Theoretical Computer Science 475 (2013) 73–81

Contents lists available at SciVerse ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

# A tiling system for the class of *L*-convex polyominoes

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ARTICLE INFO

*Article history:*
Received 11 January 2010
Received in revised form 22 October 2012
Accepted 17 December 2012
Communicated by B. Durand

ABSTRACT

A polyomino is said to be *L*-convex if any two of its cells can be connected by a path entirely contained in the polyomino, and having at most one change of direction. In this paper, answering a problem posed by Castiglione and Vaglica [6], we prove that the class of *L*-convex polyominoes is tiling recognizable. To reach this goal, first we express the *L*-convexity constraint in terms of a set of independent properties, then we show that each class of convex polyominoes having one of these properties is tiling recognizable.



## 1. Introduction

In the last two decades *two-dimensional languages* (or *picture languages*) have been studied by several authors in the field of the theory of formal languages. Among those studies, the *tiling system recognizable picture languages* (briefly tiling recognizable languages), introduced in [7], have shown theoretical importance and remarkable applications. For the main results and properties of such a topic we refer the reader to [8].

Recently Matz proposed some interesting developments concerning tiling recognizable languages [11]. While proving that some picture languages (the star-free languages) can be defined by regular expressions but are not tiling recognizable, he developed a technique to prove that a certain picture language is not tiling recognizable. Roughly speaking, he showed that the pictures of a tiling recognizable language must satisfy a special constraint, to which we will refer to as the *Matz condition*: in practice, in a picture of a tiling recognizable language, at most an exponential amount of information can get from one half of the picture to another.

Moreover, Matz remarked that such a condition is necessary, and asked to investigate if it is also sufficient. As a matter of fact, he raised the problem of determining a two-dimensional language for which non recognizability can be proved without applying his condition. He suggested that a possible candidate could be the language of squares on the alphabet $\{a, b\}$ having the same number of $a$ and $b$. Later Reinhardt proved that, instead, such a language is tiling recognizable [13].

Matz condition has been successively reconsidered and reformulated by Giammarresi and Restivo in [9], along with the treatment of picture languages by means of the associated complexity functions, which rely on the concept of Hankel matrix of a string language.

In recent years some studies related to picture languages focused on the recognizability of classes of discrete planar objects which can be defined in terms of geometric constraints, such as connectivity, directedness or convexity. We mention here Reinhardt who proved in [12] that the class of polyominoes is tiling recognizable. In [1] the authors showed that some constraints on polyominoes, such as convexity and directed growth, can be represented by means of tiling systems. For instance, they exhibited tiling systems for the classes of horizontally convex, convex, directed, and parallelogram polyominoes.

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0304-3975/$ – see front matter 
doi:10.1016/j.tcs.2012.12.033

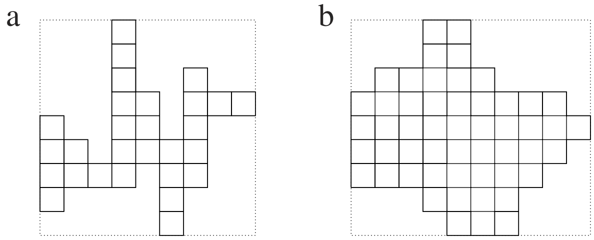


**Fig. 1.** (a) A vertically convex polyomino. (b) A convex polyomino.

The work by Castiglione and Vaglica [6] perfectly fits into this research line. They considered the class of *L*-convex polyominoes as a possible solution of the problem posed by Matz. The authors observe that

> *[...] the language of L-convex polyominoes is on the borderline between recognizability and non-recognizability, and it is an interesting object to study in the theory of picture languages.*

To certify this assertion they give a nice proof that *L*-convex polyominoes satisfy Matz condition. Moreover, if we limit the number of maximal rectangles, the class is tiling recognizable [1]. On the other side, this class can be obtained as the intersection of two picture languages, that of the horizontally convex and that of the vertically convex polyominoes that are univocally determined by their horizontal and vertical projections (as intended in Discrete Tomography, see chapter 1 of [10] for an introduction to the topic), which are non tiling recognizable. This motivates the conjecture that *the language of L-convex polyominoes is not tiling recognizable*.

In this paper we prove such a conjecture not to be true, by proving that the language of *L*-convex polyominoes is tiling recognizable. Since our method is quite elaborate, in order not to render the formalism too heavy, we start by decomposing the *L*-convexity constraint in terms of a set of independent properties. Then we show that each picture language having one only of these properties is tiling recognizable. Therefore, using intersection closure of tiling languages, we obtain the desired result. For brevity's sake, some technical proofs will be omitted.

As a consequence of our result, it turns out that the non sufficiency of the Matz condition cannot be proved by means of the *L*-convex polyominoes.

## 2. Some basic definitions and properties

In this section we briefly recall the definitions of *local picture languages* and *tiling systems*, and then we relate them with *convex polyominoes* showing some results that are useful in the rest of the paper. Finally we give evidence of the non tiling recognizability of a simple class of polyominoes. For more details on picture languages we refer to [8].

### 2.1. On tiling systems

Given an alphabet $\Sigma$, we define $\Sigma^{*,*}$ to be the set of all matrices on $\Sigma$, and $\Sigma^{m,n}$ to be those matrices having dimension $m \times n$. A picture $p$ of size $m \times n$ is simply an element of $\Sigma^{m,n}$.

We indicate by $\hat{p}$ the picture of size $(m+2) \times (n+2)$ obtained by surrounding $p$ with a special *boundary symbol* # not belonging to $\Sigma$.

Moreover, for any $h \leq m$, $k \leq n$, we denote by $\Sigma^{h,k}(\hat{p})$ the set of all sub-matrices (or sub-pictures) of $\hat{p}$ of size $h \times k$ whose alphabet is $\Sigma \cup \{\#\}$, and we call each sub-matrix of dimension $2 \times 2$ on the same alphabet a *tile*.

A two dimensional language $L \subseteq \Sigma^{*,*}$ is *local* if there exists a finite set $\theta$ of tiles such that $L = \{p \in \Sigma^{*,*} : \Sigma^{2,2}(\hat{p}) \subseteq \theta\}$. The set $\theta$ is usually called a *representation by tiles* for the local language $L$, and we write $L = L(\theta)$.

A *tiling system* is a 4-uple $\mathcal{T} = (\Sigma, \Gamma, \theta, \pi)$, where $\Sigma$ and $\Gamma$ are two finite alphabets, $\theta$ is a finite set of tiles over the alphabet $\Gamma \cup \{\#\}$, and $\pi : \Gamma \to \Sigma$ is a projection.

We say that a tiling system $\mathcal{T}$ defines the language $L = \pi(L(\theta))$, where $L(\theta)$ is a local language over $\Gamma$, called the *underlying language* for $L$, and write by convention $L = L(\mathcal{T})$. Moreover, we say that $L \subseteq \Sigma^{*,*}$ is *recognizable by tiling systems* (or *tiling recognizable*), if there is a tiling system $\mathcal{T} = (\Sigma, \Gamma, \theta, \pi)$, such that $L = L(\mathcal{T})$.

### 2.2. Convex and L-convex polyominoes

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square whose vertices have integer coordinates, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations.

A *column* (resp. *row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (resp. horizontal) line. A polyomino is *horizontally convex* (resp. *vertically convex*) if each of its rows (resp. columns) is connected (see Fig. 1(a)), and it is *convex* if it is both horizontally and vertically convex (see Fig. 1(b)).

In this paper we deal with a special class of convex polyominoes, the *L-convex polyominoes*, introduced in [4] as the first level in a classification of convex polyominoes.

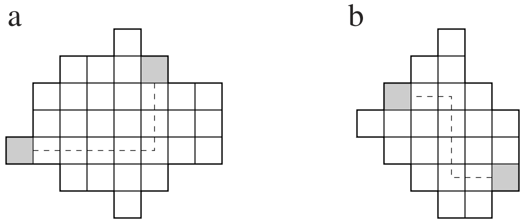


**Fig. 2.** (a) A $L$-convex polyomino, and a monotone path with a single change of direction joining two of its cells. (b) A convex but not $L$-convex polyomino: the two highlighted cells cannot be connected by a path with only one change of direction.

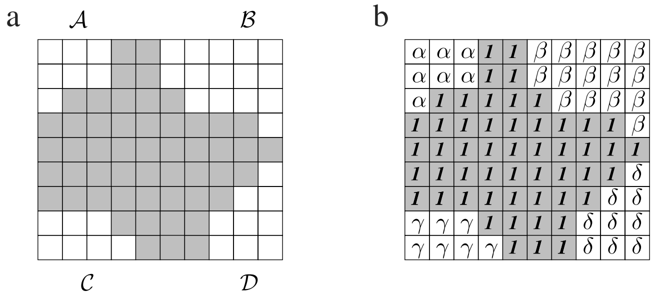


**Fig. 3.** (a) The convex polyomino of Fig. 1 individuates four disjoint exterior parts. (b) The representation of the polyomino as a picture on the alphabet $\{\alpha, \beta, \gamma, \delta, 1\}$.

To define this class we need some further basic concepts: in a polyomino an *internal path* is a self-avoiding sequence of unit steps along one of the four directions north, south, east, and west. We say that a path is *monotone* if it is made of steps of only two types. The authors of [4] observed that convex polyominoes have the property that every pair of cells is connected by a monotone path entirely contained in the polyomino. In this way each convex polyomino is characterized by a parameter $k$ that represents the minimal number of changes of direction in these paths. More precisely, a convex polyomino is called *k-convex* if, for every pair of its cells, there is at least a monotone path with at most $k$ changes of direction that connects them. When the value of $k$ is 1 we have the so called $L$-convex polyominoes, where this terminology is motivated by the $L$-shape of the path that connects any two cells (see Fig. 2).

This class of polyominoes has been successively considered by several points of view: in [5] it is shown that $L$-convex polyominoes are a well-ordering according to the sub-picture order, in [2,4] the authors have investigated some tomographical aspects of this family, and have discovered that $L$-convex polyominoes are uniquely determined by their horizontal and vertical projections. Finally, in [3] $L$-convex polyominoes were enumerated according to semi-perimeter.

There is a simple way to represent a polyomino in terms of a picture on the alphabet $\{0, 1\}$ by setting a 1 (resp. a 0) in each position of the word corresponding to a cell belonging (resp. not belonging) to the polyomino in its minimal bounding rectangle. If the language of the pictures which encode the elements of a class of polyominoes is tiling recognizable, then we say that the class of polyominoes under consideration is *tiling recognizable*.

In [1], it is proved that various classes of polyominoes, including convex and column convex polyominoes, are tiling recognizable classes. The main idea on which all these proofs rely is to control the convexity of the four disjoint (possibly empty) parts $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ which form the exterior of a convex polyomino, see Fig. 3(a). For each convex polyomino we consider its picture representation, and we mark with the symbol $\alpha$ (resp. $\beta$, $\gamma$, $\delta$) every element in $\mathcal{A}$ (resp. $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$), as depicted in Fig. 3(b). We can prove that the language of these pictures is local, hence we can easily map it into the language of convex polyominoes.

### *2.3. Non tiling recognizable languages*

Let us now present the following definition.

**Definition 1** (*Matz Condition*)**.** Let $L \subseteq \Sigma^{*,*}$ be a two-dimensional language. We say that $L$ satisfies the Matz condition if for each sequence of sets $\mathcal{M}_n \subseteq \Sigma^{n,+} \times \Sigma^{n,+}$ satisfying,

1. $\forall (P, Q) \in \mathcal{M}_n$ we have $PQ \in L$;
2. $\forall (P, Q), (P', Q') \in \mathcal{M}_n$, with $(P, Q) \neq (P', Q')$, we have $\{PQ', P'Q\} \not\subseteq L$,

then $|\mathcal{M}_n|$ is $2^{\mathcal{O}(n)}$.

As we already mentioned, Matz [11] proved the following property of tiling recognizable languages, and used it to show the non tiling recognizability of some two-dimensional languages.

**Lemma 1.** *If $L \subseteq \Sigma^{*,*}$ is tiling recognizable, then it satisfies the Matz condition.*

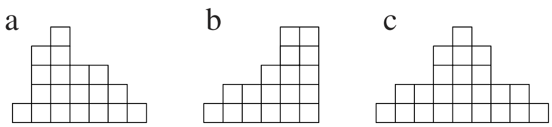


**Fig. 4.** (a) A stack polyomino. (b) A Ferrers diagram. (c) A stack polyomino symmetric according to the $y$-axis.

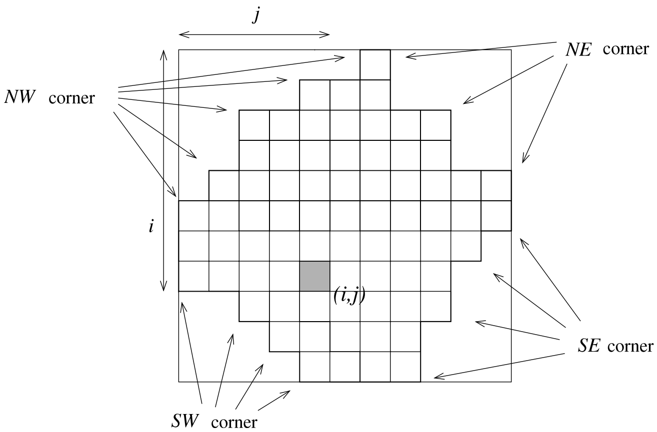


**Fig. 5.** The different types of corners in a convex polyomino. The element in position $(i, j)$ has been highlighted.

Just to give an example of the application of Lemma 1, we show that the language $\mathcal{S}$ of *stack polyominoes symmetric according to the y-axis* is not tiling recognizable. We recall that a *stack* polyomino (resp. a *Ferrers diagram*) is a convex polyomino where the regions $\mathcal{C}$ and $\mathcal{D}$ (resp. the regions $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{B}$) are empty (see Fig. 4). Moreover, given a stack polyomino $P$, let $\overline{P}$ be the symmetric of $P$ according to the $y$-axis. Now, for any $n \geq 1$ we can consider the set

$$\mathcal{M}_n = \{(P, \overline{P}) : P \text{ is a Ferrers diagram of height } n\}.$$

The set $\mathcal{M}_n$ clearly satisfies the two conditions of Definition 1, but for any $n \geq 1$, $\mathcal{M}_n$ is an infinite set. Then $\mathcal{S}$ does not satisfy the Matz condition, hence by Lemma 1 it is not tiling recognizable.

In [11] Matz himself raised the problem to establish if Lemma 1 could be inverted, i.e., if there are some non tiling recognizable languages for which his condition is satisfied. He suggested as a possible candidate the language of squares on the alphabet $\{a, b\}$ having the same number of $a$ and $b$. Later Reinhardt [13] proved that, instead, such a language is tiling recognizable.

To give an answer to Matz question, some years later Castiglione and Vaglica [6] investigated the tiling recognizability of $L$-convex polyominoes. They proved that:

i. the class of $L$-convex polyominoes satisfies Matz condition;
ii. this class can be obtained as the intersection of two picture languages, the unique $h$-convex and the unique $v$-convex polyominoes, which are non tiling recognizable languages.

Then they conjectured that *the language of L-convex polyominoes is not tiling recognizable*.

## 3. Some basic properties of *L*-convex polyominoes

In the sequel, we will always identify a polyomino $P$ with its picture representation, i.e. a binary matrix whose dimensions are the dimensions of the minimal bounding rectangle of $P$, where the value of an entry is 1 if and only if there is a cell of $P$ in the corresponding position. According to this representation, by referring to the $(i, j)$ *element of P* we will understand just the element in the $i$th row and $j$th column of the matrix corresponding to the picture representation of $P$ (see Fig. 5). So, the numbering of the rows and of the columns of the polyomino follow that of the corresponding matrix, being the cell $(1, 1)$ the upper leftmost.

Two cells of a polyomino are said to be $L$-connected if there exists a monotone path having at most one change of direction connecting them and entirely contained in the polyomino.

We say that an element 1 in position $(i, j)$ is a *SW corner* if the elements in positions $(i+1, j)$ and $(i, j-1)$ are both equal to 0 or #.

Symmetrically, we define the *NW corners*, *NE corners* and *SE corners* (see Fig. 5).

### 3.1. The L-convexity constraint

In this section we will furnish an alternative characterization of $L$-convex polyominoes in terms of a series of elementary properties which involve the corners. In the next section we show that, for each of these properties, the class of polyominoes satisfying such a property is tiling recognizable. Consequently, since tiling recognizable languages are closed under intersection, the whole class of $L$-convex polyominoes turns out to be tiling recognizable.

**Theorem 1.** *A convex polyomino is L-convex if and only if every corner is L-connected to every other cell.*

**Proof.** The condition is clearly necessary; to prove that it is also sufficient let us proceed by contradiction assuming that there exist two cells $a$ in $(i, j)$ and $b$ in $(i', j')$, with $i' > i$ and $j' > j$, that are not $L$-connected.

Let us consider a $NW$ corner $c$ in $(i'', j'')$, with $i'' \leq i$, and $j'' \leq j$. Since there exists by hypothesis an $L$-path running from $c$ to $b$, then we can easily compute an $L$-path that runs from $a$ to $b$, which is a contradiction. A similar reasoning applies after changing the mutual positions of $a$ and $b$, by using the other corners. □

**Corollary 1.** *A convex polyomino is L-convex if and only if, for every choice of two of its corners, the two corners are L-connected.*

In what follows we will exhibit a set of conditions which express that a given corner of a convex polyomino is $L$-connected to all its other cells. We will concentrate on $SW$ corners, but we advise the reader that a similar set of conditions can be easily determined for the other types of corners. Then, the union of all these conditions will be equivalent to $L$-convexity, and it will be used to express $L$-convexity in terms of a tiling system.

**Theorem 2.** *Let $P$ be a convex polyomino having a $SW$ corner $a$ in position $(i, j)$, $j'$ the index of the column of the rightmost cell of $P$ in row $i$, and $i'$ the index of the row of the uppermost cell of $P$ in column $j$.*

*The corner $a$ is $L$-connected to all the cells of $P$ if and only if there are no cells $(i'', j'')$ of $P$ satisfying one of the following three conditions:*

(1) $i'' < i'$ *and* $j'' \leq j$;
(2) $j'' > j'$ *and* $i'' < i'$;
(3) $j'' > j'$ *and* $i'' \geq i$.

*These conditions individuate three regions that we call $\mathcal{A}(a)$, $\mathcal{B}(a)$ and $\mathcal{C}(a)$, respectively, in the bounding rectangle of $P$, as shown in Fig. 6(a).*

**Proof.** ($\Leftarrow$) Let us consider a cell $b$ of $P$ in position $(i'', j'')$. Conditions (1)–(3) impose that $i' \leq i'' \leq i$ or $j \leq j'' \leq j'$. In both situations there exists an $L$-path from $a$ to $b$.
($\Rightarrow$) Let us proceed by contradiction assuming that there exists a cell $b$ of $P$ in $(i'', j'')$ such that $i'' < i'$ and $j'' \leq j$, against condition (1).

Then, an $L$-path from $a$ to $b$ changes its direction in a cell $(i''', j''')$ having either $i''' = i''$ or $j''' = j''$; both cases are not possible by definition of row $i'$ and of $SW$ corner, respectively.
A similar argument holds when the cell $b$ does not satisfy condition (3).

Finally, let us assume that the cell $b$ of $P$ in position $(i'', j'')$ violates condition (2). Since $a$ is $L$-connected to $b$ by hypothesis, then there exists an $L$-path having one change of direction in $(i'', j')$ or in $(i', j'')$, with $i'' < i'$ or $j'' > j$, against the definitions of $i'$ and $j'$. □

Now we consider the definition of $L$-convexity restricted to $SW$ corners: a convex polyomino is said to be *SW-convex* if all its $SW$ corners are $L$-connected to all its cells, i.e., they satisfy conditions (1)–(3) of Theorem 2. Analogously we can define the *NE-convex*, *NW-convex* and *SE-convex* polyominoes.

Concerning $SW$ corners, we observe that conditions (1) and (2) [resp. (2) and (3)] of Theorem 2 may be redundant. More precisely we have the following.

**Theorem 3.** *Let $a$ and $b$ be two SW corners of a convex polyomino $P$ in positions $(i, j)$ and $(i', j')$, respectively, with $i' < i$. It holds that:*
*- if the uppermost cells of $P$ in columns $j$ and $j'$ lie in the same row, then it holds $\mathcal{A}(b) \subset \mathcal{A}(a)$ and $\mathcal{B}(b) \subset \mathcal{B}(a)$ (see Fig. 6(b));*
*- if the rightmost cells of $P$ in rows $i$ and $i'$ lie in the same column, then it holds $\mathcal{C}(a) \subset \mathcal{C}(b)$ and $\mathcal{B}(a) \subset \mathcal{B}(b)$.*

The proof directly follows from Theorem 2.
We define an $SW$ corner $a$ to be *active* if there does not exist another $SW$ corner $b$ such that either $\mathcal{B}(a) \subset \mathcal{B}(b)$ or, if $\mathcal{B}(a) = \mathcal{B}(b)$ then $\mathcal{A}(a) \subset \mathcal{A}(b)$.
We say that a $SW$ corner $A$ is *inactive* otherwise (see Fig. 6(b)). It is worth mentioning that an analogous definition of active and inactive corners can be furnished for $NE$, $SE$, $NW$ corners.

The concept of active corner plays a key role in the definition of our tiling system for $SW$-convex polyominoes, since, by definition, it allows us to check Condition (2) of Theorem 2 only for those $SW$ corners that are active (while conditions (1) and (3) will be checked for all the $SW$ corners).

We explicitly state the following property that is a direct consequence of Theorem 3, and that will furnish an effective way of detecting active corners by means of a tiling system:

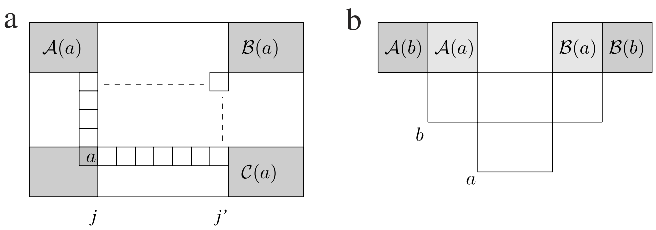


**Fig. 6.** (a) The presence of the *SW* corner $a$ imposes that there cannot be cells of the polyomino in regions $\mathcal{A}(a)$, $\mathcal{B}(a)$ and $\mathcal{C}(a)$. (b) Two *SW* corners $a$ and $b$ such that $\mathcal{A}(b) \subset \mathcal{A}(a)$, and $\mathcal{B}(b) \subset \mathcal{B}(a)$.

**Property 1.** *Let a and b be two SW corners of a convex polyomino P in positions* $(i, j)$ *and* $(i', j')$*, respectively. The corner a is non active if and only if one of the following two conditions are satisfied:*

i. *the rightmost cells of P in rows i and i′ lie in the same column, and it holds* $i > i'$*;*
ii. *the rightmost cells of P in rows i and i′ do not lie in the same column, the uppermost cells of P in columns j and j′ lie in the same row, and it holds* $j < j'$*.*

## 4. A tiling system for *L*-convex polyominoes

Our proof of the tiling recognizability of *L*-convex polyominoes can be split in four different conditions: SW-convexity, NW-convexity, NE-convexity and SE-convexity. The class of *L*-convex polyominoes is then recognized by taking the intersection of all these languages. We will only construct a tiling system recognizing SW-convex polyominoes, all other cases being symmetric.

The tiling system we construct, will consist of several layers $Z, C, D, B$, each of them serves a different purpose, the complete tiling system is included in $Z \times C \times B$:

- The first layer $Z$ has alphabet $\Sigma_Z = \{\#, 0, 1\}$ and will serve as the projection support, *i.e.* the picture language is obtained by restraining to the symbols of this layer. This layer actually represents the SW-convex polyomino.
- The second layer $C$ will be used to find the SW-corners $c$, the corresponding zones $\mathcal{A}(c)$, $\mathcal{C}(c)$ and to forbid the appearance of 1 in them. That is to say it checks conditions (1) and (3) of Theorem 2.
- The third layer $D$ will be used to find and label active and inactive corners according to Property 1.
- The last layer $B$ checks condition (2) of Theorem 2. It places labels corresponding to the lower left corners of the zones $\mathcal{B}(c)$ corresponding to each active corner $c$.

To have a valid tiling, the polyomino on layer $Z$ will need to be SW-convex.

### 4.1. *Detecting and forbidding zones* $\mathcal{A}(c)$ *and* $\mathcal{C}(c)$

We describe here the second layer, $C$, whose alphabet is

$$\Sigma_C = \left\{c, c_{\rightarrow}, c_{\uparrow}, c_{\uparrow\rightarrow}, c_u, c_e, \overset{c_e}{c}, \overset{c_u}{c}, \overline{c}_u, \overline{c}_e, \cdot\right\}.$$

The symbol $c$ is placed on top of all SW-corners, which are easily detectable. Each $c$ symbol then propagates a signal up, $c_{\uparrow}$, and right, $c_{\rightarrow}$, until it finds the last 1 before the border and marks it $c_u$ and $c_e$. Above a $c_u$, there can only be a $\overline{c}_u$. A $\overline{c}_u$ can only have other $\overline{c}_u$ to its left and above. To the right of a $c_e$ there can only be a $\overline{c}_e$, and a $\overline{c}_e$ can only have other $\overline{c}_e$ to its left and above. The two symbols $\overset{c_e}{c}$, $\overset{c_u}{c}$, must respect the same rules as their subsymbols.

In the polyomino on the left of Fig. 7, it is shown how layer $C$ detects the zones $\mathcal{A}(c)$ and $\mathcal{C}(c)$ and prevents any element of the polyomino lieing there: the symbols $\overline{c}_u$ (resp. $\overline{c}_e$) are in an $\mathcal{A}(c)$ (resp. $\mathcal{C}(c)$) zone for some corner $c$. The positions without a symbol actually have $\cdot$ on them.

The rules for superimposition are the symbols $c, c_{\rightarrow}, c_{\uparrow}, c_{\uparrow\rightarrow}, c_u, c_e$ can only be placed on top of 1, while symbols $\overline{c}_u, \overline{c}_e$ can only be placed on top of 0. The last symbol, $\cdot$ may be placed anywhere and is only used to mark those cells which do not have any special purpose.

The tiling system that characterizes the layer $C$ is really simple, and it is not furnished here.

## 5. Detecting active SW-corners

Layer $D$ detects active corners. In order to do that, it is separated in two sublayers: sublayer $E$ and sublayer $U$, which detect corners that are inactive by Property 1i, and corners that are inactive by Property 1ii, respectively. Let us describe in detail $E$ only, as $U$ is based on the same principle. The alphabet of $E$ is formed by the symbols:

$$\Sigma_E = \left\{a, i, a_{\rightarrow}, i_{\rightarrow}, A, I, \overset{A}{a}, \overset{I}{i}, A_{\uparrow}, I_{\uparrow}, \overset{A_{\uparrow}}{i_{\rightarrow}}, \overset{A_{\uparrow}}{a_{\rightarrow}}, \overset{I_{\uparrow}}{a_{\rightarrow}}, \overset{I_{\uparrow}}{i_{\rightarrow}}, \cdot\right\}.$$

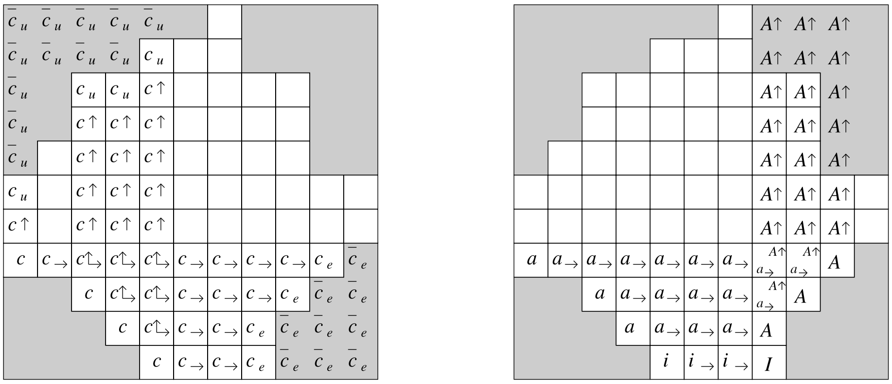

**Fig. 7.** How the different layers $C$, on the left, and $D$, on the right, are tiled. The colors gray and white denote respectively the symbols 0 and 1 of layer $Z$.

The symbols with arrows are signals that are transmitted in the same direction of the arrow. Symbols $a, i$ represent whether a corner is active or inactive, and may be superimposed them only on corners $c$ of layer $C$. A signal is sent from the $a$ and $i$ cells rightward until it reaches the border of the polyomino, this signal is represented by the symbols $a_\rightarrow$, $i_\rightarrow$, when the border is reached, it is marked with a symbol $A$ or $I$ depending on the signal. So both $A$ and $I$ can only be superimposed to 0 symbols on layer $Z$; the two symbols $\substack{A\\a}$, $\substack{I\\i}$ take care of the one cells rows, i.e. where either $a$ and $A$ or $i$ and $I$ collapse. The cells containing $A$ or $I$ also send a signal upward: $A_\uparrow$ and $I_\uparrow$ respectively. These signals are propagated upward with the rule that a symbol $A_\uparrow$ cannot be below of an $A$ and that an $I_\uparrow$ must end with a symbol $A$ above, and may not be superimposed to 0 on layer $Z$.

Note that the signals $i_\rightarrow$, $a_\rightarrow$ and $A_\uparrow$, $I_\uparrow$ may cross each other, and so, for this purpose, we have four symbols $\substack{I_\uparrow\\a_\rightarrow}$, $\substack{A_\uparrow\\a_\rightarrow}$, $\substack{A_\uparrow\\i_\rightarrow}$, $\substack{I_\uparrow\\i_\rightarrow}$ which must respect the same rules as their subsymbols. Again $\cdot$ is used on the cells that do not hold any particular meaning. See Fig. 7 on the right, for a detailed example for this layer: the corners $a$ and $i$ are represented, and the signals leading rightward from them are depicted till reaching the cells $A$ and $I$. Then, the upward signals are propagated using the labels $A_\uparrow$ and $I_\uparrow$. Labels $I_\uparrow$ may not be superimposed to cell with symbol 0 on component $Z$: it has to become $A$ before.

As an example, we furnish the details of the horizontal tilings of layer $E$ that take care of the propagation of the signals $a$ rightward, being the propagations of the symbols $i$, $A$ and $I$ similar:

$$\left\{ \begin{array}{cccc} \boxed{\cdot \mid a} & \boxed{\cdot \mid \substack{A\\a}} & \boxed{a \mid A} & \boxed{a \mid \alpha^-} \\ \boxed{\alpha \mid \alpha} & \boxed{\alpha \mid A} & \boxed{A \mid \cdot} & \boxed{\substack{A\\a} \mid \cdot} \end{array} \right\}$$

with $\alpha^- \in \{a_\rightarrow, \substack{A_\uparrow\\a_\rightarrow}\}$, $\alpha \in \alpha^- \cup \{\substack{I_\uparrow\\a_\rightarrow}\}$.

Layer $U$ detects the corners inactive by Property 1ii, and uses the same principle on the alphabet

$$\Sigma_U = \left\{a, i, a_\uparrow, i_\uparrow, A, I, \substack{A\\a}, \substack{I\\i}, A_\rightarrow, I_\rightarrow, \substack{a_\uparrow\\A_\rightarrow}, \substack{a_\uparrow\\I_\rightarrow}, \substack{i_\uparrow\\A_\rightarrow}, \substack{i_\uparrow\\I_\rightarrow}, \cdot\right\}$$

with the further addiction that the $\cdot$ is superimposed also on the i cells of layer $E$ in order to resemble the condition of Property 1ii that requires already detected inactive corners not to be considered.

The active corners are those that have a symbol $a$ on the layer $U$. One can see that layer $E$ detects inactive corners when they verify Property 1i, and that layer $U$ detects inactive corners when they verify Property 1ii.

## 6. Detecting and forbidding $\mathcal{B}(c)$ zones

The last step involving layer $B$, infers, from the active corners $a$, the zones $\mathcal{B}(a)$ by labeling $L$ the cell on the left below it. The alphabet of $B$ is the following:

$$\Sigma_B = \left\{a, U, E, \substack{U\\a}, \substack{E\\a}, u_\uparrow, e_\rightarrow, L_\rightarrow, L_\uparrow, L, \bar{L}, \substack{L_\uparrow\\L_\rightarrow}, \substack{L_\uparrow\\e_\rightarrow}, \substack{u_\uparrow\\L_\rightarrow}, \substack{u_\uparrow\\e_\rightarrow}, \cdot\right\}.$$

The active corners detected in the previous layer are given label $a$ and send a signal $e_\rightarrow$ to the right and a signal $u_\uparrow$ up. These two signals go to the border of the polyomino and mark it with $E$ and $U$ respectively; again the two symbols $\substack{E\\a}$, $\substack{U\\a}$ take care of the case where either $a$ and $E$ or $a$ and $U$ collapse. Then the two symbols $E$ and $U$ propagate respectively a signal $L_\uparrow$ to the right and a signal $L_\rightarrow$ upward, and when these symbols meet, they may combine to form a symbol $L$ and stop, or cross each other and continue. These two signals are forbidden to go on the border, *i.e.* they have to combine before reaching the end of the picture. This means that all signals $L_\rightarrow$ and $L_\uparrow$ have to combine at some point.

Now the cell above and on the right of an $L$ must contain an $\bar{L}$ or be the border of the picture. Above and to the right of an $\bar{L}$, there can only be other $\bar{L}$'s.

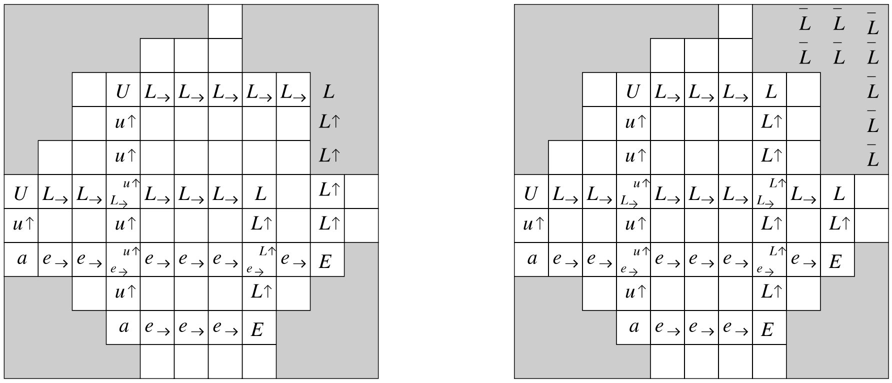

**Fig. 8.** The active corners send signals $u_\uparrow$ and $e_\rightarrow$ up and right, the border is marked with $U$ and $E$ and each of these then sends a signal $L_\rightarrow$ and $L_\uparrow$ respectively, these signals must not arrive on the borders # of the picture and may cross each other or generate an $L$ label and stop. The labeling on the left is impossible as the upper $L$ actually appears inside the zone that should be covered with $\overline{L}$ because of the other $L$. On the right the only possible labeling: here none of the $L$ appears in the $\overline{L}$ zone generated by the other.

$\overline{L}$ may only be superimposed to 0 on the $Z$ layer, while $L$, $L_\rightarrow$, $L_\uparrow$ can be superimposed to both 0 and 1. Fig. 8 shows two examples of labelings.

Again, all the signals introduced before may cross one another, for this purpose, we have the symbols $\genfrac{}{}{0pt}{}{L_\uparrow}{L_\rightarrow}, \genfrac{}{}{0pt}{}{L_\uparrow}{e_\rightarrow}, \genfrac{}{}{0pt}{}{u_\uparrow}{L_\rightarrow}, \genfrac{}{}{0pt}{}{u_\uparrow}{e_\rightarrow}$ of the alphabet which must verify the same rules as both their subsymbols.

The following two properties hold:

**Property 2.** *The defined tiling system places a number of labels L equal to the number of active SW.*

The proof is straightforward after observing that each $L$ shares its row with a label $c_u$ and its column with a label $c_e$, and no other $L$ can lie in the same row or column.

**Property 3.** *The defined tiling system places the labels L in the cells of the polyomino P that are the lower leftmost cells of the $\mathcal{B}$ zones related to the active corners.*

**Proof.** Let us arrange the $k$ active corners of $P$, with $k \geq 2$, in an increasing sequence $c_1, \ldots, c_k$ according to the row (and consequently column) indexes $i_1 < i_2 < \cdots < i_k$. Let us proceed by contradiction assuming that a cell in position $(i, j)$ and labeled with $L$ delimits a $\mathcal{B}$ zone not related to any active corner. This means that there exist two labels $c_u$ and $c_e$ that lie in row $i$ and column $j$, and that are related to two different active corners, say $c_r$ and $c_s$, with $r < s$. Let us assume that $c_u$ is related to $c_r$, and $c_e$ to $c_s$; there exist $s - 1$ cells labeled with $c_e$ whose columns are greater than $j$, and only $r - 1$ cells labeled with $c_u$ whose row is greater than $i$, so by Property 2 at least one label $L$ must lie in a column greater than $j$ and in a row smaller than $i$, against what imposed by the tiling system. If we assume that $c_u$ is related to $c_s$, and $c_e$ to $c_r$ a similar reasoning holds. □

Similar tiling systems can be defined to check the other three types of convexity, i.e. *SE*, *NW* and *NE* convexity. As already mentioned, the union of all these conditions, obtained by intersecting the correspondent tiling systems, gives the tiling recognizability of the entire class of *L*-convex polyominoes.

The particular case of the one cell polyomino can be treated as a special case.

## 7. Conclusions

In [6] the class of *L*-convex polyominoes was proposed as a possible candidate to be a non-tiling recognizable class satisfying Matz condition, while in this paper we have proved that this conjecture is not true.

We reached this aim by refining a previous result in [1], where it was shown that *L*-convex polyominoes having a fixed number of maximal rectangles are tiling system recognizable; this result relies on the possibility of checking the presence of a maximal rectangle by using an alphabet symbol. As a consequence, if the number of maximal rectangles is not fixed a priori, then the cardinality of the alphabet is not bounded as well. Here, we overcame this problem by moving the focus from the maximal rectangles to the zones that cannot be reached by *L*-path from the corners of the polyomino.

However, the original problem of the search for a class that is non tiling system recognizable, and that satisfy Matz condition remains open, and it constitutes an interesting challenge as it could give insights on the expressive power of tiling systems and on how some properties can avoid the recognizability of a language.

## Acknowledgments

We would like to thank A. Restivo, G. Castiglione, and R. Vaglica, for introducing us to the class of *L*-convex polyominoes, more than five years ago, and more recently, for pointing us the problem of the tiling recognizability of this class.


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