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Fault-tolerant path embedding in folded hypercubes with both node and edge faults

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ABSTRACT

The folded hypercube FQ_n is a well-known variation of the hypercube structure. FQ_n is superior to Q_n in many measurements, such as diameter, fault diameter, connectivity, and so on. Let $\tilde{V}(FQ_n)$ (resp. $\tilde{E}(FQ_n)$) denote the set of faulty nodes (resp. faulty edges) in FQ_n. In the case that all nodes in FQ_n are fault-free, it has been shown that FQ_n contains a fault-free path of length $2^n - 1$ (resp. $2^n - 2$) between any two nodes of odd (resp. even) distance if $|\tilde{E}(FQ_n)| \leq n - 1$, where $n \geq 1$ is odd; and FQ_n contains a fault-free path of length $2^n - 1$ between any two nodes if $|\tilde{E}(FQ_n)| \leq n - 2$, where $n \geq 2$ is even. In this paper, we extend the above result to obtain two further properties, which consider both node and edge faults, as follows:

1. FQ_n contains a fault-free path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ (resp. $2^n - 2|\tilde{V}(FQ_n)| - 2$) between any two fault-free nodes of odd (resp. even) distance if $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \leq n - 1$, where $n \geq 1$ is odd.
2. FQ_n contains a fault-free path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ between any two fault-free nodes if $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \leq n - 2$, where $n \geq 2$ is even.

1. Introduction

The *hypercube* is a well-known interconnection network model. Hypercube networks have received much attention over the past few years since they possess several attractive properties such as symmetry, recursive structure, regularity, and logarithmic diameter [7]. In order to further improve the performance of the hypercube networks, some variations of the hypercube structure have been proposed [1,2,9]. One of these variations proposed by El-Amawy and Latifi [1] is the *folded hypercube* which can be constructed from a hypercube by adding a link to every pair of nodes that are the farthest apart, i.e., two nodes with complementary addresses. The folded hypercube is superior to the hypercube in many measurements, such as diameter, fault diameter, connectivity, and so on [1,13].

An important feature of an interconnection network is its ability to efficiently simulate algorithms designed for other architectures. Such a simulation can be formulated as a *network embedding*. An *embedding* of a *guest network* G into a *host network* H is defined as a one-to-one mapping f from nodes in G into nodes in H so that an edge of G corresponds to a path of H under f [7]. Linear arrays and rings [7], whose underlying topologies are paths and cycles respectively, are two of the most popular guest networks because they are suitable for designing simple algorithms with low communication cost.

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Since faults may occur on both nodes and edges in a network, it is practically meaningful and important to consider faulty networks. The problems of embedding linear arrays or rings in faulty hypercubes and faulty folded hypercubes have been extensively studied [3–5,8,10,13]. Throughout this paper, we denote the sets of faulty nodes and faulty edges of a network G as $\tilde{V}(G)$ and $\tilde{E}(G)$, respectively.

Given an n-dimensional folded hypercube FQ_n without any faulty nodes, Hsieh [5] has shown that FQ_n contains a fault-free path of length $2^n - 1$ (resp. $2^n - 2$) between any two nodes of odd (resp. even) distance if $|\tilde{E}(FQ_n)| \le n - 1$, where n is odd; and FQ_n contains a fault-free path of length $2^n - 1$ between any two nodes if $|\tilde{E}(FQ_n)| \le n - 2$, where n is even. In this paper, we extend Hsieh's result to obtain two further properties, which consider both node and edge faults, as follows:

1. FQ_n contains a fault-free path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ (resp. $2^n - 2|\tilde{V}(FQ_n)| - 2$) between any two fault-free nodes of odd (resp. even) distance if $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n - 1$, where $n \ge 1$ is odd.
2. FQ_n contains a fault-free path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ between any two fault-free nodes if $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n - 2$, where $n \ge 2$ is even.

The rest of this paper is organized as follows. In Section 2, definitions and notations used in this paper are introduced. In Section 3, we introduce the previous results that will be employed later. In Section 4, we present our main results. Conclusions are given in Section 5.

2. Preliminaries

In this paper, a network topology is represented by a simple undirected graph, which is loopless and without multiple edges. We denote the node set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. Throughout this paper, the terms *network* and *graph*, *node* and *vertex*, *link* and *edge* are used interchangeably. Two nodes u and v are *adjacent*, if $(u, v) \in E(G)$, and u and v are the *end-nodes* of (u, v). Two adjacent nodes are called *neighbors* each other. A *path*, denoted by $\langle v_0, v_1, \ldots, v_k\rangle$, is a sequence of distinct nodes $v_0, v_1, \ldots, v_k$ in which any two consecutive nodes are adjacent. We call v_0 and v_k the *end-nodes* of the path. A path with end-nodes u and v, denoted by $P[u, v]$, is referred as *uv-path*. The *length* of a uv-path, denoted by $|P[u, v]|$, is the number of edges on the path. The *distance* between u and v is the smallest length of any uv-path in G and is denoted by $d_G(u, v)$ or simply $d(u, v)$ if there is no ambiguity. A path $\langle v_0, v_1, \ldots, v_k\rangle$ forms a *cycle* if $v_0 = v_k$. A path (resp. cycle) in G is called a *Hamiltonian path* (resp. *Hamiltonian cycle*) if it contains every node of G exactly once. G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and *Hamiltonian-connected* if there exists a Hamiltonian path between any two distinct nodes of G. A graph G is *bipartite* if $V(G)$ can be partitioned into two *partite sets* V_0 and V_1 such that $V_0 \cap V_1 = \emptyset$ and $E(G) \subseteq \{(x, y)|x \in V_0 \text{ and } y \in V_1\}$. A Hamiltonian bipartite graph G is *Hamiltonian-laceable* if there exists a Hamiltonian path between any two nodes from different partite sets [11]. An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. We say that G is *isomorphic to* H, written as $G \cong H$, if there is an isomorphism from G to H. An *automorphism* of G is an isomorphism from G to G. A graph G is *node-transitive* if for any two nodes u and v in $V(G)$, there is an automorphism that maps u to v. A graph G is *edge-transitive* if for any two edges e_1 and e_2 in $E(G)$, there is an automorphism that maps e_1 to e_2.

An *n-dimensional hypercube* Q_n is an n-regular graph with 2^n nodes and $n \cdot 2^{n-1}$ edges. Every node u in Q_n can be labelled by an n-bit binary string $u = u_n u_{n-1} \ldots u_1$ on the set $\{0, 1\}$. Two nodes are joined by an edge (also called *hypercube edge*) if and only if their binary strings differ in exactly one bit. Let $j \in \{1, 2, \ldots, n\}$. An edge in Q_n is called *j-dimensional* if the binary strings of its end-nodes differ in the j-th bit. We use E_j to denote the set of all j-dimensional edges in Q_n. For any $j \in \{1, 2, \ldots, n\}$, Q_n can be partitioned along dimension j into two $(n - 1)$-dimensional subcubes, Q^0_{n-1} and Q^1_{n-1}, which are induced by the nodes where the j-th bit is 0 and 1. For any two vertices x and y, we use $d_H(x, y)$ to denote the Hamming distance between x and y, which is the number of different positions between the binary strings of x and y. Note that an n-cube Q_n is a bipartite graph with two equal-size partite sets.

An *n-dimensional folded hypercube* (*folded n-cube* for short) FQ_n can be constructed from an n-cube Q_n by adding an edge (also called *complementary edge*) to every pair of nodes whose addresses are complementary (i.e., node $x = x_n x_{n-1} \ldots x_1$ and node $\bar{x} = \bar{x}_n \bar{x}_{n-1} \ldots \bar{x}_1$) in addition to its original n edges. We use E_c to denote the set of all complementary edges in FQ_n. Fig. 1 shows a folded 2-cube and a folded 3-cube. Notice that a folded n-cube can be partitioned into two node-disjoint $(n - 1)$-cubes by removing the hypercube edges in some dimension and all the complimentary edges. It has been shown that FQ_n is $(n+1)$-regular, $(n+1)$-connected, node-transitive and edge-transitive [14]. For convenience, let $F_j = \tilde{E}(FQ_n) \cap E_j$ for every $j \in \{1, 2, \ldots, n, c\}$ when referring to the faulty edges in FQ_n. An edge (u, v) is said to be *free* if (1) the edge (u, v) is fault-free, and (2) the end-nodes u and v are both fault-free.

3. Basic properties

This section reviews some properties of both hypercubes Q_n and folded hypercubes FQ_n which are used later on to introduce our method. The basic structural properties of hypercubes and folded hypercubes are listed as follows.

On the problem of finding a fault-free cycle in a faulty Q_n, Sengupta [10] considered the case in which both node and edge faults were allowed and showed the following result.

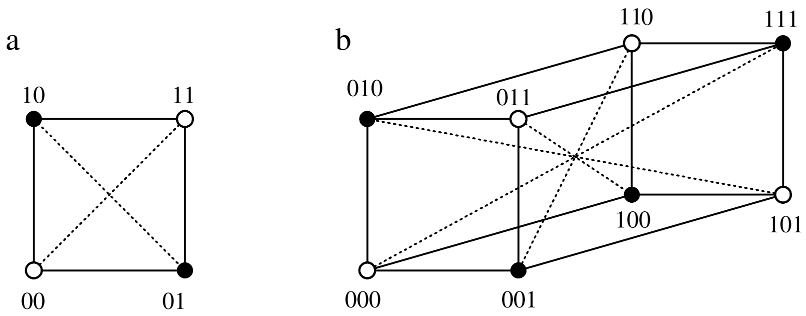

Fig. 1. Illustration of (a) FQ_2, and (b) FQ_3, where complementary edges are plotted with dotted lines.

Lemma 1 (*[10]*). *Q_n contains a fault-free cycle of length at least $2^n - 2|\tilde{V}(Q_n)|$ if (1) $|\tilde{V}(Q_n)| \geq 1$ or $|\tilde{E}(Q_n)| \leq n-2$ and (2) $|\tilde{V}(Q_n)| + |\tilde{E}(Q_n)| \leq n-1$, where $n \geq 3$.*

On the problem of finding fault-free paths in a faulty Q_n, Ma et al. [8] showed the following result.

Lemma 2 (*[8]*). *Let u and v be any two fault-free nodes in Q_n with $|\tilde{V}(Q_n)| + |\tilde{E}(Q_n)| \leq n-2$, where $n \geq 2$. Then, Q_n contains a fault-free uv-path of length l for each l satisfying $d_{Q_n}(u, v) + 2 \leq l \leq 2^n - 2|\tilde{V}(Q_n)| - 1$ and $2 \mid (l - d_{Q_n}(u, v))$. Moreover, Q_n contains a fault-free uv-path of length $d_{Q_n}(u, v)$ if $d_{Q_n}(u, v) \geq n-1$.*

The above lemma leads to the following corollary.

Corollary 1. *Let u and v be any two fault-free nodes in Q_n with $|\tilde{V}(Q_n)| + |\tilde{E}(Q_n)| \leq n-2$, where $n \geq 2$. Then, Q_n contains a fault-free uv-path of length $2^n - 2|\tilde{V}(Q_n)| - 1$ (resp. $2^n - 2|\tilde{V}(Q_n)| - 2$) when $d_H(u, v)$ is odd (resp. even).*

When $|\tilde{V}(Q_n)| = 0$ and $|\tilde{E}(Q_n)| = 0$ (i.e., Q_n contains no node and edge faults), we have the following result.

Corollary 2. *Let u and v be any two nodes in a fault-free Q_n, where $n \geq 2$. Then, Q_n contains a fault-free uv-path of length $2^n - 1$ (resp. $2^n - 2$) when $d_H(u, v)$ is odd (resp. even).*

In the case where only node faults are considered, Kueng et al. [6] showed the following result.

Lemma 3 (*[6]*). *Let u and v be any two fault-free nodes in Q_n with (1) $|\tilde{V}(Q_n)| \leq 2n-5$ and (2) every node of Q_n has at least two fault-free neighbors, where $n \geq 3$. Then, there exists a fault-free uv-path of length at least $2^n - 2|\tilde{V}(Q_n)| - 1$ (resp. $2^n - 2|\tilde{V}(Q_n)| - 2$) when $d_{Q_n}(u, v)$ is odd (resp. even).*

Tsai [12] showed the following results for finding two node-disjoint paths in a fault-free Q_n.

Lemma 4 (*[12]*). *Let X and Y be the partite sets of a fault-free Q_n, where $n \geq 2$. In addition, x and u are two distinct nodes of X; and y and v are two distinct nodes of Y. Then, there exist two node-disjoint paths $P_1[x, y]$ and $P_2[u, v]$ such that $V(P_1[x, y]) \cup V(P_2[u, v]) = V(Q_n)$.*

Lemma 5 (*[12]*). *Let X and Y be the partite sets of a fault-free Q_n, where $n \geq 3$. In addition, let x, u and v be three distinct nodes of X, and let y be a node of Y. Then, there exist two node-disjoint paths $P_1[x, y]$ and $P_2[u, v]$ such that $V(P_1[x, y]) \cup V(P_2[u, v]) = V(Q_n)$ except one node.*

Hsieh [5] showed the following result for finding fault-free paths in a faulty FQ_n.

Lemma 6 (*[5]*). *The following two statements hold:*

1. *Let u and v be any two nodes in FQ_n with $|\tilde{E}(FQ_n)| \leq n-1$, where $n \geq 1$ is odd. Then, FQ_n contains a fault-free uv-path of length $2^n - 1$ (resp. $2^n - 2$) when $d_H(u, v)$ is odd (resp. even).*
2. *Let u and v be any two nodes in FQ_n with $|\tilde{E}(FQ_n)| \leq n-2$, where $n \geq 2$ is even. Then, FQ_n contains a fault-free uv-path of length $2^n - 1$.*

Zhu et al. [15] showed the following results for finding the minimum length of a cycle in FQ_n.

Lemma 7 (*[15]*). *Any two nodes in FQ_n have two exactly common neighbors for $n \geq 4$ if they have.*

Lemma 8 (*[15]*). *The girth of FQ_n equals 4 for $n \geq 3$.*

The following lemma shows the edge-transitive property of FQ_n.

Lemma 9 (*[14]*). *There is an automorphism σ of FQ_n such that $\sigma(E_i) = E_j$ for any $i, j \in \{1, 2, \ldots, n, c\}$.*

This lemma derives the following corollary.

Corollary 3. *$FQ_n - E_j$ is isomorphic to Q_n, where $j \in \{1, 2, \ldots, n, c\}$.*

4. Fault-free paths in faulty folded hypercubes

In this section, we extend Hsieh's results described in Lemma 6.

Lemma 10. *Let u and v be any two fault-free nodes in FQ_3 with $|\tilde{V}(FQ_3)|+|\tilde{E}(FQ_3)| \le 2$. Then, FQ_3 contains a fault-free uv-path of length at least $7-2|\tilde{V}(FQ_3)|$ (resp. $6-2|\tilde{V}(FQ_3)|$) when $d_H(u,v)$ is odd (resp. even).*

Proof. The proof is presented in Appendix. □

Lemma 11. *Let u and v be any two fault-free nodes in FQ_n with $|\tilde{V}(FQ_n)| \le n-1$, where $n \ge 4$. Then, FQ_n contains a fault-free uv-path of length at least $2^n-2|\tilde{V}(FQ_n)|-1$ (resp. $2^n-2|\tilde{V}(FQ_n)|-2$) when $d_H(u,v)$ is odd (resp. even).*

Proof. First, we consider the case that $|\tilde{V}(FQ_n)| \le n-2$. Since FQ_n contains Q_n as a subgraph with extra complementary edge set, the result holds by applying Corollary 1 to FQ_n. Next, we consider the case that $|\tilde{V}(FQ_n)| = n-1$. Since FQ_n is $(n+1)$-regular and $|\tilde{V}(FQ_n)| = n-1$, every node in FQ_n has at least two fault-free neighbors in FQ_n. According to the number of fault-free neighbors of a node, we consider two scenarios:

Case 1: Every node in FQ_n has at least three fault-free neighbors in FQ_n.

By Corollary 3, we have that $FQ_n - E_c \cong Q_n$ with the same faulty nodes. Since every node in FQ_n has at least three fault-free neighbors in FQ_n, it has at least two fault-free neighbors in $FQ_n - E_c$. Since $n-1 \le 2n-5$ for all $n \ge 4$, then by Lemma 3, $FQ_n - E_c$ contains a fault-free uv-path of length at least $2^n-2|\tilde{V}(FQ_n)|-1$ (resp. $2^n-2|\tilde{V}(FQ_n)|-2$) when $d_H(u,v)$ is odd (resp. even). Since $FQ_n - E_c$ is a subgraph of FQ_n, the lemma holds.

Case 2: At least one node in FQ_n has exactly two fault-free neighbors in FQ_n.

Let x be a node with exactly two fault-free neighbors in FQ_n. Since $|\tilde{V}(FQ_n)| = n-1$, we have that x is fault-free, and all nodes in $\tilde{V}(FQ_n)$ are the faulty neighbors of x. We first claim that x is unique. Suppose, on the contrary, that there exists a node y, $y \ne x$, such that y has exactly two fault-free neighbors in FQ_n. Similar to x, we have that y is fault-free, and all nodes in $\tilde{V}(FQ_n)$ are also the faulty neighbors of y. Then, by Lemma 8, x and y are not adjacent. Moreover, by Lemma 7, x and y have exactly two common neighbors, which leads to a contradiction because $|\tilde{V}(FQ_n)| = n-1 \ge 3$ for $n \ge 4$. Therefore, such y does not exist (i.e., x is unique).

Let x' be a faulty neighbor of x and $(x,x') \in E_j$, where $j \in \{1,2,\ldots,n,c\}$. By Corollary 3, we have that $FQ_n - E_j \cong Q_n$. Moreover, every node in FQ_n has at least two fault-free neighbors in $FQ_n - E_j$. Then, by Lemma 3, $FQ_n - E_j$ contains a fault-free uv-path of length at least $2^n-2|\tilde{V}(FQ_n)|-1$ (resp. $2^n-2|\tilde{V}(FQ_n)|-2$) when $d_H(u,v)$ is odd (resp. even). Since $FQ_n - E_j$ is a subgraph of FQ_n, the lemma holds. □

Lemma 12. *Let u and v be any two fault-free nodes in FQ_n with $|\tilde{V}(FQ_n)| \ge 1$, $|\tilde{E}(FQ_n)| \ge 1$ and $|\tilde{V}(FQ_n)|+|\tilde{E}(FQ_n)| \le n-1$, where $n \ge 4$. Then, FQ_n contains a fault-free uv-path of length at least $2^n-2|\tilde{V}(FQ_n)|-1$ (resp. $2^n-2|\tilde{V}(FQ_n)|-2$) when $d_H(u,v)$ is odd (resp. even).*

Proof. Let e be a faulty edge in FQ_n. Since FQ_n is edge-transitive, without loss of generality, we can assume that $e \in E_c$. Next, since the binary strings of u and v differ in the jth bit for some $j \in \{1,2,\ldots,n\}$, we can partition FQ_n into two $(n-1)$-subcubes, Q^0_{n-1} and Q^1_{n-1}, along dimension j such that one subcube contains u and the other contains v. Without loss of generality, assume that $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Note that e remains in E_c. According to the distribution of faulty nodes and faulty edges, we consider the following cases:

Case 1: $|\tilde{V}(Q^0_{n-1})|+|\tilde{E}(Q^0_{n-1})| = n-2$.

We have that $|F_j| = |\tilde{V}(Q^1_{n-1})| = |\tilde{E}(Q^1_{n-1})| = 0$ and $|F_c| = 1$. Note that $|\tilde{V}(Q^0_{n-1})| = |\tilde{V}(FQ_n)| \ge 1$. Then, by Lemma 1, Q^0_{n-1} contains a fault-free cycle C of length at least $2^{n-1}-2|\tilde{V}(Q^0_{n-1})|$. According to whether u is contained in C, we consider two subcases:

Case 1.1: u is contained in C.

Let $0w$ be a neighbor of u in C such that $1w \ne v$, and $P_0[u,0w]$ be the path by removing $(u,0w)$ from C (see Fig. 2(a)(b)). Since $d_H(u,v)$ is odd (resp. even), $d_H(1w,v)$ is also odd (resp. even). Then, by Corollary 2, Q^1_{n-1} contains a fault-free path $P_1[1w,v]$ of length $2^{n-1}-1$ (resp. $2^{n-1}-2$) when $d_H(1w,v)$ is odd (resp. even). Therefore, $\langle u, P_0[u,0w], 0w, 1w, P_1[1w,v], v\rangle$ is a fault-free uv-path of length at least $\overbrace{2^{n-1}-2|\tilde{V}(Q^0_{n-1})|-1}^{|P_0[u,0w]|}+1+\overbrace{2^{n-1}-1}^{|P_1[1w,v]|} = 2^n-2|\tilde{V}(FQ_n)|-1$ (resp. $2^n-2|\tilde{V}(FQ_n)|-2$) when $d_H(u,v)$ is odd (resp. even).

Case 1.2: u is not contained in C.

Let $u = 0u'$. According to whether $1u'$ is v, we consider two scenarios:

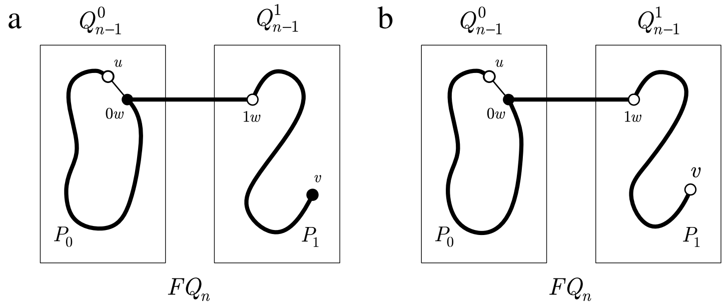

Fig. 2. Illustration of Case 1.1 in the proof of Lemma 12. Here, (a) $d_H(u, v)$ is odd; and (b) $d_H(u, v)$ is even.

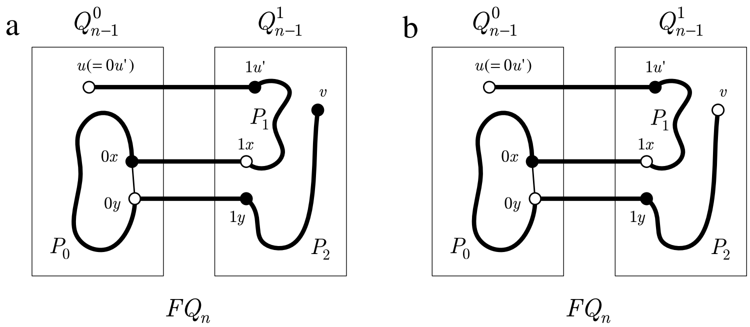

Fig. 3. Illustration of Case 1.2.1 in the proof of Lemma 12. Here, (a) $d_H(u, v)$ is odd; and (b) $d_H(u, v)$ is even.

Case 1.2.1: $1u' \neq v$.

Since $\left\lfloor \frac{2^{n-1}-2|\tilde{V}(Q^0_{n-1})|}{2} \right\rfloor \geq \frac{2^{n-1}-2(n-2)}{2} = 2^{n-2} - n + 2 > 1$ for $n \geq 4$, there exists an edge $(0x, 0y)$ in C such that $\{1x, 1y\} \cap \{v\} = \emptyset$ (see Fig. 3(a)(b)). Let $P_0[0x, 0y]$ be the path by removing $(0x, 0y)$ from C. Without loss of generality, assume that $d_H(u, 0x)$ is odd, which implies that (1) $d_H(1u', 1x)$ is odd, and (2) $d_H(1y, v)$ is even (resp. odd) when $d_H(u, v)$ is odd (resp. even). Then, by Lemma 5 (resp. Lemma 4), there exist two node-disjoint paths $P_1[1u', 1x]$ and $P_2[1y, v]$ such that the sum of their lengths equals $2^{n-1} - 3$ (resp. $2^{n-1} - 2$) when $d_H(1y, v)$ is even (resp. odd). Therefore, $\langle u, 1u', P_1[1u', 1x], 1x, 0x, P_0[0x, 0y], 0y, 1y, P_2[1y, v], v\rangle$ is a fault-free uv-path of length at least $\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[0x,0y]|} + 3 + \overbrace{2^{n-1} - 3}^{|P_1[1u',1x]|+|P_2[1y,v]|} = 2^n - 2|\tilde{V}(FQ_n)| - 1$ (resp. $2^n - 2|\tilde{V}(FQ_n)| - 2$) when $d_H(u, v)$ is odd (resp. even).

Case 1.2.2: $1u' = v$.

Since Q^0_{n-1} is $(n-1)$-regular and $|\tilde{V}(Q^0_{n-1})| + |\tilde{E}(Q^0_{n-1})| \leq n - 2$, u has neighbor $0t$ such that $(u, 0t)$ is a free edge. According to whether $0t$ is contained in C, we consider two scenarios:

Case 1.2.2.1: $0t$ is contained in C.

Let $0w$ be a neighbor of $0t$ in C, and let $P_0[0t, 0w]$ be the path by removing $(0t, 0w)$ from C (see Fig. 4(a)). Since $d_H(u, v)$ is odd, $d_H(1w, v)$ is even. Then, by Corollary 2, Q^1_{n-1} contains a fault-free path $P_1[1w, v]$ of length $2^{n-1} - 2$. Therefore, $\langle u, 0t, P_0[0t, 0w], 0w, 1w, P_1[1w, v], v\rangle$ is a fault-free uv-path of length at least $\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[0t,0w]|} + 1 + 1 + \overbrace{2^{n-1} - 2}^{|P_1[1w,v]|} = 2^n - 2|\tilde{V}(FQ_n)| - 1$.

Case 1.2.2.2: $0t$ is not contained in C.

Let $(0x, 0y)$ be an edge in C, and let $P_0[0x, 0y]$ be the path by removing $(0x, 0y)$ from C (see Fig. 4(b)). Without loss of generality, assume that $d_H(u, 0x)$ is even, which implies that $d_H(1t, 1x)$ and $d_H(1y, v)$ are both odd. Then, by Lemma 4, there exist two node-disjoint paths $P_1[1t, 1x]$ and $P_2[1y, v]$ such that the sum of their lengths equals $2^{n-1} - 2$. Therefore, $\langle u, 0t, 1t, P_1[1t, 1x], 1x, 0x, P_0[0x, 0y], 0y, 1y, P_2[1y, v], v\rangle$ is a fault-free uv-path of length at least

$$\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[0x,0y]|} + 1 + 3 + \overbrace{2^{n-1} - 2}^{|P_1[1t,1x]|+|P_2[1y,v]|} \geq 2^n - 2|\tilde{V}(FQ_n)| - 1.$$

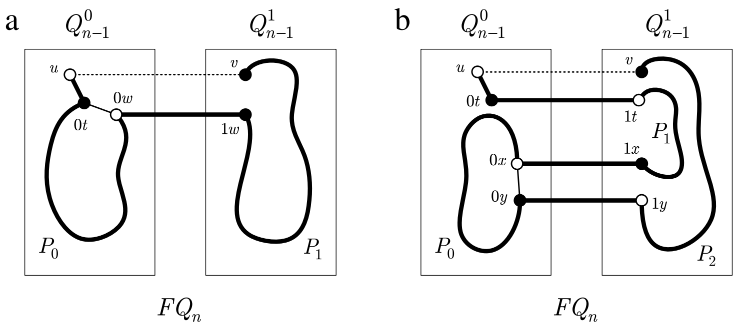

Fig. 4. Illustration of Case 1.2.2.1 and Case 1.2.2.2 in the proof of Lemma 12.

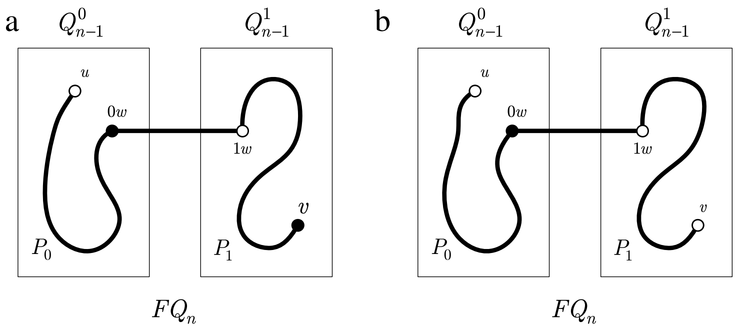

Fig. 5. Illustration of Case 3 in the proof of Lemma 12. Here, (a) $d_H(u, v)$ is odd; and (b) $d_H(u, v)$ is even.

Case 2: $|\tilde{V}(Q^1_{n-1})| + |\tilde{E}(Q^1_{n-1})| = n - 2$.

The proof is similar to that of Case 1 and hence omitted here.

Case 3: $|\tilde{V}(Q^0_{n-1})| + |\tilde{E}(Q^0_{n-1})| \le n - 3$ and $|\tilde{V}(Q^1_{n-1})| + |\tilde{E}(Q^1_{n-1})| \le n - 3$.

Let $W = \{(0w, 1w) | d_H(u, 0w) \text{ is odd}\}$ be a matching. Since $|W| = \frac{2^{n-1}}{2} = 2^{n-2} > (n-2) + \overbrace{1}^{\text{node } v} = n - 1$ for $n \ge 4$, there exists a free edge $(0w, 1w) \in W$ such that $1w \ne v$ (see Fig. 5(a)(b)). Note that $d_H(u, 0w)$ is odd and $d_H(1w, v)$ is odd (resp. even) when $d_H(u, v)$ is odd (resp. even). Then, by Corollary 1, Q^0_{n-1} contains a fault-free path $P_0[u, 0w]$ of length $2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1$, and Q^1_{n-1} contains a fault-free path $P_1[1w, v]$ of length $2^{n-1} - 2|\tilde{V}(Q^1_{n-1})| - 1$ (resp. $2^{n-1} - 2|\tilde{V}(Q^1_{n-1})| - 2$) when $d_H(1w, v)$ is odd (resp. even). Therefore, $\langle u, P_0[u, 0w], 0w, 1w, P_1[1w, v], v\rangle$ is a fault-free uv-path of length at least $\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[u,0w]|} + 1 + \overbrace{2^{n-1} - 2|\tilde{V}(Q^1_{n-1})| - 1}^{|P_1[1w,v]|} = 2^n - 2|\tilde{V}(FQ_n)| - 1$ (resp. $2^n - 2|\tilde{V}(FQ_n)| - 2$) when $d_H(u, v)$ is odd (resp. even).

Combining the above cases complete the proof. □

Based on (a) Lemmas 11 and 12, and Lemma 6(1) when $n \ge 5$ is odd, (b) Lemma 10 when $n = 3$, and (c) Lemma 6(1) when $n = 1$, we have the following result.

Theorem 1. *Let u and v be any two fault-free nodes in FQ_n with $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n - 1$, where $n \ge 1$ is odd. Then, FQ_n contains a fault-free uv-path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ (resp. $2^n - 2|\tilde{V}(FQ_n)| - 2$) when $d_H(u, v)$ is odd (resp. even).*

Lemma 13. *Let u and v be any two fault-free nodes in FQ_n with $|\tilde{V}(FQ_n)| \ge 1$ and $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n - 2$, where $n \ge 4$ is even. Then, FQ_n contains a fault-free uv-path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$.*

Proof. Since the binary strings of u and v differ in the jth bit for some $j \in \{1, 2, \ldots, n\}$, we can partition FQ_n into two $(n-1)$-subcubes, Q^0_{n-1} and Q^1_{n-1}, along dimension j such that one subcube contains u and the other contains v. Without loss of generality, assume that $u \in V(Q^0_{n-1})$ and $v \in V(Q^1_{n-1})$. Note that since $n \ge 4$ is even, it is known that for every node w in FQ_n, $d_H(w, \overline{w})$ is even. According to the distribution of faulty nodes and edges, we consider the following four cases:

Case 1: $|\tilde{V}(Q^0_{n-1})| + |\tilde{E}(Q^0_{n-1})| = n - 2$.

We have that $|F_j| = |F_c| = |\tilde{V}(Q^1_{n-1})| = |\tilde{E}(Q^1_{n-1})| = 0$. Note that $|\tilde{V}(Q^0_{n-1})| = |\tilde{V}(FQ_n)| \ge 1$. By applying Lemma 1, Q^0_{n-1} contains a fault-free cycle C of length at least $2^{n-1} - 2|\tilde{V}(Q^0_{n-1})|$. According to whether u is contained

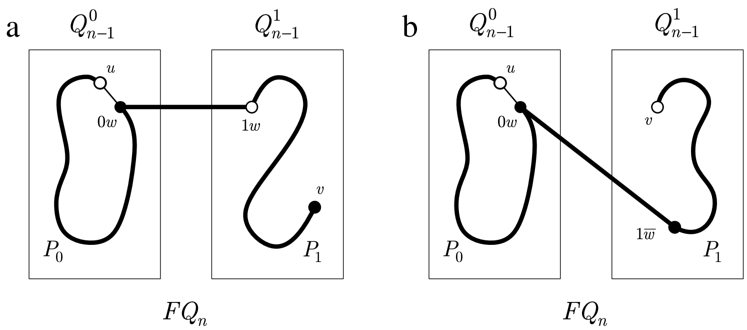

Fig. 6. Illustration of Case 1.1 in the proof of Lemma 13. Here, (a) $d_H(u, v)$ is odd; and (b) $d_H(u, v)$ is even.

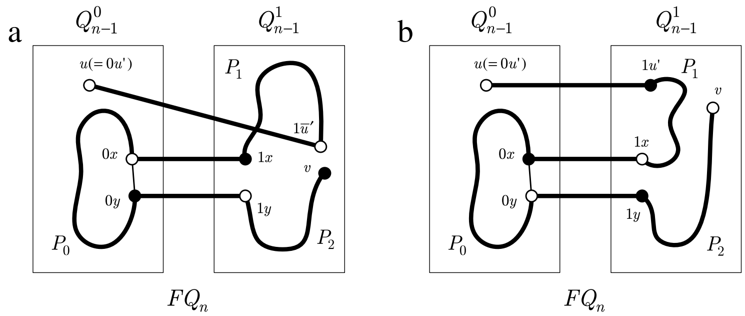

Fig. 7. Illustration of Case 1.2 in the proof of Lemma 13. Here, (a) $d_H(u, v)$ is odd; and (b) $d_H(u, v)$ is even.

in C, we consider the following two subcases:

Case 1.1: u is contained in C.

Let $0w$ be a neighbor of u in C, and let $P_0[u, 0w]$ be the path of removing the edge $(u, 0w)$ from C. Since $|F_j| = |F_c| = |\tilde{V}(Q^1_{n-1})| = |\tilde{E}(Q^1_{n-1})| = 0$, $(0w, 1w)$ and $(0w, 1\overline{w})$ are both free edges. If $d_H(0w, v)$ is even, we connect $0w$ to $1w$; otherwise, we connect $0w$ to $1\overline{w}$ (see Fig. 6(a)(b)). We observe that $d_H(1w, v)$ (resp. $d_H(1\overline{w}, v)$) is odd when $d_H(0w, v)$ is even (resp. odd). Then, by Corollary 2, Q^1_{n-1} contains a fault-free path $P_1[1w, v]$ (resp. $P_1[1\overline{w}, v]$) of length $2^n - 1$ when $d_H(0w, v)$ is even (resp. odd). Therefore, $P[u, v] = \langle u, P_0[u, 0w], 0w, 1w, P_1[1w, v], v\rangle$ (resp. $P[u, v] = \langle u, P_0[u, 0w], 0w, 1\overline{w}, P_1[1\overline{w}, v], v\rangle$) forms a fault-free uv-path of length at least $\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[u,0w]|} + 1 + \overbrace{2^{n-1} - 1}^{|P_1[1w,v]|\ (resp. P_1[1\overline{w},v])} = 2^n - 2|\tilde{V}(FQ_n)| - 1$ when $d_H(u, v)$ is odd (resp. even).

Case 1.2: u is not contained in C.

Let $u = 0u'$. Since $|F_j| = |F_c| = |\tilde{V}(Q^1_{n-1})| = |\tilde{E}(Q^1_{n-1})| = 0$, $(u, 1u')$ and $(u, 1\overline{u'})$ are both free edges. If $d_H(u, v)$ is odd, we connect u to $1\overline{u'}$; otherwise we connect u to $1\overline{u'}$ (see Fig. 7(a)(b)). We observe that $d_H(1\overline{u'}, v)$ (resp. $d_H(1u', v)$) is odd when $d_H(u, v)$ is odd (resp. even). Since (1) $\left\lfloor \frac{2^{n-1}-2|\tilde{V}(Q^0_n)|}{2} \right\rfloor \geq \frac{2^{n-1}-2(n-2)}{2} = 2^{n-2} - n + 2 \geq 2$ for $n \geq 4$ and (2) $d_H(1\overline{u'}, v)$ is odd when $d_H(u, v)$ is odd, there exists an edge $(0x, 0y)$ in C such that $\{1x, 1y\} \cap \{1\overline{u'}, v\} = \emptyset$ (resp. $\{1x, 1y\} \cap \{v\} = \emptyset$) when $d_H(u, v)$ is odd (resp. even). Let $P_0[0x, 0y]$ be the path by removing $(0x, 0y)$ from C. Without loss of generality, assume that $d_H(u, 0x)$ is even (resp. odd) when $d_H(u, v)$ is odd (resp. even), which implies that $d_H(1x, 1\overline{u'})$ (resp. $d_H(1x, 1u')$) and $d_H(1y, v)$ are both odd. Then, by Lemma 4, Q^1_{n-1} contains two node-disjoint paths $P_1[1\overline{u'}, 1x]$ (resp. $P_1[1u', 1x]$) and $P_2[1y, v]$ such that the sum of their lengths equals $2^n - 2$ when $d_H(u, v)$ is odd (resp. even). Therefore, $P[u, v] = \langle u, 1\overline{u'}, P_1[1\overline{u'}, 1x], 1x, 0x, P_0[0x, 0y], 0y, 1y, P_2[1y, v], v\rangle$ (resp. $P[u, v] = \langle u, 1u', P_1[1u', 1x], 1x, 0x, P_0[0x, 0y], 0y, 1y, P_2[1y, v], v\rangle$) is a fault-free uv-path of length at least $\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[0x,0y]|} + 3 + \overbrace{2^{n-1} - 2}^{|P_1[1u',1x]|+|P_2[1y,v]|} \geq 2^n - 2|\tilde{V}(FQ_n)| - 1$ when $d_H(u, v)$ is odd (resp. even).

Case 2: $|\tilde{V}(Q^1_{n-1})| + |\tilde{E}(Q^1_{n-1})| = n - 2$.

The proof is similar to that of case 1 and hence omitted here.

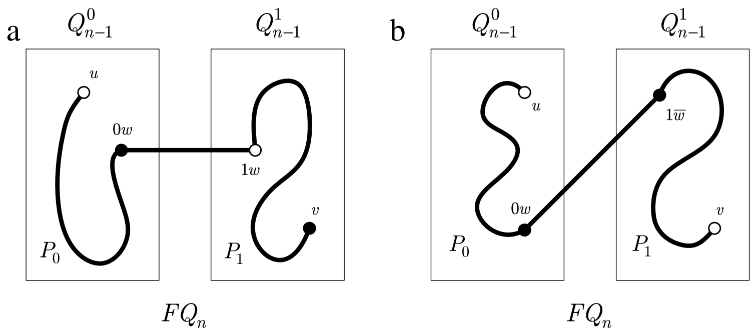

Fig. 8. Illustration of Case 3 in the proof of Lemma 13. Here, (a) $d_H(u, v)$ is odd; and (b) $d_H(u, v)$ is even.

Case 3: $|\tilde{V}(Q^0_{n-1})| + |\tilde{E}(Q^0_{n-1})| \le n-3$ and $|\tilde{V}(Q^1_{n-1})| + |\tilde{E}(Q^1_{n-1})| \le n-3$.

If $d_H(u, v)$ is odd, let $W = \{(0w, 1w) | \, d_H(u, 0w) \text{ is odd}\}$; otherwise, let $W = \{(0w, 1\overline{w}) | \, d_H(u, 0w) \text{ is odd}\}$ (see Fig. 8(a)(b)). Obviously, W is a matching in $E_j \cup E_c$. Since $|W| = \frac{2^{n-1}}{2} = 2^{n-2} > n-2$ for $n \ge 4$, there exists a free edge $(0w, 1w)$ (resp. $(0w, 1\overline{w})$) in W when $d_H(u, v)$ is odd (resp. even). Note that $d_H(u, 0w)$ and $d_H(1w, v)$ (resp. $d_H(1\overline{w}, v)$) are both odd when $d_H(u, v)$ is odd (resp. even). Then, by Corollary 1, Q^0_{n-1} contains a fault-free path $P_0[u, 0w]$ of length $2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1$, and Q^1_{n-1} contains a fault-free path $P_1[1w, v]$ (resp. $P_1[1\overline{w}, v]$) of length $2^{n-1} - 2|\tilde{V}(Q^1_{n-1})| - 1$ when $d_H(u, v)$ is odd (resp. even). Therefore, $P[u, v] = \langle u, P_0[u, 0w], 0w, 1w, P_1[1w, v], v\rangle$ (resp. $P[u, v] = \langle u, P_0[u, 0w], 0w, 1\overline{w}, P_1[1\overline{w}, v], v\rangle$) is a fault-free uv-path of length

$$\overbrace{2^{n-1} - 2|\tilde{V}(Q^0_{n-1})| - 1}^{|P_0[u,0w]|} + 1 + \overbrace{2^{n-1} - 2|\tilde{V}(Q^1_{n-1})| - 1}^{|P_1[1w,v]| \ (resp. |P_1[1\overline{w},v]|)} = 2^n - 2|\tilde{V}(FQ_n)| - 1$$ when $d_H(u, v)$ is odd (resp. even).

Combining the above three cases completes the proof. □

Based on Lemmas 6 and 13 (when $n \ge 4$ is even), and Lemma 6(2) when $n = 2$, we obtain the following result.

Theorem 2. *Let u and v be any two fault-free nodes in FQ_n with $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n-2$, where $n \ge 2$ is even. Then, FQ_n contains a fault-free uv-path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$.*

5. Concluding remarks

Fault tolerance is an important research topic in the area of the multi-process computer systems, and many studies have focused on the node-fault tolerant or edge-fault tolerant properties of some specific networks. In this paper, we extend Hsieh's result [5] to obtain two further fault-tolerant properties about fault-free paths in a faulty folded n-cube as follows:

1. FQ_n contains a fault-free path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ (resp. $2^n - 2|\tilde{V}(FQ_n)| - 2$) between any two fault-free nodes of odd (resp. even) distance if $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n-1$, where $n \ge 1$ is odd.
2. FQ_n contains a fault-free path of length at least $2^n - 2|\tilde{V}(FQ_n)| - 1$ between any two fault-free nodes if $|\tilde{V}(FQ_n)| + |\tilde{E}(FQ_n)| \le n-2$, where $n \ge 2$ is even.

Our results imply that the algorithms designed for paths can also be executed efficiently on a faulty folded hypercube with both faulty nodes and edges.

Appendix

According to the number of $\tilde{V}(FQ_3)$, we consider the following cases. First, if $|\tilde{V}(FQ_3)| = 0$ (i.e., $|\tilde{E}(FQ_3)| \le 2$), the lemma holds from Lemma 6. Next, consider the case that $|\tilde{V}(FQ_3)| = 1$, i.e., $|\tilde{E}(FQ_3)| \le 1$. Since FQ_3 is node-transitive, we assume that the faulty node is 000. By the symmetry of FQ_3, we only need to consider the faulty edge in $\{(001, 011), (110, 111), (001, 110)\}$. All fault-free uv-paths of length $7 - 2 \cdot 1 = 5$ (resp. $6 - 2 \cdot 1 = 4$) when $d_H(u, v)$ is odd (resp. even) are demonstrated in Table 1.

Lastly, consider the case that $|\tilde{V}(FQ_3)| = 2$. Since FQ_3 is also node-transitive, we assume one of the faulty nodes is 000. By the symmetry of FQ_3, we only need to consider the other faulty node in $\{001, 011, 111\}$. All fault-free uv-paths of length $7 - 2 \cdot 2 = 3$ (resp. $6 - 2 \cdot 2 = 2$) when $d_H(u, v)$ is odd (resp. even) are demonstrated in Table 2.

Table 1
uv-paths in FQ_3 with $|\tilde{V}(FQ_3)| = 1$ and $|\tilde{E}(FQ_3)| = 1$.

Faulty edge	u	v	uv-path
(001, 011)	011	001	⟨011, 010, 110, 100, 101, 001⟩
(001, 011)	011	010	⟨011, 111, 101, 100, 110, 010⟩
(001, 011)	011	100	⟨011, 010, 110, 111, 101, 100⟩
(001, 011)	011	111	⟨011, 010, 110, 100, 101, 111⟩
(001, 011)	110	001	⟨110, 010, 011, 111, 101, 001⟩
(001, 011)	110	010	⟨110, 100, 101, 111, 011, 010⟩
(001, 011)	110	100	⟨110, 010, 011, 111, 101, 100⟩
(001, 011)	110	111	⟨110, 010, 011, 100, 101, 111⟩
(001, 011)	101	001	⟨101, 111, 011, 010, 110, 001⟩
(001, 011)	101	010	⟨101, 100, 110, 111, 011, 010⟩
(001, 011)	101	100	⟨101, 111, 011, 010, 110, 100⟩
(001, 011)	101	111	⟨101, 100, 110, 010, 011, 111⟩
(001, 011)	011	110	⟨011, 111, 101, 100, 110⟩
(001, 011)	011	101	⟨011, 111, 110, 100, 101⟩
(001, 011)	110	101	⟨110, 010, 011, 111, 101⟩
(001, 011)	001	010	⟨001, 101, 100, 110, 010⟩
(001, 011)	001	100	⟨001, 101, 111, 110, 100⟩
(001, 011)	001	111	⟨001, 101, 100, 110, 111⟩
(001, 011)	010	100	⟨010, 110, 111, 101, 100⟩
(001, 011)	010	111	⟨010, 110, 100, 101, 111⟩
(001, 011)	100	111	⟨100, 110, 010, 011, 111⟩
(110, 111)	011	001	⟨011, 010, 110, 100, 101, 001⟩
(110, 111)	011	010	⟨011, 111, 101, 100, 110, 010⟩
(110, 111)	011	100	⟨011, 010, 110, 001, 101, 100⟩
(110, 111)	011	111	⟨011, 010, 110, 100, 101, 111⟩
(110, 111)	110	001	⟨110, 010, 011, 111, 101, 001⟩
(110, 111)	110	010	⟨110, 100, 101, 111, 011, 010⟩
(110, 111)	110	100	⟨110, 010, 011, 111, 101, 100⟩
(110, 111)	110	111	⟨110, 010, 011, 001, 101, 111⟩
(110, 111)	101	001	⟨101, 100, 110, 010, 011, 001⟩
(110, 111)	101	010	⟨101, 100, 110, 001, 011, 010⟩
(110, 111)	101	100	⟨101, 111, 011, 010, 110, 100⟩
(110, 111)	101	111	⟨101, 100, 110, 010, 011, 111⟩
(110, 111)	011	110	⟨011, 111, 101, 100, 110⟩
(110, 111)	011	101	⟨011, 010, 110, 100, 101⟩
(110, 111)	110	101	⟨110, 010, 011, 111, 101⟩
(110, 111)	001	010	⟨001, 101, 100, 110, 010⟩
(110, 111)	001	100	⟨001, 011, 010, 110, 100⟩
(110, 111)	001	111	⟨001, 110, 010, 011, 111⟩
(110, 111)	010	100	⟨010, 011, 111, 101, 100⟩
(110, 111)	010	111	⟨010, 110, 100, 101, 111⟩
(110, 111)	100	111	⟨100, 110, 010, 011, 111⟩
(001, 110)	011	001	⟨011, 010, 110, 100, 101, 001⟩
(001, 110)	011	010	⟨011, 111, 101, 100, 110, 010⟩
(001, 110)	011	100	⟨011, 010, 110, 111, 101, 100⟩
(001, 110)	011	111	⟨011, 010, 110, 100, 101, 111⟩
(001, 110)	110	001	⟨110, 010, 011, 111, 101, 001⟩
(001, 110)	110	010	⟨110, 100, 101, 111, 011, 010⟩
(001, 110)	110	100	⟨110, 010, 011, 111, 101, 100⟩
(001, 110)	110	111	⟨110, 010, 011, 001, 101, 111⟩
(001, 110)	101	001	⟨101, 111, 110, 010, 011, 001⟩
(001, 110)	101	010	⟨101, 100, 110, 111, 011, 010⟩
(001, 110)	101	100	⟨101, 111, 011, 010, 110, 100⟩
(001, 110)	101	111	⟨101, 100, 110, 010, 011, 111⟩
(001, 110)	011	110	⟨011, 111, 101, 100, 110⟩
(001, 110)	011	101	⟨011, 111, 110, 100, 101⟩
(001, 110)	110	101	⟨110, 010, 011, 111, 101⟩
(001, 110)	001	010	⟨001, 101, 100, 110, 010⟩
(001, 110)	001	100	⟨001, 101, 111, 110, 100⟩
(001, 110)	001	111	⟨001, 101, 100, 110, 111⟩
(001, 110)	010	100	⟨010, 110, 111, 101, 100⟩
(001, 110)	010	111	⟨010, 110, 100, 101, 111⟩
(001, 110)	100	111	⟨100, 110, 010, 011, 111⟩

Table 2
uv-paths in FQ_3 with $|\tilde{V}(FQ_3)| = 2$.

Faulty nodes	u	v	uv-path
{000, 001}	011	010	⟨011, 111, 110, 010⟩
{000, 001}	011	100	⟨011, 111, 110, 100⟩
{000, 001}	011	111	⟨011, 010, 110, 111⟩
{000, 001}	101	010	⟨101, 111, 110, 010⟩
{000, 001}	101	100	⟨101, 111, 110, 100⟩
{000, 001}	101	111	⟨101, 100, 110, 111⟩
{000, 001}	110	010	⟨110, 111, 011, 010⟩
{000, 001}	110	100	⟨110, 111, 101, 100⟩
{000, 001}	110	111	⟨110, 100, 101, 111⟩
{000, 001}	011	101	⟨011, 111, 101⟩
{000, 001}	011	110	⟨011, 111, 110⟩
{000, 001}	101	110	⟨101, 111, 110⟩
{000, 001}	010	100	⟨010, 110, 100⟩
{000, 001}	010	111	⟨010, 110, 111⟩
{000, 001}	100	111	⟨100, 110, 111⟩
{000, 011}	101	001	⟨101, 111, 110, 001⟩
{000, 011}	101	010	⟨101, 111, 110, 010⟩
{000, 011}	101	100	⟨101, 111, 110, 100⟩
{000, 011}	101	111	⟨101, 100, 110, 111⟩
{000, 011}	110	001	⟨110, 111, 101, 001⟩
{000, 011}	110	010	⟨110, 111, 101, 010⟩
{000, 011}	110	100	⟨110, 111, 101, 100⟩
{000, 011}	110	111	⟨110, 100, 101, 111⟩
{000, 001}	101	110	⟨101, 111, 110⟩
{000, 001}	001	010	⟨001, 110, 010⟩
{000, 001}	001	100	⟨001, 101, 100⟩
{000, 001}	001	111	⟨001, 101, 111⟩
{000, 001}	010	100	⟨010, 110, 100⟩
{000, 001}	010	111	⟨010, 110, 111⟩
{000, 001}	100	111	⟨100, 110, 111⟩
{000, 011}	011	001	⟨011, 100, 101, 001⟩
{000, 011}	011	010	⟨011, 100, 110, 010⟩
{000, 011}	011	100	⟨011, 010, 110, 100⟩
{000, 011}	101	001	⟨101, 100, 110, 001⟩
{000, 011}	101	010	⟨101, 001, 011, 010⟩
{000, 011}	101	100	⟨101, 010, 110, 100⟩
{000, 011}	110	001	⟨110, 010, 011, 001⟩
{000, 011}	110	010	⟨110, 100, 011, 010⟩
{000, 011}	110	100	⟨110, 001, 101, 100⟩
{000, 011}	011	101	⟨011, 001, 101⟩
{000, 011}	011	110	⟨011, 010, 110⟩
{000, 011}	101	110	⟨101, 100, 110⟩
{000, 011}	001	010	⟨001, 011, 010⟩
{000, 011}	001	100	⟨001, 101, 100⟩
{000, 011}	010	100	⟨010, 110, 100⟩

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