, some online comparison shopping search engines, such as shopping.yahoo.com, nextag.com, bitraze.com, may themselves take part at sponsored keyword auctions of search engines such as Google and Yahoo. These advertisers may bring traffic back to their own markets, which at the same time display some other sponsored keyword advertisement slots. Fig. 1 illustrates such strategic behavior of traffic arbitrage. When the keyword "rose mask" is typed into the search engine of Google, Yahoo.com and bitraze.com will be displayed along the search results as sponsored links. Actually, all the other sponsored links could also be considered as search engines in some senses.

2. Click Arbitrage: On Internet, there are many affiliates undertaking advertising business for the small companies who may not have the expertise dealing with the search engine advertisement task. These affiliates specialize in advertising their clients' websites. The commission depends on the traffic the affiliates bring to the clients' websites. Therefore, the emergence of the sponsored search auctions creates a chance for the affiliates to act on their clients' behaviors to take part in the auctions. The affiliate (maybe a search engine itself) can charge a fee for a click to its clients, at the same time to participate at the sponsored search auctions, paying less, to bring in potential consumers to the clients. For example, the affiliate could use the client's website as the advertisement's display URL which will be shown on the text description of the advertisement, but use the affiliate's webpage which has the redirection function as the advertisement's destination URL. If a potential user clicks on the advertisement on the search engines, he/she will be directed to the destination URL, then redirected to the client's webpage, which creates the impression that the traffic comes directly from the affiliate's webpage. Even in case search engines disallow, as a policy, a display URL to be different from the destination URL, there are technological tools to handle it, e.g., as in http://www.apexpacific.com/knowledgebase/bidmaximizer/faqdetail.asp?id=78.

As an example, if one auctioneer of some GSP auction bids for some advertising slot on other GSP auctions to increase the traffic to the auctioneer's search engine to increase the number of clicks of each advertising slot on the search engine, this strategic behavior belongs to *traffic arbitrage*. On the other hand, if one auctioneer of some GSP auction bids for some advertising slot on other GSP auctions to transfer the clicks won to another slot on his own GSP auction to increase the number of clicks of that slot, this strategic behavior is called *click arbitrage*.

1.2. Our contributions

We propose the first models for such arbitrage behaviors. Based on those models, we model several such sponsored auction markets, held independently by different auctioneers, participated by agents in one, or all of them as an overall

revenue consideration. We study the Nash equilibria with respect to the above types of arbitrage behaviors. Our results show that properly regulated ones can improve social efficiency, and as a byproduct, improve the revenues of all the auctioneers.

1.3. Related work

In general, the study of auction protocols has traditionally relied on a concept called incentive compatibility [14], which requires that protocols should be designed such that each participant will realize its optimal utility by speaking the truth. Such a requirement is justified by a fundamental theorem in the mechanism design, the revelation principle [12]. Informally, it states that, for any protocol that guarantees the dominant strategy, there is a corresponding incentive compatible protocol.

The reigning protocol for sponsored search markets, the GSP auction, however, is not incentive compatible. The auctioneer could have simply used the standard VCG mechanism [14,6,9] to make sure that each agent will reveal the true private value. Aggarwal, Goel and Motwani proposed an implementation [1] that would force the auctioneer to obtain the minimum VCG, but it would have to rely on the auctioneer's goodwill to make the choice. Unfortunately, this is not the case in reality. With the effect of justification of the GSP to the auctioneer, it was proven that the auctioneer gets a payoff at least as that of VCG in a subclass of Nash equilibria, called *symmetric Nash equilibrium* (SNE for short) [13] and *locally envy-free Nash equilibrium* (LEFN for short) [8], but potentially more.

In [4], Bu, Deng and Qi derived a property called *forward-looking attribute* for a bid, out of the many optimal bidding strategies of each participant, to justify its use as a *forward-looking response* for every bidder. In the relative value to the immediate higher/lower bidders, it is the same as that in the final solution of Aggarwal, Goel and Motwani [1] but different in the intermediate bids. The same bidding value is called *balanced greedy bidding strategy* (BB for short) by Cary, Das, Edelman, Giotis, Heimerl, Karlin, Mathieu and Schwarz in [5]. Both groups [2,5] proved that the strategy converges to a unique solution that gives the auctioneer exactly the same payoff as the VCG protocol. The bidders can, therefore, rationally bid their own optimal response in the forward-looking sense to limit the auctioneer's gain to that of the truthful VCG protocol. In some sense, this result restores the Revenue Equivalence Theorem of Vickrey and Myerson [14,12] under a new condition, despite the possibility for the auctioneer to avoid it for gaining potentially higher revenue in SNE or LEFN.

Our work builds upon those results for a single market model and is the first to handle arbitrage across several sponsored search markets.

1.4. Layout of presentation

In Section 2 we present the single sponsored search market model and formally introduce the GSP protocol. We also discuss the concept of forward-looking Nash equilibrium to prepare our study into the multiple markets model. In Section 3, we consider the auctioneer's strategic behavior of *traffic arbitrage* and the corresponding equilibrium across markets. In Section 4, we consider the auctioneer's strategic behavior of *click arbitrage* and the corresponding equilibrium across markets. In Section 5, we discuss the limitations of our models and future improvements.

2. The single market model

The GSP auction model was first presented in [8,13] for the sponsored search market.

For a keyword, let there be N advertisers who compete for K advertisement slots ($K < N$). $\mathcal{N}$ denotes the set of advertisers and $\mathcal{K}$ denotes the set of advertisement slots. In the online advertising market, impression is related to web traffic and usually used in the phrase Cost Per Impression (CPI), which is one of the popular pricing methods in the online advertising market. For Online Advertising, an impression can be defined as one access to the advertisement link. Therefore, once one user visits a webpage with some online advertisements such as banner and text link, the number of the impressions on the advertisement is added by one. A tracker (Web counter) can be placed in the webpage to verify how many accesses that page had and who (in terms of Internet addresses) originated those accesses. For each slot k, θ_k denotes the expected *click-through-rate* (the number of clicks/the number of impressions) of slot k. In other words, within a period of time if the keyword's searching result page is displayed I times, then slot k would be clicked $\theta_k I$ times. Further, we assume that if the indexes of slots satisfy $k_1 < k_2$, then slot k_1's *click-through-rate* (CTR for short) θ_{k_1} is larger than θ_{k_2}. Namely, $\theta_1 > \theta_2 > \cdots > \theta_K > 0$. For convenience of presentation, let $\theta_{K+1} = 0$. In addition, each bidder $i \in \mathcal{N}$ has a privately known information, v^i, which represents the maximum price he is willing to pay for each click on his advertisement link.

According to each bidder i's submitted bid $b^i \geq 0$, the auctioneer decides how to distribute the advertisement slots among the bidders and how much they should pay for per-click. For the GSP protocol, the auctioneer firstly sorts the bidders in decreasing order according to their submitted bids. The last $N–K$ bidders would lose and get nothing. Each winner would be charged the next bid in the descending bid queue. Losers would pay nothing. In the case of ties, we assume that the auctioneer would break ties according to a pre-assigned order (e.g., the time the agents register with the system).

Let b_k denote kth highest bid in the descending bid queue and v_k the true value of the kth bidder in the descending queue. So if bidder i got slot k, i's payment per click would be b_{k+1}, and i's payment per impression would be $b_{k+1} \cdot \theta_k$. Otherwise, his

payment would be zero. Hence, for any bidder $i \in \mathcal{N}$, if i were on slot $k \in \mathcal{K}$, his utility per impression could be represented as

$$u_k^i = (v^i - b_{k+1}) \cdot \theta_k.$$

If there is more than one position to sell, the GSP protocol will not be incentive compatible [8,1]. That is, bidders may benefit by not bidding their true private values. However, there exists a set of pure Nash equilibria where an advertiser will choose a strategy that maximizes his own utility with respect to a given set of strategies of other players. It is assumed that the advertiser will be left well alone if no other choices will gain him a better utility. In arguing for a more robust solution in the practical dynamic system, the advertisers are assumed to be wise enough to explore potential improvement without the possibility of making their own payoff reduced. Every bidder would take into account both his current behavior and its effect on the other bidder's future behavior. It was shown [2] that a particular response function, the forward-looking response, would implement the rationality of maximizing the bidders' utilities in such a circumstance. If all bidders apply the forward-looking best response, they will arrive at a singleton Nash equilibrium, called the forward-looking Nash equilibrium, that results in the same payment for every participant as in the celebrated VCG auction protocol [14,6,9], even though none bids his true private value for the clicks. The VCG mechanism is incentive compatible but is not applied to the sponsored search auction as it is regarded as being too complicated to the users. Therefore, their work justifies the usage of GSP as the mechanism to sell online advertisements.

The same terminal equilibrium is also reached by the Generalized English auction protocol, (not the Generalized Second Price auction protocol,) also proposed by Edelman et al. in [8], and the laddered auction of Aggarwal et al. in [1], as well as the *balanced greedy bidding strategy* named by Cary et al. in [5].

We next introduce the concept of the forward-looking Nash equilibrium for the study of the more general setting of the search engine arbitrage models of GSP auctions.

Definition 2.1 (*Forward-Looking Response Function [2]*). Given the other bidders' bidding set $\mathbf{b}^{-i}$, suppose bidder i prefers slot k, then bidder i's forward-looking response function $\mathcal{F}^i(\mathbf{b}^{-i})$ is defined as

$$\mathcal{F}^i(\mathbf{b}^{-i}) = \begin{cases} v^i - \frac{\theta_k}{\theta_{k-1}}(v^i - b_{k+1}) & 2 \leq k \leq K; \\ v^i & k = 1 \text{ or } k > K. \end{cases} \tag{2.1}$$

Definition 2.2 (*Forward-Looking Nash Equilibrium [2]*). A forward-looking response function-based equilibrium is a strategy profile $\widehat{\mathbf{b}}$ such that

$$\forall i \in \mathcal{N}, \quad \widehat{\mathbf{b}}^i = \mathcal{F}^i(\widehat{\mathbf{b}}^{-i}).$$

Theorem 2.3 (*[2]*). *The GSP auction has a unique forward-looking Nash equilibrium* $\mathbf{b}$ *satisfying*

$$\begin{cases} b^i = v^i & \text{for } i = 1 \text{ and } i > K; \\ b^i = \frac{1}{\theta_{i-1}} \left[\sum_{j=i}^{K} (\theta_{j-1} - \theta_j) v^j + \theta_K v^{K+1} \right] & \text{for } 2 \leq i \leq K. \end{cases} \tag{2.2}$$

Theorem 2.4 (*[8,1,2,5]*). *In the GSP auction, any bidder's expected* payment per impression *under the forward-looking Nash equilibrium is equal to his expected* payment per impression *under VCG mechanism.*

Corollary 2.5 (*[8,1,2,5]*). *The auctioneer's expected* revenue per impression *in the forward-looking Nash equilibrium is equal to that in the VCG mechanism.*

Theorem 2.6 (*[2,5]*). *The GSP auction converges to the forward-looking Nash equilibrium with probability one under randomized readjustment scheme.*

3. Traffic arbitrage across markets

In this section, we consider the auctioneer's strategic behavior of *traffic arbitrage*. For *traffic arbitrage*, the auctioneer of some GSP auction will bid for some advertising slot on other GSP auctions to increase the traffic to the auctioneer's own search engine. This allows the auctioneer to increase the number of clicks of each advertising slot on his own search engine. The most probable case is when he bids for a slot on a keyword from the other search auction to bring back to his webpage displaying advertisement for the same keyword.

We would study the impact of the strategy of *traffic arbitrage* on the number of impressions of the slots on the auctioneer's search engine. Further, we would estimate the auctioneer/arbitrageur's cost and benefit of *traffic arbitrage*. Regarding all the auctions as a whole, we then study the Nash equilibria in terms of all bidders' forward-looking best response strategies when the strategy of *traffic arbitrage* is allowed in cross markets. Although we allow the auctioneer to attend the other auctions, we prohibit the auctioneer from attending his own auction in the whole paper.

3.1. Model and assumptions

Consider M GSP auctions. $\mathcal{M}$ denotes the set of GSP auctions. We use G_i to denote the ith auction itself, $i \in \mathcal{M}$. A_i denotes the auctioneer of G_i and $\mathcal{A}$ denotes the set of all the auctioneers. $\mathcal{A}_j$ is the set of the auctioneers attending auction G_j, where $j \in \mathcal{M}$, $\mathcal{A}_j \subseteq \mathcal{A} \backslash A_j$. Without loss of generality, we assume all the auctions have the same numbers of advertising slots, say, K slots. Clearly, if one of the auctions has less than K slots, we could add some dummy slot(s) with CTR = 0 to that auction.

There are N advertisers who could attend all the auctions. For each advertiser i, his true value per click in the different auctions may be different. So we use v^{i,G_j} to denote advertiser i's true value per click in auction G_j. For auctioneer A_i, we use v^{A_i,G_j}, $i \neq j$ to denote auctioneer A_i's true value per click in auction G_j. As we mentioned before, we forbid the auctioneer from attending his own auction. We use $\theta_k^{G_j}$ to denote the CTR of slot k in auction G_j. Although each bidder is allowed to attend all the auctions, it is supposed that each bidder is prohibited from submitting more than one bid, i.e., multi-bidding, to the same auction.

As for the CTR of each slot in M auctions, we assume that each slot's CTR remains unchanged no matter whether any of the auctioneers uses the strategy of *traffic arbitrage* or not. However, this strategy could bring more impressions for the auctioneer's own GSP auction and therefore bring more clicks per period to all of the advertising slots in his own GSP auction.

In more detail, in this model of *traffic arbitrage*, there is an important assumption.

Assumption 3.1. For all the search engines, the CTR of any slot is not affected by any auctioneer's strategy of *traffic arbitrage*.

Remark 3.2. We assume the CTRs will not change in this model. Though for different reasons, the CTRs could increase or decrease because of traffic arbitrage. For example, consider two GSP auctions G_1 and G_2. Suppose auctioneer A_1 adopts traffic arbitrage strategy and wins a slot of auction G_2. If some user enters search engine G_1 by clicking the search engine G_1's advertisement on search engine G_2 where the user types the keyword, the user may be more interested in the sponsored results than the users who enter search engine G_1 directly. So the user may click advertisements with a higher possibility than other users. In addition, since this user is absorbed into search engine G_1, he may be satisfied with the search results or sponsored results on search engine G_1 and never return to search engine G_2 again to click the other advertisements on search engine G_2. For simplicity, however, we first consider that the CTRs will not be affected by traffic arbitrage. The general principle can also extend to more sophisticated cases.

After modeling the strategic behavior of *traffic arbitrage*, in the next part we will begin to estimate the arbitrageur's true value and payment, then study the Nash equilibria in terms of all bidders' forward-looking best response strategies when the strategy of *traffic arbitrage* is allowed.

3.2. Properties of the model

First, we give a rigorous definition of the forward-looking Nash equilibrium in cross markets when the auctioneer's strategic behavior of *traffic arbitrage* is allowed.

Definition 3.3 (*Forward-looking Response Function in the Traffic Arbitrage Model*). $\forall i \in \mathcal{N} \cup \mathcal{A}_j$, given the other bidders' bidding set $\mathbf{b}^{-i,G_j}$, if bidder i prefers slot k, then bidder i's forward-looking response function $\mathcal{F}^{i,G_j}(\mathbf{b}^{-i,G_j})$ is defined as

$$\mathcal{F}^{i,G_j}(\mathbf{b}^{-i,G_j}) = \begin{cases} v^{i,G_j} - \dfrac{\theta_k^{G_j}}{\theta_{k-1}^{G_j}}(v^{i,G_j} - b_{k+1}^{G_j}) & 2 \leq k \leq K; \\ v^{i,G_j} & k = 1 \text{ or } k > K. \end{cases} \tag{3.1}$$

Definition 3.4 (*Forward-looking Nash Equilibrium in the Traffic Arbitrage Model*). Let $\widehat{\mathbf{b}}^{G_j} = \left\{ \widehat{b}^{i,G_j} \middle| i \in \mathcal{N} \cup \mathcal{A}_j \right\}$. If traffic arbitrage is allowed, a forward-looking response function-based equilibrium is a strategy profile $\widehat{\mathbf{b}} = \left\{ \widehat{\mathbf{b}}^{G_j} \middle| j \in \mathcal{M} \right\}$ such that $\forall i \in \mathcal{N} \cup \mathcal{A}_j$, $\widehat{b}^{i,G_j} = \mathcal{F}^{i,G_j}(\widehat{\mathbf{b}}^{-i,G_j})$.

The following lemma is directly derived from the above definition.

Lemma 3.5. *In the traffic arbitrage model, a strategy profile $\widehat{\mathbf{b}}$ is a forward-looking Nash equilibrium if and only if $\forall j \in \mathcal{M}$, $\widehat{\mathbf{b}}^{G_j}$ is a forward-looking Nash equilibrium.*

The following lemma estimates the arbitrageur's true value per click in the other auction in terms of the forward-looking Nash equilibrium.

Lemma 3.6. *In the traffic arbitrage model, arbitrageur A_i's true value per click in auction G_j, where $i \neq j$, is equal to his VCG revenue per impression in his own auction G_i in terms of the forward-looking Nash equilibrium.*

Proof. If auctioneer A_i attends auction G_j, he could be regarded as a bidder in auction G_j. Hence, auctioneer A_i's traffic arbitrage behavior will increase one more bidder in auction G_j. The only difference between arbitrageur A_i and the other bidders is that arbitrageur A_i's true value per click is not a constant while the others' true values per click are constants, since arbitrageur A_i's value per click depends on all the bids in his own auction, i.e, the revenue in auction G_i.

Let I^{G_i}, I^{G_j} represent the number of impressions per period of search engine G_i, G_j respectively. If traffic arbitrageur A_i wins slot l in auction G_j, the number of impressions he gets per period is increased by $\Delta I^{G_i} = I^{G_j}\theta_l^{G_j}$ and his total revenue from auction G_i per period is increased by $\Delta R^{G_i} = \Delta I^{G_i} R_{\text{per impression}}^{G_i}$. This augmenting of revenue is due to his participating in auction G_j. And every click the traffic arbitrageur A_i earned in auction G_j could bring him one more impression.

Therefore, his value per click in auction G_j is

$$
\begin{aligned}
v^{A_i, G_j} &= \frac{\Delta R^{G_i}}{\Delta I^{G_i}} \\
&= R_{\text{per impression}}^{G_i}.
\end{aligned}
\tag{3.2}
$$

By Lemma 3.5, in the forward-looking Nash equilibrium of the traffic arbitrage model, the sub-market auction G_i is also in the forward-looking Nash equilibrium. Thus, according to Corollary 2.5, the arbitrageur A_i's revenue in GSP auction per impression is equal to his VCG revenue per impression, i.e., $R_{\text{per impression}}^{G_i} = R_{\text{VCG}}^{G_i}$, where $R_{\text{VCG}}^{G_i}$ is auctioneer A_i's VCG revenue per impression.

Therefore, $v^{A_i, G_j} = \sum_{i=1}^{K} \sum_{j=i+1}^{K+1} (\theta_{j-1}^{G_i} - \theta_j^{G_i}) v_j^{G_i}$. $\quad\square$

Theorem 3.7. *Consider the traffic arbitrage model where all advertisers and arbitrageurs are following the forward-looking response function. We have:*

1. *There always exists a forward-looking Nash equilibrium.*
2. *The model always converges to its forward-looking Nash equilibrium.*
3. *In the forward-looking Nash equilibrium, all the auctioneers' revenue will not be worse off in the presence of the traffic arbitrage behavior.*

Proof. We first prove the basic case: $M = 2$, i.e., there are two GSP auctions in the market.

- Proof of Part 1:

  First consider the case: only one of the auctioneers is a traffic arbitrageur. We prove it by construction. Without loss of generality, assume auctioneer A_2 uses the *traffic arbitrage* strategy. By Theorem 2.3, there always exists a unique forward-looking Nash equilibrium $\widehat{\mathbf{b}}^{G_2}$ in auction G_2. Given $\widehat{\mathbf{b}}^{G_2}$, by Lemma 3.6, arbitrageur A_2 could be regarded as a bidder with a constant true value $v^{A_2, G_1} = \sum_{i=1}^{K} \sum_{j=i+1}^{K+1} (\theta_{j-1}^{G_2} - \theta_j^{G_2}) v_j^{G_2}$ in auction G_1. Now consider auction G_1. In auction G_1, there are $N + 1$ bidders (N advertisers and 1 arbitrageur). Each of these bidders has a constant true value. Therefore, there exists a unique forward-looking Nash equilibrium $\widehat{\mathbf{b}}^{G_1}$ in auction G_1. By Lemma 3.5, we obtain there always exists a forward-looking Nash equilibrium in this case.

  Now consider the case: both auctioneers are *traffic arbitrageurs*. We prove it by construction. Suppose initially both auction G_1 and G_2 are in the forward-looking Nash equilibrium. Then arbitrageur A_1 attends auction G_2 and bids a high enough price to get the first slot of auction G_2. Arbitrageur A_2 can calculate his true value based on arbitrageur A_1's bid and attend auction G_1. After his attending, auction G_1 reaches a new forward-looking Nash equilibrium. Then arbitrageur A_1 could estimate his true value according to the new equilibrium and decreases his own bidding price according to the forward-looking response function. If the whole procedure terminates in finite steps, the forward-looking Nash equilibrium exists. If the procedure would not terminate in finite steps, since either arbitrageur A_1 or A_2 always decreases his bidding price in the whole procedure, the bidding prices of both arbitrageur A_1 and A_2 would decrease but tend to some values respectively. Actually, such 'two values' are the equilibrium value. Since during the whole process, the arbitrageurs and all bidders follow the forward-looking response function, the equilibrium they reached is the desired forward-looking Nash equilibrium.

- Proof of Part 2:

  We will utilize the construction in the above proof again. Obviously, initially both auction G_1 and G_2 can reach the forward-looking Nash equilibrium under the Lowest-First adjustment scheme respectively with the same argument in [2]. After arbitrageur A_1 attends auction G_2 and wins the first slot of auction G_2, arbitrageur A_2 can calculate his true value based on arbitrageur A_1 and attend auction G_1. Then locking auction G_2, auction G_1 can converge to the forward-looking Nash equilibrium under the Lowest-First adjustment scheme with the same argument. Next, locking auction G_1, auction G_2 can also converge to the equilibrium under the Lowest-First adjustment scheme. Combined with the proof of part 1, the whole process must converge to the forward-looking Nash equilibrium under the Lowest-First adjustment scheme. Though Lowest-First is a deterministic adjustment scheme, there is a nonzero probability that the deterministic scheme will occur in a randomized adjustment scheme. That means, the game will converge to the forward-looking Nash equilibrium with probability one under the randomized adjustment scheme.

- Proof of Part 3:

  In the first part, we have already proved that the forward-looking Nash equilibrium always exists even if *traffic arbitrage* is allowed.

By Lemma 3.5, the cross market is in the forward-looking Nash equilibrium if and only if both sub-markets are in the forward-looking Nash equilibrium. By Corollary 2.5, the revenue per impression of auctioneer A_2 in auction G_2 before auctioneer A_1's participation is

$$R^{G_2}_{\text{per impression}} = R^{G_2}_{\text{VCG}} = \sum_{i=1}^{K} \sum_{j=i+1}^{K+1} (\theta^{G_2}_{j-1} - \theta^{G_2}_{j}) v^{G_2}_{j}$$

where v_j is the true value of the bidder on the jth slot.

Now consider auctioneer A_1 attends auction G_2 and gets slot l. Assume the new revenue per impression of auctioneer A_2 in auction G_2 is $R'^{G_2}_{\text{per impression}}$. Since all the other N bidders' true values in auction G_2 remain unchanged, we have:

For $l > K + 1$,

$$R'^{G_2}_{\text{per impression}} = R^{G_2}_{\text{per impression}}.$$

For $l \leq K + 1$,

$$R'^{G_2}_{\text{per impression}} = \sum_{i=1}^{K} \sum_{j=i+1}^{K+1} (\theta^{G_2}_{j-1} - \theta^{G_2}_{j}) v'^{G_2}_{j}$$

where

$$v'^{G_2}_{j} = \begin{cases} v^{G_2}_{j} & \text{if } j < l; \\ v^{A_1, G_2} & \text{if } j = l; \\ v^{G_2}_{j-1} & \text{if } j > l. \end{cases} \tag{3.3}$$

Since $v^{A_1, G_2} \geq v^{G_2}_{l}$ and $v^{G_2}_{j-1} \geq v^{G_2}_{j}$, $R'^{G_2}_{\text{per impression}} \geq R^{G_2}_{\text{per impression}}.$

Therefore, auctioneer A_1's arbitrage behavior will increase auctioneer A_2's revenue of his own auction.

Similarly, we can get $R'^{G_1}_{\text{per impression}} \geq R^{G_1}_{\text{per impression}}.$

On the other hand, consider the case that auctioneer A_2 attends auction G_1. In the forward-looking Nash equilibrium, auctioneer A_2's bidding price is always less than his true value, thus he can always get a nonnegative revenue in auction G_1.

We can obtain the same result for auctioneer A_1, since auctioneer A_1 and A_2 are symmetric.

Therefore, both auctioneers' revenue will not be worse off in the presence of the traffic arbitrage behavior in terms of the forward-looking Nash equilibrium.

Next, we generalize the proof to M GSP auction case by induction.

We have already proved that there always exists a forward-looking Nash equilibrium for the basic case: $M = 2$. Now suppose it also stands for $M = k$. To finalize the proof by induction, we only need to check the case: $M = k + 1$.

Since there always exists a forward-looking Nash equilibrium for $M = k$ (with traffic arbitrage) and the auction G_{k+1} itself (without traffic arbitrage) always has a forward-looking Nash equilibrium, suppose initially $\{G_1, \ldots, G_k\}$ and G_{k+1} are in the forward-looking Nash equilibrium respectively. Then auctioneer A_{k+1} attends the set of auctions $\mathcal{G}_{k+1}$ and bids an infinite high price to get the first slot of all the auctions in $\mathcal{G}_{k+1}$, where $\mathcal{G}_{k+1}$ is the set of the auctions that auctioneer A_{k+1} attends. Based on arbitrageur A_{k+1}'s bids, the auction set $\{G_1, \ldots, G_k\}$ would move to a new forward-looking Nash equilibrium. Then arbitrageurs $\mathcal{A}_{k+1}$ can calculate their true values and attend auction G_{k+1}. After their attending, auction G_{k+1} reaches a new forward-looking Nash equilibrium. Then auctioneer A_{k+1} could estimate his true value according to the new equilibrium and decreases his own bidding price according to the forward-looking response function in $\mathcal{G}_{k+1}$. If the whole procedure terminates in finite steps, the forward-looking Nash equilibrium exists. If the procedure would not terminate in finite steps, since either arbitrageur A_{k+1} or auctioneers $\{A_1, \ldots, A_k\}$ always decrease their bidding prices in the whole procedure, the bidding prices of all the arbitrageurs would decrease but tend to some values respectively. Clearly, such a set of values would be arbitrageurs' bidding prices in the forward-looking Nash equilibrium. Hence, we finished the proof of part 1. Similarly, part 2 and part 3 of the theorem could be easily derived by induction from the basic case: $M = 2$. $\square$

4. Click arbitrage across markets

Consider an affiliate as a search engine company which itself could hold the GSP auction and sell advertising slots. Suppose the client of the affiliate is the owner of one advertising slot. If the affiliate bids a slot from another search engine's GSP auction for its client and wins the slot eventually, this event could be considered as the search engine company wins the slot of some other search engine and apportions the clicks won to his own adverting slot to increase this slot's number of clicks. If the affiliate pays less to the other search engine than it collects from its own client, this strategic behavior would gain new revenue for the affiliate.

In general, if one auctioneer/affiliate of some GSP auction bids for some advertising slot on other GSP auctions to transfer the clicks won to the slot on his own GSP auction to increase the number of clicks of that slot, this strategic behavior is called *click arbitrage*. We shall consider such a strategic behavior of the affiliate's *click arbitrage*.

4.1. Model and assumptions

Similarly to the model of traffic arbitrage, we assume there are $\mathcal{M}$ GSP auctions with the same number K of advertising slots. There are $\mathcal{N}$ advertisers who could attend all these auctions. We use $c_k^{G_j}$ to denote the number of clicks per period of slot k in auction G_j. Furthermore, we also assume that

Assumption 4.1. For all the search engines, the CTR of any slot is not affected by any auctioneer's strategy of *click arbitrage*.

4.2. Properties of the model

At first, in the forward-looking Nash equilibrium there is an interesting proposition of the *click arbitrage* model .

Proposition 4.2. *In the* click arbitrage *model, for any slot $k \neq 1$, if the number of clicks on slot k is decreased by 1 and the number of clicks on slot 1 is increased by 1, the auctioneer's revenue will be augmented in the forward-looking Nash equilibrium.*

Intuitively, the proposition looks obvious. However, it is not quite so but requires a worked-out proof. Although the first slot has the highest price, decreasing 1 click on slot k will decrease the price of slot k. Furthermore, such adjustment would also affect the bids on the slots between 1 and k. Thus, in order to complete the proof, we need to compare the auctioneer's revenue before the adjustment and the revenue after the adjustment as a whole.

Proof. According to the equation (2.1), under the forward-looking Nash equilibrium, the payment of the bidder on slot c_i could be recursively represented as follows.

$$
\begin{aligned}
p_i &= b_{i+1} \cdot c_i \\
&= \left(v^{i+1} - \tfrac{c_{i+1}}{c_i}(v^{i+1} - b_{i+2}) \right) \cdot c_i \\
&= v^{i+1} \cdot c_i - v^{i+1} \cdot c_{i+1} + b_{i+2} \cdot c_{i+1} \\
&= v^{i+1} \cdot (c_i - c_{i+1}) + p_{i+1}.
\end{aligned}
\tag{4.1}
$$

Now suppose for some slot $k \neq 1$, the number of clicks on slot k is decreased by 1, i.e., $\widetilde{c}_k = c_k - 1$. And the number of clicks on slot 1 is increased by 1, i.e., $\widetilde{c}_1 = c_1 + 1$. In the new position auction, obviously $\widetilde{p}_i = p_i$ for $i > k$.

For slot k,

$$
\begin{aligned}
\widetilde{p}_k - p_k &= v^{k+1}(\widetilde{c}_k - c_{k+1}) + \widetilde{p}_{k+1} - v^{k+1}(c_k - c_{k+1}) - p_{k+1} \\
&= v^{k+1}(c_k - 1 - c_{k+1}) + p_{k+1} - v^{k+1}(c_k - c_{k+1}) - p_{k+1} \\
&= -v^{k+1}.
\end{aligned}
$$

For slot $k - 1$,

$$
\begin{aligned}
\widetilde{p}_{k-1} - p_{k-1} &= v^k(c_{k-1} - \widetilde{c}_k) + \widetilde{p}_k - v^k(c_{k-1} - c_k) - p_k \\
&= v^k(c_{k-1} - (c_k - 1)) + \widetilde{p}_k - v^k(c_{k-1} - c_k) - p_k \\
&= v^k - v^{k+1}.
\end{aligned}
$$

Obviously, $\widetilde{p}_2 - p_2 = \cdots = \widetilde{p}_{k-2} - p_{k-2} = \widetilde{p}_{k-1} - p_{k-1} = v^k - v^{k+1}$

$$
\begin{aligned}
\widetilde{p}_1 - p_1 &= v^2(\widetilde{c}_1 - c_2) + \widetilde{p}_2 - v^2(c_1 - c_2) - p_2 \\
&= v^2(c_1 + 1 - c_2) + \widetilde{p}_2 - v^2(c_1 - c_2) - p_2 \\
&= v^2 + v^k - v^{k+1}.
\end{aligned}
$$

So, $\sum_{i=1}^{K}(\widetilde{p}_i - p_i) = (k-1)(v^k - v^{k+1}) + v^2 - v^{k+1} > 0$. Therefore, for any slot $k \neq 1$, if the number of clicks on slot k is decreased by 1 and the number of clicks on slot 1 is increased by 1, the auctioneer's revenue will be augmented in the forward-looking Nash equilibrium. □

Consequently, the proposition implies the following lemma.

Lemma 4.3. *In the* click arbitrage *model, if the auctioneer would apportion extra clicks among these K slots to maximally increase his revenue, he will apportion all the extra clicks to slot 1.*

Therefore, if the auctioneer A_i attends auction G_j ($i \neq j$) and wins some slot at low enough price, he may make an extra profit by apportioning the clicks he earns to slot 1 of his own auction. If such strategic behavior of the auctioneer is permitted, does the forward-looking Nash equilibrium still exist in the whole market? Before we reply to the question positively, we have to define the forward-looking Nash equilibrium in the context of the click arbitrage model. Actually the definition is similar to the Definition 3.4 for the traffic arbitrage model, so we omit the definition here. Furthermore, we have the following lemma corresponding to Lemma 3.5.

Lemma 4.4. *In the* click arbitrage *model, a strategy profile* $\widehat{\mathbf{b}}$ *is a forward-looking Nash equilibrium if and only if* $\forall j \in \mathcal{M}$, $\widehat{\mathbf{b}}^{G_j}$ *is a forward-looking Nash equilibrium.*

Lemma 4.5. *In the* click arbitrage *model, the click arbitrageur* A_i*'s true value per click in auction* G_j*, where* $i \neq j$*, is equal to,* $v_2^{G_i}$*, the second highest value in his own auction in the forward-looking Nash equilibrium.*

Proof. Now suppose the auctioneer A_i wins some position k at auction G_j and apportions all $c_k^{G_j}$ clicks to slot 1 in auction G_i. Clearly, for any slot $h \neq 1$ of auction G_i, under the forward-looking Nash equilibrium, $\widetilde{p}_h^{G_i} - p_h^{G_i} = 0$ according to equation (2.2), where $p_h^{G_i}$ denotes the payment on slot h in auction G_i and $\widetilde{p}_h^{G_i}$ denotes the payment on slot h when $c_k^{G_j}$ clicks are added into slot 1 in auction G_i.

For slot 1 in auction G_i,

$$\widetilde{p}_1^{G_i} - p_1^{G_i} = v_2^{G_i}\left(\left(c_1^{G_i} + c_k^{G_j}\right) - c_2^{G_i}\right) + \widetilde{p}_2^{G_i} - v_2^{G_i}\left(c_1^{G_i} - c_2^{G_i}\right) - p_2^{G_i} = v_2^{G_i}c_k^{G_j}.$$

Therefore, if the auctioneer A_i wins some position k at auction G_j and apportions all $c_k^{G_j}$ clicks to slot 1 in auction G_i, he could earn an extra profit of $v_2^{G_i}c_k^{G_j}$. In other words, every click in auction G_j could bring a profit of $v_2^{G_i}$ to the auctioneer A_i. So acting as a bidder in auction G_j, the true value of the auctioneer A_i is $v_2^{G_i}$. $\quad\square$

Since we could estimate the arbitrageur's true value, we can further discuss the equilibrium in this click arbitrage model.

Theorem 4.6. *Consider the click arbitrage model where all advertisers and arbitrageurs are following the forward-looking response function. We have:*

1. *There always exists a forward-looking Nash equilibrium.*
2. *The model always converges to its forward-looking Nash equilibrium.*
3. *In the forward-looking Nash equilibrium, all the auctioneers' revenue will not be worse off in the presence of the click arbitrage behavior.*

For simplicity, we only prove the case of two GSP auctions; generalizing to M auction case is straightforward and similar to the proof of Theorem 3.7. We use G_1 and G_2 to denote the two GSP auctions respectively.

- Proof of Part 1:

  We first consider the case: only one of the auctioneers is a click arbitrageur. Without loss of generality, suppose only auctioneer A_2 is a click arbitrageur. By Theorem 2.3, there exists a forward-looking Nash equilibrium in auction G_2 no matter whether the arbitrageur A_2 wins some position at auction G_1 or not. By Lemma 4.5, arbitrageur A_2 could be regarded as a bidder with true value $v_2^{G_2}$ in auction G_1 actually. In an $N+1$ bidders GSP auction, the forward-looking Nash equilibrium always exists. Therefore, by Lemma 4.4, the forward-looking Nash equilibrium exists even if the auctioneer A_2 could attend auction G_1 as a click arbitrageur.

  Now consider the case: both auctioneers are *click arbitrageurs*. First, we sort and relabel the bidders according to their true values in both auction G_1 and G_2, such that, $v_1^{G_1} \geq v_2^{G_1} \geq \cdots \geq v_N^{G_1}, v_1^{G_2} \geq v_2^{G_2} \geq \cdots \geq v_N^{G_2}$. Without loss of generality, we assume $v_2^{G_2} \geq v_2^{G_1}$. Then we only need to prove the following two cases: $v_2^{G_2} \geq v_1^{G_1}$ and $v_2^{G_2} < v_1^{G_1}$.

  Case I: $v_2^{G_2} \geq v_1^{G_1}$.

  Auctioneer A_2 could be regarded as a bidder with true value $v_2^{G_2}$ in auction G_1. After auctioneer A_2's participating, by Theorem 2.3 and Lemma 4.4, auction G_1 will converge to a new forward-looking Nash equilibrium where auctioneer A_2 gets slot 1 or 2 of auction G_1. No matter whether A_2 gets slot 1 or 2, now in auction G_1, the second highest bidder's true value equals $v_1^{G_1}$. So auctioneer A_1 will use the true value $v_1^{G_1}$ to participant auction $\widetilde{G}_2$. Similarly, auction G_2 will reach a new forward-looking Nash equilibrium where auctioneer A_1 gets slot 2 or 3. Since the second highest bidder's true value still equals $v_2^{G_2}$, we proved the existence of the forward-looking Nash equilibrium.

  Case II: $v_2^{G_2} < v_1^{G_1}$.

  Since $v_2^{G_2} \geq v_2^{G_1}$, it is easy to verify that there exists a forward-looking Nash equilibrium in which auctioneer A_1 gets slot 2 or 3 in auction G_2 with true value $v_2^{G_2}$ and auctioneer A_2 gets slot 2 or 3 in auction G_1 also with true value $v_2^{G_2}$.

  Therefore, the forward-looking Nash equilibrium exists even if both the auctioneers are click arbitrageurs.

- Proof of Part 2:

  This theorem can be derived directly from Theorem 3 in [2].

- Proof of Part 3:

  The proof is similar to that of Theorem 3.7.

Remark 4.7. If there exists the click arbitrage across markets, the advertisement displayed in the top slot of some auction may be displayed in some other slot in the same auction in the forward-looking Nash equilibrium. For example, A merchant wins the first slot of auction G_1 and meanwhile he wins some slot of auction G_2. Suppose auctioneer A_1 adopts click arbitrage for his first slot and wins some slot of auction G_2. Hence, there are two slots having the advertisement of the merchant in auction G_2 now. Actually, because the affiliate usually will keep this strategy of click arbitrage secret from his client, he would not attend the auction where the client's advertisement has been displayed. Furthermore, two display URLs which are the same, appearing on the same result page is forbidden in search engines, which prevents the phenomenon in the above example.

5. Conclusion

Our results derive interesting properties of cross-market auction models under GSP, utilizing the nice properties of the forward-looking Nash equilibrium. There are of course practical limitations of the models and the assumptions as well as the methodologies. First of all, the click-through-rates may not be invariable to the advertisements displayed on the slots, nor fixed even for the same advertisement [11]. Second, the forward-looking strategy may not be always followed by every player (see [10,15] for detailed discussions on this topic). Third, though the arbitrage behavior is in use in reality even for large Internet companies, there is a potential deterioration of the qualities of the advertisement slots.

Those limitations may have a potential restriction on the applicabilities of particular findings presented in this work. As a first approximation of the reality, our study makes the first viable effort to understand the cross-market phenomenon in the sponsored search market. We will further explore possibilities of further refinement to resolve the limitations pointed above. At this micro-economic level, it has not always been easy to derive a strong unification of theoretical work and practical reality. For example, the well celebrated revenue equivalence theorem [14,12] has not been found to universally hold in reality [7]. Our work as an extension in the new setting may not be possible to escape from such fallacy when human decision makers are involved. However, the fast speed at which Internet develops may rule out direct involvement of human subjects at tick level decisions and open up the room for software agents' full participation. Rationality may finally play a much more important role to prevail at Internet-based market places.

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References

[1] G. Aggarwal, A. Goel, R. Motwani, Truthful auctions for pricing search keywords, in: Proceedings of the 7th ACM Conference on Electronic Commerce, EC-06, Ann Arbor, USA, 11–15 June 2006, pp. 1–7.
[2] T.-M. Bu, X. Deng, Q. Qi, Dynamics of strategic manipulation in ad-words auction, in: 3rd Workshop on Sponsored Search Auctions, SSA2007, in conjunction with WWW2007, Banff, Canada, 8 May 2007.
[3] T.-M. Bu, X. Deng, Q. Qi, Arbitrage opportunities across search markets, in: Workshop on Targeting and Ranking for Online Advertising, TROA2008, in conjunction with WWW2008, Beijing, China, 22 April 2008.
[4] T.-M. Bu, X. Deng, Q. Qi, Forward-looking Nash equilibrium for keyword auction, Information Processing Letters 105 (2) (2008) 41–46.
[5] M. Cary, A. Das, B. Edelman, I. Giotis, K. Heimerl, A.R. Karlin, C. Mathieu, M. Schwarz, Greedy bidding strategies for keyword auctions, in: Proceedings of the 8th ACM Conference on Electronic Commerce, EC-2007, San Diego, USA, 11–15 June 2007, pp. 262–271.
[6] E.H. Clarke, Multipart pricing of public goods, Public Choice 11 (1) (1971) 11–33.
[7] J. Cox, B. Roberson, V. Smith, Theory and behavior of single object auctions, Research in Experimental Economics 2 (1982) 1–43.
[8] B. Edelman, M. Ostrovsky, M. Schwarz, Internet advertising and the generalized second price auction: Selling billions of dollars worth of keywords, American Economic Review 97 (1) (2007) 242–259.
[9] T. Groves, Incentives in teams, Econometrica 41 (4) (1973) 617–631.
[10] L. Liang, Q. Qi, Cooperative or vindictive: Bidding strategies in sponsored search auction, in: Proceedings of the 3rd International Workshop on Internet and Network Economics, WINE2007, San Diego, USA, 12–14 December 2007, pp. 167–178.
[11] M. Mahdian, H. Nazerzadeh, A. Saberi, Allocating online advertisement space with unreliable estimates, in: ACM Conference on Electronic Commerce, EC-07, 2007, pp. 288–294.
[12] R. Myerson, Incentive-compatibility and the bargaining problem, Econometrica 47 (1979) 61–73.
[13] H.R. Varian, Position auctions, International Journal of Industrial Organization 25 (6) (2007) 1163–1178.
[14] W. Vickrey, Counterspeculation, auctions, and competitive sealed tenders, Journal of Finance XVI (1) (1961) 8–37.
[15] Y. Vorobeychik, D.M. Reeves, Equilibrium analysis of dynamic bidding in sponsored search auctions, in: Proceedings of the 3rd International Workshop on Internet and Network Economics, WINE2007, San Diego, USA, 12–14 December 2007, pp. 155–166.